Congruence Lattices of Finite Algebras

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Ph.D. Dissertation Defense

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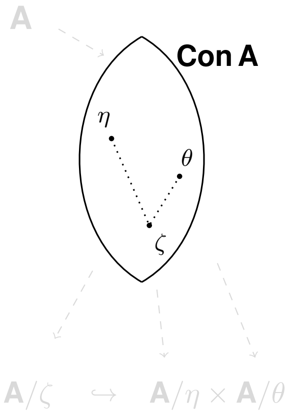
Congruence Decompositions

Congruence relations reveal structural characteristics.



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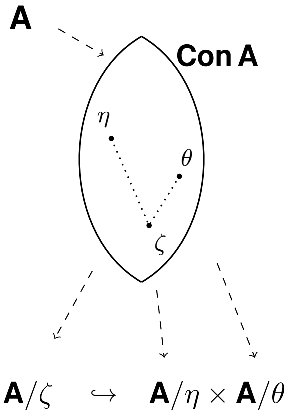

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The Finite Lattice Representation Problem

There is essentially no restriction on the shape of a congruence lattice of an arbitrary algebra.

Theorem (Grätzer-Schmidt, 1963)

Every algebraic lattice is isomorphic to the congruence lattice of an algebra.

What if the algebra is finite?

Problem: Given a finite lattice $\mathbf{L}$, does there exist a *finite* algebra $\mathbf{A}$ such that $\mathbf{Con\,A} \cong \mathbf{L}$?

Definition

We call a finite lattice ***representable*** if it is isomorphic to the congruence lattice of a finite algebra.

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Some important classes of finite lattices

- $\mathscr{L}_0 =$ all finite lattices
- $\mathscr{L}_1 =$ lattices isomorphic to sublattices of finite partition lattices
- $\mathscr{L}_2 =$ ...strong congruence lattices of finite partial algebras
- $\mathscr{L}_3 =$ ...congruence lattices of finite algebras
- $\mathscr{L}_4 =$ ...intervals in subgroup lattices of finite groups
- $\mathscr{L}_5 =$ ...subgroup lattices of finite groups

- Clearly $\mathscr{L}_0 \supseteq \mathscr{L}_1 \supseteq \mathscr{L}_2 \supseteq \mathscr{L}_3 \supseteq \mathscr{L}_4 \supseteq \mathscr{L}_5$.
- Does equality hold in each case?
- $\mathscr{L}_0 = \mathscr{L}_1$
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RECAP

THEOREM (PUDLÁK AND TŮMA, 1980)

Every finite lattice can be embedded in $\mathrm{Eq}(X)$ *with* X *finite.*

In other words, $\mathscr{L}_0 = \mathscr{L}_1$.

THEOREM (PÁLFY AND PUDLÁK, 1980)

The following statements are equivalent:

(I) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*

(II) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

In other words, $\mathscr{L}_0 = \mathscr{L}_3$ if and only if $\mathscr{L}_0 = \mathscr{L}_4$.

How to find a representation of a finite lattice

Method 1 (use closure properties)

The class $\mathscr{L}_3$ is closed under the following operations:

- lattice duals (Kurzweil and Netter, 1986)
- interval sublattices (follows from Kurzweil-Netter)
- direct products (Tůma, 1986)
- ordinal sums (McKenzie, 1984; Snow, 2000)
- parallel sums (Snow, 2000)
- certain sublattices of lattices in $\mathscr{L}_3$ (Snow, 2000)
  (namely, those obtained as a union of a filter and ideal)

How to find a representation of a finite lattice

Method 2 (use a Galois correspondence)

- Fix $\theta \subseteq X \times X$, $\quad f : X^n \to X$.

  Say that f ***respects*** θ and write $f(\theta) \subseteq \theta$ provided

  $$(x_i, y_i) \in \theta \;\Rightarrow\; (f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \theta.$$

- For $L \subseteq \mathrm{Eq}(X)$ define

  $$\lambda(L) = \{f \in X^X \mid (\forall \theta \in L)\; f(\theta) \subseteq \theta\},$$

  the set of unary maps on X which respect all relations in L.

- For $F \subseteq X^X$ define

  $$\rho(F) = \{\theta \in \mathrm{Eq}(X) \mid (\forall f \in F)\; f(\theta) \subseteq \theta\},$$

  the set of equivalence relations on X respected by all $f \in F$.

- Then $L \subseteq \rho\lambda(L)$ and $\rho\lambda$ is a *closure operator* on $\mathrm{Sub}[\mathrm{Eq}(X)]$.
  (idempotent, extensive, order preserving)

- If a lattice $L \leqslant \mathrm{Eq}(X)$ is *closed*, i.e. $\rho\lambda(L) = L$, then

  $$L = \mathrm{Con}\,\langle X, \lambda(L)\rangle$$

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METHOD 3 (SUBGROUP LATTICE INTERVAL)

Find L as an interval in a subgroup lattice of a finite group.

If $H \leqslant G$ are finite groups, then the interval above H in $Sub(G)$,

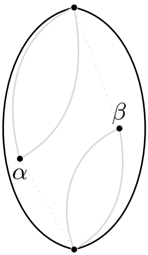
$$[H, G] := \{K \mid H \leqslant K \leqslant G\},$$

is isomorphic to $\mathrm{Con}\,\langle G/H, G\rangle$.

METHOD 4 (FILTER+IDEAL)

Find L as the union of a filter and ideal in a representable lattice.

$L \leqslant L_0$



Suppose $L_0 \cong \mathrm{Con}\,\langle A, F\rangle$, $\alpha, \beta \in L_0 \setminus \{0, 1\}$.

Consider $L = \alpha^{\uparrow} \cup \beta^{\downarrow}$.

Then there exists a set $F' \subset A^A$ such that

$$L \cong \mathrm{Con}\,\langle A, F \cup F'\rangle.$$

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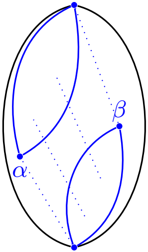

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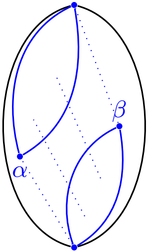
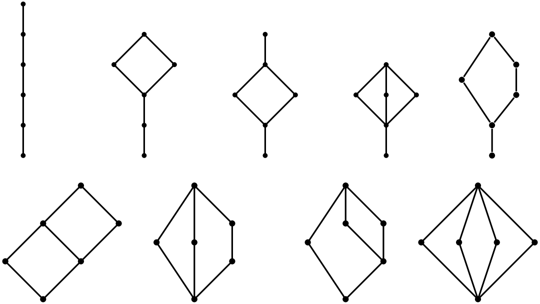
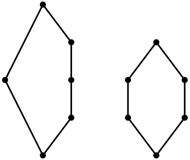


Lattices with at most 6 elements are representable.



