# CSPs of Finite Commutative Idempotent Binars

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joint work with

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BLAST 2015
University of North Texas
June 7–12

# Constraint Satisfaction Problems

**Input**

- *variables:* $V = \{v_1, v_2, \ldots\}$
- *domain:* $D$
- *constraints:* $C_1, C_2, \ldots$

**Output**

- "yes" if there is a *solution*

  $\sigma : V \to D$ (an assignment of values to variables that satisfies all $C_i$)

- "no" otherwise

**Input**

- *variables:* $V = \{v_1, \ldots, v_n\}$
- *domain:* $D = \{0, 1\}$
- *constraints:* a formula, say,

$$f(v_1, \ldots, v_n) = (v_1 \vee v_2 \vee \neg v_3) \wedge (\neg v_1 \vee v_3 \vee v_4) \wedge \cdots$$

**Output**

- "yes" if there is a solution: $\sigma : V \to D$ such that

$$f(\sigma v_1, \ldots, \sigma v_n) = 1$$

- "no" otherwise

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- *constraints:* $(s_1, C_1), (s_2, C_2), \ldots$ of the form

$$s = (i, j, k) \in \{1, \ldots, n\}^3 \qquad \text{(scopes)}$$
$$C = \neg(v_i = v_j = v_k)$$

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In terms of relational structures...

Let $S := \{(v_i, v_j, v_k) : (i, j, k) \text{ is a scope } \} \subseteq V^3$

$$R := \{(0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0)\} \subseteq D^3$$

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*Solutions are homomorphisms!*

$$\sigma : \langle V, S\rangle \to \langle D, R\rangle$$

# CSP: RELATIONAL FORMULATION

Let $\mathbb{D} = \langle D, \mathcal{R} \rangle$ be a relational structure.

$\mathrm{CSP}(\mathbb{D})$ is the decision problem with

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- A structure $\mathbb{V} = \langle V, \mathcal{C} \rangle$ *similar* to $\mathbb{D}$.

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Alternatively, let $\Rightarrow$ be the binary relation on similar structures

$$\mathbb{V} \Rightarrow \mathbb{D} \quad \text{iff there is a homomorphism } \sigma : \mathbb{V} \to \mathbb{D}$$

Then the CSP is the membership problem for the set

$$\mathrm{CSP}(\mathbb{D}) := \{\mathbb{V} : \mathbb{V} \Rightarrow \mathbb{D}\}$$

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We call $\mathbb{D}$ "tractable" if there is a polynomial-time algorithm for $\mathrm{CSP}(\mathbb{D})$.

# CSP: ALGEBRAIC FORMULATION

Let $\mathbb{D} = \langle D, \mathcal{R} \rangle$ be a relational structure.

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that is, $f \in \text{pol}(R)$ iff for every $\rho \in R$ (say, $n$-ary)

$$\begin{array}{rcc}
(a_1, b_1, \dots, z_1) & \in & \rho \\
 & \vdots & \\
(a_k, b_k, \dots, z_k) & \in & \rho \\
\hline
(f(a_1, \dots, a_k), \dots, f(z_1, \dots, z_k)) & \in & \rho
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For $F$ a set of operations on $D$, define the *relational clone* of $F$,

$$\text{rel}(F) := \{\rho \subseteq D^n \mid f(\rho) \subseteq \rho \text{ for every } f \in F\}$$

Let $\bar{R} := \text{rel}(\text{pol}(R))$ be the "closure" of $R$.

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$$\text{Then,} \quad \text{CSP}\langle D, R \rangle \leqslant_P \text{CSP}\langle D, \bar{R} \rangle$$

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THEOREM (BODNARČUK ET AL.; GEIGER, 1968)

*Let $R$ be a set of relations on a finite set.*

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**Corollary** $\langle D, \mathrm{pol}(R)\rangle = \langle D, \mathrm{pol}(S)\rangle \implies \mathrm{CSP}(R) \equiv_P \mathrm{CSP}(S)$

*The algebras determine the complexity of the corresponding constraint satisfaction problem!*

## General Problem

Find properties (of algebras) that characterize the complexity of CSPs.

## CSP dichotomy conjecture

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable} \iff \mathbf{A} \text{ has a weak-nu term operation}$$

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The left-to-right direction is known.

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A weak near unanimity (weak-nu) term operation is one that satisfies

$$t(x, x, \dots, x) \approx x \qquad \text{(idempotent)}$$

$$t(y, x, \dots, x) \approx t(x, y, \dots, x) \approx \cdots \approx t(x, x, \dots, y)$$

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A *binary* operation $t(x, y)$ is weak-nu if

$$t(x, x) \approx x \qquad \text{(idempotent)}$$

$$t(y, x) \approx t(x, y) \qquad \text{(commutative)}$$

So let's try to prove (?) for commutative idempotent binars.

