ON A PROBLEM OF PÁLFY AND SAXL

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ABSTRACT. If $\mathbf{A}$ is a finite algebra in a Taylor variety with $\operatorname{Con}\mathbf{A} \cong M_n$, $n \geqslant 4$, and if three nontrivial congruences of $\mathbf{A}$ pairwise permute, then every pair of congruences of $\mathbf{A}$ permute. Thus, for algebras satisfying a nontrivial idempotent Malcev condition, this gives a positive answer to a question asked by Pálfy and Saxl in 1990.

1. INTRODUCTION

In the paper [1], Péter Pálfy and Jan Saxl pose the following

> PROBLEM. Let $\mathbf{A}$ be a finite algebra with $\operatorname{Con}\mathbf{A} \cong M_n$, $n \geqslant 4$. If three nontrivial congruences of $\mathbf{A}$ pairwise permute, does it follow that every pair of congruences of $\mathbf{A}$ permute?

In this note we solve this problem for a special class of algebras—namely, those satisfying a nontrivial idempotent Malcev condition. We call such algebras *Taylor algebras* because they have Taylor term operations.

1.1. Notation.

Throughout, X denotes a finite set and $\operatorname{Eq}(X)$ denotes the lattice of equivalence relations on X. For $\alpha \in \operatorname{Eq}(X)$ and $x \in X$, we let x/α denote the equivalence class of α containing x, and X/α denotes the set of all equivalence classes of α. That is,

$$x/\alpha = \{y \in X : x \;\alpha\; y\} \quad \text{and} \quad X/\alpha = \{x/\alpha : x \in X\}.$$

We often refer to equivalence classes as "blocks," and we denote by $\boldsymbol{\nu}(\alpha)$ the number of blocks of the relation α. That is, $\boldsymbol{\nu}(\alpha) = |X/\alpha|$, and we may use either $\boldsymbol{\nu}(\alpha)$ or $|X/\alpha|$ to denote the number of blocks of α. For a given $\alpha \in \operatorname{Eq}(X)$ the map $\varphi_\alpha : x \mapsto x/\alpha$ is a function from X into the power set $\mathscr{P}(X)$ with kernel $\ker \varphi_\alpha = \alpha$. The *block-size function* $x \mapsto |x/\alpha|$ is a function from X into $\{1, 2, \dots, |X|\}$.

We will often abuse notation and identify an equivalence relation with the corresponding partition of the set X. For example, we identify the relation

$$\alpha = \{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0), (2,3), (3,2)\}$$

with the partition $|0,1|2,3|$, and will even write $\alpha = |0,1|2,3|$.

We say that α has *uniform blocks* if all blocks of α have the same size; or, equivalently, the block-size function is constant: for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. In this case we will use $|x/\alpha|$ without specifying a particular x to denote this block size. Thus, if α has uniform blocks, then

$$|X| = |x/\alpha||X/\alpha| = |x/\alpha|\,\boldsymbol{\nu}(\alpha) \quad (\text{for all } x \in X).$$

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We say that two equivalence relations with uniform blocks have *compatible uniform block structure* (CUBS) if the number of blocks of one is equal to the block size of the other. That is, α and β have CUBS iff $|x/\alpha||y/\beta| = |X|$ (for all x and y).

If α and β are binary relations on X, then the relation

$$\alpha \circ \beta = \{(x, y) \in X^2 : (\exists z)\, x \;\alpha\; z \;\beta\; y\} \tag{1.1}$$

is called the *composition of* α *and* β. If $\alpha \circ \beta = \beta \circ \alpha$ then we call α and β *permuting* relations and we say that α and β *permute*. It is not hard to see that $\alpha \circ \beta \subseteq \alpha \vee \beta$ with equality if and only if α and β permute.

From the definition (1.1) it is clear that $(x, y) \in \alpha \circ \beta$ if and only if $y \in z/\beta$ for some $z \in x/\alpha$. Thus, for every $x \in X$,

$$x/(\alpha \circ \beta) = \bigcup_{z \in x/\alpha} z/\beta. \tag{1.2}$$

The largest and smallest equivalence relations on X are denoted by $1_X = X^2$ and $0_X = \{(x, x) : x \in X\}$, respectively.

