DEDEKIND'S TRANSPOSITION PRINCIPLE

AND

PERMUTING SUBGROUPS & EQUIVALENCE RELATIONS

AND (MAYBE, BUT PROBABLY NOT)

ISOTOPIC ALGEBRAS WITH NONISOMORPHIC CONGRUENCE LATTICES

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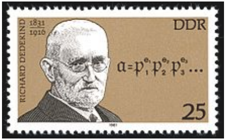
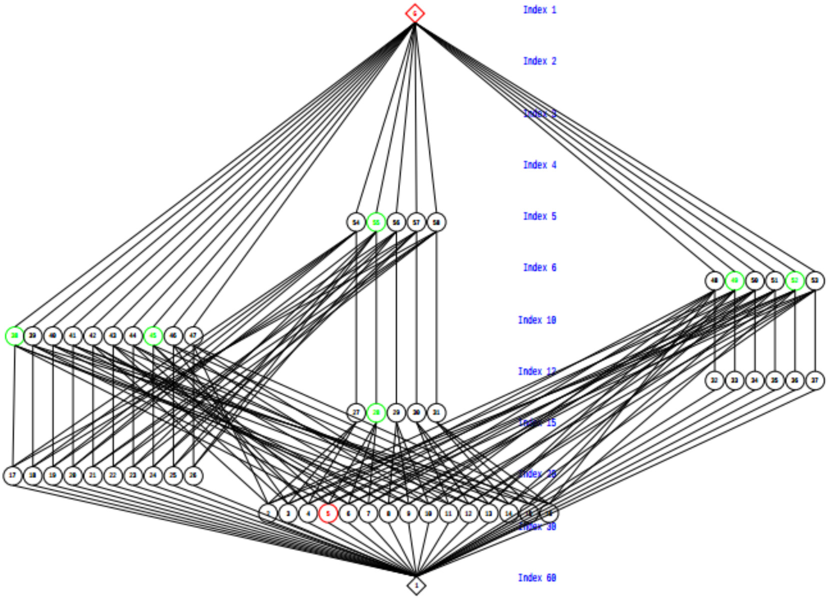
These slides and other resources are available at

http://williamdemeo.wordpress.com

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Intervals in Subgroup Lattices

- Let H, K be subgroups of a group G.
- Recall the set

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is a group if and only if $HK = KH = \langle U, H \rangle$.

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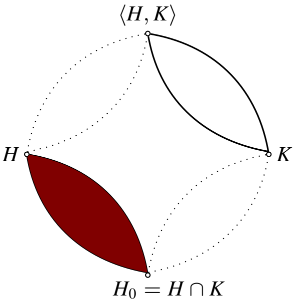

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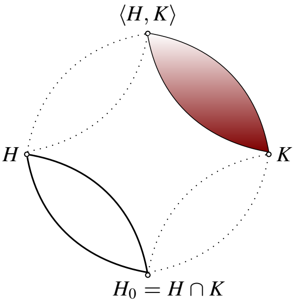

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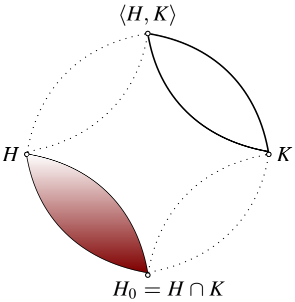

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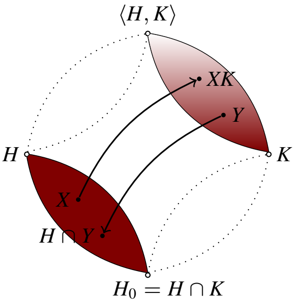

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Lemma

If $HK = KH$, then $[\![K, HK]\!] \cong [\![H_0, H]\!]^K \leqslant [\![H_0, H]\!]$.

