

# ON A PROBLEM OF PÁLFY AND SAXL

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## 1. INTRODUCTION

In the paper [1], Péter Pálfi and Jan Saxl pose the following

**PROBLEM.** Let  $\mathbf{A}$  be a finite algebra with  $\text{Con } \mathbf{A} \cong M_n$ ,  $n \geq 4$ . If three nontrivial congruences of  $\mathbf{A}$  pairwise permute, does it follow that every pair of congruences of  $\mathbf{A}$  permute?

These notes collect some notation and facts that might be useful for attacking this problem. Throughout,  $X$  denotes a finite set,  $\text{Eq}(X)$  denotes the lattice of equivalence relations on  $X$  and, for  $\alpha \in \text{Eq}(X)$  and  $x \in X$ , we denote by  $x/\alpha$  the equivalence class of  $\alpha$  containing  $x$ . We often refer to equivalence classes as “blocks,” and we denote by  $\#\text{Blocks}(\alpha)$  the number of blocks of the equivalence relation  $\alpha$ .

For a given  $\alpha \in \text{Eq}(X)$  the map  $\varphi_\alpha : x \mapsto x/\alpha$  is a function from  $X$  into the power set  $\mathcal{P}(X)$  with kernel  $\ker \varphi_\alpha = \alpha$ . The *block-size function*  $x \mapsto |x/\alpha|$  is a function from  $X$  into  $\{1, 2, \dots, |X|\}$ .

We will often abuse notation and equate an equivalence relation with the corresponding partition of the set  $X$ . For example, we will equate the relation

$$\alpha = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0), (2, 3), (3, 2)\}$$

with the partition  $|0, 1|2, 3|$ , and often we resort to writing  $\alpha = |0, 1|2, 3|$ .

We say that  $\alpha$  has *uniform blocks* if all blocks of  $\alpha$  have the same size; or, equivalently, the block-size function is constant: for all  $x, y \in X$ ,  $|x/\alpha| = |y/\alpha|$ . We will use  $|x/\alpha|$ , without specifying a particular  $x \in X$ , to denote this block size. Thus, when  $\alpha$  has uniform blocks, we have  $|X| = |x/\alpha| \cdot \#\text{Blocks}(\alpha)$ .

We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary blocks*, if the number of blocks of one is equal to the block size of the other. In other words, if  $\alpha$  and  $\beta$  are two equivalence relations on  $X$  with uniform block sizes  $|x/\alpha|$  and  $|x/\beta|$ , respectively, then  $\alpha$  and  $\beta$  have complementary blocks if and only if  $(\forall x)(\forall y) |x/\alpha| \cdot |y/\beta| = |X|$ .

Given two equivalence relations  $\alpha$  and  $\beta$  on  $X$ , the relation

$$\alpha \circ \beta = \{(x, y) \in X^2 : (\exists z) x \alpha z \beta y\}$$

is called the *composition of  $\alpha$  and  $\beta$* , and if  $\alpha \circ \beta = \beta \circ \alpha$  then  $\alpha$  and  $\beta$  are said to *permute*, or to be *permuting* equivalence relations. Note that  $\alpha \circ \beta \subseteq \alpha \vee \beta$  with equality if and only if  $\alpha$  and  $\beta$  permute.

The largest and smallest equivalence relations on  $X$  are  $1_X = X^2$  and  $0_X = \{(x, x) : x \in X\}$ , respectively.

One more piece of shorthand notation will be useful below. Suppose  $\Theta$  is a set of equivalence relations that are *pairwise permuting pairwise complements* (pairwise permuting pairwise complements (PPPC))—that is, for all  $\gamma \neq \delta$  in  $\Theta$ , we have

$$\gamma \circ \delta = \delta \circ \gamma, \quad \gamma \wedge \delta = 0_X, \quad \gamma \vee \delta = 1_X.$$

Then we say that the relations in  $\Theta$  are PPPC.

## 2. BASIC OBSERVATIONS

We say that  $\alpha$  and  $\beta$  are *complementary* equivalence relations on  $X$  provided  $\alpha \vee \beta = 1_X$  and  $\alpha \wedge \beta = 0_X$ .