We say that α and β are *complementary* equivalence relations on X provided $\alpha \vee \beta = 1_X$ and $\alpha \wedge \beta = 0_X$. If Γ is a set of equivalence relations, we say that Γ consists of *pairwise-permuting pairwise-complements* (PPPC) if the following conditions hold for all $\gamma \neq \delta$ in Γ: (i) $\gamma \circ \delta = \delta \circ \gamma$; (ii) $\gamma \wedge \delta = 0_X$; (iii) $\gamma \vee \delta = 1_X$.

1.2. **Centralizers and abelian algebras.** In later sections nonabelian algebras will play the following role: some of the theorems will begin with the assumption that a particular algebra $\mathbf{A}$ is nonabelian and then proceed to show that if the result to be proved were false, then $\mathbf{A}$ would have to be abelian. To prepare the way for such arguments, we review some basic facts about abelian algebras. (Much of the notation we adopt here is similar to that used in [2].)

Suppose $\mathbf{S}$ and $\mathbf{T}$ are tolerances of $\mathbf{A}$. An $\mathbf{S}, \mathbf{T}$*-matrix* is a 2×2 array of the form

$$\begin{bmatrix} f(a, u) & f(a, v) \\ f(b, u) & f(b, v) \end{bmatrix},$$

where f, a, b, u, v have the following properties:

(i) $f \in \mathrm{Pol}_{\ell+m}(\mathbf{A})$,
(ii) $(a, b) \in A^\ell \times A^\ell$ and $a \;\underline{\mathbf{S}}\; b$,
(iii) $(u, v) \in A^m \times A^m$ and $u \;\underline{\mathbf{T}}\; v$.

Let δ be a congruence relation of $\mathbf{A}$. If every $\mathbf{S}, \mathbf{T}$-matrix satisfies

$$f(a, u) \;\delta\; f(a, v) \quad \Longleftrightarrow \quad f(b, u) \;\delta\; f(b, v), \tag{1.3}$$

then we say that $\mathbf{S}$ *centralizes* $\mathbf{T}$ *modulo* δ and we write $\mathsf{C}(\mathbf{S}, \mathbf{T}; \delta)$. It is important to note that $\mathsf{C}(\mathbf{S}, \mathbf{T}; \delta)$ holds iff condition (1.3) is true *for all* ℓ, m, f, a, b, u, v satisfying properties (i)–(iii). The condition $\mathsf{C}(\mathbf{S}, \mathbf{T}; 0_{\mathbf{A}})$ is sometimes called the $\mathbf{S}, \mathbf{T}$*-term condition*, and when it holds we say that $\mathbf{S}$ *centralizes* $\mathbf{T}$, denoted $\mathsf{C}(\mathbf{S}, \mathbf{T})$. The *commutator* of $\mathbf{S}$ and $\mathbf{T}$, denoted by $[\mathbf{S}, \mathbf{T}]$, is the least congruence δ such that $\mathsf{C}(\mathbf{S}, \mathbf{T}; \delta)$ holds. A tolerance $\mathbf{T}$ is called *abelian* if $[\mathbf{T}, \mathbf{T}] = 0_{\mathbf{A}}$ (i.e., $\mathsf{C}(\mathbf{T}, \mathbf{T})$). An algebra $\mathbf{A}$ is called *abelian* if $1_{\mathbf{A}}$ is abelian (i.e., $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$). The *centralizer of* $\mathbf{T}$ *modulo* δ, denoted by $(\delta : \mathbf{T})$, is the largest congruence α on $\mathbf{A}$ such that $\mathsf{C}(\alpha, \mathbf{T}; \delta)$ holds.

We pause to remark that an algebra $\mathbf{A}$ is abelian iff $\mathsf{C}(1_{\mathbf{A}}, 1_{\mathbf{A}})$ iff $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$ iff for all $\ell, m \in \mathbb{N}$, $f \in \mathrm{Pol}_{\ell+m}(\mathbf{A})$, and $(a, b) \in A^{\ell} \times A^{\ell}$, we have $\ker f(a, \cdot) = \ker f(b, \cdot)$.

We now collect some well-known useful facts about centralizers of congruence relations that we need later.

