Interval Enforceable Properties of Finite Groups

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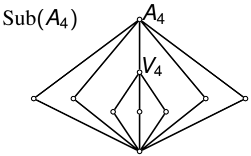
For some groups, $\mathrm{Sub}(G)$ determines G up to isomorphism.

Examples

The Klein 4-group, V_4.

The alternating groups, $A_n\,(n \geqslant 4)$.

Every finite nonabelian simple group.

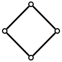


For other groups, $\mathrm{Sub}(G)$ is isomorphic to the subgroup lattices of all groups in an infinite class of nonisomorphic groups.

Examples

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of prime order.

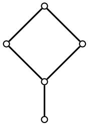
$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order p^2.

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order pq.

At the other extreme, there are finite lattices which are not subgroup lattices.

Example: For all G,

$$\mathrm{Sub}(G) \ncong$$



We are interested in the *local structure* of subgroup lattices, that is, the possible ***intervals***

$$[H, K] := \{X \mid H \leqslant X \leqslant K\} \leqslant \mathrm{Sub}(G)$$

where $H \leqslant K \leqslant G$.

We restrict our attention to ***upper intervals***, where $K = G$, and ask two questions:

1. *What intervals $[H, G]$ are possible?*
2. *What properties of a group G can be inferred from the shape of an upper interval in Sub(G)?*

1. What intervals $[H, G]$ are possible?

There is a remarkable theorem relating this question to what is perhaps the most important open problem in unversal algebra – the *finite lattice representation problem* (FLRP).

Theorem (Pálfy and Pudlák(1980))

The following statements are equivalent:

(A) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*

(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

2. What properties of G can be inferred from $[H, G]$?

A group theoretical property $\mathcal{P}$ (and the associated class $\mathscr{G}_{\mathcal{P}}$) is

- ***interval enforceable*** (IE) provided $\exists$ a lattice L such that

  if $G \in \mathfrak{G}$ and $L \cong [H, G]$, then G is a $\mathcal{P}$-group.

- ***core-free interval enforceable*** (cf-IE) provided $\exists L$ st

  if $G \in \mathfrak{G}$, $L \cong [H, G]$, H core-free, then G is a $\mathcal{P}$-group.

- ***minimal interval enforceable*** (min-IE) provided $\exists L$ st
  if $G \in \mathfrak{G}$, $L \cong [H, G]$, and if G has minimal order (wrt $L \cong [H, G]$), then G is a $\mathcal{P}$-group.

Clearly, if $\mathcal{P}$ is IE, then it is also cf-IE.

There is a simple sufficient condition under which the converse holds.

If $\mathcal{P}$ is a group property, let $\mathscr{G}_{\mathcal{P}}^{c} := \{G \in \mathfrak{G} \mid G \not\models \mathcal{P}\}$ denote the class of $(\neg\mathcal{P})$-groups.

Lemma

Suppose $\mathcal{P}$ is a core-free interval enforceable property. If

$$\mathbf{H}(\mathscr{G}_{\mathcal{P}}^{c}) = \mathscr{G}_{\mathcal{P}}^{c}$$

then $\mathcal{P}$ is an interval enforceable property.

The following are at least core-free interval enforceable:

- $\mathscr{G}_0 = \mathfrak{S}^c =$ the insoluble groups
- $\mathscr{G}_1 = \{G \in \mathfrak{G} \mid (\forall n < \omega)\ (G \neq A_n \text{ and } G \neq S_n)\}$
- $\mathscr{G}_2 =$ the subdirectly irreducible groups
- $\mathscr{G}_3 =$ groups with no nontrivial abelian normal subgroups
- $\mathscr{G}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteq G\}$.

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For $i = 2, 3, 4$,

$$\mathbf{H}(\mathscr{G}_i^c) \neq \mathscr{G}_i^c$$

Proof: If $H \in \mathscr{G}_i$, $K \in \mathscr{G}_i^c$, then, $H \times K$ belongs to $\mathscr{G}_i^c$, but $(H \times K)/(1 \times K) \cong H$ does not.

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L ***group representable***.

By the Pálfy-Pudlák Theorem, the FLRP has a negative answer if we can find a (finite) lattice that is not group representable.

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Suppose there exists property $\mathcal{P}$ such that both $\mathcal{P}$ and its negation $\neg\mathcal{P}$ are interval enforceable by the lattices L and L_c, respectively:

$$L \cong [H, G] \implies G \text{ is a } \mathcal{P}\text{-group}$$

$$L_c \cong [H_c, G_c] \implies G_c \text{ is not a } \mathcal{P}\text{-group}$$

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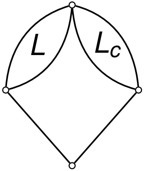
Then the lattice 

wouldn't be group representable.

