Expansions of finite algebras and their congruence lattices

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This article is dedicated to Ralph Freese and Bill Lampe.

Abstract. In this paper, we present a novel approach to the construction of new finite algebras and describe the congruence lattices of these algebras. Given a finite algebra $\langle B_0, \dots \rangle$, let $B_1, B_2, \dots, B_K$ be sets that either intersect B_0 or intersect each other at certain points. We construct an *overalgebra* $\langle A, F_A \rangle$, by which we mean an expansion of $\langle B_0, \dots \rangle$ with universe $A = B_0 \cup B_1 \cup \cdots \cup B_K$, and a certain set F_A of unary operations that includes mappings e_i satisfying $e_i^2 = e_i$ and $e_i(A) = B_i$, for $0 \leqslant i \leqslant K$. We explore two such constructions and prove results about the shape of the new congruence lattices $\mathrm{Con}\langle A, F_A \rangle$ that result. Thus, descriptions of some new classes of finitely representable lattices is one contribution of this paper. Another, perhaps more significant, contribution is the announcement of a novel approach to the discovery of new classes of representable lattices, the full potential of which we have only begun to explore.

1. Introduction

A lattice L is called *algebraic* if it is complete and if every element $x \in L$ is the supremum of the compact elements below x. Equivalently, a lattice is algebraic if and only if it is the congruence lattice of an algebra. (The fact that every algebraic lattice is the congruence lattice of an algebra was proved by Grätzer and Schmidt in [6].) An important and long-standing open problem in universal algebra is to characterize those lattices that are isomorphic to congruence lattices of *finite* algebras. We call such lattices *finitely representable*. Until this problem is resolved, our understanding of finite algebras is incomplete, since, given an arbitrary finite algebra, we cannot say whether there are any restrictions on the shape of its congruence lattice. If we find a finite lattice which does not occur as the congruence lattice of a finite algebra (as many suspect we will), then we can finally declare that such restrictions do exist.

The main contribution of this paper is the description and analysis of a new procedure for generating finite lattices which are, by construction, finitely representable. Roughly speaking, we start with an arbitrary finite algebra $\mathbf{B} = \langle B, \dots \rangle$, with known congruence lattice $\mathrm{Con}\,\mathbf{B}$, and we let $B_1, B_2, \dots, B_K$

be sets that either intersect B or intersect each other at certain points. The choice of intersection points plays an important role which we will describe in detail later. We then construct an *overalgebra* $\mathbf{A} = \langle A, F_A \rangle$, by which we mean an expansion of $\mathbf{B}$ with universe $A = B \cup B_1 \cup \cdots \cup B_K$, and a certain set F_A of unary operations that includes idempotent mappings e and e_i satisfying $e(A) = B$ and $e_i(A) = B_i$.

Given our interest in the problem mentioned above, the important consequence of this procedure is the new (finitely representable) lattice $\operatorname{Con} \mathbf{A}$ that it produces. The shape of this lattice is, of course, determined by the shape of $\operatorname{Con} \mathbf{B}$, the choice of intersection points of the B_i, and the unary operations chosen for inclusion in F_A. In this paper, we describe two constructions of this type and prove some results about the shape of the congruence lattices of the resulting overalgebras. However, it is likely that we have barely scratched the surface of useful constructions that are possible with this approach.

Before giving an overview of the paper, we describe the seminal example that provided the impetus for this work. In the spring of 2011, while our research seminar was mainly focused on the finite congruence lattice representation problem, we were visited by Peter Jipsen who initiated the ambitious project of cataloging every small finite lattice L for which there is a known finite algebra $\mathbf{A}$ with $\operatorname{Con} \mathbf{A} \cong L$. Before long, we had identified such finite representations for all lattices of order seven or less, except for the two lattices appearing in Figure 1. Ralph Freese then discovered a way to construct an
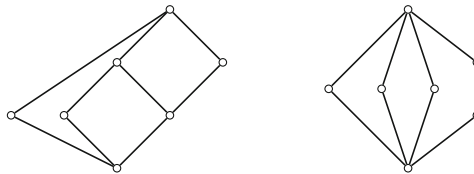

FIGURE 1. Lattices of order 7 with no obvious finite algebraic representation.

algebra that has the second of these as its congruence lattice. The idea is to start with an algebra $\mathbf{B} = \langle B, \ldots \rangle$ having congruence lattice $\operatorname{Con} \mathbf{B} \cong M_4$, expand the universe to a larger set, $A = B \cup B_1 \cup B_2$, and then define the right set F_A of operations on A so that the congruence lattice of $\mathbf{A} = \langle A, F_F \rangle$ will be an M_4 with one atom "doubled"—that is, $\operatorname{Con} \mathbf{A}$ will be the second lattice in Figure 1.

In this paper, we formalize this approach and extend it in two ways. The first is a generalization of the original overalgebra construction. The second is a construction based on one suggested by Bill Lampe that addresses a basic limitation of the original procedure. For each of these constructions, we prove results that describe the congruence lattices of the resulting overalgebras.

Here is a brief outline of the remaining sections of the paper: In Section 2, we prove a lemma that simplifies the analysis of the structure of the newly

enlarged congruence lattice and its relation to the original congruence lattice. In Section 3, we define *overalgebra* and in Section 3.1, we give a formal description of the first overalgebra construction mentioned above. We then describe Freese's example in detail before proving some general results about the congruence lattices of such overalgebras. We conclude Section 3.1, with an example demonstrating the utility of the first overalgebra construction. Section 3.2 presents a second overalgebra construction which overcomes a basic limitation of the first. The last section discusses the impact that our results have on the main problem—the finite congruence lattice representation problem—as well as the inherent limitations of this approach.

2. Residuation lemma

Let $\mathbf{A} = \langle A, \dots \rangle$ be an algebra with congruence lattice $\mathrm{Con}\langle A, \dots \rangle$. Recall that a *clone* on a non-void set A is a set of operations on A that contains the projection operations and is closed under compositions. The *clone of term operations* of the algebra $\mathbf{A}$, denoted by $\mathrm{Clo}(\mathbf{A})$, is the smallest clone on A containing the basic operations of $\mathbf{A}$. The *clone of polynomial operations* of $\mathbf{A}$, denoted by $\mathrm{Pol}(\mathbf{A})$, is the clone generated by the basic operations of $\mathbf{A}$ and the constant unary maps on A. The set of n-ary members of $\mathrm{Pol}(\mathbf{A})$ is denoted by $\mathrm{Pol}_n(\mathbf{A})$. It is not hard to show that $\mathrm{Con}\langle A, \dots \rangle = \mathrm{Con}\langle A, \mathrm{Clo}(\mathbf{A}) \rangle = \mathrm{Con}\langle A, \mathrm{Pol}(\mathbf{A}) \rangle = \mathrm{Con}\langle A, \mathrm{Pol}_1(\mathbf{A}) \rangle$; see [8, Theorem 4.18].

Suppose $e \in \mathrm{Pol}_1(\mathbf{A})$ is a unary polynomial satisfying $e^2(x) = e(x)$ for all $x \in A$. Define $B = e(A)$ and $F_B = \{ef|_B \mid f \in \mathrm{Pol}_1(\mathbf{A})\}$, and consider the unary algebra $\mathbf{B} = \langle B, F_B \rangle$. (In the definition of F_B, we could have used $\mathrm{Pol}(\mathbf{A})$ instead of $\mathrm{Pol}_1(\mathbf{A})$, and then our discussion would not be limited to unary algebras. However, we are mainly concerned with congruence lattices, so we lose nothing by restricting the scope in this way.)

Péter Pálfy and Pavel Pudlák proved in [10, Lemma 1] that the restriction mapping $|_B$, defined on $\mathrm{Con}\,\mathbf{A}$ by $\alpha|_B = \alpha \cap B^2$, is a lattice epimorphism of $\mathrm{Con}\,\mathbf{A}$ onto $\mathrm{Con}\,\mathbf{B}$. In [7], Ralph McKenzie developed the foundations of what would become tame congruence theory, and the Pálfy-Pudlák lemma played a seminal role in this development. In his presentation of the lemma, McKenzie introduced the mapping $\widehat{}$, defined on $\mathrm{Con}\,\mathbf{B}$ as follows:

$$\widehat{\beta} = \{(x, y) \in A^2 \mid (ef(x), ef(y)) \in \beta \text{ for all } f \in \mathrm{Pol}(\mathbf{A})\}.$$

Throughout this paper, we use a definition of $\widehat{}$ that is effectively the same. Whenever $\mathbf{A} = \langle A, \dots \rangle$ and $\mathbf{B} = \langle B, \dots \rangle$ are algebras with $B = e(A)$ for some $e^2 = e \in \mathrm{Pol}_1(\mathbf{A})$, we take the map $\widehat{} \colon \mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ to mean

$$\widehat{\beta} = \{(x, y) \in A^2 \mid (ef(x), ef(y)) \in \beta \text{ for all } f \in \mathrm{Pol}_1(\mathbf{A})\}. \qquad (2.1)$$

It is not hard to see that $\widehat{}$ maps $\mathrm{Con}\,\mathbf{B}$ into $\mathrm{Con}\,\mathbf{A}$. For example, if $(x, y) \in \widehat{\beta}$ and $g \in \mathrm{Pol}_1(\mathbf{A})$, then for all $f \in \mathrm{Pol}_1(\mathbf{A})$, we have $(efg(x), efg(y)) \in \beta$, so $(g(x), g(y)) \in \widehat{\beta}$.

For each $\beta \in \operatorname{Con} \mathbf{B}$, let $\beta^* = \operatorname{Cg}^{\mathbf{A}}(\beta)$. That is, $^*\colon \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ is the congruence generation operator restricted to the set $\operatorname{Con} \mathbf{B}$. The following lemma concerns the three mappings, $|_B$, $\widehat{}$, and *. The third statement of the lemma, which follows from the first two, will be useful in the later sections of the paper.

Lemma 2.1.

(i) $^*\colon \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ *is a residuated mapping with residual* $|_B$.

(ii) $|_B\colon \operatorname{Con} \mathbf{A} \to \operatorname{Con} \mathbf{B}$ *is a residuated mapping with residual* $\widehat{}$.

(iii) *For all* $\alpha \in \operatorname{Con} \mathbf{A}$ *and for all* $\beta \in \operatorname{Con} \mathbf{B}$, $\quad \beta = \alpha|_B \iff \beta^* \leqslant \alpha \leqslant \widehat{\beta}$.
    In particular, $\beta^*|_B = \beta = \widehat{\beta}|_B$.

Proof. We first recall the definition of *residuated mapping*. If X and Y are partially ordered sets, and if $f\colon X \to Y$ and $g\colon Y \to X$ are order preserving maps, then the following are equivalent:

(a) $f\colon X \to Y$ is a *residuated mapping* with *residual* $g\colon Y \to X$.

