

THE ALGEBRAIC APPROACH TO CSP  
AND  
CSPs OF COMMUTATIVE IDEMPOTENT BINARS

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joint work with

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# CONSTRAINT SATISFACTION PROBLEMS

## Input

- *variables*:  $V = \{v_1, v_2, \dots\}$
- *domain*:  $D$
- *constraints*:  $C_1, C_2, \dots$

## Output

- “yes” if there is a *solution*

$\sigma : V \rightarrow D$  (an assignment of values to variables that satisfies all  $C_i$ )

- “no” otherwise

# CONSTRAINT SATISFACTION PROBLEMS

## EXAMPLE: 3-SAT

### Input

- *variables*:  $V = \{v_1, \dots, v_n\}$
- *domain*:  $D = \{0, 1\}$
- *constraints*: a formula, say,

$$f(v_1, \dots, v_n) = (v_1 \vee v_2 \vee \neg v_3) \wedge (\neg v_1 \vee v_3 \vee v_4) \wedge \dots$$

### Output

- “yes” if there is a solution:  $\sigma : V \rightarrow D$  such that

$$f(\sigma v_1, \dots, \sigma v_n) = 1$$

- “no” otherwise

# CONSTRAINT SATISFACTION PROBLEMS

EXAMPLE: NAE-SAT

## Input

- *variables*:  $V = \{v_1, \dots, v_n\}$
- *domain*:  $D = \{0, 1\}$
- *constraints*:  $(s_1, C_1), (s_2, C_2), \dots$  of the form

$$s = (i, j, k) \in \{1, \dots, n\}^3 \quad (\text{scopes})$$

$$C = \neg(v_i = v_j = v_k)$$

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In terms of relational structures...

$$\text{Let } S := \{(v_i, v_j, v_k) : (i, j, k) \text{ is a scope}\} \subseteq V^3$$

$$R := \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0)\} \subseteq D^3$$

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*Solutions are homomorphisms!*

$$\sigma : \langle V, S \rangle \rightarrow \langle D, R \rangle$$

## CSP: RELATIONAL FORMULATION

Let  $\mathbb{D} = \langle D, \mathcal{R} \rangle$  be a relational structure.

$\text{CSP}(\mathbb{D})$  (or  $\text{CSP}(\mathcal{R})$ ) is the decision problem with

### Input

- A structure  $\mathbb{V} = \langle V, \mathcal{C} \rangle$  *similar* to  $\mathbb{D}$ .

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- “yes” if there is a homomorphism  $\sigma : \mathbb{V} \rightarrow \mathbb{D}$
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Alternatively, let  $\Rightarrow$  be the binary relation on similar structures:

$$\mathbb{V} \Rightarrow \mathbb{D} \quad \text{iff there is a homomorphism } \sigma : \mathbb{V} \rightarrow \mathbb{D}$$

Then the CSP of  $\mathbb{D}$  is the membership problem for the set

$$\text{CSP}(\mathbb{D}) := \{ \mathbb{V} : \mathbb{V} \Rightarrow \mathbb{D} \}$$

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### Output

- “yes” if there is a homomorphism  $\sigma : \mathbb{V} \rightarrow \mathbb{D}$
- “no” otherwise

We call  $\mathbb{D}$  (or  $\mathcal{R}$ ) “tractable” if there is a polynomial-time algorithm for solving  $\text{CSP}(\mathbb{D})$  (or  $\text{CSP}(\mathcal{R})$ ).

## CSP: ALGEBRAIC FORMULATION

Let  $\mathbb{D} = \langle D, \mathcal{R} \rangle$  be a relational structure.

For  $R \subseteq \mathcal{R}$  define the *polymorphisms* of  $R$ ,

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that is,  $f \in \text{pol}(R)$  iff for every  $\rho \in R$

$$(a_1, b_1, \dots, z_1) \in \rho$$

$$\vdots$$

$$(a_k, b_k, \dots, z_k) \in \rho$$

---

$$(f(a_1, \dots, a_k), \dots, f(z_1, \dots, z_k)) \in \rho$$

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Define the algebra  $\mathbf{D} := \langle D, \text{pol}(R) \rangle$ .

We call  $\mathbf{D}$  “tractable” if the corresponding structure  $\langle D, R \rangle$  is tractable.

