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WHICH PARTITION LATTICES ARE CONGRUENCE LATTICES?

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During the summer school 1970 on "Universal algebra and Mal'cev conditions" in Waterloo a group consisting of S. Burris, H. Crapo, A. Day, D. Higgs, W. Nichols and the author investigated the question which partition lattices on a set X are actually the congruence lattices of algebras with underlying set X. So they discovered the following result, which was found independently and published by R. Quackenbush and B. Wolk [10], namely that every completely distributive algebraic sublattice of the full partition lattice on X is the congruence lattice of a suitable algebra with underlying set X. But somehow this result and the results of that paper never became properly written up and published. In June 1973 the author came for a visit to Waterloo and still being interested in these problems he collected all the old notes and began to extract the results. In July he gave a report on these results on the conference on "Universal algebra" in Oberwolfach and because of the great interest of many participants in these results, he got in touch with the group to get things properly written up. Having stuck out his neck so far already, the author was made the "editorial secretary" for the present paper.

We are here mainly concerned with the concrete characterization problem for the congruence lattices of algebras. Throughout this paper an algebra is always a universal algebra with possibly infinitely many finitary full operations. Every algebra is associated with the lattices of its subalgebras and congruences, respectively, and with its group of automorphisms. As a natural question now thw abstract and the concrete characterization problems for these structures arise, abstract meaning characterization up to isomorphism among lattices and groups, respectively, and concrete meaning a characterization of the lattices Sub (A), $\mathfrak{C}(A)$ and the groups Aut (A) for all algebras A with underlying set X as sets of subsets of X, sets of partitions on X, and sets of permutations on X, respectively.

For the subalgebra lattice both problems were settled in 1948 by B. Birkhoff and O. Frink [3]. For the automorphism groups the abstract and concrete characterizations were given in 1946 by G. Birkhoff [2] and in 1968 by B. Jónsson [7]. For the congruence lattice things seem to be much more difficult. In 1963 G. Grätzer and E.T. Schmidt [6] solved the abstract problem and characterized the congruence lattices of algberas as the complete algebraic lattices. However, their proof is very complicated and hasn't been simplified since, so one wouldn't expect an easy solution to the concrete problem, which is problem 2 in Grätzer [4].

Partial solutions to the concrete characterization of congruence lattices have been given by M. Armbrust [1] in 1970 and by R. Quackenbush and B. Wolk [10] in 1971. Closest to a complete solution is the concrete characterization of the lattice of congruence classes of an algebra by R. Wille [13] in 1970. In [8] Theorem 4.4.1 of B. Jónsson gives a solution to the concrete characterization problem which was discovered independently also by the group in Waterloo (see the remarks in [8] p. 174) and his result is essentially the same as our Theorem 2.2. In this paper we present our approach to the problem and we colour it with some geometrical considerations.

1. EQUIVALENCE RELATIONS DEFINED BY GRAPHS

A first step towards a characterization of congruence lattices is the following

Theorem 1.1 (Mal'cev [9]): *Let* $(A.F)$ *be an algebra and* G *the set of all its unary algebraic functions. Then the two algebras* (A,F) *and* (A,G) *have the same congruence relations.*

Proof of this well-known result can also be found in G. Grätzer [5] §10, Thm. 3 and in R. Wille [13] Satz 1.2. As an immediate consequence we get

Corollary 1.2. *If* $\mathscr{L}$ *is the congruence lattice of an algebra then it is also the congruence lattice of a unary algebra (i.e. an algebra which has only unary operations).*

Let us now introduce the notations we use throughout this paper. Let X be a set. We denote the set (lattice) of all equivalence relations on X by $\mathscr{E}(X)$, its largest element $X \times X$ by ∇, and its smallest element Id_x by Δ. $\mathfrak{R}(X)$ denotes the set of all binary relations on X, i.e. $\mathfrak{R}(X) = \{R \mid R \subseteq X \times X\}$. For a (unary) function $f: X \to X$ we say f *preserves* $R \in \mathfrak{R}(X)$ or R *admits* f iff $\forall (x,y) \in R$: $(fx, fy) \in R$. Similary we say R admits $F = \{f, g, \ldots\}$ iff R admits every $f \in F$. In this language the congruences of a unary algebra (A,F) are exactly those equivalence relations on A which admit F. The congruence lattice of (A,F) is denoted by $\mathfrak{C}(A,F)$. If $\mathfrak{M}$ is a set of (equivalence) relations on X, then $D(\mathfrak{M})$ denotes the set of all functions $f: X \to X$ which preserve every $R \in \mathfrak{M}$, these functions are called the *dilatations* of $\mathfrak{M}$. Now Corollary 1.1 can be phrased as follows:

Corollary 1.3. *A subset* $\mathscr{L} \subseteq \mathscr{E}(X)$ *is the congruence lattice of an algebra (with underlying set* X*) iff* $\mathscr{L} = \mathfrak{C}(X, D(\mathscr{L}))$.

The relation of admitting defines a galois connection between the sets of (equivalence) relations and the sets of unary functions on a set X and thereby defines a closure operator $[\ ]$ on $\mathfrak{R}(X)$, namely for $\mathfrak{M} \subseteq \mathfrak{R}(X)$ $[\mathfrak{M}]$ is the set of all relations $R \in \mathfrak{R}(X)$ such that R admits $D(\mathfrak{M})$.

The restriction of this closure operator to $\mathscr{E}(X)$ is the mapping $\mathscr{L} \mapsto$ $\mapsto \mathfrak{C}(X, D(\mathscr{L}))$ and so Corollary 1.3 just says that the congruence lattices are exactly those subsets of $\mathscr{E}(X)$ which are closed under this particular closure operator, This already gives an external description of the congruence lattices on the set X but it tells us moreover that the congruence lattices on X form a closure system on $\mathscr{E}(X)$. Every closure system is known to be the subalgebra system of a suitable possibly infinitary algebra on the underlying set. So we known that there are operations on $\mathscr{E}(X)$ such that the congruence lattices are exactly the subalgebras of $\mathscr{E}(X)$ together with these operations. Among these operations belong the infinite meet and the infinite join in $\mathscr{E}(X)$. In this section we give a big family of operations on $\mathscr{E}(X)$ under which all congruence lattices have to be closed and in the next section we give the proof that these operations in fact suffice the characterize the congruence lattices.

