

ON A PROBLEM OF PÁLFY AND SAXL

WILLIAM DEMEO

1. INTRODUCTION

In the paper [1], Péter Pálfi and Jan Saxl pose the following

PROBLEM. Let \mathbf{A} be a finite algebra with $\text{Con } \mathbf{A} \cong M_n$, $n \geq 4$. If three nontrivial congruences of \mathbf{A} pairwise permute, does it follow that every pair of congruences of \mathbf{A} permute?

These notes collect some notation and facts that might be useful for attacking this problem. Throughout, X denotes a finite set, $\text{Eq}(X)$ denotes the lattice of equivalence relations on X and, for $\alpha \in \text{Eq}(X)$ and $x \in X$, we denote by x/α the equivalence class of α containing x . We often refer to equivalence classes as “blocks,” and we denote by $\nu(\alpha)$ the number of blocks of the relation α .

For a given $\alpha \in \text{Eq}(X)$ the map $\varphi_\alpha : x \mapsto x/\alpha$ is a function from X into the power set $\mathcal{P}(X)$ with kernel $\ker \varphi_\alpha = \alpha$. The *block-size function* $x \mapsto |x/\alpha|$ is a function from X into $\{1, 2, \dots, |X|\}$.

We will often abuse notation and identify an equivalence relation with the corresponding partition of the set X . For example, we identify the relation

$$\alpha = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0), (2, 3), (3, 2)\}$$

with the partition $|0, 1|2, 3|$, and will even resort to writing $\alpha = |0, 1|2, 3|$.

We say that α has *uniform blocks* if all blocks of α have the same size; or, equivalently, the block-size function is constant: for all $x, y \in X$, $|x/\alpha| = |y/\alpha|$. In this case we will use $|x/\alpha|$ without specifying a particular x to denote this block size. Thus, when α has uniform blocks, we have $|X| = |x/\alpha| \nu(\alpha)$.

We say that two equivalence relations with uniform blocks have *complementary uniform block structure*, or simply *complementary blocks*, if the number of blocks of one is equal to the block size of the other. In other words, if α and β are two equivalence relations on X with uniform block sizes $|x/\alpha|$ and $|x/\beta|$, respectively, then α and β have complementary blocks if and only if $(\forall x)(\forall y) |x/\alpha| |y/\beta| = |X|$.

Given two equivalence relations α and β on X , the relation

$$(1.1) \quad \alpha \circ \beta = \{(x, y) \in X^2 : (\exists z) x \alpha z \beta y\}$$

is called the *composition of α and β* , and if $\alpha \circ \beta = \beta \circ \alpha$ then α and β are said to *permute*, or to be *permuting* equivalence relations. Note that $\alpha \circ \beta \subseteq \alpha \vee \beta$ with equality if and only if α and β permute.

From the definition (1.1) it is clear that $(x, y) \in \alpha \circ \beta$ if and only if $y \in z/\beta$ for some $z \in x/\alpha$. In other terms, for every $x \in X$ we have

$$(1.2) \quad x/(\alpha \circ \beta) = \bigcup_{z \in x/\alpha} z/\beta.$$

The largest and smallest equivalence relations on X are $1_X = X^2$ and $0_X = \{(x, x) : x \in X\}$, respectively.

We say that α and β are *complementary* equivalence relations on X provided $\alpha \vee \beta = 1_X$ and $\alpha \wedge \beta = 0_X$. If Γ is a set of equivalence relations, we say that Γ consists of *pairwise-permuting pairwise-complements (PPPC)* if the following conditions hold for all $\gamma \neq \delta$ in Γ : (i) $\gamma \circ \delta = \delta \circ \gamma$; (ii) $\gamma \wedge \delta = 0_X$; (iii) $\gamma \vee \delta = 1_X$.