(b) For all $x \in X$ and $y \in Y$, $f(x) \leqslant y \iff x \leqslant g(y)$.

(c) $g \circ f \geqslant \operatorname{id}_X$ and $f \circ g \leqslant \operatorname{id}_Y$, where id_S denotes the identity map on the set S.

The definition says that for each $y \in Y$, there is a unique $x \in X$ that is maximal with respect to the property $f(x) \leqslant y$, and the maximum is given by $x = g(y)$. Thus, (i) is equivalent to: for all $\alpha \in \operatorname{Con} \mathbf{A}$ and $\beta \in \operatorname{Con} \mathbf{B}$,

$$\beta^* \leqslant \alpha \iff \beta \leqslant \alpha|_B. \tag{2.2}$$

This is easily verified, as follows: If $\beta^* \leqslant \alpha$ and $(x, y) \in \beta$, then $(x, y) \in \beta^* \leqslant \alpha$ and $(x, y) \in B^2$, so $(x, y) \in \alpha|_B$. If $\beta \leqslant \alpha|_B$, then $\beta^* \leqslant (\alpha|_B)^* \leqslant \operatorname{Cg}^{\mathbf{A}}(\alpha) = \alpha$.

Statement (ii) is equivalent to: for all $\alpha \in \operatorname{Con} \mathbf{A}$ and $\beta \in \operatorname{Con} \mathbf{B}$,

$$\alpha|_B \leqslant \beta \iff \alpha \leqslant \widehat{\beta}. \tag{2.3}$$

This is also easy to check. For suppose $\alpha|_B \leqslant \beta$ and $(x, y) \in \alpha$. Then $(ef(x), ef(y)) \in \alpha$ for all $f \in \operatorname{Pol}_1(\mathbf{A})$ and $(ef(x), ef(y)) \in B^2$; therefore, $(ef(x), ef(y)) \in \alpha|_B \leqslant \beta$, so $(x, y) \in \widehat{\beta}$. Suppose $\alpha \leqslant \widehat{\beta}$ and $(x, y) \in \alpha|_B$. Then $(x, y) \in \alpha \leqslant \widehat{\beta}$, so $(ef(x), ef(y)) \in \beta$ for all $f \in \operatorname{Pol}_1(\mathbf{A})$, including $f = \operatorname{id}_A$, so $(e(x), e(y)) \in \beta$. But $(x, y) \in B^2$, so $(x, y) = (e(x), e(y)) \in \beta$.

Combining (2.2) and (2.3), we obtain statement (iii) of the lemma. $\square$

The above Lemma 2.1 was inspired by the two approaches to the proof of [10, Lemma 1]. In the original paper, * is used, while McKenzie uses the $\widehat{}$ operator. Both β^* and $\widehat{\beta}$ are mapped onto β by the restriction map $|_B$, so the restriction map is indeed onto $\operatorname{Con} \mathbf{B}$. However, our lemma emphasizes the fact that the interval $[\beta^*, \widehat{\beta}] = \{\alpha \in \operatorname{Con} \mathbf{A} \mid \beta^* \leqslant \alpha \leqslant \widehat{\beta}\}$ is precisely the set of congruences for which $\alpha|_B = \beta$. In other words, the $|_B$-inverse image of β is $[\beta^*, \widehat{\beta}]$. This fact plays a central role in the theory developed in this paper. For the sake of completeness, we conclude this section by verifying that we can obtain [10, Lemma 1] from Lemma 2.1.

Corollary 2.2 (cf. [10, Lemma 1]). *The mapping $|_B \colon \operatorname{Con} \mathbf{A} \to \operatorname{Con} \mathbf{B}$ is onto and preserves meets and joins.*

Proof. Given $\beta \in \operatorname{Con} \mathbf{B}$, each $\theta \in \operatorname{Con} \mathbf{A}$ in the interval $[\beta^*, \widehat{\beta}]$ is mapped to $\theta|_B = \beta$, so $|_B$ is clearly onto. That $|_B$ preserves meets is obvious, so we just check that $|_B$ is join preserving. Since $|_B$ is order preserving, we have, for all $S \subseteq \operatorname{Con} \mathbf{A}$, $\bigvee \theta|_B \leqslant (\bigvee \theta)|_B$, where joins are over all $\theta \in S$. The opposite inequality follows from (2.3) above. Indeed, by (2.3), we have

$$\left(\bigvee \theta\right)\Big|_B \leqslant \bigvee \theta|_B \quad \Longleftrightarrow \quad \bigvee \theta \leqslant \left(\bigvee \theta|_B\right)^{\widehat{}}$$

and the last inequality holds by another application of (2.3): if $\eta \in S$, then

$$\eta \leqslant \left(\bigvee \theta|_B\right)^{\widehat{}} \quad \Longleftrightarrow \quad \eta|_B \leqslant \bigvee \theta|_B. \qquad \square$$

3. Overalgebras

In the previous section, we started with an algebra $\mathbf{A}$ and considered a subreduct $\mathbf{B}$ with universe $B = e(A)$, the image of an idempotent unary polynomial of $\mathbf{A}$. In this section, we start with a fixed finite algebra $\mathbf{B} = \langle B, \ldots \rangle$ and consider various ways to construct an *overalgebra*, that is, an algebra $\mathbf{A} = \langle A, F_A \rangle$ having $\mathbf{B}$ as a subreduct where $B = e(A)$ for some $e^2 = e \in F_A$. Beginning with a specific finite algebra $\mathbf{B}$, our goal is to understand what congruence lattices $\operatorname{Con} \mathbf{A}$ can be built up from $\operatorname{Con} \mathbf{B}$ by expanding the algebra $\mathbf{B}$ in this way.

3.1. Overalgebras I. Let B be a finite set and let F be a set of unary maps that take B into itself. Consider the algebra $\mathbf{B} = \langle B, F \rangle$ with universe B and basic operations F. Let T denote a finite sequence $t_1, t_2, \ldots, t_K$ of elements of B (possibly with repetitions). For each $1 \leqslant i \leqslant K$, let B_i be a set of the same cardinality as B and intersecting B at the i-th point of the sequence T; that is, $B_i \cap B = \{t_i\}$. Assume also that $B_i \cap B_j = \emptyset$ if $t_i \neq t_j$, and $B_i \cap B_j = \{t_i\}$ if $t_i = t_j$. Occasionally, for notational convenience, we use the label $B_0 = B$, and for extra clarity we may write F_B in place of F.

For each $1 \leqslant i \leqslant K$, let $\pi_i \colon B \to B_i$ be a bijection that leaves t_i fixed. That is, $\pi_i(t_i) = t_i$ and otherwise, π_i is an arbitrary (but fixed) bijection. Let $\pi_0 = \operatorname{id}_B$, the identity map on B. For $b \in B$, we often use the label b^i to denote $\pi_i(b)$. The map π_i and the operations F induce a set F_i of unary operations on B_i as follows: to each $f \in F$ corresponds the operation $f^{\pi_i} \colon B_i \to B_i$ defined by $f^{\pi_i} = \pi_i f \pi_i^{-1}$. It is easy to see that $\mathbf{B}_i = \langle B_i, F_i \rangle$ and $\mathbf{B} = \langle B, F \rangle$ are isomorphic algebras. (Indeed, π_i is a bijection of the universes that respects the interpretation of the basic operations: $\pi_i f(b) = \pi_i f(\pi_i^{-1} \pi_i b) = f^{\pi_i}(\pi_i b)$.)

Let $A = \bigcup_{i=0}^{K} B_i$ and for $0 \leqslant k \leqslant K$, define the maps $e_k \colon A \to A$ by $e_0(x) = \pi_{\iota x}^{-1}(x)$ and $e_k(x) = \pi_k e_0(x)$, where ιx denotes the smallest index k such that $x \in B_k$. Thus, e_0 acts as the identity on B_0 ($= B$) and maps

each $b^i \in B_i$ onto the corresponding element $b \in B$. Similarly, e_k acts as the identity on B_k and maps each $b^i \in B_i$ onto the corresponding element $b^k \in B_k$.

The next set of maps that we define on A will be based on a given partition of the elements of T. Actually, since T is a sequence, possibly with repetitions, we must consider instead a partition of the *indices* of T. Let $\mathscr{T} = |\mathscr{T}_1|\mathscr{T}_2|\cdots|\mathscr{T}_N|$ denote this partition of the index set $\{1, 2, \ldots, K\}$. We now define maps $s_n \colon A \to A$ based on the partition $\mathscr{T}$, as follows: for each $1 \leqslant n \leqslant N$, let

$$s_n(x) = \begin{cases} t_i, & \text{if } x \in B_i \text{ for some } i \in \mathscr{T}_n, \\ x, & \text{otherwise.} \end{cases} \tag{3.1}$$

(Recall, B_i intersects B at the point t_i. If the index of this point belongs to block $\mathscr{T}_n$, then s_n collapses B_i onto t_i; otherwise, s_n acts as the identity on B_i. A typical special case of our construction takes the partition $\mathscr{T}$ to be the trivial partition consisting of a single block $\mathscr{T}_1 = 1, 2, \ldots, K$. In this case, $N = 1$; the map s_1 acts as the identity on B and collapses each set B_i onto the point at which it intersects B.)

Let $F_A = \{fe_0 \mid f \in F_B\} \cup \{e_k \mid 0 \leqslant k \leqslant K\} \cup \{s_n \mid 1 \leqslant n \leqslant N\}$; we define the algebra $\mathbf{A} = \langle A, F_A \rangle$, which we call an *overalgebra of* $\mathbf{B}$. Occasionally, we refer to subsets $B_i \subseteq A$ as the *subreduct universes* of $\mathbf{A}$; we call $\mathbf{B}$ the *base algebra* of $\mathbf{A}$ and T the sequence of *tie-points* of $\mathbf{A}$.

Before proving some general results about the basic structure of the congruence lattice of an overalgebra, we present the first example, discovered by Ralph Freese, of a finite algebra with congruence lattice isomorphic to the second lattice in Figure 1. (All computational experiments described in this paper rely on two open source programs, GAP [5] and the Universal Algebra Calculator [4]. Source code for reproducing our results is available at http://williamdemeo.wordpress.com/software/overalgebras/.)