Things become much smoother if we consider the [ ]-closed subsets of $\mathfrak{R}(X)$ rather than congruence lattices, and this amounts to characterizing for a set F of unary functions on X the set $\mathfrak{A}(F)$ of all $R \in \mathfrak{R}(X)$ which admit F.

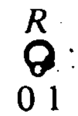
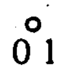
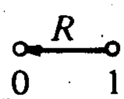
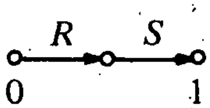
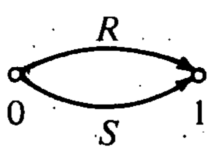
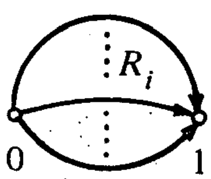
Observe that R admits F iff R is a subalgebra of $(X, F) \times (X, F)$. As this algebra is unary, $\mathfrak{A}(F)$ is closed under arbitrary intersections and unions. Obviously, Δ and ∇ admit every F, so we have $\nabla, \Delta \in \mathfrak{A}(F)$. Now if relations R, S admit F then the relations $R^{-1} := \{(x, y) \mid (x, y) \in$ $\in R\}$ and $R \circ S := \{(x, y) \mid \exists z \ (x, z) \in R \ \& \ (z, y) \in S\}$ admit F, hence $\mathfrak{A}(F)$ is closed under the operations $^{-1}, \circ$. From these remarks it follows that for every relation $R \in \mathfrak{A}(F)$ the equivalence relation $\mathrm{Eq}(R)$ generated by R admits F, too, hence, $\mathfrak{A}(F)$ is closed under the operation Eq.

As a consequence we get the well-known result that $\mathfrak{C}(X, F) =$ $= \mathfrak{A}(F) \cap \mathscr{E}(X)$ is a complete (hence algebraic) sublattice of $\mathscr{E}(X)$. As we will see below, all these conditions are not sufficient to characterize $\mathfrak{A}(F)$. Most of the above operations on $\mathfrak{A}(F)$ can be associated with a graph and vice versa every graph can be associated with an operation on $\mathfrak{A}(F)$ in the manner described below. The point is that we can make these operations on $\mathfrak{A}(F)$ into operations on $\mathscr{E}(X)$, where we call them *graphical compositions* and the congruence lattices on the set X turn out to be exactly the subalgebras of the algebra $\mathscr{E}(X)$ with the infinite join and all graphical compositions as operations. Before we give the precise definition of graphical compositions, we want to give the intuitive meaning and some examples which hopefully will make the definition more comprehensible.

Let G be a directed graph with two distinguished vertices $0, 1$. Let ψ be a labeling of the edges of G by elements of $\mathfrak{A}(F)$ (i.e. ψ is a mapping from the edges of G into $\mathfrak{A}(F)$). Let φ be a labeling of the vertices of G by elements of X. Then φ is *compatible* with ψ if $(\varphi x, \varphi y) \in \psi e$ for every edge e from x to y. For fixed $G, 0, 1, \psi$ define the relation S on X by $(a, b) \in S$ iff there is a φ compatible with ψ such that $\varphi 0 = a$, $\varphi 1 = b$. The claim is that this relation, and hence also the equivalence relation $\mathrm{Eq}\,(S)$, admits F, too.

This follows easily from the following observation: If $f \in F$ and φ is compatible with ψ when $f \circ \varphi$ is compatible, too, because f preserves all the relations the edges are labelled with. To illustrate this intuitive definition we give the following example.

Example 1.4.

Graph; 0, 1; Labelling of the edges	S	Eq (S)
	$R \wedge \Delta$	Δ
	Δ	Δ
	Δ	Δ
	R^{-1}	Eq (R)
	$R \circ S$	$R \vee S$ if $R, S \in \mathscr{E}(X)$
	$R \cap S$	$R \cap S$ if $R, S \in \mathscr{E}(X)$
	$\bigcap_{i \in I} R_i$	$\bigcap_{i \in I} R_i$ if $R_i \in \mathscr{E}(X)$

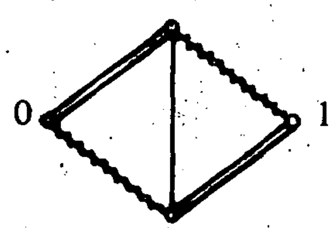
It appears to be impossible to express the union of two relations as an operation of this kind and it would be interesting either to see a proof of this conjecture or to see the graph which defines the union.

It is obvious that in the case that all relations which appear as labels of edges are reflexive or symmetric, respectively, we can forget about all loops in the graph or about the directedness of the graph, respectively. As in this context we are only interested in the effect of these operations on equivalence relations, we can restrict our attention to undirected graphs without loops.

So let G be such a graph with two distinguished edges 0, 1. We define an operation $P_{G,0,1}$ on $\mathscr{E}(X)$ whose arity is the number of edges in G by defining for every labeling ψ of the edges with elements of $\mathscr{E}(X)$ the relation S as above and setting: $P_{G,0,1}(\psi) := \mathrm{Eq}\,(S)$.

We can reduce the arity of this operation by starting out with a coloured graph and by insisting that the labeling is such that two edges with the same colour have the same label. The arity of this operation is just the number of colours appearing in the graph.

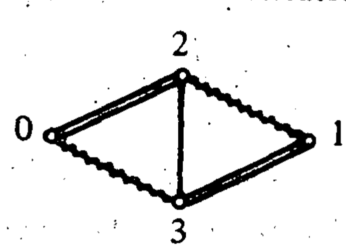
From what we said before it follows that to be a congruence lattice, a subset $\mathscr{L} \subseteq \mathscr{E}(X)$ has to be closed under all these graphical compositions $P_{G,0,1}$. Now before we give the abstract definition we want to give some examples of graphical composition of more complicated type than in 1.4. So, for example, what operation is defined by the 3-coloured graph

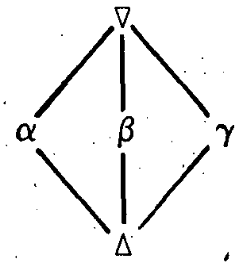


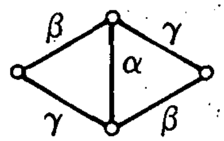
which we call the *"Wheatsone Bridge"*.

The next example gives a complete sublattice of some $\mathscr{E}(X)$ which is not closed under this graphical composition, so this is an essentially new operation on $\mathscr{E}(X)$.