2. BASIC OBSERVATIONS

Lemma 1. Suppose α and β are complementary equivalence relations on X . Then α and β permute if and only if they have complementary blocks. That is,

$$\alpha \circ \beta = 1_X \iff (\forall x)(\forall y) |x/\alpha||y/\beta| = |X|.$$

Corollary 1. Suppose $\alpha_1, \alpha_2, \alpha_3$ are pairwise complementary equivalence relations on X . Then $\alpha_1, \alpha_2, \alpha_3$ pairwise permute if and only if they all have uniform blocks of size $\sqrt{|X|}$. In other words,

$$(\forall i)(\forall j) (i \neq j \longrightarrow \alpha_i \circ \alpha_j = 1_X) \iff (\forall i)(\forall x) |x/\alpha_i| = \sqrt{|X|}.$$

In this case, $|x/\alpha_i| = \nu(\alpha_i)$.

Proof of Lemma 1. Suppose α and β are complementary equivalence relations. Then the union in (1.2) is disjoint since $\alpha \wedge \beta = 0_X$. We denote this by writing

$$(2.1) \quad x/(\alpha \circ \beta) = \coprod_{z \in x/\alpha} z/\beta.$$

Also, since $\alpha \circ \beta = \alpha \vee \beta = 1_X$, we have $x/(\alpha \circ \beta) = X$ for every $x \in X$. Thus the union in (2.1) is all of X , so every block of β appears in this union. It follows that the size of the block x/α is exactly $\nu(\beta)$. As x was arbitrary, α has uniform blocks of size $\nu(\beta)$. The same argument with the roles of α and β reversed gives $x/(\beta \circ \alpha) = \coprod_{z \in x/\beta} z/\alpha = X$, and $|x/\beta| = \nu(\alpha)$ for all $x \in X$. Therefore, for all $x, z \in X$ we have

$$|x/\alpha||z/\beta| = |x/\alpha|\nu(\alpha) = |X|.$$

To prove the converse, suppose α and β are pairwise complements with complementary blocks. Then $|x/\alpha||y/\beta| = |X|$, so $|y/\beta| = |x/\alpha|^{-1}|X| = \nu(\alpha)$. Therefore, for all $x \in X$,

$$\begin{aligned} |x/(\alpha \circ \beta)| &= \left| \coprod_{y \in x/\alpha} y/\beta \right| = \sum_{y \in x/\alpha} |y/\beta| \\ &= \sum_{y \in x/\alpha} \nu(\alpha) \\ &= |x/\alpha|\nu(\alpha) = |X|. \end{aligned}$$

This proves that $\alpha \circ \beta = 1_X$, as desired. \square

Proof of Corollary 1. Since α_1 and α_2 permute and are complements, Lemma 1 implies they have complementary blocks, so

$$(2.2) \quad |x/\alpha_1| = |x/\alpha_2|^{-1}|X| = \nu(\alpha_2).$$



FIGURE 1. The Wheatstone Bridge which defines the relation $\tau(\alpha_1, \alpha_2, \beta)$ as follows: $(x, y) \in \tau(\alpha_1, \alpha_2, \beta)$ if and only if there exist $a, b \in X$ satisfying the relations in the diagram.

(This holds for all $x \in X$. Recall that complementary blocks are always uniform.) Similarly, since α_1 and α_3 permute, we have $|x/\alpha_1| = |x/\alpha_3|^{-1}|X| = \nu(\alpha_3)$. Therefore, $\nu(\alpha_2) = \nu(\alpha_3)$. Since α_2 and α_3 permute, we have

$$(2.3) \quad |x/\alpha_2| = |x/\alpha_3|^{-1}|X| = \nu(\alpha_3),$$

and the latter is equal to $\nu(\alpha_2)$. Therefore, $|X| = |x/\alpha_2| \nu(\alpha_2) = |x/\alpha_2| |x/\alpha_2|$, so $|x/\alpha_2| = \sqrt{|X|}$. By (2.2) and (2.3), it follows that $|x/\alpha_i| = \sqrt{|X|} = \nu(\alpha_i)$ for $i = 1, 2, 3$.

The converse is obvious since, if α_i and α_j are complementary equivalence relations on X with $|x/\alpha_i| = \sqrt{|X|}$, then $\nu(\alpha_i) = \sqrt{|X|}$, so $\alpha_i \circ \alpha_j = 1_X$. \square

Define the set $\mathcal{S} \subseteq \text{Eq}(X)$ as follows:

$$\mathcal{S} = \{\alpha \in \text{Eq}(X) : (\forall x) |x/\alpha| = \sqrt{|X|} = \nu(\alpha)\}.$$

From Corollary 1 we see that the Pálfi-Saxl problem can be rephrased as follows:

PROBLEM. Let \mathbf{A} be a finite algebra with $\text{Con } \mathbf{A} \cong M_n$, $n \geq 4$. If the set \mathcal{S} contains three atoms of $\text{Con } \mathbf{A}$, does it follow that \mathcal{S} contains every atom of $\text{Con } \mathbf{A}$?