As the next result shows, however, if a group property and its negation are interval enforceable by L and L_c, then already at least one of these lattices is not group representable.

Lemma

If $\mathcal{P}$ is a group property that is interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not interval enforceable by a group representable lattice.

Insolubility is interval enforceable, but solubility is not.

For if $L \cong [H, G]$, then for any insoluble group K we have $L \cong [H \times K, G \times K]$, and $G \times K$ is insoluble.

Note that the group $H \times K$ at the bottom of the interval is not core-free. So a more interesting question is whether a property and its negation could both be *core-free* IE.

Conjecture

If $\mathcal{P}$ is core-free interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not core-free interval enforceable by a group representable lattice.

The following lemma shows that any class of groups that omits certain wreath products cannot be core-free interval enforceable by a group representable lattice.

Lemma

Suppose $\mathcal{P}$ is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group S, there exists a wreath product group of the form $W = S \wr U$ that is a $\mathcal{P}$-group.

Corollary

Solubility is not core-free interval enforceable.

Proof Sketch

Let L be a group representable lattice such that if $L \cong [H, G]$ and $\mathrm{core}_G(H) = 1$ then G is a $\mathcal{P}$-group.

Since L is group representable, $\exists$ $\mathcal{P}$-group G with $L \cong [H, G]$.

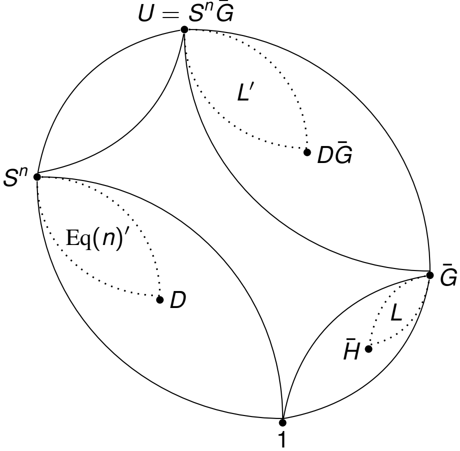
Proof Sketch

Let L be a group representable lattice such that if $L \cong [H, G]$ and $\mathrm{core}_G(H) = 1$ then G is a $\mathcal{P}$-group.

Since L is group representable, $\exists$ $\mathcal{P}$-group G with $L \cong [H, G]$.

We apply the idea of Hans Kurzweil twice:

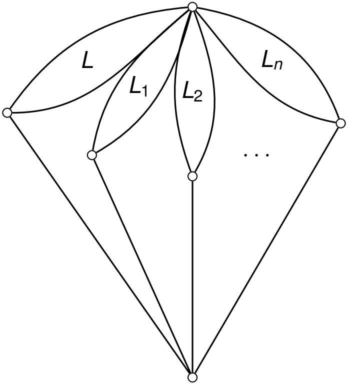
- Fix a finite nonabelian simple group S.
- Suppose the index of H in G is $|G : H| = n$.
- Then the action of G on the cosets of H induces an automorphism of the group S^n by permutation of coordinates.
- Denote this by $\varphi : G \to \mathrm{Aut}(S^n)$, and let $\varphi(G) = \bar{G} \leqslant \mathrm{Aut}(S^n)$.



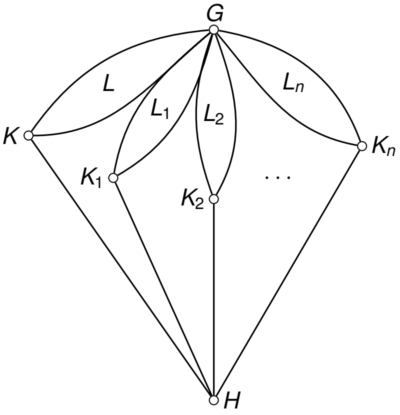
The interval $[D, S^n]$ is isomorphic to $\mathrm{Eq}(n)'$, the dual of the lattice of partitions of an n-element set.

The dual lattice L' is an upper interval of $\mathrm{Sub}(U)$, namely, $L' \cong [D\bar{G}, U]$.

We conclude that a class of groups that does not include wreath products of the form $S \wr G$, where S is an arbitrary finite nonabelian simple group, is not a core-free interval enforceable class. The class of soluble groups is an example.



L
L1
L2
Ln
. . .