Example 3.1. Consider a finite permutational algebra $\mathbf{B} = \langle B, F \rangle$ with congruence lattice $\mathrm{Con}\,\mathbf{B} \cong M_4$ (Figure 2). The right regular S_3-set—i.e., the group S_3 acting on itself by right multiplication—is one such algebra. In GAP,

```
gap> g:=SymmetricGroup(3); # The symmetric group on 3 letters.
gap> B:=[(),(1,2,3),(1,3,2),(1,2),(1,3),(2,3)]; # The elements of g.
gap> G:=Action(g,B,OnRight); # Group([(1,2,3)(4,5,6), (1,4)(2,6)(3,5)])
```

In our computational examples, we prefer to use 0-offset notation since this is the convention used by the UACalc software. Thus, in the present example, we define the universe of the S_3-set described above to be $B = \{0, 1, \ldots, 5\}$, instead of $\{1, 2, \ldots, 6\}$. As such, the partitions of B corresponding to nontrivial congruence relations of the algebra are given as follows:

```
gap> for b in AllBlocks(G) do Print(Orbit(G,b,OnSets)-1,"\n"); od;
[ [ 0, 1, 2 ], [ 3, 4, 5 ] ]
[ [ 0, 3 ], [ 1, 4 ], [ 2, 5 ] ]
[ [ 0, 4 ], [ 1, 5 ], [ 2, 3 ] ]
[ [ 0, 5 ], [ 1, 3 ], [ 2, 4 ] ]
```

Next, we create an algebra in UACalc format using the two generators of the group as basic operations. This can be accomplished using our GAP script `gap2uacalc.g` as follows:

```
gap> Read("gap2uacalc.g");
gap> gset2uacalc([G,"S3action"]);
```

This creates a UACalc file specifying an algebra which has universe $B = \{0, 1, \ldots, 5\}$ and two basic unary operations $g_0 = (1\ 2\ 0\ 4\ 5\ 3)$ and $g_1 = (3\ 5\ 4\ 0\ 2\ 1)$. These operations are the permutations $(0, 1, 2)(3, 4, 5)$ and $(0, 3)(1, 5)(2, 4)$, which are (0-offset versions of) the generators of the S_3-set appearing in the GAP output above. Figure 2 displays the congruence lattice of this algebra.
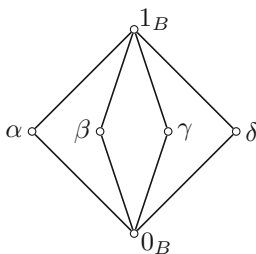

FIGURE 2. Congruence lattice of the right regular S_3-set, where congruences α, β, γ, and δ correspond to the partitions $|0, 1, 2|3, 4, 5|$, $|0, 3|1, 4|2, 5|$, $|0, 4|1, 5|2, 3|$, and $|0, 5|1, 3|2, 4|$, respectively.

We now construct an overalgebra that "doubles" the congruence α corresponding to the partition $|0, 1, 2|3, 4, 5|$. (From now on, we will identify a congruence relation with the corresponding partition of the underlying set, and write, for example, $\alpha = |0, 1, 2|3, 4, 5|$.) Note that α can be generated by the pair $(0, 2)$—that is, $\alpha = \mathrm{Cg}^{\mathbf{B}}(0, 2)$—and the desired doubling of α can be achieved by choosing the tie-point sequence $t_1, t_2 = 0, 2$ in the overalgebra construction. (Theorem 3.2 below will reveal why this choice works. Note that, in this simple example, no non-trivial partition of the tie-point indices is required.)

Our GAP function `Overalgebra` carries out the construction, and is invoked as follows:

```
gap> Read("Overalgebras.g");
gap> Overalgebra([G, [0,2]]);
```

This results in an algebra with universe $A = B \cup B_1 \cup B_2 = \{0, 1, 2, 3, 4, 5\} \cup \{0, 6, 7, 8, 9, 10\} \cup \{11, 12, 2, 13, 14, 15\}$, and the following operations:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
e_0	0	1	2	3	4	5	1	2	3	4	5	0	1	3	4	5
e_1	0	6	7	8	9	10	6	7	8	9	10	0	6	8	9	10
e_2	11	12	2	13	14	15	12	2	13	14	15	11	12	13	14	15
s_1	0	1	2	3	4	5	0	0	0	0	0	2	2	2	2	2
$g_0 e_0$	1	2	0	4	5	3	2	0	4	5	3	1	2	4	5	3
$g_1 e_0$	3	5	4	0	2	1	5	4	0	2	1	3	5	0	2	1

If $F_A = \{e_0, e_1, e_2, s_1, g_0 e_0, g_1 e_0\}$, then the algebra $\langle A, F_A \rangle$ has the congruence lattice shown in Figure 3.
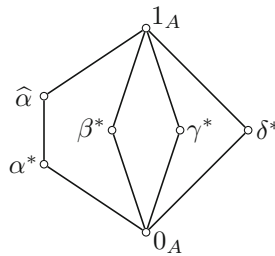

FIGURE 3. Congruence lattice of the overalgebra of the S_3-set with intersection points 0 and 2.

The congruence relations in Figure 3 are as follows:

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$
$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$
$$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$$
$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$
$$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|.$$

It is important to note that the resulting congruence lattice depends on our choice of which congruence to "expand," which is controlled by our specification of the tie-points of the overalgebra. For example, suppose we want one of the congruences having three blocks, say, $\beta = \mathrm{Cg}^{\mathbf{B}}(0, 3) = |0, 3|1, 4|2, 5|$, to have a non-trivial $|_B$-inverse image $[\beta^*, \widehat{\beta}]$. Then we could specify the sequence t_1, t_2 to be $0, 3$ by invoking the command `Overalgebra([G, [0,3]])`. (In this instance, we arrive at the same congruence lattice if we let t_1, t_2 be $2, 5$ or $1, 4$.) This produces an overalgebra with universe

$$A = B \cup B_1 \cup B_2 = \{0, 1, 2, 3, 4, 5\} \cup \{0, 6, 7, 8, 9, 10\} \cup \{11, 12, 13, 3, 14, 15\}$$

and congruence lattice shown in Figure 4, where

$$\alpha^* = |0, 1, 2, 6, 7|3, 4, 5, 14, 15|8, 9, 10|11, 12, 13|$$
$$\widehat{\beta} = |0, 3, 8, 11|1, 4|2, 5|6, 9, 12, 14|7, 10, 13, 15|$$
$$\beta_\varepsilon = |0, 3, 8, 11|1, 4|2, 5|6, 9, 12, 14|7, 10|13, 15|$$
$$\beta_{\varepsilon'} = |0, 3, 8, 11|1, 4|2, 5|6, 9|7, 10, 13, 15|12, 14|$$
$$\beta^* = |0, 3, 8, 11|1, 4|2, 5|6, 9|7, 10|12, 14|13, 15|$$
$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$
$$\delta^* = |0, 5, 10|1, 3, 12|2, 4|6, 8|7, 9|11, 15|13, 14|.$$
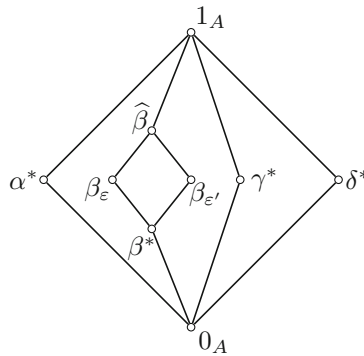

FIGURE 4. Congruence lattice of the overalgebra of the S_3-set with intersection points 0 and 3.

Before proceeding, we require some more notation. Given a set X, we let $\mathrm{Eq}(X)$ denote the set of equivalence relations on the set X, and we let $\mathbf{1}, \mathbf{2}$, and $\mathbf{3}$ denote the abstract one, two, and three element lattices, respectively. For example, $\mathrm{Eq}(1) \cong \mathbf{1}$ and $\mathrm{Eq}(2) \cong \mathbf{2}$. Also, let us agree that $\mathrm{Eq}(0) \cong \mathbf{1}$.

Below, we present a theorem that describes the basic structure of the congruence lattice of an overalgebra constructed as described in this section. In particular, the theorem explains why the interval $[\alpha^*, \widehat{\alpha}] \cong \mathbf{2}$ appears in the first example above, while $[\beta^*, \widehat{\beta}] \cong \mathbf{2} \times \mathbf{2}$ appears in the second.

If $\beta \in \mathrm{Con}\,\mathbf{B}$ and if we denote by $C_1, \ldots, C_m$ the congruence classes of β, then for each $1 \leqslant i \leqslant K$, the isomorphism π_i yields a corresponding congruence relation $\beta^i \in \mathrm{Con}\,\mathbf{B}_i$, which partitions the set B_i into congruence classes $\pi_i(C_1), \ldots, \pi_i(C_m)$. We denote the r-th class of β^i by $C_r^i = \pi_i(C_r)$.

If $t_1, t_2, \ldots, t_K$ is the sequence of tie-points on which an overalgebra is based, we let $\mathcal{I}_r = \{i \mid t_i \in C_r\}$ denote the indices of those tie-points that lie in the r-th congruence class of β. Note that if $i \in \mathcal{I}_r$, then the r-th congruence class of β^i intersects the r-th congruence class of β at the tie-point t_i, and we have $\{t_i\} = C_r^i \cap C_r$, and $t_i/\beta^i = C_r^i$. Finally, we let β^0 denote β.

Theorem 3.2. *Let* $\mathbf{B}$ *be a finite unary algebra and let* $\mathbf{A}$ *be the overalgebra of* $\mathbf{B}$ *constructed, as described above, from the sequence* $t_1, t_2, \ldots, t_K$ *and the partition* $|\mathcal{I}_1|\mathcal{I}_2|\cdots|\mathcal{I}_N|$ *of the indices* $\{1, 2, \ldots, K\}$. *Given* $\beta \in \mathrm{Con}\,\mathbf{B}$ *with congruence classes* $C_1, \ldots, C_m$, *define the relations*

$$\beta^\star = \bigcup_{k=0}^{K} \beta^k \cup \bigcup_{r=1}^{m} \left(C_r \cup \bigcup_{i \in \mathcal{I}_r} C_r^i \right)^2, \tag{3.2}$$

and

$$\widetilde{\beta} = \beta^\star \cup \bigcup_{n=1}^{N} \bigcup_{r=1}^{m} \bigcup_{\substack{\ell=1 \\ \ell \neq r}}^{m} \left(\bigcup_{i \in \mathcal{I}_n \cap \mathcal{I}_r} C_\ell^i \right)^2. \tag{3.3}$$

Then

  (i) $\beta^\star = \beta^*$, *the minimal* $\theta \in \mathrm{Con}\,\mathbf{A}$ *such that* $\theta|_B = \beta$;
 (ii) $\widetilde{\beta} = \widehat{\beta}$, *the maximal* $\theta \in \mathrm{Con}\,\mathbf{A}$ *such that* $\theta|_B = \beta$;
(iii) *the interval* $[\beta^\star, \widetilde{\beta}]$ *in* $\mathrm{Con}\,\mathbf{A}$ *contains every equivalence relation on* A *between* $\beta^\star$ *and* $\widetilde{\beta}$, *and satisfies*

$$[\beta^\star, \widetilde{\beta}] = \{\theta \in \mathrm{Eq}(A) \mid \beta^\star \subseteq \theta \subseteq \widetilde{\beta}\} \cong \prod_{r=1}^{m} \prod_{n=1}^{N} (\mathrm{Eq}\,|\mathcal{I}_n \cap \mathcal{I}_r|)^{m-1}. \tag{3.4}$$

Remarks. If a block C_r of β contains no tie-point, then $\mathcal{I}_r = \emptyset$. In such cases, for all n, $\mathrm{Eq}\,|\mathcal{I}_n \cap \mathcal{I}_r| = \mathrm{Eq}\,|\mathcal{I}_r| = \mathrm{Eq}(0) \cong \mathbf{1}$. Similarly, if the block C_r contains a single tie-point, then $|\mathcal{I}_n \cap \mathcal{I}_r| \leqslant 1$ for all n, and again the corresponding terms in (3.4) contribute nothing to the product. In fact, the term of index (n, r) in (3.4) is nontrivial if and only if there are at least two tie-points (say, t_i, t_j) in block C_r with indices (say, i, j) in block $\mathcal{I}_n$.

