Constraint Satisfaction Problems, Graph Theory, and Universal Algebra

Cliff Bergman

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What is a CSP?

Informally, a **C**onstraint **S**atisfaction **P**roblem consists of

- a list of variables ranging over a finite domain and
- a set of constraints on those variables.

Problem: can we assign values to all the variables so that all of the constraints are satisfied?

Examples

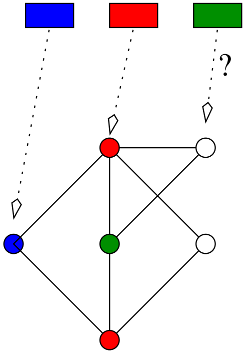
A system of linear equations is a CSP

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Also, a system of nonlinear equations is a CSP

$$
\begin{aligned}
a_{11}x_1^2x_3 + a_{12}x_2x_3x_7 + \cdots + a_{1n}x_4x_n^3 &= b_1 \\
a_{21}x_2x_5 + a_{22}x_2 \qquad + \cdots + a_{2n}x_4^3 \quad &= b_2 \\
&\vdots \\
a_{m1}x_3x_5x_8 + a_{m2}x_2 \qquad + \cdots + a_{mn}x_n \quad &= b_m
\end{aligned}
$$

For a fixed k, determining whether a graph is k-colorable is a CSP



Given a propositional formula $\phi(x_1, \ldots, x_n)$, determine whether ϕ is satisfiable

$$\phi(x, y, z) = (x \vee y \vee z') \wedge (x' \vee y \vee z')$$

then

$$\phi(0, 0, 1) = 1$$

Algorithms

There is an efficient algorithm (Gaussian elimination) for solving any linear system. That is

There is an algorithm that accepts as input a linear system and decides whether that system has a solution.

The running time of the algorithm is bounded above by $f(s)$ where f is a polynomial and s is the size of the system.

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The particular system is an instance of the problem LINEAR SYSTEM

Similarly

There is an algorithm that accepts as input a graph and decides whether the graph is 2-colorable.

The running time is bounded by $f(s)$ where f is a polynomial and s is the size of the graph.

The graph is an instance of the problem 2-COLORABILITY.

There is an algorithm that accepts as input a formula, $\phi = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k$, in which each ϕ_i is bijunctive, and decides whether ϕ is satisfiable.

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We say these algorithms run in polynomial time.

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Thus these problems are solvable in nondeterministic polynomial time.

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It is possible for $X \leq_{\mathrm{p}} Y \leq_{\mathrm{p}} X$. In that case, write $X \equiv_{\mathrm{p}} Y$.

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- $\mathbb{P} \subseteq \mathbb{NP}$
- Both $\mathbb{P}$ and $\mathbb{NP}$ are downsets, i.e.,
  $Y \in \mathbb{P} \;\&\; X \leq_{\mathrm{p}} Y \implies X \in \mathbb{P}$

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The maximal members of $\mathbb{NP}$ are called $\mathbb{NP}$-complete.

3-COLORABILITY, NONLINEAR SYSTEM, and 3-SAT are known to be $\mathbb{NP}$-complete.

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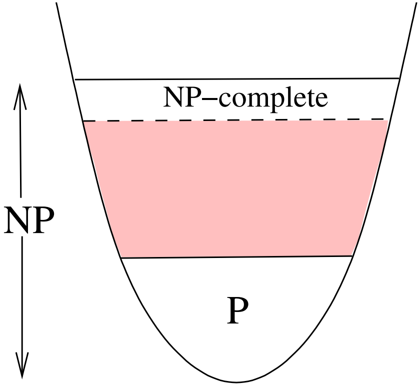
Theorem (Ladner, 1975)

If $\mathbb{P} \neq \mathbb{NP}$ then there are problems in $\mathbb{NP} - \mathbb{P}$ that are not $\mathbb{NP}$-complete.



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Formal Definition of CSP

Let D be a set, n a positive integer
An n-ary relation on D is a subset of D^n

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An *n-ary relation on D* is a subset of D^n

$\mathrm{Rel}_n(D)$ denotes the set of all n-ary relations on D

$$\mathrm{Rel}(D) = \bigcup_{n>0} \mathrm{Rel}_n(D)$$

Let D be a finite set and $\Delta \subseteq \mathrm{Rel}(D)$

CSP($\langle D, \Delta \rangle$) is the problem:
instance: A finite set $V = \{ v_1, \ldots, v_n \}$ of variables and a finite set $\{ C_1, \ldots, C_m \}$ of constraints

Each constraint C_i is a pair $(\langle x_{i1}, \ldots, x_{i\,p_i} \rangle , \delta_i)$ in which $x_{i1}, \ldots, x_{i\,p_i} \in V$ and $\delta_i \in \Delta$

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Question: Does there exist a mapping $f\colon V \to D$ such that for all $i \leq m$, $\langle f(x_{i1}), \ldots, f(x_{i\,p}) \rangle \in \delta_i$?

