CSPs of Finite Commutative Idempotent Binars

William DeMeo
williamdemeo@gmail.com
Iowa State University

joint work with

Cliff Bergman
Jiali Li

Shanks Workshop
Vanderbilt University
May 30, 2015

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable } \iff \mathbf{A} \text{ has a weak-nu term operation}$$

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable } \implies \mathbf{A} \text{ has a weak-nu term operation } \quad \checkmark$$

The left-to-right direction is known.

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable} \quad \Longleftarrow \quad \mathbf{A} \text{ has a weak-nu term operation} \qquad (?)$$

The right-to-left direction is open.

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable} \quad \Longleftarrow \quad \mathbf{A} \text{ has a weak-nu term operation} \qquad (?)$$

A weak near unanimity (weak-nu) term operation is one that satisfies

$$t(x, x, \dots, x) \approx x \qquad \text{(idempotent)}$$

$$t(y, x, \dots, x) \approx t(x, y, \dots, x) \approx \cdots \approx t(x, x, \dots, y)$$

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable} \quad \Longleftarrow \quad \mathbf{A} \text{ has a weak-nu term operation} \qquad (?)$$

A weak near unanimity (weak-nu) term operation is one that satisfies

$$t(x, x, \ldots, x) \approx x \qquad \text{(idempotent)}$$

$$t(y, x, \ldots, x) \approx t(x, y, \ldots, x) \approx \cdots \approx t(x, x, \ldots, y)$$

A *binary* operation $t(x, y)$ is weak-nu if

$$t(x, x) \approx x \qquad \text{(idempotent)}$$

$$t(y, x) \approx t(x, y) \qquad \text{(commutative)}$$

So let's try to prove (?) for commutative idempotent binars.

Commutative Idempotent Binars

A CIB is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.

Commutative Idempotent Binars

A CIB is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.

Question

Is every finite commutative idempotent binar tractable?

Commutative Idempotent Binars

A CIB is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.

Question

Is every finite commutative idempotent binar tractable?

First Example: a semilattice is an associative CIB.

Semilattices are tractable (in fact, they have *finite width*).

Commutative Idempotent Binars

A CIB is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.

Question

Is every finite commutative idempotent binar tractable?

First Example: a semilattice is an associative CIB.

Semilattices are tractable (in fact, they have *finite width*).

Pause to consider more general case for a minute...

General case

Some well known facts

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

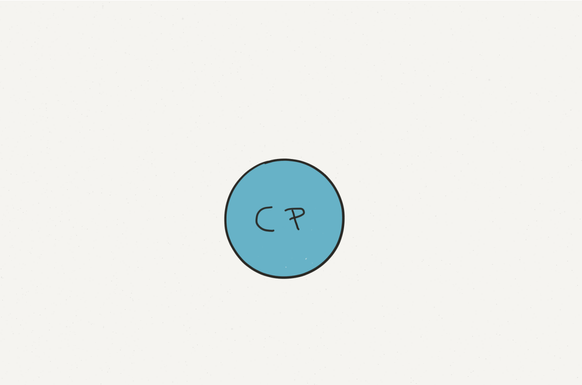
$$\mathrm{V}(\mathbf{A}) \text{ is CP} \iff \mathbf{A} \text{ has Malcev term}$$

General case

Some well known facts

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

$$\mathrm{V}(\mathbf{A}) \text{ is CP} \iff \mathbf{A} \text{ has Malcev term}$$

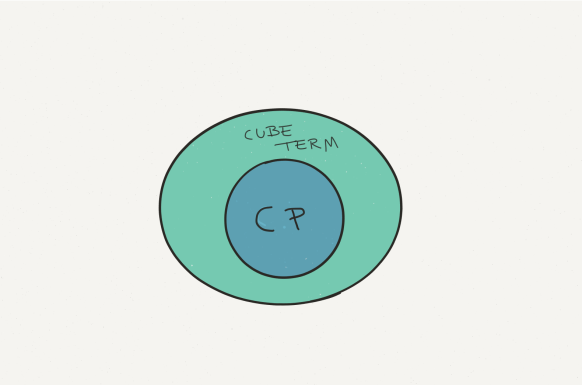


General case

Some well known facts

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

$$\begin{aligned} \mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has Malcev term} \\ &\implies \mathbf{A} \text{ has cube term} \end{aligned}$$