Lemma 1. Let $\mathbf{A}$ be an algebra with congruences $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$. Then,

(1) there exists a largest congruence that centralizes α, denoted by $(0 : \alpha)$ and called the *centralizer* of α;
(2) if $\alpha \wedge \beta = 0_{\mathbf{A}}$ then $\mathsf{C}(\beta, \alpha)$ and $\mathsf{C}(\alpha, \beta)$;
(3) $\mathsf{C}(\beta, \alpha)$ if and only if $\beta \leqslant (0 : \alpha)$.

2. Basic observations

Lemma 2. Suppose α and β are complementary equivalence relations on X. Then α and β permute if and only if they have CUBS. That is,

$$\alpha \circ \beta = 1_X \quad \Longleftrightarrow \quad (\forall x)(\forall y)\, |x/\alpha||y/\beta| = |X|.$$

Corollary 1. Suppose α_1, α_2, α_3 are pairwise complementary equivalence relations on the finite set X. Then α_1, α_2, α_3 pairwise permute if and only if they all have uniform blocks of size $\sqrt{|X|}$. That is,

$$(\forall i)(\forall j)\,(i \neq j \longrightarrow \alpha_i \circ \alpha_j = 1_X) \quad \Longleftrightarrow \quad (\forall i)(\forall x)\, |x/\alpha_i| = \sqrt{|X|}.$$

In this case, $|x/\alpha_i| = \boldsymbol{\nu}(\alpha_i)$.

Proof of Lemma 2. Suppose α and β are complementary equivalence relations. Then, since $\alpha \wedge \beta = 0_X$, the union in (1.2) is disjoint; we denote this by writing

$$x/(\alpha \circ \beta) = \coprod_{z \in x/\alpha} z/\beta. \tag{2.1}$$

Also, since $\alpha \circ \beta = \alpha \vee \beta = 1_X$, we have $x/(\alpha \circ \beta) = X$ for every $x \in X$. Thus the union in (2.1) is all of X, so every block of β appears in this union. It follows that the size of the block x/α is exactly $\boldsymbol{\nu}(\beta)$. As x was arbitrary, α has uniform blocks of size $\boldsymbol{\nu}(\beta)$. The same argument with the roles of α and β reversed gives $x/(\beta \circ \alpha) = \coprod_{z \in x/\beta} z/\alpha = X$, and $|x/\beta| = \boldsymbol{\nu}(\alpha)$ for all $x \in X$. Therefore, for all $x, z \in X$ we have

$$|x/\alpha||z/\beta| = |x/\alpha|\, \boldsymbol{\nu}(\alpha) = |X|.$$

To prove the converse, suppose α and β are pairwise complements with complementary blocks. Then $|x/\alpha||y/\beta| = |X|$, so $|y/\beta| = |x/\alpha|^{-1}|X| = \boldsymbol{\nu}(\alpha)$. Therefore, for all $x \in X$,

$$\begin{aligned} \bigl|x/(\alpha \circ \beta)\bigr| &= \Bigl|\coprod_{y \in x/\alpha} y/\beta\Bigr| = \sum_{y \in x/\alpha} |y/\beta| \\ &= \sum_{y \in x/\alpha} \boldsymbol{\nu}(\alpha) \\ &= |x/\alpha|\, \boldsymbol{\nu}(\alpha) = |X|. \end{aligned}$$

This proves that $\alpha \circ \beta = 1_X$, as desired. $\square$

Proof of Corollary 1. Since α_1 and α_2 permute and are complements, Lemma 2 implies they have complementary blocks, so

$$|x/\alpha_1| = |x/\alpha_2|^{-1}|X| = \boldsymbol{\nu}(\alpha_2). \tag{2.2}$$

(This holds for all $x \in X$. Recall that complementary blocks are always uniform.) Similarly, since α_1 and α_3 permute, we have $|x/\alpha_1| = |x/\alpha_3|^{-1}|X| = \boldsymbol{\nu}(\alpha_3)$. Therefore, $\boldsymbol{\nu}(\alpha_2) = \boldsymbol{\nu}(\alpha_3)$. Since α_2 and α_3 permute, we have

$$|x/\alpha_2| = |x/\alpha_3|^{-1}|X| = \boldsymbol{\nu}(\alpha_3), \tag{2.3}$$

and the latter is equal to $\boldsymbol{\nu}(\alpha_2)$. Therefore, $|X| = |x/\alpha_2|\,\boldsymbol{\nu}(\alpha_2) = |x/\alpha_2||x/\alpha_2|$, so $|x/\alpha_2| = \sqrt{|X|}$. By (2.2) and (2.3), it follows that $|x/\alpha_i| = \sqrt{|X|} = \boldsymbol{\nu}(\alpha_i)$ for $i = 1, 2, 3$.