Example

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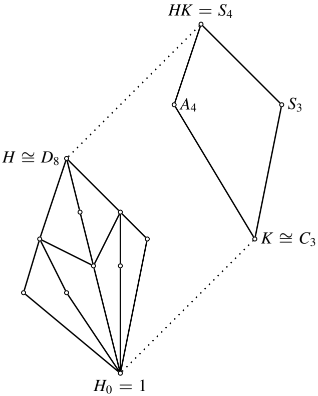

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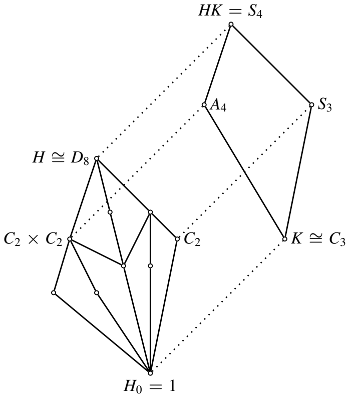

- Only four subgroups of H permute with K, including

$$H \cap A_4 \cong C_2 \times C_2, \qquad H \cap S_3 \cong C_2.$$

Notation

Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice with $a \in L$.

Let φ_a and ψ_a be the "perspectivity maps"

$$\varphi_a(x) = x \wedge a \quad \text{and} \quad \psi_a(x) = x \vee a$$

For $x, y \in L$, let $[\![x, y]\!]_L = \{z \in L \mid x \leqslant z \leqslant y\}$.

Dedekind's Transposition Principle

for modular lattices

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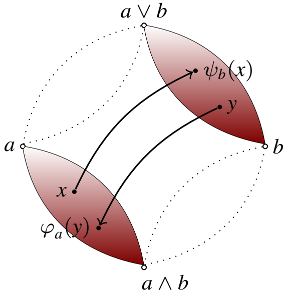

Theorem (Dedekind's Transposition Principle)

$\mathbf{L}$ is modular iff for all $a, b \in L$ the maps φ_a and ψ_b are inverse lattice isomorphisms of $[\![a \wedge b, a]\!]$ and $[\![b, a \vee b]\!]$.

Another Transposition Principle

for lattices of equivalence relations

Let X be a set and let $\operatorname{Eq} X$ be the lattice of equivalence relations on X.

Given $\alpha, \beta \in \operatorname{Eq} X$, define the *composition* of α and β to be the binary relation

$$\alpha \circ \beta = \{(x, y) \in X^2 \mid (\exists z \in X)\, x \; \alpha \; z \; \beta \; y\}.$$

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i.e., the relations in $[\![\eta, \theta]\!]_L$ that permute with β.

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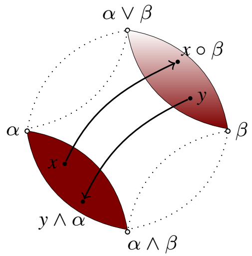

Lemma

Suppose α and β are permuting relations in $L \leqslant \operatorname{Eq} X$.

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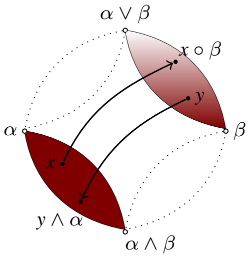

Question: Does this generalize the subgroup lattice lemma?

Answer

Yes!

Answer

Yes!

<insert G-set stuff here>

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Yes!

<insert G-set stuff here>

Lemma

In $\mathrm{Con}\langle G\backslash H, \bar{G}\rangle$, two congruences θ_{K_1} and θ_{K_2} permute if and only if the corresponding subgroups K_1 and K_2 permute.

Questions

Recall that $HK = \langle H, K \rangle$ if and only if $HK = KH$.

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$H \circ^n K = \langle H, K \rangle$ if and only if $H \circ^n K = K \circ^n H$?

QUESTIONS

Denote by $H \circ^n K$ the *n-fold composition of H and K*.

$$\begin{aligned}
H \circ^1 K &= H, \\
H \circ^2 K &= HK, \\
H \circ^3 K &= HKH, \\
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We say H and K are *n-permuting* if $H \circ^n K = K \circ^n H$.