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CSP($\langle D, \Delta \rangle$) always lies in $\mathbb{NP}$.

Example: Linear Equations over $\mathbb{F}_2$

$D = \{0, 1\}$
Δ consists of all relations

$$\delta^b_{n,\mathbf{a}} = \{\, \langle x_1, \ldots, x_n \rangle \in D^n : a_1 x_1 + \cdots a_n x_n = b \}$$

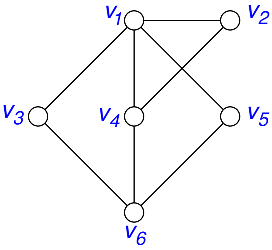
Here, $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle \in D^n$, $b \in D$.

Then CSP($\langle D, \Delta \rangle$) is LINEAR SYSTEM

Example: 3-colorability

$$D = \{r, g, b\}, \qquad \Delta = \{\kappa_3\}$$
$$\kappa_3 = \{\, (x, y) \in D : x \neq y \,\}$$

Then CSP($\langle D, \Delta \rangle$) is the 3-colorability problem



$$V = \{v_1, \ldots, v_6\}$$
$$\langle v_1, v_2 \rangle \in \kappa$$
$$\langle v_1, v_3 \rangle \in \kappa$$
$$\langle v_1, v_4 \rangle \in \kappa$$
$$\langle v_2, v_4 \rangle \in \kappa$$
$$\vdots$$
$$\langle v_5, v_6 \rangle \in \kappa$$

Example: 2-SAT

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For a bijunctive clause $\phi(x, y)$,

$$\delta_\phi = \{ \langle a, b \rangle \in D^2 : \phi(a, b) = 1 \}$$

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x	y	$x \vee y'$
0	0	1
0	1	0
1	0	1
1	1	1

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$\Delta = \{\delta_{x\vee y},\ \delta_{x\vee y'},\ \delta_{x'\vee y},\ \delta_{x'\vee y'}\}$

Example: Horn-SAT

Horn formula:

$(x_0 \wedge x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{n-1}) \rightarrow x_i$

Equivalently:

$x'_0 \vee x'_1 \vee \ldots x'_{i-1} \vee x_i \vee \cdots \vee x'_{n-1}$

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Corresponding relation $\gamma_i^n = \{0, 1\}^n - \{\langle 111 \ldots 0 \ldots 111 \rangle\}$

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Horn-SAT is CSP$\langle D, \Gamma \rangle$
$D = \{0,1\}, \quad \Gamma = \{\, \gamma_i^n : 0 \leq i < n \,\}$

Schaefer's Dichotomy

Theorem (Schaefer, 1978)

Let $D = \{0, 1\}$. There are six families $\Delta_0, \ldots, \Delta_5$ such that

$$\mathrm{CSP}(\langle D, \Delta\rangle) \in \mathbb{P} \iff \Delta \subseteq \Delta_i, \textit{ some } i < 6$$

Otherwise $\mathrm{CSP}(\langle D, \Delta\rangle)$ *is* $\mathbb{NP}$*-complete.*

The six families

$\Delta_0 = \{\, \delta : \langle 0, 0, \ldots, 0 \rangle \in \delta \,\}$ ("All False")
$\Delta_1 = \{\, \delta : \langle 1, 1, \ldots, 1 \rangle \in \delta \,\}$ ("All True")
$\Delta_2 = \{\delta_{x \vee y},\ \delta_{x \vee y'},\ \delta_{x' \vee y},\ \delta_{x' \vee y'}\}$ (bijunctive)
$\Delta_3 = \Gamma$ (Horn)
$\Delta_4 = \Gamma^{\partial}$ (dual-Horn)
Δ_5 (affine, i.e., linear system over $\mathbb{F}_2$)

Two Motivating Questions

1. Dichotomy Conjecture
   Every CSP($\langle D, \Delta \rangle$) either lies in $\mathbb{P}$ or is $\mathbb{NP}$-complete.

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   Characterize those CSPs that lie in $\mathbb{P}$.

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1. Dichotomy Conjecture
   Every CSP($\langle D, \Delta \rangle$) either lies in $\mathbb{P}$ or is $\mathbb{NP}$-complete.
2. Tractability Problem
   Characterize those CSPs that lie in $\mathbb{P}$.

