The Algebraic Approach to CSP
and
CSPs of Commutative Idempotent Binars

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joint work with

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slides available at
https://github.com/williamdemeo/Talks

Constraint Satisfaction Problems

Input

- *variables:* $V = \{v_1, v_2, \dots\}$
- *domain:* D
- *constraints:* $C_1, C_2, \dots$

Output

- "yes" if there is a *solution*

  $\sigma : V \to D$ (an assignment of values to variables that satisfies all C_i)
- "no" otherwise

Constraint Satisfaction Problems
Example: 3-SAT

Input

- *variables:* $V = \{v_1, \dots, v_n\}$
- *domain:* $D = \{0, 1\}$
- *constraints:* a formula, say,

$$f(v_1, \dots, v_n) = (v_1 \vee v_2 \vee \neg v_3) \wedge (\neg v_1 \vee v_3 \vee v_4) \wedge \cdots$$

Output

- "yes" if there is a solution: $\sigma : V \to D$ such that

$$f(\sigma v_1, \dots, \sigma v_n) = 1$$

- "no" otherwise

Constraint Satisfaction Problems
Example: NAE-SAT

Input

- *variables:* $V = \{v_1, \dots, v_n\}$
- *domain:* $D = \{0, 1\}$
- *constraints:* $(s_1, C_1), (s_2, C_2), \dots$ of the form

$$s = (i, j, k) \in \{1, \dots, n\}^3 \qquad \text{(scopes)}$$
$$C = \neg(v_i = v_j = v_k)$$

In terms of relational structures...

Let $S := \{(v_i, v_j, v_k) : (i, j, k) \text{ is a scope }\} \subseteq V^3$

$R := \{(0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0)\} \subseteq D^3$

Then a solution σ must satisfy "$\sigma S \subseteq R$"

that is, $(x, y, z) \in S \implies (\sigma x, \sigma y, \sigma z) \in R$

Solutions are homomorphisms!

$$\sigma : \langle V, S\rangle \to \langle D, R\rangle$$

CSP: relational formulation

Let $\mathbb{D} = \langle D, \mathcal{R} \rangle$ be a relational structure.

$\mathrm{CSP}(\mathbb{D})$ (or $\mathrm{CSP}(\mathcal{R})$) is the decision problem with

Input

- A structure $\mathbb{V} = \langle V, \mathcal{C} \rangle$ *similar* to $\mathbb{D}$.

Output

- "yes" if there is a homomorphism $\sigma : \mathbb{V} \to \mathbb{D}$
- "no" otherwise

Alternatively, let $\Rightarrow$ be the binary relation on similar structures:

$$\mathbb{V} \Rightarrow \mathbb{D} \quad \text{iff there is a homomorphism } \sigma : \mathbb{V} \to \mathbb{D}$$

Then the CSP of $\mathbb{D}$ is the membership problem for the set

$$\mathrm{CSP}(\mathbb{D}) := \{\mathbb{V} : \mathbb{V} \Rightarrow \mathbb{D}\}$$

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- A structure $\mathbb{V} = \langle V, \mathcal{C} \rangle$ *similar* to $\mathbb{D}$.

Output

- "yes" if there is a homomorphism $\sigma : \mathbb{V} \to \mathbb{D}$
- "no" otherwise

We call $\mathbb{D}$ (or $\mathcal{R}$) "tractable" if there is a polynomial-time algorithm for solving $\mathrm{CSP}(\mathbb{D})$ (or $\mathrm{CSP}(\mathcal{R})$).

CSP: algebraic formulation

Let $\mathbb{D} = \langle D, \mathcal{R} \rangle$ be a relational structure.

For $R \subseteq \mathcal{R}$ define the *polymorphisms* of R,

$$\mathsf{pol}(R) := \{f : D^k \to D \mid f(\rho) \subseteq \rho \text{ for every } \rho \in R\}$$

that is, $f \in \mathsf{pol}(R)$ iff for every $\rho \in R$

$$\begin{array}{rcc} (a_1, b_1, \dots, z_1) & \in & \rho \\ & \vdots & \\ (a_k, b_k, \dots, z_k) & \in & \rho \\ \hline (f(a_1, \dots, a_k), \dots, f(z_1, \dots, z_k)) & \in & \rho \end{array}$$

CSP: algebraic formulation

Let $\mathbb{D} = \langle D, \mathcal{R} \rangle$ be a relational structure.

