CSP Theory of Commutative Idempotent Binars

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joint work with

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Jiali Li

May 25, 2015

General Problem: Find Maltsev conditions that characterize complexity of CSPs of universal algebras.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable} \iff \mathbf{A} \text{ has a wnu term operation}$$

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A term $t(x_1, \ldots, x_n)$ is a weak near unanimity term operation if it satisfies

$$t(x, x, \ldots, x) \approx x \quad \text{(idempotent)}$$

$$t(y, x, \ldots, x) \approx t(x, y, \ldots, x) \approx \cdots \approx t(x, x, \ldots, y).$$

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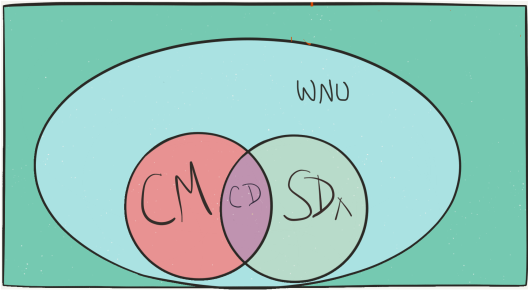

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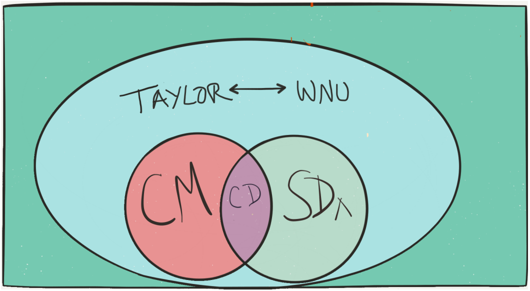

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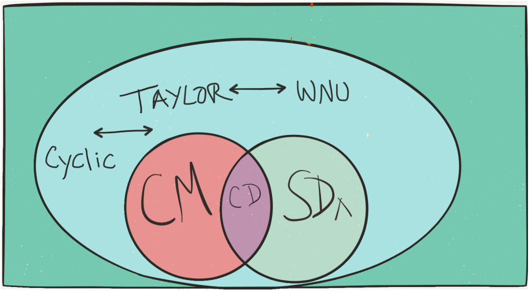

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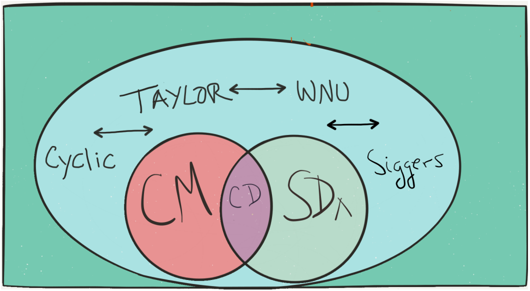

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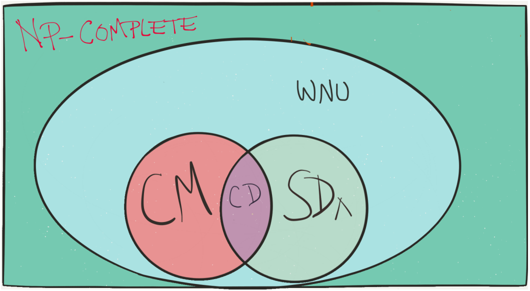

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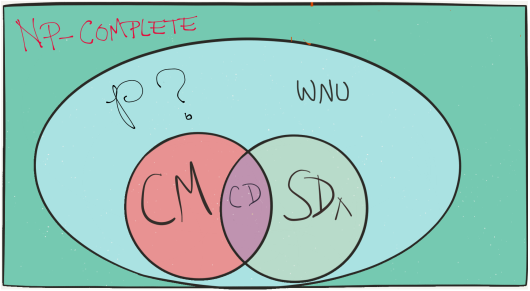

Commutative Idempotent Binars

Some more definitions.

- A set A together with a single binary operation is called a binar.
- A commutative idempotent binar is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.
- A binary operation $x \cdot y = t(x, y)$ is a WNU term if and only if it is idempotent and commutative. This suggests the following

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A semilattice is an associative CIB.

Semilattices are tractable (in fact, they have *finite width*).

