

ON A PROBLEM OF PÁLFY AND SAXL

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1. INTRODUCTION

In the paper [1], Péter Pálfi and Jan Saxl pose the following

PROBLEM. Let \mathbf{A} be a finite algebra with $\text{Con } \mathbf{A} \cong M_n$, $n \geq 4$. If three nontrivial congruences of \mathbf{A} pairwise permute, does it follow that every pair of congruences of \mathbf{A} permute?

These notes collect some notation and facts that might be useful for attacking this problem. Throughout, X denotes a finite set, $\text{Eq}(X)$ denotes the lattice of equivalence relations on X and, for $\alpha \in \text{Eq}(X)$ and $x \in X$, we denote by x/α the equivalence class of α containing x . We often refer to equivalence classes as “blocks,” and we denote by $\#\text{Blocks}(\alpha)$ the number of blocks of the equivalence relation α .

For a given $\alpha \in \text{Eq}(X)$ the map $\varphi_\alpha : x \mapsto x/\alpha$ is a function from X into the power set $\mathcal{P}(X)$ with kernel $\ker \varphi_\alpha = \alpha$. The *block-size function* $x \mapsto |x/\alpha|$ is a function from X into $\{1, 2, \dots, |X|\}$.

We will often abuse notation and equate an equivalence relation with the corresponding partition of the set X . For example, we will equate the relation

$$\alpha = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0), (2, 3), (3, 2)\}$$

with the partition $|0, 1|2, 3|$, and often we resort to writing $\alpha = |0, 1|2, 3|$.

We say that α has *uniform blocks* if all blocks of α have the same size; or, equivalently, the block-size function is constant: for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. We will use $|x/\alpha|$, without specifying a particular $x \in X$, to denote this block size. Thus, when α has uniform blocks, we have $|X| = |x/\alpha| \cdot \#\text{Blocks}(\alpha)$.

We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary blocks*, if the number of blocks of one is equal to the block size of the other. In other words, if α and β are two equivalence relations on X with uniform block sizes $|x/\alpha|$ and $|x/\beta|$, respectively, then α and β have complementary blocks if and only if $(\forall x)(\forall y) |x/\alpha| \cdot |y/\beta| = |X|$.

Given two equivalence relations α and β on X , the relation

$$\alpha \circ \beta = \{(x, y) \in X^2 : (\exists z) x \alpha z \beta y\}$$

is called the *composition of α and β* , and if $\alpha \circ \beta = \beta \circ \alpha$ then α and β are said to *permute*, or to be *permuting* equivalence relations. Note that $\alpha \circ \beta \subseteq \alpha \vee \beta$ with equality if and only if α and β permute.

The largest and smallest equivalence relations on X are $1_X = X^2$ and $0_X = \{(x, x) : x \in X\}$, respectively.

One more piece of shorthand notation will be useful below. Suppose Θ is a set of equivalence relations that are *pairwise permuting pairwise complements*—that is, for all $\gamma \neq \delta$ in Θ , we have

$$\gamma \circ \delta = \delta \circ \gamma, \quad \gamma \wedge \delta = 0_X, \quad \gamma \vee \delta = 1_X.$$

Then we say that the relations in Θ are PPPC.

2. BASIC OBSERVATIONS

We say that α and β are *complementary* equivalence relations on X provided $\alpha \vee \beta = 1_X$ and $\alpha \wedge \beta = 0_X$.

Lemma 1. Suppose α and β are complementary equivalence relations on X . Then α and β permute if and only if they have complementary blocks. That is,

$$\alpha \circ \beta = 1_X \iff (\forall x)(\forall y) |x/\alpha| \cdot |y/\beta| = |X|.$$

Corollary 1. Suppose $\alpha_1, \alpha_2, \alpha_3$ are pairwise complementary equivalence relations on X . Then $\alpha_1, \alpha_2, \alpha_3$ pairwise permute if and only if they all have uniform blocks of size $\sqrt{|X|}$. In other words,

$$(\forall i)(\forall j) (i \neq j \longrightarrow \alpha_i \circ \alpha_j = 1_X) \iff (\forall i)(\forall x) |x/\alpha_i| = \sqrt{|X|}.$$

In this case, we clearly have $|x/\alpha_i| = \#\text{Blocks}(\alpha_i)$.

