THE FINITE LATTICE REPRESENTATION PROBLEM AND INTERVALS IN SUBGROUP LATTICES THE PROOFS

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joint work with

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These slides and other resources are available at

http://www.math.sc.edu/~demeow/FLRP.html ⟶

The P^5 Lemma

Lemma (Pálfy-Pudlák, 1980)

Let $\mathbf{A} = \langle A, F \rangle$ be a unary algebra where F is a monoid and let $e \in F$ be an idempotent operation. Define $\mathbf{B} = \langle B, G \rangle$ as follows:

$$B = e(A) \quad \text{and} \quad G = \{ef|_B \mid f \in F\}.$$

Let $|_B : Con(\mathbf{A}) \to Con(\mathbf{B})$ be the restriction of congruences to the set B:

$$\theta|_B = \theta \cap B^2$$

Then $|_B$ is a surjective homomorphism (even for arbitrary meets and joins).

Péter Pál Pálfy and Pavel Pudlák: *Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups.*

Algebra Universalis **11**(1), 22–27 (1980).

URL `http://dx.doi.org/10.1007/BF02483080`

Star map and hat map

star map $^*: \operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ is the congruence generation operator restricted to the set $\operatorname{Con}\mathbf{B}$:

$$\beta^* = \operatorname{Cg}^{\mathbf{A}}(\beta) \qquad (\forall\, \beta \in \operatorname{Con}\mathbf{B})$$

STAR MAP AND HAT MAP

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HAT MAP $\widehat{\ }$: $\mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ is

$$\widehat{\beta} = \{(x, y) \in A^2 \mid (ef(x), ef(y)) \in \beta \text{ for all } f \in F\},$$

(Used by McKenzie (1982) in an alternative proof of the P^5 Lemma.)

Ralph McKenzie: *Finite forbidden lattices.*

In: Universal algebra and lattice theory (Puebla, 1982), *Lecture Notes in Math.*, vol. 1004, pp. 176–205. Springer, Berlin (1983).

URL `http://dx.doi.org/10.1007/BFb0063438`

Residuation lemma

A little lemma relating the three maps *, $|_B$ and $\widehat{\ }$.

Lemma

(I) $^* : \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ *is a residuated mapping with residual* $|_B$.

(II) $|_B : \operatorname{Con} \mathbf{A} \to \operatorname{Con} \mathbf{B}$ *is a residuated mapping with residual* $\widehat{\ }$.

(III) *For all* $\alpha \in \operatorname{Con} \mathbf{A}$, $\beta \in \operatorname{Con} \mathbf{B}$,

$$\beta = \alpha|_B \quad \Leftrightarrow \quad \beta^* \leqslant \alpha \leqslant \widehat{\beta}.$$

In particular, $\beta^*|_B = \beta = \widehat{\beta}|_B$.

PROOF OF THE P^5 LEMMA

LEMMA (PÁLFY-PUDLÁK, 1980)

The restriction mapping

$$\operatorname{Con}\mathbf{A} \ni \alpha \mapsto \alpha|_B = \alpha \cap B^2 \in \operatorname{Con}\mathbf{B}$$

is a complete lattice epimorphism.

Overalgebras

- Let $\mathbf{B} = \langle B, F_B \rangle$ be a finite algebra, and suppose

$$\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), (a_2, b_2), \ldots, (a_{K-1}, b_{K-1})).$$

OVERALGEBRAS

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$$\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), (a_2, b_2), \ldots, (a_{K-1}, b_{K-1})).$$

- Fix $u \geqslant 1$ and let $B_1, B_2, \ldots, B_{uK}$ be sets of cardinality $|B|$.
- Fix bijections $\pi_i : B \to B_i$ and let $x^i = \pi_i(x)$, the element of B_i corresponding to $x \in B$.

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- Arrange the sets so they intersect as follows:

  For $\ell \in \{0, K, 2K, \ldots, (u-1)K\}$ and $1 \leqslant i < K$,

$$B_{\ell-1} \cap B_\ell = B_\ell \cap B_{\ell+1} = \{b_{K-1}^{\ell-1}\} = \{a_1^\ell\} = \{a_1^{\ell+1}\},$$
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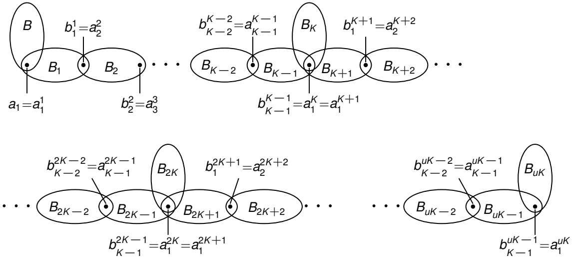

Overalgebras

Let $A = B_0 \cup \cdots \cup B_{uK}$ and define some unary operations on A.

First, for $0 \leqslant i, j \leqslant uK$, let $S_{i,j} : B_i \to B_j$ be the bijection $S_{i,j}(x^i) = x^j$.

