Universal Algebraic Methods for Constraint Satisfaction Problems

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What is a CSP?

Informally, a **C**onstraint **S**atisfaction **P**roblem consists of

- a list of variables ranging over a finite domain and
- a set of constraints on those variables.

Problem: can we assign values to all the variables so that all of the constraints are satisfied?

More specifically...

Let D be a finite set and $\mathcal{R} \subseteq \mathrm{Rel}(D) = \bigcup_{n<\omega} \mathcal{P}(D^n)$

CSP$(D, \mathcal{R})$ is the following decision problem:

Instance:

- variables: $V = \{ v_1, \dots, v_n \}$ (a finite set)
- constraints: $(C_1, \dots, C_m)$ (a finite list)

Each C_i is a pair $(\mathbf{s}_i,\ R_i)$, where

$$\mathbf{s}_i(j) \in V \quad \text{and} \quad R_i \in \mathcal{R}$$

Question: Does there exist a solution?

an assignment $f \colon V \to D$ of values to variables satisfying

$$\forall i \quad f \circ \mathbf{s}_i = (\ f\,\mathbf{s}_i(1),\ f\,\mathbf{s}_i(2), \dots,\ f\,\mathbf{s}_i(p)\ ) \in R_i$$

The CSP-Dichotomy Conjecture

Conjecture of Feder and Vardi

Every $CSP(D, \mathcal{R})$ either lies in $\mathbb{P}$ or is $\mathbb{NP}$-complete.

Polymorphisms

Definition

Let $R \in \mathrm{Rel}_k(D)$ and $f\colon D^n \to D$. We say f *preserves* R if

$$(a_{11}, \ldots, a_{1k}), \ldots, (a_{n1}, \ldots, a_{nk}) \in R \implies (f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1k}, \ldots, a_{nk})) \in R$$

$$\begin{array}{ccccccc} a_{11} & a_{12} & \ldots & a_{1k} & \in & R \\ a_{21} & a_{22} & \ldots & a_{2k} & \in & R \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nk} & \in & R \\ \downarrow & \downarrow & & \downarrow & & \\ (\ f(\mathbf{a}_1) & f(\mathbf{a}_2) & \ldots & f(\mathbf{a}_k)\ ) & \in & R \end{array}$$

Galois Connection

Let $\mathcal{R}$ be a set of relations on D.

Poly($\mathcal{R}$) = set of all operations preserving all relations in $\mathcal{R}$.

These are the polymorphisms of $\mathcal{R}$.

Let $\mathcal{F}$ be a set of operations on D.

Inv($\mathcal{F}$) = set of all relations preserved by all operations in $\mathcal{F}$.

Galois Connection...

...from relational to algebraic structures, and back.

$$\begin{array}{ccc} \textbf{Relational} & & \textbf{Algebraic} \\ (D, \mathcal{R}) & \longrightarrow & (D, \mathrm{Poly}(\mathcal{R})) \\ (D, \mathrm{Inv}(\mathcal{F})) & \longleftarrow & (D, \mathcal{F}) \end{array}$$

$\mathrm{CSP}(D, \mathcal{R}) \equiv_{\mathrm{p}} \mathrm{CSP}(D, \mathrm{Inv}(\mathrm{Poly}(\mathcal{R})))$

We can use algebra to help classify CSPs!

Algebraic CSP

For an algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$ *define* $\mathrm{CSP}(\mathbf{A}) = \mathrm{CSP}(A, \mathrm{Inv}(\mathcal{F}))$

Informal algebraic CSP dichotomy conjecture

If Poly($\mathbf{A}$) is rich, then CSP($\mathbf{A}$) is tractable.

If Poly($\mathbf{A}$) is poor, then CSP($\mathbf{A}$) is $\mathbb{NP}$-complete.

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What does it mean to be rich?

Two General Techniques/Algorithms

Method 1

If $\mathrm{Poly}(\mathcal{R})$ contains a "cube term" then $\mathrm{CSP}(\mathcal{R}) \in \mathbb{P}$

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Examples of cube terms:

$P(x, y, z) = x - y + z$
$M(x, y, z) = \text{majority}$

Algebras with a cube term operation possess "few subpowers."