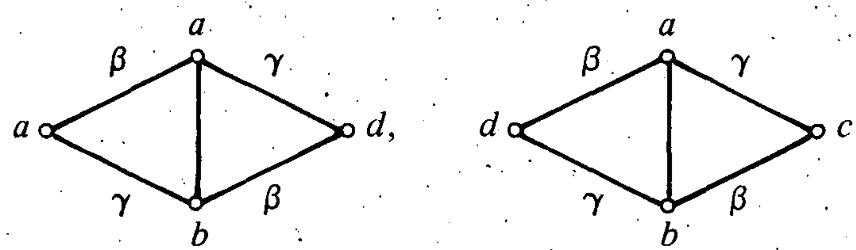
Example 1.5. Let us consider the *"Wheatstone Bridge"*



Let X be the set $\{a, b, c, d\}$ and consider the partitions $\alpha = (ab)(c)(d)$, $\beta = (ac)(bd)$, and $\gamma = (ad)(bc)$ on X, then we have the sublattice $\mathscr{L} =$  of $\mathscr{E}(X)$. Now take the following labeling

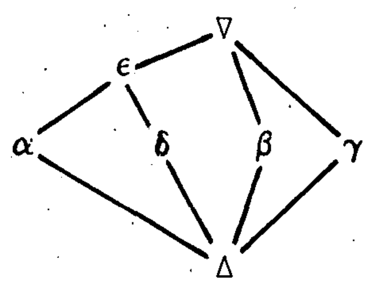
of the wheatstone bridge .

One easily checks that all compatible labelings that can occur are



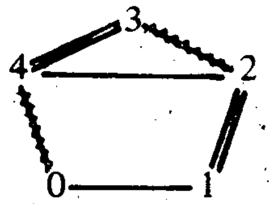
and all trivial labelings (i.e. all vertices have the same label).

So our operation defines the new equivalence relation $\delta = (a)(b)(cd)$, hence in the closure of $\mathscr{L}$ there must also be the join of α and δ, namely $\epsilon = \alpha \vee \delta = (ab)(cd)$. The sublattice

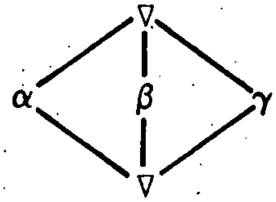


of $\mathscr{E}(X)$ turns out to be the congruence lattice of the algebra $(X; f)$, where $f\colon X \to X$ is defined by $f(1) = 2$, $f(2) = 1$, $f(3) = 4$, and $f(4) = 3$, so this lattice is the closure of $\mathscr{L}$ in $\mathscr{E}(X)$.

Example 1.6. In this example we consider a partition lattice on the five element set $X = \{a, b, c, d, e\}$ which is closed under all operations defined by the wheatstone bridge but not under the operations defined by the graph

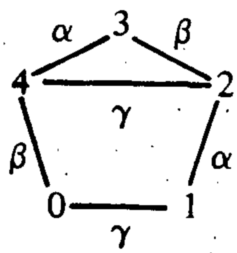


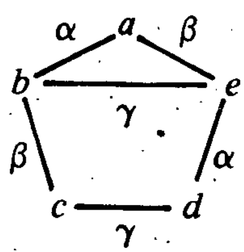
Let $\alpha = (ab)(c)(de)$, $\beta = (ae)(bc)(d)$, and $\gamma = (a)(be)(cd)$. Then

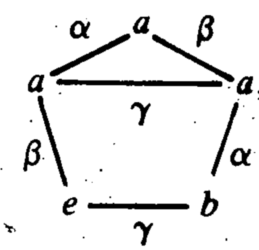
$\mathscr{L} =$ 

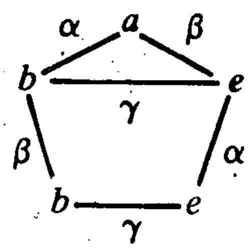
is a sublattice of $\mathscr{E}(X)$.

It turns out that the wheatstone bridge does not define any new equivalence relations and the only nontrivial compatible labelings of the graph

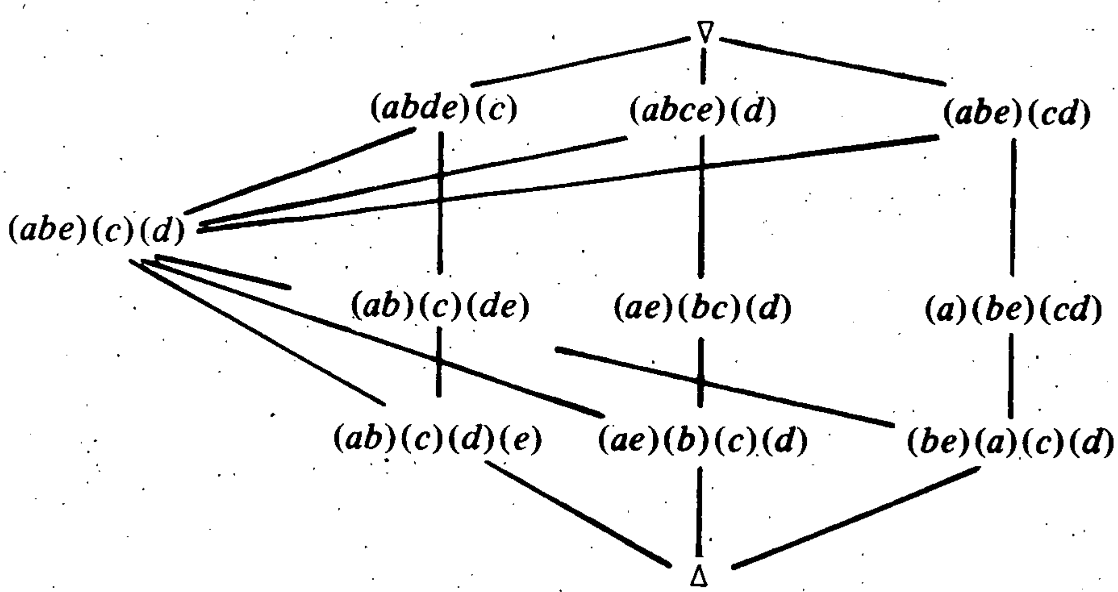


are




, and


Now for the particular choice of two vertices $(0, 2), (1, 4), (2, 3)$ $(3, 4), (4, 2)$ we get the relations $(a, b, c, e)(d)$, $(abde)(c)$, $(ae)(b)(c)(d)$, and $(a)(be)(c)(d)$ and so we get as the closure of $\mathscr{L}$ the lattice



which is the congruence lattice of the algebra $(X;\ f, g)$, where $f, g\colon X \to X$ are defined as follows:

	a	b	c	d	e
f	a	a	e	b	a
g	a	b	b	e	e

Now we are ready to give the formal definition for the graphical compositions and in the next section we are going to prove that the congruecne lattices are exactly those subsets of $\mathscr{E}(X)$ which are closed under the infinite join and all graphical compositions.