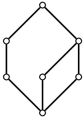
Watatani (1996) *J. Funct. Anal.*



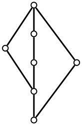
Aschbacher (2008) *JAMS*

Theorem: *Every lattice with at most 6 elements is an interval in the subgroup lattice of a finite group.*

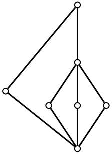
Are all lattices with at most 7 elements representable?



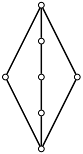
L_{19}



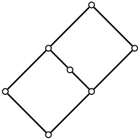
L_{20}



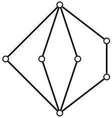
L_{17}



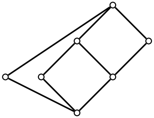
L_{13}



L_{11}



L_9



L_7

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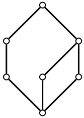
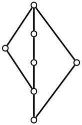
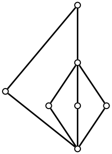


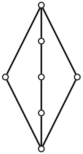
L_{19} ✓



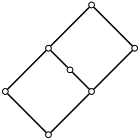
L_{20} ✓



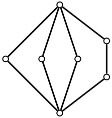
L_{17}



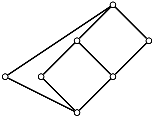
L_{13}



L_{11}



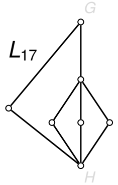
L_9



L_7

Finding representations…

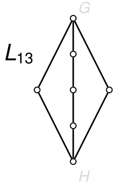
…as intervals in subgroup lattices



SmallGroup(288,1025)

$|G : H| = 48$

- The group $G = (A_4 \times A_4) \rtimes C_2$ has a subgroup $H \cong S_3$ such that $[H, G] \cong L_{17}$.
- …so the dual L_{16} is also representable.



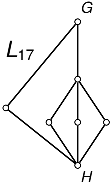
SmallGroup(960,11358)

$|G : H| = 80$

- The group $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ has a subgroup $H \cong A_4$ such that $[H, G] \cong L_{13}$.

FINDING REPRESENTATIONS…

…AS INTERVALS IN SUBGROUP LATTICES



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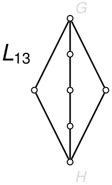

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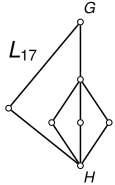

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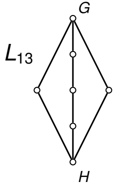

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Finding representations...

...using subgroup lattice intervals and the filter+ideal lemma.

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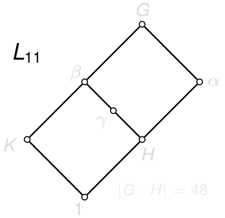

- Let $G = (A_4 \times A_4) \rtimes C_2$.
- G has a subgroup $H \cong C_6$ with $[H, G] \cong N_5$.
- Let $[H, G] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.
- $\mathrm{Sub}(G)$ is a congruence lattice, so if there exists a subgroup $K \succ 1$, below β and not below γ, then

$$L_{11} \cong K^{\downarrow} \cup H^{\uparrow}.$$

- $\mathrm{Sub}(A_4)$ is a congruence lattice (of A_4 acting regularly on itself).
- Therefore,

$$L_{17} \cong V_4^{\downarrow} \cup P^{\uparrow}$$

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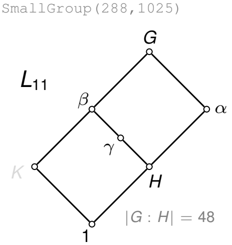

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FINDING REPRESENTATIONS...

...USING SUBGROUP LATTICE INTERVALS AND THE FILTER+IDEAL LEMMA.

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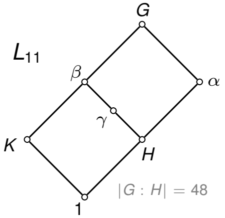

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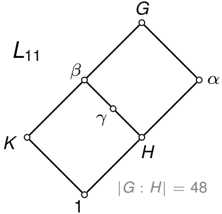
L_{11}



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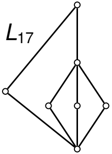

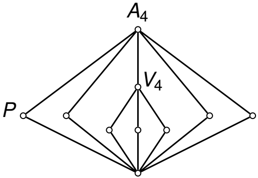

- $\mathrm{Sub}(A_4)$ is a congruence lattice (of A_4 acting regularly on itself).
- Therefore,

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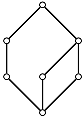
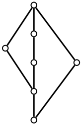
is a congruence lattice.

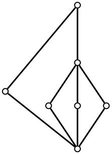
ARE ALL LATTICES WITH AT MOST 7 ELEMENTS REPRESENTABLE?



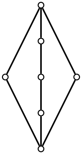
L_{19}



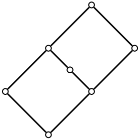
L_{20}



L_{17}



L_{13}



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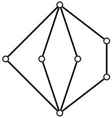
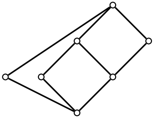
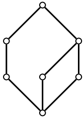


L_9

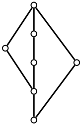


L_7

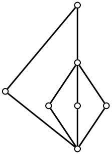
Are all lattices with at most 7 elements representable?



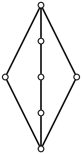
L_{19} ✓



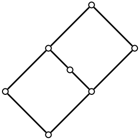
L_{20} ✓



L_{17} ✓



L_{13} ✓



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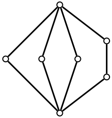

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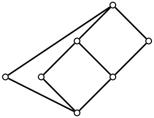
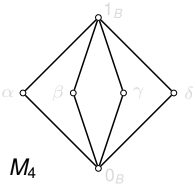
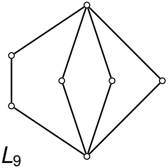


L_7

CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\mathrm{Con}\,\mathbf{A} \cong L_9$.



M_4



L_9

STEP 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\mathrm{Con}\,\mathbf{B} \cong M_4$.

Example:

- Let $B = \{0, 1, \dots, 5\}$ index the elements of S_3 and consider the right regular action of S_3 on itself.
- $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- $\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle \cong M_4$ with congruences

$\alpha = |012|345|,\ \beta = |03|14|25|,\ \gamma = |04|15|23|,\ \delta = |05|13|24|.$

Goal: *expand* $\mathbf{B}$ *to an algebra* $\mathbf{A}$ *that has* α *"doubled" in* $\mathrm{Con}\,\mathbf{A}$.

STEP 2 Since $\alpha = \mathrm{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

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STEP 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0 e_0$, and $g_1 e_0$.

Contruction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.

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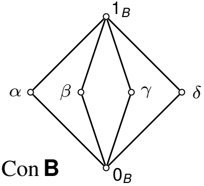

$\operatorname{Con}\mathbf{B}$

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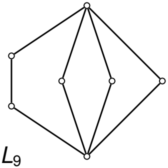

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CONTRUCTION OF AN ALGEBRA **A** WITH Con **A** $\cong L_9$.

