Algebraic Methods for Deciding Complexity of Constraint Satisfaction Problems

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What is a CSP?

Informally, a **C**onstraint **S**atisfaction **P**roblem consists of

- a list of variables ranging over a finite domain and
- a set of constraints on those variables.

Problem: can we assign values to all the variables so that all of the constraints are satisfied?

Examples

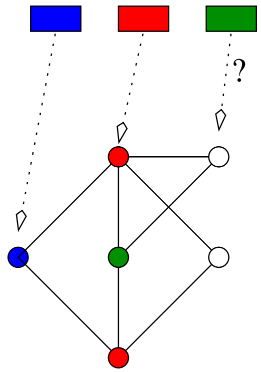
A system of linear equations is a CSP

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Also, a system of nonlinear equations is a CSP

$$\begin{aligned} a_{11}x_1^2x_3 + a_{12}x_2x_3x_7 + \cdots + a_{1n}x_4x_n^3 &= b_1 \\ a_{21}x_2x_5 + a_{22}x_2 \quad + \cdots + a_{2n}x_4^3 &= b_2 \\ &\vdots \\ a_{m1}x_3x_5x_8 + a_{m2}x_2 \quad + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

For a fixed k, determining whether a graph is k-colorable is a CSP



Determining whether a given formula $\varphi(x_1, \ldots, x_n)$ is satisfiable is a CSP For example,

$$\varphi(x, y, z) = (x \vee y \vee \neg z) \wedge (\neg x \vee y \vee \neg z)$$

is satisfiable (by $(x, y, z) = (0, 0, 1)$)

Algorithms

There is an efficient algorithm (Gaussian elimination) for solving any linear system. That is

There is an algorithm that accepts as input a linear system and decides whether that system has a solution.

The running time of the algorithm is bounded above by $f(s)$ where f is a *polynomial* and s is the size of the system.

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The input, a particular system, is an instance of the problem LINEAR SYSTEM.

Similarly

There is an algorithm that accepts as input a graph and decides whether the graph is 2-colorable.
Running time bounded by $f(s)$, a *polynomial* in size s.

The input, a particular graph, is an instance of the problem 2-COLORABILITY.

There is an algorithm that accepts as input a formula, $\varphi = \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k$ (each φ_i bijunctive) and decides whether φ is satisfiable.
Running time bounded by $f(k)$, a *polynomial* in size k.

The intput formula φ is an instance of the problem 2-SAT.

We say that all these algorithms run in polynomial time.

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Thus these problems are solvable in nondeterministic polynomial time.

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It is possible for $X \leq_{\mathrm{p}} Y \leq_{\mathrm{p}} X$. In that case, write $X \equiv_{\mathrm{p}} Y$.

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- $\mathbb{P} \subseteq \mathbb{NP}$
- Both $\mathbb{P}$ and $\mathbb{NP}$ are downsets, i.e.,
  $Y \in \mathbb{P}$ & $X \leq_p Y \implies X \in \mathbb{P}$

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The maximal members of $\mathbb{NP}$ are called $\mathbb{NP}$-complete.

3-COLORABILITY, NONLINEAR SYSTEM, and 3-SAT are known to be $\mathbb{NP}$-complete.

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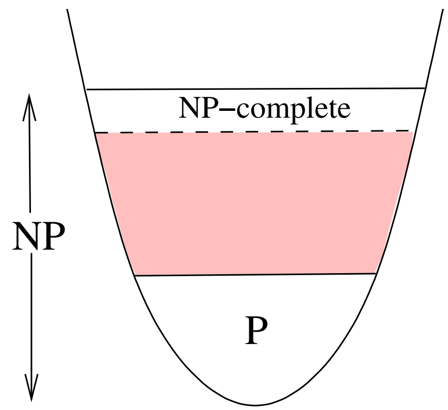
Theorem (Ladner, 1975)

If $\mathbb{P} \neq \mathbb{NP}$ then there are problems in $\mathbb{NP} - \mathbb{P}$ that are not $\mathbb{NP}$-complete.



If $\mathbb{P} \neq \mathbb{NP}$ then the pink area is nonempty.

Formal Definition of CSP

Let D be a set, n a positive integer
An *n-ary relation on D* is a subset of D^n

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$\mathrm{Rel}_n(D)$ denotes the set of all n-ary relations on D

$\mathrm{Rel}(D) = \bigcup_{n>0} \mathrm{Rel}_n(D)$

Let D be a finite set and $\Delta \subseteq \mathrm{Rel}(D)$

$\mathrm{CSP}(\langle D, \Delta\rangle)$ is the following decision problem:

Instance. A finite set $V = \{ v_1, \dots, v_n \}$ of variables and a finite set $\{ C_1, \dots, C_m \}$ of constraints;

each constraint C_i is a pair $(\langle x_{i1}, \dots, x_{i\,p_i} \rangle, \delta_i)$ in which $x_{i1}, \dots, x_{i\,p_i} \in V$ and $\delta_i \in \Delta$

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Question. Does there exist a solution, that is, a “context” $\rho \colon V \to D$, such that for all $i \leq m$, $\langle \rho(x_{i1}), \dots, \rho(x_{i\,p})\rangle \in \delta_i$?

