ON A PROBLEM OF PALFY AND SAXL

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1. Introduction

In the paper [1], Peter Palfy and Jan Saxl pose the following

> **PROBLEM.** Let $\mathbf{A}$ be a finite algebra with $\operatorname{Con}\mathbf{A} \cong M_n$, $n \geqslant 4$. If three nontrivial congruences of $\mathbf{A}$ pairwise permute, does it follow that every pair of congruences of $\mathbf{A}$ permute?

These notes address answer this question affirmatively.

First, we establish some basic facts and notation. Throughout, X denotes a finite set, $\mathrm{Eq}(X)$ denotes the lattice of equivalence relations on X and, for $\alpha \in \mathrm{Eq}(X)$ and $x \in X$, we denote by x/α the equivalence class of α containing x. We often refer to equivalnce classes as "blocks," and we say that α has *uniform blocks* if, for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. In other words, all equivalence classes of α have the same size.

If α has uniform blocks of size r, then the number of blocks of α is $m = |X|/r$. We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary blocks*, if the number of blocks of one is equal to the block size of the other. In other words, if α and β are two equivalence relations on X with uniform block sizes r_α and r_β, respectively, then α and β have complementary block structure if and only if $r_\alpha r_\beta = |X|$.

Given two equivalence relations α and β on X, the relation

$$\alpha \circ \beta = \{(x, y) \in X^2 : (\exists z) x \; \alpha \; z \; \beta \; y\}$$

is called the *composition of* α *and* β, and if $\alpha \circ \beta = \beta \circ \alpha$ then α and β are said to *permute*, or to be *permuting* equivalence relations. Note that $\alpha \circ \beta \subseteq \alpha \vee \beta$ with equility if and only if α and β permute.

We denote the largest and smallest equivalence relations on X by 1_X and 0_X, respectively. That is, $1_X = X^2$ and $0_X = \{(x, x) : x \in X\}$. We say that α and β are *complementary* equivalence relations on X provided $\alpha \vee \beta = 1_X$ and $\alpha \wedge \beta = 0_X$.

Lemma 1. Let $\{\alpha_i\}$ be some pairwise complementary equivalence relations on X.

(1) If $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$, then α_1 and α_2 have uniform blocks.
(2) If α_1 and α_2 have uniform blocks of size $|X|^{1/2}$, then $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$.
(3) Three pairwise complementary equivalence relations are pairwise permuting if and only if all three have uniform blocks of size $|X|^{1/2}$.

References

[1] P. P. Pálfy and J. Saxl. Congruence lattices of finite algebras and factorizations of groups. *Comm. Algebra*, 18(9):2783–2790, 1990.

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