Definition 1.7. A *coloured graph* (undirected, without loops) is a triple (V, E, C) of sets V, E, and C whose elements are called *vertices*, and *edges*, and *colours*, respectively, together with maps $v\colon E \to \mathfrak{P}_2(V) = \{P \mid P \subseteq V\ \&\ |P| = 2\}$ and $c\colon E \to C$ which is onto. If $v(e) = \{x, y\}$, we say e *is an edge between* x *and* y.

Let $G = (V, E, C)$ be a coloured graph and $0, 1 \in V$ two distinguished vertices. For every mapping $\varphi\colon C \to \mathscr{E}(X)$ we define a relation $S_{G,0,1}(\varphi) := \{(f0, f1) \mid f\colon V \to X\ \&\ \forall e \in E\ \ v(e) = \{x, y\} \Rightarrow (fx, fy) \in \varphi c(e)\}$ and $P_{G,0,1}(\varphi) := \mathrm{Eq}\,(S_{G,0,1}(\varphi))$. This defines a mapping $P_{G,0,1}\colon \mathscr{E}(X)^C \to \mathscr{E}(X)$, which we call a *graphical composition.*

2. THE CONCRETE CHARACTERIZATION

In this section we prove that a complete sublattice $\mathscr{L}$ of $\mathscr{E}(X)$ is a congruence lattice if and only if it is closed under all graphical compositions, in fact we shall prove that one graphical composition will do, namely the one defined by the complete graph on the set X. In order to formulate this result properly, we need the concept of a relational system (which turns out to be equivalent to the notion of a coloured graph) and the notion of homomorphisms between them.

Definition 2.1. A *relational system* is a set X together with a set $\mathfrak{M}$ of binary relations on X (i.e. $\mathfrak{M} \subseteq \mathfrak{R}(X)$). In the sequel we always assume all relations in $\mathfrak{M}$ to be reflexive and symmetric. If $(X, \mathfrak{M})$ and

$(Y, \mathfrak{N})$ are relational systems and $\varphi\colon \mathfrak{M} \to \mathfrak{N}$ is a mapping then we call a function $f\colon X \to Y$ a *φ-homomorphism* (Notation: $f \in \mathrm{Hom}_\varphi (X, Y)$) if $\forall x, y \in X\colon (x, y) \in R \in \mathfrak{M} \Rightarrow (fx, fy) \in \varphi R$.

The homomorphisms are essentially those maps which preserve the relations, so if $(X, \mathfrak{M})$ is a relational system and $\iota\colon \mathfrak{M} \to \mathfrak{M}$ is the identity map then $\mathrm{Hom}_\iota (X, X) = D(\mathfrak{M})$, that is the endomorphisms of a relational system are exactly the dilatations of the given set of relations.

This new concept is linked to the notions of the preceding sections by the

Remark 2.2.

(1) There is a 1-1-correspondence between relational systems and coloured graphs. This 1-1-correspondence is established in the following manner:

For a graph (V, E, C) we introduce for every colour $u \in C$ a (reflexive symmetrical) relation $R_u := \Delta \cup \{(x, y) \mid \exists e \in E\colon v(e) = \{x, y\}\ \&\ c(e) = u\}$ on V and thus get a relational system $(V, \mathfrak{M})$.

For a relational system $(V, \mathfrak{M})$ we define the set of colours $C := \mathfrak{M}$ and introduce $E := \{(\{x, y\}, R) \mid x \neq y\ \&\ (x, y) \in R \in \mathfrak{M}\}$ with the obvious mappings $v\colon E \to \mathfrak{P}(V)$ and $c\colon E \to C \to \mathfrak{M}$.

(2) If $G = (V, E, C)$ is a coloured graph, $0, 1 \in V$, and if $(V, \mathfrak{M})$ is the corresponding relational system then considering the relational system $(X, (X))$ and observing that every map $\varphi\colon C \to \mathscr{E}(X)$ can also be considered as a map $\varphi\colon \mathfrak{M} \to \mathscr{E}(X)$, we get $S_{G,0,1}(\varphi) = \{(f0, f1) \mid f \in \mathrm{Hom}_\varphi (V, X)\}$.

The following theorem gives a characterization of the closure $[\mathfrak{M}] = \{R \in \mathfrak{R}(X) \mid R$ admits $D(\mathfrak{M})\}$ for every $\mathfrak{M} \subseteq \mathfrak{R}(X)$.

Theorem 2.3. *Let $(X, \mathfrak{M})$ be a rational system and $R \in \mathfrak{R}(X)$. Then the following conditions are equivalent:*

(1) $R \in [\mathfrak{M}]$;

(2) *There is a relation $S \subseteq X \times X$ such that*

$$R = \bigcup_{f \in D(\mathfrak{M})} (f \times f)S, \quad \textit{where} \quad (f \times f)S = \{(fx, fy) \mid (x, y) \in S\};$$

(3) *There is a relational system* $(Y, \mathfrak{N})$, *a mapping* $\varphi\colon \mathfrak{N} \to \mathfrak{M}$, *and a relation* $S \subseteq Y \times Y$ *such that*

$$R = \bigcup_{f \in \mathrm{Hom}_\varphi(X, Y)} (f \times f)S.$$

Proof.

(1) $\Rightarrow$ (2). Choose $S := R$. As R admits $D(\mathfrak{M})$, we get "$\supseteq$" and we get the other direction "$\subseteq$" as $\mathrm{Id}_x \in D(\mathfrak{M})$.