For $R \subseteq \mathcal{R}$ define the *polymorphisms* of R,

$$\mathsf{pol}(R) := \{f : D^k \to D \mid f(\rho) \subseteq \rho \text{ for every } \rho \in R\}$$

Define the algebra $\mathbf{D} := \langle D, \mathsf{pol}(R) \rangle$.

We call $\mathbf{D}$ "tractable" if the corresponding structure $\langle D, R \rangle$ is tractable.

CSP: ALGEBRAIC FORMULATION

For F a set of operations on D, define the *relational clone* of F,

$$\text{rel}(F) := \{\rho \subseteq D^n \mid f(\rho) \subseteq \rho \text{ for every } f \in F\}$$

Let $\bar{R} := \text{rel}(\text{pol}(R))$ be the "closure" of R.

Then, $\mathrm{CSP}\langle D, R\rangle$ is *poly-time reducible* to $\mathrm{CSP}\langle D, \bar{R}\rangle$. In fact,

THEOREM

$\mathrm{CSP}\langle D, R\rangle \equiv_P \mathrm{CSP}\langle D, \bar{R}\rangle$

Corollary $\quad \text{pol}(R) = \text{pol}(S) \quad \Longrightarrow \quad \mathrm{CSP}(R) \equiv_P \mathrm{CSP}(S)$

The algebra $\langle D, \text{pol}(R)\rangle$ determines the complexity of the corresponding CSP!

GENERAL PROBLEM

Find properties (of algebras) that characterize the complexity of CSPs.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$\mathrm{CSP}(\mathbf{A})$ is tractable $\iff$ $\mathbf{A}$ has a weak-nu term operation

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The left-to-right direction is known.

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CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$\mathrm{CSP}(\mathbf{A})$ is tractable $\Longleftarrow$ $\mathbf{A}$ has a weak-nu term operation (?)

The right-to-left direction is open.

General Problem

Find properties (of algebras) that characterize the complexity of CSPs.

CSP dichotomy conjecture

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable} \quad \Longleftarrow \quad \mathbf{A} \text{ has a weak-nu term operation} \qquad (?)$$

A weak near unanimity (weak-nu) term operation is one that satisfies

$$t(x, x, \ldots, x) \approx x \qquad \text{(idempotent)}$$

$$t(y, x, \ldots, x) \approx t(x, y, \ldots, x) \approx \cdots \approx t(x, x, \ldots, y)$$

A *binary* operation $t(x, y)$ is weak-nu if

$$t(x, x) \approx x \qquad \text{(idempotent)}$$

$$t(y, x) \approx t(x, y) \qquad \text{(commutative)}$$

So let's try to prove (?) for commutative idempotent binars.

Commutative Idempotent Binars

A CIB is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.

Question

Is every finite commutative idempotent binar tractable?

First Example: a semilattice is an associative CIB.

Semilattices are tractable.

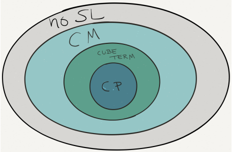
Pause to consider more general case for a minute...

General case

Some well known facts

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

$$\begin{aligned} \mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has Malcev term} \\ &\implies \mathbf{A} \text{ has cube term} \\ &\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\ &\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A}) \end{aligned}$$



First Reduction

by Cube-Term Blockers

Marković, M. Maróti, McKenzie (M^4)
"Finitely related clones and algebras with cube terms" (2012)

A cube-term blocker (CTB) is a pair (C, B) of subuniverses satisfying $\emptyset < C < B \leqslant A$ and for every $t(x_1, \ldots, x_n)$ there is an index $i \in [n]$ with

$$(\forall (b_1, \ldots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \ldots, b_n) \in C)$$

$$t(b_1, \ldots, b_{i-1}, c, b_{i+1}, \ldots, b_n) \in C$$

M^4 prove a finite idempotent algebra has a cube term iff it has no CTB.

Lemma

A finite CIB $\mathbf{A}$ has a CTB if and only if $\mathbf{S}_2 \in \mathrm{HS}(\mathbf{A})$.

Proof.