To prove that the answer is “yes,” it suffices to show that whenever $M_n \cong L \leq \text{Eq}(X)$ has 3 atoms in \mathcal{S} and an atom β with $|x/\beta| < \sqrt{|X|}$, then L is not a congruence lattice.

3. GRAPHICAL COMPOSITIONS

Suppose α_1 , α_2 , and α_3 are PPPC in $\text{Eq}(X)$, and let $\beta \in \text{Eq}(X)$ be complementary to each α_i , so that

$$L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4.$$

Define the relation $\tau = \tau(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ as follows:

$$x \tau y \iff (\exists(a, b) \in \beta) x \alpha_1 a \alpha_2 y \alpha_1 b \alpha_2 x.$$

Graphically, $x \tau y$ if and only if there exist $a, b \in X$ satisfying the relations depicted in Figure 1.

It is clear that τ is a *tolerance*, that is, a reflexive and symmetric binary relation. Let $f \in X^X$ be a unary function and suppose that f is *compatible* with each relation $\theta \in \{\alpha_1, \alpha_2, \beta\}$, that is, $(u, v) \in \theta \implies (f(u), f(v)) \in \theta$. Then f is also compatible with τ . (Consider the diagram in Figure 1, and give each vertex u the label $f(u)$.)

Fact 3.1. If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$, then

$$\begin{aligned}\alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta) &= 0_X = \alpha_2 \cap \tau(\alpha_1, \alpha_2, \beta), \\ \alpha_1 \cap \tau(\alpha_1, \alpha_3, \beta) &= 0_X = \alpha_3 \cap \tau(\alpha_1, \alpha_3, \beta), \\ \alpha_2 \cap \tau(\alpha_2, \alpha_3, \beta) &= 0_X = \alpha_3 \cap \tau(\alpha_2, \alpha_3, \beta).\end{aligned}$$

Proof. Fix $(x, y) \in \alpha_1 \cap \tau(\alpha_1, \alpha_2, \beta)$ and suppose a and b satisfy the diagram in Figure 1. Then $(x, y) \in \alpha_1$ implies $(a, b) \in \alpha_1 \wedge \beta = 0_X$, so $a = b$. Therefore, $(x, y) \in \alpha_1 \wedge \alpha_2 = 0_X$, so $x = y$. Proofs of the other identities are similar. \square

4. FUNCTIONS DERIVED FROM GRAPHICAL COMPOSITIONS

Let R_{12}^β be the relation on $X^2 \times X^2$ defined by

$$(a, b) R_{12}^\beta (x, y) \iff (a, b) \in \beta \text{ and } x \alpha_1 a \alpha_2 y \alpha_1 b \alpha_2 x.$$

Define R_{13}^β and R_{23}^β similarly. Graphically, $(a, b) R_{12}^\beta (x, y)$ holds if and only if the relations in Figure 1 are satisfied.

Lemma 2. Suppose α_i and α_j are complementary equivalence relations on X with uniform blocks of size $\sqrt{|X|}$. Then the relation R_{ij}^β restricted to $\beta \times X^2$ is a one-to-one function from β into X^2 .

Proof. First we note that each pair $(a, b) \in \beta$ has at most one image. For if $(a, b) R_{ij}^\beta (x, y)$ and $(a, b) R_{ij}^\beta (u, v)$, then $(x, u) \in \alpha_i \wedge \alpha_j = 0_X$ and $(y, v) \in \alpha_i \wedge \alpha_j = 0_X$, so $(x, y) = (u, v)$.