(2) $\Rightarrow$ (3) being trivial, we have to show (3) $\Rightarrow$ (1), so let $g \in D(\mathfrak{M})$. As $g \in \mathrm{Hom}_\iota(X, X)$, we get for every $f \in \mathrm{Hom}_\varphi(Y, X)$ $gf \in \mathrm{Hom}_\varphi(Y, X)$.

$$(g \times g)R = (g \times g) \bigcup_{f \in \mathrm{Hom}_\varphi(Y,X)} (f \times f)S =$$

$$= \bigcup_{f \in \mathrm{Hom}_\varphi(Y,X)} (gf \times gf)S \subseteq \bigcup_{h \in \mathrm{Hom}_\varphi(Y,X)} (h \times h)S = R.$$

Let us remark that (3) $\Rightarrow$ (1) is also the proof that the relation $S_{G,0,1}(\varphi)$ in Def. 1.7 admits all functions which preserve every relation $\varphi(c)$ $(c \in C)$.

As a byproduct we can prove in a very similar fashion the following theorem characterizing those functions which preserve a given equivalence relation.

For $f\colon X \to X$ $\mathrm{Ker}\, f$ denotes the relation $\mathrm{Ker}\, f := \{(x, y) \mid fx = fy\}$.

Theorem 2.4. *For* $\Theta \in \mathscr{E}(X)$ *and* $f\colon X \to X$ *the following conditions are equivalent:*

(1) f *preserves* Θ;

(2) *For every map* $g\colon X \to X$ *with* $\mathrm{Ker}\, g = \Theta$, $\Theta \subseteq \mathrm{Ker}\, gf$;

(3) *There is a map* $g\colon X \to X$ *such that* $\mathrm{Ker}\, g \subseteq \Theta \subseteq \mathrm{Ker}\, gf$.

Proof.

(1) $\Rightarrow$ (2). If $(x, y) \in \Theta$ then by (1) $(fx, fy) \in \Theta = \operatorname{Ker} g$, hence $gfx = gfy$.

(2) $\Rightarrow$ (3) being trivial, we prove (3) $\Rightarrow$ (1). Let $(x, y) \in \Theta$, then $gfx = gfy$, hence $(fx, fy) \in \operatorname{Ker} g \subseteq \Theta$.

Returning to the graphical compositions we first prove in the following lemma that all graphical compositions are generated by those stemming from the complete graphs. The complete graph on the set X, denoted by X^*, has vertices X and exactly one edge between each two distinct vertices and all edges have different colours, so $X^* = (X, \mathfrak{P}_2(X), \mathfrak{P}_2(X))$.

Lemma 2.5. *Let $\mathscr{L} \subseteq \mathscr{E}(X)$ be a closure system and let V be a set. Then the following conditions are equivalent.*

(1) *$\mathscr{L}$ is closed under $P_{G,0,1}$ for every coloured graph with vertex set V.*

(2) *$\mathscr{L}$ is closed under $P_{V^*,0,1}$ for an arbitrary choice of $0, 1 \in V$, $0 \neq 1$.*

Proof.

(1) $\Rightarrow$ (2) is trivial, so we have to prove (2) $\Rightarrow$ (1). Let $G = (V, E, C)$ be a coloured graph and $\varphi\colon C \to \mathscr{E}(X)$ a mapping.

Define $\psi\colon \mathfrak{P}_2(V) \to \mathscr{E}(X)$ by $\psi(\{x, y\})| = \bigcap \{\varphi c(e) \mid e \in E \;\&\; \nu(e) = \{x, y\}\}$. It can be easily seen that $\operatorname{Hom}_\varphi (V, X) = \operatorname{Hom}_\psi (V, X)$ and hence $P_{G,0,1}(\varphi) = P_{V^*,0,1}(\psi)$. As the graph V^* is completely symmetric, $P_{V^*,0,1}$ is the same operation up to the sequence of arguments for every choice of $0, 1 \in V$, $0 \neq 1$. Now we are ready to give the characterization theorem.

Theorem 2.6. *For a complete sublattice $\mathscr{L}$ of $\mathscr{E}(X)$ the following conditions are equivalent:*

(1) *$\mathscr{L}$ is the congruence lattice of a suitable algebra (X, F).*

(2) *$\mathscr{L}$ is closed under all graphical compositions.*

(3) $\mathscr{L}$ *is closed under the graphical composition* $P_{X^*,0,1}$ *for some* $0, 1 \in X$, $0 \neq 1$.

Proof. (1) ⇒ (2) is Theorem 2.3 (3) ⇒ (1). (2) ⇒ (3) is trivial, so we have to show (3) ⇒ (1). By 1.3 we can formulate (1) as follows: $\Theta \in \mathfrak{C}(X, D(\mathscr{L})) = [\mathscr{L}] \cap \mathscr{E}(X) \Rightarrow \Theta \in \mathscr{L}$. Assuming (3) we get by 2.5 that $\mathscr{L}$ is closed under all graphical compositions stemming from coloured graphs with vertex set X. As $\Theta \in [\mathscr{L}]$, we can find a relation $S \subseteq X \times X$ such that $\Theta = \bigcup_{f \in D(\mathscr{L})} (f \times f) S$ by 2.3 (2).

$$\Theta = \bigcup_{f \in D(\mathscr{L})} \bigcup_{(x,y) \in S} \{(fx, fy)\} = \bigcup_{(x,y) \in S} S_{G,x,y}(\iota),$$

where G is the graph on X corresponding to $(X, \mathscr{L})$ and $\iota\colon \mathfrak{M} \to \mathfrak{M}$ is the identity function (see 2.2). As Θ is an equivalence realtion, we get $\Theta = \bigcup_{(x,y) \in S} S_{G,x,y}(\iota) = \bigcup_{(x,y) \in S} P_{G,x,y}(i)$. All $P_{G,x,y}(\iota)$ are in $\mathscr{L}$ and $\mathscr{L}$ is closed under joins, hence $\Theta \in \mathscr{L}$.

We have now characterized the congruence lattices as the complete sublattices of $\mathscr{E}(X)$ which are closed under P_{X^*}. The operation P_{X^*}, however, is already of high arity on comparatively small sets and on infinite sets even infinitary. So the following questions arise immediately:

Problem 1. Do the finitary graphical compositions together with the infinite meet and join generate all graphical compositions?

Problem 2. Is there a finite set of finitary graphical compositions which together with the infinite meet and join generate all finitary graphical compositions?

