OVERALGEBRAS:

EXPANSIONS AND EXTENSIONS OF FINITE ALGEBRAS

William DeMeo

williamdemeo@gmail.com

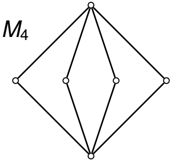
Iowa State University
Algebra & Combinatorics Seminar

22 Feb 2016

These slides and other resources are available at

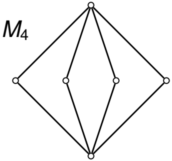
https://github.com/williamdemeo/Talks

REPRESENTING LATTICES



Question: Is this a congruence lattice?

Representing lattices



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Answer: Yes! ...of which algebra?

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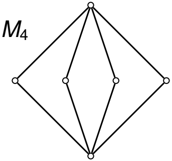

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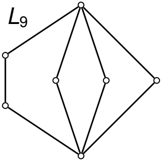

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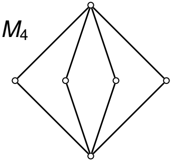

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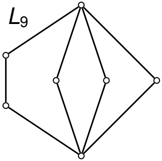

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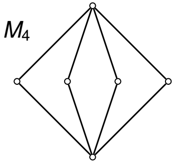
Answer: Yes! ...but this one is harder.

Construction of an algebra $\mathbf{A}$ with $\mathrm{Con}\,\mathbf{A} \cong L_9$.



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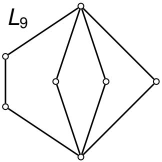
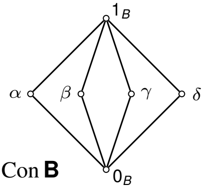
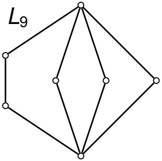


CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con} \mathbf{A} \cong L_9$.



Con **B**



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- the right regular action of S_3 on itself has generators $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$.
- $\operatorname{Con} \langle B, \{g_0, g_1\} \rangle \cong M_4$ has elements

$\alpha = |012|345|,\ \beta = |03|14|25|, \gamma = |04|15|23|,\ \delta = |05|13|24|$.

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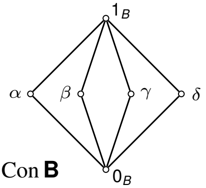

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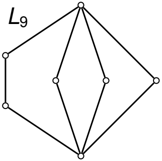

STEP 2 Define $A = B_0 \cup B_1 \cup B_2$ where

$$B_0 = \{0, 1, 2, 3, 4, 5\}$$
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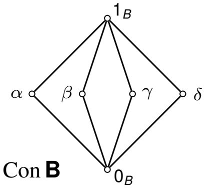

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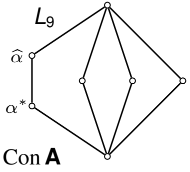

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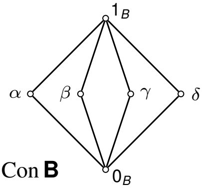
Construction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.



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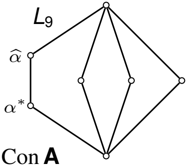

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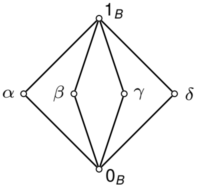
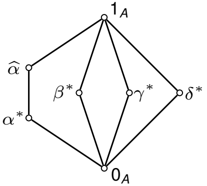


$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$



$\mathrm{Con}\,\langle A, F_A\rangle$

$$\alpha = |0, 1, 2|3, 4, 5|$$

$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$

$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$

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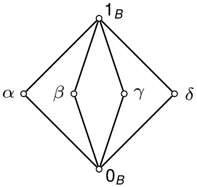

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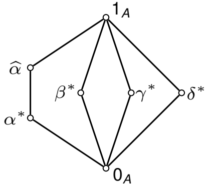

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$$\alpha = \alpha^* \cap B^2 = \widehat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \ldots$$

Expansion & Extension

Expanded Universe

$A = B_0 \cup B_1 \cup B_2$ where

$$B_0 = \{0, 1, 2, 3, 4, 5\}$$
$$B_1 = \{0, 6, 7, 8, 9, 10\}$$
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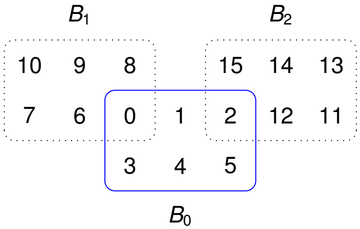
New Operations

$$e_0 : A \twoheadrightarrow B_0$$
$$e_1 : A \twoheadrightarrow B_1$$
$$e_2 : A \twoheadrightarrow B_2$$
$$s : A \twoheadrightarrow B_0,$$
$$ge_0 : A \overset{e_0}{\twoheadrightarrow} B_0 \xrightarrow{g} B_0,$$

for each $g \in F^B$.