The converse is obvious since, if α_i and α_j are complementary equivalence relations on X with $|x/\alpha_i| = \sqrt{|X|}$, then $\boldsymbol{\nu}(\alpha_i) = \sqrt{|X|}$, so $\alpha_i \circ \alpha_j = 1_X$. □

Define the set $\mathcal{S} \subseteq \mathrm{Eq}(X)$ as follows:

$$\mathcal{S} = \{\alpha \in \mathrm{Eq}(X) : (\forall x)\ |x/\alpha| = \sqrt{|X|} = \boldsymbol{\nu}(\alpha)\}.$$

From Corollary 1 we see that the Pálfy-Saxl problem can be rephrased as follows:

> PROBLEM. Let $\mathbf{A}$ be a finite algebra with $\mathrm{Con}\,\mathbf{A} \cong M_n$, $n \geqslant 4$. If the set $\mathcal{S}$ contains three atoms of $\mathrm{Con}\,\mathbf{A}$, does it follow that $\mathcal{S}$ contains every atom of $\mathrm{Con}\,\mathbf{A}$?

To prove that the answer is "yes," it suffices to show that whenever $M_n \cong L \leqslant \mathrm{Eq}(X)$ has 3 atoms in $\mathcal{S}$ and an atom β with $|x/\beta| < \sqrt{|X|}$, then L is not a congruence lattice.

3. Graphical Compositions

Let X be a nonempty set. Given α, β, $\gamma \in \mathrm{Eq}(X)$, define the relation

$$W(\alpha, \beta; \gamma) := \{(x, y) \in X^2 : \exists (a, b) \in \gamma \ .\ x\ \alpha\ a\ \beta\ y\ \alpha\ b\ \beta\ x\}.$$

That is, $(x, y) \in W(\alpha, \beta; \gamma)$ iff there exists $(a, b) \in \gamma$ such that the relations in Figure 1 hold.

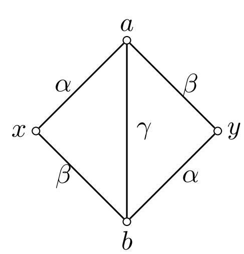


FIGURE 1. The Wheatstone Bridge defines the relation $W(\alpha, \beta; \gamma)$.

It is not hard to see that $W(\alpha, \beta; \gamma)$ is a *tolerance* (that is, a reflexive symmetric binary relation) on X. Also obvious is the fact that $W(\alpha, \beta; \gamma) = W(\beta, \alpha; \gamma)$.

Suppose α_1, α_2, and α_3 are PPPC in $\mathrm{Eq}(X)$, and let $\beta \in \mathrm{Eq}(X)$ be complementary to each α_i, so that $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$. (See Figure 2.)

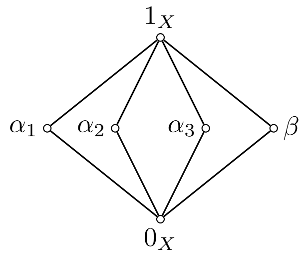


FIGURE 2. The lattice M_4.

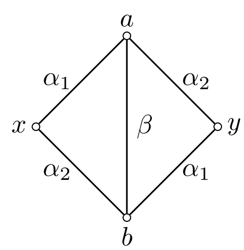


FIGURE 3. The Wheatstone Bridge which defines the relation $\tau(\alpha_1, \alpha_2, \beta)$ as follows: $(x, y) \in \tau(\alpha_1, \alpha_2, \beta)$ if and only if there exist $a, b \in X$ satisfying the relations in the diagram.