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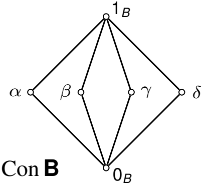

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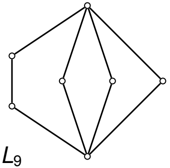

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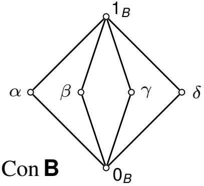

$\mathrm{Con}\,\mathbf{B}$

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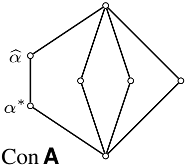

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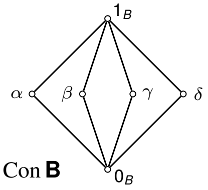

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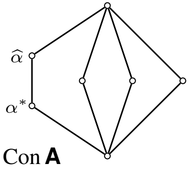

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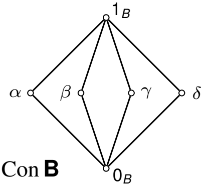
Contruction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.



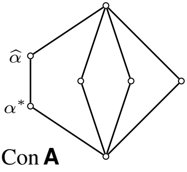
Step 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\operatorname{Con}\mathbf{B} \cong M_4$.

Example:

- Let $B = \{0, 1, \dots, 5\}$ index the elements of S_3 and consider the right regular action of S_3 on itself.
- $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- $\operatorname{Con}\langle B, \{g_0, g_1\}\rangle \cong M_4$ with congruences

$\alpha = |012|345|$, $\beta = |03|14|25|, \gamma = |04|15|23|$, $\delta = |05|13|24|$.

Goal: *expand* $\mathbf{B}$ *to an algebra* $\mathbf{A}$ *that has* α *"doubled" in* $\operatorname{Con}\mathbf{A}$.

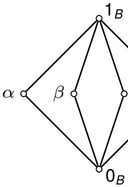


Step 2 Since $\alpha = \operatorname{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

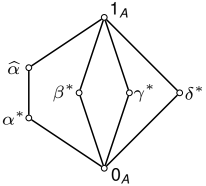
$$B_0 = \{0, 1, 2, 3, 4, 5\} = B$$
$$B_1 = \{0, 6, 7, 8, 9, 10\}$$
$$B_2 = \{11, 12, 2, 13, 14, 15\}.$$

Step 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0 e_0$, and $g_1 e_0$.

CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\mathrm{Con}\,\mathbf{A} \cong L_9$.



$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$



$\mathrm{Con}\,\langle A, F_A\rangle$

$$\alpha = |0,1,2|3,4,5|$$

$$\beta = |0,3|1,4|2,5|$$

$$\gamma = |0,4|1,5|2,3|$$

$$\delta = |0,5|1,3|2,4|$$

$$\widehat{\alpha} = |0,1,2,6,7,11,12|3,4,5|8,9,10,13,14,15|$$

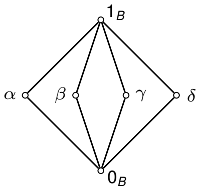
$$\alpha^* = |0,1,2,6,7,11,12|3,4,5|8,9,10|13,14,15|$$

$$\beta^* = |0,3,8|1,4|2,5,15|6,9|7,10|11,13|12,14|$$

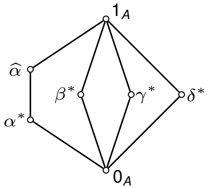
$$\gamma^* = |0,4,9|1,5|2,3,13|6,10|7,8|11,14|12,15|$$

$$\delta^* = |0,5,10|1,3|2,4,14|6,8|7,9,11,15|12,13|$$

CONTRUCTION OF AN ALGEBRA **A** WITH Con **A** $\cong L_9$.



Con $\langle B, \{g_0, g_1\}\rangle$



Con $\langle A, F_A\rangle$

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$$\delta = |0, 5|1, 3|2, 4|$$

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$

$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$

$$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$$

$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$

$$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$$

$$\alpha = \alpha^* \cap B^2 = \widehat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \ldots$$

Why does it work?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

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$$\delta = |0,5|1,3|2,4|$$

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \end{array}$$

B_0

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

$$e_0 \colon A \twoheadrightarrow B_0$$
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$$e_2 \colon A \twoheadrightarrow B_2$$
$$s \colon A \twoheadrightarrow B_0$$

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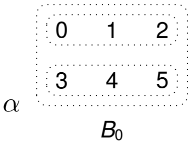

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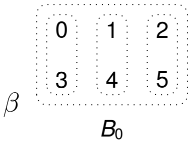

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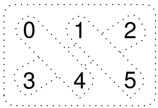
γ 

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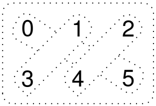
δ 

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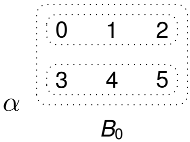

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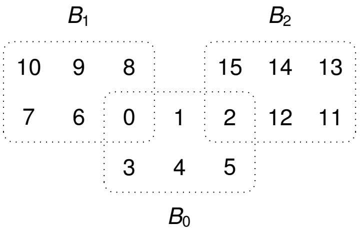

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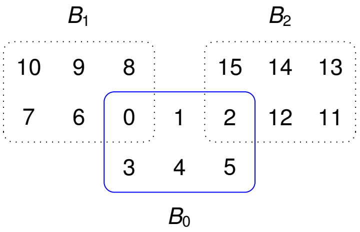

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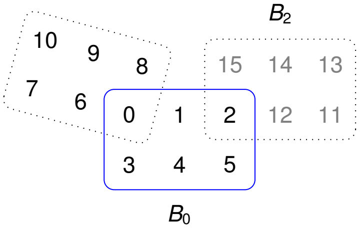

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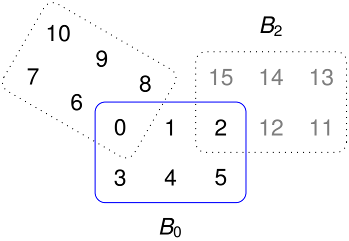

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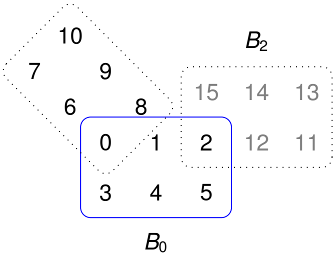

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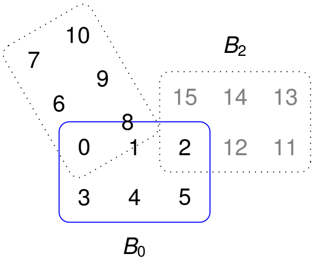

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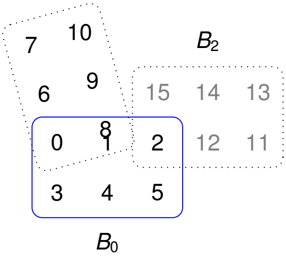

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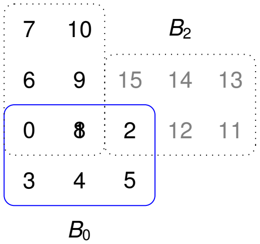