3. CONNECTIONS WITH KNOWN RESULTS

It is well known that every congruence lattice of an algebra with underlying set X is an inductive closure system $\mathscr{L}$ on $\mathscr{E}(X)$ and, moreover, it is completely determined by its principal congruences in a sense described below. So we start this section by reformulating the characterization Theorem 2.6 in terms of principal congruences only.

If $\mathscr{L} \subseteq \mathscr{E}(x)$ is a closure system then we have for $M \subseteq X$ the smallest member of $\mathscr{L}$ which identifies M, namely $\mathscr{L}(M) := \bigcap \{\Theta \in \mathscr{L} \mid \forall x \in M \; M \subseteq [x]\Theta\}$. In the case $M = \{x, y\}$ we write $\mathscr{L}(x, y)$ instead of $\mathscr{L}(\{x, y\})$ and these relations $\mathscr{L}(x, y)$ $x, y \in X$ are called the *principal $\mathscr{L}$-congruences*. If $\mathscr{L}$ is a congruence lattice, then it has the following two additional properties

(P1) $\forall \Theta \in \mathscr{E}(x)$ $[\forall (x, y) \in \Theta \;\; \mathscr{L}(x, y) \subseteq \Theta \Rightarrow \Theta \in \mathscr{L}]$

(P2) $\forall x, y, z \in X$ $\mathscr{L}(x, z) \subseteq \mathscr{L}(x, y) \vee \mathscr{L}(y, z)$ (the join $\vee$ is taken in $\mathscr{E}(X)$!).

In the sequel $\vee$ and $\bigvee$ will always denote the join in $\mathscr{E}(X)$.

Property (P1) expresses the fact that $\mathscr{L}$ is completely determined by its principal congruences, in fact every $\Theta \in \mathscr{L}$ is the union of principal congruences, hence, in particular $\mathscr{L}$ is inductive. (P2) says that $\mathscr{L}$ is a sublattice of $\mathscr{E}(X)$ and it is equivalent to either of the statements $\mathscr{L}(\{x, y, z\}) = \mathscr{L}(x, y) \vee \mathscr{L}(x, z)$ for every $x, y, z \in X$, or $\mathscr{L}(M) = \bigvee \{\mathscr{L}(x, y) \mid x, y \in M\}$ for every $M \subseteq X$.

Closure systems $\mathscr{L} \subseteq \mathscr{E}(X)$ with property (P1) have been studied in [12] section 2, where they were called *equivalence class geometries* and they were identified as the congruence lattices of finitary *partial* algebras. If one defines a mapping Π: $X \times \mathfrak{P}(X) \to \mathfrak{P}(X)$ by $\Pi(x \mid M) := [x]\mathscr{L}(M)$ then by [12] Cor. 2.4 property (P1) is equivalent to the equations $\Pi(x \mid \phi) = \{x\}$, $y \in \Pi(x \mid \{x, y\})$, $\Pi(x \mid \Pi(y \mid M)) \subseteq \Pi(x \mid M)$, and $\Pi(x \mid M) = \bigcup \{\Pi(x \mid F) \mid F \subseteq M \text{ finite}\}$. We want to give a characterization of those equivalence class geometries which stem from full algebras, i.e. which are congruence class geometries in the sense of Wille [13].

For his characterization of congruence class geometries Wille introduced for a submonoid D of X^X and a subset M of X the notion of congruence modulo M and D by $x \doteq y \pmod{M; D}$ iff x and y are in the same class of the partition generated by $\{\delta(M) \mid \delta \in D\}$. We are going to prove the following

Theorem 3.1. *Let be $\mathscr{L} \subseteq \mathscr{E}(X)$ a closure system. Then the following conditions are equivalent*

(1) $\mathscr{L}$ *is the congruence lattice of some algebra* (X, f);

(2) (a) $[\forall x \in N \forall y,\ z \in M\ \ \Pi(x \mid y, z) \subseteq N] \Rightarrow \forall x \in N\ \ \Pi(x \mid M) \subseteq N$

(b) $v \in \Pi(u \mid x, y) \Rightarrow v \equiv u \pmod{\{x, y\}, D(\mathscr{L})}$;

(3) $\mathscr{L}$ *has the properties* (P1) *and*

(P3) $P_{X^*, x, y}(\lambda) = \mathscr{L}(x, y)$ *where* $\lambda\colon \mathfrak{P}_2(X) \to \mathscr{L}$ *is defined by* $\lambda\colon \{x, y\} \mapsto \mathscr{L}(x, y)$.

For the proof of this theorem we need a few auxiliary lemmas.

Lemma 3.2. *Let* $\mathscr{L} \subseteq \mathscr{E}(X)$ *be a closure system with property* (P1) *and* (P2). *If* $R \subseteq X \times X$ *has property*

(*) $\forall (x, y) \in R\colon\ \mathscr{L}(x, y) \subseteq \mathrm{Eq}\,(R)$, *then* $\mathrm{Eq}\,(R) \in \mathscr{L}$.

Proof. (P2) has a consequence that if R, S have (*) then so do R^{-1} and $R \circ S$. Now by (P1) the assertion follows.

Lemma 3.3. *Let* $\mathscr{L} \subseteq \mathscr{E}(X)$ *be a closure system and* $\lambda\colon \mathfrak{P}_2(X) \to \mathscr{L}$, $\lambda\colon \{x, y\} \mapsto \mathscr{L}(x, y)$. *Then the following conditions are equivalent*

(1) $P_{X^*, x, y}(\lambda) \in \mathscr{L}$

(2) $\mathscr{L}(x, y) \subseteq P_{X^*, x, y}(\lambda)$

(3) $\mathscr{L}(x, y) = P_{X^*, x, y}(\lambda)$

Proof. Observe that $\mathrm{Hom}_\lambda(X, X) = D(\mathscr{L})$. As $\mathrm{Id}_X \in D(\mathscr{L})$, we have $(x, y) \in P_{X^*, x, y}(\lambda)$, hence (1) $\Rightarrow$ (2). $\lambda\{x, y\} = \mathscr{L}(x, y)$, so $P_{X^*, x, y}(\lambda) \subseteq \mathscr{L}(x, y)$ and therefore (2) $\Rightarrow$ (3). (3) $\Rightarrow$ (1) is trivial.

Lemma 3.4. *For every closure system* $\mathscr{L} \subseteq \mathscr{E}(X)$ *property* (P1) *is equivalent to property* (2.a) *in Theorem* 3.1.

Proof.

