OVERALGEBRAS:
EXPANSIONS AND EXTENSIONS OF FINITE ALGEBRAS

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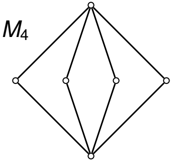
Algebra & Combinatorics Seminar

22 Feb 2016

These slides and other resources are available at

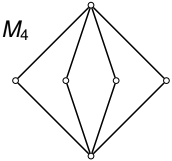
https://github.com/williamdemeo/Talks

Representing lattices



Question: Is this a congruence lattice?

Representing lattices



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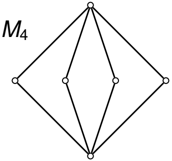

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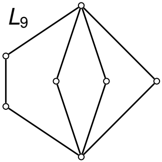

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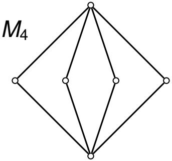

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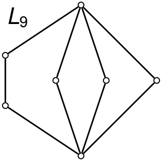

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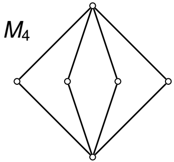
Answer: Yes! ...but this one is harder.

Construction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.



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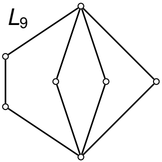
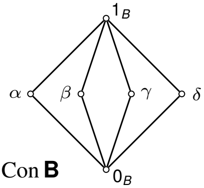


CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.



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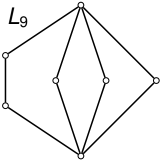

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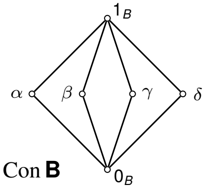

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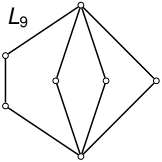

STEP 2 Define $A = B_0 \cup B_1 \cup B_2$ where

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\begin{aligned}
B_0 &= \{0, 1, 2, 3, 4, 5\}\\
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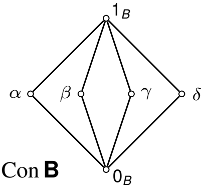

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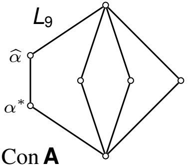

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Step 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0 e_0$, and $g_1 e_0$.

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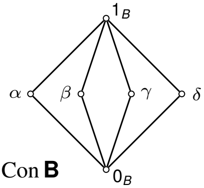

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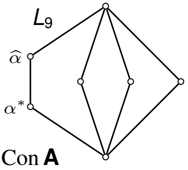

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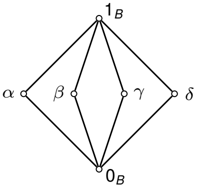
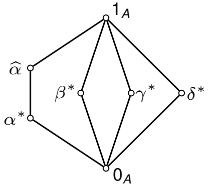


$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$



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$$\alpha = |0,1,2|3,4,5|$$

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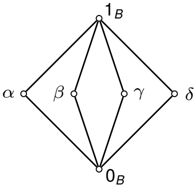

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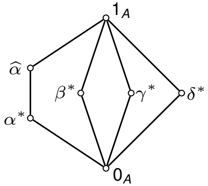

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$$\alpha = \alpha^* \cap B^2 = \widehat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \ldots$$

Expansion & Extension

<u>Expanded Universe</u>

$A = B_0 \cup B_1 \cup B_2$ where

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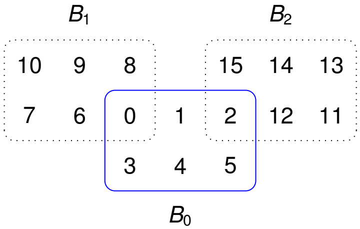

<u>New Operations</u>

$$e_0 : A \twoheadrightarrow B_0$$
$$e_1 : A \twoheadrightarrow B_1$$
$$e_2 : A \twoheadrightarrow B_2$$
$$s : A \twoheadrightarrow B_0,$$
$$ge_0 : A \overset{e_0}{\twoheadrightarrow} B_0 \xrightarrow{g} B_0,$$

for each $g \in F^B$.