Next, we remark that the blocks of the relation $\beta^\star$ are, for $1 \leqslant r \leqslant m$,

$$C_r \cup \bigcup_{i \in \mathcal{I}_r} C_r^i \quad \text{and} \quad C_r^j \quad (1 \leqslant j \leqslant K, \; j \notin \mathcal{I}_r). \tag{3.5}$$

The blocks of $\widetilde{\beta}$ are, for $1 \leqslant r \leqslant m$ and $1 \leqslant n \leqslant N$,

$$C_r \cup \bigcup_{i \in \mathcal{I}_r} C_r^i \quad \text{and} \quad \bigcup_{j \in \mathcal{I}_n \cap \mathcal{I}_r} C_s^j \quad (1 \leqslant s \leqslant m, \; s \neq r). \tag{3.6}$$

In the special case $N = 1$ (the trivial partition), (3.6) is simply

$$C_r \cup \bigcup_{i \in \mathcal{I}_r} C_r^i \quad \text{and} \quad \bigcup_{j \in \mathcal{I}_r} C_s^j \quad (1 \leqslant s \leqslant m, \; s \neq r).$$

Figures 5 and 6 depict the blocks of $\beta^\star$ and $\widetilde{\beta}$ for a small example illustrating the special case $N = 1$. These diagrams serve as a rough guide to intuition, and make the proof of Theorem 3.2 easier to follow.

Proof of Theorem 3.2. (i): We first check that $\beta^\star \in \mathrm{Con}\,\mathbf{A}$. It is easy to see that $\beta^\star$ is an equivalence relation on A. To see that it is a congruence relation of $\mathbf{A}$, we prove $f(\beta^\star) \subseteq \beta^\star$ for all $f \in F_A$, where

$$F_A = \{ge_0 \mid g \in F_B\} \cup \{e_k \mid 0 \leqslant k \leqslant K\} \cup \{s_n \mid 1 \leqslant n \leqslant N\}.$$

That is, we prove: if $(x, y) \in \beta^\star$ and $f \in F_A$, then $(f(x), f(y)) \in \beta^\star$.
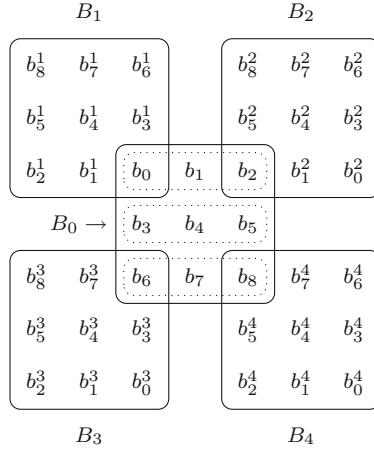
FIGURE 5. The universe $A = B_0 \cup \cdots \cup B_4$ of an example overalgebra; dotted lines surround each congruence class C_i of β; the tie-points are $t_1, t_2, t_3, t_4 = b_0, b_2, b_6, b_8$; if we have $\mathcal{I}_r = \{i \mid t_i \in C_r\}$, then $\mathcal{I}_1 = \{1,2\}$, $\mathcal{I}_2 = \emptyset$, and $\mathcal{I}_3 = \{3,4\}$.
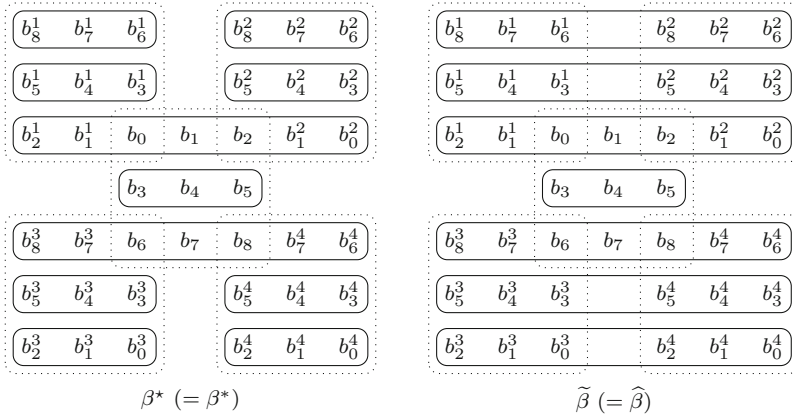

FIGURE 6. Solid lines delineate the congruence classes of $\beta^\star$ (left) and $\widetilde{\beta}$ (right); dotted lines surround the sets B_i.

Fix $0 \leqslant k \leqslant K$ and consider the action of the map e_k on the congruence classes of $\beta^\star$ described in (3.5). For all $1 \leqslant r \leqslant m$, we have $e_k(C_r^j) = C_r^k$, for all $0 \leqslant j \leqslant K$, and so $e_k(C_r \cup \bigcup_{i \in \mathcal{I}_r} C_r^i) = C_r^k$. It follows that for all $0 \leqslant k \leqslant K$, the map e_k takes blocks of $\beta^\star$ into blocks of $\beta^\star$; thus, $e_k(\beta^\star) \subseteq \beta^\star$. In particular, e_0 takes blocks of $\beta^\star$ into blocks of β, so $ge_0(\beta^\star) \subseteq \beta^\star$ for all $g \in F_B$. Next, note that for $1 \leqslant n \leqslant N$, $0 \leqslant r \leqslant K$, and $0 \leqslant j \leqslant K$, the map s_n acts as the identity on C_r^j when $j \notin \mathcal{T}_n$, otherwise it maps C_r^j to the point t_j. It follows that $s_n(C_r \cup \bigcup_{i \in \mathcal{I}_r} C_r^i) \subseteq C_r \cup \bigcup_{i \in \mathcal{I}_r} C_r^i$, since the union is over

$\mathcal{I}_r = \{i \mid t_i \in C_r\}$. Therefore, $s_n(\beta^\star) \subseteq \beta^\star$ for all $1 \leqslant n \leqslant N$, and we have proved $\beta^\star \in \mathrm{Con}\,\mathbf{A}$.

Since $\beta = \beta_0 \subseteq \beta^\star$, we have $\beta^\star|_B = \beta$. Therefore, $\beta^\star \geqslant \beta^*$, by the residuation lemma of Section 2. To complete the proof of (i), we show that $\beta \subseteq \eta \in \mathrm{Con}\,\mathbf{A}$ implies $\beta^\star \leqslant \eta$. If $\beta \subseteq \eta \in \mathrm{Con}\,\mathbf{A}$, then $\bigcup \beta^k \subseteq \eta$, since for each $0 \leqslant k \leqslant K$ and for each pair $(u, v) \in \beta^k$ there exists a pair $(x, y) \in \beta$ with $(e_k(x), e_k(y)) = (u, v)$, and $(x, y) \in \beta \subseteq \eta$ implies $(u, v) = (e_k(x), e_k(y)) \in \eta$. The second term of (3.2) belongs to η by transitivity. Indeed, suppose (x, y) is an arbitrary element of that term, with, say, $(x, t_i) \in \beta^i$, $(y, t_j) \in \beta^j$, and $(t_i, t_j) \in \beta$. As we just observed, β, β^i, and β^j are subsets of η, so $x \, \beta^i \, t_i \, \beta \, t_j \, \beta^j \, y$ implies $(x, y) \in \eta$.

(ii): Clearly, $\widetilde{\beta}$ is an equivalence on A. To see that it is a congruence relation of $\mathbf{A}$, we prove $f(\widetilde{\beta}) \subseteq \widetilde{\beta}$ for all $f \in F_A$. Fix $(x, y) \in \widetilde{\beta}$. If $(x, y) \in \beta^\star$, then $(f(x), f(y)) \in \beta^\star$ holds for all $f \in F_A$, as in part (i). If $(x, y) \notin \beta^\star$, then $x \in C_\ell^j$ and $y \in C_\ell^k$ for some $j, k \in \mathcal{T}_n \cap \mathcal{I}_r$ with $1 \leqslant n \leqslant N$, $1 \leqslant r \leqslant m$, and $\ell \neq r$. In this case, x and y are in the ℓ-th blocks of B_j and B_k, respectively, so for each $0 \leqslant i \leqslant K$, both $e_i(x)$ and $e_i(y)$ belong to C_ℓ^i, so $(e_i(x), e_i(y)) \in \beta^i$. In particular, $(e_0(x), e_0(y)) \in \beta$, so $(ge_0(x), ge_0(y)) \in \beta$ for all $g \in F_B$. Also, $(s_n(x), s_n(y)) = (t_j, t_k) \in C_r \times C_r \subseteq \beta$, while $(s_{n'}(x), s_{n'}(y)) = (x, y) \in \widetilde{\beta}$ when $n' \neq n$. This proves that $(f(x), f(y)) \in \widetilde{\beta}$ for all $f \in F_A$. Whence, $\widetilde{\beta} \in \mathrm{Con}\,\mathbf{A}$. (Note that we have proved: $(x, y) \in \widetilde{\beta}$ implies $(f(x), f(y)) \in \beta^\star$ for all $f \in F_A$, except when $f = s_n$ acts as the identity on both x and y. This will be useful in the proof of part (iii) below.)

