ON A PROBLEM OF PÁLFY AND SAXL

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1. Introduction

In the paper [1], Péter Pálfy and Jan Saxl pose the following

> Problem. Let $\mathbf{A}$ be a finite algebra with $\operatorname{Con}\mathbf{A} \cong M_n$, $n \geqslant 4$. If three nontrivial congruences of $\mathbf{A}$ pairwise permute, does it follow that every pair of congruences of $\mathbf{A}$ permute?

These notes collect some notation and facts that might be useful for attacking this problem. Throughout, X denotes a finite set and $\operatorname{Eq}(X)$ denotes the lattice of equivalence relations on X. For $\alpha \in \operatorname{Eq}(X)$ and $x \in X$, we let x/α denote the equivalence class of α containing x, and X/α denotes the set of all equivalence classes of α. That is,

$$x/\alpha = \{y \in X : x \;\alpha\; y\} \quad \text{and} \quad X/\alpha = \{x/\alpha : x \in X\}.$$

We often refer to equivalence classes as "blocks," and we denote by $\boldsymbol{\nu}(\alpha)$ the number of blocks of the relation α. That is, $\boldsymbol{\nu}(\alpha) = |X/\alpha|$, and we may use either $\boldsymbol{\nu}(\alpha)$ or $|X/\alpha|$ to denote the number of blocks of α. For a given $\alpha \in \operatorname{Eq}(X)$ the map $\varphi_\alpha : x \mapsto x/\alpha$ is a function from X into the power set $\mathscr{P}(X)$ with kernel $\ker\varphi_\alpha = \alpha$. The *block-size function* $x \mapsto |x/\alpha|$ is a function from X into $\{1, 2, \dots, |X|\}$.

We will often abuse notation and identify an equivalence relation with the corresponding partition of the set X. For example, we identify the relation

$$\alpha = \{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0), (2,3), (3,2)\}$$

with the partition $|0,1|2,3|$, and will even write $\alpha = |0,1|2,3|$.

We say that α has *uniform blocks* if all blocks of α have the same size; or, equivalently, the block-size function is constant: for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. In this case we will use $|x/\alpha|$ without specifying a particular x to denote this block size. Thus, if α has uniform blocks, then

$$|X| = |x/\alpha||X/\alpha| = |x/\alpha|\,\boldsymbol{\nu}(\alpha) \quad (\text{for all } x \in X).$$

We say that two equivalence relations with uniform blocks have *compatible uniform block structure* (CUBS) if the number of blocks of one is equal to the block size of the other. That is, α and β have CUBS iff $|x/\alpha||y/\beta| = |X|$ (for all x and y).

If α and β are binary relations on X, then the relation

$$\alpha \circ \beta = \{(x, y) \in X^2 : (\exists z)\, x \;\alpha\; z \;\beta\; y\} \tag{1.1}$$

is called the *composition of α and β*. If $\alpha \circ \beta = \beta \circ \alpha$ then we call α and β *permuting* relations and we say that α and β *permute*. It is not hard to see that $\alpha \circ \beta \subseteq \alpha \vee \beta$ with equality if and only if α and β permute.

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From the definition (1.1) it is clear that $(x,y) \in \alpha \circ \beta$ if and only if $y \in z/\beta$ for some $z \in x/\alpha$. Thus, for every $x \in X$,

$$x/(\alpha \circ \beta) = \bigcup_{z \in x/\alpha} z/\beta. \tag{1.2}$$

The largest and smallest equivalence relations on X are denoted by $1_X = X^2$ and $0_X = \{(x,x) : x \in X\}$, respectively.

We say that α and β are *complementary* equivalence relations on X provided $\alpha \vee \beta = 1_X$ and $\alpha \wedge \beta = 0_X$. If Γ is a set of equivalence relations, we say that Γ consists of *pairwise-permuting pairwise-complements* (PPPC) if the following conditions hold for all $\gamma \neq \delta$ in Γ: (i) $\gamma \circ \delta = \delta \circ \gamma$; (ii) $\gamma \wedge \delta = 0_X$; (iii) $\gamma \vee \delta = 1_X$.

