The Finite Lattice Representation Problem

William DeMeo

joint work with

Ralph Freese, Peter Jipsen, Bill Lampe, J.B. Nation

University of Hawaiʻi at Mānoa

ARCS Presentation
April 23, 2011

Apology

I am sorry...

Apology

I am sorry...

...this talk is about math.

What is an algebra?

...an utterly fundamental object in mathematics.

Definition

An algebra $\mathbf{A}$ is an ordered pair $\mathbf{A} = \langle A, F \rangle$ where

What is an algebra?

...an utterly fundamental object in mathematics.

Definition

An algebra $\mathbf{A}$ is an ordered pair $\mathbf{A} = \langle A, F \rangle$ where

- A is a nonempty set, called the *universe* of $\mathbf{A}$
- F is a family of operations acting on $\mathbf{A}$

What is an algebra?

...an utterly fundamental object in mathematics.

Definition

An algebra $\mathbf{A}$ is an ordered pair $\mathbf{A} = \langle A, F \rangle$ where

A is a nonempty set, called the *universe* of $\mathbf{A}$

F is a family of operations acting on $\mathbf{A}$

An algebra $\langle A, F \rangle$ is called finite if $|A|$ is finite.

What is an algebra?

...an utterly fundamental object in mathematics.

Definition

An algebra $\mathbf{A}$ is an ordered pair $\mathbf{A} = \langle A, F \rangle$ where

A is a nonempty set, called the *universe* of $\mathbf{A}$

F is a family of operations acting on $\mathbf{A}$

An algebra $\langle A, F \rangle$ is called finite if $|A|$ is finite.

- **Example:** The set of integers

$$A = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

What is an algebra?

...an utterly fundamental object in mathematics.

Definition

An algebra $\mathbf{A}$ is an ordered pair $\mathbf{A} = \langle A, F \rangle$ where

A is a nonempty set, called the *universe* of $\mathbf{A}$

F is a family of operations acting on $\mathbf{A}$

An algebra $\langle A, F \rangle$ is called finite if $|A|$ is finite.

- **Example:** The set of integers

$$A = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

along with the operations $F = \{+, -, \times\}$.

What is an algebra?

...an utterly fundamental object in mathematics.

Definition

An algebra $\mathbf{A}$ is an ordered pair $\mathbf{A} = \langle A, F \rangle$ where

A is a nonempty set, called the *universe* of $\mathbf{A}$

F is a family of operations acting on $\mathbf{A}$

An algebra $\langle A, F \rangle$ is called finite if $|A|$ is finite.

- **Example:** The set of integers

$$A = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

along with the operations $F = \{+, -, \times\}$.

Question: Is $\div$ an operation on this set, A?

What is an algebra?

...an utterly fundamental object in mathematics.

Definition

An algebra $\mathbf{A}$ is an ordered pair $\mathbf{A} = \langle A, F \rangle$ where

A is a nonempty set, called the *universe* of $\mathbf{A}$

F is a family of operations acting on $\mathbf{A}$

An algebra $\langle A, F \rangle$ is called finite if $|A|$ is finite.

- **Example:** The set of integers

$$A = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

along with the operations $F = \{+, -, \times\}$.

- **Other Examples:** semigroups, groups, quasigroups, rings, modules, lattices, Boolean algebras, $*$-algebras, etc.

Examples

- A lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ with universe L and binary operations:

Examples

- A lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ with universe L and binary operations:

$$\begin{array}{ll} x \wedge y = \text{g.l.b.}(x, y) & \text{the "meet" of } x \text{ and } y \\ x \vee y = \text{l.u.b.}(x, y) & \text{the "join" of } x \text{ and } y \end{array}$$

Examples

- A lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ with universe L and binary operations:

  $x \wedge y = \text{g.l.b.}(x, y)$ the "meet" of x and y
  $x \vee y = \text{l.u.b.}(x, y)$ the "join" of x and y

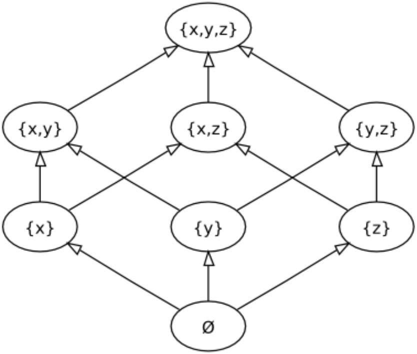
- *Examples of lattices:*
  - subsets of a set
  - closed subsets of a topology
  - subgroups of a group, normal subgroups of a group
  - ideals of a ring
  - submodules of a module
  - invariant subspaces of an operator or operator algebra

