

ON A PROBLEM OF PALFY AND SAXL

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1. INTRODUCTION

In the paper [1], Peter Palfy and Jan Saxl pose the following

PROBLEM. Let \mathbf{A} be a finite algebra with $\text{Con } \mathbf{A} \cong M_n$, $n \geq 4$.
If three nontrivial congruences of \mathbf{A} pairwise permute, does it follow
that every pair of congruences of \mathbf{A} permute?

These notes collect some notation and facts that might be useful for attacking this problem. Throughout, X denotes a finite set, $\text{Eq}(X)$ denotes the lattice of equivalence relations on X and, for $\alpha \in \text{Eq}(X)$ and $x \in X$, we denote by x/α the equivalence class of α containing x . We often refer to equivalence classes as “blocks,” and we denote by $\#\text{Blocks}(\alpha)$ the number of blocks of the equivalence relation α .

For a given $\alpha \in \text{Eq}(X)$ the map $\varphi_\alpha : x \mapsto x/\alpha$ is a function from X into the power set $\mathcal{P}(X)$ with kernel $\ker \varphi_\alpha = \alpha$. The *block-size function* $x \mapsto |x/\alpha|$ is a function from X into $\{1, 2, \dots, |X|\}$.

We will often abuse notation and equate an equivalence relation with the corresponding partition of the set X . For example, we will equate the relation

$$\alpha = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0), (2, 3), (3, 2)\}$$

with the partition $[0, 1][2, 3]$, and often we resort to writing $\alpha = [0, 1][2, 3]$.

We say that α has *uniform blocks* if all blocks of α have the same size; or, equivalently, the block-size function is constant: for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. We will use $|x/\alpha|$, without specifying a particular $x \in X$, to denote this block size.¹ Thus, when α has uniform blocks, we have $|X| = |x/\alpha| \cdot \#\text{Blocks}(\alpha)$.

We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary blocks*, if the number of blocks of one is equal to the block size of the other. In other words, if α and β are two equivalence relations on X with uniform block sizes $|x/\alpha|$ and $|x/\beta|$, respectively, then α and β have complementary blocks if and only if $(\forall x)(\forall y) |x/\alpha| \cdot |y/\beta| = |X|$.

Given two equivalence relations α and β on X , the relation

$$\alpha \circ \beta = \{(x, y) \in X^2 : (\exists z) x \alpha z \beta y\}$$

is called the *composition of α and β* , and if $\alpha \circ \beta = \beta \circ \alpha$ then α and β are said to *permute*, or to be *permuting* equivalence relations. Note that $\alpha \circ \beta \subseteq \alpha \vee \beta$ with equality if and only if α and β permute.

The largest and smallest equivalence relations on X are given by $1_X = X^2$ and $0_X = \{(x, x) : x \in X\}$, respectively.

Date: November 13, 2013.

¹Alternatively, we might consider using $|x./\alpha|$ to emphasize that every $x \in X$ can be substituted for x . without changing the value of $|x./\alpha|$, but this notation may be too cumbersome.

We say that α and β are *complementary* equivalence relations on X provided $\alpha \vee \beta = 1_X$ and $\alpha \wedge \beta = 0_X$.

Lemma 1. Suppose α and β are complementary equivalence relations on X . Then α and β permute if and only if they have complementary blocks. That is,

$$\alpha \circ \beta \iff (\forall x)(\forall y) |x/\alpha| \cdot |y/\alpha| = |X|.$$

Corollary 1. Suppose $\alpha_1, \alpha_2, \alpha_3$ are pairwise complementary equivalence relations on X . Then $\alpha_1, \alpha_2, \alpha_3$ are pairwise permuting if and only if they all have uniform blocks of size $\sqrt{|X|}$. In other words,

$$(\forall i)(\forall j) (i \neq j \longrightarrow \alpha_i \circ \alpha_j = 1_X) \iff (\forall i)(\forall x) |x/\alpha_i| = \sqrt{|X|}.$$

In this case, we clearly have $|x/\alpha_i| = \#\text{Blocks}(\alpha_i)$.

Proof of Lemma 1. Assume $\alpha \circ \beta = \alpha \vee \beta = 1_X$. Then, for all $x \in X$ we have

$$(1.1) \quad x/(\alpha \circ \beta) = \coprod_{y \in x/\alpha} y/\beta = X,$$

where \coprod denotes disjoint union. The union is disjoint since $\alpha \wedge \beta = 0_X$. Since the union in (1.1) is all of X , every block of β must appear in the union, so the block x/α has exactly $\#\text{Blocks}(\beta)$ elements. Since x was arbitrary, α has uniform blocks of size $|x/\alpha| = \#\text{Blocks}(\beta)$. Similarly, $x/(\beta \circ \alpha) = \coprod_{y \in x/\beta} y/\alpha = X$, so $|x/\beta| = \#\text{Blocks}(\alpha)$ holds for all $x \in X$. Therefore, for all $x, y \in X$, we have

$$|x/\alpha| \cdot |y/\beta| = |x/\alpha| \cdot \#\text{Blocks}(\alpha) = |X|.$$

To prove the converse, suppose α and β are pairwise complements with complementary blocks. Then $|x/\alpha| \cdot |y/\beta| = |X|$, thus $|y/\beta| = |x/\alpha|^{-1} \cdot |X| = \#\text{Blocks}(\alpha)$ hold for all $x, y \in X$. Therefore, for all $x \in X$, we have

$$\begin{aligned} |x/(\alpha \circ \beta)| &= \left| \coprod_{y \in x/\alpha} y/\beta \right| = \sum_{y \in x/\alpha} |y/\beta| \\ &= \sum_{y \in x/\alpha} \#\text{Blocks}(\alpha) \\ &= |x/\alpha| \cdot \#\text{Blocks}(\alpha) = |X|. \end{aligned}$$

This proves that $\alpha \circ \beta = 1_X$. □

Proof of Corollary 1. □

REFERENCES

- [1] P. P. Pálffy and J. Saxl. Congruence lattices of finite algebras and factorizations of groups. *Comm. Algebra*, 18(9):2783–2790, 1990.