(P1) $\Rightarrow$ (2.a). Let be $M, N \subseteq X$, $\forall u \in N\ \forall y, z \in M\ \ \Pi(u \mid y, z) \subseteq N$, $x \in N$, $v \in \Pi(x \mid M)$. To show $v \in N$. $v \in \Pi(x \mid M) = [x]\, \mathscr{L}(M) = [x] \bigvee_{y, z \in M} \mathscr{L}(y, z)$ by (P1).

$\Rightarrow \exists u_0, \ldots, u_n \in X$, $y_1, \ldots, y_n, z_1, \ldots, z_n \in M$: $x = u_0$, $u_i \in \Pi(u_{i-1} \mid y_i, z_i)$, $u_n = v$. Now for every $i \leqslant n$ it follows that $u_n = v$, thus $v \in N$.

(2.a) $\Rightarrow$ (P1). For every closure system $\mathscr{L} \subseteq \mathscr{E}(X)$ the following hold: $\Pi(x \mid \phi) = \{x\}$, $y \in \Pi(x \mid x, y)$, $\Pi(x \mid \Pi(y \mid M)) \subseteq \Pi(x \mid M)$ and $F \subseteq M \Rightarrow \Pi(x \mid F) \subseteq \Pi(x \mid M)$, so by [12] 2.4 it remains to show $\Pi(x \mid M) \subseteq \cup \{\Pi(x \mid F) \mid F \subseteq M$ finite$\}$ but that is an easy consequence of (2.a).

Lemma 3.5. *Let* $\mathscr{L} \subseteq \mathscr{E}(X)$ *be a closure system with* (P1) *and let* $\lambda: \mathfrak{P}_2(X) \to \mathscr{L}$ $\lambda: \{x, y\} \mapsto \mathscr{L}(x, y)$. *Then for every* $\varphi: \mathfrak{P}_2(X) \to \mathscr{L}$ *and* $f \in \mathrm{Hom}_\varphi(X, X)$ *we have* $P_{X^*, f0, f1}(\lambda) \subseteq P_{X^*, 0, 1}(\varphi)$.

Proof. We first make the observation $g \in \mathrm{Hom}_\lambda(X, X) \Rightarrow gf \in \mathrm{Hom}_\varphi(X, X)$. Namely, if $x, y \in X$, $x \neq y$ then $(gfx, gfy) \in \mathscr{L}(fx, fy) \subseteq \varphi(\{x, y\})$, as $(fx, fy) \in \varphi(\{x, y\}) \in \mathscr{L}$. Now assume $(x, y) \in S_{X^*, f0, f1}(\lambda)$. Then there is a $g \in \mathrm{Hom}_\lambda(X, X)$ such that $(gf0, gf1) = (x, y)$, but by the above observation $(x, y) = (gf0, gf1) \in S_{X^*, 0, 1}(\varphi)$. Now $S_{X^*, f0, f1}(\lambda) \subseteq S_{X^*, 0, 1}(\varphi)$ implies $P_{X^*, f0, f1}(\lambda) \subseteq P_{X^*, 0, 1}(\varphi)$.

Now we can proceed to the *proof of Theorem* 3.1.

It is easy to derive the following two statements just from the definitions.

(A) $\qquad \mathrm{Hom}_\lambda(X, X) = D(\mathscr{L})$.

(B) $\qquad P_{X^*, x, y}(\lambda) = \{(u, v) \mid u \equiv v \pmod{\{x, y\};\ D(\mathscr{L})}\}$.

In particular, (B) together with Lemma 3.4 shows the equivalence of (2) and (3). From (A) we get for every $x, y, z \in X$ $S_{X^*, x, z}(\lambda) \in S_{X^*, x, y}(\lambda) \circ S_{X^*, y, z}(\lambda)$, hence $P_{X^*, x, z}(\lambda) \subseteq P_{X^*, x, y}(\lambda) \vee P_{X^*, y, z}(\lambda)$, so (P3) implies (P2). By Theorem 2.6, to show (3) $\Rightarrow$ (1) we have only to show $\varphi: \mathfrak{P}_2(X) \to \mathscr{L}$ implies $P_{X^*, 0, 1}(\varphi) \in \mathscr{L}$. By Lemma 3.2 it suffices to show that the relation $R = S_{X^*, 0, 1}(\varphi)$ has property $(*)$, i.e., $\forall (x, y) \in R$: $\mathscr{L}(x, y) \subseteq \mathrm{Eq}(R) = P_{X^*, 0, 1}(\varphi)$.

If $(x, y) \in R$ then there is an $f \in \mathrm{Hom}_\varphi(X, X)$ such that $(f0, f1) = (x, y)$. By Lemma 3.5 we have $P_{X^*, f0, f1}(\lambda) \subseteq P_{X^*, 0, 1}(\varphi)$, but (P3) says $P_{X^*,f0,f1}(\lambda) = \mathscr{L}(x, y)$, and this finishes the proof of Theorem 3.1.

In [12] 2.4 closure systems with property (P1) have been characterized by equations for the corresponding operator $\Pi\colon X \times \mathfrak{P}(X) \to \mathfrak{P}(X)$.

Problem 3. Is it possible to characterize property (P3) in a similar fashion by equations for Π?

Before we show the connections of this result and the results of Wille, Armbrust, and Quackenbush – Wolk we deduce a corollary from Theorem 3.1, which tells us how to get the smallest congruence lattice containing a given closure system $\mathscr{L}$ on $\mathscr{E}(X)$.

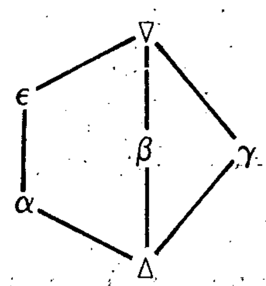
Corollary 3.6. *Let* $\mathscr{L} \subseteq \mathscr{E}(X)$ *be a closure system,* $\lambda\colon \mathfrak{P}_2(X) \to \mathscr{L}$, $\lambda\colon \{x, y\} \mapsto \mathscr{L}(x, y)$. *Define for* $x, y \in X$ $\mathscr{L}'(x, y) := P_{X^*,x,y}(\lambda)$ *and* $\mathscr{L}' := \{\Theta \in \mathscr{E}(X) \mid (x, y) \in \Theta \Rightarrow \mathscr{L}'(x, y) \subseteq \Theta\}$. *Then* $\mathscr{L}' = \mathfrak{C}(X, D(\mathscr{L}))$.