2. Basic observations

Lemma 1. Suppose α and β are complementary equivalence relations on X. Then α and β permute if and only if they have CUBS. That is,

$$\alpha \circ \beta = 1_X \quad \Longleftrightarrow \quad (\forall x)(\forall y)\, |x/\alpha||y/\beta| = |X|.$$

Corollary 1. Suppose α_1, α_2, α_3 are pairwise complementary equivalence relations on the finite set X. Then α_1, α_2, α_3 pairwise permute if and only if they all have uniform blocks of size $\sqrt{|X|}$. That is,

$$(\forall i)(\forall j)\,(i \neq j \longrightarrow \alpha_i \circ \alpha_j = 1_X) \quad \Longleftrightarrow \quad (\forall i)(\forall x)\, |x/\alpha_i| = \sqrt{|X|}.$$

In this case, $|x/\alpha_i| = \boldsymbol{\nu}(\alpha_i)$.

Proof of Lemma 1. Suppose α and β are complementary equivalence relations. Then, since $\alpha \wedge \beta = 0_X$, the union in (1.2) is disjoint; we denote this by writing

$$x/(\alpha \circ \beta) = \coprod_{z \in x/\alpha} z/\beta. \tag{2.1}$$

Also, since $\alpha \circ \beta = \alpha \vee \beta = 1_X$, we have $x/(\alpha \circ \beta) = X$ for every $x \in X$. Thus the union in (2.1) is all of X, so every block of β appears in this union. It follows that the size of the block x/α is exactly $\boldsymbol{\nu}(\beta)$. As x was arbitrary, α has uniform blocks of size $\boldsymbol{\nu}(\beta)$. The same argument with the roles of α and β reversed gives $x/(\beta \circ \alpha) = \coprod_{z \in x/\beta} z/\alpha = X$, and $|x/\beta| = \boldsymbol{\nu}(\alpha)$ for all $x \in X$. Therefore, for all $x, z \in X$ we have

$$|x/\alpha||z/\beta| = |x/\alpha|\, \boldsymbol{\nu}(\alpha) = |X|.$$

To prove the converse, suppose α and β are pairwise complements with complementary blocks. Then $|x/\alpha||y/\beta| = |X|$, so $|y/\beta| = |x/\alpha|^{-1}|X| = \boldsymbol{\nu}(\alpha)$. Therefore, for all $x \in X$,

$$\begin{aligned} \big|x/(\alpha \circ \beta)\big| &= \big|\coprod_{y \in x/\alpha} y/\beta\big| = \sum_{y \in x/\alpha} |y/\beta| \\ &= \sum_{y \in x/\alpha} \boldsymbol{\nu}(\alpha) \\ &= |x/\alpha|\, \boldsymbol{\nu}(\alpha) = |X|. \end{aligned}$$

This proves that $\alpha \circ \beta = 1_X$, as desired. ∎

Proof of Corollary 1. Since α_1 and α_2 permute and are complements, Lemma 1 implies they have complementary blocks, so

$$|x/\alpha_1| = |x/\alpha_2|^{-1}|X| = \boldsymbol{\nu}(\alpha_2). \tag{2.2}$$

(This holds for all $x \in X$. Recall that complementary blocks are always uniform.) Similarly, since α_1 and α_3 permute, we have $|x/\alpha_1| = |x/\alpha_3|^{-1}|X| = \boldsymbol{\nu}(\alpha_3)$. Therefore, $\boldsymbol{\nu}(\alpha_2) = \boldsymbol{\nu}(\alpha_3)$. Since α_2 and α_3 permute, we have

$$|x/\alpha_2| = |x/\alpha_3|^{-1}|X| = \boldsymbol{\nu}(\alpha_3), \tag{2.3}$$

and the latter is equal to $\boldsymbol{\nu}(\alpha_2)$. Therefore, $|X| = |x/\alpha_2|\,\boldsymbol{\nu}(\alpha_2) = |x/\alpha_2||x/\alpha_2|$, so $|x/\alpha_2| = \sqrt{|X|}$. By (2.2) and (2.3), it follows that $|x/\alpha_i| = \sqrt{|X|} = \boldsymbol{\nu}(\alpha_i)$ for $i = 1, 2, 3$.