This is used to prove the algorithm is poly-time.

Two General Techniques for Tractable Algorithms

Method 2

If $\mathrm{Poly}(\mathcal{R})$ contains WNU terms $v(x, y, z)$ and $w(x, y, z, u)$ satisfying $v(y, x, x) = w(y, x, x, x)$, then $\mathrm{CSP}(\mathcal{R}) \in \mathbb{P}$.

Two General Techniques for Tractable Algorithms

Method 2

If Poly($\mathcal{R}$) contains WNU terms $v(x, y, z)$ and $w(x, y, z, u)$ satisfying $v(y, x, x) = w(y, x, x, x)$, then CSP($\mathcal{R}$) $\in \mathbb{P}$.

Examples: majority, semilattice

Algebras with these operations are congruence SD-$\wedge$

Current State of Affairs

The two general techniques do not cover all cases of a WNU term.

Two possible directions:

1. Find a completely new algorithm.
2. Combine the two existing algorithms.

We describe some progress in the second direction.

A Motivating Example

Let $\mathbf{A} = \langle \{0, 1, 2, 3\}, \cdot \rangle$, have the following Cayley table:

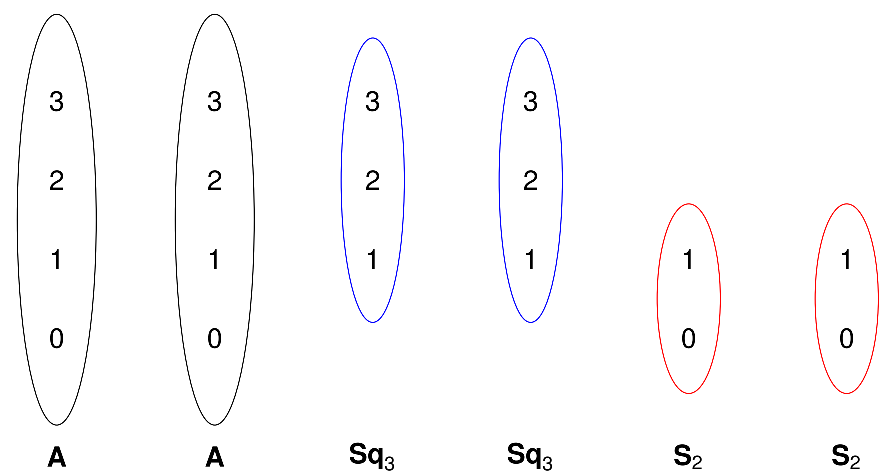
$\cdot$	0	1	2	3
0	0	0	3	2
1	0	1	3	2
2	3	3	2	1
3	2	2	1	3

What is an instance of CSP(S($\mathbf{A}$))?

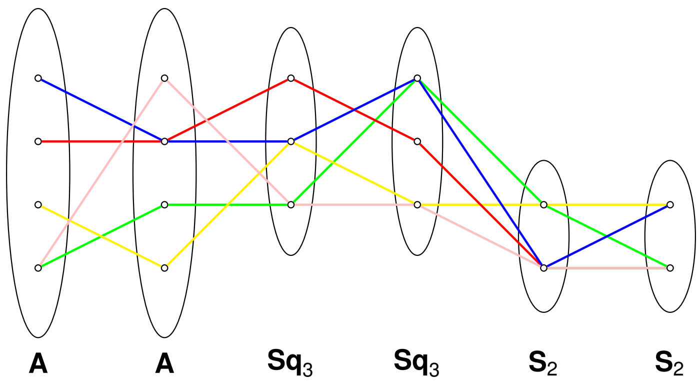
Contraint relations are subdirect products of subalgebras of $\mathbf{A}$.

The proper nontrivial subuniverses of $\mathbf{A}$ are $\{0, 1\}$ and $\{1, 2, 3\}$.