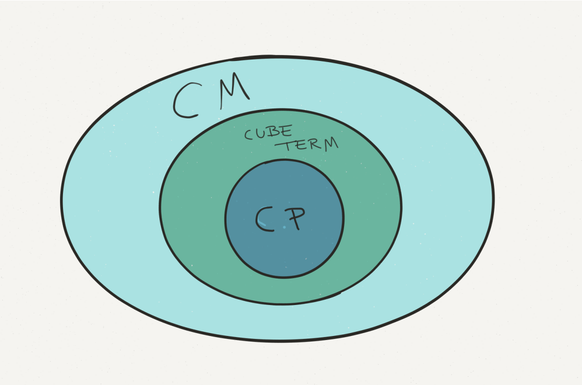


GENERAL CASE

SOME WELL KNOWN FACTS

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

$$\begin{aligned} \mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has Malcev term} \\ &\implies \mathbf{A} \text{ has cube term} \\ &\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \end{aligned}$$

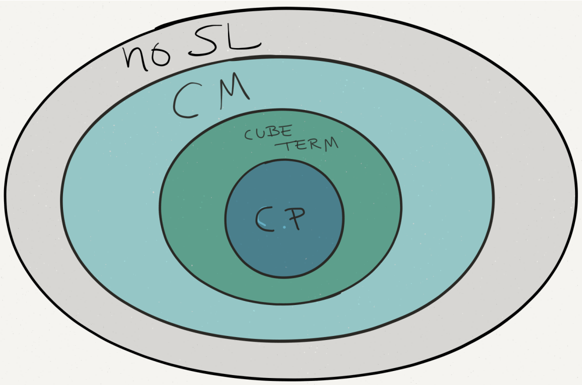


General case

Some well known facts

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

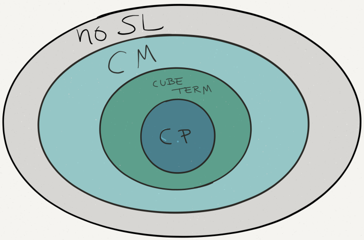
$$\begin{aligned} \mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has Malcev term} \\ &\implies \mathbf{A} \text{ has cube term} \\ &\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\ &\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A}) \end{aligned}$$



Some well known facts

$\mathbf{A}$ = a finite idempotent algebra
$\mathbf{S}_2$ = the 2-elt semilattice.

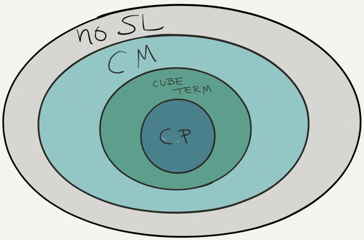
$$
\begin{aligned}
\mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has a Malcev term} \\
&\implies \mathbf{A} \text{ has a cube term} \\
&\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\
&\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A})
\end{aligned}
$$



SOME WELL KNOWN FACTS

$\mathbf{A}$ = a finite idempotent algebra
$\mathbf{S}_2$ = the 2-elt semilattice.

$$\begin{aligned} \mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has a Malcev term} \\ &\implies \mathbf{A} \text{ has a cube term} \\ &\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\ &\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A}) \end{aligned}$$



- cube term $\implies$ CM

  (Berman, Idziak, Marković, McKenzie, Valeriote, Willard 2010)

- CM $\implies$ $\mathbf{S}_2$ is not in $\mathrm{V}(\mathbf{A})$

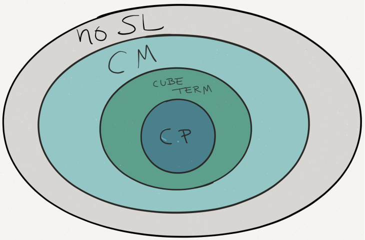
  Proof: $\mathbf{S}_2 \in \mathrm{V}(\mathbf{A}) \;\Rightarrow\; \mathbf{S}_2^2 \in \mathrm{V}(\mathbf{A})$;

  $\mathrm{Con}\,(\mathbf{S}_2^2)$ is not modular.

Some well known facts

$\mathbf{A}$ = a finite **CIB**
$\mathbf{S}_2$ = the 2-elt semilattice.

$$
\begin{aligned}
\mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has a Malcev term} \\
&\implies \mathbf{A} \text{ has a cube term} \\
&\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\
&\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A})
\end{aligned}
$$



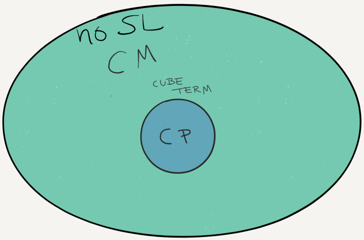
<u>CIB case</u>

- 1st reduction by cube-term blockers.

