ON A PROBLEM OF PÁLFY AND SAXL

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1. Introduction

In the paper [1], Péter Pálfy and Jan Saxl pose the following

> Problem. Let $\mathbf{A}$ be a finite algebra with $\operatorname{Con}\mathbf{A} \cong M_n$, $n \geqslant 4$. If three nontrivial congruences of $\mathbf{A}$ pairwise permute, does it follow that every pair of congruences of $\mathbf{A}$ permute?

These notes collect some notation and facts that might be useful for attacking this problem. Throughout, X denotes a finite set, $\mathrm{Eq}(X)$ denotes the lattice of equivalence relations on X and, for $\alpha \in \mathrm{Eq}(X)$ and $x \in X$, we denote by x/α the equivalence class of α containing x. We often refer to equivalence classes as "blocks," and we denote by $\#\mathrm{Blocks}(\alpha)$ the number of blocks of the equivalence relation α.

For a given $\alpha \in \mathrm{Eq}(X)$ the map $\varphi_\alpha : x \mapsto x/\alpha$ is a function from X into the power set $\mathscr{P}(X)$ with kernel $\ker \varphi_\alpha = \alpha$. The *block-size function* $x \mapsto |x/\alpha|$ is a function from X into $\{1, 2, \dots, |X|\}$.

We will often abuse notation and equate an equivalence relation with the corresponding partition of the set X. For example, we will equate the relation

$$\alpha = \{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0), (2,3), (3,2)\}$$

with the partition $|0,1|2,3|$, and often we resort to writing $\alpha = |0,1|2,3|$.

We say that α has *uniform blocks* if all blocks of α have the same size; or, equivalently, the block-size function is constant: for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. We will use $|x/\alpha|$, without specifying a particular $x \in X$, to denote this block size. Thus, when α has uniform blocks, we have $|X| = |x/\alpha| \cdot \#\mathrm{Blocks}(\alpha)$.

We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary blocks*, if the number of blocks of one is equal to the block size of the other. In other words, if α and β are two equivalence relations on X with uniform block sizes $|x/\alpha|$ and $|x/\beta|$, respectively, then α and β have complementary blocks if and only if $(\forall x)(\forall y)\, |x/\alpha| \cdot |y/\beta| = |X|$.

Given two equivalence relations α and β on X, the relation

$$\alpha \circ \beta = \{(x,y) \in X^2 : (\exists z) x \; \alpha \; z \; \beta \; y\}$$

is called the *composition of* α *and* β, and if $\alpha \circ \beta = \beta \circ \alpha$ then α and β are said to *permute*, or to be *permuting* equivalence relations. Note that $\alpha \circ \beta \subseteq \alpha \vee \beta$ with equality if and only if α and β permute.

The largest and smallest equivalence relations on X are $1_X = X^2$ and $0_X = \{(x,x) : x \in X\}$, respectively.

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One more piece of shorthand notation will be useful below. Suppose Θ is a set of equivalence relations that are *pairwise permuting pairwise complements*—that is, for all $\gamma \neq \delta$ in Θ, we have

$$\gamma \circ \delta = \delta \circ \gamma, \quad \gamma \wedge \delta = 0_X, \quad \gamma \vee \delta = 1_X.$$

Then we say that the relations in Θ are PPPC.

2. Basic observations

We say that α and β are *complementary* equivalence relations on X provided $\alpha \vee \beta = 1_X$ and $\alpha \wedge \beta = 0_X$.

Lemma 1. Suppose α and β are complementary equivalence relations on X. Then α and β permute if and only if they have complementary blocks. That is,

$$\alpha \circ \beta = 1_X \quad \iff \quad (\forall x)(\forall y)\, |x/\alpha| \cdot |y/\alpha| = |X|.$$

Corollary 1. Suppose α_1, α_2, α_3 are pairwise complementary equivalence relations on X. Then α_1, α_2, α_3 pairwise permute if and only if they all have uniform blocks of size $\sqrt{|X|}$. In other words,

$$(\forall i)(\forall j)\,(i \neq j \longrightarrow \alpha_i \circ \alpha_j = 1_X) \quad \iff \quad (\forall i)(\forall x)\, |x/\alpha_i| = \sqrt{|X|}.$$

In this case, we clearly have $|x/\alpha_i| = \#\text{Blocks}(\alpha_i)$.

