The Finite Lattice Representation Problem

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Background

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Every algebraic lattice is isomorphic to the congruence lattice of an algebra.

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status: open

age: 45+ years

What is an algebra?

Definition (algebra)

An algebra $\mathbf{A}$ is an ordered pair $\mathbf{A} = \langle A, F \rangle$ where

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- A **variety** $\mathcal{K}$ of algebras is a class of (similar) algebras defined by equations. They are closed under homomorphic images, subalgebras and direct products, and in fact

$$\boldsymbol{V}(\mathcal{K}) = \boldsymbol{HSP}(\mathcal{K})$$

is the variety generated by a class $\mathcal{K}$ of algebras.

Examples

- A group is an algebra $\mathbf{G} = \langle G, \cdot, ^{-1}, 1 \rangle$ with binary, unary, and nullary operations satisfying, $\forall x, y, z \in G$,

  G1: $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$
  G2: $x \cdot 1 \approx 1 \cdot x \approx x$
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- A lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ with universe L, a partially ordered set, and binary operations:

$$x \wedge y = \text{g.l.b.}(x, y) \quad \text{the "meet" of } x \text{ and } y$$
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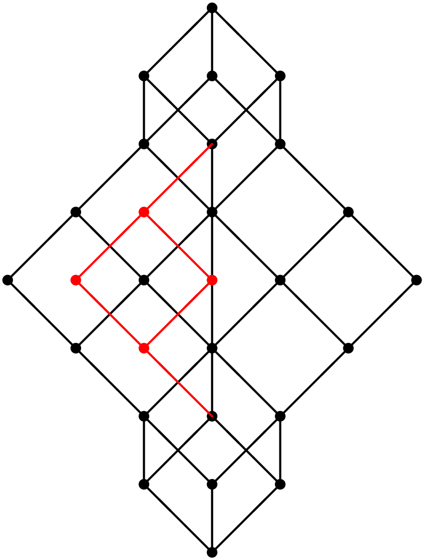
- *Examples of lattices:*
  - subsets of a set
  - closed subsets of a topology
  - subgroups of a group, normal subgroups of a group
  - ideals of a ring
  - submodules of a module
  - invariant subspaces of an operator or operator algebra

Lattices



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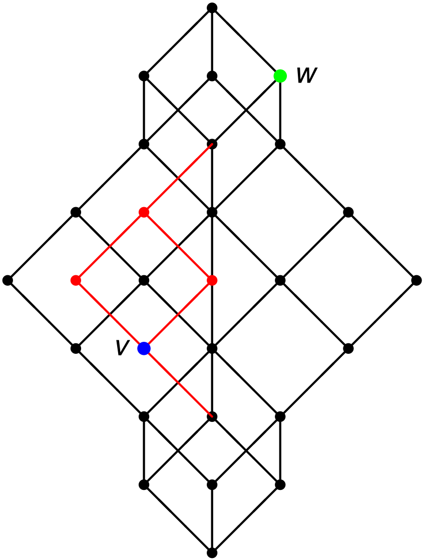

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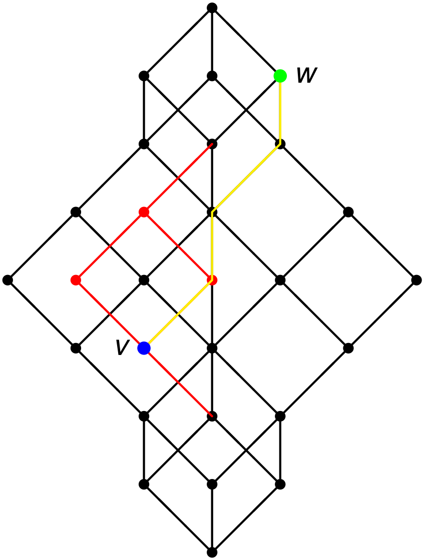
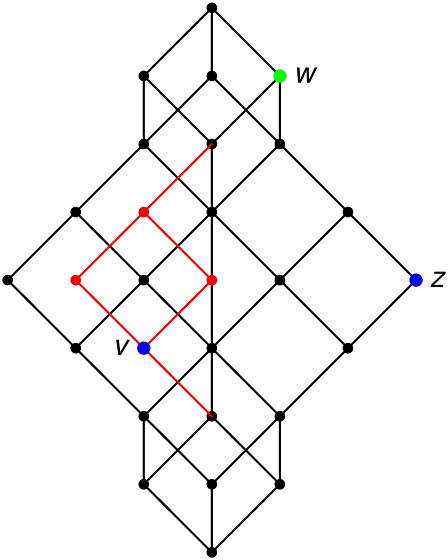


