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CONGRUENCE LATTICES OF FINITE ALGEBRAS AND FACTORIZATIONS OF GROUPS

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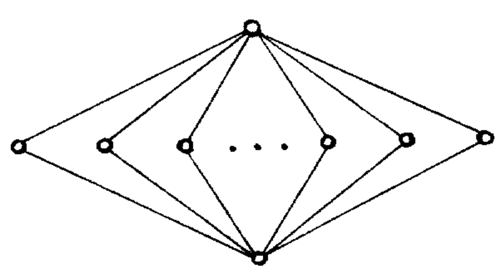
ABSTRACT. Making use of deep results from group theory we prove that if a finite algebra has permutable congruences and its congruence lattice is M_n, then $n-1$ is a prime power.

It is not known whether every finite lattice is representable as the congruence lattice of a finite algebra. For finite distributive lattices it is easy to find such representations (cf. [13]). The simplest case when the problem is unsolved is the case of modular lattices of height two. Let M_n denote the lattice consisting of a largest, a smallest, and n pairwise incomparable elements:



The congruence lattice of the 2-dimensional vector space over the q-element field is isomorphic to M_{q+1}. The attention was focused on the lattices M_n with $n-1$ not a prime power by P. Goralčík [3]. For the time being only M_7 and M_{11} are known to be representable among them (W. Feit [2], see also [14]). Feit's examples arise from groups, since the problem is essentially group theoretic: M_n is representable as a congruence lattice of a finite algebra if and only if there is a finite group G and a subgroup H such that the interval $[H;G]$ in the subgroup lattice of G is isomorphic to M_n (see [15]).

Negative results have been proved under various additional assumptions. The first of this kind was obtained by R.W. Quackenbush [16]:

THEOREM 1. *If $\mathcal{A}$ is a finite algebra in a congruence permutable variety and the congruence lattice of $\mathcal{A}$ is M_n $(n \geq 3)$ then $n-1$ is a prime power.*

The proof actually made use of the permutability of the congruences of $\mathcal{A} \times \mathcal{A}$ only.

In the present paper we will assume the permutability of the congruences only for the algebra $\mathcal{A}$ itself. In fact, all we need is that three of the nontrivial congruences permute. We will prove:

THEOREM 2. *Let $\mathcal{A}$ be a finite algebra with $\mathrm{Con}\mathcal{A} \cong M_n$, $n \geq 4$. If at least three nontrivial congruences of $\mathcal{A}$ pairwise permute then $n-1$ is a prime power.*

The proof will be mainly group theoretic, since we will quickly reduce the statement to the corresponding group theoretic one:

THEOREM 3. *Let G be a finite group, H a subgroup of G such that the interval $[H;G]$ in the subgroup lattice of G is isomorphic to M_n, $n \geq 4$. If at least three intermediate subgroups $(H < K_i < G, \quad i = 1,2,3)$ pairwise permute $(K_iK_j = K_jK_i, \quad 1 \leq i < j \leq 3)$ then $n-1$ is a prime power.*

It will turn out that any group satisfying the assumptions of Theorem 3 contains a nontrivial abelian normal subgroup, and we can make use of the following result of P. Pudlák and the first author [15]:

THEOREM 4. *If $[H;G] \cong M_n$ $(n \geq 3)$ and G contains an abelian normal subgroup A, $A \not\leq H$, then $n-1$ is a prime power.*

We will also refer to a result of P. Köhler [9]:

THEOREM 5. *Let $[H;G] \cong M_n$, $n \geq 4$, and assume that H contains no nontrivial normal subgroup of G. If G has more than one minimal normal subgroups then $n-1$ is a prime power.*

In fact, Köhler proves that the minimal normal subgroups of G are abelian and so Theorem 4 applies.

By virtue of these results it suffices to deal with groups G with a unique minimal normal subgroup N, which is nonabelian and $N \not\leq H$. (If $N \leq H$, we could take the interval $[H/N;\, G/N] \cong [H;G]$.) With the help of some consequences of the classification of finite simple groups we will prove that such groups cannot satisfy the assumptions of Theorem 3. Fortunately, the hardest case when N is simple, can be solved by referring to the works of C. Hering, M.W. Liebeck, C.E. Praeger, and the second author [6], [11] on factorizations of simple groups. For our purposes the following consequence of these difficult results will be crucial.

