

# ON A PROBLEM OF PÁLFY AND SAXL

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## 1. INTRODUCTION

In the paper [1], Péter Pálfi and Jan Saxl pose the following

**PROBLEM.** Let  $\mathbf{A}$  be a finite algebra with  $\text{Con } \mathbf{A} \cong M_n$ ,  $n \geq 4$ . If three nontrivial congruences of  $\mathbf{A}$  pairwise permute, does it follow that every pair of congruences of  $\mathbf{A}$  permute?

These notes collect some notation and facts that might be useful for attacking this problem. Throughout,  $X$  denotes a finite set,  $\text{Eq}(X)$  denotes the lattice of equivalence relations on  $X$  and, for  $\alpha \in \text{Eq}(X)$  and  $x \in X$ , we denote by  $x/\alpha$  the equivalence class of  $\alpha$  containing  $x$ . We often refer to equivalence classes as “blocks,” and we denote by  $\#\text{Blocks}(\alpha)$  the number of blocks of the equivalence relation  $\alpha$ .

For a given  $\alpha \in \text{Eq}(X)$  the map  $\varphi_\alpha : x \mapsto x/\alpha$  is a function from  $X$  into the power set  $\mathcal{P}(X)$  with kernel  $\ker \varphi_\alpha = \alpha$ . The *block-size function*  $x \mapsto |x/\alpha|$  is a function from  $X$  into  $\{1, 2, \dots, |X|\}$ .

We will often abuse notation and equate an equivalence relation with the corresponding partition of the set  $X$ . For example, we will equate the relation

$$\alpha = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0), (2, 3), (3, 2)\}$$

with the partition  $|0, 1|2, 3|$ , and often we resort to writing  $\alpha = |0, 1|2, 3|$ .

We say that  $\alpha$  has *uniform blocks* if all blocks of  $\alpha$  have the same size; or, equivalently, the block-size function is constant: for all  $x, y \in X$ ,  $|x/\alpha| = |y/\alpha|$ . We will use  $|x/\alpha|$ , without specifying a particular  $x \in X$ , to denote this block size. Thus, when  $\alpha$  has uniform blocks, we have  $|X| = |x/\alpha| \cdot \#\text{Blocks}(\alpha)$ .

We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary blocks*, if the number of blocks of one is equal to the block size of the other. In other words, if  $\alpha$  and  $\beta$  are two equivalence relations on  $X$  with uniform block sizes  $|x/\alpha|$  and  $|x/\beta|$ , respectively, then  $\alpha$  and  $\beta$  have complementary blocks if and only if  $(\forall x)(\forall y) |x/\alpha| \cdot |y/\beta| = |X|$ .

Given two equivalence relations  $\alpha$  and  $\beta$  on  $X$ , the relation

$$\alpha \circ \beta = \{(x, y) \in X^2 : (\exists z) x \alpha z \beta y\}$$

is called the *composition of  $\alpha$  and  $\beta$* , and if  $\alpha \circ \beta = \beta \circ \alpha$  then  $\alpha$  and  $\beta$  are said to *permute*, or to be *permuting* equivalence relations. Note that  $\alpha \circ \beta \subseteq \alpha \vee \beta$  with equality if and only if  $\alpha$  and  $\beta$  permute.

The largest and smallest equivalence relations on  $X$  are  $1_X = X^2$  and  $0_X = \{(x, x) : x \in X\}$ , respectively.

## 2. BASIC OBSERVATIONS

We say that  $\alpha$  and  $\beta$  are *complementary* equivalence relations on  $X$  provided  $\alpha \vee \beta = 1_X$  and  $\alpha \wedge \beta = 0_X$ .

**Lemma 1.** Suppose  $\alpha$  and  $\beta$  are complementary equivalence relations on  $X$ . Then  $\alpha$  and  $\beta$  permute if and only if they have complementary blocks. That is,

$$\alpha \circ \beta = 1_X \iff (\forall x)(\forall y) |x/\alpha| \cdot |y/\alpha| = |X|.$$

**Corollary 1.** Suppose  $\alpha_1, \alpha_2, \alpha_3$  are pairwise complementary equivalence relations on  $X$ . Then  $\alpha_1, \alpha_2, \alpha_3$  pairwise permute if and only if they all have uniform blocks of size  $\sqrt{|X|}$ . In other words,

$$(\forall i)(\forall j) (i \neq j \longrightarrow \alpha_i \circ \alpha_j = 1_X) \iff (\forall i)(\forall x) |x/\alpha_i| = \sqrt{|X|}.$$

In this case, we clearly have  $|x/\alpha_i| = \# \text{Blocks}(\alpha_i)$ .