Why does it work?

Con $\langle B, \{g_0, g_1\}\rangle$

$$\alpha = |0,1,2|3,4,5|$$

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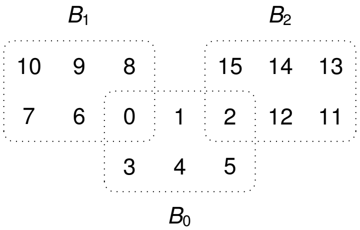

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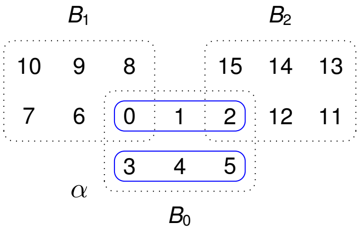

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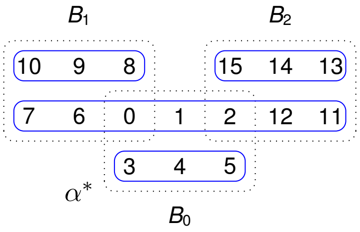

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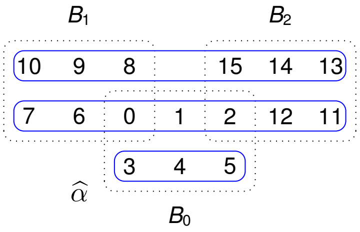

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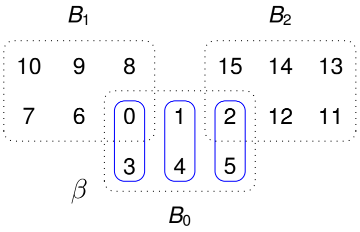

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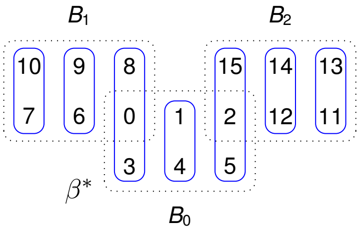

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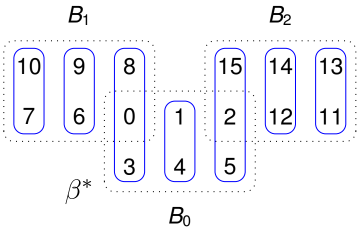

$\mathrm{Con}\,\langle A, F_A\rangle$

Why don't β classes of B_1, B_2 mix?

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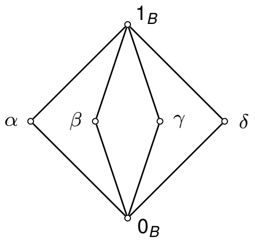
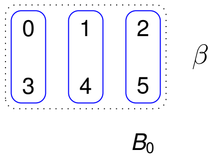
VARIATIONS ON THE SAME EXAMPLE...

- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\![\beta^*, \widehat{\beta}]\!]$.





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- Select elements 0 and 3 as intersection points:

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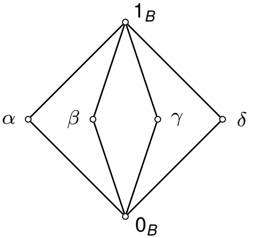

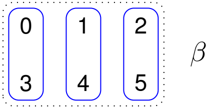

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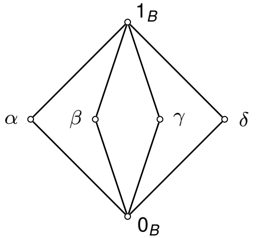

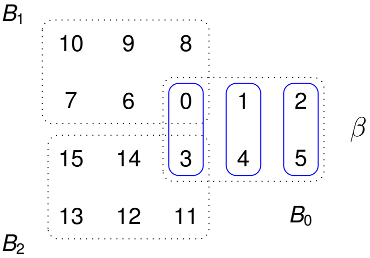

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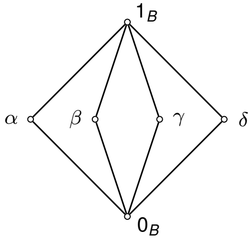

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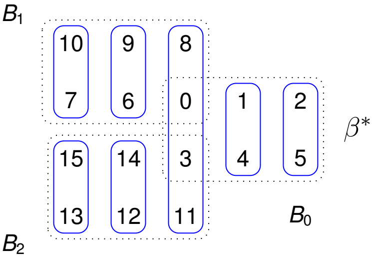

