Part III: Interval Enforceable Properties of Finite Groups

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Algebra & Logic Seminar

September 28, 2012

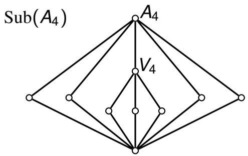
For some groups, $\mathrm{Sub}(G)$ determines G up to isomorphism.

Examples

The Klein 4-group, V_4.

The alternating groups, A_n ($n \geqslant 4$).

Every finite nonabelian simple group.

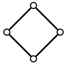


For other groups, $\mathrm{Sub}(G)$ is isomorphic to the subgroup lattices of all groups in an infinite class of nonisomorphic groups.

EXAMPLES

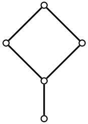
$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of prime order.

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order p^2.

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order pq.

At the other extreme, there are finite lattices that are not subgroup lattices.

Example: For all G,

$\mathrm{Sub}(G) \ncong$ 

We are interested in the *local structure* of subgroup lattices, that is, the possible ***intervals***

$$[H, K] := \{X \mid H \leqslant X \leqslant K\} \leqslant \mathrm{Sub}(G)$$

where $H \leqslant K \leqslant G$.

We restrict our attention to ***upper intervals***, where $K = G$, and ask two questions:

1. *What intervals $[H, G]$ are possible?*
2. *What properties of a group G can be inferred from the shape of an upper interval in $Sub(G)$?*

1. What intervals $[H, G]$ are possible?

There is a remarkable theorem relating this question to the *finite lattice representation problem* (FLRP).

Theorem (Pálfy and Pudlák(1980))

The following statements are equivalent:

(A) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*

(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

2. What properties of G can be inferred from $[H, G]$?

A group theoretical property $\mathfrak{X}$ (and the associated class $\mathscr{G}_{\mathfrak{X}}$) is

- ***interval enforceable*** (IE) provided there exists a lattice L such that

  $$\text{if } G \in \mathfrak{G} \text{ and } L \cong [H, G], \text{ then } G \text{ is a } \mathfrak{X}\text{-group.}$$

- ***core-free interval enforceable*** (cf-IE) provided $\exists L$ st

  $$\text{if } G \in \mathfrak{G},\ L \cong [H, G],\ H \text{ core-free, then } G \text{ is a } \mathfrak{X}\text{-group.}$$

- ***minimal interval enforceable*** (min-IE) provided $\exists L$ st
  if $G \in \mathfrak{G}$, $L \cong [H, G]$, and if G has minimal order (wrt $L \cong [H, G]$), then G is a $\mathfrak{X}$-group.

Clearly, if $\mathfrak{X}$ is IE, then it is also cf-IE.

There is a simple sufficient condition under which the converse holds.

If $\mathfrak{X}$ is a group property, let $\mathscr{G}_{\mathfrak{X}}^{c} := \{G \in \mathfrak{G} \mid G \not\models \mathfrak{X}\}$ denote the class of $(\neg\mathfrak{X})$-groups.

Lemma

Suppose $\mathfrak{X}$ is a core-free interval enforceable property. If

$$\mathbf{H}(\mathscr{G}_{\mathfrak{X}}^{c}) = \mathscr{G}_{\mathfrak{X}}^{c}$$

then $\mathfrak{X}$ is an interval enforceable property.

The following are at least core-free interval enforceable:

- $\mathscr{G}_0 = \mathfrak{S}^c =$ the insoluble groups
- $\mathscr{G}_1 = \{G \in \mathfrak{G} \mid (\forall n < \omega)\ (G \neq A_n \text{ and } G \neq S_n)\}$
- $\mathscr{G}_2 =$ the subdirectly irreducible groups
- $\mathscr{G}_3 =$ groups with no nontrivial abelian normal subgroups
- $\mathscr{G}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteq G\}$.

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For $i = 2, 3, 4$,

$$\mathbf{H}(\mathscr{G}_i^c) \neq \mathscr{G}_i^c$$

Proof: If $H \in \mathscr{G}_i$, $K \in \mathscr{G}_i^c$, then, $H \times K$ belongs to $\mathscr{G}_i^c$, but $(H \times K)/(1 \times K) \cong H$ does not.

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L ***group representable***.

By the Pálfy-Pudlák Theorem, the FLRP has a negative answer (i.e. $\mathscr{L}_0 \neq \mathscr{L}_3$) if we can find a finite lattice that is not group representable.

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Suppose there exists property $\mathfrak{X}$ such that both $\mathfrak{X}$ and its negation $\neg\mathfrak{X}$ are interval enforceable by the lattices L and L_c, respectively:

$$L \cong [H, G] \implies G \text{ is a } \mathfrak{X}\text{-group}$$

$$L_c \cong [H_c, G_c] \implies G_c \text{ is not a } \mathfrak{X}\text{-group}$$

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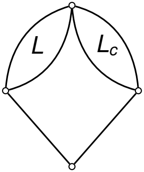
Then the lattice



wouldn't be group representable.

As the next result shows, however, if a group property and its negation are interval enforceable by L and L_c, then already at least one of these lattices is not group representable.