Notice that $\widetilde{\beta}|_B = \beta$. Therefore, by the residuation lemma of Section 2, we have $\widetilde{\beta} \leqslant \widehat{\beta}$. To prove $\widetilde{\beta} = \widehat{\beta}$, we suppose $(x, y) \notin \widetilde{\beta}$ and show $(x, y) \notin \widehat{\beta}$. Recall that the map $\widehat{} \colon \mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ is given by

$$\widehat{\beta} = \{(x, y) \in A^2 \mid (e_0 f(x), e_0 f(y)) \in \beta \text{ for all } f \in \mathrm{Pol}_1(\mathbf{A})\}.$$

Let $x \in C_p^j$ and $y \in C_q^k$ for some $1 \leqslant p, q \leqslant m$ and $1 \leqslant j, k \leqslant K$. If $p \neq q$, then $e_0(x) \in C_p$ and $e_0(y) \in C_q$—distinct β classes—so $(x, y) \notin \widehat{\beta}$. Suppose $p = q$. If both j and k belong to $\mathcal{T}_n \cap \mathcal{I}_r$ for some $1 \leqslant n \leqslant N$ and $1 \leqslant r \leqslant m$, then $(x, y) \in \widetilde{\beta}$, contradicting our assumption. If $(j, k) \in \mathcal{T}_n^2$ for some n, then for all r, we have $(j, k) \notin \mathcal{I}_r^2$, so $(e_0 s_n(x), e_0 s_n(y)) = (t_j, t_k) \notin \beta$ and $(x, y) \notin \widehat{\beta}$. If $(j, k) \in \mathcal{I}_r^2$ for some r, then for all n, we have $(j, k) \notin \mathcal{T}_n^2$. Without loss of generality, assume $j \in \mathcal{T}_n$. Then $(e_0 s_n(x), e_0 s_n(y)) = (t_j, e_0(y))$, which does not belong to β. For $t_j \in C_r$, while $e_0(y) \in C_q$, and $q \neq r$ (otherwise $(x, y) \in \beta^\star \leqslant \widetilde{\beta}$). Thus, $(x, y) \notin \widehat{\beta}$. Finally, suppose that for all n and r, we have $(j, k) \notin \mathcal{T}_n^2$ and $(j, k) \notin \mathcal{I}_r^2$. Let $j \in \mathcal{T}_n$ and $k \in \mathcal{T}_{n'}$. Then $(e_0 s_n s_{n'}(x), e_0 s_n s_{n'}(y)) = (t_j, t_k) \notin \beta$, so $(x, y) \notin \widehat{\beta}$.

(iii): Note that every equivalence relation θ on A with $\beta^\star \subseteq \theta \subseteq \widetilde{\beta}$ satisfies $f(\theta) \subseteq \theta$ for all $f \in F_A$, and is therefore a congruence relation of $\mathbf{A}$. Indeed, in proving that $\widetilde{\beta}$ is a congruence of $\mathbf{A}$, we saw that $f(\widetilde{\beta}) \subseteq \beta^\star$ for all $f \in F_A$, except when $f = s_n$, in which case f acts as the identity on some blocks of

$\widetilde{\beta}$ and maps other blocks of $\widetilde{\beta}$ to tie-points. It follows that $f(\theta) \subseteq \theta$ for all equivalence relations $\beta^\star \subseteq \theta \subseteq \widetilde{\beta}$. Therefore, the interval $[\beta^\star, \widetilde{\beta}]$ in Con $\mathbf{A}$ is $\{\theta \in \text{Eq}(A) \mid \beta^\star \subseteq \theta \subseteq \widetilde{\beta}\}$. To complete the proof, we must show that this interval is isomorphic to the lattice $\prod_{r=1}^m \prod_{n=1}^N (\text{Eq} \,|\, \mathscr{T}_n \cap \mathcal{I}_r|)^{m-1}$. This follows from a standard fact about intervals $[\zeta, \eta]$ for $\zeta \leqslant \eta$ in the lattice of equivalence relations on a set S. (See, e.g., Birkhoff [1], Exercise 10b, page 98.) Specifically, if the equivalence classes of η are $S_1, \ldots, S_N$ and if each S_j $(1 \leqslant j \leqslant N)$ is the union of n_j equivalence classes of ζ, then

$$[\zeta, \eta] \cong \prod_{j=1}^N \text{Eq}(n_j). \qquad (3.7)$$

Now, if a block of $\widetilde{\beta}$ contains C_r for some $1 \leqslant r \leqslant m$, then it consists of a single block of $\beta^\star$; otherwise, it consists of $q = |\mathscr{T}_n \cap \mathcal{I}_r|$ blocks of $\beta^\star$, say, $C_\ell^{i_1}, C_\ell^{i_2}, \ldots, C_\ell^{i_q}$, for some $1 \leqslant \ell \leqslant m$ with $\ell \neq r$. Indeed, we have, for each $1 \leqslant r \leqslant m$, one block of $\widetilde{\beta}$ consisting of a single block of $\beta^\star$, and for each $1 \leqslant n \leqslant N$, $m-1$ blocks of $\widetilde{\beta}$ consisting of $|\mathscr{T}_n \cap \mathcal{I}_r|$ blocks of $\beta^\star$. Therefore, by (3.7), we arrive at $[\beta^\star, \widetilde{\beta}] \cong \prod_{r=1}^m \prod_{n=1}^N (\text{Eq} \,|\, \mathscr{T}_n \cap \mathcal{I}_r|)^{m-1}$.

$\square$

We now describe a situation in which the foregoing construction is particularly useful and easy to apply. Given an algebra $\mathbf{B}$ and a pair $(x, y) \in B^2$, the unique smallest congruence relation of $\mathbf{B}$ containing (x, y) is called the *principal congruence generated by* (x, y), denoted by $\text{Cg}^{\mathbf{B}}(x, y)$. Given a finite congruence lattice Con $\mathbf{B}$, let $\beta = \text{Cg}^{\mathbf{B}}(x, y)$, and consider the overalgebra $\mathbf{A}$ constructed from base algebra $\mathbf{B}$ and tie-points $T : x, y$. Then by Theorem 3.2, the interval consisting of all $\theta \in \text{Con}\,\mathbf{A}$ for which $\theta|_B = \beta$ is given by $[\beta^*, \widehat{\beta}] \cong \text{Eq}(2)^{m-1} \cong \mathbf{2}^{m-1}$, where m is the number of congruence classes of β. Also, since β is the unique smallest congruence containing (x, y), we have $(x, y) \notin \theta$ for all $\theta \not\geqslant \beta$. Since x, y are the only tie-points, Theorem 3.2 implies that for all $\theta \not\geqslant \beta$, the interval $[\theta^*, \widehat{\theta}]$ is trivial (since $(x, y) \notin \theta$ implies no block of θ contains more than one tie-point). Thus, $\theta^* = \widehat{\theta}$. It is also immediate from Theorem 3.2 that if $\theta \geqslant \beta$ and if θ has r congruence classes, then $[\theta^*, \widehat{\theta}] \cong \mathbf{2}^{r-1}$.

Example 3.3. Theorem 3.2 explains the shapes of the two congruence lattices that we observed in Example 3.1. Returning to that example, with base algebra $\mathbf{B}$ equal to the right regular S_3-set, we now show some other congruence lattices that result by simply changing the sequence of tie-points. Recall that the partitions of B corresponding to nontrivial congruence relations of $\mathbf{B}$ are $\alpha = |0, 1, 2|3, 4, 5|$, $\beta = |0, 3|1, 4|2, 5|$, $\gamma = |0, 4|1, 5|2, 3|$, and $\delta = |0, 5|1, 3|2, 4|$.

It is clear from Theorem 3.2 that choosing the sequence T to be $0, 1$, or $0, 1, 2$, or $0, 2, 3$, and taking the trivial partition of the tie-point indices $(N = 1)$, yields the congruence lattices appearing in Figure 7. When $0, 2, 3, 5$ is chosen

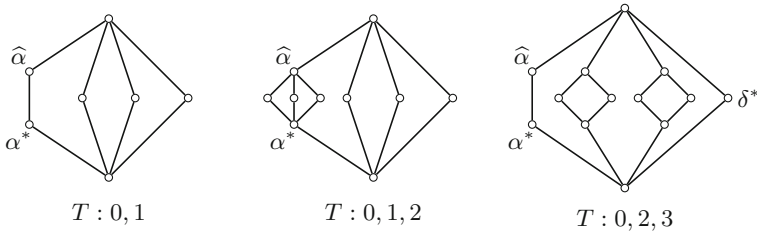FIGURE 7. Congruence lattices of overalgebras of the S_3-set for various tie-point sequences.
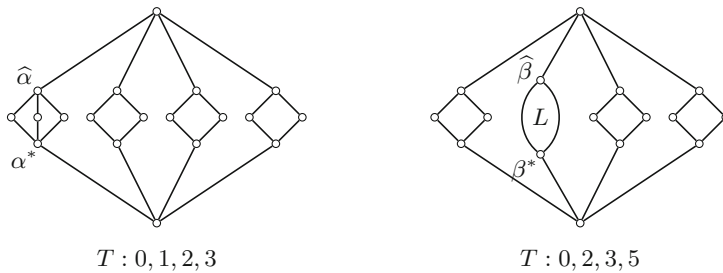

FIGURE 8. Congruence lattices of overalgebras of the S_3-set for various tie-point sequences; here, $L \cong \mathbf{2}^2 \times \mathbf{2}^2$.

(Figure 8, right) we have $[\beta^*, \widehat{\beta}] \cong \mathbf{2}^2 \times \mathbf{2}^2$, which we represent abstractly by L instead of drawing all 16 points in this interval.

Consider again the situation depicted in the congruence lattice on the right of Figure 8, where $[\beta^*, \widehat{\beta}] = L \cong \mathbf{2}^2 \times \mathbf{2}^2$, and suppose we prefer that all the other $|_B$-inverse images be trivial; that is,

$$[\beta^*, \widehat{\beta}] \cong \mathbf{2}^2 \times \mathbf{2}^2, \ \alpha^* = \widehat{\alpha}, \ \gamma^* = \widehat{\gamma}, \ \delta^* = \widehat{\delta}.$$

In other words, we seek a finite algebra with a congruence lattice isomorphic to the lattice in Figure 9. This is easy to achieve by selecting an appropriate partition of the indices of the tie-points. Let $t_1, t_2, t_3, t_4 = 0, 2, 3, 5$, and let the partition of the tie-point indices be $|\mathscr{T}_1|\mathscr{T}_2| = |1, 3|2, 4|$. Then $\beta = |C_1|C_2|C_3| = |0, 3|1, 4|2, 5|$ is the only nontrivial congruence of $\mathbf{B}$ having blocks containing multiple tie-points with indices in a single block of the partition $|\mathscr{T}_1|\mathscr{T}_2|$. Specifically, $(t_1, t_3) \in \beta$ and $\mathscr{T}_1 \cap \mathcal{I}_1 = \{1, 3\}$ (recall, $\mathcal{I}_1 = \{i \mid t_i \in C_1\}$), and $(t_2, t_4) \in \beta$ and $\mathscr{T}_2 \cap \mathcal{I}_2 = \{2, 4\}$. Since the number of congruence classes of β is $m = 3$, we have

$$[\beta^*, \widehat{\beta}] \cong \prod_{r=1}^{m} \prod_{n=1}^{N} (\mathrm{Eq}\,|\mathscr{T}_n \cap \mathcal{I}_r|)^{m-1}$$
$$= (\mathrm{Eq}\,|\mathscr{T}_1 \cap \mathcal{I}_1|)^{m-1} \times (\mathrm{Eq}\,|\mathscr{T}_2 \cap \mathcal{I}_2|)^{m-1} = (\mathrm{Eq}\,2)^2 \times (\mathrm{Eq}\,2)^2.$$
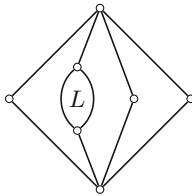
FIGURE 9. A lattice obtained as a congruence lattice of an
overalgebra by specifying a suitable partition of the indices of
the sequence of tie-points.