The converse is obvious since, if α_i and α_j are complementary equivalence relations on X with $|x/\alpha_i| = \sqrt{|X|}$, then $\boldsymbol{\nu}(\alpha_i) = \sqrt{|X|}$, so $\alpha_i \circ \alpha_j = 1_X$. □

Define the set $\mathcal{S} \subseteq \mathrm{Eq}(X)$ as follows:

$$\mathcal{S} = \{\alpha \in \mathrm{Eq}(X) : (\forall x)\ |x/\alpha| = \sqrt{|X|} = \boldsymbol{\nu}(\alpha)\}.$$

From Corollary 1 we see that the Pálfy-Saxl problem can be rephrased as follows:

> Problem. Let $\mathbf{A}$ be a finite algebra with $\mathrm{Con}\,\mathbf{A} \cong M_n$, $n \geqslant 4$. If the set $\mathcal{S}$ contains three atoms of $\mathrm{Con}\,\mathbf{A}$, does it follow that $\mathcal{S}$ contains every atom of $\mathrm{Con}\,\mathbf{A}$?

To prove that the answer is "yes," it suffices to show that whenever $M_n \cong L \leqslant \mathrm{Eq}(X)$ has 3 atoms in $\mathcal{S}$ and an atom β with $|x/\beta| < \sqrt{|X|}$, then L is not a congruence lattice.

3. Graphical Compositions

Let X be a nonempty set. Given α, β, $\gamma \in \mathrm{Eq}(X)$, define the relation

$$W(\alpha, \beta; \gamma) := \{(x, y) \in X^2 : \exists (a, b) \in \gamma\ .\ x\ \alpha\ a\ \beta\ y\ \alpha\ b\ \beta\ x\}.$$

That is, $(x, y) \in W(\alpha, \beta; \gamma)$ iff there exists $(a, b) \in \gamma$ such that the relations in Figure 1 hold.

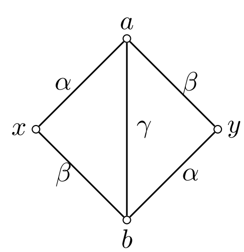


Figure 1. The Wheatstone Bridge defines the relation $W(\alpha, \beta; \gamma)$.

Suppose α_1, α_2, and α_3 are PPPC in $\mathrm{Eq}(X)$, and let $\beta \in \mathrm{Eq}(X)$ be complementary to each α_i, so that $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$. (See Figure 2.)

Define the relation $\tau = \tau(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ as $R(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ as follows:

$$x\ \tau\ y \quad \longleftrightarrow \quad (\exists (a, b) \in \beta)\ x\ \alpha_1\ a\ \alpha_2\ y\ \alpha_1\ b\ \alpha_2\ x.$$

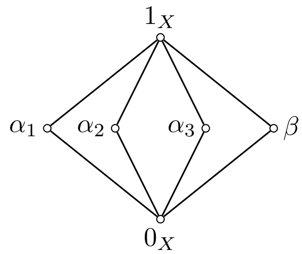


FIGURE 2. The lattice M_4.

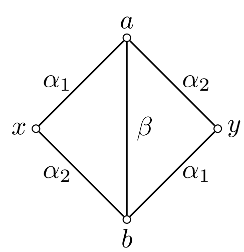


FIGURE 3. The Wheatstone Bridge which defines the relation $\tau(\alpha_1, \alpha_2, \beta)$ as follows: $(x, y) \in \tau(\alpha_1, \alpha_2, \beta)$ if and only if there exist $a, b \in X$ satisfying the relations in the diagram.

Graphically, $x \;\tau\; y$ if and only if there exist $a, b \in X$ satisfying the relations depicted in Figure 3. It is clear that τ is a *tolerance*, that is, a reflexive and symmetric binary relation. Let $f : X \to X$ be a unary operation and suppose that f is *compatible* with each relation $\theta \in \{\alpha_1, \alpha_2, \beta\}$, that is, $(u, v) \in \theta \longrightarrow (f(u), f(v)) \in \theta$. Then f is also compatible with τ.

