DEDEKIND'S TRANSPOSITION PRINCIPLE
AND
PERMUTING SUBGROUPS & EQUIVALENCE RELATIONS

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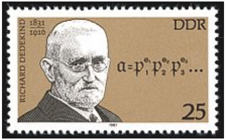
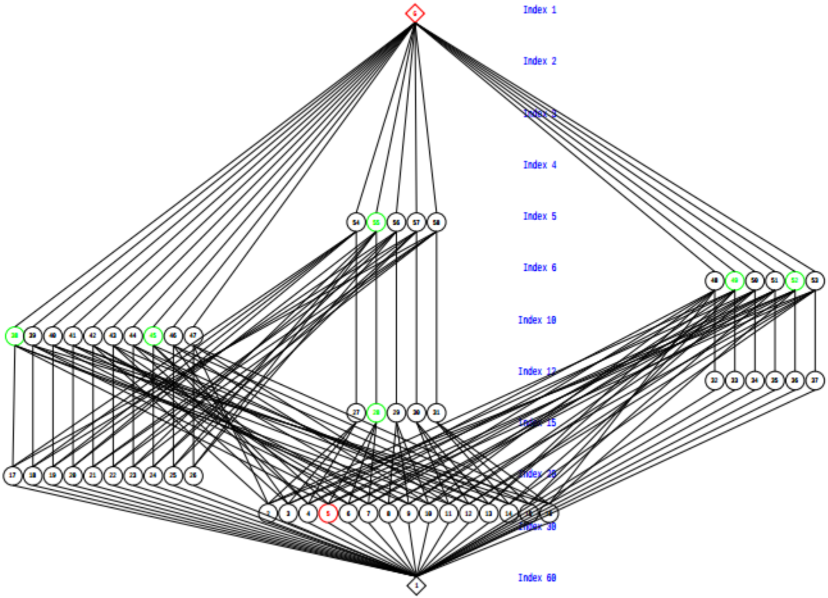
These slides and other resources are available at

http://williamdemeo.wordpress.com





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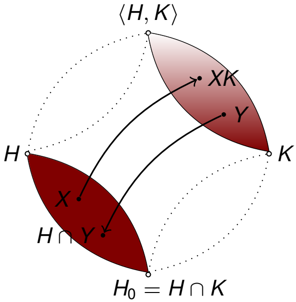
Intervals in Subgroup Lattices



- Let H, K be subgroups of a group G.
- Recall the set

$$HK = \{hk \mid h \in H,\ k \in K\}$$

is a group if and only if $HK = KH = \langle U, H\rangle$.

- Let $H_0 = H \cap K$ and define

$$[\![H_0, H]\!] := \{X \mid H_0 \leqslant X \leqslant H\},$$

$$[\![K, \langle H, K\rangle]\!] := \{X \mid K \leqslant X \leqslant \langle H, K\rangle\}.$$

- Define

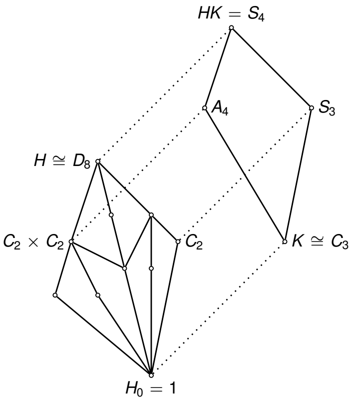
$$[\![H_0, H]\!]^K := \{X \in [\![H_0, H]\!] \mid XK = KX\}.$$

Lemma

If $HK = KH$, then $[\![K, HK]\!] \cong [\![H_0, H]\!]^K \leqslant [\![H_0, H]\!]$.

EXAMPLE

- The group S_4 has permuting subgroups $H \cong D_8$ and $K \cong C_3$.
  (neither one normalizes the other)



- Only four subgroups of H permute with K, including

$$H \cap A_4 \cong C_2 \times C_2, \qquad H \cap S_3 \cong C_2.$$

DEDEKIND'S TRANSPOSITION PRINCIPLE

FOR MODULAR LATTICES

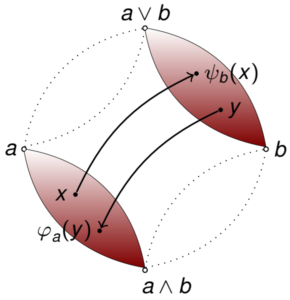
Notation

Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice with $a \in L$.

Let φ_a and ψ_a be the "perspectivity maps"

$$\varphi_a(x) = x \wedge a \quad \text{and} \quad \psi_a(x) = x \vee a$$

For $x, y \in L$, let $[\![x, y]\!]_L = \{z \in L \mid x \leqslant z \leqslant y\}$.