Why does it work?

Con $\langle B, \{g_0, g_1\}\rangle$

$$\alpha = |0,1,2|3,4,5|$$

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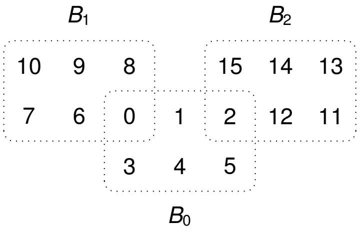

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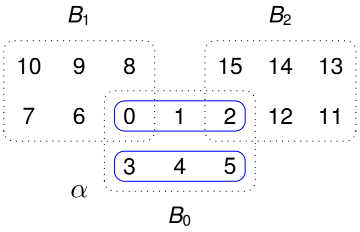

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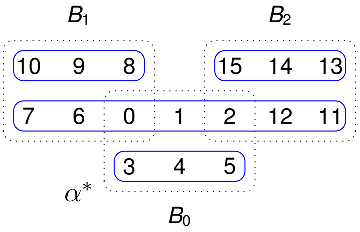

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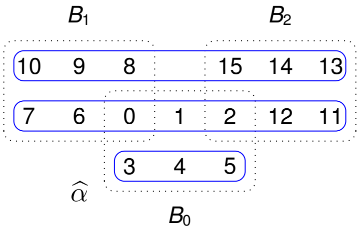

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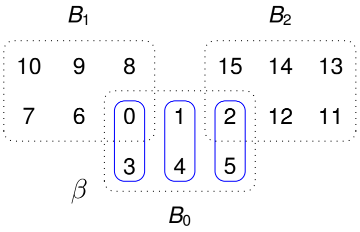

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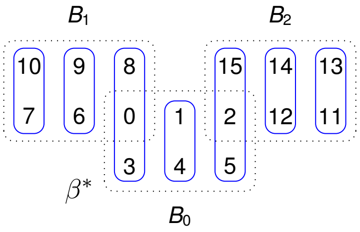

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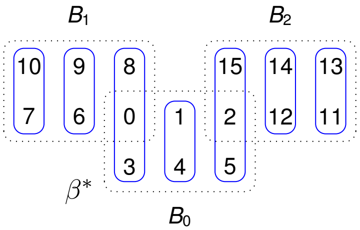

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Why don't β classes of B_1, B_2 mix?

VARIATIONS ON THE SAME EXAMPLE...

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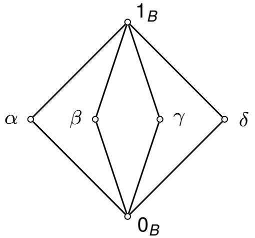

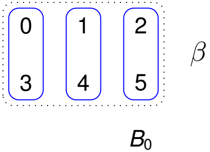

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- Select elements 0 and 3 as intersection points:

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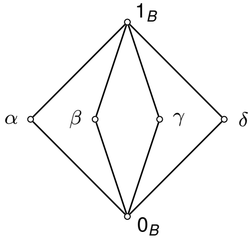

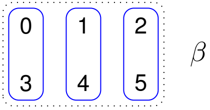

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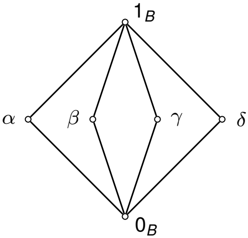

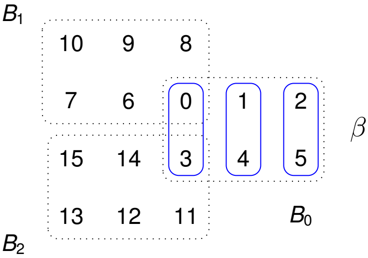

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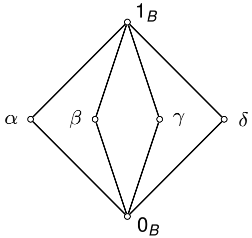

