CSPs of Finite Commutative Idempotent Binars

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joint work with

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Vanderbilt University
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General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable} \iff \mathbf{A} \text{ has a weak-nu term operation}$$

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The left-to-right direction is known.

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The right-to-left direction is open.

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A weak near unanimity (weak-nu) term operation is one that satisfies

$$t(x, x, \dots, x) \approx x \qquad \text{(idempotent)}$$

$$t(y, x, \dots, x) \approx t(x, y, \dots, x) \approx \cdots \approx t(x, x, \dots, y)$$

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A *binary* operation $t(x, y)$ is weak-nu if

$$t(x, x) \approx x \qquad \text{(idempotent)}$$

$$t(y, x) \approx t(x, y) \qquad \text{(commutative)}$$

So let's try to prove (?) for commutative idempotent binars.

Commutative Idempotent Binars

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Pause to consider more general case for a minute...

General case

Some well known facts

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

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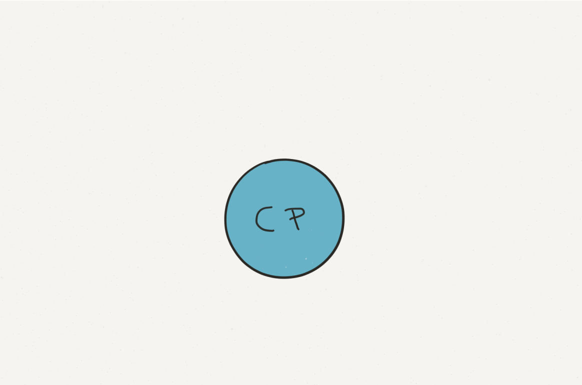

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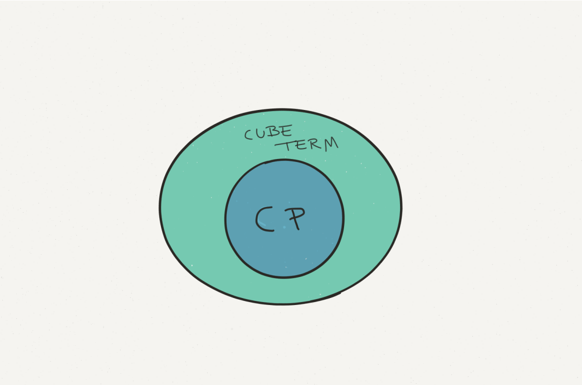

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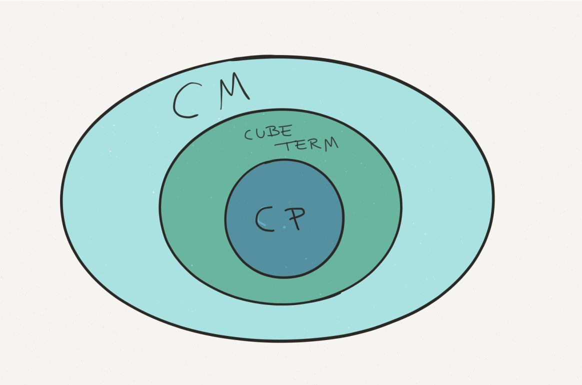

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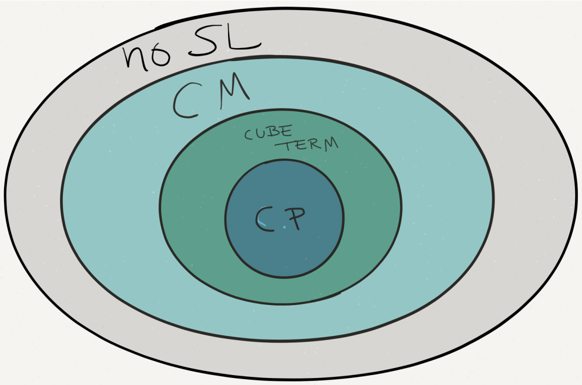

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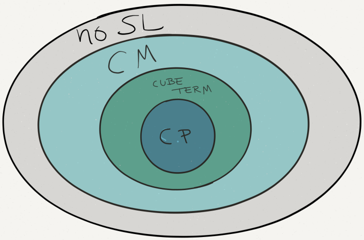

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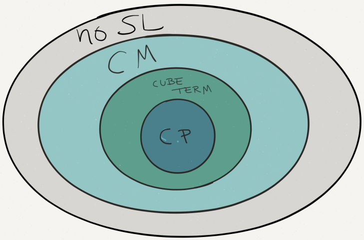

- cube term $\Longrightarrow$ CM

  (Berman, Idziak, Marković, McKenzie, Valeriote, Willard 2010)

- CM $\Longrightarrow$ $\mathbf{S}_2$ is not in $\mathrm{V}(\mathbf{A})$

  Proof: $\mathbf{S}_2 \in \mathrm{V}(\mathbf{A}) \ \Rightarrow \ \mathbf{S}_2^2 \in \mathrm{V}(\mathbf{A})$;

  $\mathrm{Con}\,(\mathbf{S}_2^2)$ is not modular.