$v \leq w$

Lattices



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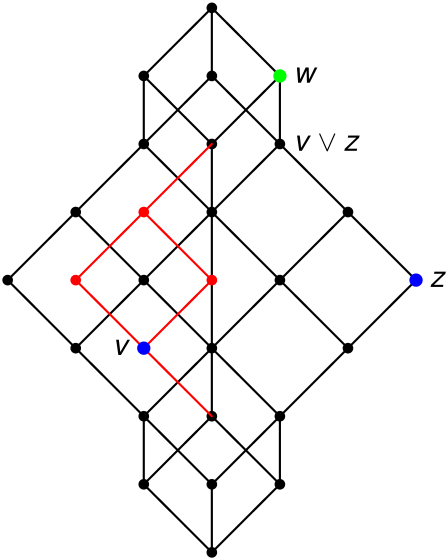
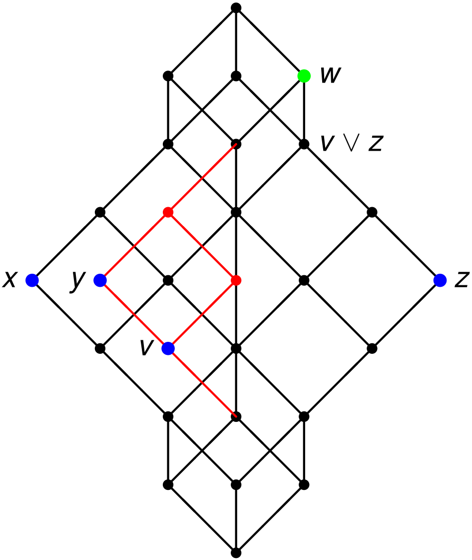
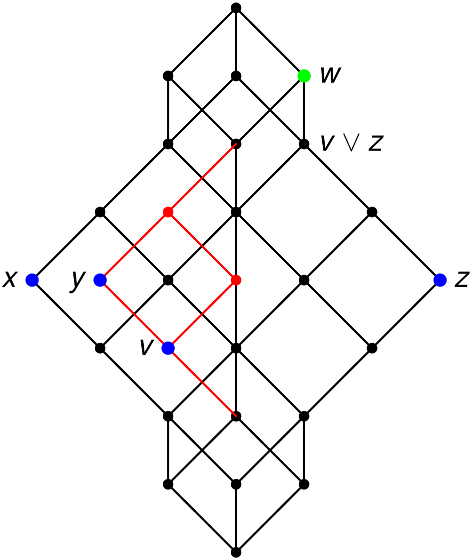




x, y, z **generate**

Lattices



$$v = y \wedge (x \vee z)$$

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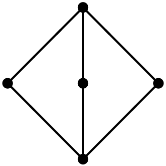

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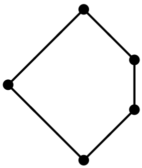
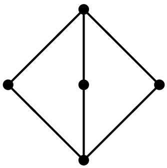
- $\mathbf{N}_5$ is not even modular.

$\mathbf{N}_5$: 

$\mathbf{M}_3$: 

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- Then $\mathbf{CSub}[\mathbf{X}] = \langle \mathrm{CSub}[\mathbf{X}], \wedge, \vee \rangle$ is a lattice.
- It is *modular* if and only if $\mathbf{X}$ is finite dimensional.
- It is *distributive* if and only if $\mathbf{X}$ has dimension 0 or 1.

  (See e.g. Halmos, "A Hilbert Space Problem Book," Springer, 1984.)

Example: **Sub**[**G**]

- The **lattice of subgroups** of a group **G**

$$\mathbf{Sub}[\mathbf{G}] = \langle \mathrm{Sub}[\mathbf{G}], \subseteq \rangle = \langle \mathrm{Sub}[\mathbf{G}], \wedge, \vee \rangle$$

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- For subgroups $H, K \in \mathrm{Sub}[\mathbf{G}]$,

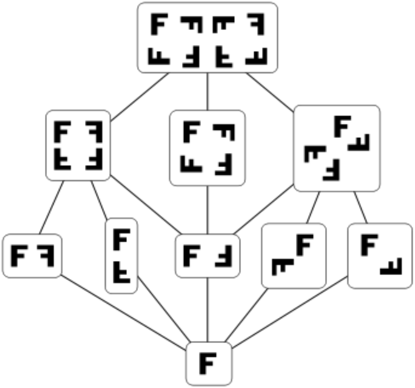
  meet is set intersection:

$$H \wedge K = H \cap K$$

  join is the subgroup generated by set union:

$$H \vee K = \bigcap \{ J \in \mathrm{Sub}[\mathbf{G}] \mid H \cup K \subseteq J \}$$

Example: Hasse diagram of **Sub**$[D_4]$



The lattice of subgroups of the dihedral group D_4, represented as groups of rotations and reflections of a plane figure.

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- Similar lattice-theoretic characterizations exist for solvable and perfect groups.