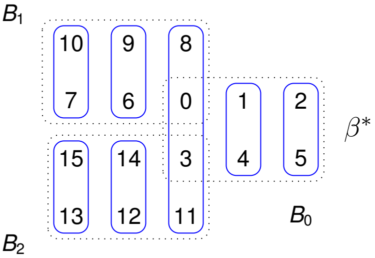

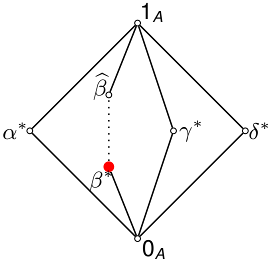

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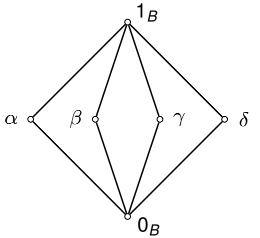

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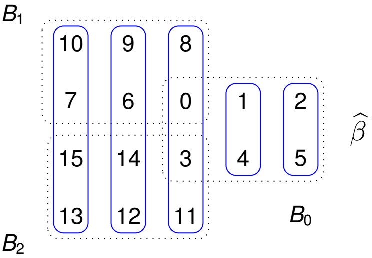

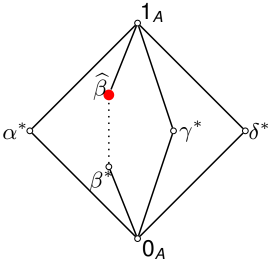

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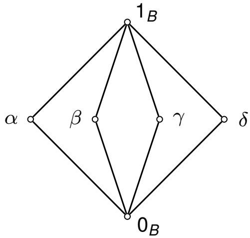

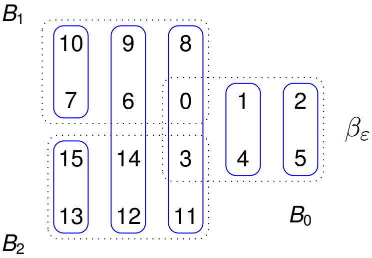

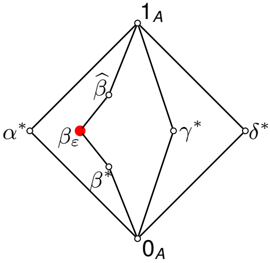

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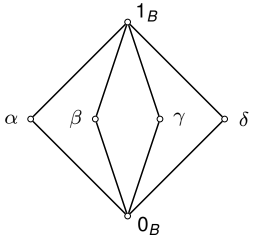

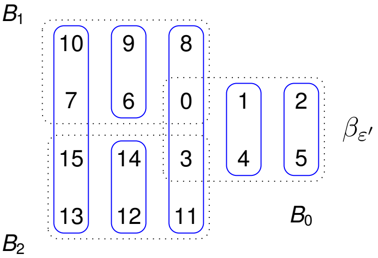

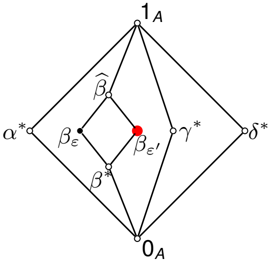

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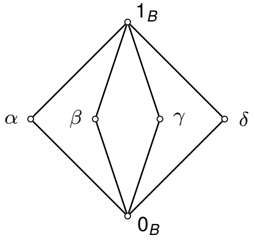
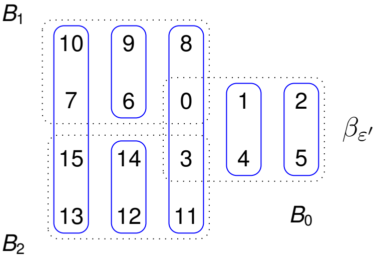
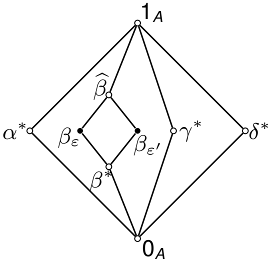