THEOREM 6. *Let G be a finite simple group, A a group of automorphisms of G, K_1 and K_2 maximal A-invariant subgroups in G of equal order. If $G = K_1K_2$ then one of the following holds:*

(1) $G = M_{12}$, $K_1 \cong M_{11} \cong K_2$ *(where* M_{11}, M_{12} *are the Mathieu groups), and* $[K_1 \cap K_2; G] \cong M_2$;

(2) $G = Sp_4(q)$ *with* q *even,* $q > 2$, $K_1 = Sp_2(q^2).2$, $K_2 = O_4^-(q)$, *and* $[K_1 \cap K_2; G] \cong M_2$;

(2') $G = A_6$, $K_1 = A_5$, $K_2 = L_2(5)$, *and* $[K_1 \cap K_2; G] \cong M_2$;

(3) $G = P\Omega_8^+(q)$, K_1, K_2, *and a third subgroup* K_3 *are isomorphic to* $\Omega_7(q)$ *and are permuted cyclically by a triality automorphism, and* $[K_1 \cap K_2; G] = \{K_1 \cap K_2, K_1, K_2, K_3, G\} \cong M_3$.

The equivalence of Theorems 2 and 3

Let G be a group and H a subgroup of G. Denote by $G : H$ the set of left cosets with respect to H in G and let G act on $G : H$ by left multiplication, $g : xH \mapsto gxH$. Then the congruence lattice of the unary algebra $(G : H; G)$ is isomorphic to the interval $[H; G]$ of the subgroup lattice of G (cf. [15], Lemma 3). If $H \leq K \leq G$ then the congruence corresponding to K is $\Theta_K = \{(xH, yH) : xK = yK\}$. It is straightforward to check:

LEMMA 1. *Let* $H < K, L < G$. *The congruences* Θ_K *and* Θ_L *permute if and only if the subgroups* K *and* L *permute.*

Assume now that Theorem 2 holds true. Let G be a finite group, $H < G$, $[H; G] \cong M_n$, $n \geq 4$, and suppose that three intermediate subgroups $H < K_1, K_2, K_3 < G$ pairwise permute. Then the unary algebra $(G : H; G)$ meets the assumptions of Theorem 2, and in particular $\mathrm{Con}(G : H; G) \cong [H; G] \cong M_n$. Hence $n - 1$ is a prime power, as it is claimed in Theorem 3.

The converse does not follow so immediately. Let us now assume that Theorem 3 holds true, and let $\mathcal{A}$ be a finite algebra with $\mathrm{Con}\mathcal{A} \cong M_n$, $n \geq 4$, with at least three pairwise permuting nontrivial congruences. Since M_n $(n \geq 4)$ is a tight lattice, $\mathcal{A}$ is a tame algebra (see R. McKenzie [12]). Let $B = e(A)$ be a minimal set, where e is a unary polynomial of $\mathcal{A}$ and $e^2 = e$. By Pudlák's lemma ([15], Lemma 1) the restriction to B yields an isomorphism between $\mathrm{Con}\mathcal{A}$ and the congruence lattice of the induced algebra on B, as $\mathrm{Con}\mathcal{A}$ is simple. We claim that if $\Theta, \Psi \in \mathrm{Con}\mathcal{A}$ permute then so do their restrictions to B as well. To show this let $(a, b) \in \Theta_B \vee \Psi_B$. Then $(a, b) \in \Theta \vee \Psi = \Theta \circ \Psi$, so there exists a $c \in A$ such that $(a, c) \in \Theta$ and $(c, b) \in \Psi$. Hence $(a, e(c)) = (e(a), e(c)) \in \Theta$ and similarly $(e(c), b) \in \Psi$, where $e(c) \in e(A) = B$, so $(a, b) \in \Theta_B \circ \Psi_B$, as we wanted. Therefore, our assumptions for $\mathcal{A}$ are inherited by the algebra $\mathcal{B}$ induced on the minimal set B. The nonconstant unary polynomials of $\mathcal{B}$ form a permutation group G on B. This group is transitive, see [15], Theorem 1(iii). Let H be the stabilizer of an arbitrarily chosen point of B. Then $[H; G] \cong \mathrm{Con}\mathcal{B} \cong \mathrm{Con}\mathcal{A} \cong M_n$, and by Lemma 1 subgroups corresponding to permuting congruences themselves permute. Our assumption of the validity of Theorem 3 then yields that $n - 1$ is a prime power, so Theorem 2 holds as well.