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Question 2. *Is the following true?*

> *If H and K are n-permuting, then interval $[\![K, \langle H, K\rangle]\!]$ is isomorphic to the lattice of subgroups in $[\![H_0, H]\!]$ that n-permute with K.*

CONNECTION WITH EQUIVALENCE RELATIONS

Let $\mathbf{A} = \langle H\backslash G, \bar{G}\rangle$ be the algebra with

- universe: the right cosets $H\backslash G = \{Hx \mid x \in G\}$
- operations: $\bar{G} = \{g^{\mathbf{A}} : g \in G\}$, where $g^{\mathbf{A}}(Hx) = Hxg$.

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Lemma

The subgroups K_1 and K_2 are n-permuting if and only if their corresponding congruences θ_{K_1} and θ_{K_2} are n-permuting. That is,

$$K_1 \circ^n K_2 = K_2 \circ^n K_1 \iff \theta_{K_1} \circ^n \theta_{K_2} = \theta_{K_2} \circ^n \theta_{K_1}.$$

Answer to Question 1.

Lemma

For $\alpha, \beta \in \mathrm{Eq}\, X$, and for every even integer $n > 1$, TFAE:

(i) $\alpha \circ^n \beta = \alpha \vee \beta$

(ii) $\alpha \circ^n \beta = \beta \circ^n \alpha$

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For $n = 3$,

$$\alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta \quad \Longrightarrow \quad \alpha \circ \beta \circ \alpha = \alpha \vee \beta$$

but the converse is false.

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Corollary

For $H, K \leqslant G$, and for every even integer $n > 1$, TFAE:

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For $n = 3$,

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but the converse is false.
Question 1.' What are conditions on G under which the converse is true?

Answer to Question 1.

case $n = 5$

Question 1. Is it true that

$H \circ^5 K = \langle H, K \rangle$ *if and only if* $H \circ^5 K = K \circ^5 H$*?*

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$$H \circ^5 K = \langle H, K \rangle \text{ if and only if } H \circ^5 K = K \circ^5 H?$$

Answer. No.

Example. Let $G = (C_3 \times C_3) : C_4$.

This is a group of order 36 with generators f_1, f_2, f_3, f_4.

Let $H = \langle f_1 \rangle \cong C_2$, and $K = \langle f_1 \cdot f_3 \cdot f_4^2, f_2 \cdot f_4^2 \rangle \cong C_4$. Then,

- $H \cap K = 1$
- $\langle H, K \rangle = K \circ^5 H$ has order 36 so it is the whole group.
- The *set* $H \circ^5 K$ has size 34, so does not generate $\langle H, K \rangle$.
- H covers 1.

Answer to Question 2.

No.

In general, it is not true that if H and K are n-permuting, then the interval $[\![K, \langle H, K\rangle]\!]$ is isomorphic to the lattice of those subgroups in $[\![H_0, H]\!]$ that n-permute with K.

ANSWER TO QUESTION 2.

No.

> *In general, it is not true that if H and K are n-permuting, then the interval $[\![K, \langle H, K\rangle]\!]$ is isomorphic to the lattice of those subgroups in $[\![H_0, H]\!]$ that n-permute with K.*

Example. The group A_5 has subgroups $H \cong D_{10}$, and $K \cong C_2$ such that