Examples

- A lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ with universe L and binary operations:

  $x \wedge y = \text{g.l.b.}(x, y)$ the "meet" of x and y
  $x \vee y = \text{l.u.b.}(x, y)$ the "join" of x and y

- *Examples of lattices:*

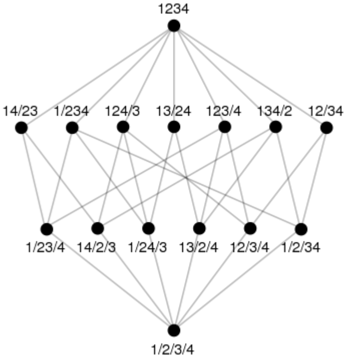


Examples

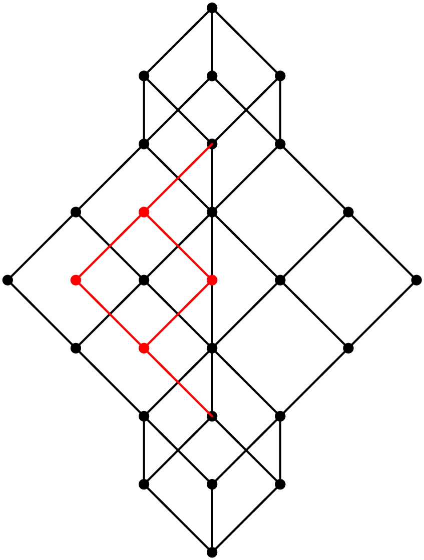
- A lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ with universe L and binary operations:

$$\begin{array}{ll} x \wedge y = \text{g.l.b.}(x, y) & \text{the "meet" of } x \text{ and } y \\ x \vee y = \text{l.u.b.}(x, y) & \text{the "join" of } x \text{ and } y \end{array}$$