A second version of our GAP function used to construct overalgebras allows
the user to specify an arbitrary partition of the tie-points, and the associated
operations $\{s_n \mid 1 \leqslant n \leqslant N\}$ will be defined accordingly, as in (3.1). Con-
tinuing with our running example with tie-points $t_1, t_2, t_3, t_4 = 0, 2, 3, 5$, we
now introduce the partition $|1, 3|2, 4|$ of the tie-point indices by invoking the
following commands:

```
gap> g:=SymmetricGroup(3);;
gap> B:=[(),(1,2,3),(1,3,2),(1,2),(1,3),(2,3)];;
gap> G:=Action(g,B,OnRight);;
gap> OveralgebraXO([ G, [[0,3], [2,5]] ]);
```

The resulting overalgebra has a congruence lattice isomorphic to the lattice in
Figure 9, with $L \cong \mathbf{2}^2 \times \mathbf{2}^2$. Similarly,

```
gap> OveralgebraXO([ G, [[0,1,2], [3,4,5]] ]);
```

produces an overalgebra with congruence lattice isomorphic to the lattice in
Figure 9, but with $L = [\alpha^*, \widehat{\alpha}] \cong \mathrm{Eq}(3) \times \mathrm{Eq}(3)$. This time, α is the only
congruence with nontrivial $|_B$-inverse image.

We conclude this section by noting that, as mentioned at the outset, terms
in the sequence of tie-points may be repeated, and this gives us control over the
number of terms that appear in the product in (3.4). For example, consider
the sequence of tie-points $t_1, t_2, \ldots, t_9 = 0, 1, 2, 0, 1, 2, 3, 4, 5$ and the partition
$|1, 2, 3|4, 5, 6|7, 8, 9|$ of the tie-point indices. The command

```
gap> OveralgebraXO([ G, [[0,1,2], [0,1,2], [3,4,5]] ]);
```

produces an overalgebra with a 130 element congruence lattice like the one in
Figure 9, with $L = [\alpha^*, \widehat{\alpha}] \cong \mathrm{Eq}(3) \times \mathrm{Eq}(3) \times \mathrm{Eq}(3)$. Similarly,

```
gap> OveralgebraXO([ G, [[0,3], [0,3], [0,3], [0,3]] ]);
```

gives a 261 element congruence lattice with $L = [\beta^*, \widehat{\beta}] \cong \mathbf{2}^{16}$.

Indeed, there is no bound on the number of terms that can be inserted
in the $|_B$-inverse images in $\mathrm{Con}\,\mathbf{A}$. However, as Theorem 3.2 makes clear,
the shape of each term is invariably a power of a partition lattice. It seems
the only way to alter this outcome would be to add more operations to the
overalgebra. Determining how to further expand an overalgebra in order to
achieve such specific goals is one aspect of the theory that could benefit from
further research.

3.2. Overalgebras II. Consider a finite algebra $\mathbf{B}$ with $\beta \in \mathrm{Con}\,\mathbf{B}$ and suppose β is not a principal congruence. If the method described in Section 3.1 is used to construct an overalgebra $\mathbf{A}$ in such a way that $[\beta^*, \widehat{\beta}]$ is a nontrivial interval in $\mathrm{Con}\,\mathbf{A}$, then there will invariably be a principal congruence $\theta < \beta$ that also has nontrivial $|_B$-inverse image $[\theta^*, \widehat{\theta}]$. (This follows from Theorem 3.2.) It is natural to ask whether there is an alternative overalgebra construction that might result in a nontrivial interval $[\beta^*, \widehat{\beta}]$ such that $\theta^* = \widehat{\theta}$ for all $\theta \not\geq \beta$. Bill Lampe proposed an ingenious construction to answer this question. In this section, we present a generalization of Lampe's construction and prove two theorems which describe the resulting congruence lattices.

Let $\mathbf{B} = \langle B, F_B \rangle$ be a finite algebra, and suppose

$$\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \ldots, (a_{K-1}, b_{K-1}))$$

for some $a_1, \ldots, a_{K-1}, b_1, \ldots, b_{K-1} \in B$. Fix an arbitrary integer $u \geq 1$, let $B_1, B_2, \ldots, B_{uK}$ be sets of cardinality $|B|$, and fix a set of bijections $\pi_i \colon B \to B_i$. As in the Overalgebras I construction, we use the label x^i to denote $\pi_i(x)$, the element of B_i corresponding to $x \in B$ under the bijection π_i, and we use B_0 to denote B. Arrange the sets so that they intersect as follows (see Figure 10):

$$B_0 \cap B_1 = \{a_1\} = \{a_1^1\},\ B_1 \cap B_2 = \{b_1^1\} = \{a_2^2\},\ B_2 \cap B_3 = \{b_2^2\} = \{a_3^3\}, \ldots$$

$$\ldots, B_{K-2} \cap B_{K-1} = \{b_{K-2}^{K-2}\} = \{a_{K-1}^{K-1}\},$$
$$B_{K-1} \cap B_K = B_K \cap B_{K+1} = \{b_{K-1}^{K-1}\} = \{a_1^K\} = \{a_1^{K+1}\},$$
$$B_{K+1} \cap B_{K+2} = \{b_1^{K+1}\} = \{a_2^{K+2}\}, \ldots$$

$$\ldots, B_{2K-2} \cap B_{2K-1} = \{b_{K-2}^{2K-2}\} = \{a_{K-1}^{2K-1}\},$$
$$B_{2K-1} \cap B_{2K} = B_{2K} \cap B_{2K+1} = \{b_{K-1}^{2K-1}\} = \{a_1^{2K}\} = \{a_1^{2K+1}\},$$
$$B_{2K+1} \cap B_{2K+2} = \{b_1^{2K+1}\} = \{a_2^{2K+2}\}, \ldots$$

$$\ldots, B_{uK-2} \cap B_{uK-1} = \{b_{K-2}^{uK-2}\} = \{a_{K-1}^{uK-1}\},$$
$$B_{uK-1} \cap B_{uK} = \{b_{K-1}^{uK-1}\} = \{a_1^{uK}\}.$$

All other intersections are empty. In general, for ℓ a multiple of K, and for $1 \leq i < K$, we have

$$B_{\ell-1} \cap B_\ell = B_\ell \cap B_{\ell+1} = \{b_{K-1}^{\ell-1}\} = \{a_1^\ell\} = \{a_1^{\ell+1}\},$$
$$B_{\ell+i} \cap B_{\ell+i+1} = \{b_i^{\ell+i}\} = \{a_{i+1}^{\ell+i+1}\}.$$

As usual, we put $A = B_0 \cup \cdots \cup B_{uK}$ and proceed to define some unary operations on A. First, for $0 \leq i, j \leq uK$, let $S_{i,j} \colon B_i \to B_j$ be the bijection $S_{i,j}(x^i) = x^j$, and note that $S_{i,i} = \mathrm{id}_{B_i}$. Let $\mathscr{T} = |\mathscr{T}_1|\mathscr{T}_2|\cdots|\mathscr{T}_N|$ be an
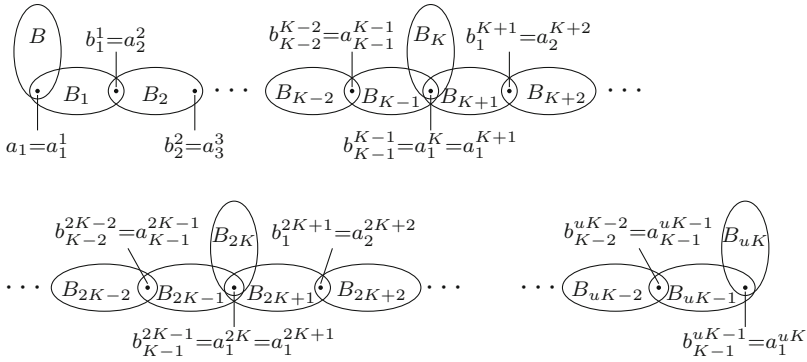
B $b_1^1=a_2^2$ $b_{K-2}^{K-2}=a_{K-1}^{K-1}$ B_K $b_1^{K+1}=a_2^{K+2}$

B_1 B_2 $\cdots$ B_{K-2} B_{K-1} B_{K+1} B_{K+2} $\cdots$

$a_1=a_1^1$ $b_2^2=a_3^3$ $b_{K-1}^{K-1}=a_1^K=a_1^{K+1}$

$b_{K-2}^{2K-2}=a_{K-1}^{2K-1}$ B_{2K} $b_1^{2K+1}=a_2^{2K+2}$ $b_{K-2}^{uK-2}=a_{K-1}^{uK-1}$ B_{uK}

$\cdots$ B_{2K-2} B_{2K-1} B_{2K+1} B_{2K+2} $\cdots$ $\cdots$ B_{uK-2} B_{uK-1}

$b_{K-1}^{2K-1}=a_1^{2K}=a_1^{2K+1}$ $b_{K-1}^{uK-1}=a_1^{uK}$

FIGURE 10. The universe of an overalgebra of the second type.

arbitrarily chosen partition of the set $\{0, K, 2K, \ldots, uK\}$. For each $1 \leqslant n \leqslant N$ and each $\ell \in \mathscr{T}_n$, define

$$e_\ell(x) = \begin{cases} S_{j,\ell}(x), & \text{if } x \in B_j \text{ for some } j \in \mathscr{T}_n, \\ a_1^\ell, & \text{otherwise.} \end{cases}$$

For each $\ell \in \{0, K, 2K, \ldots, (u-1)K\}$ and each $1 \leqslant i < K$, define

$$e_{\ell+i}(x) = \begin{cases} a_i^{\ell+i}, & \text{if } x \in B_j \text{ for some } j < \ell+i, \\ x, & \text{if } x \in B_{\ell+i}, \\ b_i^{\ell+i}, & \text{if } x \in B_j \text{ for some } j > \ell+i. \end{cases}$$

In other words, if ℓ is a multiple of K, then e_ℓ maps each up-pointing set in Figure 10 in the same $\mathscr{T}$-block as ℓ bijectively onto the up-pointing set B_ℓ, and maps all other points of A to the tie-point $a_1^\ell \in B_\ell$. For each set $B_{\ell+i}$ in between—represented in the figure by an ellipse with horizontal major axis— there is a map $e_{\ell+i}$ which act as the identity on $B_{\ell+i}$ and maps all points in A left of $B_{\ell+i}$ to the left tie-point of $B_{\ell+i}$ and all points to the right of $B_{\ell+i}$ to the right tie-point of $B_{\ell+i}$. Finally, for $0 \leqslant i, j \leqslant uK$, we define $q_{i,j} = S_{i,j} \circ e_i$ and take the set of basic operations on A to be

$$F_A = \{fe_0 \mid f \in F_B\} \cup \{q_{i,0} \mid 0 \leqslant i \leqslant uK\} \cup \{q_{0,j} \mid 1 \leqslant j \leqslant uK\}.$$

We are now ready to define the overalgebra as $\mathbf{A} = \langle A, F_A \rangle$.