Proof of Lemma 1. Assume $\alpha \circ \beta = \alpha \vee \beta = 1_X$. Then, for all $x \in X$ we have

$$x/(\alpha \circ \beta) = \coprod_{y \in x/\alpha} y/\beta = X, \tag{2.1}$$

where $\coprod$ denotes disjoint union. The union is disjoint since $\alpha \wedge \beta = 0_X$. Since the union in (2.1) is all of X, every block of β must appear in the union, so the block x/α has exactly $\#\text{Blocks}(\beta)$ elements. Since x was arbitrary, α has uniform blocks of size $|x/\alpha| = \#\text{Blocks}(\beta)$. Similarly, $x/(\beta \circ \alpha) = \coprod_{y \in x/\beta} y/\alpha = X$, so $|x/\beta| = \#\text{Blocks}(\alpha)$ holds for all $x \in X$. Therefore, for all $x, y \in X$, we have

$$|x/\alpha| \cdot |y/\beta| = |x/\alpha| \cdot \#\text{Blocks}(\alpha) = |X|.$$

To prove the converse, suppose α and β are pairwise complements with complementary blocks. Then $|x/\alpha| \cdot |y/\beta| = |X|$, thus $|y/\beta| = |x/\alpha|^{-1} \cdot |X| = \#\text{Blocks}(\alpha)$ hold for all $x, y \in X$. Therefore, for all $x \in X$, we have

$$\begin{aligned}
\big|x/(\alpha \circ \beta)\big| &= \Big| \coprod_{y \in x/\alpha} y/\beta \Big| = \sum_{y \in x/\alpha} |y/\beta| \\
&= \sum_{y \in x/\alpha} \#\text{Blocks}(\alpha) \\
&= |x/\alpha| \#\text{Blocks}(\alpha) = |X|.
\end{aligned}$$

This proves that $\alpha \circ \beta = 1_X$, as desired. $\square$

Proof of Corollary 1. Since α_1 and α_2 permute and are complements, Lemma 1 implies they have complementary blocks, so

$$|x/\alpha_1| = |x/\alpha_2|^{-1} \cdot |X| = \#\text{Blocks}(\alpha_2). \tag{2.2}$$

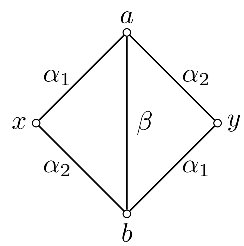

FIGURE 1. The graph defining the relation $\tau(\alpha_1, \alpha_2, \beta)$; that is, $(x, y) \in \tau(\alpha_1, \alpha_2, \beta)$ if and only if there exist $a, b \in X$ satisfying the relations in the diagram.

(This holds for all $x \in X$. Recall that complementary blocks are always uniform.) Similarly, since α_1 and α_3 permute, we have $|x/\alpha_1| = |x/\alpha_3|^{-1} \cdot |X| = \#\text{Blocks}(\alpha_3)$. Therefore, $\#\text{Blocks}(\alpha_2) = \#\text{Blocks}(\alpha_3)$. Since α_2 and α_3 permute, we have

$$|x/\alpha_2| = |x/\alpha_3|^{-1} \cdot |X| = \#\text{Blocks}(\alpha_3), \tag{2.3}$$

and the latter is equal to $\#\text{Blocks}(\alpha_2)$. Therefore,

$$|X| = |x/\alpha_2| \cdot \#\text{Blocks}(\alpha_2) = |x/\alpha_2| \cdot |x/\alpha_2|.$$

Thus, $|x/\alpha_2| = \sqrt{|X|}$, so by (2.2) and (2.3) we have $|x/\alpha_i| = \sqrt{|X|} = \#\text{Blocks}(\alpha_i)$ for $i = 1, 2, 3$.

The converse is obvious, since if α_i and α_j are complementary equivalence relations on X with $|x/\alpha_i| = \sqrt{|X|}$, then $\#\text{Blocks}(\alpha_i) = \sqrt{|X|}$, so $\alpha_i \circ \alpha_j = 1_X$. □

From Corollary 1 we see that the Pálfy-Saxl problem can be stated as

> PROBLEM. Let $\mathbf{A}$ be a finite algebra with $\operatorname{Con} \mathbf{A} \cong M_n$, $n \geqslant 4$. If three atoms of $\mathbf{A}$ have Property (2.4) below, does it follow that every atom has Property (2.4)?

$$(\forall x)\ |x/\alpha| = \sqrt{|X|} = \#\text{Blocks}(\alpha) \tag{2.4}$$

To prove that the answer is "yes," it will suffice to prove that if $M_n \leqslant \text{Eq}(X)$ has 3 atoms with Property (2.4) and an atom β with $|x/\beta| < \sqrt{|X|}$, then this M_n is not a congruence lattice.