Define the relation $\tau = \tau(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ as $R(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ as follows:

$$x \; \tau \; y \quad \longleftrightarrow \quad (\exists (a, b) \in \beta) \; x \; \alpha_1 \; a \; \alpha_2 \; y \; \alpha_1 \; b \; \alpha_2 \; x.$$

Graphically, $x \; \tau \; y$ if and only if there exist $a, b \in X$ satisfying the relations depicted in Figure 3. It is clear that τ is a *tolerance*, that is, a reflexive and symmetric binary relation. Let $f : X \to X$ be a unary operation and suppose that f is *compatible* with each relation $\theta \in \{\alpha_1, \alpha_2, \beta\}$, that is, $(u, v) \in \theta \longrightarrow (f(u), f(v)) \in \theta$. Then f is also compatible with τ.

Fact 3.1. If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$, then

$$\alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta) = 0_X = \alpha_2 \cap \tau(\alpha_1, \alpha_2, \beta),$$
$$\alpha_1 \cap \tau(\alpha_1, \alpha_3, \beta) = 0_X = \alpha_3 \cap \tau(\alpha_1, \alpha_3, \beta),$$
$$\alpha_2 \cap \tau(\alpha_2, \alpha_3, \beta) = 0_X = \alpha_3 \cap \tau(\alpha_2, \alpha_3, \beta).$$

Proof. Fix $(x, y) \in \alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta)$ and suppose a and b satisfy the diagram in Figure 3. Then $(x, y) \in \alpha_1$ implies $(a, b) \in \alpha_1 \wedge \beta = 0_X$, so $a = b$. Therefore, $(x, y) \in \alpha_1 \wedge \alpha_2 = 0_X$, so $x = y$. Proofs of the other identities are similar. $\square$

4. FUNCTIONS DERIVED FROM GRAPHICAL COMPOSITIONS

Let R^{β}_{12} be the relation on $X^2 \times X^2$ defined by

$$(a, b) \; R^{\beta}_{12} \; (x, y) \quad \longleftrightarrow \quad (a, b) \in \beta \;\text{ and }\; x \; \alpha_1 \; a \; \alpha_2 \; y \; \alpha_1 \; b \; \alpha_2 \; x.$$

Define R^{β}_{13} and R^{β}_{23} similarly. Graphically, $(a, b) \; R^{\beta}_{12} \; (x, y)$ holds if and only if the relations in Figure 3 are satisfied.

Lemma 3. Suppose α_i and α_j are complementary equivalence relations on X with uniform blocks of size $\sqrt{|X|}$. Then the relation R^β_{ij} restricted to $\beta \times X^2$ is a one-to-one function from β into X^2.

Proof. First we note that each pair $(a,b) \in \beta$ has at most one image. For if $(a,b)\ R^\beta_{ij}\ (x,y)$ and $(a,b)\ R^\beta_{ij}\ (u,v)$, then $(x,u) \in \alpha_i \wedge \alpha_j = 0_X$ and $(y,v) \in \alpha_i \wedge \alpha_j = 0_X$, so $(x,y) = (u,v)$.

Next, since both α_i and α_j have $\sqrt{|X|}$ blocks, and since each of these blocks has size $\sqrt{|X|}$, we see that each block of α_i intersects each block of α_j at exactly one point. That is, for all $a, b \in X$, the set $a/\alpha_i \cap b/\alpha_j$ is a singleton. Therefore, to each $(a,b) \in \beta$ there corresponds precisely one $(x,y) \in X^2$ such that $(a,b)\ R^\beta_{ij}\ (x,y)$ holds. Specifically, $\{x\} = a/\alpha_i \cap b/\alpha_j$ and $\{y\} = b/\alpha_i \cap a/\alpha_j$. Thus, R^β_{ij} is a function.

From now on, we let $R^\beta_{ij}((a,b))$ denote the image of (a,b) under R^β_{ij}; that is, $R^\beta_{ij}((a,b))$ denotes the ordered pair (x,y) satisfying $(a,b)\ R^\beta_{ij}\ (x,y)$.