*Proof of Lemma 1.* Assume  $\alpha \circ \beta = \alpha \vee \beta = 1_X$ . Then, for all  $x \in X$  we have

$$(2.1) \quad x/(\alpha \circ \beta) = \coprod_{y \in x/\alpha} y/\beta = X,$$

where  $\coprod$  denotes disjoint union. The union is disjoint since  $\alpha \wedge \beta = 0_X$ . Since the union in (2.1) is all of  $X$ , every block of  $\beta$  must appear in the union, so the block  $x/\alpha$  has exactly  $\# \text{Blocks}(\beta)$  elements. Since  $x$  was arbitrary,  $\alpha$  has uniform blocks of size  $|x/\alpha| = \# \text{Blocks}(\beta)$ . Similarly,  $x/(\beta \circ \alpha) = \coprod_{y \in x/\beta} y/\alpha = X$ , so  $|x/\beta| = \# \text{Blocks}(\alpha)$  holds for all  $x \in X$ . Therefore, for all  $x, y \in X$ , we have

$$|x/\alpha| \cdot |y/\beta| = |x/\alpha| \cdot \# \text{Blocks}(\alpha) = |X|.$$

To prove the converse, suppose  $\alpha$  and  $\beta$  are pairwise complements with complementary blocks. Then  $|x/\alpha| \cdot |y/\beta| = |X|$ , thus  $|y/\beta| = |x/\alpha|^{-1} \cdot |X| = \# \text{Blocks}(\alpha)$  hold for all  $x, y \in X$ . Therefore, for all  $x \in X$ , we have

$$\begin{aligned} |x/(\alpha \circ \beta)| &= \left| \coprod_{y \in x/\alpha} y/\beta \right| = \sum_{y \in x/\alpha} |y/\beta| \\ &= \sum_{y \in x/\alpha} \# \text{Blocks}(\alpha) \\ &= |x/\alpha| \# \text{Blocks}(\alpha) = |X|. \end{aligned}$$

This proves that  $\alpha \circ \beta = 1_X$ , as desired.  $\square$

*Proof of Corollary 1.* Since  $\alpha_1$  and  $\alpha_2$  permute and are complements, Lemma 1 implies they have complementary blocks, so

$$(2.2) \quad |x/\alpha_1| = |x/\alpha_2|^{-1} \cdot |X| = \# \text{Blocks}(\alpha_2).$$

(This holds for all  $x \in X$ . Recall that complementary blocks are always uniform.) Similarly, since  $\alpha_1$  and  $\alpha_3$  permute, we have  $|x/\alpha_1| = |x/\alpha_3|^{-1} \cdot |X| = \# \text{Blocks}(\alpha_3)$ . Therefore,  $\# \text{Blocks}(\alpha_2) = \# \text{Blocks}(\alpha_3)$ . Since  $\alpha_2$  and  $\alpha_3$  permute, we have

$$(2.3) \quad |x/\alpha_2| = |x/\alpha_3|^{-1} \cdot |X| = \# \text{Blocks}(\alpha_3),$$

and the latter is equal to  $\# \text{Blocks}(\alpha_2)$ . Therefore,

$$|X| = |x/\alpha_2| \cdot \# \text{Blocks}(\alpha_2) = |x/\alpha_2| \cdot |x/\alpha_2|.$$

Thus,  $|x/\alpha_2| = \sqrt{|X|}$ , so by (2.2) and (2.3) we have  $|x/\alpha_i| = \sqrt{|X|} = \# \text{Blocks}(\alpha_i)$  for  $i = 1, 2, 3$ .



FIGURE 1. The graph defining the relation  $\rho(\alpha_1, \alpha_2, \beta)$ ; that is,  $(x, y) \in \rho(\alpha_1, \alpha_2, \beta)$  if and only if there exist  $a, b \in X$  satisfying the relations in the diagram.

The converse is obvious, since if  $\alpha_i$  and  $\alpha_j$  are complementary equivalence relations on  $X$  with  $|x/\alpha_i| = \sqrt{|X|}$ , then  $\#\text{Blocks}(\alpha_i) = \sqrt{|X|}$ , so  $\alpha_i \circ \alpha_j = 1_X$ .  $\square$

From Corollary 1 we see that the Pálfi-Saxl problem can be stated as

**PROBLEM.** Let  $\mathbf{A}$  be a finite algebra with  $\text{Con } \mathbf{A} \cong M_n$ ,  $n \geq 4$ . If three atoms of  $\mathbf{A}$  have Property (2.4) below, does it follow that every atom has Property (2.4)?

$$(2.4) \quad (\forall x) |x/\alpha| = \sqrt{|X|} = \#\text{Blocks}(\alpha)$$

To prove that the answer is “yes,” it will suffice to prove that if  $M_n \leq \text{Eq}(X)$  has 3 atoms with Property (2.4) and an atom  $\beta$  with  $|x/\beta| < \sqrt{|X|}$ , then this  $M_n$  is not a congruence lattice.

### 3. GRAPHICAL COMPOSITIONS

Suppose  $\alpha_1, \alpha_2$ , and  $\alpha_3$  are pairwise permuting pairwise complements (PPPC) in  $\text{Eq}(X)$ , and let  $\beta \in \text{Eq}(X)$  be complementary to each  $\alpha_i$ , so that

$$L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4.$$

Define the relation  $\rho = \rho(\alpha_1, \alpha_2, \beta) \subseteq X \times X$  as follows:

$$x \rho y \iff (\exists(a, b) \in \beta) x \alpha_1 a \alpha_2 y \alpha_1 b \alpha_2 x.$$

Graphically,  $x \rho y$  if and only if there exist  $a, b \in X$  satisfying the relations depicted in Figure 1.