Potatoes of a six-variables instance of CSP(S($\mathbf{A}$))



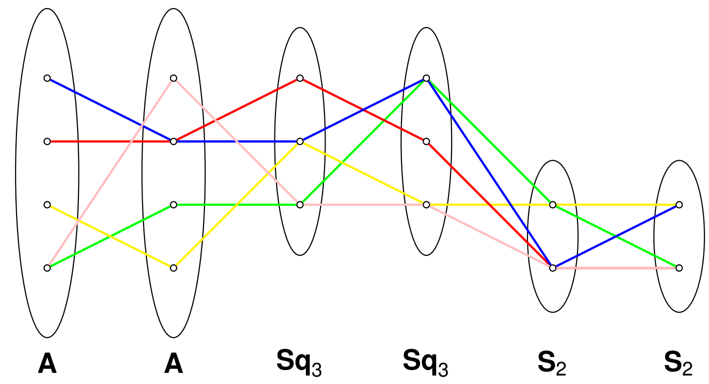
Subuniverse of Product = Constraint Relation



Each colored line represents a tuple in the relation R

$R \subseteq A \times A \times Sq_3 \times Sq_3 \times S_2 \times S_2$

Subuniverse of Product = Constraint Relation



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Question: Does this R form a subuniverse?

Theorem 1

Let $\mathbf{A}_i$, $\mathbf{B}_j$ be finite algebras in a Taylor variety. Assume

- each $\mathbf{A}_i$ is abelian
- each $\mathbf{B}_j$ has a sink s_j

Suppose

$$\mathbf{R} \leq_{\mathrm{sd}} \mathbf{A}_1 \times \cdots \times \mathbf{A}_J \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_K$$

Then

$$\mathrm{Proj}_{1\ldots J} R \times \{s_1\} \times \{s_2\} \times \cdots \times \{s_K\} \subseteq R$$

By *Taylor variety* we mean an idempotent variety with a Taylor term.

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By *Taylor variety* we mean an idempotent variety with a Taylor term.

$s \in B$ is called a sink if for all $t \in Clo_k(\mathbf{B})$ and $1 \leq j \leq k$, if t depends on its j-th argument, then $t(b_1, \ldots, b_{j-1}, s, b_{j+1}, \ldots, b_k) = s$ for all $b_i \in B$.

Theorem 2

Let $\mathbf{A}_i$, $\mathbf{B}_j$ be finite algebras in a Taylor variety. Assume

- each $\mathbf{A}_i$ has a cube term operation
- each $\mathbf{B}_j$ has a sink s_j

Suppose

$$\mathbf{R} \leq_{\text{sd}} \mathbf{A}_1 \times \cdots \times \mathbf{A}_J \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_K$$

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The proof depends on the following result of Barto, Kozik, Stanovsky: a finite idempotent algebra has a cube term iff every one of its subalgebras has a so called transitive term operation.

Application

Corollary

Suppose every algebra in the set $\mathcal{A}$ contains either a cube terms or a sink. Then CSP$(\mathcal{A})$ is tractable.

Algorithm:

Restrict the given instance to potatoes with cube terms.

Find a solution to the restricted instance (in poly-time by few subpowers).

If a restricted solution exists, then there is a full solution (by Thm 2).

If no restricted solution exists, then no full solution exists.

Quotient strategy

Start with

$$\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_n$$

Choose a tuple of congruence relations

$$\Theta = (\theta_1, \theta_2, \dots, \theta_n) \in \prod \text{Con}\, \mathbf{A}_i$$

so that $\mathcal{A} := \{\mathbf{A}_1/\theta_0, \dots, \mathbf{A}_n/\theta_n\}$ is a "jointly tractable" set of algebras.

That is, CSP$(\mathcal{A})$ is tractable.

Obvious fact: a solution to I is a solution to I/Θ.

For some problems, we have the following converse:

$(\star)$ a solution to I/Θ can *always* be extended to a solution to I.

Problem: For what algebras does the $\star$-converse hold?