DEDEKIND'S TRANSPOSITION PRINCIPLE

AND

ISOTOPIC ALGEBRAS WITH NONISOMORPHIC CONGRUENCE LATTICES

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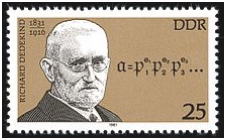
AMS Spring Western Sectional Meeting
University of Colorado, Boulder, CO

April 13-14, 2013

These slides and other resources are available at

http://williamdemeo.wordpress.com



Notation

Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice with $a \in L$.

Let φ_a and ψ_a be the *perspectivity maps*

$$\varphi_a(x) = x \wedge a \quad \text{and} \quad \psi_a(x) = x \vee a$$

For $x, y \in L$, let $[\![x, y]\!]_L = \{z \in L \mid x \leqslant z \leqslant y\}$.

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Theorem (Dedekind's Transposition Principle)

$\mathbf{L}$ is modular iff for all $a, b \in L$ the maps φ_a and ψ_b are inverse lattice isomorphisms of $[\![a \wedge b, a]\!]$ and $[\![b, a \vee b]\!]$.

Dedekind's Transposition Principle

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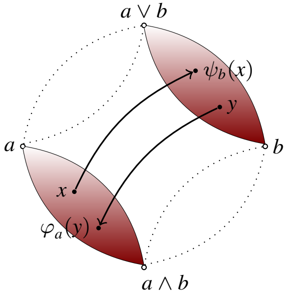

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Another transposition principle

for lattices of equivalence relations

Let X be a set and let $\operatorname{Eq} X$ be the lattice of equivalence relations on X.

If L is a sublattice of $\operatorname{Eq} X$ with $\eta, \theta \in L$, then we define

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For $\beta \in \mathrm{Eq}\,X$, let $[\![\eta, \theta]\!]_L^\beta$ be the set of relations in $[\![\eta, \theta]\!]_L$ that permute with β,

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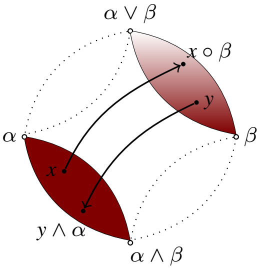
Lemma

Suppose α and β are permuting relations in $L \leqslant \operatorname{Eq} X$.

Then $[\![\beta, \alpha \vee \beta]\!]_L \cong [\![\alpha \wedge \beta, \alpha]\!]_L^\beta \leqslant [\![\alpha \wedge \beta, \alpha]\!]_L$.



Dedekind's Rule

The proof requires the following version of *Dedekind's Rule:*

Lemma

Suppose $\alpha, \beta, \gamma \in L \leqslant \mathrm{Eq}\, X$ *and* $\alpha \leqslant \beta$*.*

Then the following identities of subsets of X^2 *hold:*

$$\alpha \circ (\beta \cap \gamma) = \beta \cap (\alpha \circ \gamma)$$

$$(\beta \cap \gamma) \circ \alpha = \beta \cap (\gamma \circ \alpha)$$

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be algebras of the same type.

$\mathbf{A}$ and $\mathbf{B}$ are *isotopic over* $\mathbf{C}$, denoted $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$, if there is an isomorphism

$$\varphi : \mathbf{A} \times \mathbf{C} \xrightarrow{\cong} \mathbf{B} \times \mathbf{C} \quad \text{that leaves the second coordinate fixed}$$

$$\text{i.e.} \quad (\forall a \in A)\,(\forall c \in C) \quad \varphi(a, c) = (\varphi_1(a, c), c)$$

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It is easy to verify that $\sim$ is an equivalence relation.

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Lemma 11. If $\mathbf{A} \sim^{\mathrm{mod}} \mathbf{B}$ then $\operatorname{Con}\mathbf{A} \cong \operatorname{Con}\mathbf{B}$.

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But this only shows that the same argument doesn't work...

COUNTEREXAMPLES

We describe a class of examples in which $\mathbf{A} \sim \mathbf{B}$ and $\operatorname{Con}\mathbf{A} \not\cong \operatorname{Con}\mathbf{B}$.

The examples show that congruence lattices of isotopic algebras can differ arbitrarily in size.

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A group G is called a *Dedekind group* if every subgroup of G is normal.

Let S be any group and let D denote the *diagonal subgroup* of $S \times S$,

$$D = \{(x, x) \mid x \in S\}$$

The interval $[\![D, S \times S]\!] \leqslant \operatorname{Sub}(S \times S)$ is described by the following

LEMMA

The filter above the diagonal subgroup of $S \times S$ is isomorphic to the lattice of normal subgroups of S.

THE EXAMPLE

Let S be a group, and let $G = S_1 \times S_2$, where $S_1 \cong S_2 \cong S$.

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Then $D \cong T_1 \cong T_2$, and these are pair-wise compliments:

$$\langle T_1, T_2 \rangle = \langle T_1, D \rangle = \langle D, T_2 \rangle = G$$

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Let $\mathbf{A} = \langle G/T_1, G^{\mathbf{A}} \rangle$ = the algebra with universe the left cosets of T_1 in G, and basic operations the left multiplications by elements of G.

For each $g \in G$ the operation $g^{\mathbf{A}} \in G^{\mathbf{A}}$ is defined by

$$g^{\mathbf{A}}(xT_1) = (gx)T_1 \qquad (xT_1 \in G/T_1).$$

Define the algebra $\mathbf{C} = \langle G/T_2, G^{\mathbf{C}} \rangle$ similarly.

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Consider the binary relation $\varphi \subseteq (A \times C) \times (B \times C)$ that associates to each ordered pair

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$$\operatorname{Con}\mathbf{B} \cong \operatorname{NSub}(S) \leqslant \operatorname{Sub}(S) \cong \operatorname{Con}\mathbf{A}$$

So, if S is any non-Dedekind group, $\operatorname{Con}\mathbf{B} \ncong \operatorname{Con}\mathbf{A}$.

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So, if S is any non-Dedekind group, $\operatorname{Con} \mathbf{B} \ncong \operatorname{Con} \mathbf{A}$.

If S is a nonabelian simple group, then $\operatorname{Con} \mathbf{B} \cong \mathbf{2}$, while $\operatorname{Con} \mathbf{A} \cong \operatorname{Sub}(S)$ can be arbitrarily large.