# The Rectangularity Theorem of Libor Barto and Marcin Kozik

## with applications to small CIBs

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joint work with

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Iowa State University

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slides available at

https://github.com/williamdemeo/Talks

# Definition of CSP
## (naive version)

**Input**

- *variables:* $\mathcal{V} = \{v_1, v_2, \ldots\}$
- *domain:* $\mathcal{D}$
- *constraints:* $C_1, C_2, \ldots$

**Output**

- "yes" if there is a *solution*

  $f : \mathcal{V} \to \mathcal{D}$ (an assignment of values to variables that satisfies all $C_i$)

- "no" otherwise

# Definition of CSP

(jaded version)

$\mathbf{A} = \langle A, \mathcal{F} \rangle$ is a finite idempotent algebra, $\mathsf{Sub}(\mathbf{A})$ is all subuniverses of $\mathbf{A}$.

*In this talk* $\mathrm{CSP}(\mathbf{A})$ denotes the following decision problem:

An *instance of degree* $n$ of $\mathrm{CSP}(\mathbf{A})$ is the tuple $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$

- *variables* $\mathcal{V} = \{0, 1, \dots, n-1\}$;
- *domains* $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1}\} \subset \mathsf{Sub}(\mathbf{A})$ (one for each variable)
- *scope functions* $\mathcal{S} = (\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{p-1})$ with *constraint arities* $\mathrm{ar}(\mathcal{S}) = (m_0, m_1, \dots, m_{p-1})$;
- *constraint relations* $\mathcal{R} = (\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_{p-1})$, where

$$\mathbf{R}_i \leqslant \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}.$$

A *solution* to $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$ is an assignment $f : \mathcal{V} \to A$ of values to variables that satisfies all constraints. That is,

$$f \in \Pi_{\mathcal{V}} A_j \quad \text{and} \quad \mathrm{Proj}_{\mathbf{s}_i} f \in \mathbf{R}_i, \ \text{for each } 0 \leqslant i < p.$$

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**Notation:** $\underline{n} = \{0, 1, \dots, n-1\}$, so the $i$-th scope has type $\mathbf{s}_i : \underline{m_i} \to \underline{n}$ and

$$\mathrm{Proj}_{\mathbf{s}_i} f = f \circ \mathbf{s}_i$$

# Example 1
## ...thanks, Ross!

Let $\mathbf{A} = \langle \{0, 1\}, \{f\} \rangle$, where

$$f(x, y, z) = x + y + z \pmod 2.$$

Consider the ternary relations

$$R_0 = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$$

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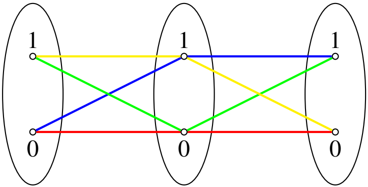
So we have a degree 3 instance of $\text{CSP}(\mathbf{A})$, where

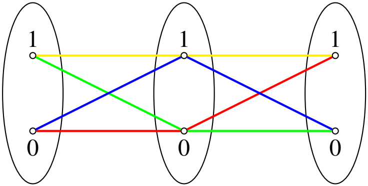
- variables: $\mathcal{V} = \{0,1,2\}$
- domains: $A_i = \{0,1\}, \quad i = 0,1,2$
- scope functions: the identity on $\{0,1,2\}$
- constraint relations: $\mathbf{R}_0$ and $\mathbf{R}_1$

# Example 1

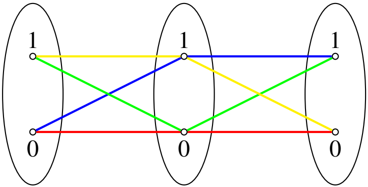
## ∩ and potatoes




(lines are elements of $R_0$)




(lines are elements of $R_1$)

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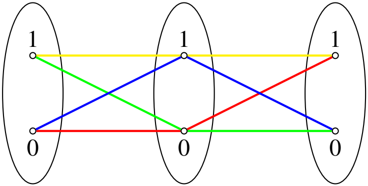


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Notice for all $i, j \in \{0, 1, 2\}$,

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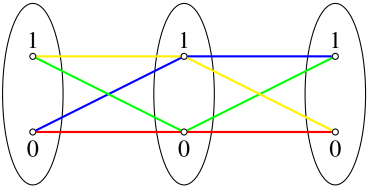


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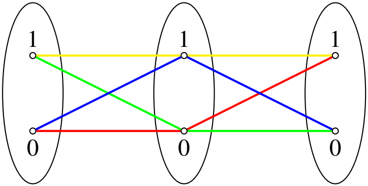


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Notice for all $i, j \in \{0, 1, 2\}$,

$$\mathrm{Proj}_{ij} R_0 = \mathrm{Proj}_{ij} R_1$$

...yet $R_0 \cap R_1 = \emptyset$.