Proof of Lemma 1. Assume $\alpha \circ \beta = \alpha \vee \beta = 1_X$. Then, for all $x \in X$ we have

$$(2.1) \quad x/(\alpha \circ \beta) = \coprod_{y \in x/\alpha} y/\beta = X,$$

where \coprod denotes disjoint union. The union is disjoint since $\alpha \wedge \beta = 0_X$. Since the union in (2.1) is all of X , every block of β must appear in the union, so the block x/α has exactly $\#\text{Blocks}(\beta)$ elements. Since x was arbitrary, α has uniform blocks of size $|x/\alpha| = \#\text{Blocks}(\beta)$. Similarly, $x/(\beta \circ \alpha) = \coprod_{y \in x/\beta} y/\alpha = X$, so $|x/\beta| = \#\text{Blocks}(\alpha)$ holds for all $x \in X$. Therefore, for all $x, y \in X$, we have

$$|x/\alpha| \cdot |y/\beta| = |x/\alpha| \cdot \#\text{Blocks}(\alpha) = |X|.$$

To prove the converse, suppose α and β are pairwise complements with complementary blocks. Then $|x/\alpha| \cdot |y/\beta| = |X|$, thus $|y/\beta| = |x/\alpha|^{-1} \cdot |X| = \#\text{Blocks}(\alpha)$ hold for all $x, y \in X$. Therefore, for all $x \in X$, we have

$$\begin{aligned} |x/(\alpha \circ \beta)| &= \left| \coprod_{y \in x/\alpha} y/\beta \right| = \sum_{y \in x/\alpha} |y/\beta| \\ &= \sum_{y \in x/\alpha} \#\text{Blocks}(\alpha) \\ &= |x/\alpha| \cdot \#\text{Blocks}(\alpha) = |X|. \end{aligned}$$

This proves that $\alpha \circ \beta = 1_X$, as desired. \square

Proof of Corollary 1. Since α_1 and α_2 permute and are complements, Lemma 1 implies they have complementary blocks, so

$$(2.2) \quad |x/\alpha_1| = |x/\alpha_2|^{-1} \cdot |X| = \#\text{Blocks}(\alpha_2).$$



FIGURE 1. The graph defining the relation $\rho(\alpha_1, \alpha_2, \beta)$; that is, $(x, y) \in \rho(\alpha_1, \alpha_2, \beta)$ if and only if there exist $a, b \in X$ satisfying the relations in the diagram.

(This holds for all $x \in X$. Recall that complementary blocks are always uniform.) Similarly, since α_1 and α_3 permute, we have $|x/\alpha_1| = |x/\alpha_3|^{-1} \cdot |X| = \#Blocks(\alpha_3)$. Therefore, $\#Blocks(\alpha_2) = \#Blocks(\alpha_3)$. Since α_2 and α_3 permute, we have

$$(2.3) \quad |x/\alpha_2| = |x/\alpha_3|^{-1} \cdot |X| = \#Blocks(\alpha_3),$$

and the latter is equal to $\#Blocks(\alpha_2)$. Therefore,

$$|X| = |x/\alpha_2| \cdot \#Blocks(\alpha_2) = |x/\alpha_2| \cdot |x/\alpha_2|.$$

Thus, $|x/\alpha_2| = \sqrt{|X|}$, so by (2.2) and (2.3) we have $|x/\alpha_i| = \sqrt{|X|} = \#Blocks(\alpha_i)$ for $i = 1, 2, 3$.

The converse is obvious, since if α_i and α_j are complementary equivalence relations on X with $|x/\alpha_i| = \sqrt{|X|}$, then $\#Blocks(\alpha_i) = \sqrt{|X|}$, so $\alpha_i \circ \alpha_j = 1_X$. \square

From Corollary 1 we see that the Pálffy-Saxl problem can be stated as

PROBLEM. Let \mathbf{A} be a finite algebra with $\text{Con } \mathbf{A} \cong M_n$, $n \geq 4$. If three atoms of \mathbf{A} have Property (2.4) below, does it follow that every atom has Property (2.4)?

$$(2.4) \quad (\forall x) |x/\alpha| = \sqrt{|X|} = \#Blocks(\alpha)$$

To prove that the answer is “yes,” it will suffice to prove that if $M_n \leq \text{Eq}(X)$ has 3 atoms with Property (2.4) and an atom β with $|x/\beta| < \sqrt{|X|}$, then this M_n is not a congruence lattice.