## CSP: ALGEBRAIC FORMULATION

For  $F$  a set of operations on  $D$ , define the *relational clone* of  $F$ ,

$$\text{rel}(F) := \{\rho \subseteq D^n \mid f(\rho) \subseteq \rho \text{ for every } f \in F\}$$

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### THEOREM

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### THEOREM

$$\text{CSP}\langle D, R \rangle \equiv_P \text{CSP}\langle D, \bar{R} \rangle$$

$$\text{Corollary} \quad \text{pol}(R) = \text{pol}(S) \implies \text{CSP}(R) \equiv_P \text{CSP}(S)$$

*The algebra  $\langle D, \text{pol}(R) \rangle$  determines the complexity of the corresponding CSP!*

## GENERAL PROBLEM

Find properties (of algebras) that characterize the complexity of CSPs.

## CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra  $\mathbf{A}$ ...

$\text{CSP}(\mathbf{A})$  is tractable  $\iff \mathbf{A}$  has a weak-nu term operation

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The left-to-right direction is known.

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The right-to-left direction is open.

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A **weak near unanimity** (weak-nu) term operation is one that satisfies

$$t(x, x, \dots, x) \approx x \quad (\text{idempotent})$$

$$t(y, x, \dots, x) \approx t(x, y, \dots, x) \approx \dots \approx t(x, x, \dots, y)$$

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A *binary* operation  $t(x, y)$  is weak-nu if

$$t(x, x) \approx x \quad (\text{idempotent})$$

$$t(y, x) \approx t(x, y) \quad (\text{commutative})$$

So let's try to prove (?) for **commutative idempotent binars**.

## COMMUTATIVE IDEMPOTENT BINARS

A **CIB** is an algebra  $\mathbf{A} = \langle A, \cdot \rangle$  satisfying  $x \cdot y \approx y \cdot x$  and  $x \cdot x \approx x$ .

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Semilattices are tractable.

Pause to consider more general case for a minute...

## GENERAL CASE

### SOME WELL KNOWN FACTS

Let  $\mathbf{A}$  be a finite idempotent algebra. Let  $\mathbf{S}_2$  be the 2-elt semilattice.

$$\mathbf{V}(\mathbf{A}) \text{ is CP} \iff \mathbf{A} \text{ has Malcev term}$$

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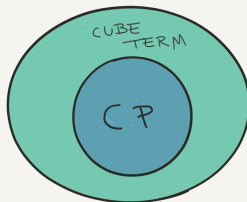


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$$\begin{aligned} \mathbf{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has Malcev term} \\ &\implies \mathbf{A} \text{ has cube term} \end{aligned}$$



## GENERAL CASE

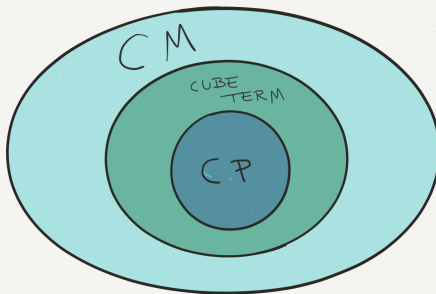
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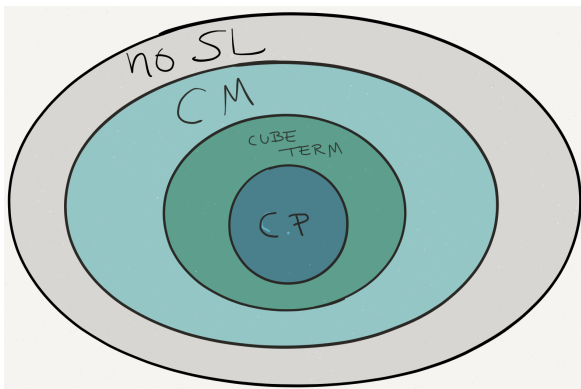
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$\implies V(\mathbf{A})$  is CM

$\implies \mathbf{S}_2$  is not in  $V(\mathbf{A})$



# FIRST REDUCTION

BY CUBE-TERM BLOCKERS

Marković, M. Maróti, McKenzie ( $M^4$ )

“Finitely related clones and algebras with cube terms” (2012)

A **cube-term blocker** (CTB) is a pair  $(C, B)$  of subuniverses satisfying  $\emptyset < C < B \leq A$  and for every  $t(x_1, \dots, x_n)$  there is an index  $i \in [n]$  with

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## LEMMA

*A finite CIB  $\mathbf{A}$  has a CTB if and only if  $\mathbf{S}_2 \in \text{HS}(\mathbf{A})$ .*

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## PROOF.

$(C, B)$  a CTB implies  $\theta = C^2 \cup (B - C)^2$  a congruence with  $\mathbf{B}/\theta \cong \mathbf{S}_2$ .