Some well known facts

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

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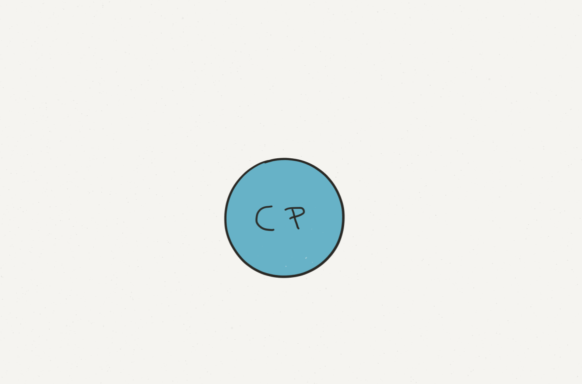

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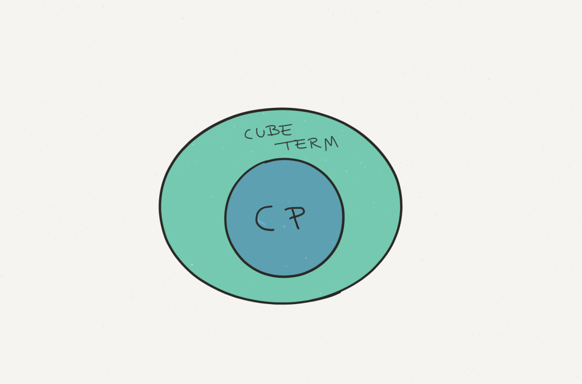

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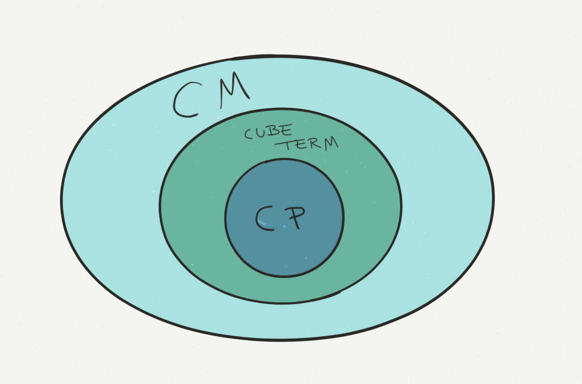

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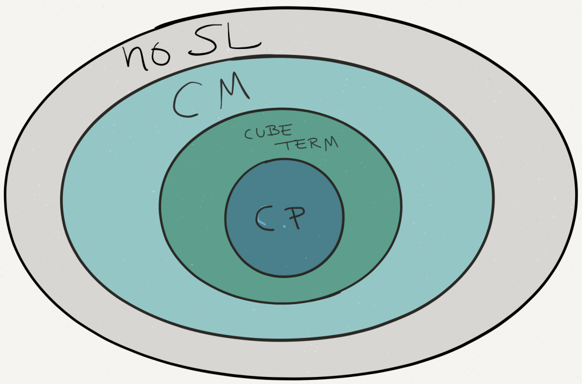

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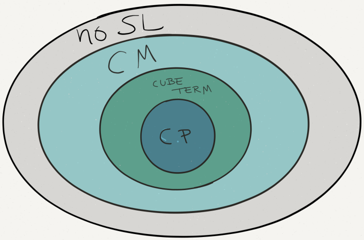

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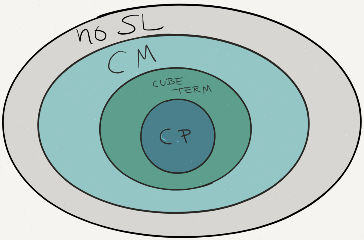

- cube term $\Longrightarrow$ CM

  Proof: few subalgebras of powers

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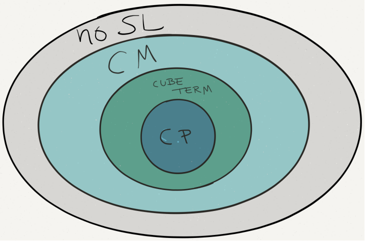

- cube term $\implies$ CM

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- CM $\implies$ $\mathbf{S}_2$ is not in $\mathrm{V}(\mathbf{A})$

  Proof: $\mathbf{S}_2 \in \mathrm{V}(\mathbf{A}) \Rightarrow \mathbf{S}_2^2 \in \mathrm{V}(\mathbf{A})$;

  $\mathrm{Con}\,(\mathbf{S}_2^2)$ is not modular.

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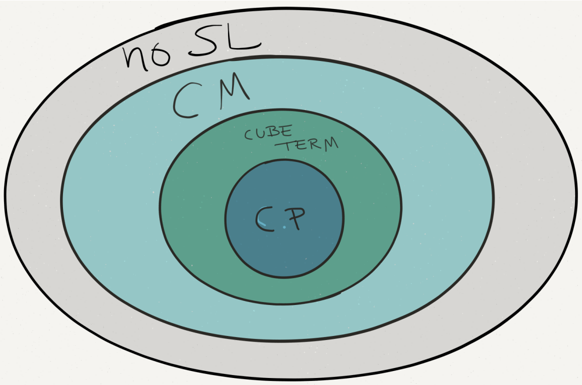

