ON A PROBLEM OF PÁLFY AND SAXL

WILLIAM DEMEO

1. Introduction

In the paper [1], Peter Pálfy and Jan Saxl pose the following

> **PROBLEM.** Let $\mathbf{A}$ be a finite algebra with $\operatorname{Con}\mathbf{A} \cong M_n$, $n \geqslant 4$. If three nontrivial congruences of $\mathbf{A}$ pairwise permute, does it follow that every pair of congruences of $\mathbf{A}$ permute?

These notes collect some notation and facts that might be useful for attacking this problem. Throughout, X denotes a finite set, $\operatorname{Eq}(X)$ denotes the lattice of equivalence relations on X and, for $\alpha \in \operatorname{Eq}(X)$ and $x \in X$, we denote by x/α the equivalence class of α containing x. We often refer to equivalnce classes as "blocks," and we denote by $\#\text{Blocks}(\alpha)$ the number of blocks of the equivalence relation α.

For a given $\alpha \in \operatorname{Eq}(X)$ the map $\varphi_\alpha : x \mapsto x/\alpha$ is a function from X into the power set $\mathscr{P}(X)$ with kernel $\ker \varphi_\alpha = \alpha$. The *block-size function* $x \mapsto |x/\alpha|$ is a function from X into $\{1, 2, \dots, |X|\}$.

We will often abuse notation and equate an equivalence relation with the corresponding partition of the set X. For example, we will equate the relation

$$\alpha = \{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0), (2,3), (3,2)\}$$

with the partition $|0,1|2,3|$, and often we resort to writing $\alpha = |0,1|2,3|$.

We say that α has *uniform blocks* if all blocks of α have the same size; or, equivalently, the block-size function is constant: for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. We will use $|x/\alpha|$, without specifying a particular $x \in X$, to denote this block size.[1] Thus, when α has uniform blocks, we have $|X| = |x/\alpha| \cdot \#\text{Blocks}(\alpha)$.

We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary blocks*, if the number of blocks of one is equal to the block size of the other. In other words, if α and β are two equivalence relations on X with uniform block sizes $|x/\alpha|$ and $|x/\beta|$, respectively, then α and β have complementary blocks if and only if $(\forall x)(\forall y)\, |x/\alpha| \cdot |y/\beta| = |X|$.

Given two equivalence relations α and β on X, the relation

$$\alpha \circ \beta = \{(x, y) \in X^2 : (\exists z) x \;\alpha\; z \;\beta\; y\}$$

is called the *composition of* α *and* β, and if $\alpha \circ \beta = \beta \circ \alpha$ then α and β are said to *permute*, or to be *permuting* equivalence relations. Note that $\alpha \circ \beta \subseteq \alpha \vee \beta$ with equility if and only if α and β permute.

The largest and smallest equivalence relations on X are given by $1_X = X^2$ and $0_X = \{(x, x) : x \in X\}$, respectively.

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[1]Alternatively, we might consider using $|x./\alpha|$ to emphasize that every $x \in X$ can be substituted for $x.$ without changing the value of $|x./\alpha|$, but this notation may be too cumbersome.

We say that α and β are *complementary* equivalence relations on X provided $\alpha \vee \beta = 1_X$ and $\alpha \wedge \beta = 0_X$.

Lemma 1. Suppose α and β are complementary equivalence relations on X. Then α and β permute if and only if they have complementary blocks. That is,

$$\alpha \circ \beta = 1_X \quad \Longleftrightarrow \quad (\forall x)(\forall y)\, |x/\alpha| \cdot |y/\alpha| = |X|.$$

Corollary 1. Suppose α_1, α_2, α_3 are pairwise complementary equivalence relations on X. Then α_1, α_2, α_3 pairwise permute if and only if they all have uniform blocks of size $\sqrt{|X|}$. In other words,

$$(\forall i)(\forall j)\,(i \neq j \longrightarrow \alpha_i \circ \alpha_j = 1_X) \quad \Longleftrightarrow \quad (\forall i)(\forall x)\, |x/\alpha_i| = \sqrt{|X|}.$$

In this case, we clearly have $|x/\alpha_i| = \#\text{Blocks}(\alpha_i)$.