What would a characterization look like? What language could we use?

Why is 2-SAT tractable, but 3-SAT is not?

2-SAT: $\Delta_2 = \{\delta_{x \vee y},\, \delta_{x \vee y'},\, \delta_{x' \vee y},\, \delta_{x' \vee y'}\}$

3-SAT: $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_7\}$

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$$M(x, y, z) = \begin{cases} 0 & \text{if at least 2 of } x, y, z \text{ equal } 0 \\ 1 & \text{otherwise} \end{cases}$$

"Majority Operation"

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"Majority Operation"

M preserves each $\delta \in \Delta_2$:

$$\begin{array}{ccc} \langle a_1, & b_1 \rangle & \in \delta \\ \langle a_2, & b_2 \rangle & \in \delta \\ \langle a_3, & b_3 \rangle & \in \delta \text{ implies} \\ \langle M(a_1, a_2, a_3), & M(b_1, b_2, b_3) \rangle & \in \delta \end{array}$$

But M fails to preserve each $\lambda \in \Lambda$

For example, with $\lambda = \lambda_{x \vee y \vee z'} = \{0, 1\}^3 - \{\langle 001 \rangle\}$

$$\begin{aligned}
&\langle 1, 0, 0 \rangle \in \lambda \\
&\langle 0, 0, 1 \rangle \in \lambda \\
&\langle 0, 1, 1 \rangle \in \lambda \text{ but} \\
&\langle 0, 0, 1 \rangle \notin \lambda
\end{aligned}$$

Polymorphisms

Definition

Let $\delta \in \mathrm{Rel}_k(D)$ and $f\colon D^n \to D$. We say *f* preserves δ if

$$(a_{11}, \ldots, a_{1k}), \ldots, (a_{n1}, \ldots, a_{nk}) \in \delta \implies \\ \big(f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1k}, \ldots, a_{nk})\big) \in \delta$$

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$$\begin{array}{cccccc}
a_{11} & a_{12} & \dots & a_{1k} & \in & \delta \\
a_{21} & a_{22} & \dots & a_{2k} & \in & \delta \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{n1} & a_{n2} & \dots & a_{nk} & \in & \delta \\
\downarrow_f & \downarrow_f & & \downarrow_f & & \\
\star & \star & \dots & \star & \in & \delta
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Important point: $\langle D, \mathrm{Pol}(\Delta)\rangle$ is an algebraic structure

Theorem

Let $\Gamma, \Delta \subseteq \mathrm{Rel}(D)$*. Then*

$$\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Delta) \implies \mathrm{CSP}(\Delta) \leq_{\mathrm{p}} \mathrm{CSP}(\Gamma).$$

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Let $\Gamma, \Delta \subseteq \mathrm{Rel}(D)$. *Then*

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Thus, the richer the algebraic structure, the easier the corresponding CSP

Schaefer proved that on $D = \{0, 1\}$, there are 4 key polymorphisms:

$$M(x, y, z) \text{ (majority)}$$
$$x \wedge y$$
$$x \vee y$$
$$P(x, y, z) = x \oplus y \oplus z = x - y + z$$

$\langle \{0, 1\}, \Delta \rangle$ is tractable iff one of these four is a polymorphism of Δ

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Unfortunately, things become much more complicated when $|D| > 2$.

One can go back and forth between relational and algebraic structures

$$\begin{array}{ccc} \textbf{Relational} & & \textbf{Algebraic} \\ \langle D, \Delta \rangle & \longrightarrow & \langle D, \mathrm{Pol}(\Delta) \rangle \\ \langle D, \mathrm{Inv}(F) \rangle & \longleftarrow & \langle D, F \rangle \end{array}$$

$$\mathrm{CSP}\langle D, \Delta \rangle \equiv_{\mathrm{p}} \mathrm{CSP}\langle D, \mathrm{Inv}(\mathrm{Pol}(\Delta)) \rangle$$

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Perhaps the expressive power of algebra can be used to classify CSPs.

For a set of relations, Δ, on D, $\mathrm{Inv}\big(\mathrm{Pol}(\Delta)\big)$ coincides with the set of relations definable from Δ by a positive primitive formula.

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$$\phi(x_1, \ldots, x_n) =$$
$$(\exists y_1)(\exists y_2)\cdots(\exists y_m)\big(\delta_1(z_{1_1}, \ldots, z_{1_k}) \wedge \ldots \wedge \delta_t(z_{t_1}, \ldots, z_{t_j})\big)$$

Here $\delta_1 \ldots, \delta_t \in \Delta$ and every $z_{i_j} \in \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$

Algebraic Facts

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras

$\mathbf{B}$ a subalgebra of $\mathbf{A} \implies \mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{p}} \mathrm{CSP}(\mathbf{A})$.