Fact 3.1. If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$, then

$$\alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta) = 0_X = \alpha_2 \cap \tau(\alpha_1, \alpha_2, \beta),$$
$$\alpha_1 \cap \tau(\alpha_1, \alpha_3, \beta) = 0_X = \alpha_3 \cap \tau(\alpha_1, \alpha_3, \beta),$$
$$\alpha_2 \cap \tau(\alpha_2, \alpha_3, \beta) = 0_X = \alpha_3 \cap \tau(\alpha_2, \alpha_3, \beta).$$

Proof. Fix $(x, y) \in \alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta)$ and suppose a and b satisfy the diagram in Figure 3. Then $(x, y) \in \alpha_1$ implies $(a, b) \in \alpha_1 \wedge \beta = 0_X$, so $a = b$. Therefore, $(x, y) \in \alpha_1 \wedge \alpha_2 = 0_X$, so $x = y$. Proofs of the other identities are similar. $\square$

4. FUNCTIONS DERIVED FROM GRAPHICAL COMPOSITIONS

Let R^β_{12} be the relation on $X^2 \times X^2$ defined by

$$(a, b)\; R^\beta_{12}\; (x, y) \quad \longleftrightarrow \quad (a, b) \in \beta \;\text{ and }\; x\; \alpha_1\; a\; \alpha_2\; y\; \alpha_1\; b\; \alpha_2\; x.$$

Define R^β_{13} and R^β_{23} similarly. Graphically, $(a, b)\; R^\beta_{12}\; (x, y)$ holds if and only if the relations in Figure 3 are satisfied.

Lemma 2. Suppose α_i and α_j are complementary equivalence relations on X with uniform blocks of size $\sqrt{|X|}$. Then the relation R^β_{ij} restricted to $\beta \times X^2$ is a one-to-one function from β into X^2.

Proof. First we note that each pair $(a,b) \in \beta$ has at most one image. For if $(a,b)\ R^{\beta}_{ij}$ (x,y) and $(a,b)\ R^{\beta}_{ij}\ (u,v)$, then $(x,u) \in \alpha_i \wedge \alpha_j = 0_X$ and $(y,v) \in \alpha_i \wedge \alpha_j = 0_X$, so $(x,y) = (u,v)$.

Next, since both α_i and α_j have $\sqrt{|X|}$ blocks, and since each of these blocks has size $\sqrt{|X|}$, we see that each block of α_i intersects each block of α_j at exactly one point. That is, for all $a,b \in X$, the set $a/\alpha_i \cap b/\alpha_j$ is a singleton. Therefore, to each $(a,b) \in \beta$ there corresponds precisely one $(x,y) \in X^2$ such that $(a,b)\ R^{\beta}_{ij}\ (x,y)$ holds. Specifically, $\{x\} = a/\alpha_i \cap b/\alpha_j$ and $\{y\} = b/\alpha_i \cap a/\alpha_j$. Thus, R^{β}_{ij} is a function.

From now on, we let $R^{\beta}_{ij}((a,b))$ denote the image of (a,b) under R^{β}_{ij}; that is, $R^{\beta}_{ij}((a,b))$ denotes the ordered pair (x,y) satisfying $(a,b)\ R^{\beta}_{ij}\ (x,y)$.

Suppose $R^{\beta}_{ij}((a,b)) = R^{\beta}_{ij}((c,d))$. Then $(a,c) \in \alpha_i \wedge \alpha_j = 0_X$ and $(b,d) \in \alpha_i \wedge \alpha_j = 0_X$, so $(a,b) = (c,d)$. Therefore, R^{β}_{ij} is one-to-one. □

If, in addition to the assumptions of Lemma 2, we assume that the image of β under R^{β}_{ij} is contained in β, then $R^{\beta}_{ij} : \beta \to \beta$ is a bijective involution. That is, R^{β}_{ij} is one-to-one and onto, and $R^{\beta}_{ij} \circ R^{\beta}_{ij}$ is the identity map.