Next, since both α_i and α_j have $\sqrt{|X|}$ blocks, and since each of these blocks has size $\sqrt{|X|}$, we see that each block of α_i intersects each block of α_j at exactly one point. That is, for all $a, b \in X$, the set $a/\alpha_i \cap b/\alpha_j$ is a singleton. Therefore, to each $(a, b) \in \beta$ there corresponds precisely one $(x, y) \in X^2$ such that $(a, b) R_{ij}^\beta (x, y)$ holds. Specifically, $\{x\} = a/\alpha_i \cap b/\alpha_j$ and $\{y\} = b/\alpha_i \cap a/\alpha_j$. Thus, R_{ij}^β is a function.

From now on, we let $R_{ij}^\beta((a, b))$ denote the image of (a, b) under R_{ij}^β ; that is, $R_{ij}^\beta((a, b))$ denotes the ordered pair (x, y) satisfying $(a, b) R_{ij}^\beta (x, y)$.

Suppose $R_{ij}^\beta((a, b)) = R_{ij}^\beta((c, d))$. Then $(a, c) \in \alpha_i \wedge \alpha_j = 0_X$ and $(b, d) \in \alpha_i \wedge \alpha_j = 0_X$, so $(a, b) = (c, d)$. Therefore, R_{ij}^β is one-to-one. \square

If, in addition to the assumptions of Lemma 2, we assume that the image of β under R_{ij}^β is contained in β , then $R_{ij}^\beta : \beta \rightarrow \beta$ is a bijective involution. That is, R_{ij}^β is one-to-one and onto, and $R_{ij}^\beta \circ R_{ij}^\beta$ is the identity map.

5. FINAL PIECE OF THE PUZZLE

As above, suppose $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice and suppose $\{\alpha_i\}_{i=1}^3$ is PPPC. Suppose $R_{ij}^\beta : \beta \rightarrow \beta$ holds for all $i, j \in \{1, 2, 3\}$.

Lemma 3. If $a \alpha_1 z \beta w$, then one of the following holds:

- (1) $(a, w) \in \alpha_2$,
- (2) $(a, w) \in \alpha_3$,
- (3) $(a, w) \in \beta$,
- (4) $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$,

$$(5) \ a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset.$$

If Lemma 3 is true, then we can prove the following:

Theorem 1. *If $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, \beta, 1_X\} \cong M_4$ is a congruence lattice with α_i PPC, then β permutes with each α_i .*

Proof. We will show $\alpha_1 \circ \beta \subseteq \beta \circ \alpha_1$. Assume $a \alpha_1 z \beta w$. We consider each of the cases in Lemma 3 in turn and, in each case, find b satisfying $a \beta b \alpha_1 w$.

- (1) If $(a, w) \in \alpha_2$, then let $b = z/\alpha_2 \cap w/\alpha_1$. Then $R_{12}^\beta(z, w) = (a, b)$ and since $R_{12}^\beta : \beta \rightarrow \beta$, we have $(a, b) \in \beta$, so $a \beta b \alpha_1 w$, as desired.
- (2) If $(a, w) \in \alpha_3$, then let $b = z/\alpha_3 \cap w/\alpha_1$. Use the same argument as in the first case, but replace R_{12}^β with R_{13}^β .
- (3) If $(a, w) \in \beta$, then let $b = a$.
- (4) If $a/\alpha_2 \cap z/\alpha_3 \cap w/\alpha_1 \neq \emptyset$, then let y denote the element in this set. Let $x = z/\alpha_1 \cap w/\alpha_3$, and let $b = x/\alpha_2 \cap y/\alpha_1$. Then $(R_{12}^\beta \circ R_{13}^\beta)(z, w) = R_{12}^\beta(x, y) = (a, b)$, so $(a, b) \in \beta$. Now, $b \alpha_1 y \alpha_1 w$, so $a \beta b \alpha_1 w$, as desired.
- (5) If $a/\alpha_3 \cap z/\alpha_2 \cap w/\alpha_1 \neq \emptyset$, then let y denote this element, let $x = z/\alpha_1 \cap w/\alpha_2$, and let $b = x/\alpha_3 \cap y/\alpha_1$. Then $(R_{13}^\beta \circ R_{12}^\beta)(z, w) = R_{12}^\beta(x, y) = (a, b)$, so $(a, b) \in \beta$. Now, $b \alpha_1 y \alpha_1 w$, so $a \beta b \alpha_1 w$, as desired.