LEMMA

If $\mathfrak{X}$ is a group property that is interval enforceable by a group representable lattice, then $\neg\mathfrak{X}$ is not interval enforceable by a group representable lattice.

Insolubility is interval enforceable, but solubility is not.

For if $L \cong [H, G]$, then for any insoluble group K we have $L \cong [H \times K, G \times K]$, and $G \times K$ is insoluble.

Note that the group $H \times K$ at the bottom of the interval is not core-free. So a more interesting question is whether a property and its negation could both be *core-free* IE.

Conjecture

If $\mathfrak{X}$ is core-free interval enforceable by a group representable lattice, then $\neg\mathfrak{X}$ is not core-free interval enforceable by a group representable lattice.

The following lemma shows that any class of groups that omits certain wreath products cannot be core-free interval enforceable by a group representable lattice.

Lemma

Suppose $\mathfrak{X}$ is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group S, there exists a wreath product group of the form $W = S \wr U$ that is a $\mathfrak{X}$-group.

Corollary

Solubility is not core-free interval enforceable.

Proof Sketch

Let L be a group representable lattice such that if $L \cong [H, G]$ and $\mathrm{core}_G(H) = 1$ then G is a $\mathfrak{X}$-group.

Since L is group representable, $\exists$ $\mathfrak{X}$-group G with $L \cong [H, G]$.

Proof Sketch

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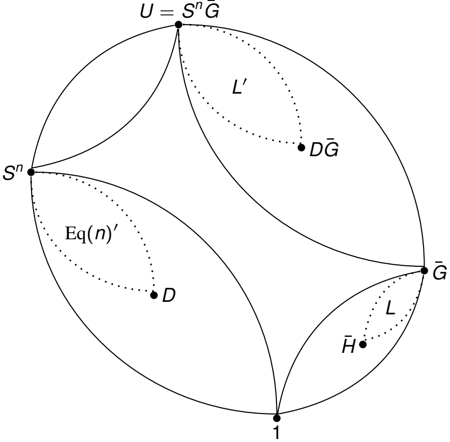
Since L is group representable, $\exists$ $\mathfrak{X}$-group G with $L \cong [H, G]$.

We apply the idea of Hans Kurzweil twice:

- Fix a finite nonabelian simple group S.
- Suppose the index of H in G is $|G : H| = n$.
- Then the action of G on the cosets of H induces an automorphism of the group S^n by permutation of coordinates.
- Denote this by $\varphi : G \to \mathrm{Aut}(S^n)$, and let $\varphi(G) = \bar{G} \leqslant \mathrm{Aut}(S^n)$.



The interval $[D, S^n]$ is isomorphic to $\mathrm{Eq}(n)'$, the dual of the lattice of partitions of an n-element set.

The dual lattice L' is an upper interval of $\mathrm{Sub}(U)$, namely, $L' \cong [D\bar{G}, U]$.

We conclude that a class of groups that does not include wreath products of the form $S \wr G$, where S is an arbitrary finite nonabelian simple group, is not a core-free interval enforceable class. The class of soluble groups is an example.

Theorem

The following statements are equivalent:

(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

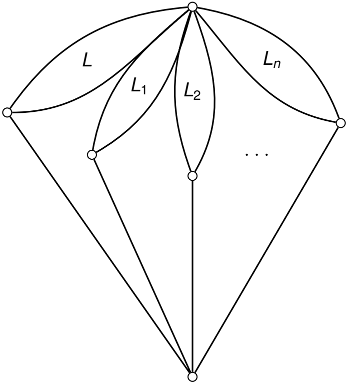
(C) *For every finite lattice L and every finite collection $\mathscr{G}_1, \ldots, \mathscr{G}_n$ of cf-IE classes of groups,*

$$\exists\ G \in \bigcap_{i=1}^{n} \mathscr{G}_i \text{ such that } L \cong [H, G] \text{ and } \operatorname{core}_G(H) = 1.$$

(D) *For every finite collection $\mathscr{L}$ of finite lattices, there exists a finite group G such that each $L_i \in \mathscr{L}$ is isomorphic to $[H_i, G]$ for some core-free subgroup $H_i \leqslant G$.*

By (C), the FLRP would have a negative answer if we could find a collection $\mathscr{G}_1, \ldots, \mathscr{G}_n$ of cf-IE classes such that $\bigcap\limits_{i=1}^{n} \mathscr{G}_i$ is empty.

By (D), it makes sense to consider finite collections of finite lattices and ask what can be proved about a group G if one assumes that all of these lattices are isomorphic to upper intervals of $\mathrm{Sub}(G)$.



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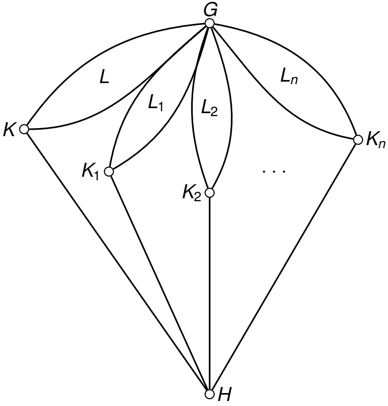
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