In this overalgebra construction, the significance of a particular congruence of $\mathbf{B}$—namely, $\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \ldots, (a_{K-1}, b_{K-1}))$—is more explicit than in the overalgebra construction of Section 3.1. The following theorem describes the $|_{\scriptscriptstyle B}$-inverse image of this special congruence, that is, the interval $[\beta^*, \widehat{\beta}]$ in $\mathrm{Con}\,\mathbf{A}$. As above, assume β has m congruence classes, denoted by C_r ($1 \leqslant r \leqslant m$); let C_r^j denote $S_{0,j}(C_r)$, and let β^j denote $S_{0,j}(\beta)$.

Theorem 3.4. *Let* $\mathbf{A} = \langle A, F_A \rangle$ *be the overalgebra described above, and for each* $0 \leqslant j \leqslant uK$, *let* t_j *denote a tie-point of the set* B_j. *Define*

$$\beta^\star = \bigcup_{j=0}^{uK} \beta^j \cup \Big(\bigcup_{j=0}^{uK} t_j/\beta^j \Big)^2. \tag{3.8}$$

and

$$\widetilde{\beta} = \beta^\star \cup \bigcup_{r=1}^{m-1} \bigcup_{n=1}^{N} \Big(\bigcup_{\ell \in \mathscr{I}_n} C_r^\ell \Big)^2 \tag{3.9}$$

Then

(i) $\beta^\star = \beta^*$, *the minimal* $\theta \in \mathrm{Con}\,\mathbf{A}$ *such that* $\theta|_B = \beta$;

(ii) $\widetilde{\beta} = \widehat{\beta}$, *the maximal* $\theta \in \mathrm{Con}\,\mathbf{A}$ *such that* $\theta|_B = \beta$;

(iii) *the interval* $[\beta^\star, \widetilde{\beta}]$ *in* $\mathrm{Con}\,\mathbf{A}$ *satisfies* $[\beta^\star, \widetilde{\beta}] \cong \prod_{n=1}^{N} (\mathrm{Eq}\,|\mathscr{I}_n|)^{m-1}$.

Proof. (i): It is easy to see that $\beta^\star$ is an equivalence relation on A, so we first check that $f(\beta^\star) \subseteq \beta^\star$ for all $f \in F_A$, thereby establishing that $\beta^\star \in \mathrm{Con}\,\mathbf{A}$. Then we show that $\beta \subseteq \eta \in \mathrm{Con}\,\mathbf{A}$ implies $\beta^\star \leqslant \eta$.

For all $0 \leqslant i, j \leqslant uK$ and $1 \leqslant r \leqslant m$, either $e_i(C_r^j)$ is a singleton or, in case i and j are multiples of K in the same block of $\mathscr{T}$, we have $e_i(C_r^j) = S_{j,i}(C_r^j) = C_r^i$. Therefore, $q_{i,0}(C_r^j) = S_{i,0}e_i(C_r^j)$ is either a singleton or C_r, so $q_{i,0}(\bigcup_{j=0}^{uK} \beta^j) \subseteq \beta$. Similarly, since $e_0(C_r^j)$ is either $\{a_1\}$ or C_r, we see that $q_{0,i}(C_r^j) = S_{0,i}e_0(C_r^j)$ is either $\{a_1^i\}$ or C_r^i. Also, for each $g \in F_B$, we have $ge_0(C_r^j) \subseteq C_k$ for some $0 \leqslant k \leqslant m$. Therefore, $f(\bigcup_{j=0}^{uK} \beta^j) \subseteq \beta^\star$ for all $f \in F_A$.

Let $\mathcal{B} = \bigcup_{j=0}^{uK} t_j/\beta^j$. Then for each $0 \leqslant i \leqslant uK$, we have $e_i(\mathcal{B}) = t_i/\beta^i$, so $q_{i,0}(\mathcal{B}) = S_{i,0}(t_i/\beta^i) = C_k$ for some $1 \leqslant k \leqslant m$. Since $e_0(\mathcal{B}) = a_1/\beta = C_1$, it is clear that $q_{0,j}(\mathcal{B}) = S_{0,j}e_0(\mathcal{B}) = C_1^j$. Also, for each $g \in F_B$, we have $ge_0(\mathcal{B}) = g(C_1) \subseteq C_k$ for some $1 \leqslant k \leqslant m$. Therefore, $f(\mathcal{B})^2 \subseteq \beta^\star$ for all $f \in F_A$. We conclude that $\beta^\star \in \mathrm{Con}\,\mathbf{A}$.

Next, suppose $\beta \subseteq \eta \in \mathrm{Con}\,\mathbf{A}$. Then for all $0 \leqslant j \leqslant uK$, we have $q_{0,j}(\beta) = S_{0,j}e_0(\beta) = \beta^j \subseteq \eta$. For each $\ell \in \{0, K, 2K, \ldots, (u-1)K\}$ and $0 \leqslant i < K$, the tie-points of $B_{\ell+i}$ are a_1^ℓ if $i = 0$, and $\{a_i^{\ell+i}, b_i^{\ell+i}\}$ if $i > 0$. Thus, from the fact that $(a_i^{\ell+i}, b_i^{\ell+i}) \in \beta^{\ell+i}$ and from the overlapping of $B_{\ell+i}$ and $B_{\ell+i+1}$, it follows by transitivity that any equivalence relation on A containing all β^j, for $0 \leqslant j \leqslant uK$, must have a single equivalence class containing all the tie-points of A. We have just seen that η contains all β^j, so $\big(\bigcup_{j=0}^{uK} t_j/\beta^j \big)^2 \subseteq \eta$. Therefore, $\beta^\star \leqslant \eta$.

(ii): We first show $\widetilde{\beta} \in \mathrm{Con}\,\mathbf{A}$. Fix $1 \leqslant r \leqslant m$ and $1 \leqslant n \leqslant N$, and let $\mathcal{B}_r^n = \bigcup_{\ell \in \mathscr{I}_n} C_r^\ell$. Thus, $\mathcal{B}_r^n$ is the join of corresponding β blocks of up-pointing sets in Figure 10 from the same block $\mathscr{I}_n$.

If $\ell \in \{0, K, 2K, \ldots, (u-1)K\}$ and $1 \leqslant i < K$, then $B_{\ell+i}$ has tie-points $\{a_i^{\ell+i}, b_i^{\ell+i}\}$, and the mapping $e_{\ell+i}$ takes the set $\mathcal{B}_r^n$ onto this pair of tie-points. Thus, if $\ell \in \{0, K, 2K, \ldots, (u-1)K\}$ for $1 \leqslant i < K$ and if $k = \ell + i$, then $q_{k,0}(\mathcal{B}_r^n) = \{a_i, b_i\}$. It follows that $q_{k,0}(\widetilde{\beta}) \subseteq \beta$ for all $k \notin \{0, K, 2K, \ldots, uK\}$.

If $k \in \{0, K, 2K, \dots, uK\} \setminus \mathscr{T}_n$, then $e_k(\mathcal{B}_r^n) = \{a_1^k\}$, so $q_{k,0}(\mathcal{B}_r^n) = \{a_1\}$. If $k \in \mathscr{T}_n$, then $e_k(\mathcal{B}_r^n) = C_r^k$, so $q_{k,0}(\mathcal{B}_r^n) = C_r$. It follows that $q_{k,0}(\widetilde{\beta}) \subseteq \beta$ for all $0 \leqslant k \leqslant uK$.

Next, if $g \in F_B$ then, since $e_0(\mathcal{B}_r^n)$ is either C_r or $\{a_1\}$, the operation ge_0 takes $\mathcal{B}_r^n$ to a single β class. Finally, for each $0 \leqslant k \leqslant uK$, the map $q_{0,k}$ takes the set $\mathcal{B}_r^n$ to either C_r^k or $\{a_1^k\}$. Therefore, $q_{0,k}(\widetilde{\beta}) \subseteq \widetilde{\beta}$, and we have proved $f(\widetilde{\beta}) \subseteq \widetilde{\beta}$ for all $f \in F_A$.

Since the restriction of $\widetilde{\beta}$ to B is clearly $\widetilde{\beta}|_B = \beta$, the residuation lemma yields $\widetilde{\beta} \leqslant \widehat{\beta}$. On the other hand, it is easy to verify that for each $(x, y) \notin \widetilde{\beta}$, there is an operation $f \in \mathrm{Pol}_1(\mathbf{A})$ such that $(e_0 f(x), e_0 f(y)) \notin \beta$, and thus $(x, y) \notin \widehat{\beta}$. Therefore, $\widetilde{\beta} \geqslant \widehat{\beta}$.