SOME WELL KNOWN FACTS

$\mathbf{A}$ = a finite **CIB**
$\mathbf{S}_2$ = the 2-elt semilattice.

$$
\begin{aligned}
\mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has a Malcev term} \\
&\implies \mathbf{A} \text{ has a cube term} \\
&\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\
&\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A}) \\
&\implies \mathbf{A} \text{ has a cube term}
\end{aligned}
$$



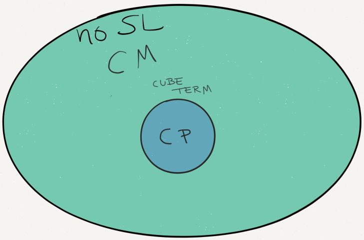
CIB case

- 1st reduction by cube-term blockers.

SOME WELL KNOWN FACTS

$\mathbf{A}$ = a finite **CIB**
$\mathbf{S}_2$ = the 2-elt semilattice.

$$
\begin{aligned}
\mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has a Malcev term} \\
&\implies \mathbf{A} \text{ has a cube term} \\
&\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\
&\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A}) \\
&\implies \mathbf{A} \text{ has a cube term}
\end{aligned}
$$



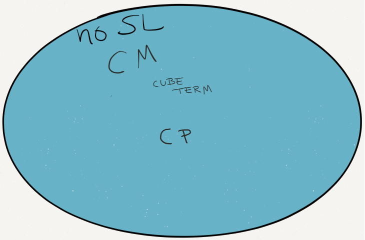
CIB case

- 1st reduction by cube-term blockers.
- 2nd reduction by Kearnes-Tschantz.

SOME WELL KNOWN FACTS

$\mathbf{A}$ = a finite **CIB**
$\mathbf{S}_2$ = the 2-elt semilattice.

$$
\begin{aligned}
\mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has a Malcev term} \\
&\implies \mathbf{A} \text{ has a cube term} \\
&\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\
&\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A}) \\
&\implies \mathbf{A} \text{ has a cube term} \\
&\implies \mathrm{V}(\mathbf{A}) \text{ is CP}
\end{aligned}
$$



CIB case

- 1st reduction by cube-term blockers.
- 2nd reduction by Kearnes-Tschantz.

First Reduction

by Cube-Term Blockers

Marković, M. Maróti, McKenzie (M^4)
"Finitely related clones and algebras with cube terms" (2012)

A cube-term blocker (CTB) is a pair (C, B) of subuniverses satisfying $\emptyset < C < B \leqslant A$ and for every $t(x_1, \ldots, x_n)$ there is an index $i \in [n]$ with

$$(\forall (b_1, \ldots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \ldots, b_n) \in C).$$

First Reduction

by Cube-Term Blockers

Marković, M. Maróti, McKenzie (M^4)
"Finitely related clones and algebras with cube terms" (2012)

A cube-term blocker (CTB) is a pair (C, B) of subuniverses satisfying $\emptyset < C < B \leqslant A$ and for every $t(x_1, \ldots, x_n)$ there is an index $i \in [n]$ with

$$(\forall (b_1, \ldots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \ldots, b_n) \in C).$$

M^4 prove a finite idempotent algebra has a cube term iff it has no CTB.

First Reduction
by Cube-Term Blockers

Marković, M. Maróti, McKenzie (M^4)
"Finitely related clones and algebras with cube terms" (2012)

A cube-term blocker (CTB) is a pair (C, B) of subuniverses satisfying $\emptyset < C < B \leqslant A$ and for every $t(x_1, \ldots, x_n)$ there is an index $i \in [n]$ with

$$(\forall (b_1, \ldots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \ldots, b_n) \in C).$$

M^4 prove a finite idempotent algebra has a cube term iff it has no CTB.

Lemma

A finite CIB $\mathbf{A}$ has a CTB if and only if $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$.

First Reduction

by Cube-Term Blockers

Marković, M. Maróti, McKenzie (M^4)
"Finitely related clones and algebras with cube terms" (2012)

A cube-term blocker (CTB) is a pair (C, B) of subuniverses satisfying $\emptyset < C < B \leqslant A$ and for every $t(x_1, \ldots, x_n)$ there is an index $i \in [n]$ with

$$(\forall (b_1, \ldots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \ldots, b_n) \in C).$$

M^4 prove a finite idempotent algebra has a cube term iff it has no CTB.

Lemma

A finite CIB $\mathbf{A}$ *has a CTB if and only if* $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$.

Proof.

(C, B) a CTB implies $\theta = C^2 \cup (B - C)^2$ a congruence with $\mathbf{B}/\theta \cong \mathbf{S}_2$.