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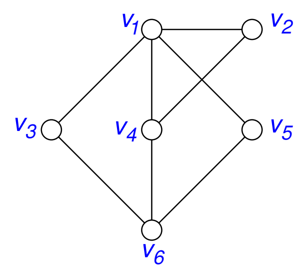
CSP($\langle D, \Delta \rangle$) always lies in $\mathbb{NP}$.

Example: 3-colorability

$D = \{r, g, b\}, \quad \Delta = \{\kappa_3\}$
$\kappa_3 = \{ (x, y) \in D : x \neq y \}$

Then CSP($\langle D, \Delta \rangle$) is the 3-colorability problem



$$
\begin{aligned}
&V = \{v_1, \dots, v_6\} \\
&\langle v_1, v_2 \rangle \in \kappa \\
&\langle v_1, v_3 \rangle \in \kappa \\
&\langle v_1, v_4 \rangle \in \kappa \\
&\langle v_2, v_4 \rangle \in \kappa \\
&\quad \vdots \\
&\langle v_5, v_6 \rangle \in \kappa
\end{aligned}
$$

Two Motivating Questions

1. Dichotomy Conjecture
   Every CSP($\langle D, \Delta \rangle$) either lies in $\mathbb{P}$ or is $\mathbb{NP}$-complete.

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   Characterize those CSPs that lie in $\mathbb{P}$.

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1. Dichotomy Conjecture
   Every CSP($\langle D, \Delta\rangle$) either lies in $\mathbb{P}$ or is $\mathbb{NP}$-complete.
2. Tractability Problem
   Characterize those CSPs that lie in $\mathbb{P}$.

What would a characterization look like? What language could we use?

Polymorphisms

Definition

Let $\delta \in \mathrm{Rel}_k(D)$ and $f\colon D^n \to D$. We say f *preserves* δ if

$$(a_{11}, \dots, a_{1k}), \dots, (a_{n1}, \dots, a_{nk}) \in \delta \implies (f(a_{11}, \dots, a_{n1}), \dots, f(a_{1k}, \dots, a_{nk})) \in \delta$$

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f is an *n-ary operation* on D.

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$$\begin{array}{cccccc}
a_{11} & a_{12} & \dots & a_{1k} & \in & \delta \\
a_{21} & a_{22} & \dots & a_{2k} & \in & \delta \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{n1} & a_{n2} & \dots & a_{nk} & \in & \delta \\
\downarrow_f & \downarrow_f & & \downarrow_f & & \\
\star & \star & \dots & \star & \in & \delta
\end{array}$$

Definition

Let Δ be a set of relations on D. Then $\mathrm{Pol}(\Delta)$ denotes the set of all operations preserving all members of Δ. These are the *polymorphisms* of Δ.

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Let F be a set of operations on D. Then $\mathrm{Inv}(F)$ denotes the set of all relations preserved by all operations in F.

Important point: $\langle D, \mathrm{Pol}(\Delta)\rangle$ is an algebraic structure

Theorem

Let $\Gamma, \Delta \subseteq \mathrm{Rel}(D)$. *Then*

$$\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Delta) \implies \mathrm{CSP}(\Delta) \leq_{\mathrm{p}} \mathrm{CSP}(\Gamma).$$

Theorem

Let $\Gamma, \Delta \subseteq \mathrm{Rel}(D)$. *Then*

$$\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Delta) \implies \mathrm{CSP}(\Delta) \leq_{\mathrm{p}} \mathrm{CSP}(\Gamma).$$

Thus, the richer the algebraic structure, the easier the corresponding CSP

One can go back and forth between relational and algebraic structures

$$\begin{array}{ccc} \textbf{Relational} & & \textbf{Algebraic} \\ \langle D, \Delta \rangle & \longrightarrow & \langle D, \mathrm{Pol}(\Delta) \rangle \\ \langle D, \mathrm{Inv}(F) \rangle & \longleftarrow & \langle D, F \rangle \end{array}$$

$$\mathrm{CSP}\langle D, \Delta \rangle \equiv_{\mathrm{p}} \mathrm{CSP}\langle D, \mathrm{Inv}(\mathrm{Pol}(\Delta)) \rangle$$

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Perhaps the expressive power of algebra can be used to classify CSPs.

Algebraic Facts

Let **A** and **B** be algebras

B a subalgebra of **A** $\implies$ $\mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{p}} \mathrm{CSP}(\mathbf{A})$.

B a homomorphic image of **A** $\implies$ $\mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{p}} \mathrm{CSP}(\mathbf{A})$.

$\mathrm{CSP}(\mathbf{A}^n) \equiv_{\mathrm{p}} \mathrm{CSP}(\mathbf{A})$

Theorem (Bulatov, Jeavons, Krokhin, 2000)

If $\langle D, \Delta \rangle$ is a core and every polymorphism is essentially unary, then CSP(Δ) *is* $\mathbb{NP}$*-complete.*

f is *essentially unary* if $f(x_1, \ldots, x_n) = g(x_j)$ for some unary g and some $j \leq n$.

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Corollary

3-COLORABILITY, NONLINEAR SYSTEM, and 3-SAT are $\mathbb{NP}$*-complete.*

Informal reformulation of the dichotomy conjecture
If **A** has some kind of decent algebraic structure then CSP(**A**) $\in \mathbb{P}$ otherwise CSP(**A**) is $\mathbb{NP}$-complete.