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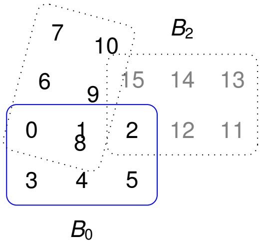

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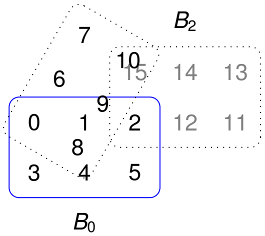

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$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

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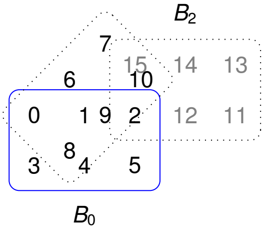

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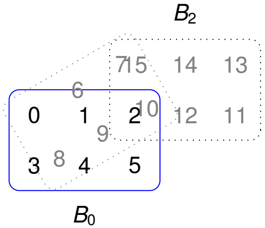

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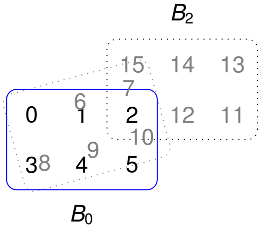

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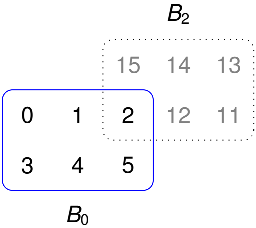

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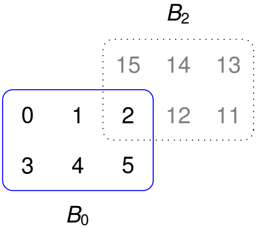

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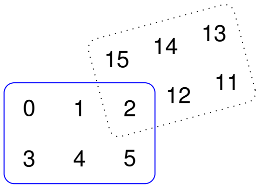

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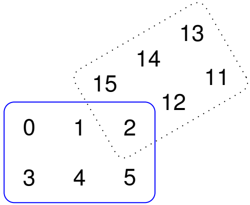

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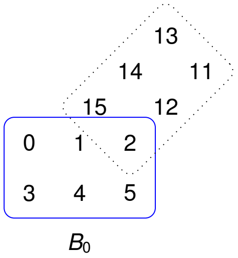

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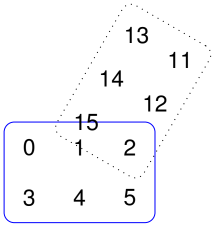

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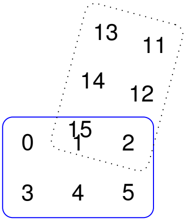

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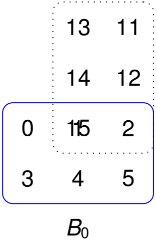

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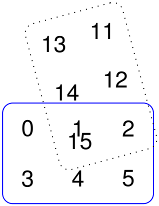

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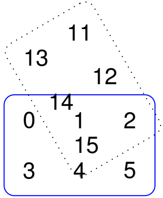

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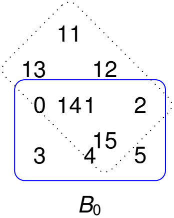

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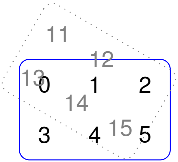

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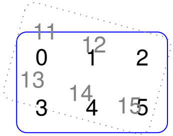

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 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
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\end{array}$$

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Why does it work?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

$$\alpha = |0,1,2|3,4,5|$$

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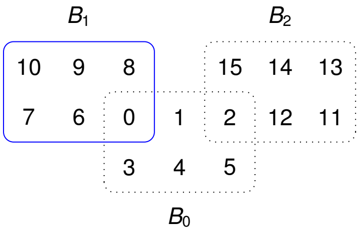

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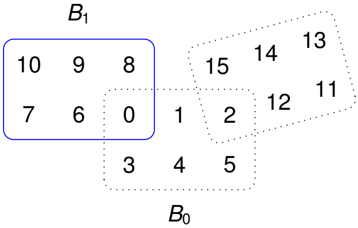

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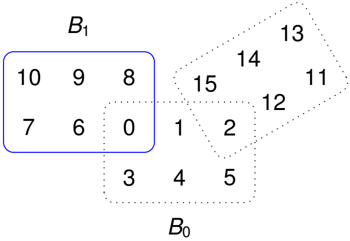

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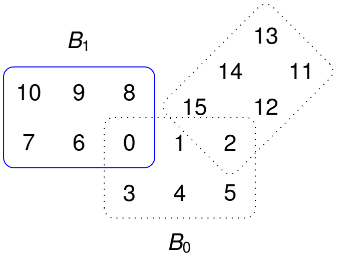

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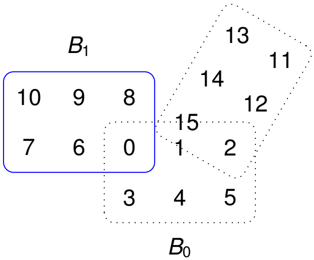

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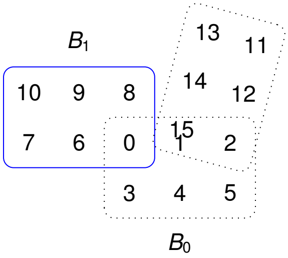

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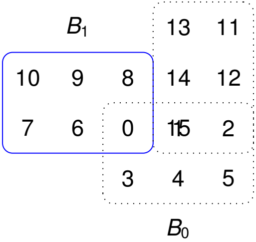

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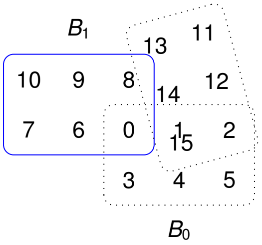

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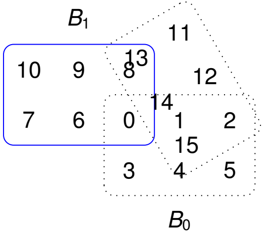

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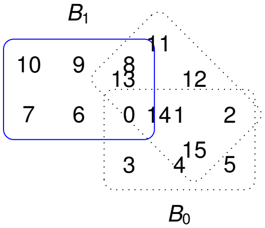

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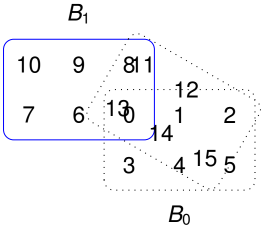

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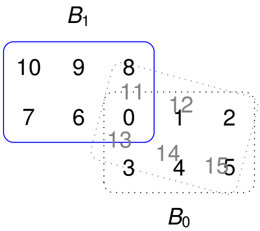

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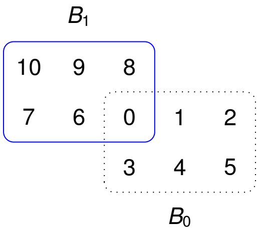