Proof of Theorem 6

Suppose that $G = K_1 K_2$ with $|K_1| = |K_2|$. Taking $K_i \leq X_i < G$, we obtain a factorization $G = X_1 X_2$ satisfying for any prime number p the following condition:

(∗) If p^a divides $|G|$ and p^b is the highest power of p dividing both $|X_1|$ and $|X_2|$ then $b \geq a/2$.

The maximal factorizations are listed in [11], and we inspect the lists there, using (∗).

Assume first that G is a sporadic simple group; then the maximal factorizations are in [11], Table 6. The only factorization satisfying (∗) is $M_{12} = M_{11} \cdot M_{11}$ (note that no proper subgroup of M_{11} will do). Then $K_1 \cap K_2 = L_2(11)$. There are two conjugacy classes of subgroups M_{11} in M_{12} (see [1], p. 33). No two M_{11} in the same conjugacy class can intersect in $L_2(11)$, since no group can factorize as the product of two proper subgroups in the same conjugacy class. It follows that $K_1 \cap K_2$ lies in precisely two subgroups M_{11}, and since it lies in no other maximal subgroup of M_{12} (cf. [1], p. 33), the lattice $[K_1 \cap K_2; M_{12}]$ is M_2 and we are in case (1) of Theorem 6.

Next take G to be an alternating group A_n with $n \geq 5$. By [11], Theorem D, with a few exceptions for $n \in \{6, 8, 10\}$, we have $A_{n-k} \triangleleft K_1 \leq S_{n-k} \times S_k$ and K_2 is k-homogeneous of degree n for some k with $k \leq 5$, $k \leq n/2$. The exceptions for $n \in \{6, 8, 10\}$ are easily handled, so we concentrate on the general case. If $n \geq 8$, there is a prime p with $n/2 < p < n-2$ (cf. [5], pp. 343, 373); then by (∗) p divides $|K_2|$ and it follows applying Jordan's theorem that $K_2 = A_n$ (cf. [11], p. 123), a contradiction. If n is a prime, K_1 is not divisible by n contradicting (∗). So let $n = 6$. Here we get $K_1 = A_5$, $K_2 = L_2(5)$ as the only possibility, with $K_1 \cap K_2 = D_{10}$, and the corresponding lattice is M_2 as before. This is case (2') of Theorem 6.

Finally let G be a simple group of Lie type. The maximal factorizations are listed in [11], Tables 1-5, and again we examine the lists using (∗) (in fact in almost all cases it is sufficient to concentrate on p being the primitive prime divisor of $q^m - 1$ for the largest possible m — cf. [11], Section 2.4). The only maximal factorizations that satisfy (∗) are

(i) $\Omega_8^+(2) = A_9 \cdot \Omega_7(2)$;

(ii) $Sp_4(q) = (Sp_2(q^2).2) \cdot O_4^-(q)$, with q even (and $q > 2$ for simplicity of G);

(iii) $P\Omega_8^+(q) = \Omega_7(q) \cdot \Omega_7(q)$.

In (i), however, we would need subgroups K_i of order divisible by $2^6 \cdot 3^3 \cdot 5 \cdot 7$; but $\Omega_7(2)$ has no proper subgroup of index dividing 24 (cf. [1], p. 46).

In (ii), we are in case (2) of Theorem 6. For, there are precisely two conjugacy classes in G of subgroups isomorphic to $L_2(q^2).2$, and we see as before that $K_1 \cap K_2 = (q^2+1).4$ lies precisely in K_1 and K_2 (there are no other subgroups containing it, e.g. by [8]). Also, no proper subgroups of K_i will do: the only subgroups of small enough index in K_i are K_i' of index 2, but $G \neq K_1' \cdot K_2'$ since $K_1' \cap K_2' = (q^2+1).2$ (as both K_i' contain an involution inverting the torus of order q^2+1).