5. Final piece of the puzzle

As above, suppose $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice and suppose $\{\alpha_i\}_{i=1}^3$ is PPPC. Suppose $R^{\beta}_{ij} : \beta \to \beta$ holds for all $i,j \in \{1,2,3\}$.

Lemma 3. If $a\ \alpha_1\ z\ \beta\ w$, then one of the following holds:

(1) $(a,w) \in \alpha_2$,
(2) $(a,w) \in \alpha_3$,
(3) $(a,w) \in \beta$,
(4) $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$,
(5) $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$.

If Lemma 3 is true, then we can prove the following:

Theorem 1. *If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice with α_i PPPC, then β permutes with each α_i.*

Proof. We will show $\alpha_1 \circ \beta \subseteq \beta \circ \alpha_1$. Assume $a\ \alpha_1\ z\ \beta\ w$. We consider each of the cases in Lemma 3 in turn and, in each case, find b satisfying $a\ \beta\ b\ \alpha_1\ w$.

(1) If $(a,w) \in \alpha_2$, then let $b = z/\alpha_2 \cap w/\alpha_1$. Then $R^{\beta}_{12}(z,w) = (a,b)$ and since $R^{\beta}_{12} : \beta \to \beta$, we have $(a,b) \in \beta$, so $a\ \beta\ b\ \alpha_1\ w$, as desired.
(2) If $(a,w) \in \alpha_3$, then let $b = z/\alpha_3 \cap w/\alpha_1$. Use the same argument as in the first case, but replace R^{β}_{12} with R^{β}_{13}.
(3) If $(a,w) \in \beta$, then let $b = a$.
(4) If $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$, then let y denote the element in this set. Let $x = z/\alpha_1 \cap w/\alpha_3$, and let $b = x/\alpha_2 \cap y/\alpha_1$. Then $(R^{\beta}_{12} \circ R^{\beta}_{13})(z,w) = R^{\beta}_{12}(x,y) = (a,b)$, so $(a,b) \in \beta$. Now, $b\ \alpha_1\ y\ \alpha_1\ w$, so $a\ \beta\ b\ \alpha_1\ w$, as desired.
(5) If $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$, then let y denote this element, let $x = z/\alpha_1 \cap w/\alpha_2$, and let $b = x/\alpha_3 \cap y/\alpha_1$. Then $(R^{\beta}_{13} \circ R^{\beta}_{12})(z,w) = R^{\beta}_{12}(x,y) = (a,b)$, so $(a,b) \in \beta$. Now, $b\ \alpha_1\ y\ \alpha_1\ w$, so $a\ \beta\ b\ \alpha_1\ w$, as desired.

$\square$

6. Proof of Lemma 3

Consider the relation θ_{ij} defined as follows:

$$x\ \theta_{ij}\ y \quad \longleftrightarrow \quad (\exists a, b)\ a\ \alpha_i\ x\ \alpha_j\ b\ \beta\ y\ \alpha_j\ a.$$

Easy arguments similar to those above establish that

$$\theta_{ij} \cap \alpha_i = \theta_{ij} \cap \alpha_j = \theta_{ij} \cap \beta = 0_X.$$

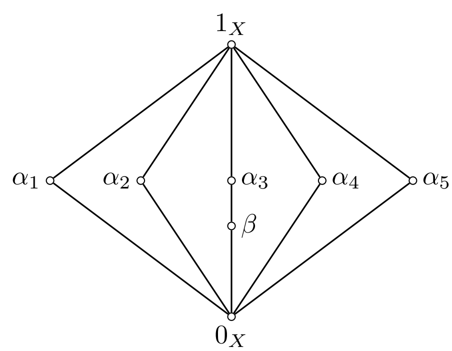
On the other hand, since L is a congruence lattice, it must be the case that the transtive closure of θ_{ij} is contained in L.

TODO: complete proof of Lemma 3 (if possible).