**Lemma 1.** Suppose  $\alpha$  and  $\beta$  are complementary equivalence relations on  $X$ . Then  $\alpha$  and  $\beta$  permute if and only if they have complementary blocks. That is,

$$\alpha \circ \beta = 1_X \iff (\forall x)(\forall y) |x/\alpha| \cdot |y/\beta| = |X|.$$

**Corollary 1.** Suppose  $\alpha_1, \alpha_2, \alpha_3$  are pairwise complementary equivalence relations on  $X$ . Then  $\alpha_1, \alpha_2, \alpha_3$  pairwise permute if and only if they all have uniform blocks of size  $\sqrt{|X|}$ . In other words,

$$(\forall i)(\forall j) (i \neq j \longrightarrow \alpha_i \circ \alpha_j = 1_X) \iff (\forall i)(\forall x) |x/\alpha_i| = \sqrt{|X|}.$$

In this case, we clearly have  $|x/\alpha_i| = \#\text{Blocks}(\alpha_i)$ .

*Proof of Lemma 1.* Assume  $\alpha \circ \beta = \alpha \vee \beta = 1_X$ . Then, for all  $x \in X$  we have

$$(2.1) \quad x/(\alpha \circ \beta) = \coprod_{y \in x/\alpha} y/\beta = X,$$

where  $\coprod$  denotes disjoint union. The union is disjoint since  $\alpha \wedge \beta = 0_X$ . Since the union in (2.1) is all of  $X$ , every block of  $\beta$  must appear in the union, so the block  $x/\alpha$  has exactly  $\#\text{Blocks}(\beta)$  elements. Since  $x$  was arbitrary,  $\alpha$  has uniform blocks of size  $|x/\alpha| = \#\text{Blocks}(\beta)$ . Similarly,  $x/(\beta \circ \alpha) = \coprod_{y \in x/\beta} y/\alpha = X$ , so  $|x/\beta| = \#\text{Blocks}(\alpha)$  holds for all  $x \in X$ . Therefore, for all  $x, y \in X$ , we have

$$|x/\alpha| \cdot |y/\beta| = |x/\alpha| \cdot \#\text{Blocks}(\alpha) = |X|.$$

To prove the converse, suppose  $\alpha$  and  $\beta$  are pairwise complements with complementary blocks. Then  $|x/\alpha| \cdot |y/\beta| = |X|$ , thus  $|y/\beta| = |x/\alpha|^{-1} \cdot |X| = \#\text{Blocks}(\alpha)$  hold for all  $x, y \in X$ . Therefore, for all  $x \in X$ , we have

$$\begin{aligned} |x/(\alpha \circ \beta)| &= \left| \coprod_{y \in x/\alpha} y/\beta \right| = \sum_{y \in x/\alpha} |y/\beta| \\ &= \sum_{y \in x/\alpha} \#\text{Blocks}(\alpha) \\ &= |x/\alpha| \cdot \#\text{Blocks}(\alpha) = |X|. \end{aligned}$$

This proves that  $\alpha \circ \beta = 1_X$ , as desired.  $\square$

*Proof of Corollary 1.* Since  $\alpha_1$  and  $\alpha_2$  permute and are complements, Lemma 1 implies they have complementary blocks, so

$$(2.2) \quad |x/\alpha_1| = |x/\alpha_2|^{-1} \cdot |X| = \#\text{Blocks}(\alpha_2).$$



FIGURE 1. The graph defining the relation  $\tau(\alpha_1, \alpha_2, \beta)$ ; that is,  $(x, y) \in \tau(\alpha_1, \alpha_2, \beta)$  if and only if there exist  $a, b \in X$  satisfying the relations in the diagram.

(This holds for all  $x \in X$ . Recall that complementary blocks are always uniform.) Similarly, since  $\alpha_1$  and  $\alpha_3$  permute, we have  $|x/\alpha_1| = |x/\alpha_3|^{-1} \cdot |X| = \# \text{Blocks}(\alpha_3)$ . Therefore,  $\# \text{Blocks}(\alpha_2) = \# \text{Blocks}(\alpha_3)$ . Since  $\alpha_2$  and  $\alpha_3$  permute, we have

$$(2.3) \quad |x/\alpha_2| = |x/\alpha_3|^{-1} \cdot |X| = \# \text{Blocks}(\alpha_3),$$

and the latter is equal to  $\# \text{Blocks}(\alpha_2)$ . Therefore,

$$|X| = |x/\alpha_2| \cdot \# \text{Blocks}(\alpha_2) = |x/\alpha_2| \cdot |x/\alpha_2|.$$

Thus,  $|x/\alpha_2| = \sqrt{|X|}$ , so by (2.2) and (2.3) we have  $|x/\alpha_i| = \sqrt{|X|} = \# \text{Blocks}(\alpha_i)$  for  $i = 1, 2, 3$ .