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$$\begin{array}{ccccc}
 & B_1 & & & \\
10 & 9 & 8 & & \\
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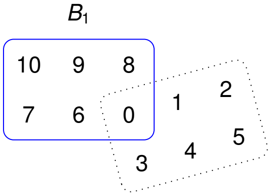

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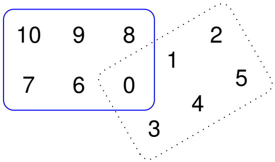

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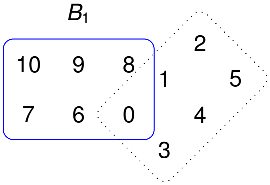

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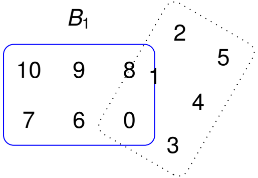

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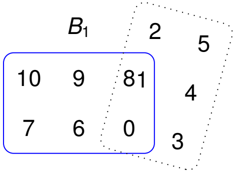

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$e_1 \colon A \twoheadrightarrow B_1$

$e_2 \colon A \twoheadrightarrow B_2$

$s \colon A \twoheadrightarrow B_0$

$$B_1 = \{\ 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10\ \}$$

$$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$$

$$B_2 = \{\ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15\ \}$$

$$ge_0 \colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\rightarrow} B_0$$

WHY DOES IT WORK?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

$$\alpha = |0,1,2|3,4,5|$$

$$\beta = |0,3|1,4|2,5|$$

$$\gamma = |0,4|1,5|2,3|$$

$$\delta = |0,5|1,3|2,4|$$

$$\begin{array}{cccc} & B_1 & 2 & 5 \\ 10 & 9 & 8 & 4 \\ 7 & 6 & 0 & 3 \end{array}$$

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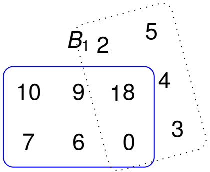

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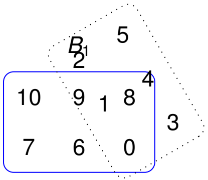

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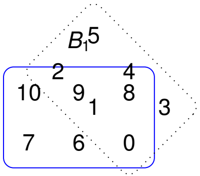

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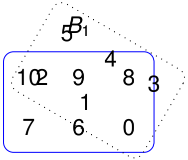

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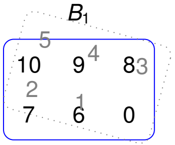

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B_1

$$\boxed{\begin{matrix} 10 & 9 & 8 \\ 7 & 6 & 0 \end{matrix}}$$

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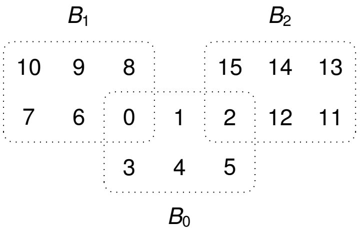

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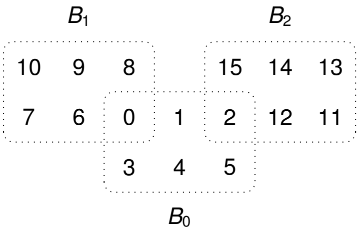

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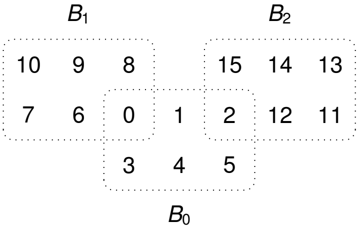

$$\mathrm{Con}\,\langle A, F_A\rangle$$

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$$\alpha^* = |0,1,2,6,7,11,12|3,4,5|8,9,10|13,14,15|$$
$$\beta^* = |0,3,8|1,4|2,5,15|6,9|7,10|11,13|12,14|$$
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Con $\langle B, \{g_0, g_1\}\rangle$

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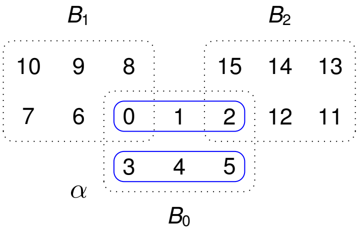

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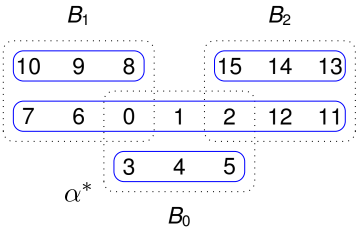

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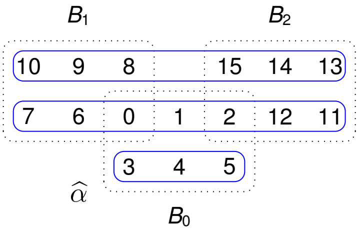

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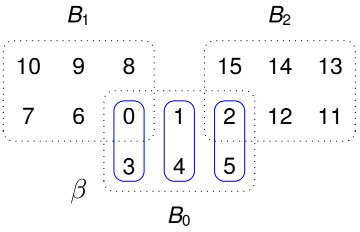

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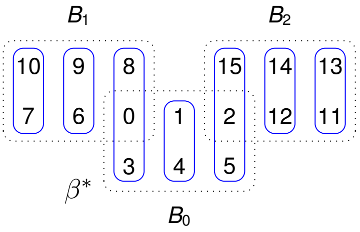

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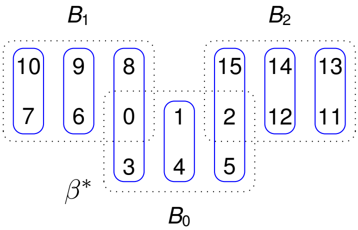

Con $\langle A, F_A\rangle$

Why don't the β classes of B_1 and B_2 mix?

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Variations on the same example...

- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\beta^*, \widehat{\beta}]$.
- Select elements 0 and 3 as intersection points:

$$A = B_0 \cup B_1 \cup B_2 \quad \text{where}$$

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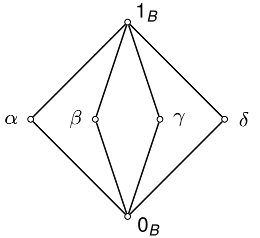

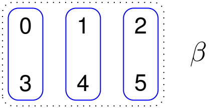

B_0

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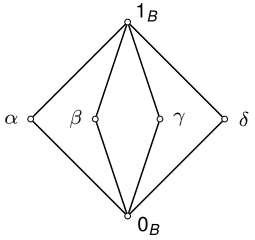

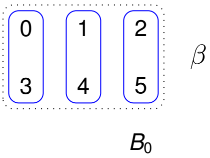

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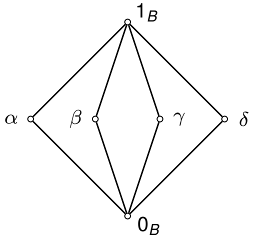