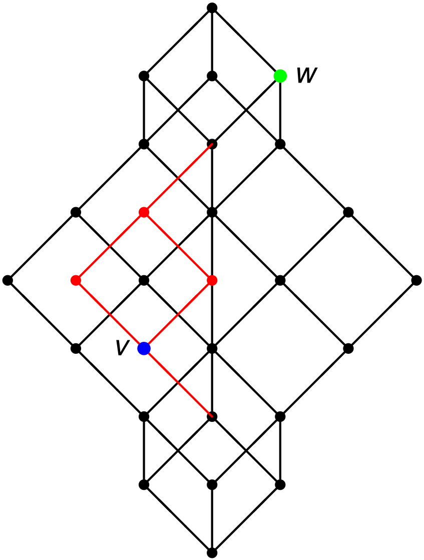
- *Examples of lattices:*



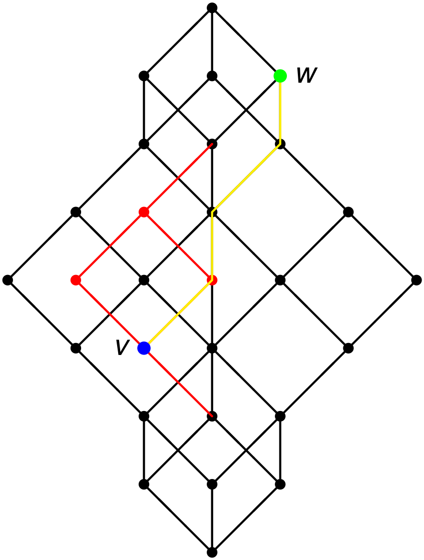
Lattices



Lattices

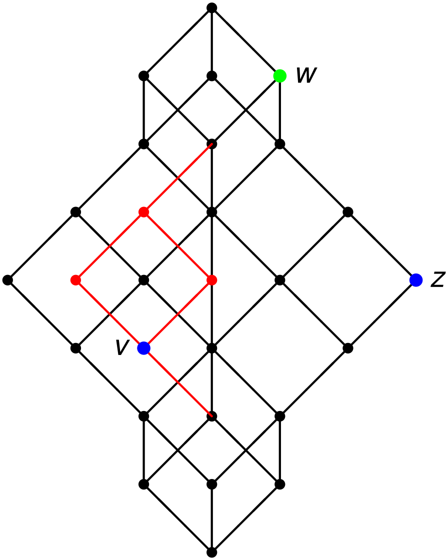


Lattices



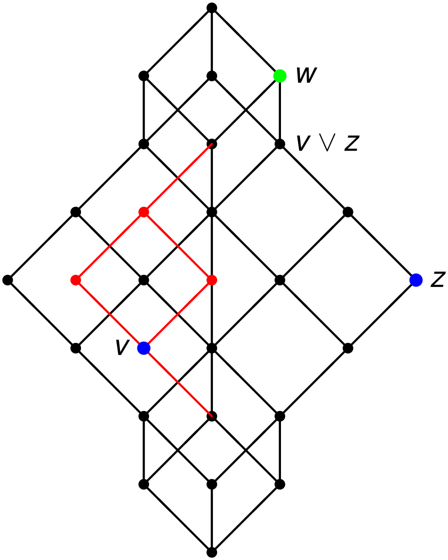
$v \leq w$

Lattices

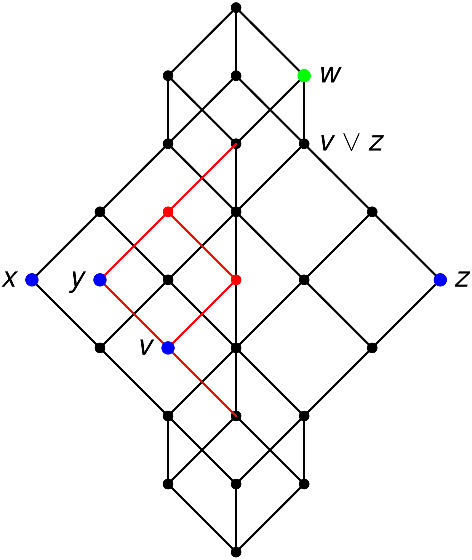


$v \nleq z$

Lattices

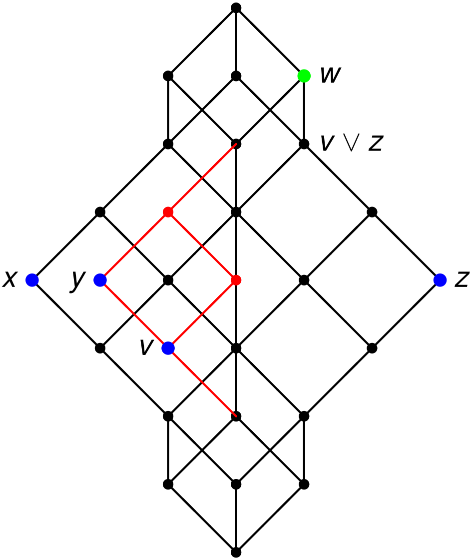


Lattices



x, y, z **generate**

Lattices



$$v = y \wedge (x \vee z)$$

How do we study algebras?

The Key Features

Given an algebra **A**, there are three other important algebras which enable us to characterize and understand **A**. These are...

How do we study algebras?

The Key Features

Given an algebra **A**, there are three other important algebras which enable us to characterize and understand **A**. These are...

- **SubA**, the lattice of subalgebras of **A**.

How do we study algebras?

The Key Features

Given an algebra **A**, there are three other important algebras which enable us to characterize and understand **A**. These are...

- **SubA**, the lattice of subalgebras of **A**.
- **EndA** and **AutA**, the endomorphisms and automorphisms of **A**.

How do we study algebras?

The Key Features

Given an algebra **A**, there are three other important algebras which enable us to characterize and understand **A**. These are...

- **SubA**, the lattice of subalgebras of **A**.
- **EndA** and **AutA**, the endomorphisms and automorphisms of **A**.
- **ConA**, the lattice of **congruence relations** of **A**.

How do we study algebras?

The Key Features

Given an algebra **A**, there are three other important algebras which enable us to characterize and understand **A**. These are...

- **SubA**, the lattice of subalgebras of **A**.
- **EndA** and **AutA**, the endomorphisms and automorphisms of **A**.
- **ConA**, the lattice of **congruence relations** of **A**.