G
L
L_1
L_2
L_n
K
K_1
K_2
K_n
...
H

Theorem

The following statements are equivalent:

(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

(C) *For every finite lattice L and every finite collection $\mathscr{G}_1, \ldots, \mathscr{G}_n$ of cf-IE classes of groups,*

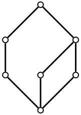
$$\exists\ G \in \bigcap_{i=1}^{n} \mathscr{G}_i \text{ such that } L \cong [H, G] \text{ and } \mathrm{core}_G(H) = 1.$$

(D) *For every finite collection $\mathscr{L}$ of finite lattices, there exists a finite group G such that each $L_i \in \mathscr{L}$ is isomorphic to $[H_i, G]$ for some core-free subgroup $H_i \leqslant G$.*

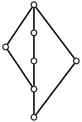
By (C), the FLRP would have a negative answer if we could find a collection $\mathscr{G}_1, \ldots, \mathscr{G}_n$ of cf-IE classes such that $\bigcap\limits_{i=1}^{n} \mathscr{G}_i$ is empty.

By (D), it makes sense to consider finite collections of finite lattices and ask what can be proved about a group G if one assumes that all of these lattices are isomorphic to upper intervals of $\mathrm{Sub}(G)$.

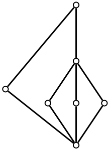
Are all lattices with at most 7 elements representable?



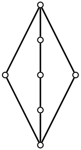
L_{19} ✓ Galois correspondence



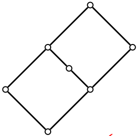
L_{20} ✓ Galois correspondence



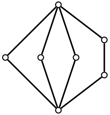
L_{17} ✓ filter+ideal



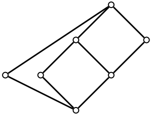
L_{13}



L_{11} ✓ filter+ideal

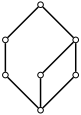


L_9 ✓ overalgebra construction



L_7

Are all lattices with at most 7 elements representable?



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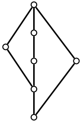

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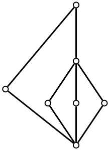
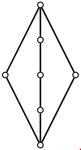


L_{17} ✓ interval in $\mathrm{Sub}(G)$



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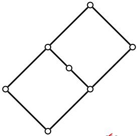
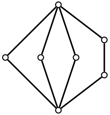


L_{11} ✓ filter+ideal



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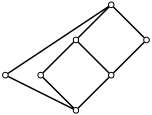

L_7

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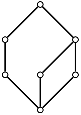
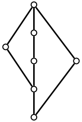
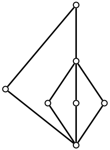


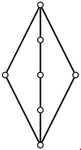
L_{19} Galois correspondence



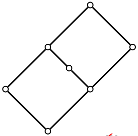
L_{20} Galois correspondence



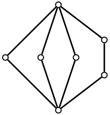
L_{17} interval in $\mathrm{Sub}(G)$



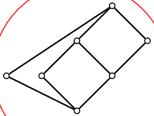
L_{13} interval in $\mathrm{Sub}(G)$



L_{11} filter+ideal



L_9 interval in $\mathrm{Sub}(G)$



L_7

?

Has anyone seen this lattice?

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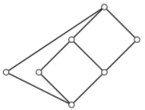
Given a lattice L with n elements, are there finite groups H < G such that L ≅ the lattice of subgroups between H and G?

13

5

If there is no restriction on n, this is a famous open problem. I'm wondering if any recent work has been done for small $n > 6$. I believe the question is answered (positively) for $n = 6$ by Watatani (1996) MR1409040 and Aschbacher (2008) MR2393428. I also believe we can answer it for $n = 7$, with one possible exception. The exceptional case is shown below.



So my two questions are these:

1) Does anyone know of recent work on this special case of the problem (specifically for $n = 7$ or $n = 8$)?

2) Has anyone found a finite group G with a subgroup H such that the interval

$$[H, G] = \{K : H \leq K \leq G\}$$

is the lattice shown above?

tagged

finite-groups × 277

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universal-algebra × 53

congruences × 6

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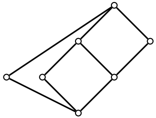
Tip: You can search for questions with arbitrary boolean combinations of tags (like **this**). See tip 12 for details on how. **See more tips and tricks.**

MathJax trouble? (Re)process math with jsMath.

The exceptional seven element lattice



L_7

Theorem

Suppose $H < G$, $\operatorname{core}_G(H) = 1$, $L_7 \cong [H, G]$.