3. GRAPHICAL COMPOSITIONS

Suppose α_1, α_2, and α_3 are pairwise permuting pairwise complements (PPPC) in $\text{Eq}(X)$, and let $\beta \in \text{Eq}(X)$ be complementary to each α_i, so that

$$L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4.$$

Define the relation $\tau = \tau(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ as follows:

$$x\ \tau\ y \quad \longleftrightarrow \quad (\exists (a, b) \in \beta)\ x\ \alpha_1\ a\ \alpha_2\ y\ \alpha_1\ b\ \alpha_2\ x.$$

Graphically, $x\ \tau\ y$ if and only if there exist $a, b \in X$ satisfying the relations depicted in Figure 1.

It is clear that τ is a a *tolerance*, that is, a reflexive and symmetric binary relation. Let $f \in X^X$ be a unary function and suppose that f is *compatible* with each relation $\theta \in \{\alpha_1, \alpha_2, \beta\}$, that is, $(u, v) \in \theta \longrightarrow (f(u), f(v)) \in \theta$. Then f is also compatible with τ. (Consider the diagram in Figure 1, and give each vertex u the label $f(u)$.)

Fact 1. If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$, then

$$\begin{aligned}
\alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta) = 0_X &= \alpha_2 \cap \tau(\alpha_1, \alpha_2, \beta),\\
\alpha_1 \cap \tau(\alpha_1, \alpha_3, \beta) = 0_X &= \alpha_3 \cap \tau(\alpha_1, \alpha_3, \beta),\\
\alpha_2 \cap \tau(\alpha_2, \alpha_3, \beta) = 0_X &= \alpha_3 \cap \tau(\alpha_2, \alpha_3, \beta).
\end{aligned}$$

Proof. Fix $(x, y) \in \alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta)$ and suppose a and b satisfy the diagram in Figure 1. Then $(x, y) \in \alpha_1$ implies $(a, b) \in \alpha_1 \wedge \beta = 0_X$, so $a = b$. Therefore, $(x, y) \in \alpha_1 \wedge \alpha_2 = 0_X$, so $x = y$. Proofs of the other identities are similar. $\square$

4. Functions Derived from Graphical Compositions

Let R_{12}^{β} be the relation on $X^2 \times X^2$ defined by

$$(a, b)\ R_{12}^{\beta}\ (x, y) \quad \longleftrightarrow \quad (a, b) \in \beta \ \text{ and } \ x\ \alpha_1\ a\ \alpha_2\ y\ \alpha_1\ b\ \alpha_2\ x.$$

Define R_{13}^{β} and R_{23}^{β} similarly. Graphically, $(a, b)\ R_{12}^{\beta}\ (x, y)$ holds if and only if the relations in Figure 1 are satisfied.

Lemma 2. Suppose α_i and α_j are complementary equivalence relations on X with uniform blocks of size $\sqrt{|X|}$. Then the relation R_{ij}^{β} restricted to $\beta \times X^2$ is a one-to-one function from β into X^2.

Proof. First we note that each pair $(a, b) \in \beta$ has at most one image. For if $(a, b)\ R_{ij}^{\beta}$ (x, y) and $(a, b)\ R_{ij}^{\beta}\ (u, v)$, then $(x, u) \in \alpha_i \wedge \alpha_j = 0_X$ and $(y, v) \in \alpha_i \wedge \alpha_j = 0_X$, so $(x, y) = (u, v)$.

Next, since both α_i and α_j have $\sqrt{|X|}$ blocks, and since each of these blocks has size $\sqrt{|X|}$, we see that each block of α_i intersects each block of α_j at exactly one point. That is, for all $a, b \in X$, the set $a/\alpha_i \cap b/\alpha_j$ is a singleton. Therefore, to each $(a, b) \in \beta$ there corresponds precisely one $(x, y) \in X^2$ such that $(a, b)\ R_{ij}^{\beta}\ (x, y)$ holds. Specifically, $\{x\} = a/\alpha_i \cap b/\alpha_j$ and $\{y\} = b/\alpha_i \cap a/\alpha_j$. Thus, R_{ij}^{β} is a function.