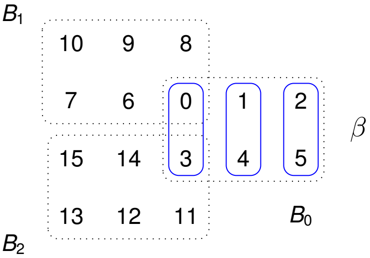

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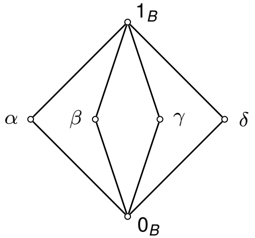

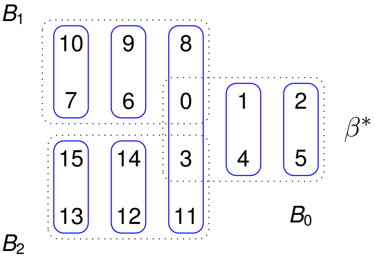

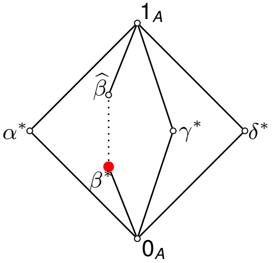

Variations on the same example...

- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\beta^*, \widehat{\beta}]$.
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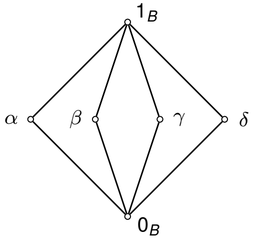

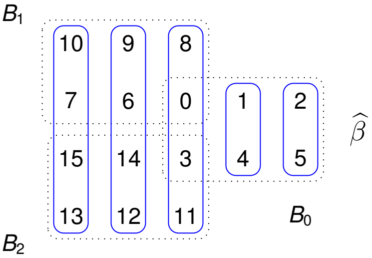

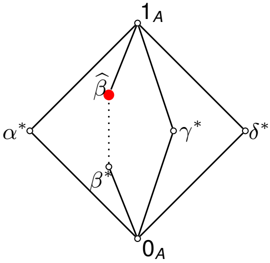

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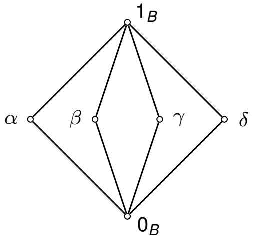

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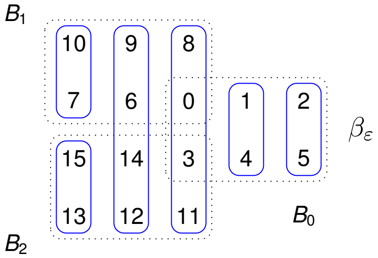

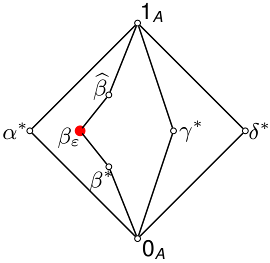

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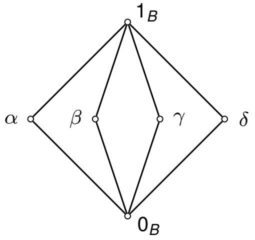

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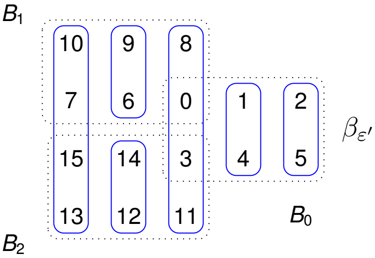

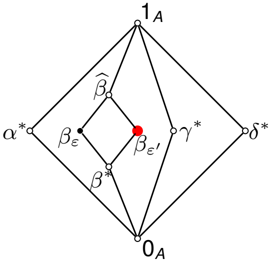

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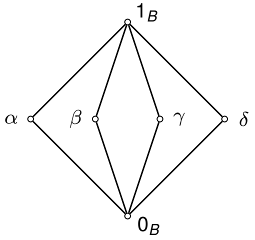

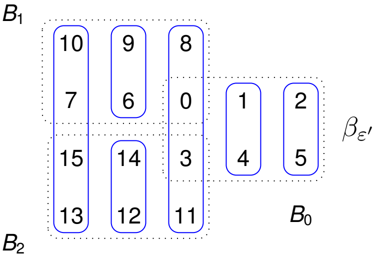

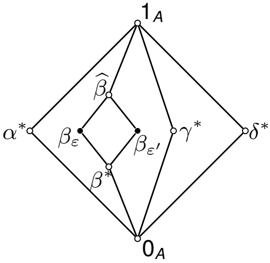

Residuation Lemma

- Define $\widehat{\ }: \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ by

$$\widehat{\beta} = \{(x, y) \in A^2 : (ef(x), ef(y)) \in \beta \text{ for all } f \in \operatorname{Pol}_1(\mathbf{A})\}.$$

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$$^* : \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$$

is the congruence generation operator restricted to $\operatorname{Con} \mathbf{B}$.

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(i) $^* : \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ *is a residuated mapping with residual* $|_B$.

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(iii) *For all* $\alpha \in \operatorname{Con} \mathbf{A}$, $\beta \in \operatorname{Con} \mathbf{B}$,

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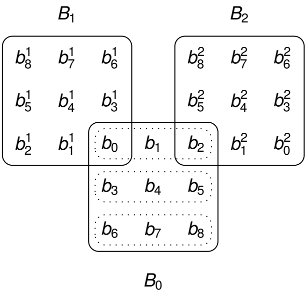
THE STRUCTURE OF THE INTERVAL $[\beta^*, \widehat{\beta}] \leqslant$ **Con A**.

- If $\beta \in \mathrm{Con}\,\mathbf{B}$ is a coatom of **Con B** with m congruence classes then the interval $[\beta^*, \widehat{\beta}]$ in **Con A** is $\mathbf{2}^{m-1}$.



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$$T_r = \bigcup_{k=1}^{K} B_k \cap b_{\beta(r)}/\beta.$$

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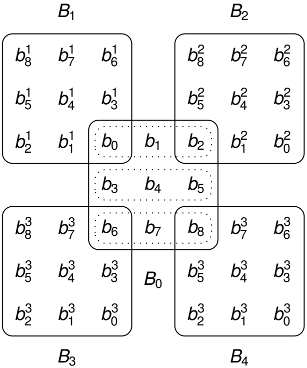

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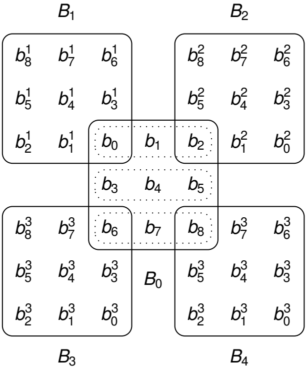

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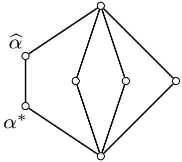
Then $\quad [\beta^*, \widehat{\beta}] = \{\theta \in \mathrm{Eq}(A) : \beta^* \subseteq \theta \subseteq \widehat{\beta}\} \cong \prod_{r=1}^{m} (\mathrm{Eq}|T_r|)^{m-1}.$