  Michio Suzuki, “On the lattice of subgroups of finite groups,” *Trans. AMS* (1951)

  , “Structure of a group and the structure of its lattice of subgroups,” *Springer* (1956)

Example: equivalence relations

- The set of equivalence relations on a set is a 0-1 lattice:

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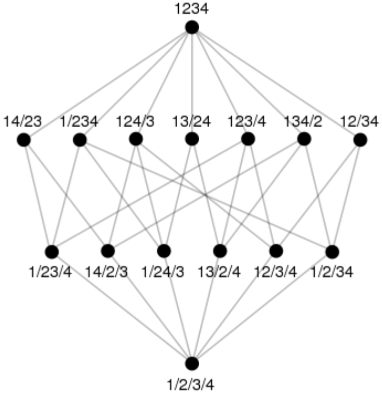
- The greatest equivalence is the *all* relation:

$$\nabla = A \times A$$

- The least equivalence is the *diagonal* relation:

$$\Delta = \{(x, y) \in A \times A \mid x = y\}$$

Example: Eq(4)



The lattice of equivalence relations on the set of four elements.

Congruence Lattices

- If $f : A \to B$ is a mapping of one set to another, the relation θ on A,
$$x \, \theta \, y \quad \Leftrightarrow \quad f(x) = f(y)$$
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- For lattices, and the algebras of logic, **ConA** is distributive.

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Theorem (Pálfy and Pudlák, AU 11, 1980)

The following statements are equivalent:

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- If $h \, R \, \theta$ we say "h respects θ" or "θ admits h"

YAGC

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- (λ, ρ) is a *Galois correspondence* between $\mathrm{Eq}(X)$ and X^X
- Easy consequences:
  $\rho\lambda$ and $\lambda\rho$ are idempotent; $\rho\lambda\rho = \rho$ and $\lambda\rho\lambda = \lambda$;
  $F \subseteq \rho\lambda(F)$, for any set $F \in \mathcal{E}$.

Closure operator

- The map $\rho\lambda$ is idempotent, extensive, and order preserving; i.e.

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  We call F dense iff $\rho\lambda(F) = \mathrm{Eq}(X)$
- If $\mathbf{L} \cong \mathbf{L}' \leq \mathbf{Eq}(X)$ and if $\rho\lambda(L') = \mathrm{Eq}(X)$, then we say $\mathbf{L}$ can be densely embedded in $\mathbf{Eq}(X)$.

A density result

Theorem

If $\mathbf{L} \leq \mathbf{Eq}(X)$*, then* $\mathbf{L} = \mathbf{Con}\mathbf{A}$ *for some algebra* $\mathbf{A} = \langle X, F \rangle$ *if and only if* $\mathbf{L}$ *is closed; that is, iff* $\rho\lambda(L) = L$*.*

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Idea of proof: Find an $\mathbf{L} \cong \mathbf{M}_3$ in $\mathbf{Eq}(X)$ such that every non-trivial operation in X^X violates some equivalence in the universe L of $\mathbf{L}$. Then $\lambda(L)$ is trivial, so the closure $\rho\lambda(L)$ is all of Eq(X). John Snow proved this for $|X|$ odd.

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So, for any $n \geq 3$, $\mathbf{M}_n$ can be densely embedded in $\mathbf{Eq}(X)$, for some finite set X.

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Lemma

Suppose $\mathbf{L} = \langle L, \wedge, \vee \rangle$ *is a complete* $0, 1$*-lattice. TFAE*

(i) *There is an element* $\alpha \in L \setminus \{0_L\}$ *such that* $\bigvee\{\gamma \in L : \gamma \ngeq \alpha\} < 1_L$

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If $\mathbf{L} \not\cong \mathbf{2}$ *is a sublattice of* $\mathbf{Eq}(X)$ *satisfying conditions of the lemma, then* $\lambda(L)$ *contains a non-trivial unary function.*

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Corollary

If $\mathbf{L} \ncong \mathbf{2}$ *is a lattice satisfying conditions of the lemma, then* $\mathbf{L}$ *cannot be densely embedded in* $\mathbf{Eq}(X)$.

More non-density consequences...

Corollary

If $\mathbf{L} \ncong \mathbf{2}$ is a finite lattice with a prime element and X is any set, then $\mathbf{L}$ cannot be densely embedded in $\mathbf{Eq}(X)$.

Corollary

If $\mathbf{L} \in SD_{\wedge}$ is a finite semi-distributive lattice with $\mathbf{L} \ncong \mathbf{2}$, and X is any set, then $\mathbf{L}$ cannot be densely embedded in $\mathbf{Eq}(X)$.

Finally, a closure result

Theorem (Snow 2009)

Suppose $\mathbf{L} \le \mathbf{Eq}(X)$ *is a closed sublattice and* $\mathbf{L}' \le \mathbf{L}$ *is a sublattice with universe* $A \cup B$*, where* $A = \{x \in L \mid x \le \alpha\}$ *and* $B = \{x \in L \mid x \le \beta\}$ *for some* $\alpha, \beta \in L$*. Then* $\mathbf{L}'$ *is closed.*

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- This is another recent result of John Snow, which he proved using *primitive positive formulas.*
- An easy consequence is that all hexagons are congruence hereditary. That is, if a hexagon is closed, so are its sublattices.

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- The problem can be stated very concretely in terms of partitions of a set allowing us to analyze many concrete examples with the computer and locate specific representable lattices.
- In recent years, the partial results have gathered significant momentum, and there is some hope that the full solution is forthcoming.


감사합니다
Thank You