THEOREM (DEDEKIND'S TRANSPOSITION PRINCIPLE)

$\mathbf{L}$ *is modular iff for all* $a, b \in L$ *the maps* φ_a *and* ψ_b *are inverse lattice isomorphisms of* $[\![a \wedge b, a]\!]$ *and* $[\![b, a \vee b]\!]$.

Another Transposition Principle

for lattices of equivalence relations

Let X be a set and let $\operatorname{Eq} X$ be the lattice of equivalence relations on X.

Given $\alpha, \beta \in \operatorname{Eq} X$, define the *composition* of α and β to be the binary relation

$$\alpha \circ \beta = \{(x, y) \in X^2 \mid (\exists z \in X)\, x \mathrel{\alpha} z \mathrel{\beta} y\}.$$

For a sublattice $L \leqslant \operatorname{Eq} X$, with $\eta, \theta \in L$, define

$$[\![\eta, \theta]\!]_L = \{\gamma \in L \mid \eta \leqslant \gamma \leqslant \theta\},$$

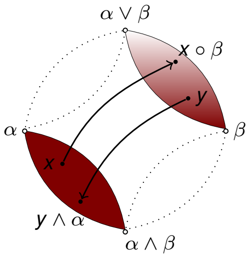
$$[\![\eta, \theta]\!]_L^\beta = \{\gamma \in L \mid \eta \leqslant \gamma \leqslant \theta \text{ and } \gamma \circ \beta = \beta \circ \gamma\},$$

i.e., the relations in $[\![\eta, \theta]\!]_L$ that permute with β.

Lemma

Suppose α and β are permuting relations in $L \leqslant \operatorname{Eq} X$.

Then $[\![\beta, \alpha \vee \beta]\!]_L \cong [\![\alpha \wedge \beta, \alpha]\!]_L^\beta \leqslant [\![\alpha \wedge \beta, \alpha]\!]_L$.



Question: Does this generalize the subgroup lattice lemma?

Answer: Yes!

- For groups $H \leqslant G$, the algebra $\mathbf{A} = \langle G \backslash H, \bar{G} \rangle$ has
  - universe: the right cosets $H \backslash G = \{Hx \mid x \in G\}$
  - operations: $\bar{G} = \{g^{\mathbf{A}} : g \in G\}$, where $g^{\mathbf{A}}(Hx) = Hxg$.
- A standard result is $\operatorname{Con} \mathbf{A} \cong [\![H, G]\!]$.

  The isomorphism $[\![H, G]\!] \ni K \mapsto \theta_K \in \operatorname{Con} \mathbf{A}$ is given by

  $$\theta_K = \{(Hx, Hy) \mid xy^{-1} \in K\}.$$

  The inverse isomorphism $\operatorname{Con} \mathbf{A} \ni \theta \mapsto K_\theta \in [\![H, G]\!]$ is

  $$K_\theta = \{g \in G \mid (H, Hg) \in \theta\}.$$

- So every lattice property of congruence lattices is also a lattice property of (intervals of) subgroup lattices. Moreover, it's easy to prove:

Lemma

In $\operatorname{Con}\langle G \backslash H, \bar{G} \rangle$, two congruences θ_{K_1} and θ_{K_2} permute if and only if the corresponding subgroups K_1 and K_2 permute.

Questions

Recall that $HK = \langle H, K \rangle$ if and only if $HK = KH$.

Question 1. Is it true that

$HKH = \langle H, K \rangle$ if and only if $HKH = KHK$?

What about

$HKHK = \langle H, K \rangle$ if and only if $HKHK = KHKH$?

$$\vdots$$

$H \circ^n K = \langle H, K \rangle$ if and only if $H \circ^n K = K \circ^n H$?

QUESTIONS

Denote by $H \circ^n K$ the *n-fold composition of H and K*.

$$
\begin{aligned}
H \circ^1 K &= H, \\
H \circ^2 K &= HK, \\
H \circ^3 K &= HKH, \\
H \circ^4 K &= HKHK, \\
&\vdots \\
H \circ^n K &= H \circ^2 K \circ^{n-1} H.
\end{aligned}
$$

We say H and K are *n-permuting* if $H \circ^n K = K \circ^n H$.

Question 2. *Is the following true?*

> *If H and K are n-permuting, then interval $[\![K, \langle H, K\rangle]\!]$ is isomorphic to the lattice of subgroups in $[\![H_0, H]\!]$ that n-permute with K.*

Connection with equivalence relations

Let $\mathbf{A} = \langle H\backslash G, \bar{G}\rangle$ be the algebra with

- universe: the right cosets $H\backslash G = \{Hx \mid x \in G\}$
- operations: $\bar{G} = \{g^{\mathbf{A}} : g \in G\}$, where $g^{\mathbf{A}}(Hx) = Hxg$.