Suppose $R^\beta_{ij}((a,b)) = R^\beta_{ij}((c,d))$. Then $(a,c) \in \alpha_i \wedge \alpha_j = 0_X$ and $(b,d) \in \alpha_i \wedge \alpha_j = 0_X$, so $(a,b) = (c,d)$. Therefore, R^β_{ij} is one-to-one. □

If, in addition to the assumptions of Lemma 3, we assume that the image of β under R^β_{ij} is contained in β, then $R^\beta_{ij} : \beta \to \beta$ is a bijective involution. That is, R^β_{ij} is one-to-one and onto, and $R^\beta_{ij} \circ R^\beta_{ij}$ is the identity map.

5. Final piece of the puzzle

As above, suppose $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice and suppose $\{\alpha_i\}_{i=1}^3$ is PPPC. Suppose $R^\beta_{ij} : \beta \to \beta$ holds for all $i, j \in \{1,2,3\}$.

Lemma 4. If $a\ \alpha_1\ z\ \beta\ w$, then one of the following holds:

(1) $(a,w) \in \alpha_2$,
(2) $(a,w) \in \alpha_3$,
(3) $(a,w) \in \beta$,
(4) $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$,
(5) $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$.

If Lemma 4 is true, then we can prove the following:

Theorem 1. *If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice with α_i PPPC, then β permutes with each α_i.*

Proof. We will show $\alpha_1 \circ \beta \subseteq \beta \circ \alpha_1$. Assume $a\ \alpha_1\ z\ \beta\ w$. We consider each of the cases in Lemma 4 in turn and, in each case, find b satisfying $a\ \beta\ b\ \alpha_1\ w$.

(1) If $(a,w) \in \alpha_2$, then let $b = z/\alpha_2 \cap w/\alpha_1$. Then $R^\beta_{12}(z,w) = (a,b)$ and since $R^\beta_{12} : \beta \to \beta$, we have $(a,b) \in \beta$, so $a\ \beta\ b\ \alpha_1\ w$, as desired.
(2) If $(a,w) \in \alpha_3$, then let $b = z/\alpha_3 \cap w/\alpha_1$. Use the same argument as in the first case, but replace R^β_{12} with R^β_{13}.
(3) If $(a,w) \in \beta$, then let $b = a$.
(4) If $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$, then let y denote the element in this set. Let $x = z/\alpha_1 \cap w/\alpha_3$, and let $b = x/\alpha_2 \cap y/\alpha_1$. Then $(R^\beta_{12} \circ R^\beta_{13})(z,w) = R^\beta_{12}(x,y) = (a,b)$, so $(a,b) \in \beta$. Now, $b\ \alpha_1\ y\ \alpha_1\ w$, so $a\ \beta\ b\ \alpha_1\ w$, as desired.

(5) If $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$, then let y denote this element, let $x = z/\alpha_1 \cap w/\alpha_2$, and let $b = x/\alpha_3 \cap y/\alpha_1$. Then $(R^{\beta}_{13} \circ R^{\beta}_{12})(z, w) = R^{\beta}_{12}(x, y) = (a, b)$, so $(a, b) \in \beta$. Now, $b \; \alpha_1 \; y \; \alpha_1 \; w$, so $a \; \beta \; b \; \alpha_1 \; w$, as desired.

□

6. Proof of Lemma 3

Consider the relation θ_{ij} defined as follows:

$$x \; \theta_{ij} \; y \quad \longleftrightarrow \quad (\exists a, b) \; a \; \alpha_i \; x \; \alpha_j \; b \; \beta \; y \; \alpha_j \; a.$$

Easy arguments similar to those above establish that

$$\theta_{ij} \cap \alpha_i = \theta_{ij} \cap \alpha_j = \theta_{ij} \cap \beta = 0_X.$$

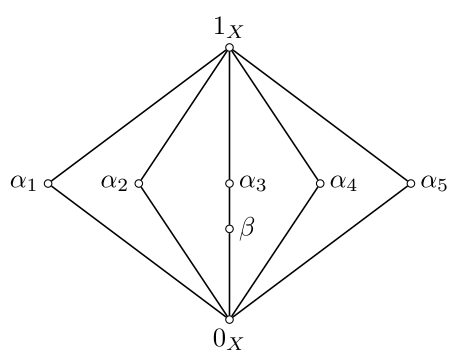
On the other hand, since L is a congruence lattice, it must be the case that the transtive closure of θ_{ij} is contained in L.

TODO: complete proof of Lemma 3 (if possible).