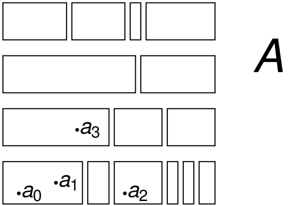
...okay, there are four.

Congruence Relations

A congruence of an algebra $\mathbf{A} = \langle A, F \rangle$ is a partition of the set A into blocks that are preserved by the operations in F.

Congruence Relations

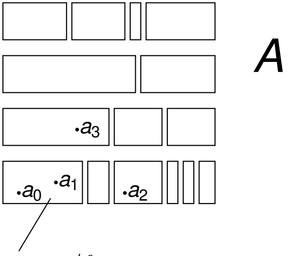
A congruence of an algebra $\mathbf{A} = \langle A, F \rangle$ is a partition of the set A into blocks that are preserved by the operations in F.



$$\theta_1 = (a_0, a_1, \dots | a_2, a_8, \dots | a_3 \dots)$$

Congruence Relations

A congruence of an algebra $\mathbf{A} = \langle A, F \rangle$ is a partition of the set A into blocks that are preserved by the operations in F.

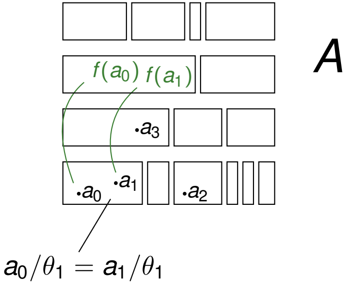


$$a_0/\theta_1 = a_1/\theta_1$$

$$\theta_1 = (a_0, a_1, \ldots | a_2, a_8, \ldots | a_3 \ldots)$$

Congruence Relations

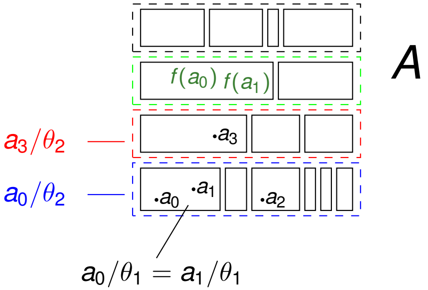
A congruence of an algebra $\mathbf{A} = \langle A, F \rangle$ is a partition of the set A into blocks that are preserved by the operations in F.



$$\theta_1 = (a_0, a_1, \ldots | a_2, a_8, \ldots | a_3 \ldots)$$

Congruence Relations

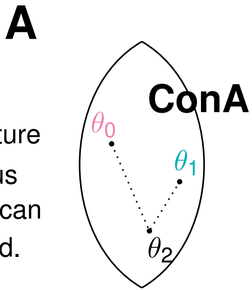
A congruence of an algebra $\mathbf{A} = \langle A, F \rangle$ is a partition of the set A into blocks that are preserved by the operations in F.



$$\theta_2 = (a_0, a_1, a_2, \dots | a_3, \dots | \dots), \quad \theta_1 = (a_0, a_1, \dots | a_2, a_8, \dots | a_3 \dots)$$

Congruence Decompositions

We know an algebra by the congruences it keeps.

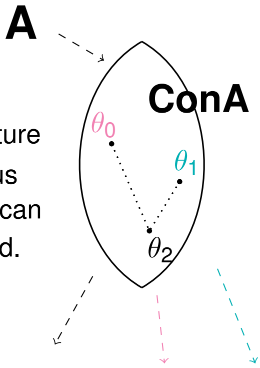


The shape/structure of **ConA** tells us how the algebra can be decomposed.

Congruence Decompositions

We know an algebra by the congruences it keeps.

The shape/structure of **ConA** tells us how the algebra can be decomposed.



$$\mathbf{A}/\theta_2 \hookrightarrow \mathbf{A}/\theta_0 \times \mathbf{A}/\theta_1$$

The Problem

What are the possible shapes of congruence lattices?

The Problem

Theorem (Grätzer-Schmidt, 1963)

Every algebraic lattice is isomorphic to the congruence lattice of an algebra.

Theorem (Grätzer-Schmidt, 1963)

Every algebraic lattice is isomorphic to the congruence lattice of an algebra.

Thus, there is essentially no restriction on the shape of a congruence lattice of an *infinite* algebra.

The Problem

What are the possible shapes of congruence lattices?