# EXAMPLE 2

## ...THANKS, CLIFF!

Let $\mathbf{A} = \langle\{0,1\},\{m\}\rangle$, where $m : A^3 \to A$ is a majority operation,

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x.$$

Let $\mathbf{R}_0, \mathbf{R}_1 \leqslant_{\text{sd}} \mathbf{A}^3$ with universes

$$R_0 = \{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\},$$
$$R_1 = \{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}.$$

This describes the instance of $CSP(\mathbf{A})$ with

- variables: $\mathcal{V} = \{0,1,2\}$
- domains: $A_i = \{0,1\}, \quad i = 0,1,2$
- scope functions: the identity on $\{0,1,2\}$
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## SOME CONVENIENCES

Retrict attention to instances where all constraint relation are subdirect,

$$\mathbf{R}_i \leqslant_{\mathrm{sd}} \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}$$

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Could visualize $(\mathbf{s}_i, \mathbf{R}_i)$ as specifying a subalgebra of the full product $\Pi_{\mathcal{V}} \mathbf{A}_j$

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Convenient because now solutions are the elements in $\bigcap_{i \in V} [\![\mathbf{s}_i, \mathbf{R}_i]\!]$.

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BUT input size is not a function of these “full” subdirect products!

(Input size could be defined as the length of a string of all tuples in scopes and constraint relations of the instance.)

# END OF ACT I

## FIRST INTERMISSION

pause...

...draw more potatoes...

...give audience chance to escape.

# Absorption Theory (for mortals)

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be a finite algebra in a Taylor variety.

Let $t \in \mathrm{Clo}(\mathbf{A})$ be a $k$-ary term operation.

A subalgebra $\mathbf{B} \leqslant \mathbf{A}$ is *absorbing in* $\mathbf{A}$ *with respect to* $t$ if

$$a \in A,\ b_i \in B \quad \Longrightarrow \quad t^{\mathbf{A}}(b_0, \dots, b_{j-1}, a, b_{j+1}, \dots, b_{k-1}) \in B \quad (\text{all } j \in \underline{k})$$

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Equivalently, $t^{\mathbf{A}}[B^{j-1} \times A \times B^{k-j}] \subseteq B$, for all $0 \leqslant j < k$, that is,

$$(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B^{j-1} \times A \times B^{k-j} \quad \Longrightarrow \quad t^{\mathbf{A}}(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B.$$

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**Notation:**

$\mathbf{B} \triangleleft \mathbf{A}$ means $\mathbf{B}$ is absorbing in $\mathbf{A}$ with respect to some term.

To be explicit about the term, $\mathbf{B} \triangleleft_t \mathbf{A}$.

$\mathbf{B} \triangleleft\!\triangleleft \mathbf{A}$ means $\mathbf{B} \triangleleft \mathbf{A}$ and $B$ is minimal (with respect to inclusion) among absorbing subuniverses of $\mathbf{A}$.

An algebra is *absorption-free* (AF) if it has no proper absorbing subalgebras.

# Where are we going with this?

"The Absorption Theorem" of Barto and Kozik (LMCS 2012)

Concerns the special class of "linked" subdirect products.

*Identifies some special cases in which a subdirect product is the full product!*

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### Theorem (Absorption Theorem)

*If $V$ is an idempotent locally finite variety, then TFAE*

- *$V$ is a Taylor variety;*
- *if $\mathbf{A}_0, \mathbf{A}_1 \in V$ are finite idempotent absorption-free algebras and $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked, then $\mathbf{R} = \mathbf{A}_0 \times \mathbf{A}_1$.*

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At Vanderbilt Shanks Workshop (2015), Barto presented more joint work with Kozik generalizing the Absorption Theorem to more than two factors.

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*assuming suitable conditions under which the theorem is true.

# LINKED SUBDIRECT PRODUCTS

A subdirect product $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked if for all $a, a' \in \mathrm{Proj}_0 R$,

$$\exists\, c_0, c_2, \ldots, c_{2n} \in A_0, \quad \exists\, c_1, c_3, \ldots, c_{2n+1} \in A_1$$

such that

$$a = c_0, \quad (c_{2i}, c_{2i+1}) \in R, \quad (c_{2i+2}, c_{2i+1}) \in R, \quad c_{2n} = a'$$

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[todo: insert potato diagram]

**Notation:**

For $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$, let $\eta_i$ denote the kernel of the $i$-th projection of $\mathbf{R}$. That is,

$$\eta_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid \operatorname{Proj}_i \mathbf{r} = \operatorname{Proj}_i \mathbf{r}'\}$$

Let $R^{-1} = \{(y, x) \in A_1 \times A_0 \mid (x, y) \in R\}$.