Appendix A. Example

Let X be a set. It is useful to represent partitions of X as lists of lists, and write them as (possibly nonrectangular) arrays, where each row represents a single block. We do this in the following example, which aids our intuition when thinking about the Palfy-Saxl problem.

Let $X = \{0, 1, 2, \dots, 15\}$, and consider the equivalence relations $\alpha_1, \dots, \alpha_5$ and β, generating the following sublattice of $\mathrm{Eq}(X)$:



where $\alpha_1, \dots, \alpha_5$, and β correspond to the following partitions of X:

$$\begin{array}{cccc} & \alpha_1 & & \\ [0 & 1 & 2 & 3] \\ [4 & 5 & 6 & 7] \\ [8 & 9 & 10 & 11] \\ [12 & 13 & 14 & 15] \end{array} \qquad \begin{array}{cccc} & \alpha_2 & & \\ [0 & 4 & 8 & 12] \\ [1 & 5 & 9 & 13] \\ [2 & 6 & 10 & 14] \\ [3 & 7 & 11 & 15] \end{array} \qquad \begin{array}{cccc} & \alpha_3 & & \\ [0 & 5 & 10 & 15] \\ [1 & 4 & 11 & 14] \\ [2 & 7 & 8 & 13] \\ [3 & 6 & 9 & 12] \end{array}$$

$$
\begin{array}{c}
\alpha_4\\
[0\quad 7\quad 9\quad 14]\\
[1\quad 6\quad 8\quad 15]\\
[2\quad 5\quad 10\quad 12]\\
[3\quad 4\quad 11\quad 13]
\end{array}
\qquad
\begin{array}{c}
\alpha_5\\
[0\quad 6\quad 11\quad 13]\\
[1\quad 7\quad 10\quad 12]\\
[2\quad 4\quad 9\quad 15]\\
[3\quad 5\quad 8\quad 14]
\end{array}
\qquad
\begin{array}{c}
\beta\\
[0\quad 5\quad 10\quad 15]\\
[1\quad 4]\\
[2\quad 8]\\
[3\quad 12]\\
[6\quad 9]\\
[7\quad 13]\\
[11\quad 14]
\end{array}
$$

The relations $\alpha_1, \dots, \alpha_5$ are PPPC. Also, for each α_i, with $i \neq 3$, it's clear that β and α_i are nonpermuting complements. Here are some other facts that aid intuition.

Fact A.1. Each M_3 sublattice with all α's for atoms is a congruence lattice. In other words, if i, j, k are three distinct numbers in $\{1, 2, \dots, 5\}$, then the sublattice $\{0_X, \alpha_i, \alpha_j, \alpha_k, 1_X\}$ is closed.

Fact A.2. Consider any M_4 having all α's for atoms. The closure is the M_5 lattice $\{0_X, \alpha_1, \dots, \alpha_5, 1_X\}$.

Fact A.3. Each M_4 generated by β and three α's complementary to β is not closed. The closure will have many relations in it.

Regarding the last fact, I've forgotten how many relations are in the closure.

TODO: Check this; also check whether α_3 and the other omitted α always end up in the closure.

Fact A.4. The M_3 sublattice $\{0_X, \alpha_1, \alpha_2, \beta, 1_X\}$ is closed.

Fact A.5. The relation $\tau = \tau(\alpha_1, \alpha_2, \beta)$ defined via the Wheatstone Bridge (Figure 4) is a subset of β.

What follows is an informal discussion of the motivation that led to the relation β given in this example. (This and other parts of the Appendix are verbose and inelegant; all of this will be removed eventually.)

Regarding Fact A.5, β was constructed specifically to provide a nontrivial example where this fact might hold. That is, we wanted to know if an example existed in which β has smaller height than α_i (so that $|x/\beta| \leqslant |y/\alpha_i| < \boldsymbol{\nu}(\beta)$, and so β would not permute with α_1 and α_2), and such that $\tau(\alpha_1, \alpha_2, \beta) \subseteq \beta$, so that the Wheatstone Bridge of Figure 4 would not generate an equivalence relation that isn't already contained in $\{0_X, \alpha_1, \alpha_2, \beta, 1_X\}$.