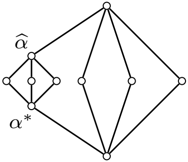
Slightly more general examples…

Returning to our original example, the base algebra $\mathbf{B}$ is the right regular S_3-set, and the nontrivial relations in $\operatorname{Con}\mathbf{B}$ are

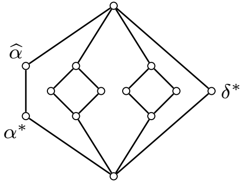
$$\alpha = |0,1,2|3,4,5| \quad \beta = |0,3|1,4|2,5| \quad \gamma = |0,4|1,5|2,3| \quad \delta = |0,5|1,3|2,4|$$



$T = \{0,1\}$



$T = \{0,1,2\}$



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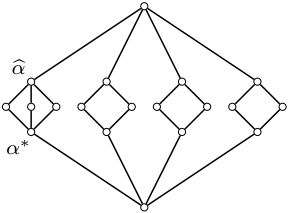

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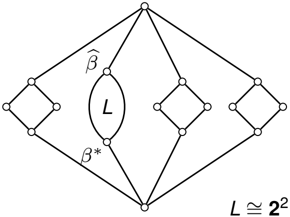

$L \cong \mathbf{2}^2 \times \mathbf{2}^2$

$T = \{0,2,3,5\}$

LIMITATIONS

Two limitations of the foregoing construction:

1. The sizes $|T_r|$ of the partition lattice factors in

$$[\beta^*, \widehat{\beta}] \cong \prod_{r=1}^{m} (\mathrm{Eq}|T_r|)^{m-1}$$

   are limited by the size of the blocks of β.

2. If β is not principal, $[\theta^*, \hat{\theta}]$ may be non-trivial for some $\theta \not\geqslant \beta$.

A GENERALIZATION

THEOREM

Let $\mathbf{B} = \langle B, F \rangle$ *be a finite algebra. Suppose*

$$\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \ldots, (a_{K-1}, b_{K-1}))$$

has m blocks and fix $N < \infty$*.*

There exists an overalgebra $\langle A, F_A \rangle$ *such that the interval* $\beta|_B^{-1} \leqslant \mathrm{Con}\,\mathbf{A}$ *is*

$$[\beta^*, \widehat{\beta}] \cong (\mathrm{Eq}(N))^{m-1}.$$

Moreover, we can arrange it so that $\theta^* = \widehat{\theta}$ *for all* $\theta \not\geqslant \beta$ *in* $\mathrm{Con}\,\mathbf{A}$*.*

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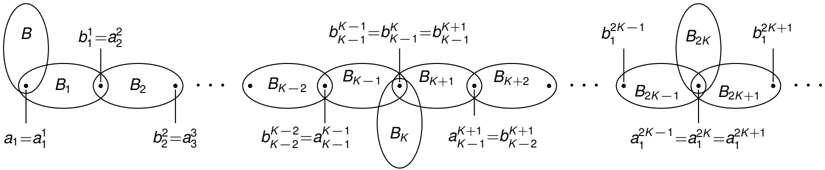
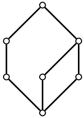
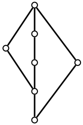


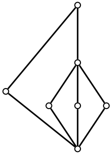
Seven element lattices: summary



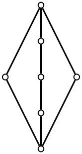
L_{19} ✓



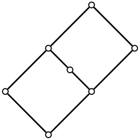
L_{20} ✓



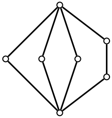
L_{17} ✓



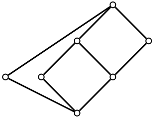
L_{13} ✓



L_{11} ✓

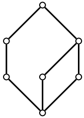


L_9 ✓



L_7

Seven element lattices: summary



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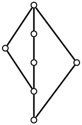
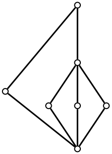
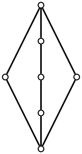


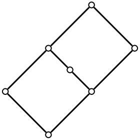
L_{20}



L_{17}



L_{13}



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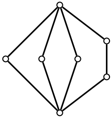

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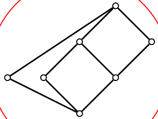


L_{7}

HAS ANYONE SEEN THIS LATTICE?

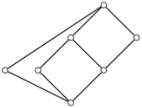
mathoverflow | Questions | Tags | Users | Badges | Unanswered | Ask Question

Given a lattice L with n elements, are there finite groups H < G such that L ≅ the lattice of subgroups between H and G?

13

5

If there is no restriction on n, this is a famous open problem. I'm wondering if any recent work has been done for small $n > 6$. I believe the question is answered (positively) for $n = 6$ by Watatani (1996) MR1409040 and Aschbacher (2008) MR2393428. I also believe we can answer it for $n = 7$, with one possible exception. The exceptional case is shown below.



So my two questions are these:

1) Does anyone know of recent work on this special case of the problem (specifically for $n = 7$ or $n = 8$)?

2) Has anyone found a finite group G with a subgroup H such that the interval

$$[H, G] = \{K : H \le K \le G\}$$

is the lattice shown above?

tagged

finite-groups × 277

open-problem × 195

lattices × 129

universal-algebra × 53

congruences × 6

asked

3 months ago

viewed

401 times

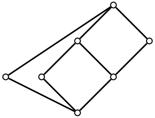
Tip: You can search for questions with arbitrary boolean combinations of tags (like this). See tip 12 for details on how. See more tips and tricks.

MathJax trouble? (Re)process math with jsMath.

Interval sublattice enforceable properties

- L_7 cannot be obtained using the overalgebra construction.
- A minimal representation of L_7 must come from a transitive G-set.
- Suppose $L_7 \cong [H, G]$ for some finite groups $H < G$.

  What can we say about the group G?
- If we prove G must have certain properties, then FLRP has a positive answer iff every finite lattice is an interval in the subgroup lattice of a group *satisfying all of these properties*.



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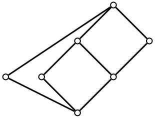

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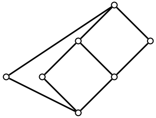

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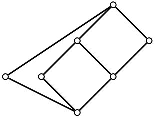

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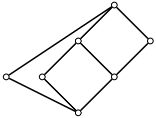

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Suppose $H < G$, $\operatorname{core}_G(H) = 1$, $L_7 \cong [H, G]$.

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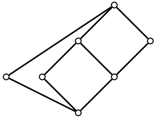

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Subgroup Lattice Basics

Let U and H be subgroups of a finite group.