(C, B) a CTB implies $\theta = C^2 \cup (B - C)^2$ a congruence with $\mathbf{B}/\theta \cong \mathbf{S}_2$.

Conversely, suppose $\mathbf{S}_2 \in \mathrm{HS}(\mathbf{A})$, and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ with $\mathbf{B}/\theta$ a meet-SL for some θ. Let C/θ be the bottom of $\mathbf{B}/\theta$, then (C, B) is a CTB. $\square$

Second Reduction

Kearnes and Tschantz
"Automorphism groups of squares and of free algebras" (2007)

Lemma

If V is an idempotent variety that is not congruence permutable, then there are subuniverses U and W of $\mathbf{F} := \mathbf{F}_V\{x, y\}$ satisfying

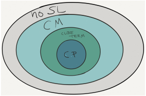
1. $x \in U \cap W$
2. $y \in U^c \cap W^c$
3. $(U \times F) \cup (F \times W) \leqslant \mathbf{F}^2$

For CIB's, either U or W will be an ideal.

This implies a CTB and a semilattice.

$\mathbf{A}$ = a finite CIB $\qquad$ $\mathbf{S}_2$ = the 2-elt semilattice.

$$
\begin{aligned}
\mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has a Malcev term}\\
&\implies \mathbf{A} \text{ has a cube term}\\
&\implies \mathrm{V}(\mathbf{A}) \text{ is CM}\\
&\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A})
\end{aligned}
$$



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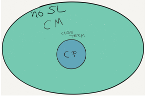

- 1st reduction by cube-term blockers.

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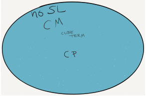


- 1st reduction by cube-term blockers.
- 2nd reduction by Kearnes-Tschantz.

Remaining Questions for Finite CIBs

Conclusion

Let $\mathbf{A}$ be a finite CIB. Then

$$\mathbf{S}_2 \notin \mathsf{HS}(\mathbf{A}) \text{ if and only if } \mathrm{V}(\mathbf{A}) \text{ is congruence permutable.}$$

(so $\mathrm{CSP}(\mathbf{A})$ tractable in this case)

Open Question

Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in $\mathsf{HS}(\mathbf{A})$. Is $\mathrm{CSP}(\mathbf{A})$ tractable?

Recall, if $\mathrm{V}(\mathbf{A})$ is $\mathrm{SD}_\wedge$, then $\mathrm{CSP}(\mathbf{A})$ is tractable.

Revised Question

Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in $\mathsf{HS}(\mathbf{A})$, and $\mathrm{V}(\mathbf{A})$ not $\mathrm{SD}_\wedge$.

Is $\mathrm{CSP}(\mathbf{A})$ tractable?

Example 1

$\cdot$	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

$*$	0	1	2	3
0	0	0	0	0
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Cliff's trick: replace binary operation with a term from clo$(\mathbf{A})$, say

$$x * y = (x \cdot (x \cdot y)) \cdot (y \cdot (x \cdot y))$$

If $\langle A, *\rangle$ tractable, then so is $\mathbf{A} = \langle A, \cdot\rangle$.

$$\{*\} \subseteq \mathrm{clo}(\mathbf{A}) \implies \mathrm{rel}(\mathrm{clo}(\mathbf{A})) \subseteq \mathrm{rel}(\{*\})$$
$$\implies \mathrm{CSP}(\mathbf{A}) \leqslant_P \mathrm{CSP}\langle A, *\rangle$$

$\langle A, *\rangle$ tractable $\implies$ $\mathbf{A}$ tractable

Example 2

$\cdot$	0	1	2	3
0	0	0	1	1
1	0	1	3	2
2	1	3	2	1
3	1	2	1	3

t	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

Let $t(x, y) = x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y))$.

$\langle A, t\rangle$ tractable

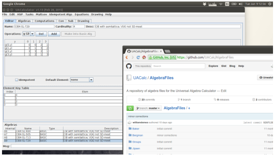
Example 3

$\cdot$	0	1	2	3
0	0	0	2	1
1	0	1	3	2
2	2	3	2	1
3	1	2	1	3

Let $t_2(x, y) = \ldots$?

Let $t_3(x, y, z) = \ldots$?

...and about 25 others.



To see them, load UACalc with files from the Bergman directory at

https://github.com/UACalc/AlgebraFiles

Thank you for listening!