The converse is obvious, since if  $\alpha_i$  and  $\alpha_j$  are complementary equivalence relations on  $X$  with  $|x/\alpha_i| = \sqrt{|X|}$ , then  $\# \text{Blocks}(\alpha_i) = \sqrt{|X|}$ , so  $\alpha_i \circ \alpha_j = 1_X$ .  $\square$

From Corollary 1 we see that the Pálffy-Saxl problem can be stated as

PROBLEM. Let  $\mathbf{A}$  be a finite algebra with  $\text{Con } \mathbf{A} \cong M_n$ ,  $n \geq 4$ . If three atoms of  $\mathbf{A}$  have Property (2.4) below, does it follow that every atom has Property (2.4)?

$$(2.4) \quad (\forall x) |x/\alpha| = \sqrt{|X|} = \# \text{Blocks}(\alpha)$$

To prove that the answer is “yes,” it will suffice to prove that if  $M_n \leq \text{Eq}(X)$  has 3 atoms with Property (2.4) and an atom  $\beta$  with  $|x/\beta| < \sqrt{|X|}$ , then this  $M_n$  is not a congruence lattice.

### 3. GRAPHICAL COMPOSITIONS

Suppose  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  are pairwise permuting pairwise complements (PPPC) in  $\text{Eq}(X)$ , and let  $\beta \in \text{Eq}(X)$  be complementary to each  $\alpha_i$ , so that

$$L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4.$$

Define the relation  $\tau = \tau(\alpha_1, \alpha_2, \beta) \subseteq X \times X$  as follows:

$$x \tau y \iff (\exists (a, b) \in \beta) x \alpha_1 a \alpha_2 y \alpha_1 b \alpha_2 x.$$

Graphically,  $x \tau y$  if and only if there exist  $a, b \in X$  satisfying the relations depicted in Figure 1.

It is clear that  $\tau$  is a *tolerance*, that is, a reflexive and symmetric binary relation. Let  $f \in X^X$  be a unary function and suppose that  $f$  is *compatible* with each relation  $\theta \in \{\alpha_1, \alpha_2, \beta\}$ , that is,  $(u, v) \in \theta \implies (f(u), f(v)) \in \theta$ . Then  $f$  is

also compatible with  $\tau$ . (Consider the diagram in Figure 1, and give each vertex  $u$  the label  $f(u)$ .)

**Fact 3.1.** If  $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ , then

$$\begin{aligned}\alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta) &= 0_X = \alpha_2 \cap \tau(\alpha_1, \alpha_2, \beta), \\ \alpha_1 \cap \tau(\alpha_1, \alpha_3, \beta) &= 0_X = \alpha_3 \cap \tau(\alpha_1, \alpha_3, \beta), \\ \alpha_2 \cap \tau(\alpha_2, \alpha_3, \beta) &= 0_X = \alpha_3 \cap \tau(\alpha_2, \alpha_3, \beta).\end{aligned}$$

*Proof.* Fix  $(x, y) \in \alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta)$  and suppose  $a$  and  $b$  satisfy the diagram in Figure 1. Then  $(x, y) \in \alpha_1$  implies  $(a, b) \in \alpha_1 \wedge \beta = 0_X$ , so  $a = b$ . Therefore,  $(x, y) \in \alpha_1 \wedge \alpha_2 = 0_X$ , so  $x = y$ . Proofs of the other identities are similar.  $\square$

#### 4. FUNCTIONS DERIVED FROM GRAPHICAL COMPOSITIONS

Let  $R_{12}^\beta$  be the relation on  $X^2 \times X^2$  defined by

$$(a, b) R_{12}^\beta (x, y) \iff (a, b) \in \beta \text{ and } x \alpha_1 a \alpha_2 y \alpha_1 b \alpha_2 x.$$

Define  $R_{13}^\beta$  and  $R_{23}^\beta$  similarly. Graphically,  $(a, b) R_{12}^\beta (x, y)$  holds if and only if the relations in Figure 1 are satisfied.

**Lemma 2.** Suppose  $\alpha_i$  and  $\alpha_j$  are complementary equivalence relations on  $X$  with uniform blocks of size  $\sqrt{|X|}$ . Then the relation  $R_{ij}^\beta$  restricted to  $\beta \times X^2$  is a one-to-one function from  $\beta$  into  $X^2$ .

*Proof.* First we note that each pair  $(a, b) \in \beta$  has at most one image. For if  $(a, b) R_{ij}^\beta (x, y)$  and  $(a, b) R_{ij}^\beta (u, v)$ , then  $(x, u) \in \alpha_i \wedge \alpha_j = 0_X$  and  $(y, v) \in \alpha_i \wedge \alpha_j = 0_X$ , so  $(x, y) = (u, v)$ .