- By UH we mean the *set* $\{uh \mid u \in U, h \in H\}$.
- $U \vee V = \langle U, H \rangle$ means the group generated by U and H.
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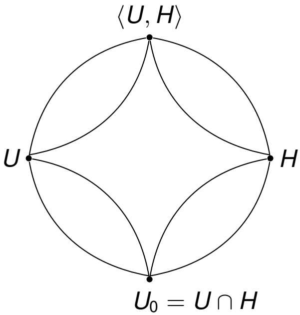

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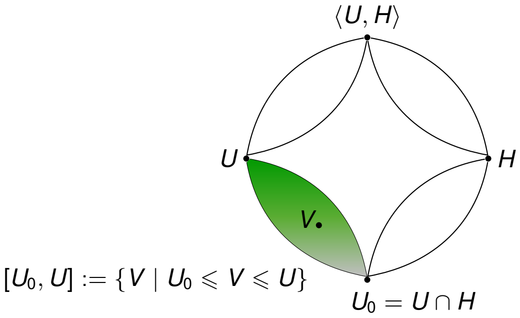

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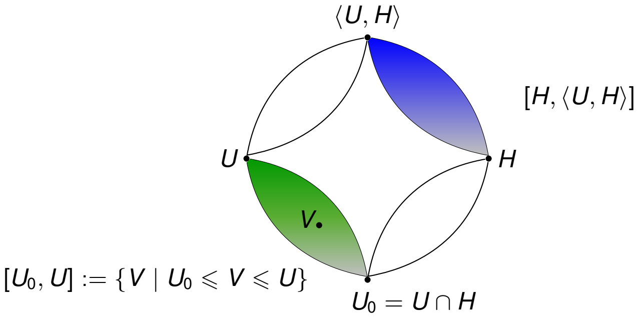

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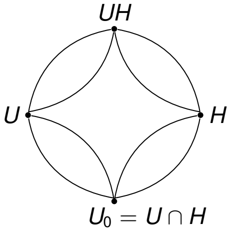

- Assume $H \trianglelefteq \langle U, H \rangle$.

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- Now drop assumption $H \trianglelefteq \langle U, H \rangle$, assume $UH = \langle U, H \rangle$, and define

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- If $U \trianglelefteq UH$, define

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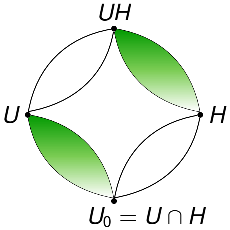

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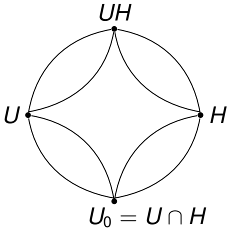

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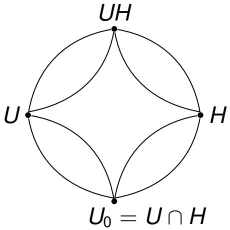

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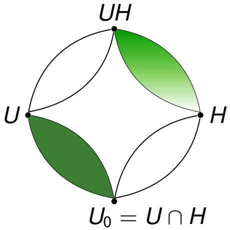

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Lemma

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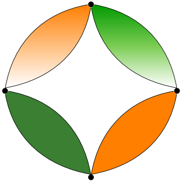

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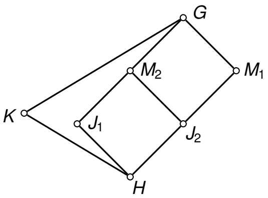

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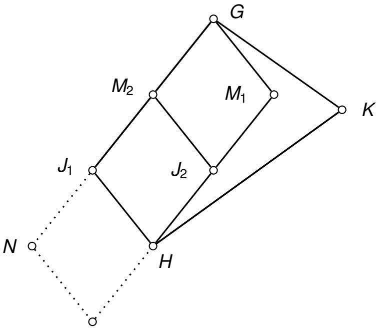

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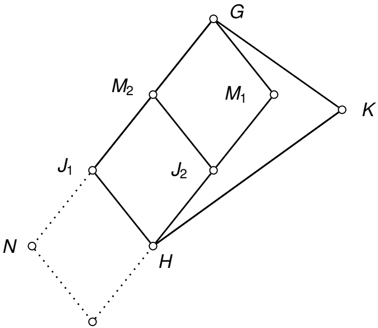

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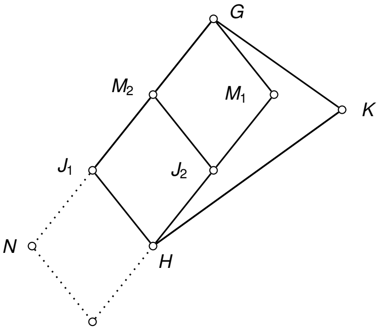

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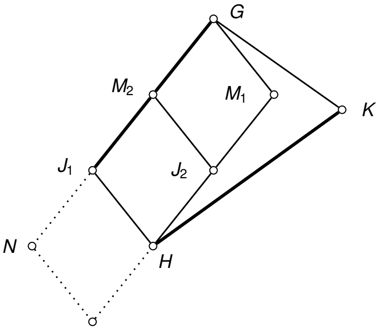

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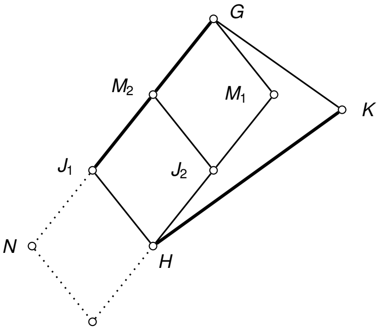

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ACHBACHER-O'NAN-SCOTT THEOREM

Let G be a primitive permutation group of degree d, and let $N := \mathrm{Soc}(G) \cong T^m$ with $m \geqslant 1$. Then one of the following holds.

1. N is regular and
   - ***Affine type*** T is cyclic of order p, so $|N| = p^m$. Then $d = p^m$ and G is permutation isomorphic to a subgroup of the affine general linear group $\mathrm{AGL}(m, p)$.
   - ***Twisted wreath product type*** $m \geqslant 6$, the group T is nonabelian and G is a group of *twisted wreath product type*, with $d = |T|^m$.
2. N is non-regular, non-abelian, and
   - ***Almost simple*** $m = 1$ and $T \leqslant G \leqslant \mathrm{Aut}(T)$.
   - ***Product action*** $m \geqslant 2$ and G is permutation isomorphic to a subgroup of the product action wreath product $P \wr S_{m/l}$ of degree $d = nm/l$. The group P is primitive of type 2.(a) or 2.(c), P has degree n and $\mathrm{Soc}(P) \cong T^l$, where $l \geqslant 1$ divides m.
   - ***Diagonal type*** $m \geqslant 2$ and $T^m \leqslant G \leqslant T^m.(\mathrm{Out}(T) \times S_m)$, with the diagonal action. The degree $d = |T|^{m-1}$.

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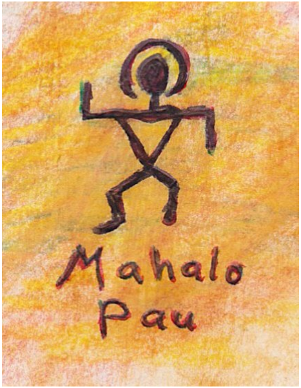
- Future work: Restrict to almost simple groups and then solve the problem using the CFSG Theorem.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