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The following are equivalent:

- $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked;
- $\eta_0 \vee \eta_1 = 1_R$;
- if $a, a' \in \mathrm{Proj}_0\, R$, then $(a, a')$ is in the transitive closure of $R \circ R^{-1}$.

# Properties of Absorption I

Absorption has nice properties...

- (transitivity) $\mathbf{C} \lhd \mathbf{B} \lhd \mathbf{A} \implies \mathbf{C} \lhd \mathbf{A}$
- (closure under nonempty $\cap$ and finite products)

  If $\mathbf{B} \lhd_f \mathbf{A}$ and $\mathbf{C} \lhd_g \mathbf{A}$ and $B \cap C \neq \emptyset$, then $\mathbf{B} \cap \mathbf{C} \lhd \mathbf{A}$.

  If $\mathbf{B}_0 \lhd_f \mathbf{A}_0$ and $\mathbf{B}_1 \lhd_g \mathbf{A}_1$, then $\mathbf{B}_0 \times \mathbf{B}_1 \lhd_t \mathbf{A}_0 \times \mathbf{A}_1$.

  ...with respect to $t = f \star g$ in both cases.

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If $f : A^\ell \to A$ and $g : A^m \to A$, then $f \star g$ is the $\ell m$-ary operation

$$f(g(a_{11}, \ldots, a_{1m}), g(a_{21}, \ldots, a_{2m}), \ldots, g(a_{\ell 1}, \ldots, a_{\ell m}))$$

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More generally, if $\mathbf{B}_i \triangleleft_{t_i} \mathbf{A}_i$ for $0 \leqslant i < n$, then $\Pi\mathbf{B}_i \triangleleft_s \Pi\mathbf{A}_i$.

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An obvious but important consequence:

*A finite product of finite idempotent algebras is AF if each factor is AF.*

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**Restriction Lemma.**
If $\mathbf{B} \lhd_t \mathbf{A}$ and $\mathbf{C} \leqslant \mathbf{A}$ and $D = B \cap C \neq \emptyset$, then $\mathbf{D} \lhd \mathbf{C}$ with respect to the restriction of $t$ to $C$.

### Lemma (LSD 1)

*If* $\mathbf{B}_i \triangleleft \mathbf{A}_i$ *and* $\mathbf{R} \leqslant \Pi_i \mathbf{A}_i$ *and* $R' := R \cap \Pi_i B_i \neq \emptyset$*, then* $\mathbf{R}' \triangleleft \mathbf{R}$*.*

*Proof.* $\Pi \mathbf{B}_i \triangleleft_t \Pi \mathbf{A}_i$, so follows Restriction Lemma if we put $C = R$.

### Lemma (LSD 2)

*Suppose* $\mathbf{B}_i \triangleleft\!\triangleleft \mathbf{A}_i$ *and* $\mathbf{R} \leqslant_{\text{sd}} \Pi \mathbf{A}_i$*. If* $R' := R \cap \Pi B_i \neq \emptyset$*, then* $\mathbf{R}' \leqslant_{\text{sd}} \Pi \mathbf{B}_i$*.*

### Lemma (LSD 2)

*If* $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ *is linked and* $\mathbf{S} \triangleleft \mathbf{R}$*, then* $\mathbf{S}$ *is linked.*

In some simple cases we get linking from LSD Lemmas along with the following elementary

**Fact.** Suppose $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and let $\eta_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i)$.

1. If $\mathbf{A}_0$ is simple, then either $\eta_0 \vee \eta_1 = 1_R$ or $\eta_0 \geqslant \eta_1$.
2. If $\mathbf{A}_0$ and $\mathbf{A}_1$ are both simple, then either $\eta_0 \vee \eta_1 = 1_R$ or $\eta_0 = 0_R = \eta_1$.

...so, if both factors are simple, then $\eta_0 \neq \eta_1$ gives the linking...

**Cor 1.** Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be simple. If $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$, then $\mathbf{R}$ is linked.

...and if one factor is simple nonabelian and the other abelian, linking is free!

**Cor 2.** If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ abelian, then every subdirect product of $\mathbf{A}_0 \times \mathbf{A}_1$ is linked.