Next, since both  $\alpha_i$  and  $\alpha_j$  have  $\sqrt{|X|}$  blocks, and since each of these blocks has size  $\sqrt{|X|}$ , we see that each block of  $\alpha_i$  intersects each block of  $\alpha_j$  at exactly one point. That is, for all  $a, b \in X$ , the set  $a/\alpha_i \cap b/\alpha_j$  is a singleton. Therefore, to each  $(a, b) \in \beta$  there corresponds precisely one  $(x, y) \in X^2$  such that  $(a, b) R_{ij}^\beta (x, y)$  holds. Specifically,  $\{x\} = a/\alpha_i \cap b/\alpha_j$  and  $\{y\} = b/\alpha_i \cap a/\alpha_j$ . Thus,  $R_{ij}^\beta$  is a function.

From now on, we let  $R_{ij}^\beta((a, b))$  denote the image of  $(a, b)$  under  $R_{ij}^\beta$ ; that is,  $R_{ij}^\beta((a, b))$  denotes the ordered pair  $(x, y)$  satisfying  $(a, b) R_{ij}^\beta (x, y)$ .

Suppose  $R_{ij}^\beta((a, b)) = R_{ij}^\beta((c, d))$ . Then  $(a, c) \in \alpha_i \wedge \alpha_j = 0_X$  and  $(b, d) \in \alpha_i \wedge \alpha_j = 0_X$ , so  $(a, b) = (c, d)$ . Therefore,  $R_{ij}^\beta$  is one-to-one.  $\square$

If, in addition to the assumptions of Lemma 2, we assume that the image of  $\beta$  under  $R_{ij}^\beta$  is contained in  $\beta$ , then  $R_{ij}^\beta : \beta \rightarrow \beta$  is a bijective involution. That is,  $R_{ij}^\beta$  is one-to-one and onto, and  $R_{ij}^\beta \circ R_{ij}^\beta$  is the identity map.

#### 5. FINAL PIECE OF THE PUZZLE

As above, suppose  $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$  is a congruence lattice with  $\alpha_i$  PPPC. Suppose  $R_{ij}^\beta : \beta \rightarrow \beta$  holds for all  $i, j \in \{1, 2, 3\}$ .

**Lemma 3.** If  $a \alpha_1 z \beta w$ , then one of the following holds:

- (1)  $(a, w) \in \alpha_2$ ,
- (2)  $(a, w) \in \alpha_3$ ,

- (3)  $(a, w) \in \beta$ ,
- (4)  $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$ ,
- (5)  $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$ .

If Lemma 3 is true, then we can prove the following:

**Theorem 1.** *If  $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$  is a congruence lattice with  $\alpha_i$  PPPC, then  $\beta$  permutes with each  $\alpha_i$ .*

*Proof.* We will show  $\alpha_1 \circ \beta \subseteq \beta \circ \alpha_1$ . Assume  $a \alpha_1 z \beta w$ . We consider each of the cases in Lemma 3 in turn and, in each case, find  $b$  satisfying  $a \beta b \alpha_1 w$ .

- (1) If  $(a, w) \in \alpha_2$ , then let  $b = z/\alpha_2 \cap w/\alpha_1$ . Then  $R_{12}^\beta(z, w) = (a, b)$  and since  $R_{12}^\beta : \beta \rightarrow \beta$ , we have  $(a, b) \in \beta$ , so  $a \beta b \alpha_1 w$ , as desired.
- (2) If  $(a, w) \in \alpha_3$ , then let  $b = z/\alpha_3 \cap w/\alpha_1$ . Use the same argument as in the first case, but replace  $R_{12}^\beta$  with  $R_{13}^\beta$ .
- (3) If  $(a, w) \in \beta$ , then let  $b = a$ .
- (4) If  $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$ , then let  $y$  denote the element in this set. Let  $x = z/\alpha_1 \cap w/\alpha_3$ , and let  $b = x/\alpha_2 \cap y/\alpha_1$ . Then  $(R_{12}^\beta \circ R_{13}^\beta)(z, w) = R_{12}^\beta(x, y) = (a, b)$ , so  $(a, b) \in \beta$ . Now,  $b \alpha_1 y \alpha_1 w$ , so  $a \beta b \alpha_1 w$ , as desired.
- (5) If  $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$ , then let  $y$  denote this element, let  $x = z/\alpha_1 \cap w/\alpha_2$ , and let  $b = x/\alpha_3 \cap y/\alpha_1$ . Then  $(R_{13}^\beta \circ R_{12}^\beta)(z, w) = R_{12}^\beta(x, y) = (a, b)$ , so  $(a, b) \in \beta$ . Now,  $b \alpha_1 y \alpha_1 w$ , so  $a \beta b \alpha_1 w$ , as desired.