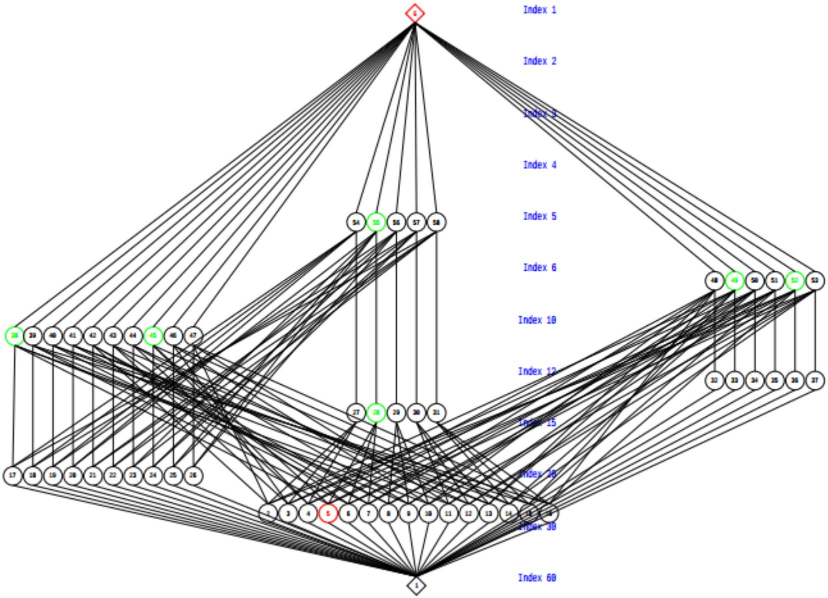
$$H \circ^4 K = K \circ^4 H = A_5,$$

but the map

$$[\![K, A_5]\!] \ni J \mapsto J \cap H \in [\![1, H]\!]$$

is not one-to-one.

Examples



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REVISED QUESTION 2.

Question 2.'

What are conditions on the group G so that

if H, K are n-permuting subgroups of G, then

$$[\![K, \langle H, K\rangle]\!] \cong [\![H_0, H]\!]^{K\circ^n} \leqslant [\![H_0, H]\!]\,?$$

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be algebras of the same type.

$\mathbf{A}$ and $\mathbf{B}$ are *isotopic over* $\mathbf{C}$, denoted $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$, if there is an isomorphism

$$\varphi : \mathbf{A} \times \mathbf{C} \xrightarrow{\cong} \mathbf{B} \times \mathbf{C} \quad \text{that leaves the second coordinate fixed}$$

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It is easy to verify that $\sim$ is an equivalence relation.

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We say that $\mathbf{A}$ and $\mathbf{B}$ are *isotopic*, denoted $\mathbf{A} \sim \mathbf{B}$, if $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ for some $\mathbf{C}$.

If $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ and $\mathrm{Con}(\mathbf{A} \times \mathbf{C})$ happens to be modular, then we write $\mathbf{A} \sim_{\mathbf{C}}^{\mathrm{mod}} \mathbf{B}$ and say that $\mathbf{A}$ and $\mathbf{B}$ are *modular isotopic over* $\mathbf{C}$.

Lemma. If $\mathbf{A} \sim_{\mathbf{C}}^{\mathrm{mod}} \mathbf{B}$ then $\operatorname{Con}\mathbf{A} \cong \operatorname{Con}\mathbf{B}$.

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Lemma. If $\mathbf{A} \sim_{\mathbf{C}}^{\text{mod}} \mathbf{B}$ then $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$.

The proof is a nice/easy application of Dedekind's Transposition Principle.

Could we use the same strategy with the non-modular version of the transposition principle to show that $\mathbf{A} \sim \mathbf{B}$ implies $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$?

No!

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But this only shows that the same argument doesn't work...

Examples

in which $\mathbf{A} \sim \mathbf{B}$ and $\mathrm{Con}\,\mathbf{A} \not\cong \mathrm{Con}\,\mathbf{B}$

EXAMPLES

IN WHICH $\mathbf{A} \sim \mathbf{B}$ AND $\operatorname{Con}\mathbf{A} \ncong \operatorname{Con}\mathbf{B}$

For any group G, let $\operatorname{Sub}(G)$ denote the lattice of subgroups of G.

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Let S be any group and let D denote the *diagonal subgroup* of $S \times S$,

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The interval $[\![D, S \times S]\!] \leqslant \operatorname{Sub}(S \times S)$ is described by the following

Lemma

The filter above the diagonal subgroup of $S \times S$ is isomorphic to the lattice of normal subgroups of S.

EXAMPLES

IN WHICH $\mathbf{A} \sim \mathbf{B}$ AND $\operatorname{Con} \mathbf{A} \ncong \operatorname{Con} \mathbf{B}$

Let S be a group, and let $G = S_1 \times S_2$, where $S_1 \cong S_2 \cong S$.

Let $D = \{(x_1, x_2) \in G \mid x_1 = x_2\}, \quad T_1 = S_1 \times \langle 1 \rangle, \quad T_2 = \langle 1 \rangle \times S_2$.