Proof of Lemma 1. Assume $\alpha \circ \beta = \alpha \vee \beta = 1_X$. Then, for all $x \in X$ we have

$$x/(\alpha \circ \beta) = \coprod_{y \in x/\alpha} y/\beta = X, \tag{1.1}$$

where $\coprod$ denotes disjoint union. The union is disjoint since $\alpha \wedge \beta = 0_X$. Since the union in (1.1) is all of X, every block of β must appear in the union, so the block x/α has exactly $\#\text{Blocks}(\beta)$ elements. Since x was arbitrary, α has uniform blocks of size $|x/\alpha| = \#\text{Blocks}(\beta)$. Similarly, $x/(\beta \circ \alpha) = \coprod_{y \in x/\beta} y/\alpha = X$, so $|x/\beta| = \#\text{Blocks}(\alpha)$ holds for all $x \in X$. Therefore, for all $x, y \in X$, we have

$$|x/\alpha| \cdot |y/\beta| = |x/\alpha| \cdot \#\text{Blocks}(\alpha) = |X|.$$

To prove the converse, suppse α and β are pairwise complements with complementary blocks. Then $|x/\alpha| \cdot |y/\beta| = |X|$, thus $|y/\beta| = |x/\alpha|^{-1} \cdot |X| = \#\text{Blocks}(\alpha)$ hold for all $x, y \in X$. Therefore, for all $x \in X$, we have

$$\begin{aligned} \big|x/(\alpha \circ \beta)\big| &= \big|\coprod_{y \in x/\alpha} y/\beta\big| = \sum_{y \in x/\alpha} |y/\beta| \\ &= \sum_{y \in x/\alpha} \#\text{Blocks}(\alpha) \\ &= |x/\alpha| \#\text{Blocks}(\alpha) = |X|. \end{aligned}$$

This proves that $\alpha \circ \beta = 1_X$, as desired. □

Proof of Corollary 1. Since α_1 and α_2 permute and are complements, Lemma 1 implies they have complmentary blocks, so

$$|x/\alpha_1| = |x/\alpha_2|^{-1} \cdot |X| = \#\text{Blocks}(\alpha_2). \tag{1.2}$$

(This holds for all $x \in X$. Recall that complemntary blocks are always uniform.) Similarly, since α_1 and α_3 permute, we have $|x/\alpha_1| = |x/\alpha_3|^{-1} \cdot |X| = \#\text{Blocks}(\alpha_3)$. Therefore, $\#\text{Blocks}(\alpha_2) = \#\text{Blocks}(\alpha_3)$. Since α_2 and α_3 permute, we have

$$|x/\alpha_2| = |x/\alpha_3|^{-1} \cdot |X| = \#\text{Blocks}(\alpha_3), \tag{1.3}$$

and the latter is equal to $\#\text{Blocks}(\alpha_2)$. Therefore,

$$|X| = |x/\alpha_2| \cdot \#\text{Blocks}(\alpha_2) = |x/\alpha_2| \cdot |x/\alpha_2|.$$

Thus, $|x/\alpha_2| = \sqrt{|X|}$, so by (1.2) and (1.3) we have $|x/\alpha_i| = \sqrt{|X|} = \#\text{Blocks}(\alpha_i)$ for $i = 1, 2, 3$.

The converse is obvious, since if α_i and α_j are complementary equivalence relations on X with $|x/\alpha_i| = \sqrt{|X|}$, then $\#\text{Blocks}(\alpha_i) = \sqrt{|X|}$, so $\alpha_i \circ \alpha_j = 1_X$. $\square$

References

[1] P. P. Pálfy and J. Saxl. Congruence lattices of finite algebras and factorizations of groups. *Comm. Algebra*, 18(9):2783–2790, 1990.