It is clear that  $\rho$  is reflexive and symmetric but not transitive. Suppose  $f \in X^X$  is a unary function that respects each relation  $\theta \in \{\alpha_1, \alpha_2, \beta\}$ —that is,  $(u, v) \in \theta \implies (f(u), f(v)) \in \theta$ . Then it is clear that  $f$  must also respect  $\rho$ . (Use the same diagram as above, replacing each vertex label  $u$  with the label  $f(u)$ .)

**Fact 1.** If  $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ , then

$$\begin{aligned} \alpha_1 \cap \rho(\alpha_1, \alpha_2, \beta) &= 0_X = \alpha_2 \cap \rho(\alpha_1, \alpha_2, \beta), \\ \alpha_1 \cap \rho(\alpha_1, \alpha_3, \beta) &= 0_X = \alpha_3 \cap \rho(\alpha_1, \alpha_3, \beta), \\ \alpha_2 \cap \rho(\alpha_2, \alpha_3, \beta) &= 0_X = \alpha_3 \cap \rho(\alpha_2, \alpha_3, \beta). \end{aligned}$$

*Proof.* Fix  $(x, y) \in \alpha_1 \cap \rho(\alpha_1, \alpha_2, \beta)$  and suppose  $a$  and  $b$  satisfy the diagram in Figure 1. Then  $(x, y) \in \alpha_1$  implies  $(a, b) \in \alpha_1 \wedge \beta = 0_X$ , so  $a = b$ . Therefore,  $(x, y) \in \alpha_1 \wedge \alpha_2 = 0_X$ , so  $x = y$ . Proofs of the other identities are similar.  $\square$

## 4. FUNCTIONS DERIVED FROM GRAPHICAL COMPOSITIONS

Let  $R_{1,2}^\beta$  be the relation on  $X^2 \times X^2$  defined by

$$(a, b) R_{1,2}^\beta (x, y) \iff (a, b) \in \beta \text{ and } x \alpha_1 a \alpha_2 y \alpha_1 b \alpha_2 x.$$

Define  $R_{1,3}^\beta$  and  $R_{2,3}^\beta$  similarly. Graphically,  $(a, b) R_{1,2}^\beta (x, y)$  holds if and only if the relations in Figure 1 are satisfied.

**Fact 2.** Suppose  $\alpha_i$  and  $\alpha_j$  are complementary equivalence relations on  $X$  with uniform blocks of size  $\sqrt{|X|}$ . Then the relation  $R_{i,j}^\beta$  restricted to  $\beta \times X^2$  is a one-to-one function from  $\beta$  to  $X^2$ .

*Proof.* First we note that there is at most one image. For if  $(a, b) R_{i,j}^\beta (x, y)$  and  $(a, b) R_{i,j}^\beta (u, v)$ , then  $(x, u) \in \alpha_i \wedge \alpha_j = 0_X$  and  $(y, v) \in \alpha_i \wedge \alpha_j = 0_X$ , so  $(x, y) = (u, v)$ .

Next, since both  $\alpha_i$  and  $\alpha_j$  have  $\sqrt{|X|}$  blocks, and since each of these blocks has size  $\sqrt{|X|}$ , we see that each block of  $\alpha_i$  intersects each block of  $\alpha_j$  at exactly one point. That is, for all  $a, b \in X$ , the set  $a/\alpha_i \cap b/\alpha_j$  is a singleton. Therefore, for each  $(a, b) \in \beta$ , there is precisely one  $(x, y) \in X^2$  such that  $(a, b) R_{i,j}^\beta (x, y)$  holds. Specifically,  $\{x\} = a/\alpha_i \cap b/\alpha_j$  and  $\{y\} = b/\alpha_i \cap a/\alpha_j$ . Thus,  $R_{i,j}^\beta$  is a function.

From now on, we let  $R_{i,j}^\beta((a, b))$  denote the image of  $(a, b)$  under  $R_{i,j}^\beta$ ; that is, the ordered pair  $(x, y)$  satisfying  $(a, b) R_{i,j}^\beta (x, y)$ .

Suppose  $R_{i,j}^\beta((a, b)) = R_{i,j}^\beta((c, d))$ . Then  $(a, c) \in \alpha_i \wedge \alpha_j = 0_X$  and  $(b, d) \in \alpha_i \wedge \alpha_j = 0_X$ , so  $(a, b) = (c, d)$ . Therefore,  $R_{i,j}^\beta$  is one-to-one.  $\square$

## REFERENCES

- [1] P. P. Pálfi and J. Saxl. Congruence lattices of finite algebras and factorizations of groups. *Comm. Algebra*, 18(9):2783–2790, 1990.