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Let $\mathbf{A} = \langle G/T_1, G^{\mathbf{A}} \rangle =$ the algebra with universe the left cosets of T_1 in G, and basic operations the left multiplications by elements of G.

For each $g \in G$ the operation $g^{\mathbf{A}} \in G^{\mathbf{A}}$ is defined by

$$g^{\mathbf{A}}(xT_1) = (gx)T_1 \qquad (xT_1 \in G/T_1).$$

Define the algebra $\mathbf{C} = \langle G/T_2, G^{\mathbf{C}} \rangle$ similarly.

Examples

in which $\mathbf{A} \sim \mathbf{B}$ and $\operatorname{Con} \mathbf{A} \ncong \operatorname{Con} \mathbf{B}$

The algebra $\mathbf{B}$ will have universe $B = G/D$, but we define the action of G on B with a twist.

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The algebra $\mathbf{B}$ will have universe $B = G/D$, but we define the action of G on B with a twist.

For each $g = (g_1, g_2) \in G$, for each $(x_1, x_2)D \in G/D$, define

$$g^{\mathbf{B}}((x_1, x_2)D) = (g_2 x_1, g_1 x_2)D.$$

Let $\mathbf{B} = \langle G/D, G^{\mathbf{B}} \rangle$, where $G^{\mathbf{B}} = \{g^{\mathbf{B}} \mid g \in G\}$.

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Let $\mathbf{B} = \langle G/D, G^{\mathbf{B}} \rangle$, where $G^{\mathbf{B}} = \{g^{\mathbf{B}} \mid g \in G\}$.

Consider the binary relation $\varphi \subseteq (A \times C) \times (B \times C)$ that associates to each ordered pair

$$((x_1, x_2)T_1, (y_1, y_2)T_2) \in A \times C$$

the pair

$$((x_2, y_1)D, (y_1, y_2)T_2) \in B \times C$$

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It is easy to verify that this relation is a function, and in fact

$$\varphi \colon \mathbf{A} \times \mathbf{C} \to \mathbf{B} \times \mathbf{C} \text{ is an isomorphism.}$$

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Since φ leaves second coordinates fixed, $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$.

CONCLUSION

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$$\operatorname{Con}\mathbf{B} \cong \operatorname{NSub}(S) \leqslant \operatorname{Sub}(S) \cong \operatorname{Con}\mathbf{A}$$

So, if S is any *non-Dedekind* group, $\operatorname{Con}\mathbf{B} \ncong \operatorname{Con}\mathbf{A}$.

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So, if S is any *non-Dedekind* group, $\operatorname{Con}\mathbf{B} \ncong \operatorname{Con}\mathbf{A}$.

If S is a nonabelian simple group, then $\operatorname{Con}\mathbf{B} \cong \mathbf{2}$, while $\operatorname{Con}\mathbf{A} \cong \operatorname{Sub}(S)$ can be arbitrarily large.

Answer

- For groups $H \leqslant G$, the algebra $\mathbf{A} = \langle G \backslash H, \bar{G} \rangle$ has
  - universe: the right cosets $H \backslash G = \{Hx \mid x \in G\}$
  - operations: $\bar{G} = \{g^{\mathbf{A}} : g \in G\}$, where $g^{\mathbf{A}}(Hx) = Hxg$.

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- A standard result is $\operatorname{Con} \mathbf{A} \cong [\![H, G]\!]$.

  The isomorphism $[\![H, G]\!] \ni K \mapsto \theta_K \in \operatorname{Con} \mathbf{A}$ is given by

$$\theta_K = \{(Hx, Hy) \mid xy^{-1} \in K\}.$$

  The inverse isomorphism $\operatorname{Con} \mathbf{A} \ni \theta \mapsto K_\theta \in [\![H, G]\!]$ is

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- So every lattice property of congruence lattices is also a lattice property of (intervals of) subgroup lattices. Moreover, it's easy to prove:

LEMMA

In $\mathrm{Con}\langle G\backslash H, \bar{G}\rangle$, two congruences θ_{K_1} and θ_{K_2} permute if and only if the corresponding subgroups K_1 and K_2 permute.