(iii): It remains to prove $[\beta^\star, \widetilde{\beta}] \cong \prod_{n=1}^{N}(\mathrm{Eq}\,|\mathscr{T}_n|)^{m-1}$. This follows easily from the proof of (ii). For in proving that $\widetilde{\beta}$ is a congruence, we showed that each operation $f \in F_A$ maps blocks of $\widetilde{\beta}$ into blocks of $\beta^\star$. That is, each operation collapses the interval $[\beta^\star, \widetilde{\beta}]$. Therefore, every equivalence relation on the set A that lies between $\beta^\star$ and $\widetilde{\beta}$ is respected by every operation of $\mathbf{A}$. In other words, as an interval in $\mathrm{Con}\,\mathbf{A}$, $[\beta^\star, \widetilde{\beta}] = \{\theta \in \mathrm{Eq}(A) \mid \beta^\star \leqslant \theta \leqslant \widetilde{\beta}\}$. In view of the configuration of the universe of $\mathbf{A}$, as shown in Figure 10, and by the same argument used to prove (iii) of Theorem 3.2 (cf. (3.7)), it is clear that the interval sublattice $\{\theta \in \mathrm{Eq}(A) \mid \beta^\star \leqslant \theta \leqslant \widetilde{\beta}\}$ is isomorphic to $\prod_{n=1}^{N}(\mathrm{Eq}\,|\mathscr{T}_n|)^{m-1}$. $\square$

In the next theorem, we continue to assume that $\mathbf{A} = \langle A, F_A \rangle$ is an over-algebra of the second type—as illustrated in Figure 10—based on the algebra $\mathbf{B}$, the congruence $\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \dots, (a_{K-1}, b_{K-1}))$, and the partition $\mathscr{T} = |\mathscr{T}_1|\mathscr{T}_2|\cdots|\mathscr{T}_N|$ of the set $\{0, K, 2K, \dots, uK\}$. We remind the reader that θ^* denotes $\mathrm{Cg}^{\mathbf{A}}(\theta)$ for $\theta \in \mathrm{Con}\,\mathbf{B}$.

Theorem 3.5. *Let $\theta \in \mathrm{Con}\,\mathbf{B}$ and suppose θ has r congruence classes. Then $\theta^* < \widehat{\theta}$ if and only if $\beta \leqslant \theta < 1_B$, in which case $[\theta^*, \widehat{\theta}] \cong \prod_{n=1}^{N}(\mathrm{Eq}\,|\mathscr{T}_n|)^{r-1}$.*

Consequently, if $\theta \not\geqslant \beta$, then $\widehat{\theta} = \theta^*$. The proof of the theorem follows easily from the next lemma.

Lemma 3.6. *Suppose $\eta \in \mathrm{Con}\,\mathbf{A}$ satisfies $\eta|_B = \theta \in \mathrm{Con}\,\mathbf{B}$ and $(x, y) \in \eta \setminus \theta^*$ for some $x \in B_i$ and $y \in B_j$. Then i and j are distinct multiples of K belonging to the same block of $\mathscr{T}$, and $\theta \geqslant \beta$.*

Proof. Assume $\eta \in \mathrm{Con}\,\mathbf{A}$ satisfies $\eta|_B = \theta$ and $(x, y) \in \eta \setminus \theta^*$ for some $x \in B_i, y \in B_j$. If $i = j$, then $(x, y) \in \eta \cap B_i^2$ and $(q_{i,0}(x), q_{i,0}(y)) \in \eta|_B = \theta \leqslant \theta^*$, so $(x, y) = (q_{0,i}q_{i,0}(x), q_{0,i}q_{i,0}(y)) \in \theta^*$, a contradiction. In fact, we will reach this same contradiction, $(x, y) \in \theta^*$, as long as i and j do not belong to the same block of $\mathscr{T}$.

We have already handled the case $i = j$, so we may suppose $0 \leqslant i < j \leqslant uK$. In order to make the simple idea of the proof more transparent, we first consider the special case in which $1 \leqslant i < j < K$. In this case, we have $q_{i,0}(y) = b_i$,
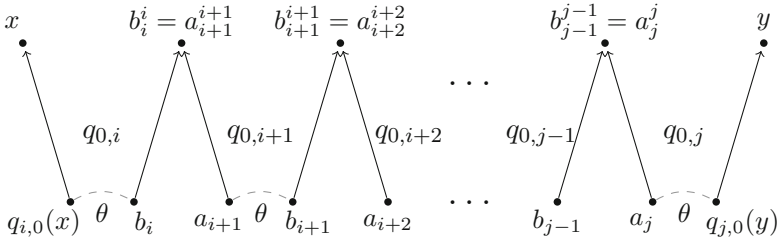
so $(x, y) \in \eta$ implies $(q_{i,0}(x), q_{i,0}(y)) = (q_{i,0}(x), b_i) \in \eta|_B = \theta$. Similarly, $q_{j,0}(x) = a_j$, so $(q_{j,0}(x), q_{j,0}(y)) = (a_j, q_{j,0}(y)) \in \theta$. If $j = i + 1$, then

$$x = q_{0,i}q_{i,0}(x) \ \theta^* \ q_{0,i}(b_i) = b_i^i = a_j^j = q_{0,j}(a_j) \ \theta^* \ q_{0,j}q_{j,0}(y) = y, \qquad (3.10)$$

so $(x, y) \in \theta^*$. The relations (3.10) are illustrated by the following diagram:
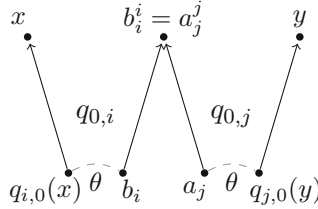
In case $j > i + 1$, we obtain the diagram below.

Here too we could write out a line analogous to (3.10), but it is obvious from the diagram that $(x, y) \in \theta^*$.

To handle the general case, we let $i = vK + p$ and $j = wK + q$ for some $0 \leqslant v \leqslant w \leqslant u$ and $0 \leqslant p, q < K$. Assume for now that $p, q \geqslant 1$, so that neither i nor j is a multiple of K. Then $q_{i,0}(y) = q_{vK+p,0}(y) = b_p$, so $(q_{i,0}(x), q_{i,0}(y)) = (q_{i,0}(x), b_p)$ belongs to $\eta|_B = \theta$. Similarly, $q_{j,0}(x) = a_q$, so $(q_{i,0}(x), q_{i,0}(y)) = (a_q, q_{j,0}(y)) \in \theta$. In this case, we have

$$x = q_{0,i}q_{i,0}(x) \ \theta^* \ q_{0,i}(b_p) = b_p^{vK+p} = a_{p+1}^{vK+p+1} = q_{0,i+1}(a_{p+1})$$

$$\theta^* \ q_{0,i+1}(b_{p+1}) = b_{p+1}^{vK+p+1} = a_{p+2}^{vK+p+2} = q_{0,i+2}(a_{p+2})$$

$$\vdots \qquad\qquad\qquad\qquad (3.11)$$

$$\theta^* \ q_{0,(v+1)K-1}(b_{K-1}) = b_{K-1}^{(v+1)K-1} = a_1^{(v+1)K}.$$

Now $a_1^{(v+1)K} = a_1^{(v+1)K+1} = q_{0,(v+1)K+1}(a_1)$, so the relations in (3.11) imply $x \ \theta^* \ q_{0,(v+1)K+1}(a_1)$. Continuing, we have

$$x \ \theta^* \ q_{0,(v+1)K+1}(a_1)$$

$$\theta^* \ q_{0,(v+1)K+1}(b_1) = b_1^{(v+1)K+1} = a_2^{(v+1)K+2} = q_{0,(v+1)K+1}(a_2)$$

$$\vdots$$

$$\theta^* \ q_{0,wK+q-1}(b_{q-1}) = b_{q-1}^{wK+q-1} = a_q^{wK+q} = q_{0,j}(a_q) \ \theta^* \ q_{0,j}q_{j,0}(y) = y.$$

Again, we arrive at the desired contradiction, $(x, y) \in \theta^*$.

The case in which exactly one of i, j is a multiple of K is almost identical, so we omit the derivation. Finally, suppose i and j are distinct multiples of K. If i and j belong to distinct blocks of the partition $\mathcal{T}$, then $(q_{i,0}(x), q_{i,0}(y)) = (q_{i,0}(x), a_1) \in \theta$ and $(q_{j,0}(x), q_{j,0}(y)) = (a_1, q_{j,0}(y)) \in \theta$, and the foregoing argument will again lead to the contradiction $(x, y) \in \theta^*$.

The only remaining possibility is $i, j \in \mathcal{T}_n$ for some $1 \leqslant n \leqslant N$. In this case, there is no contradiction. Rather, we observe that (as in a number of the contradictory cases) there is at least one $v \in \{0, 1, \dots, u - 1\}$ such that, for all $1 \leqslant k < K$, we have $i < vK + k < j$, and thus $(q_{vK+k,0}(x), q_{vK+k,0}(y)) = (a_k, b_k) \in \eta|_B = \theta$. Therefore, $\theta \geqslant \beta = \mathrm{Cg}^{\mathbf{A}}((a_1, b_1), \dots, (a_{K-1}, b_{K-1}))$. $\square$

Proof of Theorem 3.5. Lemma 3.6 implies that $\theta^* < \widehat{\theta}$ only if $\beta \leqslant \theta < 1_B$. On the other hand, if $\beta \leqslant \theta < 1_B$, then we obtain $[\theta^*, \widehat{\theta}] \cong \prod_{n=1}^{N}(\mathrm{Eq} \,|\mathcal{T}_n|)^{r-1}$ by the argument used to prove the same fact about $[\beta^\star, \widetilde{\beta}]$ in Theorem 3.4. $\square$

4. Conclusion

We have described an approach to building new finite algebras out of old which is useful in the following situation: given an algebra $\mathbf{B}$, we seek an algebra $\mathbf{A}$ with congruence lattice $\mathrm{Con}\,\mathbf{A}$ such that $\mathrm{Con}\,\mathbf{B}$ is a (nontrivial) homomorphic image of $\mathrm{Con}\,\mathbf{A}$; specifically, we construct $\mathbf{A}$ so that $|_B \colon \mathrm{Con}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{B}$ is a lattice epimorphism. We described the original example—the "triple-winged pentagon" shown on the right of Figure 1—found by Ralph Freese, which motivated us to develop a general procedure for finding such finite algebraic representations.

We mainly focused on two specific overalgebra constructions. In each case, the congruence lattice that results has the same basic shape as the one with which we started, except that some congruences are replaced with intervals that are direct products of powers of partition lattices. Thus, we have identified a broad new class of finitely representable lattices. However, the fact that the new intervals in these lattices must be products of partition lattices seems quite limiting, and this is the first limitation that we think future research might aim to overcome.

We envision variations on the constructions described in this paper which might bring us closer to the goal of replacing certain congruences, $\beta \in \mathrm{Con}\,\mathbf{B}$, with more general finite lattices, $L \cong [\beta^*, \widehat{\beta}] \leqslant \mathrm{Con}\,\mathbf{A}$. However, using the constructions described above, we have found examples of overalgebras for which it is not possible to simply add operations in order to eliminate *all* relations θ such that $\beta^* < \theta < \widehat{\beta}$. Nonetheless, we remain encouraged by some modest progress we have recently made in this direction.

As a final remark, we call attention to another obvious limitation of the methods described in this paper—they cannot be used to find an algebra with a *simple* congruence lattice. For example, the lattice on the left in Figure 1

is a simple lattice, so it is certainly not the $\vert_B$-inverse image of some smaller lattice. As of this writing, we do not know how to represent this lattice as the congruence lattice of a finite algebra. More details about this aspect of the problem appear in [2].

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