3. GRAPHICAL COMPOSITIONS

Suppose α_1, α_2 , and α_3 are pairwise permuting pairwise complements (PPPC) in $\text{Eq}(X)$, and let $\beta \in \text{Eq}(X)$ be complementary to each α_i , so that

$$L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4.$$

Define the relation $\rho = \rho(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ as follows:

$$x \rho y \iff (\exists(a, b) \in \beta) x \alpha_1 a \alpha_2 y \alpha_1 b \alpha_2 x.$$

Graphically, $x \rho y$ if and only if there exist $a, b \in X$ satisfying the relations depicted in Figure 1.

It is clear that ρ is reflexive and symmetric but not transitive. Suppose $f \in X^X$ is a unary function that respects each relation $\theta \in \{\alpha_1, \alpha_2, \beta\}$ —that is, $(u, v) \in$

$\theta \longrightarrow (f(u), f(v)) \in \theta$. Then f also respects ρ . (Consider the diagram in Figure 1, and give each vertex u the label $f(u)$.)

Fact 1. If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$, then

$$\begin{aligned}\alpha_1 \cap \rho(\alpha_1, \alpha_2, \beta) &= 0_X = \alpha_2 \cap \rho(\alpha_1, \alpha_2, \beta), \\ \alpha_1 \cap \rho(\alpha_1, \alpha_3, \beta) &= 0_X = \alpha_3 \cap \rho(\alpha_1, \alpha_3, \beta), \\ \alpha_2 \cap \rho(\alpha_2, \alpha_3, \beta) &= 0_X = \alpha_3 \cap \rho(\alpha_2, \alpha_3, \beta).\end{aligned}$$

Proof. Fix $(x, y) \in \alpha_1 \cap \rho(\alpha_1, \alpha_2, \beta)$ and suppose a and b satisfy the diagram in Figure 1. Then $(x, y) \in \alpha_1$ implies $(a, b) \in \alpha_1 \wedge \beta = 0_X$, so $a = b$. Therefore, $(x, y) \in \alpha_1 \wedge \alpha_2 = 0_X$, so $x = y$. Proofs of the other identities are similar. \square

4. FUNCTIONS DERIVED FROM GRAPHICAL COMPOSITIONS

Let R_{12}^β be the relation on $X^2 \times X^2$ defined by

$$(a, b) R_{12}^\beta (x, y) \iff (a, b) \in \beta \text{ and } x \alpha_1 a \alpha_2 y \alpha_1 b \alpha_2 x.$$

Define R_{13}^β and R_{23}^β similarly. Graphically, $(a, b) R_{12}^\beta (x, y)$ holds if and only if the relations in Figure 1 are satisfied.

Lemma 2. Suppose α_i and α_j are complementary equivalence relations on X with uniform blocks of size $\sqrt{|X|}$. Then the relation R_{ij}^β restricted to $\beta \times X^2$ is a one-to-one function from β into X^2 .

Proof. First we note that each pair $(a, b) \in \beta$ has at most one image. For if $(a, b) R_{ij}^\beta (x, y)$ and $(a, b) R_{ij}^\beta (u, v)$, then $(x, u) \in \alpha_i \wedge \alpha_j = 0_X$ and $(y, v) \in \alpha_i \wedge \alpha_j = 0_X$, so $(x, y) = (u, v)$.

Next, since both α_i and α_j have $\sqrt{|X|}$ blocks, and since each of these blocks has size $\sqrt{|X|}$, we see that each block of α_i intersects each block of α_j at exactly one point. That is, for all $a, b \in X$, the set $a/\alpha_i \cap b/\alpha_j$ is a singleton. Therefore, to each $(a, b) \in \beta$ there corresponds precisely one $(x, y) \in X^2$ such that $(a, b) R_{ij}^\beta (x, y)$ holds. Specifically, $\{x\} = a/\alpha_i \cap b/\alpha_j$ and $\{y\} = b/\alpha_i \cap a/\alpha_j$. Thus, R_{ij}^β is a function.