# Commutative Idempotent Binars

A CIB is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.

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Pause to consider more general case for a minute...

# General case

## Some well known facts

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

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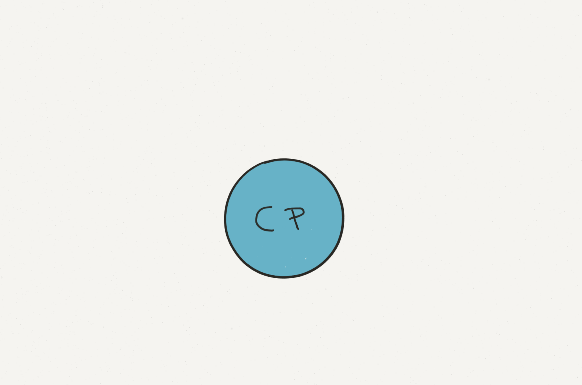

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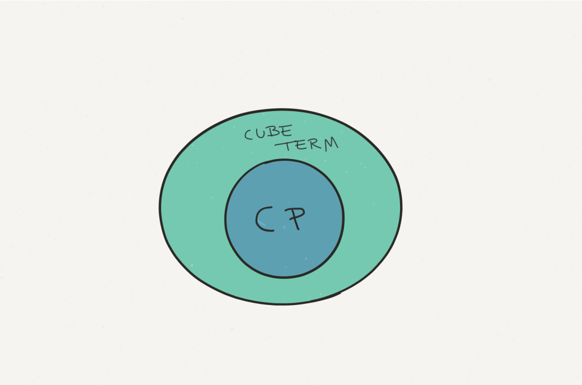

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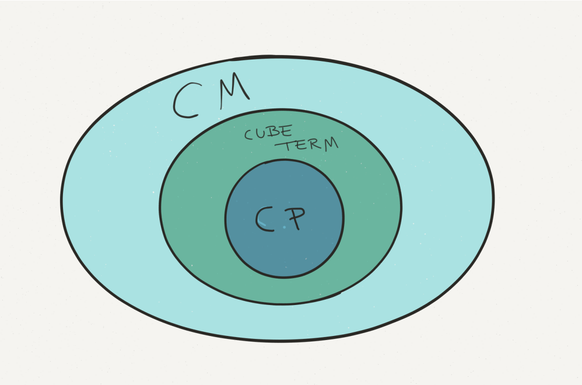

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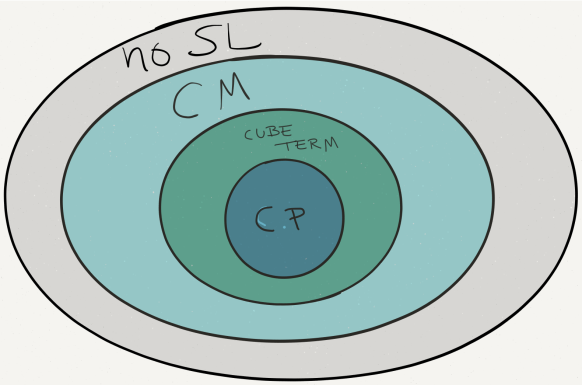

# First Reduction
## by Cube-Term Blockers

Marković, M. Maróti, McKenzie ($M^4$)
"Finitely related clones and algebras with cube terms" (2012)

A cube-term blocker (CTB) is a pair $(C, B)$ of subuniverses satisfying $\emptyset < C < B \leqslant A$ and for every $t(x_1, \ldots, x_n)$ there is an index $i \in [n]$ with

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### Lemma

*A finite CIB $\mathbf{A}$ has a CTB if and only if $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$.*

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### Lemma

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### Proof.

$(C, B)$ a CTB implies $\theta = C^2 \cup (B - C)^2$ a congruence with $\mathbf{B}/\theta \cong \mathbf{S}_2$.

Conversely, suppose $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$, and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ with $\mathbf{B}/\theta$ a meet-SL for some $\theta$. Let $C/\theta$ be the bottom of $\mathbf{B}/\theta$, then $(C, B)$ is a CTB. $\square$

## SECOND REDUCTION

Kearnes and Tschantz
"Automorphism groups of squares and of free algebras" (2007)

### LEMMA

*If $\mathcal{V}$ is an idempotent variety that is not congruence permutable, then there are subuniverses $U$ and $W$ of* $\mathbf{F} := \mathbf{F}_{\mathcal{V}}\{x, y\}$ *satisfying*

1. $x \in U \cap W$
2. $y \in U^c \cap W^c$
3. $(U \times F) \cup (F \times W) \leqslant \mathbf{F}^2$

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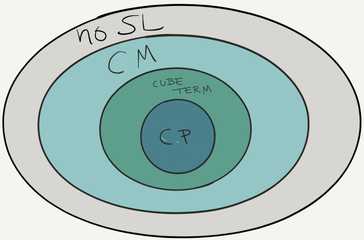
2. $y \in U^c \cap W^c$
3. $(U \times F) \cup (F \times W) \leqslant \mathbf{F}^2$

For CIB's, either $U$ or $W$ will be an ideal.