Conversely, suppose  $\mathbf{S}_2 \in \text{HS}(\mathbf{A})$ , and  $\mathbf{B}$  is a subalgebra of  $\mathbf{A}$  with  $\mathbf{B}/\theta$  a meet-SL for some  $\theta$ . Let  $C/\theta$  be the bottom of  $\mathbf{B}/\theta$ , then  $(C, B)$  is a CTB. □

## SECOND REDUCTION

Kearnes and Tschantz

“Automorphism groups of squares and of free algebras” (2007)

### LEMMA

*If  $V$  is an idempotent variety that is not congruence permutable, then there are subuniverses  $U$  and  $W$  of  $\mathbf{F} := \mathbf{F}_V\{x, y\}$  satisfying*

1.  $x \in U \cap W$
2.  $y \in U^c \cap W^c$
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For CIB's, either  $U$  or  $W$  will be an ideal.

This implies a CTB and a semilattice.

$\mathbf{A}$  = a finite CIB

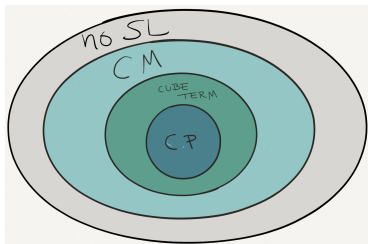
$\mathbf{S}_2$  = the 2-elt semilattice.

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$\implies$   $\mathbf{A}$  has a cube term

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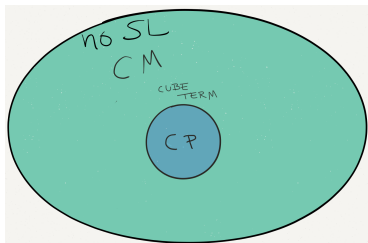
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■ 1st reduction by cube-term blockers.



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■ 1st reduction by cube-term blockers.

■ 2nd reduction by Kearnes-Tschantz.

## REMAINING QUESTIONS FOR FINITE CIBs

### CONCLUSION

Let  $\mathbf{A}$  be a finite CIB. Then

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### OPEN QUESTION

Let  $\mathbf{A}$  be a finite CIB with  $S_2$  in  $HS(\mathbf{A})$ . Is  $CSP(\mathbf{A})$  tractable?

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Recall, if  $V(\mathbf{A})$  is  $\text{SD}_\wedge$ , then  $\text{CSP}(\mathbf{A})$  is tractable.

## REMAINING QUESTIONS FOR FINITE CIBS

### CONCLUSION

Let  $\mathbf{A}$  be a finite CIB. Then

$S_2 \notin \text{HS}(\mathbf{A})$  if and only if  $V(\mathbf{A})$  is congruence permutable.

(so  $\text{CSP}(\mathbf{A})$  tractable in this case)

### OPEN QUESTION

Let  $\mathbf{A}$  be a finite CIB with  $S_2$  in  $\text{HS}(\mathbf{A})$ . Is  $\text{CSP}(\mathbf{A})$  tractable?

Recall, if  $V(\mathbf{A})$  is  $\text{SD}_\wedge$ , then  $\text{CSP}(\mathbf{A})$  is tractable.

### REVISED QUESTION

Let  $\mathbf{A}$  be a finite CIB with  $S_2$  in  $\text{HS}(\mathbf{A})$ , and  $V(\mathbf{A})$  not  $\text{SD}_\wedge$ .

Is  $\text{CSP}(\mathbf{A})$  tractable?

## EXAMPLE 1

$\cdot$	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

## EXAMPLE 1

*Cliff's trick:* replace binary operation with a term from  $\text{clo}(\mathbf{A})$ , say

$$x * y = (x \cdot (x \cdot y)) \cdot (y \cdot (x \cdot y))$$

If  $\langle A, * \rangle$  tractable, then so is  $\mathbf{A} = \langle A, \cdot \rangle$ .

$\cdot$	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
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If  $\langle A, * \rangle$  tractable, then so is  $\mathbf{A} = \langle A, \cdot \rangle$ .

$$\begin{aligned} \{*\} \subseteq \text{clo}(\mathbf{A}) &\implies \text{rel}(\text{clo}(\mathbf{A})) \subseteq \text{rel}(\{*\}) \\ &\implies \text{CSP}(\mathbf{A}) \leq_P \text{CSP}\langle A, * \rangle \end{aligned}$$

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$$\langle A, * \rangle \text{ tractable} \implies \mathbf{A} \text{ tractable}$$

$\cdot$	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

$*$	0	1	2	3
0	0	0	0	0
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

## EXAMPLE 2

$\cdot$	0	1	2	3
0	0	0	1	1
1	0	1	3	2
2	1	3	2	1
3	1	2	1	3

Let  $t(x, y) = x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y))$ .