Finally, in (iii) we are in case (3) of Theorem 6. Here the reference is [7], in particular p. 219. Now $K_i \cong \Omega_7(q)$, and $K_1 \cap K_2 = G_2(q)$. There are 1 or 4 conjugacy classes of $G_2(q)$ in G, as q is even or odd, respectively; these are all conjugate in Aut$G$, and each $G_2(q)$ is the centralizer in G of a triality automorphism τ of order 3. Take τ to centralize $K_1 \cap K_2$. Then $G = K_1 K_1^\tau = K_1 K_1^{\tau^2} = K_1^\tau K_1^{\tau^2}$. Noting further that if $G = X_1 X_2$ with $X_i \cong \Omega_7(q)$ then *not* both X_1 and X_2 fix a 1-space in the same "natural" module of G (there are three such modules), we see that K_2 is in fact K_1^τ or $K_1^{\tau^2}$, and $K_1 \cap K_1^\tau$ lies precisely in the three proper subgroups K_1, K_1^τ, $K_1^{\tau^2}$ (it lies in no other proper subgroups by [7]). Also, no proper subgroups of K_i will do: this follows from the list of maximal subgroups of $\Omega_7(q)$ in [8], using (∗).

This completes the proof of Theorem 6.

Proof of Theorem 3

We will often need the following simple fact. (In particular, it explains the assumption $|K_1| = |K_2|$ in Theorem 6.)

LEMMA 2. *If $K_iK_j = K_jK_i = G$ and $K_i \cap K_j = H$ $(1 \le i < j \le 3)$ then $|K_1| = |K_2| = |K_3|$ and $|G : K_i| = |K_i : H|$ for $i = 1,2,3$.*

Proof. It follows from $G = K_iK_j$ that $|G| = |K_i| \cdot |K_j| \cdot |K_i \cap K_j|^{-1}$, so $|K_i| \cdot |K_j| = |G| \cdot |H|$. This holds for all pairs of indices $1 \le i < j \le 3$, hence $|K_i| = (|G| \cdot |H|)^{1/2}$ for $i = 1,2,3$.

Now we turn to the proof of Theorem 3. As we have already mentioned, Theorem 4 and Theorem 5 enable us to consider only such finite groups G which have a unique minimal normal subgroup N, N is nonabelian and $N \not\le H$. Then N is the direct product of isomorphic nonabelian simple groups, $N = T_1 \times ... \times T_k$, $k \ge 1$. Moreover, as $C_G(N) \cap N = 1$ and N is the only minimal normal subgroup of G, it follows that G acts faithfully by conjugation on N, that is, G is isomorphic to a subgroup of $\mathrm{Aut}N \cong (\mathrm{Aut}T_1) \text{ wr } S_k$. Let $K_1, K_2, K_3, ..., K_n$ denote the intermediate subgroups, $H < K_i < G$.

First we settle the case when $NH < G$. Then let, say, $NH = K_1$. For any $1 < i \le n$ we have $N \cap H = N \cap K_1 \cap K_i = N \cap K_i \triangleleft K_i$, so $N \cap H \triangleleft \langle K_2, K_3, ..., K_n\rangle = G$. Since N is a minimal normal subgroup of G and $N \not\le H$, we get $1 = N \cap H = N \cap K_i$ for $i = 2, ..., n$. We also have $K_1K_i = NHK_i = NK_i = G = K_iK_1$. By our hypothesis there must be a permutable pair among the subgroups $K_2, ..., K_n$ as well. Then Lemma 2 gives $|G : K_1| = |K_1 : H| = |NH : H| = |N : N \cap H| = |N|$, so $|G| = |N|^2 \cdot |H|$. Thus we have

$$|N|^2 \,\big|\, |G| \,\big|\, |\mathrm{Aut}N|.$$

Here $|N| = |T_1|^k$ and $|\mathrm{Aut}N| = |\mathrm{Aut}T_1|^k \cdot k!$, so

$$|T_1|^{2k} \mid |\mathrm{Aut}T_1|^k \cdot k!.$$

This is impossible, for the following reason. As a consequence of the classification of finite simple groups, it is known that $|\mathrm{Aut}T| < |T|^2$ for every finite simple group T (see [4], Theorem 1.47). Thus there is a prime power p^m which divides $|T_1|^2$ but does not divide $|\mathrm{Aut}T_1|$. On the other hand, $k!$ cannot be divisible by p^k for any prime p.