From now on, we let $R_{ij}^{\beta}((a, b))$ denote the image of (a, b) under R_{ij}^{β}; that is, $R_{ij}^{\beta}((a, b))$ denotes the ordered pair (x, y) satisfying $(a, b)\ R_{ij}^{\beta}\ (x, y)$.

Suppose $R_{ij}^{\beta}((a, b)) = R_{ij}^{\beta}((c, d))$. Then $(a, c) \in \alpha_i \wedge \alpha_j = 0_X$ and $(b, d) \in \alpha_i \wedge \alpha_j = 0_X$, so $(a, b) = (c, d)$. Therefore, R_{ij}^{β} is one-to-one. $\square$

If, in addition to the assumptions of Lemma 2, we assume that the image of β under R_{ij}^{β} is contained in β, then $R_{ij}^{\beta} : \beta \to \beta$ is a bijective involution. That is, R_{ij}^{β} is one-to-one and onto, and $R_{ij}^{\beta} \circ R_{ij}^{\beta}$ is the identity map.

5. Final piece of the puzzle

As above, suppose $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice with α_i PPPC. Suppose $R_{ij}^{\beta} : \beta \to \beta$ holds for all $i, j \in \{1, 2, 3\}$.

Lemma 3. If $a\ \alpha_1\ z\ \beta\ w$, then one of the following holds:

(1) $(a, w) \in \alpha_2$,
(2) $(a, w) \in \alpha_3$,

(3) $(a, w) \in \beta$,
(4) $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$,
(5) $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$.

If Lemma 3 is true, then we can prove the following:

Theorem 1. *If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice with α_i PPPC, then β permutes with each α_i.*

Proof. We will show $\alpha_1 \circ \beta \subseteq \beta \circ \alpha_1$. Assume $a\ \alpha_1\ z\ \beta\ w$. We consider each of the cases in Lemma 3 in turn and, in each case, find b satisfying $a\ \beta\ b\ \alpha_1\ w$.

(1) If $(a, w) \in \alpha_2$, then let $b = z/\alpha_2 \cap w/\alpha_1$. Then $R^\beta_{12}(z, w) = (a, b)$ and since $R^\beta_{12} : \beta \to \beta$, we have $(a, b) \in \beta$, so $a\ \beta\ b\ \alpha_1\ w$, as desired.
(2) If $(a, w) \in \alpha_3$, then let $b = z/\alpha_3 \cap w/\alpha_1$. Use the same argument as in the first case, but replace R^β_{12} with R^β_{13}.
(3) If $(a, w) \in \beta$, then let $b = a$.
(4) If $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$, then let y denote the element in this set. Let $x = z/\alpha_1 \cap w/\alpha_3$, and let $b = x/\alpha_2 \cap y/\alpha_1$. Then $(R^\beta_{12} \circ R^\beta_{13})(z, w) = R^\beta_{12}(x, y) = (a, b)$, so $(a, b) \in \beta$. Now, $b\ \alpha_1\ y\ \alpha_1\ w$, so $a\ \beta\ b\ \alpha_1\ w$, as desired.
(5) If $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$, then let y denote this element, let $x = z/\alpha_1 \cap w/\alpha_2$, and let $b = x/\alpha_3 \cap y/\alpha_1$. Then $(R^\beta_{13} \circ R^\beta_{12})(z, w) = R^\beta_{12}(x, y) = (a, b)$, so $(a, b) \in \beta$. Now, $b\ \alpha_1\ y\ \alpha_1\ w$, so $a\ \beta\ b\ \alpha_1\ w$, as desired.

□

6. Proof of Lemma 3

Consider the relation θ_{ij} defined as follows:

$$x\ \theta_{ij}\ y \quad \longleftrightarrow \quad (\exists a, b)\ a\ \alpha_i\ x\ \alpha_j\ b\ \beta\ y\ \alpha_j\ a.$$

Easy arguments similar to those above establish that

$$\theta_{ij} \cap \alpha_i = \theta_{ij} \cap \alpha_j = \theta_{ij} \cap \beta = 0_X.$$

On the other hand, since L is a congruence lattice, it must be the case that the transtive closure of θ_{ij} is contained in L.

References

[1] P. P. Pálfy and J. Saxl. Congruence lattices of finite algebras and factorizations of groups. *Comm. Algebra*, 18(9):2783–2790, 1990.