## EXAMPLE 2

$\cdot$	0	1	2	3
0	0	0	1	1
1	0	1	3	2
2	1	3	2	1
3	1	2	1	3

$t$	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

Let  $t(x, y) = x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y))$ .

$\langle A, t \rangle$  tractable

### EXAMPLE 3

$\cdot$	0	1	2	3
0	0	0	2	1
1	0	1	3	2
2	2	3	2	1
3	1	2	1	3

### EXAMPLE 3

$\cdot$	0	1	2	3
0	0	0	2	1
1	0	1	3	2
2	2	3	2	1
3	1	2	1	3

Let  $t_2(x, y) = \dots$  ?

### EXAMPLE 3

$\cdot$	0	1	2	3
0	0	0	2	1
1	0	1	3	2
2	2	3	2	1
3	1	2	1	3

Let  $t_2(x, y) = \dots$  ?

Let  $t_3(x, y, z) = \dots$  ?

...and about 25 others.

The image shows two overlapping windows. The background window is the UACalculator v1.13 application, which has a menu bar (File, Edit, HSP, Tasks, Maltsev, Idempotent Algs, Equations, Drawing, Help) and a toolbar. The 'Editor' tab is active, showing a form for defining an algebra with fields for Name (CB4-SL-729), Cardinality (4), and Desc (CB with semilattice; V(A) not SD-meet). Below this is a table for operations (g(2)) and a section for the Element Key Table. The foreground window is a web browser displaying the GitHub repository for UACalc/AlgebraFiles. The repository page shows 23 commits, 1 branch, 0 releases, and 2 contributors. A list of commits is visible, including one by 'willanderson' and another by 'Baker'.

UACalculator v1.13 (Feb 28, 2015)

File Edit HSP Tasks Maltsev Idempotent Algs Equations Drawing Help

Editor Algebras Computations Con Sub Drawing

Name: CB4-SL-729 Cardinality: 4 Desc: CB with semilattice; V(A) not SD-meet

Operations: g(2) Del Add Make Into Basic Alg

y	0	1	2	3
g(0,y)	0	0	2	1
g(1,y)	0	1	3	2
g(2,y)	2	3	2	1
g(3,y)	1	2	1	3

☐ Idempotent Default Element: none

Element Key Table

Index	Elem
0	0
1	1
2	2
3	3

Algebras

Internal	Name	Type	Description
A6	CB4-SL-439	BASIC	CB with semilattice; V(A) not SD-meet
A7	CB4-SL-505	BASIC	CB with semilattice; V(A) not SD-meet
A8	CB4-SL-713	BASIC	CB with semilattice; V(A) not SD-meet
A9	CB4-SL-729	BASIC	CB with semilattice; V(A) not SD-meet

Msg:

UACalc/AlgebraFiles

https://github.com/UACalc/AlgebraFiles

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UACalc / AlgebraFiles

A repository of algebra files for the Universal Algebra Calculator — Edit

23 commits 1 branch 0 releases 2 contributors

branch: master AlgebraFiles / +

minor corrections

willanderson authored 10 days ago latest commit: 926f135

- Baker initial commit 11 months ago
- Bergman minor corrections 10 days ago
- Groups initial commit 11 months ago
- Jipsen initial commit 11 months ago

To see them, load UACalc with files from the **Bergman** directory at  
**<https://github.com/UACalc/AlgebraFiles>**



...and about 25 others.

The image shows two overlapping windows. The background window is the UACalc software interface, titled 'UACalc v1.13 (Feb 28, 2015)'. It has a menu bar (File, Edit, HSP, Tasks, Maltsev, Idempotent Algs, Equations, Drawing, Help) and a toolbar. The 'Editor' tab is active, showing a form for defining an algebra with fields for Name, Cardinality, and Description. Below this is a table for operations (q(0,y) to q(3,y)) and a section for 'Element Key Table' with columns for Index and Elem. At the bottom is a table of 'Algebras' with columns for Internal, Name, Type, and Description.

The foreground window is a web browser showing the GitHub repository for 'UACalc / AlgebraFiles'. The repository has 23 commits, 1 branch, 0 releases, and 2 contributors. The 'AlgebraFiles' directory is selected. A list of contributors is shown, including 'Baker', 'Bergman', 'Groups', and 'Jipson'. The 'Bergman' entry is highlighted, showing a commit for 'minor corrections'.

To see them, load UACalc with files from the **Bergman** directory at

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Thank you for listening!