From now on, we let $R_{ij}^\beta((a, b))$ denote the image of (a, b) under R_{ij}^β ; that is, $R_{ij}^\beta((a, b))$ denotes the ordered pair (x, y) satisfying $(a, b) R_{ij}^\beta (x, y)$.

Suppose $R_{ij}^\beta((a, b)) = R_{ij}^\beta((c, d))$. Then $(a, c) \in \alpha_i \wedge \alpha_j = 0_X$ and $(b, d) \in \alpha_i \wedge \alpha_j = 0_X$, so $(a, b) = (c, d)$. Therefore, R_{ij}^β is one-to-one. \square

If, in addition to the assumptions of Lemma 2, we assume that the image of β under R_{ij}^β is contained in β , then $R_{ij}^\beta : \beta \rightarrow \beta$ is a bijective involution. That is, R_{ij}^β is one-to-one and onto, and $R_{ij}^\beta \circ R_{ij}^\beta$ is the identity map.

5. FINAL PIECE OF THE PUZZLE

As above, suppose $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice with α_i PPPC. Suppose $R_{ij}^\beta : \beta \rightarrow \beta$ holds for all $i, j \in \{1, 2, 3\}$.

Lemma 3. If $a \alpha_1 z \beta w$, then one of the following holds:

- (1) $(a, w) \in \alpha_2$,
- (2) $(a, w) \in \alpha_3$,

- (3) $(a, w) \in \beta$,
- (4) $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$,
- (5) $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$.

If Lemma 3 is true, then we can prove the following:

Theorem 1. *If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice with α_i PPC, then β permutes with each α_i .*

Proof. We will show $\alpha_1 \circ \beta \subseteq \beta \circ \alpha_1$. Assume $a \alpha_1 z \beta w$. We consider each of the cases in Lemma 3 in turn and, in each case, find b satisfying $a \beta b \alpha_1 w$.

- (1) If $(a, w) \in \alpha_2$, then let $b = z/\alpha_2 \cap w/\alpha_1$. Then $R_{12}(z, w) = (a, b)$ and since $R_{12} : \beta \rightarrow \beta$, we have $(a, b) \in \beta$, so $a \beta b \alpha_1 w$, as desired.
- (2) If $(a, w) \in \alpha_3$, then let $b = z/\alpha_3 \cap w/\alpha_1$. Use the same argument as in the first case, but replace R_{12} with R_{13} .
- (3) If $(a, w) \in \beta$, then let $b = a$.
- (4) If $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$, then let y denote the element in this set. Let $x = z/\alpha_1 \cap w/\alpha_3$, and let $b = x/\alpha_2 \cap y/\alpha_1$. Then $(R_{12} \circ R_{13})(z, w) = R_{12}(x, y) = (a, b)$, so $(a, b) \in \beta$. Now, $b \alpha_1 y \alpha_1 w$, so $a \beta b \alpha_1 w$, as desired.
- (5) If $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$, then let y denote this element, let $x = z/\alpha_1 \cap w/\alpha_2$, and let $b = x/\alpha_3 \cap y/\alpha_1$. Then $(R_{13} \circ R_{12})(z, w) = R_{12}(x, y) = (a, b)$, so $(a, b) \in \beta$. Now, $b \alpha_1 y \alpha_1 w$, so $a \beta b \alpha_1 w$, as desired.

□

6. PROOF OF LEMMA 3

Consider the relation θ_{ij} defined as follows:

$$x \theta_{ij} y \iff (\exists a, b) a \alpha_i x \alpha_j b \beta y \alpha_j a.$$

Easy arguments similar to those above establish that

$$\theta_{ij} \cap \alpha_i = \theta_{ij} \cap \alpha_j = \theta_{ij} \cap \beta = 0_X.$$

On the other hand, since L is a congruence lattice, it must be the case that the transitive closure of θ_{ij} is contained in L .

REFERENCES

- [1] P. P. Pálfi and J. Saxl. Congruence lattices of finite algebras and factorizations of groups. *Comm. Algebra*, 18(9):2783–2790, 1990.