□

6. PROOF OF LEMMA 3

Consider the relation θ_{ij} defined as follows:

$$x \theta_{ij} y \iff (\exists a, b) \ a \alpha_i x \alpha_j b \beta y \alpha_j a.$$

Easy arguments similar to those above establish that

$$\theta_{ij} \cap \alpha_i = \theta_{ij} \cap \alpha_j = \theta_{ij} \cap \beta = 0_X.$$

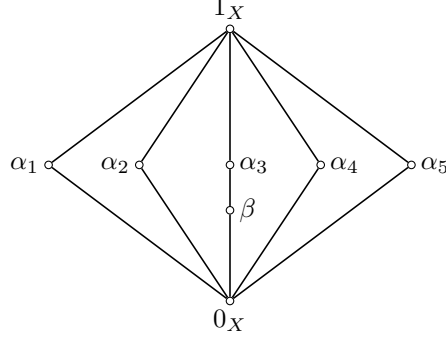
On the other hand, since L is a congruence lattice, it must be the case that the transitive closure of θ_{ij} is contained in L .

TODO: complete proof of Lemma 3 (if possible).

APPENDIX A. EXAMPLE

Let X be a set. It is useful to represent partitions of X as lists of lists, and write them as (possibly nonrectangular) arrays, where each row represents a single block. We do this in the following example, which aids our intuition when thinking about the Palfy-Saxl problem.

Let $X = \{0, 1, 2, \dots, 15\}$, and consider the equivalence relations $\alpha_1, \dots, \alpha_5$ and β , generating the following sublattice of $\text{Eq}(X)$:



where $\alpha_1, \dots, \alpha_5$, and β correspond to the following partitions of X :

α_1	α_2	α_3
[0 1 2 3]	[0 4 8 12]	[0 5 10 15]
[4 5 6 7]	[1 5 9 13]	[1 4 11 14]
[8 9 10 11]	[2 6 10 14]	[2 7 8 13]
[12 13 14 15]	[3 7 11 15]	[3 6 9 12]

α_4	α_5	β
[0 7 9 14]	[0 6 11 13]	[0 5 10 15]
[1 6 8 15]	[1 7 10 12]	[1 4]
[2 5 10 12]	[2 4 9 15]	[2 8]
[3 4 11 13]	[3 5 8 14]	[3 12]
		[6 9]
		[7 13]
		[11 14]

The relations $\alpha_1, \dots, \alpha_5$ are pairwise-permuting pairwise-complements (PPPC). Also, for each α_i , with $i \neq 3$, it's clear that β and α_i are nonpermuting complements. Here are some other facts that aid intuition.

Fact A.1. Each M_3 sublattice with all α 's for atoms is a congruence lattice. In other words, if i, j, k are three distinct numbers in $\{1, 2, \dots, 5\}$, then the sublattice $\{0_X, \alpha_i, \alpha_j, \alpha_k, 1_X\}$ is closed.

Fact A.2. Consider any M_4 having all α 's for atoms. The closure is the M_5 lattice $\{0_X, \alpha_1, \dots, \alpha_5, 1_X\}$.

Fact A.3. Each M_4 generated by β and three α 's complementary to β is not closed. The closure will have many relations in it.

Regarding the last fact, I've forgotten how many relations are in the closure.

TODO: Check this; also check whether α_3 and the other omitted α always end up in the closure.

Fact A.4. The M_3 sublattice $\{0_X, \alpha_1, \alpha_2, \beta, 1_X\}$ is closed.

Fact A.5. The relation $\tau = \tau(\alpha_1, \alpha_2, \beta)$ defined via the Wheatstone Bridge (Figure 2) is a subset of β .

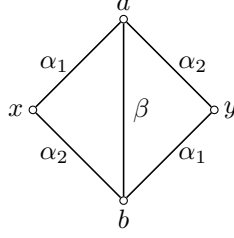


FIGURE 2. The Wheatstone Bridge which defines the relation $\tau(\alpha_1, \alpha_2, \beta)$ as follows: $(x, y) \in \tau(\alpha_1, \alpha_2, \beta)$ if and only if there exist $a, b \in X$ satisfying the relations in the diagram.