THE P^5 LEMMA

LEMMA (PÁLFY-PUDLÁK, 1980)

Let $\mathbf{A} = \langle A, F \rangle$ *be a unary algebra where* F *is a monoid.*

Suppose $e \in F$ *satisfies* $e \circ e = e$*.*

Define $\mathbf{B} = \langle B, G \rangle$

$$B = e(A) \quad \textit{and} \quad G = \{e\,f|_B \mid f \in F\}.$$

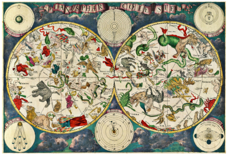
Let $|_B : Con(\mathbf{A}) \to Con(\mathbf{B})$ *be the restriction mapping:*

$$\theta|_B = \theta \cap B^2$$

Then $|_B$ *is a surjective homomorphism (even for arbitrary meets and joins).*

Péter Pál Pálfy and Pavel Pudlák: *Congruence lattices of finite algebras* AU (1980).

The Star Map and Hat Map



star map?

hat map?

The Star Map and Hat Map

STAR MAP * : Con $\mathbf{B} \to$ Con $\mathbf{A}$ is congruence generation:

$$\beta^* = \mathrm{Cg}^{\mathbf{A}}(\beta) \qquad (\forall\, \beta \in \mathrm{Con}\,\mathbf{B})$$

HAT MAP $\widehat{\ }$: Con $\mathbf{B} \to$ Con $\mathbf{A}$ is

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The hat map appears in McKenzie's "Finite Forbidden Lattices" paper (Puebla, 1982) where he gives an alternative proof of the P^5 Lemma.

Residuation lemma

A lemma relating the three maps *, $|_B$, and $\widehat{\ }$.

Lemma

(i) ${}^* : \mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ *is a **residuated mapping** with **residual*** $|_B$.

(ii) $|_B : \mathrm{Con}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{B}$ *is a **residuated mapping** with **residual*** $\widehat{\ }$.

(iii) *For all* $\alpha \in \mathrm{Con}\,\mathbf{A}$, $\beta \in \mathrm{Con}\,\mathbf{B}$,

$$\beta = \alpha|_B \quad \Leftrightarrow \quad \beta^* \leqslant \alpha \leqslant \widehat{\beta}.$$

In particular, $\beta^*|_B = \beta = \widehat{\beta}|_B$.

Residuation/adjunction lemma

New version...

Lemma

$$^{*} \qquad \dashv \qquad \big|_{B} \qquad \dashv \qquad \widehat{\;}$$

Residuation/adjunction lemma

New version...

Lemma

$$^* \quad \dashv \quad |_{B} \quad \dashv \quad \widehat{\ }$$

...that is...

(i) * : $\mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ is ***left adjoint*** to $|_B$;

(ii) $|_B$: $\mathrm{Con}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{B}$ is ***left adjoint*** to $\widehat{\ }$;

(iii) For all $\alpha \in \mathrm{Con}\,\mathbf{A}$, $\beta \in \mathrm{Con}\,\mathbf{B}$,

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Lemma (Pálfy-Pudlák, 1980)

The restriction mapping

$$\operatorname{Con} \mathbf{A} \ni \alpha \mapsto \alpha|_B = \alpha \cap B^2 \in \operatorname{Con} \mathbf{B}$$

is a complete lattice epimorphism.

Proof.

Recall, for $f : X \to Y$ a monotone function on preorders X and Y,
if f has a right (left) adjoint, then f preserves all joins (meets) that exist in X.

By the lemma $|_B$ has both a left and right adjoint. $\square$