Theorem (Grätzer-Schmidt, 1963)

Every algebraic lattice is isomorphic to the congruence lattice of an algebra.

Thus, there is essentially no restriction on the shape of a congruence lattice of an *infinite* algebra.

What if the algebra is finite?

Problem: Given a finite lattice $\mathbf{L}$, does there exist a *finite* algebra $\mathbf{A}$ such that $\mathbf{Con}\mathbf{A} \cong \mathbf{L}$?

The Problem

What are the possible shapes of congruence lattices?

Theorem (Grätzer-Schmidt, 1963)

Every algebraic lattice is isomorphic to the congruence lattice of an algebra.

Thus, there is essentially no restriction on the shape of a congruence lattice of an *infinite* algebra.

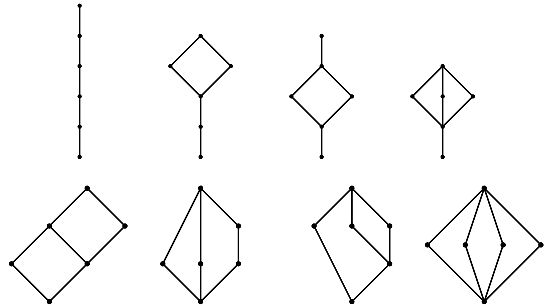
What if the algebra is finite?

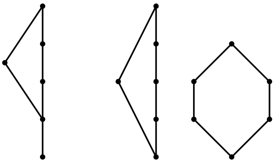
Problem: Given a finite lattice $\mathbf{L}$, does there exist a *finite* algebra $\mathbf{A}$ such that $\mathbf{Con A} \cong \mathbf{L}$?

status: open

age: 50+ years

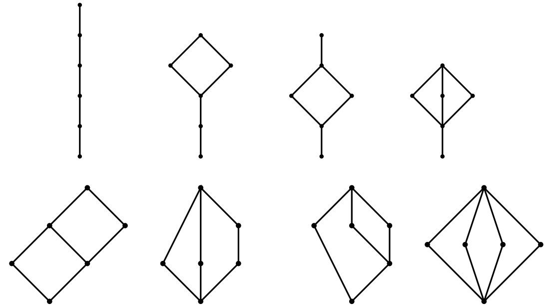
Known Results: lattices of size ≤ 6 are congruence lattices.

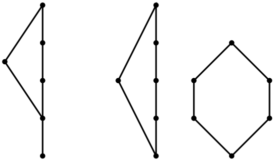




Aschbacher recently found the last two of these as intervals in subgroup lattices.

Known Results: lattices of size ≤ 6 are congruence lattices.





Aschbacher recently found the last two of these as intervals in subgroup lattices.

Theorem: *Every lattice with at most 6 elements is a congruence lattice of a finite algebra.*

LATTICES OF SIZE ≤ 7 NOT YET KNOWN TO BE CONGRUENCE LATTICES OF FINITE ALGEBRAS

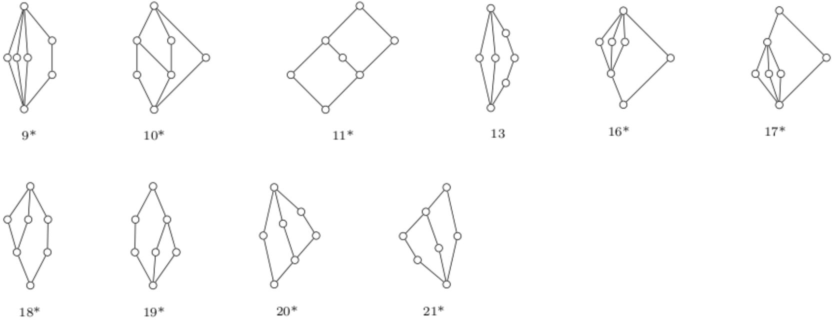
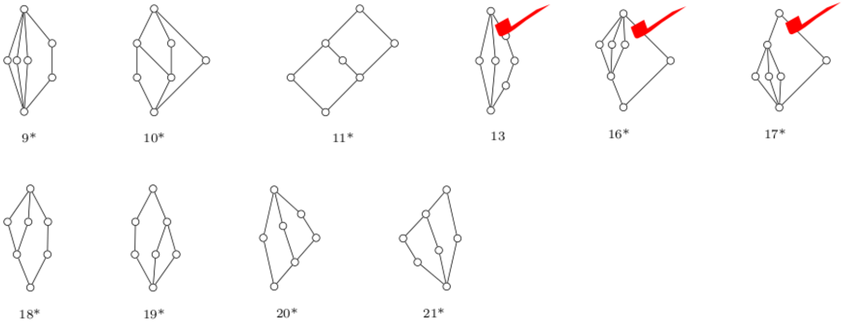