What follows is an informal discussion of the motivation that led to the relation β given in this example. (This and other parts of the Appendix are verbose and inelegant; all of this will be removed eventually.)

Regarding Fact A.5, β was constructed specifically to provide a nontrivial example where this fact might hold. That is, we wanted to know if an example existed in which β has smaller height than α_i (so that $|x/\beta| \leq |y/\alpha_i| < \nu(\beta)$, and so β would not permute with α_1 and α_2), and such that $\tau(\alpha_1, \alpha_2, \beta) \subseteq \beta$, so that the Wheatstone Bridge of Figure 2 would not generate an equivalence relation that isn't already contained in $\{0_X, \alpha_1, \alpha_2, \beta, 1_X\}$.

To construct β , we started by assuming $0/\beta = \{0, 5, 10, 15\}$, which is the main diagonal of both α_1 and α_2 . Then we considered the Wheatstone Bridge involving α_1 and α_2 and noticed that, if $\tau \subseteq \beta$, then β must contain all pairs that are at “opposite corners” (defined below) relative to pairs on the main diagonal $\{0, 5, 10, 15\}$.

By “opposite corners” we mean the following. Fix a pair in β , say, $(0, 10) \in \beta$, and consider the squares this pair generates in α_1 and α_2 ; that is, the squares with 0 and 10 at diagonal corners. We see that 2 and 8 appear at the remaining corners of such squares. We call the corners labeled 2 and 8 the “opposite corners” relative to 0 and 10.

The relation τ defined by the Wheatstone Bridge satisfies

$$0 \beta 10 \quad \longrightarrow \quad 2 \tau 8,$$

and, by symmetry of α_1 and α_2 ,

$$2 \beta 8 \quad \longrightarrow \quad 0 \tau 10.$$

Let us make this more general and precise. Recall the relation $\tau = \tau(\alpha_1, \alpha_2, \beta) \subseteq X \times X$ is defined by

$$(A.1) \quad x \tau y \quad \longleftrightarrow \quad (\exists (a, b) \in \beta) \ x \alpha_1 a \ \alpha_2 y \ \alpha_1 b \ \alpha_2 x.$$

Graphically, $x \tau y$ if and only if there exist $a, b \in X$ satisfying the relations depicted in Figure 2.

Let us order the elements of the equivalence classes of α_1 and α_2 according to the row-column arrangements given in the array representations above, and denote by $\alpha_1(i, j)$ the j -th element of the i -th equivalence class of α_1 —that is $\alpha_1(i, j)$ is the element in row i and column j of the array representation of α_1 .

Consider the Wheatstone Bridge diagram and note that, if (x, y) and (a, b) satisfy this diagram, so that (A.1) holds, then we have

$$(A.2) \quad x \in a/\alpha_1 \cap b/\alpha_2 \quad \text{and} \quad y \in b/\alpha_1 \cap a/\alpha_2.$$

Suppose $a = \alpha_1(i, j)$ and $b = \alpha_2(k, \ell)$. Then, by (A.2), x is the point where the i -th row of α_1 intersects the k -th row of α_2 . But notice that, in this example, the array representing α_2 happens to be the transpose of the array representing α_1 . Therefore, the k -th row of α_2 is the k -th column of α_1 , so x is the element contained in the i -th row and k -th column of α_1 , that is, $x = \alpha_1(i, k)$. Similarly, $y = \alpha_1(j, \ell)$. More generally, for all i, j, r, s in $\{1, 2, 3, 4\}$, we have

$$\alpha_1(i, j) \beta \alpha_1(r, s) \quad \longrightarrow \quad \alpha_1(i, s) \tau \alpha_1(j, r).$$

For example, looking at the array representing α_1 , we see that if, say, $(2, 15)$ were to belong to β , then the pair $(3, 15)$ at the opposite corners must belong to $\tau(\alpha_1, \alpha_2, \beta)$.

REFERENCES

- [1] P. P. Pálffy and J. Saxl, “Congruence lattices of finite algebras and factorizations of groups,” *Comm. Algebra*, vol. 18, no. 9, pp. 2783–2790, 1990, ISSN: 0092-7872.