# Absorption Theorem: Application

Suppose we add to the respective contexts of the last three results the hypothesis that the algebras live in an idempotent variety with a Taylor term...

(We will refer to such varieties as "Taylor varieties" and we call the algebras they contain "Taylor algebras.")

...then the Absorption Theorem (in combination with facts above) yields

**Lemma:** Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be finite Taylor algebras with $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ $(i = 0, 1)$ and suppose $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$.

(I) If $\mathbf{A}_0$ and $\mathbf{A}_1$ are simple and $R \cap (B_0 \times B_1) \neq \emptyset$, then $\mathbf{B}_0 \times \mathbf{B}_1 \leqslant \mathbf{R}$.

(II) If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ is abelian, then $\mathbf{B}_0 \times \mathbf{A}_1 \leqslant \mathbf{R}$.

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**Lemma:** Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be finite Taylor algebras with $\mathbf{B}_i \lll \mathbf{A}_i$ $(i = 0, 1)$ and suppose $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$.

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...simple nonabelian potatoes cannot.

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**The Rectangularity Theorem.**

Let $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ be finite algebras in a Taylor variety, $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$, and suppose

- at most one $\mathbf{A}_i$ is abelian,
- all nonabelian factors are simple,
- $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$.
- $\mathbf{R}' = \mathbf{R} \cap (\mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1})$ is nonempty.

$$\text{Then } \mathbf{R}' = \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}.$$

**Notation:**

Let $\underline{n} = \{0, 1, 2, \dots, n-1\}$.

Let $\sigma' = \underline{n} - \sigma$, when $\sigma$ is a subset of $\underline{n}$.

For $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi_{\underline{n}} \mathbf{A}_i$ let

$$\eta_\sigma = \ker(R \twoheadrightarrow \Pi_\sigma A_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid \mathrm{Proj}_\sigma \mathbf{r} = \mathrm{Proj}_\sigma \mathbf{r}'\},$$

If $\sigma \subseteq \underline{n}$, then by $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi_\sigma \mathbf{A}_i \times \Pi_{\sigma'} \mathbf{A}_i$ we mean

$$\mathbf{R} \leqslant \Pi_{\underline{n}} \mathbf{A}_i, \quad \mathrm{Proj}_\sigma \mathbf{R} = \Pi_\sigma \mathbf{A}_i, \quad \text{and} \quad \mathrm{Proj}_{\sigma'} \mathbf{R} = \Pi_{\sigma'} \mathbf{A}_i.$$

and we say that $\mathbf{R}$ is a *subdirect product of* $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$ in this case.

The subdirect product $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi_\sigma \mathbf{A}_i \times \Pi_{\sigma'} \mathbf{A}_i$ is said to be *linked* if $\eta_\sigma \vee \eta_{\sigma'} = 1_R$.

We may use $\mathbf{R}_\sigma$ for $\mathrm{Proj}_\sigma \mathbf{R}$, the projection of $\mathbf{R}$ onto coordinates in $\sigma$.

# Rectangularity Theorem

## Proof Sketch

From now on, *all algebras are finite and belong to the same Taylor variety*.

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**Lemma 1.**

Let $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ for each $i \in \underline{n}$, and let $\underline{n} = \sigma \cup \sigma'$ be a disjoint union.
Assume $\mathbf{R}$ is a *linked* subdirect product of $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$.
Suppose $R' = R \cap \Pi_i B_i \neq \emptyset$. Then $\mathbf{R}' = \Pi_i \mathbf{B}_i$.

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**Lemma 2.** [Kearnes-Kiss, Thm 3.27]
Suppose $\alpha$ and $\beta$ are congruences of a Taylor algebra. Then

$$\mathsf{C}(\alpha, \alpha; \alpha \wedge \beta) \quad \Longleftrightarrow \quad \mathsf{C}(\alpha \vee \beta, \alpha \vee \beta; \beta).$$

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**Lemma 3.** [Linking Lemma]
Let $n \geqslant 2$ and $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ for all $i \in \underline{n}$. Suppose

- at most one $\mathbf{A}_i$ is abelian
- all nonabelian factors are simple
- $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$.

Then there exists $k$ such that $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_k \times \mathbf{R}_{k'}$ is linked.

**The Rectangularity Theorem.**

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**Proof.** Induct on the number of factors in the product $\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$.

For $n = 2$ the result holds by earlier Lemma (slide 16).

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3. **Intersecting mass products.** RT and corollaries assume that the universe $R$ of the subdirect product in question intersects nontrivially with a product $\Pi B_i$ of minimal absorbing subuniverses.