This implies a CTB and a semilattice.

$\mathbf{A}$ = a finite CIB $\qquad$ $\mathbf{S}_2$ = the 2-elt semilattice.

$$\begin{aligned} \mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has a Malcev term} \\ &\implies \mathbf{A} \text{ has a cube term} \\ &\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\ &\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A}) \end{aligned}$$



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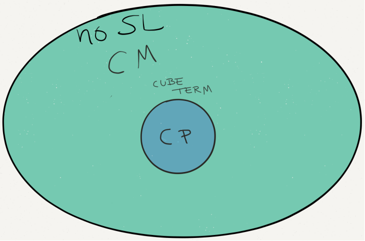


- 1st reduction by cube-term blockers.

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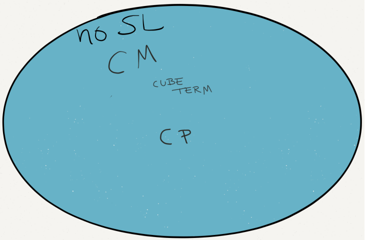


- 1st reduction by cube-term blockers.
- 2nd reduction by Kearnes-Tschantz.

# Example 1

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 0 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

# Example 1

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 0 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

*Cliff's trick:* replace binary operation with a term from $\mathrm{clo}(\mathbf{A})$, say

$$x * y = (x \cdot (x \cdot y)) \cdot (y \cdot (x \cdot y))$$

If $\langle A, * \rangle$ tractable, then so is $\mathbf{A} = \langle A, \cdot \rangle$.

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|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 0 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
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| 1 | 0 | 1 | 3 | 2 |
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$\langle A, * \rangle$ tractable $\quad \Longrightarrow \quad$ $\mathbf{A}$ tractable

# Example 2

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 1 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

Let $t(x, y) = x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y))$.

## EXAMPLE 2

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 1 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

Let $t(x,y) = x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y))$.

## EXAMPLE 2

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 1 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

Let $t(x,y) = x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y))$.

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 0 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

$\langle A, t \rangle$ tractable

# Example 3

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 2 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

# Example 3

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 2 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

Let $t_2(x, y) = \ldots$ ?

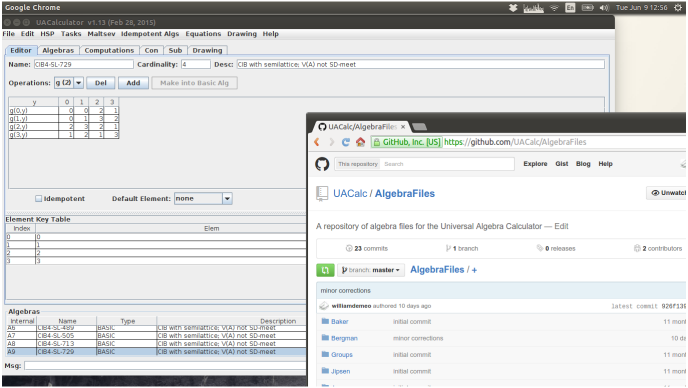
# Example 3

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 1 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 2 | 3 | 2 | 1 |
| 3 | 1 | 2 | 1 | 3 |

Let $t_2(x, y) = \ldots$ ?

Let $t_3(x, y, z) = \ldots$ ?

...and about 25 others.




To see them, load UACalc with files from the Bergman directory at

**`https://github.com/UACalc/AlgebraFiles`**

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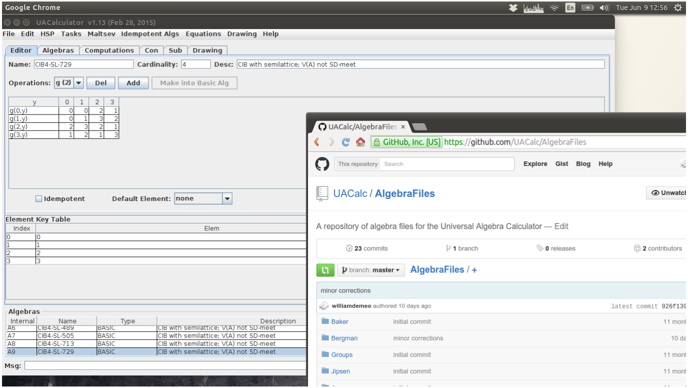


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Thank you for listening!