(i) *G is a primitive permutation group.*

(ii) *If $N \lhd G$, then $C_G(N) = 1$.*

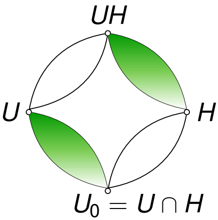
(iii) *G contains no non-trivial abelian normal subgroup.*

(iv) *G is not solvable.*

(v) *G is subdirectly irreducible.*

(vi) *With the possible exception of at most one maximal subgroup, all proper subgroups in the interval $[H, G]$ are core-free.*

Interval Isomorphisms



- If $H \trianglelefteqslant \langle U, H\rangle$, then
  $UH = \langle U, H\rangle$ and $[U_0, U] \cong [H, UH]$.
- Instead of $H \trianglelefteqslant \langle U, H\rangle$, assume only
  $UH = \langle U, H\rangle$ and define

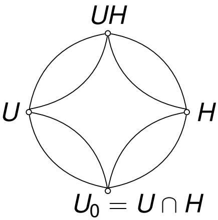
  $$[U_0, U]^H := \{V \in [U_0, U] \mid VH = HV\},$$

  the H-permuting subgroups.
- If $U \trianglelefteqslant UH$, define

  $$[U_0, U]_H := \{V \in [U_0, U] \mid H \leqslant N_{UH}(V)\},$$

  the H-invariant subgroups: $V^h = V$ $(\forall h \in H)$.

Interval Isomorphisms



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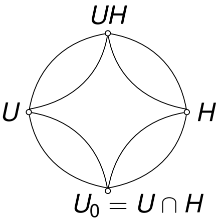

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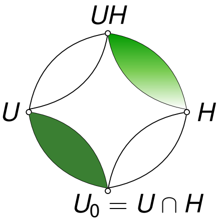

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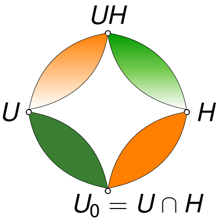
  $$[U_0, U]_H := \{V \in [U_0, U] \mid H \leqslant N_{UH}(V)\},$$

  the *H*-invariant subgroups: $V^h = V \ (\forall h \in H)$.

Lemma

1. $[H, UH] \cong [U_0, U]^H \leqslant [U_0, U]$
2. *If* $U \trianglelefteq UH$*, then* $[U_0, U]_H = [U_0, U]^H \leqslant [U_0, U]$.
3. *If* $H \trianglelefteq UH$*, then* $[U_0, U]_H = [U_0, U]^H = [U_0, U]$.

Interval Isomorphisms



- If $H \trianglelefteq \langle U, H\rangle$, then
  $UH = \langle U, H\rangle$ and $[U_0, U] \cong [H, UH]$.
- Instead of $H \trianglelefteq \langle U, H\rangle$, assume only
  $UH = \langle U, H\rangle$ and define

  $$[U_0, U]^H := \{V \in [U_0, U] \mid VH = HV\},$$

  the *H*-permuting subgroups.
- If $U \trianglelefteq UH$, define

  $$[U_0, U]_H := \{V \in [U_0, U] \mid H \leqslant N_{UH}(V)\},$$

  the *H*-invariant subgroups: $V^h = V$ $(\forall h \in H)$.

Lemma

1. $[H, UH] \cong [U_0, U]^H \leqslant [U_0, U]$
2. *If $U \trianglelefteq UH$, then $[U_0, U]_H = [U_0, U]^H \leqslant [U_0, U]$.*
3. *If $H \trianglelefteq UH$, then $[U_0, U]_H = [U_0, U]^H = [U_0, U]$.*

Example 1

- Consider $G \cong C_3 \times S_3$, say,

$$G = \langle a, b, c \mid a^2, b^3, c^3, [b, a], [c, b], c^{-1}a^{-1}a^c \rangle$$

- The subgroups

$$U = \langle a, b \rangle \cong C_6, \qquad H = \langle bc \rangle \cong C_3$$

  permute ($UH = HU$) but neither one normalizes the other.

- Three of the four subgroups of U permute with H.
  As the lemma predicts, $U \cap \langle b, c \rangle = \langle b \rangle$.