Lemma

The subgroups K_1 and K_2 are n-permuting if and only if their corresponding congruences θ_{K_1} and θ_{K_2} are n-permuting. That is,

$$K_1 \circ^n K_2 = K_2 \circ^n K_1 \iff \theta_{K_1} \circ^n \theta_{K_2} = \theta_{K_2} \circ^n \theta_{K_1}.$$

Answer to Question 1.

Lemma

For $\alpha, \beta \in \operatorname{Eq} X$, and for every even integer $n > 1$, TFAE:

(I) $\alpha \circ^n \beta = \alpha \vee \beta$
(II) $\alpha \circ^n \beta = \beta \circ^n \alpha$
(III) $\alpha \circ^n \beta \subseteq \beta \circ^n \alpha$

For $n = 3$,

$$\alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta \quad \Longrightarrow \quad \alpha \circ \beta \circ \alpha = \alpha \vee \beta$$

but the converse is false.

Corollary

For $H, K \leqslant G$, and for every even integer $n > 1$, TFAE:

(I) $H \circ^n K = \langle H, K \rangle$
(II) $H \circ^n K = K \circ^n H$
(III) $H \circ^n K \subseteq K \circ^n H$

For $n = 3$,

$$HKH = KHK \quad \Longrightarrow \quad HKH = \langle H, K \rangle$$

but the converse is false.
Question 1.' What are conditions on G under which the converse is true?

Question 1. Is it true that

$$H \circ^5 K = \langle H, K\rangle \textit{ if and only if } H \circ^5 K = K \circ^5 H?$$

Answer. No.

Example. Let $G = (C_3 \times C_3) : C_4$.

This is a group of order 36 with generators f_1, f_2, f_3, f_4.

Let $H = \langle f_1 \rangle \cong C_2$, and $K = \langle f_1 \cdot f_3 \cdot f_4^2, f_2 \cdot f_4^2 \rangle \cong C_4$. Then,

- $H \cap K = 1$
- $\langle H, K\rangle = K \circ^5 H$ has order 36 so it is the whole group.
- The *set* $H \circ^5 K$ has size 34, so does not generate $\langle H, K\rangle$.
- H covers 1.

Answer to Question 2.

No.

In general, it is not true that if H and K are n-permuting, then the interval $[\![K, \langle H, K\rangle]\!]$ is isomorphic to the lattice of those subgroups in $[\![H_0, H]\!]$ that n-permute with K.

Example. The group A_5 has subgroups $H \cong D_{10}$, and $K \cong C_2$ such that

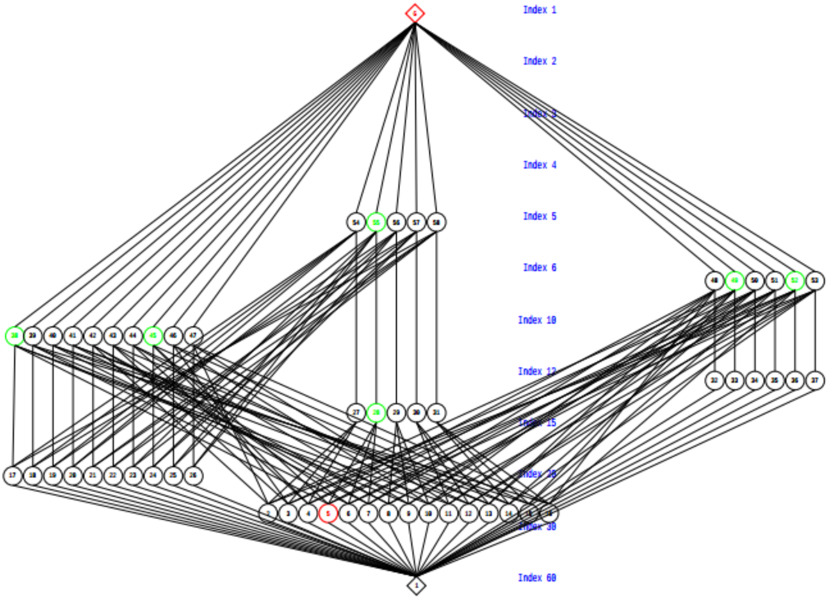
$$H \circ^4 K = K \circ^4 H = A_5,$$

but the map

$$[\![K, A_5]\!] \ni J \mapsto J \cap H \in [\![1, H]\!]$$

is not one-to-one.

EXAMPLES



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Revised Question 2.

Question 2.'

What are conditions on the group G so that

if H, K are n-permuting subgroups of G, then

$$[\![K, \langle H, K\rangle]\!] \cong [\![H_0, H]\!]^{K\circ^n} \leqslant [\![H_0, H]\!]\,?$$