Let $\mathscr{T} = |\mathscr{T}_1|\mathscr{T}_2|\cdots|\mathscr{T}_N|$ be a partition of $\{0, K, 2K, \ldots, uK\}$.

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- For each $1 \leqslant n \leqslant N$, for each $\ell \in \mathscr{T}_n$, define

$$e_\ell(x) = \begin{cases} S_{j,\ell}(x), & \text{if } x \in B_j \text{ for some } j \in \mathscr{T}_n, \\ a_1^\ell, & \text{otherwise.} \end{cases}$$

  For each $\ell \in \{0, K, 2K, \dots, (u-1)K\}$, for each $1 \leqslant i < K$, define

$$e_{\ell+i}(x) = \begin{cases} a_i^{\ell+i}, & \text{if } x \in B_j \text{ for some } j < \ell + i, \\ x, & \text{if } x \in B_{\ell+i}, \\ b_i^{\ell+i}, & \text{if } x \in B_j \text{ for some } j > \ell + i. \end{cases}$$

- For $0 \leqslant i, j \leqslant uK$, let $q_{i,j} = S_{i,j} \circ e_i$.

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Take the set of basic operations on A to be

$$F_A = \{fe_0 \mid f \in F_B\} \cup \{q_{i,0} \mid 0 \leqslant i \leqslant uK\} \cup \{q_{0,j} \mid 1 \leqslant j \leqslant uK\}.$$

and define the *overalgebra* of $\mathbf{B}$ (wrt $\beta, u, \mathscr{T}$) as $\mathbf{A} = \langle A, F_A \rangle$.

STRUCTURE OF THE INTERVAL $[\beta^*, \widehat{\beta}]$ IN $\operatorname{Con} \mathbf{A}$

Assume β has m blocks, denoted by C_r $(1 \leqslant r \leqslant m)$, let C_r^j denote $S_{0,j}(C_r)$, and let β^j denote $S_{0,j}(\beta)$.

Structure of the interval $[\beta^*, \widehat{\beta}]$ in $\operatorname{Con}\mathbf{A}$

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Theorem

In the overalgebra $\mathbf{A} = \langle A, F_A \rangle$ described above, for each $0 \leqslant j \leqslant uK$ let t_j be a "tie-point" of the set B_j.

Define

$$\beta^\star = \bigcup_{j=0}^{uK} \beta^j \cup \Big(\bigcup_{j=0}^{uK} t_j/\beta^j\Big)^2.$$

and

$$\widetilde{\beta} = \beta^\star \cup \bigcup_{r=1}^{m-1} \bigcup_{n=1}^{N} \Big(\bigcup_{\ell \in \mathscr{T}_n} C_r^\ell\Big)^2$$

Then,

(I) $\beta^\star = \beta^*$, *the minimal* $\theta \in \operatorname{Con}\mathbf{A}$ *such that* $\theta|_B = \beta$;

(II) $\widetilde{\beta} = \widehat{\beta}$, *the maximal* $\theta \in \operatorname{Con}\mathbf{A}$ *such that* $\theta|_B = \beta$;

(III) *the interval* $[\beta^\star, \widetilde{\beta}]$ *in* $\operatorname{Con}\mathbf{A}$ *satisfies* $[\beta^\star, \widetilde{\beta}] \cong \prod_{n=1}^{N} (\operatorname{Eq}|\mathscr{T}_n|)^{m-1}$.

BASIC STRUCTURE RESULT FOR Con $\mathbf{A}$

Continue to assume $\mathbf{A} = \langle A, F_A \rangle$ is an overalgebra of $\mathbf{B}$ based on:

- $\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \dots, (a_{K-1}, b_{K-1}))$ and
- $\mathscr{T} = |\mathscr{T}_1|\mathscr{T}_2| \cdots |\mathscr{T}_N|$, the partition of $\{0, K, 2K, \dots, uK\}$.

Basic structure result for Con $\mathbf{A}$

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Theorem

Let $\theta \in \operatorname{Con} \mathbf{B}$ and suppose θ has r congruence classes. Then, $\theta^* < \widehat{\theta}$ if and only if $\beta \leqslant \theta < 1_B$, in which case $[\theta^*, \widehat{\theta}] \cong \prod_{n=1}^{N} (\mathrm{Eq}|\mathscr{T}_n|)^{r-1}$.

Consequently, if $\theta \not\geqslant \beta$, then $\widehat{\theta} = \theta^*$.

Basic structure result for Con **A**

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Consequently, if $\theta \ngeqslant \beta$, then $\widehat{\theta} = \theta^*$.

The proof follows easily from the next lemma.

Lemma

Suppose $\eta \in \mathrm{Con}\,\mathbf{A}$ satisfies $\eta|_B = \theta \in \mathrm{Con}\,\mathbf{B}$ and $(x, y) \in \eta \setminus \theta^*$ for some $x \in B_i$, and $y \in B_j$. Then i and j are distinct multiples of K belonging to the same block of $\mathscr{T}$, and $\theta \geqslant \beta$.