To construct β, we started by assuming $0/\beta = \{0, 5, 10, 15\}$, which is the main diagonal of both α_1 and α_2. Then we considered the Wheatstone Bridge involving α_1 and α_2 and noticed that, if $\tau \subseteq \beta$, then β must contain all pairs that are at "opposite corners" (defined below) relative to pairs on the main diagonal $\{0, 5, 10, 15\}$.

By "opposite corners" we mean the following. Fix a pair in β, say, $(0, 10) \in \beta$, and consider the squares this pair generates in α_1 and α_2; that is, the squares with 0 and 10 at diagonal corners. We see that 2 and 8 appear at the remaining corners of such squares. We call the corners labeled 2 and 8 the "opposite corners" relative to 0 and 10.

The relation τ defined by the Wheatstone Bridge satisfies

$$0 \,\beta\, 10 \quad \longrightarrow \quad 2 \,\tau\, 8,$$

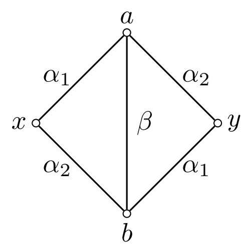


FIGURE 4. The Wheatstone Bridge which defines the relation $\tau(\alpha_1, \alpha_2, \beta)$ as follows: $(x, y) \in \tau(\alpha_1, \alpha_2, \beta)$ if and only if there exist $a, b \in X$ satisfying the relations in the diagram.

and, by symmetry of α_1 and α_2,

$$2 \; \beta \; 8 \quad \longrightarrow \quad 0 \; \tau \; 10.$$

Let us make this more general and precise. Recall the relation $\tau = \tau(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ is defined by

$$x \; \tau \; y \quad \longleftrightarrow \quad (\exists (a, b) \in \beta) \; x \; \alpha_1 \; a \; \alpha_2 \; y \; \alpha_1 \; b \; \alpha_2 \; x. \tag{A.1}$$

Graphically, $x \; \tau \; y$ if and only if there exist $a, b \in X$ satisfying the relations depicted in Figure 4.

Let us order the elements of the equivalence classes of α_1 and α_2 according to the row-column arrangements given in the array representations above, and denote by $\alpha_1(i, j)$ the j-th element of the i-th equivalence class of α_1—that is $\alpha_1(i, j)$ is the element in row i and column j of the array representation of α_1.

Consider the Wheatstone Bridge diagram and note that, if (x, y) and (a, b) satisfy this diagram, so that (A.1) holds, then we have

$$x \in a/\alpha_1 \cap b/\alpha_2 \quad \text{and} \quad y \in b/\alpha_1 \cap a/\alpha_2. \tag{A.2}$$

Suppose $a = \alpha_1(i, j)$ and $b = \alpha_2(k, \ell)$. Then, by (A.2), x is the point where the i-th row of α_1 intersects the k-th row of α_2. But notice that, in this example, the array representing α_2 happens to be the transpose of the array representing α_1. Therefore, the k-th row of α_2 is the k-th column of α_1, so x is the element contained in the i-th row and k-th column of α_1, that is, $x = \alpha_1(i, k)$. Similarly, $y = \alpha_1(j, \ell)$. More generally, for all i, j, r, s in $\{1, 2, 3, 4\}$, we have

$$\alpha_1(i, j) \; \beta \; \alpha_1(r, s) \quad \longrightarrow \quad \alpha_1(i, s) \; \tau \; \alpha_1(j, r).$$

For example, looking at the array representing α_1, we see that if, say, $(2, 15)$ were to belong to β, then the pair $(3, 15)$ at the opposite corners must belong to $\tau(\alpha_1, \alpha_2, \beta)$.

REFERENCES

[1] P. P. Pálfy and J. Saxl, "Congruence lattices of finite algebras and factorizations of groups," *Comm. Algebra*, vol. 18, no. 9, pp. 2783–2790, 1990, ISSN: 0092-7872.

[2] K. A. Kearnes and E. W. Kiss, "The shape of congruence lattices," *Mem. Amer. Math. Soc.*, vol. 222, no. 1046, pp. viii+169, 2013, ISSN: 0065-9266. DOI: 10.1090/S0065-9266-2012-00667-8. [Online]. Available: http://dx.doi.org/10.1090/S0065-9266-2012-00667-8.