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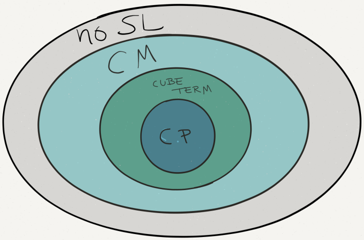

<u>CIB case</u>

- 1st reduction by cube-term blockers.

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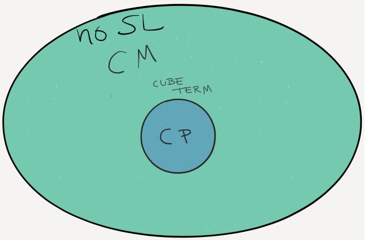

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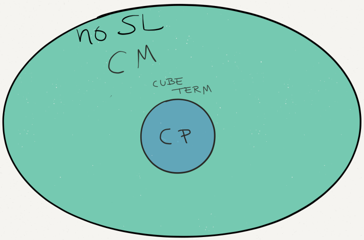

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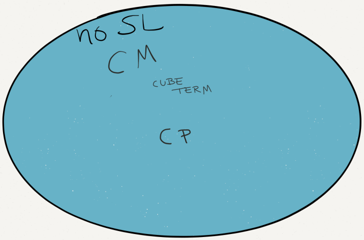

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First Reduction

by Cube-Term Blockers

Marković, M. Maróti, McKenzie (M^4)
"Finitely related clones and algebras with cube terms" (2012)

A cube-term blocker (CTB) is a pair (C, B) of subuniverses satisfying $\emptyset < C < B \leqslant A$ and for every $t(x_1, \ldots, x_n)$ there is an index $i \in [n]$ with

$$(\forall (b_1, \ldots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \ldots, b_n) \in C).$$

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Lemma

A finite CIB $\mathbf{A}$ *has a CTB if and only if* $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$.

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Proof.

(C, B) a CTB implies $\theta = C^2 \cup (B - C)^2$ a congruence with $\mathbf{B}/\theta \cong \mathbf{S}_2$.

Conversely, suppose $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$, and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ with $\mathbf{B}/\theta$ a meet-SL for some θ. Let C/θ be the bottom of $\mathbf{B}/\theta$, then (C, B) is a CTB. $\square$

Second Reduction

Kearnes and Tschantz
"Automorphism groups of squares and of free algebras" (2007)

Lemma

If V is an idempotent variety that is not congruence permutable, then there are subuniverses U and W of $\mathbf{F} := \mathbf{F}_V\{x, y\}$ satisfying

1. $x \in U \cap W$
2. $y \in U^c \cap W^c$
3. $(U \times F) \cup (F \times W) \leqslant \mathbf{F}^2$

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LEMMA

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For CIB's, either U or W will be an ideal.

This implies a CTB and a semilattice.

Remaining Questions for Finite CIBs

Conclusion

Let $\mathbf{A}$ be a finite CIB and $\mathbf{S}_2 \notin \mathsf{HS}(\mathbf{A})$. Then $\mathrm{CSP}(\mathbf{A})$ is tractable.

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Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in $\mathsf{HS}(\mathbf{A})$. Is $\mathrm{CSP}(\mathbf{A})$ tractable?

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Recall, if $\mathrm{V}(\mathbf{A})$ is $\mathrm{SD}_\wedge$, then $\mathrm{CSP}(\mathbf{A})$ is tractable.

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Revised Question

Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in $\mathrm{HS}(\mathbf{A})$, and $\mathrm{V}(\mathbf{A})$ not $\mathrm{SD}_\wedge$.

Is $\mathrm{CSP}(\mathbf{A})$ tractable?

EXAMPLES

$\cdot$	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

$*$	0	1	2	3
0	0	0	1	1
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$\circ$	0	1	2	3
0	0	0	2	1
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2	2	3	2	1
3	1	2	1	3

Maroti's idea:

Bergman's idea: replace basic binary operation with a term from $\mathrm{Clo}(\mathbf{A})$, say $t(x,y) = (x * y) * x$.

If $\langle A, t\rangle$ tractable, then so is $\langle A, *\rangle$