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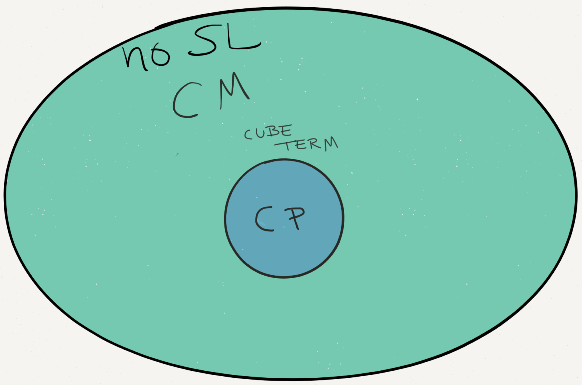

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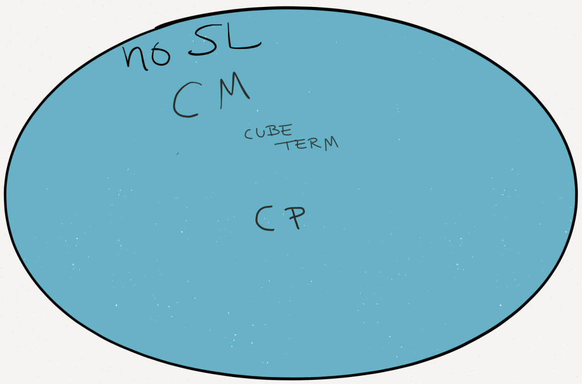

Cube Terms

A cube operation is a function $c : A^n \to A$ satisfying for each $1 \leqslant i \leqslant n$ $c(w_1, \ldots, w_n) = x$ where $\{w_1, \ldots, w_n\} \subseteq \{x, y\}$ and $w_i = y$.

Here x and y are distinct variables.

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Cube terms were introduced in... ?

Berman, Idziak, Marković, McKenzie, Valeriote, Willard,
"Varieties with few subalgebras of powers," 2010.

Marković, Maróti, McKenzie,
"Finitely related clones & algebras with cube terms," 2012.

Cube Term Blockers

A cube term blocker (CTB) for $\mathbf{A}$ is a pair (C, B) of subuniverses of $\mathbf{A}$ satisfying $\emptyset < C < B \leqslant A$ and for every term $t(x_1, \ldots, x_n)$ of $\mathbf{A}$ there is an index $i \in [n]$ such that

$$(\forall (b_1, \ldots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \ldots, b_n) \in C).$$

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Lemma

A finite CIB $\mathbf{A} = \langle A, \cdot \rangle$ *has a CTB if and only if* $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$.

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Proof.

If (C, B) is a CTB, then $\theta = C^2 \cup (B - C)^2$ is a congruence of $\mathbf{B} = \langle B, \cdot \rangle$ and $\mathbf{B}/\theta \cong \mathbf{S}_2$.

Conversely, suppose $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$, and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ with $\mathbf{B}/\theta$ a meet-SL for some θ. Let C/θ be the bottom of $\mathbf{B}/\theta$, then (C, B) is a CTB. $\square$

Kearnes and Tschantz, “Automorphism groups of squares and of free algebras,” 2007.

Lemma

If V is an idempotent variety that is not congruence permutable, then there are subuniverses U and W of $\mathbf{F} := \mathbf{F}_V\{x, y\}$ (the 2-generated free algebra) satisfying

1. $x \in U \cap W$
2. $y \in U^c \cap W^c$
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Collapse for CIBs

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For CIB's, U or W will be an ideal.

This implies a CTB and a semilattice.

Remaining Question for CIBs

Conclusion

Let $\mathbf{A}$ be a CIB and $\mathbf{S}_2 \notin \mathrm{V}(\mathbf{A})$. Then $\mathrm{CSP}(\mathbf{A})$ is tractable.

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Let $\mathbf{A}$ be a CIB and $\mathbf{S}_2 \in \mathrm{V}(\mathbf{A})$. Is $\mathrm{CSP}(\mathbf{A})$ tractable?

Recall, for every $\mathbf{A}$,

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Recall, for every $\mathbf{A}$,

- if $\mathbf{S}_2 \in \mathrm{V}(\mathbf{A})$, then $\mathrm{V}(\mathbf{A})$ is not CM;
- if $\mathrm{V}(\mathbf{A})$ is $\mathrm{SD}_\wedge$, then $\mathrm{CSP}(\mathbf{A})$ is tractable (in fact, always has a solution).

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Revised Question

Let $\mathbf{A}$ be a CIB with $\mathbf{S}_2$ in $\mathrm{V}(\mathbf{A})$, not $\mathrm{SD}_\wedge$. Is $\mathrm{CSP}(\mathbf{A})$ tractable?

EXAMPLES

$\cdot$	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

$*$	0	1	2	3
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Maroti's idea:

$\circ$	0	1	2	3
0	0	0	2	1
1	0	1	3	2
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3	1	2	1	3

Bergman's idea: replace basic binary operation with a term from $\mathrm{Clo}(\mathbf{A})$, say $t(x, y) = (x * y) * x$.

If $\langle A, t\rangle$ tractable, then so is $\langle A, *\rangle$