Conversely, suppose $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$, and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ with $\mathbf{B}/\theta$ a meet-SL for some θ. Let C/θ be the bottom of $\mathbf{B}/\theta$, then (C, B) is a CTB. $\square$

Second Reduction

Kearnes and Tschantz
"Automorphism groups of squares and of free algebras" (2007)

Lemma

If V is an idempotent variety that is not congruence permutable, then there are subuniverses U and W of $\mathbf{F} := \mathbf{F}_V\{x, y\}$ satisfying

1. $x \in U \cap W$
2. $y \in U^c \cap W^c$
3. $(U \times F) \cup (F \times W) \leqslant \mathbf{F}^2$

Second Reduction

Kearnes and Tschantz
"Automorphism groups of squares and of free algebras" (2007)

Lemma

If V is an idempotent variety that is not congruence permutable, then there are subuniverses U and W of $\mathbf{F} := \mathbf{F}_V\{x, y\}$ satisfying

1. $x \in U \cap W$
2. $y \in U^c \cap W^c$
3. $(U \times F) \cup (F \times W) \leqslant \mathbf{F}^2$

For CIB's, either U or W will be an ideal.

This implies a CTB and a semilattice.

Remaining Questions for Finite CIBs

Conclusion

Let $\mathbf{A}$ be a finite CIB. Then

$$\mathbf{S}_2 \notin \mathsf{HS}(\mathbf{A}) \text{ if and only if } \mathrm{V}(\mathbf{A}) \text{ is congruence permutable.}$$

Remaining Questions for Finite CIBs

Conclusion

Let $\mathbf{A}$ be a finite CIB. Then

$$\mathbf{S}_2 \notin \mathsf{HS}(\mathbf{A}) \text{ if and only if } \mathrm{V}(\mathbf{A}) \text{ is congruence permutable.}$$

(so $\mathrm{CSP}(\mathbf{A})$ tractable in this case)

Remaining Questions for Finite CIBs

Conclusion

Let $\mathbf{A}$ be a finite CIB. Then

$$\mathbf{S}_2 \notin \mathsf{HS}(\mathbf{A}) \text{ if and only if } \mathrm{V}(\mathbf{A}) \text{ is congruence permutable.}$$

(so $\mathrm{CSP}(\mathbf{A})$ tractable in this case)

Open Question

Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in $\mathsf{HS}(\mathbf{A})$. Is $\mathrm{CSP}(\mathbf{A})$ tractable?

Remaining Questions for Finite CIBs

Conclusion

Let $\mathbf{A}$ be a finite CIB. Then

$$\mathbf{S}_2 \notin \text{HS}(\mathbf{A}) \text{ if and only if } \mathrm{V}(\mathbf{A}) \text{ is congruence permutable.}$$

(so $\mathrm{CSP}(\mathbf{A})$ tractable in this case)

Open Question

Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in HS$(\mathbf{A})$. Is $\mathrm{CSP}(\mathbf{A})$ tractable?

Recall, if $\mathrm{V}(\mathbf{A})$ is $\mathrm{SD}_\wedge$, then $\mathrm{CSP}(\mathbf{A})$ is tractable.

Remaining Questions for Finite CIBs

Conclusion

Let $\mathbf{A}$ be a finite CIB. Then

$$\mathbf{S}_2 \notin \mathsf{HS}(\mathbf{A}) \text{ if and only if } \mathrm{V}(\mathbf{A}) \text{ is congruence permutable.}$$

(so $\mathrm{CSP}(\mathbf{A})$ tractable in this case)

Open Question

Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in $\mathsf{HS}(\mathbf{A})$. Is $\mathrm{CSP}(\mathbf{A})$ tractable?

Recall, if $\mathrm{V}(\mathbf{A})$ is $\mathrm{SD}_\wedge$, then $\mathrm{CSP}(\mathbf{A})$ is tractable.

Revised Question

Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in $\mathsf{HS}(\mathbf{A})$, and $\mathrm{V}(\mathbf{A})$ not $\mathrm{SD}_\wedge$.

Is $\mathrm{CSP}(\mathbf{A})$ tractable?

EXAMPLES

$\cdot$	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

$*$	0	1	2	3
0	0	0	1	1
1	0	1	3	2
2	1	3	2	1
3	1	2	1	3

$\circ$	0	1	2	3
0	0	0	2	1
1	0	1	3	2
2	2	3	2	1
3	1	2	1	3

Maroti's idea:

Bergman's idea: replace basic binary operation with a term from $\mathrm{Clo}(\mathbf{A})$, say $t(x,y) = (x * y) * x$.

If $\langle A, t\rangle$ tractable, then so is $\langle A, *\rangle$