Proof. Define $\bar{\lambda}\colon \mathfrak{P}_2(X) \to \mathscr{L}'$ by $\bar{\lambda}(\{x, y\}) := \mathscr{L}'(x, y)$. $\mathrm{Hom}_\lambda(X, X) = \mathrm{Hom}_{\bar{\lambda}}(X, X)$, hence $P_{X^*,x,y}(\bar{\lambda}) = P_{X^*,x,y}(\lambda) = \mathscr{L}'(x, y)$, and by 3.1 $\mathscr{L}'$ is a congruence lattice, and as $D(\mathscr{L}) = D(\mathscr{L}')$, we even have $\mathscr{L}' = \mathfrak{C}(X, D(\mathscr{L}))$.

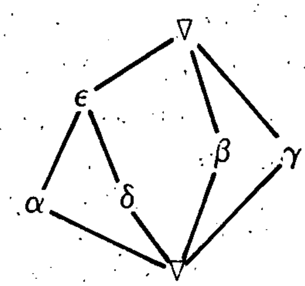
This corollary also provides a proof for the claims in Examples 1.5 and 1.6 that the given lattices are in fact congruence lattices and the closures of the lattices we started with.

In [13] Satz 3.5 R. Wille gave a characterization of the system of congruence classes of an algebra but his result does not lead to a characterization of congruence lattices.

Take for example the lattices



and



of Example 1.5.

These two systems of equivalence realtions have the same dilatations and the same system of classes, so both are congruence class geometries, but one is a congruence lattice while the other isn't.

Taking the definition $[\phi] := \phi$, $[N] := \Pi(x \mid N)$ $x \in N$, and observing $\Pi(x \mid N) \subseteq [N \cup \{x\}]$ one easily deduces from 3.1. (2) the conditions in [13] Satz 3.5.

In [10] R. Quackenbush & B. Wolk proved that any finite distributive sublattice of $\mathscr{E}(X)$ is a congruence lattice and this result can be extended to arbitrary completely distributive complete sublattices of $\mathscr{E}(X)$ as it is done by H. Draškovičová in [4]. In fact, in these cases the algebra can be chosen in such a way that all operations take at most two different values. M. Armbrust [1] has given a characterization of the congruence lattice of such "2-*valued algebras*" as follows:

Theorem 3.7 (Armbrust). *Let $\mathscr{L}$ be a subset of $\mathscr{E}(X)$ and define for $x, y \in X$ $\alpha(x,y) := \bigvee \{\Theta \in \mathscr{L} \mid (x,y) \notin \Phi\}$. $\mathscr{L}$ is the congruence lattice of a 2-valued unary algebra iff $\mathscr{L} = \{\Theta \in \mathscr{E}(X) \mid \forall x, y \in X \ (x,y) \in \Theta$ or $\Theta \subseteq \alpha(x,y)\}$.*

For completeness sake we want to give short proofs for

Corollary 3.8. *Let $\mathscr{L}$ be a subset of $\mathscr{E}(X)$ and define for $x, y \in X$ $\alpha(x,y) := \bigvee \{\Theta \in \mathscr{L} \mid (x,y) \notin \Theta\}$. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3), where*

(1) *$\mathscr{L}$ is a completely distributive complete sublattice of $\mathscr{E}(X)$.*

(2) $\mathscr{L} = \{\Theta \in \mathscr{E}(X) \mid \forall x, y \in X \ (x,y) \in \Theta$ *or* $\Theta \subseteq \alpha(x,y)\}$.

(3) *$\mathscr{L}$ is the congruence lattice of a 2-valued algebra (X, F).*

Proof.

(1) $\Rightarrow$ (2) (H. Draškovičová). Obviously, every $\Theta \in \mathscr{L}$ satisfies $(x,y) \in \Theta$ or $\Theta \subseteq \alpha(x,y)$.

Let $\Theta \in \mathscr{E}(X)$ satisfy $(x,y) \notin \Theta \Rightarrow \Theta \subseteq \alpha(x,y)$, hence $\Theta \subseteq \bigcap_{(x,y) \notin \Theta} \alpha(x,y)$. Now $\alpha(x,y) = \bigvee \{\Phi \in \mathscr{L} \mid (x,y) \notin \Phi\}$.
Let $J := \bigtimes_{(x,y) \notin \Theta} \{\Phi \in \mathscr{L} \mid (x,y) \notin \Phi\}$, then by complete distributivity

$\bigcap_{(x,y)\notin\Theta} \alpha(x, y) = \bigvee_{\varphi\in J} \bigcap_{(x,y)\notin\Theta} \varphi(x, y)$. As for every $\varphi\in J$ $\bigcap_{(x,y)\notin\Theta} \varphi(x, y) \subseteq \Theta$, we get $\Theta \subseteq \bigcap_{(x,y)\notin\Theta} \alpha(x, y) \subseteq \Theta$ and thus $\Theta \in \mathscr{L}$.

(2) $\Rightarrow$ (3) (Armbrust). For $x, y \subset X$ put δ: $X \to X$,

$$\delta a = \begin{cases} x & \text{if} \quad (x, a) \in \alpha(x, y) \\ y & \text{otherwise.} \end{cases}$$

An equivalence Θ preserves all these δ iff $(x, y) \notin \Theta \Rightarrow \Theta \subseteq \alpha(x, y)$, thus $\mathscr{L}$ is the congruence lattice of (X, F), where F is the set of all these δ's.

As a last problem the simultaneous concrete representation of different structures associated with algebras arises. In [11] M. Stone has given a simultaneous concrete representation for subalgebra lattices together with automorphism groups.

Problem 4. Give simultaneous concrete characterization of

(a) subalgebra & congruence lattices,

(b) automorphism group & congruence lattice,

(c) all three structures together.

Conjecture. Let $\mathscr{L} \subseteq \mathscr{E}(X)$ be a complete sublattice and G a permutation group on X. There is an algebra (X, F) such that $\mathscr{L} = \mathfrak{C}(X, F)$, $G = \operatorname{Aut}(X, F)$ iff

(a) $\mathscr{L}$ is closed under $P_{X^*, x, y}$.

(b) G is locally closed (see [7]).

(c) If $g \in G$ and $\Theta \in \mathscr{L}$ then $(g \times g)\Theta \in \mathscr{L}$.

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