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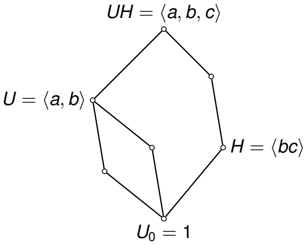

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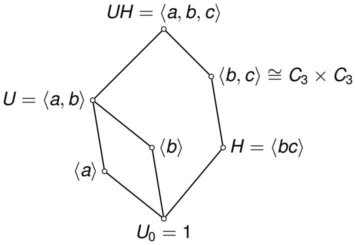

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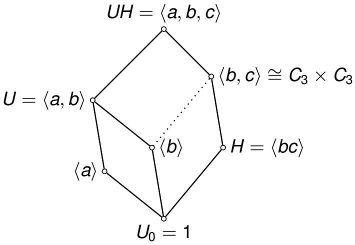

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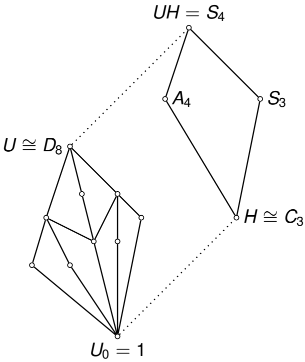

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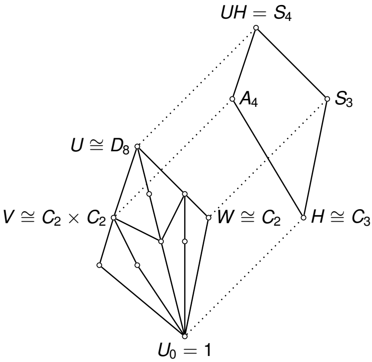

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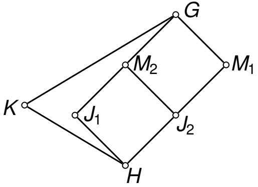
$$U \cap A_4 \cong C_2 \times C_2, \qquad U \cap S_3 \cong C_2.$$



Theorem

Suppose $H < G$, $\mathrm{core}_G(H) = 1$, and $L_7 \cong [H, G]$. Then

(i) *G is a primitive permutation group.*

(ii) *If $N \lhd G$, then $C_G(N) = 1$.*

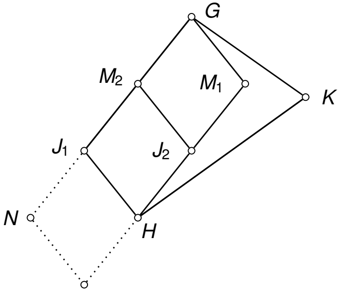
(iii) *G contains no non-trivial abelian normal subgroup.*

(iv) *G is not solvable.*

(v) *G is subdirectly irreducible.*

(vi) *With the possible exception of at most one maximal subgroup, M_1 or M_2, all proper subgroups in the interval $[H, G]$ are core-free.*

Example Application



Claim: J_1 and J_2 are core-free subgroups of G.

Proof:

- If $N \lhd G$ then NH permutes with each subgroup containing H.
- If $1 \neq N \leqslant J_1$, then $NH = J_1$, so J_1 and K permute.
- Since $J_1 K = G$ and $J_1 \cap K = H$, our lemma yields

$$[J_1, G] \cong [H, K]^{J_1} = \{X \in [H, K] \mid J_1 X = X J_1\}.$$

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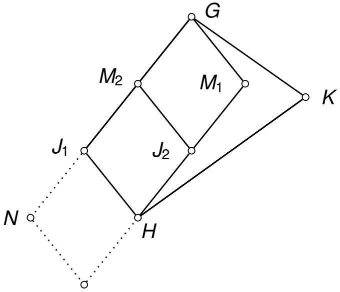

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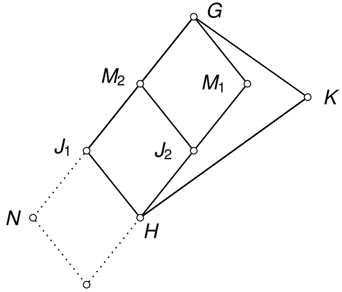

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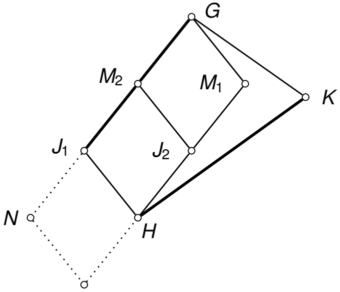

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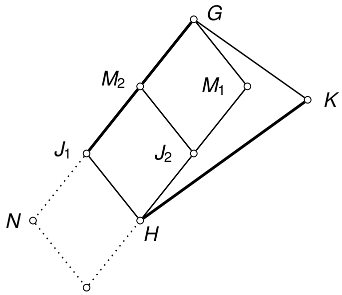

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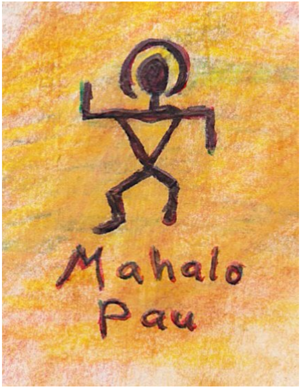
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