Appendix A. Example

Let X be a set. It is useful to represent partitions of X as lists of lists, and write them as (possibly nonrectangular) arrays, where each row represents a single block. We do this in the following example, which aids our intuition when thinking about the Palfy-Saxl problem.

Let $X = \{0, 1, 2, \dots, 15\}$, and consider the equivalence relations $\alpha_1, \dots, \alpha_5$ and β, generating the following sublattice of $\mathrm{Eq}(X)$:



where $\alpha_1, \dots, \alpha_5$, and β correspond to the following partitions of X:

$$\begin{array}{cccc} & \alpha_1 & & \\ [0 & 1 & 2 & 3] \\ [4 & 5 & 6 & 7] \\ [8 & 9 & 10 & 11] \\ [12 & 13 & 14 & 15] \end{array} \qquad \begin{array}{cccc} & \alpha_2 & & \\ [0 & 4 & 8 & 12] \\ [1 & 5 & 9 & 13] \\ [2 & 6 & 10 & 14] \\ [3 & 7 & 11 & 15] \end{array} \qquad \begin{array}{cccc} & \alpha_3 & & \\ [0 & 5 & 10 & 15] \\ [1 & 4 & 11 & 14] \\ [2 & 7 & 8 & 13] \\ [3 & 6 & 9 & 12] \end{array}$$

$$\begin{array}{cccc} & \alpha_4 & & \\ [0 & 7 & 9 & 14] \\ [1 & 6 & 8 & 15] \\ [2 & 5 & 10 & 12] \\ [3 & 4 & 11 & 13] \end{array} \qquad \begin{array}{cccc} & \alpha_5 & & \\ [0 & 6 & 11 & 13] \\ [1 & 7 & 10 & 12] \\ [2 & 4 & 9 & 15] \\ [3 & 5 & 8 & 14] \end{array} \qquad \begin{array}{cccc} & \beta & & \\ [0 & 5 & 10 & 15] \\ & [1 & 4] & \\ & [2 & 8] & \\ & [3 & 12] & \\ & [6 & 9] & \\ & [7 & 13] & \\ & [11 & 14] & \end{array}$$

The relations $\alpha_1, \dots, \alpha_5$ are PPPC. Also, for each α_i, with $i \neq 3$, it's clear that β and α_i are nonpermuting complements. Here are some other facts that aid intuition.

Fact A.1. Each M_3 sublattice with all α's for atoms is a congruence lattice. In other words, if i, j, k are three distinct numbers in $\{1, 2, \dots, 5\}$, then the sublattice $\{0_X, \alpha_i, \alpha_j, \alpha_k, 1_X\}$ is closed.

Fact A.2. Consider any M_4 having all α's for atoms. The closure is the M_5 lattice $\{0_X, \alpha_1, \dots, \alpha_5, 1_X\}$.

Fact A.3. Each M_4 generated by β and three α's complementary to β is not closed. The closure will have many relations in it.

Regarding the last fact, I've forgotten how many relations are in the closure.

TODO: Check this; also check whether α_3 and the other omitted α always end up in the closure.

Fact A.4. The M_3 sublattice $\{0_X, \alpha_1, \alpha_2, \beta, 1_X\}$ is closed.

Fact A.5. The relation $\tau = \tau(\alpha_1, \alpha_2, \beta)$ defined via the Wheatstone Bridge (Figure 4) is a subset of β.

What follows is an informal discussion of the motivation that led to the relation β given in this example. (This and other parts of the Appendix are verbose and inelegant; all of this will be removed eventually.)

Regarding Fact A.5, β was constructed specifically to provide a nontrivial example where this fact might hold. That is, we wanted to know if an example existed in which β has smaller height than α_i (so that $|x/\beta| \leqslant |y/\alpha_i| < \boldsymbol{\nu}(\beta)$, and so β would not permute with α_1 and α_2), and such that $\tau(\alpha_1, \alpha_2, \beta) \subseteq \beta$, so that the Wheatstone Bridge of Figure 4 would not generate an equivalence relation that isn't already contained in $\{0_X, \alpha_1, \alpha_2, \beta, 1_X\}$.

