ON A PROBLEM OF PALFY AND SAXL

WILLIAM DEMEO

1. Introduction

In the paper [1], Peter Palfy and Jan Saxl pose the following

> **PROBLEM.** Let $\mathbf{A}$ be a finite algebra with $\operatorname{Con}\mathbf{A} \cong M_n$, $n \geqslant 4$. If three nontrivial congruences of $\mathbf{A}$ pairwise permute, does it follow that every pair of congruences of $\mathbf{A}$ permute?

These notes address answer this question affirmatively.

First, we establish some basic facts and notation. Throughout, X denotes a finite set, $\mathrm{Eq}(X)$ denotes the lattice of equivalence relations on X and, for $\alpha \in \mathrm{Eq}(X)$ and $x \in X$, we denote by x/α the equivalence class of α containing x. We often refer to equivalnce classes as "blocks," and we say that α has *uniform blocks* if, for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. In other words, all equivalence classes of α have the same size.

If α has uniform blocks of size r, then the number of blocks of α is $m = |X|/r$. We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary structure*, if the number of blocks of one is equal to the block size of the other. In other words, if α and β are two equivalence relations on X with uniform block sizes r_α and r_β, respectively, then α and β have complementary block structure if and only if $r_\alpha r_\beta = |X|$.

We denote the largest and smallest equivalence relations on X by 1_X and 0_X, respectively. That is, $1_X = \{(x, y) : x \in X,\, y \in X\}$ and $0_X = \{(x, x) : x \in X\}$.

Fact 1. If α_1, α_2, α_3 are three distinct pairwise permuting, pairwise complementary equivalence relations on X, then these three relations have complementary structure, and each α_i has block size $|X|^{1/2} = n$, for some positive integer n. Thus, the number of blocks of each α_i is $|X|^{1/2}$, and $|X| = n^2$.

References

[1] P. P. Pálfy and J. Saxl. Congruence lattices of finite algebras and factorizations of groups. *Comm. Algebra*, 18(9):2783–2790, 1990.

Date: November 13, 2013.