$\mathbf{B}$ a homomorphic image of $\mathbf{A} \implies \mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{p}} \mathrm{CSP}(\mathbf{A})$.

$\mathrm{CSP}(\mathbf{A}^n) \equiv_{\mathrm{p}} \mathrm{CSP}(\mathbf{A})$

Theorem (Bulatov, Jeavons, Krokhin, 2000)

If $\langle D, \Delta \rangle$ is a core and every polymorphism is essentially unary, then CSP(Δ) *is* $\mathbb{NP}$*-complete.*

f is essentially unary if $f(x_1, \ldots, x_n) = g(x_j)$ for some unary g and some $j \leq n$.

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Corollary

3-COLORABILITY, NONLINEAR SYSTEM, and 3-SAT are $\mathbb{NP}$*-complete.*

Informal reformulation of the dichotomy conjecture

If $\mathbf{A}$ has some kind of decent algebraic structure then CSP($\mathbf{A}$) $\in \mathbb{P}$ otherwise CSP($\mathbf{A}$) is $\mathbb{NP}$-complete.

Definition

Let $n > 1$. An n-ary operation f is called a weak near-unanimity operation if

$$f(x, x, \dots, x) = x \text{ and}$$
$$f(y, x, x, x, \dots, x) = f(x, y, x, x, \dots, x) = \cdots$$
$$= f(x, x, \dots, x, y)$$

Note: no essentially unary operation is WNU

Theorem (Bulatov, Larose, Zádori, McKenzie, Maróti)

If Δ is a core and $\mathrm{Pol}(\Delta)$ *has no WNU operation then* $\mathrm{CSP}(\Delta)$ *is* $\mathbb{NP}$*-complete.*

Reformuated Dichotomy Conjecture

Let Δ be a core. Then CSP(Δ) is tractable if and only if it has a WNU polymorphism. Otherwise, it is $\mathbb{NP}$-complete.

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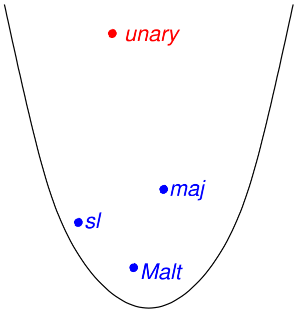

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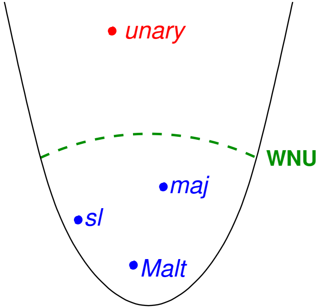

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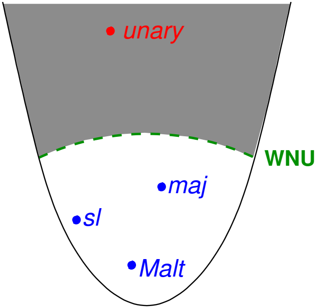

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Let Δ be a core. Then CSP(Δ) is tractable if and only if it has a WNU polymorphism. Otherwise, it is $\mathbb{NP}$-complete.

Supporting Examples

- 2-SAT, 2-COLARABILITY, LINEAR SYSTEM have a WNU.

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Supporting Examples

- 2-SAT, 2-COLARABILITY, LINEAR SYSTEM have a WNU.
- Let **A** be an abelian group, $n = |A|$. Choose integers k, l with $kl \equiv 1 \pmod{n}$. Then

  $$f(x_1, \dots, x_k) = l(x_1 + \cdots + x_k)$$

  is a WNU operation.

Two General Techniques for Tractable Algorithms

Method 1

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Examples of cube operations:

$P(x, y, z) = x - y + z$
$M(x, y, z) =$ majority

Essentially a generalization of Gaussian elimination.

Algebras with a cube operation possess "few subpowers". This algebraic property is used to prove that the algorithm terminates in polynomial time.

Method 2

If Pol(Δ) contains WNU operations $v(x, y, z)$ and $w(x, y, z, u)$ satisfying $v(y, x, x) = w(y, x, x, x)$, then CSP($\Delta$) $\in \mathbb{P}$.

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Examples: majority, semilattice

Algebras with these operations have a property called "congruence meet-semidistributivity."

Current State of Affairs

The two general techniques do not cover all cases of a WNU. What to do next?

Two possible directions:

1. Find a completely new algorithm.
2. Combine the two existing algorithms.

I am exploring the second approach.