So from now on we may assume $NH = G$. Then $[H; G]$ is isomorphic to the interval $[N \cap H; N]_H$ in the lattice of H- invariant subgroups of N. Namely, $X \mapsto N \cap X$ is a lattice isomorphism between the two intervals. Moreover, the subgroups $N \cap K_i$ and $N \cap K_j$ permute if and only if K_i and K_j do.

The description of the maximal H- invariant subgroups in N is one of the basic ingredients in the O'Nan-Scott Theorem on primitive permutation groups (see [17], p. 328 or [10]). We need some notation. Let $\overline{H}$ be the permutation group induced by H on the set $\{T_1, ..., T_k\}$. (Sometimes we will identify this set with $\{1, ..., k\}$.) Since $N = T_1 \times ... \times T_k$ is a minimal normal subgroup in $G = NH$, $\overline{H}$ is transitive. Furthermore, let $H_1 = N_H(T_1)$ and let A be the group of automorphisms of T_1 induced by H_1. Then $|H : H_1| = k$ and for a system of left coset representatives of H_1 in H, $h_1 = 1, h_2, ..., h_k$, we have $h_jT_1h_j^{-1} = T_j$ $(j = 1, ..., k)$. We will make use of the following part from the proof of the O'Nan-Scott Theorem:

LEMMA 3. *Any maximal H-invariant subgroup L in N is of one of the following two types:*

(i) ("product type") $L = U_1 \times ... \times U_k$, where U_1 is a maximal A-invariant subgroup of T_1 and $U_j = h_j U_1 h_j^{-1}$ $(j = 2, ..., k)$;

(ii) ("diagonal type") $L = \{(\varphi_1(x_{\beta 1}), ..., \varphi_k(x_{\beta k})) : x_1, ..., x_r \in T\}$, where $\beta : \{1, ..., k\} \to \{1, ..., r\}$ is a surjective mapping such that the kernel of β is an $\overline{H}$- invariant equivalence relation, $T \cong T_1 \cong ... \cong T_k$, and $\varphi_i : T \to T_i$ $(i = 1, ..., k)$ are suitable isomorphisms.

First we show that two maximal H- invariant subgroups of diagonal type, $L = \{(\varphi_1(x_{\beta 1}), ..., \varphi_k(x_{\beta k})) : x_1, ..., x_r \in T\}$ and $L' = \{(\varphi_1'(x_{\beta' 1}), ..., \varphi_k'(x_{\beta' k})) : x_1, ..., x_{r'} \in T\}$, cannot permute. Assume the contrary. The order of L is $|T|^r$ where r is the number of blocks in the kernel of β. Since $\ker\beta$ is invariant under the action of the transitive group $\overline{H}$, its blocks have equal size, hence r is a (proper) divisor of k, in particular $r \leq k/2$. If $N = LL'$ then $|T|^k = |N| = |L| \cdot |L'| \cdot |L \cap L'|^{-1} = |T|^r \cdot |T|^{r'} \cdot |L \cap L'|^{-1}$. So we must have $r = r' = k/2$ and $L \cap L' = 1$. The first equalities mean that the kernel of both β and β' has 2-element blocks. We will show that in this case the intersection $L \cap L'$ is always nontrivial. We define a sequence of indices $i_0, i_1, ...$ and isomorphisms $\alpha_0 : T_1 \to T_{i_0}, \alpha_1 : T_1 \to T_{i_1}, ...$ recursively. Let $i_0 = 1$ and $\alpha_0 : T_1 \to T_1$ be the identity. If i_j and α_j have already been defined then we choose i_{j+1} and α_{j+1} as follows. If j is even then let i_{j+1} be the (uniquely determined) other member of the block of $\ker\beta$ containing i_j. (Formally: $i_{j+1} \neq i_j$, $\beta i_{j+1} = \beta i_j$.) Now set $\alpha_{j+1} = \varphi_{i_{j+1}} \varphi_{i_j}^{-1} \alpha_j$. If j is odd then we do the same with β' and φ_m' instead of β and φ_m, i.e., $i_{j+1} \neq i_j$, $\beta' i_{j+1} = \beta' i_j$, $\alpha_{j+1} = \varphi'_{i_{j+1}} \varphi'^{-1}_{i_j} \alpha_j$. Let m be the first positive number for which i_m already occurs among $i_0, ..., i_{m-1}$. It is easy to see, that m is even and $i_m = 1 = i_0$. Then α_m is an automorphism of the simple group T_1, hence by a well-known consequence of the classification of finite simple groups (see [4], Theorem 1.48) there is an element $t \in T_1, t \neq 1$, such that $\alpha_m(t) = t$. Let us define $t^* \in T_1 \times ... \times T_k$ so that its i-th component is $\alpha_j(t)$ if $i = i_j$ for some $0 \leq j \leq m-1$, and it is 1 if i does not occur among $i_0, ..., i_{m-1}$. Then by our construction $1 \neq t^* \in L \cap L'$, which proves $LL' \neq N$.

