ON A PROBLEM OF PALFY AND SAXL

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1. Introduction

In the paper [1], Peter Palfy and Jan Saxl pose the following

> **PROBLEM.** Let $\mathbf{A}$ be a finite algebra with $\operatorname{Con}\mathbf{A} \cong M_n$, $n \geqslant 4$. If three nontrivial congruences of $\mathbf{A}$ pairwise permute, does it follow that every pair of congruences of $\mathbf{A}$ permute?

These notes collect some notation and facts that might be useful for attacking this problem. Throughout, X denotes a finite set, $\operatorname{Eq}(X)$ denotes the lattice of equivalence relations on X and, for $\alpha \in \operatorname{Eq}(X)$ and $x \in X$, we denote by x/α the equivalence class of α containing x. We often refer to equivalnce classes as "blocks," and we denote by $\#\text{Blocks}(\alpha)$ the number of blocks of the equivalence relation α.

For a given $\alpha \in \operatorname{Eq}(X)$ the map $\varphi_\alpha : x \mapsto x/\alpha$ is a function from X into the power set $\mathscr{P}(X)$ with kernel $\ker\varphi_\alpha = \alpha$. The *block-size function* $x \mapsto |x/\alpha|$ is a function from X into $\{1, 2, \dots, |X|\}$.

We will often abuse notation and equate an equivalence relation with the corresponding partition of the set X. For example, we will equate the relation

$$\alpha = \{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0), (2,3), (3,2)\}$$

with the partition $|0,1|2,3|$, and often we resort to writing $\alpha = |0,1|2,3|$.

We say that α has *uniform blocks* if all blocks of α have the same size; or, equivalently, the block-size function is constant: for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. We will use $|x/\alpha|$, without specifying a particular $x \in X$, to denote this block size.[1] Thus, when α has uniform blocks, we have $|X| = |x/\alpha| \cdot \#\text{Blocks}(\alpha)$.

We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary blocks*, if the number of blocks of one is equal to the block size of the other. In other words, if α and β are two equivalence relations on X with uniform block sizes $|x/\alpha|$ and $|x/\beta|$, respectively, then α and β have complementary blocks if and only if $(\forall x)(\forall y)|x/\alpha| \cdot |y/\beta| = |X|$.

Given two equivalence relations α and β on X, the relation

$$\alpha \circ \beta = \{(x,y) \in X^2 : (\exists z) x \;\alpha\; z \;\beta\; y\}$$

is called the *composition of* α *and* β, and if $\alpha \circ \beta = \beta \circ \alpha$ then α and β are said to *permute*, or to be *permuting* equivalence relations. Note that $\alpha \circ \beta \subseteq \alpha \vee \beta$ with equility if and only if α and β permute.

The largest and smallest equivalence relations on X are given by $1_X = X^2$ and $0_X = \{(x,x) : x \in X\}$, respectively.

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[1] Alternatively, we might consider using $|x./\alpha|$ to emphasize that every $x \in X$ can be substituted for $x.$ without changing the value of $|x./\alpha|$, but this notation may be too cumbersome.

We say that α and β are *complementary* equivalence relations on X provided $\alpha \vee \beta = 1_X$ and $\alpha \wedge \beta = 0_X$.

Lemma 1. Suppose α and β are complementary equivalence relations on X. Then α and β permute if and only if they have complementary blocks. That is,

$$\alpha \circ \beta \quad \Longleftrightarrow \quad (\forall x)(\forall y)|x/\alpha| \cdot |y/\alpha| = |X|.$$

Lemma 2. Let $\{\alpha_i : 0 \leqslant i < r\}$ be a set of pairwise complementary equivalence relations on X.

(1) If $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$, then α_1 and α_2 have uniform blocks.
(2) If α_1 and α_2 have uniform blocks of size $|X|^{1/2}$, then $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$.
(3) Three pairwise complementary equivalence relations are pairwise permuting if and only if all three have uniform blocks of size $|X|^{1/2}$.

References

[1] P. P. Pálfy and J. Saxl. Congruence lattices of finite algebras and factorizations of groups. *Comm. Algebra*, 18(9):2783–2790, 1990.