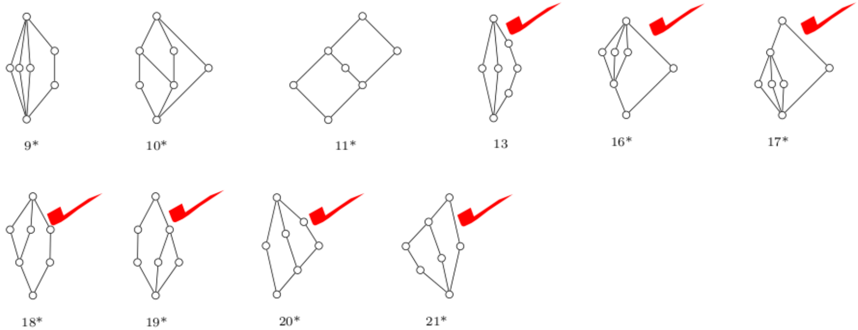
Figure courtesy of Peter Jipsen.

Recent Results

LATTICES OF SIZE ≤ 7 NOT YET KNOWN TO BE CONGRUENCE LATTICES OF FINITE ALGEBRAS

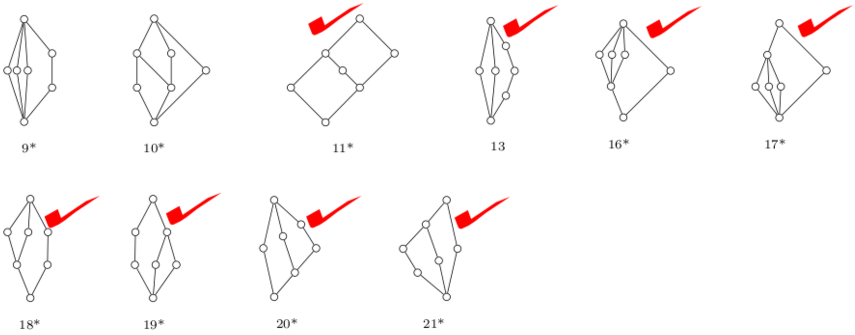


LATTICES OF SIZE ≤ 7 NOT YET KNOWN TO BE CONGRUENCE LATTICES OF FINITE ALGEBRAS



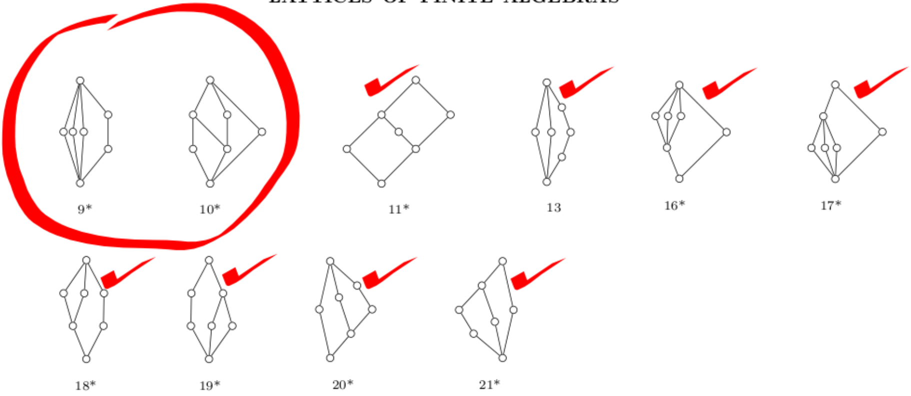
Recent Results

LATTICES OF SIZE ≤ 7 NOT YET KNOWN TO BE CONGRUENCE LATTICES OF FINITE ALGEBRAS



Recent Results

LATTICES OF SIZE ≤ 7 NOT YET KNOWN TO BE CONGRUENCE LATTICES OF FINITE ALGEBRAS




감사합니다
Thank You


Mahalo