Hence at least two of the three pairwise permuting maximal H-invariant subgroups, $L_i = K_i \cap N$ $(i = 1, 2, 3)$ must be of product type, say, $L_i = U_1 \times ... \times U_k$ and $L_j = V_1 \times ... \times V_k$ $(1 \leq i < j \leq 3)$. Then $L_i L_j = U_1 V_1 \times ... \times U_k V_k$ and $L_i \cap L_j = (U_1 \cap V_1) \times ... \times (U_k \cap V_k)$. Moreover, $K_i = L_i H$, therefore $|K_i| = |U_1|^k \cdot |H : N \cap H|$, and similarly $|K_j| = |V_1|^k \cdot |H : N \cap H|$, so $|K_i| = |K_j|$ (see Lemma 2) implies $|U_1| = |V_1|$. Thus U_1 and V_1 are maximal A- invariant subgroups of equal order in the simple group T_1 and $U_1 V_1 = T_1$ (as $L_i L_j = N$). Hence Theorem 6 is applicable. In particular, observe that in any case (1)-(3) there we have $U_1 \cap V_1 \neq 1$, hence $N \cap H = (U_1 \cap V_1) \times ... \times (U_k \cap V_k)$ contains an element which has a single nontrivial component in the direct decomposition. Such an element is not contained in any maximal H-invariant subgroup of diagonal type, since each block of $\ker\beta$ has at least two elements, and the components corresponding to the same block are either all trivial or none of them is trivial. So in our case all maximal H-invariant subgroups in the interval $[N \cap H; N]_H$ are of product type. For the number of these subgroups Theorem 6 yields $n \leq 3$, giving the final contradiction needed to prove Theorem 3.

Final remarks

PROPOSITION. *The assumptions of Theorem 3 imply that any pair of intermediate subgroups permute and $|G : K_i| = |K_i : H| = p^e$ for some prime power, $n - 1 | p^e$.*

Proof. As always, we may assume that H does not contain any nontrivial normal subgroup of G. Our proof of Theorem 3 yielded that G must have an abelian minimal normal subgroup A. Obviously, the order of A is a prime power, p^e. We have the following (see [15]): $AH < G$, say, $AH = K_1$; $AK_2 = ... = AK_n = G$, and $A \cap H = A \cap K_2 = ... = A \cap K_n = 1$. On one hand, Lemma 2 gives $|G : K_1| = |K_1 : H| = |A|$; on the other hand, $AK_i = G$, $A \cap K_i = 1$ $(i = 2, ..., n)$ implies $|G : K_2| = ... = |G : K_n| = |A|$. Hence $|K_iK_j| = |K_i| \cdot |K_j| \cdot |K_i \cap K_j|^{-1} = |G|$, that is, any two intermediate subgroups permute. Finally, choose an $x \in K_2 \backslash H$. Then the set $A_0 = \{a \in A : \exists 1 < i \leq n : ax \in K_i\}$ is in one-to-one correspondence with the set $\{K_2, ..., K_n\}$ and A_0 is a subgroup of A, see [15]. This proves $n - 1 | p^e$.

Notice that the assumption $n \geq 4$ is essential, since for the groups in Theorem 6(3) the index $|G : K_i| = |K_i : H| = q^3(q^4 - 1)/(2, q - 1)$ is not a prime power.

We wonder if an analogous statement is true for Theorem 2. Perhaps the cardinality of the algebra need not be a prime power, p^{2e}, but the permutability of all pairs of congruences may be expected.

PROBLEM. Let $\mathcal{A}$ be a finite algebra with $\mathrm{Con}\mathcal{A} \cong M_n$, $n \geq 4$. If three nontrivial congruences of $\mathcal{A}$ pairwise permute, does it follow that any pair of congruences of $\mathcal{A}$ permute?

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