□

## 6. PROOF OF LEMMA 3

Consider the relation  $\theta_{ij}$  defined as follows:

$$x \theta_{ij} y \iff (\exists a, b) a \alpha_i x \alpha_j b \beta y \alpha_j a.$$

Easy arguments similar to those above establish that

$$\theta_{ij} \cap \alpha_i = \theta_{ij} \cap \alpha_j = \theta_{ij} \cap \beta = 0_X.$$

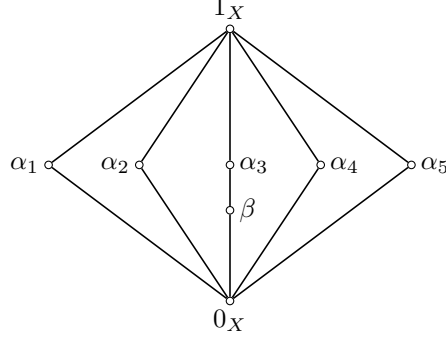
On the other hand, since  $L$  is a congruence lattice, it must be the case that the transitive closure of  $\theta_{ij}$  is contained in  $L$ .

*TODO: prove of Lemma 3 (if possible).*

## APPENDIX A. EXAMPLE

Let  $X$  be a set. It is useful to represent partitions of  $X$  as lists of lists, and write them as (possibly nonrectangular) arrays, where each row represents a single block. We do this in the following example, which seems to aid intuition when thinking about the Palfy-Saxl problem.

Let  $X = \{0, 1, 2, \dots, 15\}$ , and consider the equivalence relation  $\alpha_1, \dots, \alpha_5$  and  $\beta$ , generating the following sublattice of  $\text{Eq}(X)$ :



where  $\alpha_1, \dots, \alpha_5$ , and  $\beta$  correspond to partitions of  $X$  as follows:

| $\alpha_1$    | $\alpha_2$  | $\alpha_3$  |
|---------------|-------------|-------------|
| [0 1 2 3]     | [0 4 8 12]  | [0 5 10 15] |
| [4 5 6 7]     | [1 5 9 13]  | [1 4 11 14] |
| [8 9 10 11]   | [2 6 10 14] | [2 7 8 13]  |
| [12 13 14 15] | [3 7 11 15] | [3 6 9 12]  |

| $\alpha_4$  | $\alpha_5$  | $\beta$     |
|-------------|-------------|-------------|
| [0 7 9 14]  | [0 6 11 13] | [0 5 10 15] |
| [1 6 8 15]  | [1 7 10 12] | [1 4]       |
| [2 5 10 12] | [2 4 9 15]  | [2 8]       |
| [3 4 11 13] | [3 5 8 14]  | [3 12]      |
|             |             | [6 9]       |
|             |             | [7 13]      |
|             |             | [11 14]     |

It is clear that the relations  $\alpha_1, \dots, \alpha_5$  are pairwise permuting pairwise complements (PPPC). Also  $\beta$  is pairwise complementary by not permuting with each  $\alpha_i$ , for  $i \neq 3$ . Here are some other facts which can be helpful aids to intuition.

**Fact A.1.** Each  $M_3$  sublattice with all  $\alpha$ 's for atoms is a congruence lattice. In other words, if  $i, j, k$  are three distinct numbers from the set  $\{1, 2, \dots, 5\}^3$ , then the sublattice  $\{0_X, \alpha_i, \alpha_j, \alpha_k, 1_X\}$  is closed.

**Fact A.2.** Consider any  $M_4$  having all  $\alpha$ 's for atoms. The closure is the  $M_5$  lattice  $\{0_X, \alpha_1, \dots, \alpha_5, 1_X\}$ .

**Fact A.3.** The  $M_3$  sublattice  $\{0_X, \alpha_1, \alpha_2, \beta, 1_X\}$  is closed.

**Fact A.4.** Any  $M_4$  generated by  $\beta$  and three  $\alpha$ 's complementary to  $\beta$  is not closed. The closure will have many relations in it.

Regarding the last fact, I've forgotten how many relations are in the closure.

TODO: Check this; also check whether  $\alpha_3$  and the other omitted  $\alpha$  always end up in the closure.

## APPENDIX B. LIST OF ACRONYMS

**PPPC:** pairwise permuting pairwise complements

## REFERENCES

- [1] P. P. Pálffy and J. Saxl. Congruence lattices of finite algebras and factorizations of groups.  
*Comm. Algebra*, 18(9):2783–2790, 1990.