To construct β, we started by assuming $0/\beta = \{0, 5, 10, 15\}$, which is the main diagonal of both α_1 and α_2. Then we considered the Wheatstone Bridge involving α_1 and α_2 and noticed that, if $\tau \subseteq \beta$, then β must contain all pairs that are at "opposite corners" (defined below) relative to pairs on the main diagonal $\{0, 5, 10, 15\}$.

By "opposite corners" we mean the following. Fix a pair in β, say, $(0, 10) \in \beta$, and consider the squares this pair generates in α_1 and α_2; that is, the squares with 0 and 10 at diagonal corners. We see that 2 and 8 appear at the remaining corners of such squares. We call the corners labeled 2 and 8 the "opposite corners" relative to 0 and 10.

The relation τ defined by the Wheatstone Bridge satisfies

$$0 \mathrel{\beta} 10 \quad \longrightarrow \quad 2 \mathrel{\tau} 8,$$

and, by symmetry of α_1 and α_2,

$$2 \mathrel{\beta} 8 \quad \longrightarrow \quad 0 \mathrel{\tau} 10.$$

Let us make this more general and precise. Recall the relation $\tau = \tau(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ is defined by

$$x \mathrel{\tau} y \quad \longleftrightarrow \quad (\exists (a, b) \in \beta)\; x \mathrel{\alpha_1} a \mathrel{\alpha_2} y \mathrel{\alpha_1} b \mathrel{\alpha_2} x. \tag{A.1}$$

Graphically, $x \mathrel{\tau} y$ if and only if there exist $a, b \in X$ satisfying the relations depicted in Figure 4.

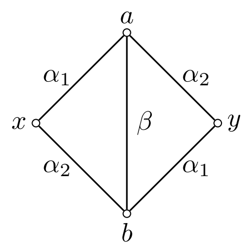


FIGURE 4. The Wheatstone Bridge which defines the relation $\tau(\alpha_1, \alpha_2, \beta)$ as follows: $(x, y) \in \tau(\alpha_1, \alpha_2, \beta)$ if and only if there exist $a, b \in X$ satisfying the relations in the diagram.

Let us order the elements of the equivalence classes of α_1 and α_2 according to the row-column arrangements given in the array representations above, and denote by $\alpha_1(i, j)$ the j-th element of the i-th equivalence class of α_1—that is $\alpha_1(i, j)$ is the element in row i and column j of the array representation of α_1.

Consider the Wheatstone Bridge diagram and note that, if (x, y) and (a, b) satisfy this diagram, so that (A.1) holds, then we have

$$x \in a/\alpha_1 \cap b/\alpha_2 \quad \text{and} \quad y \in b/\alpha_1 \cap a/\alpha_2. \tag{A.2}$$

Suppose $a = \alpha_1(i, j)$ and $b = \alpha_2(k, \ell)$. Then, by (A.2), x is the point where the i-th row of α_1 intersects the k-th row of α_2. But notice that, in this example, the array representing α_2 happens to be the transpose of the array representing α_1. Therefore, the k-th row of α_2 is the k-th column of α_1, so x is the element contained in the i-th row and k-th column of α_1, that is, $x = \alpha_1(i, k)$. Similarly, $y = \alpha_1(j, \ell)$. More generally, for all i, j, r, s in $\{1, 2, 3, 4\}$, we have

$$\alpha_1(i, j)\ \beta\ \alpha_1(r, s) \quad \longrightarrow \quad \alpha_1(i, s)\ \tau\ \alpha_1(j, r).$$

For example, looking at the array representing α_1, we see that if, say, $(2, 15)$ were to belong to β, then the pair $(3, 15)$ at the opposite corners must belong to $\tau(\alpha_1, \alpha_2, \beta)$.

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[1] P. P. Pálfy and J. Saxl, "Congruence lattices of finite algebras and factorizations of groups," *Comm. Algebra*, vol. 18, no. 9, pp. 2783–2790, 1990, ISSN: 0092-7872.