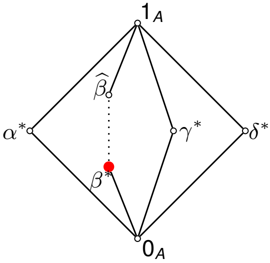

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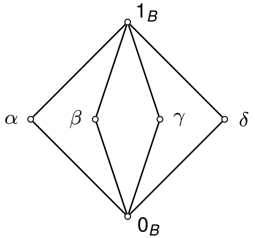

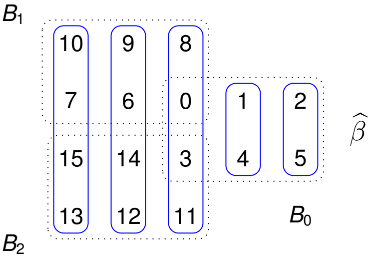

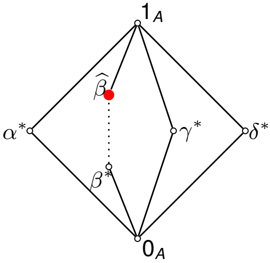

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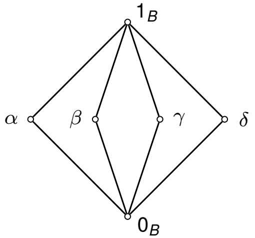

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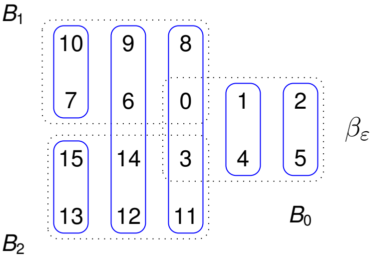

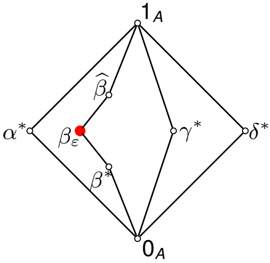

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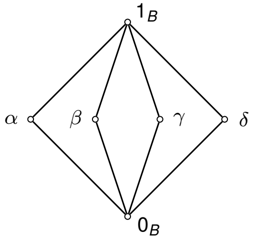

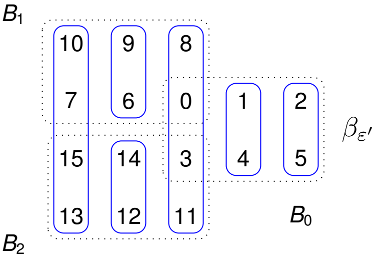

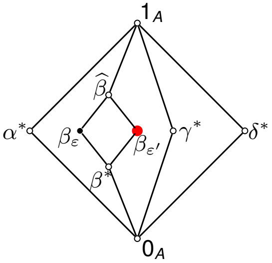

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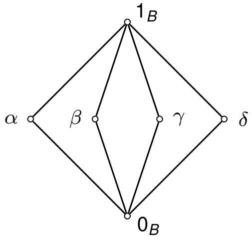

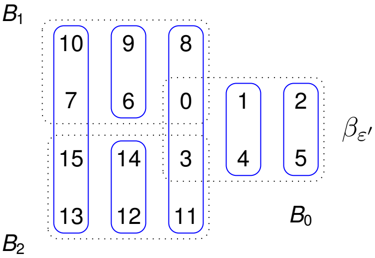

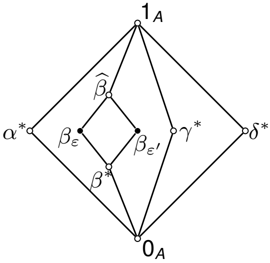

THE P^5 LEMMA

LEMMA (PÁLFY-PUDLÁK, 1980)

Let $\mathbf{A} = \langle A, F \rangle$ *be a unary algebra where* F *is a monoid.*

Suppose $e \in F$ *satisfies* $e \circ e = e$.

Define $\mathbf{B} = \langle B, G \rangle$

$$B = e(A) \quad \text{and} \quad G = \{ e f|_B \mid f \in F \}.$$

Let $|_B : Con(\mathbf{A}) \to Con(\mathbf{B})$ *be the restriction mapping:*

$$\theta|_B = \theta \cap B^2$$

Then $|_B$ *is a surjective homomorphism (even for arbitrary meets and joins).*

Péter Pál Pálfy and Pavel Pudlák: *Congruence lattices of finite algebras* AU (1980).

The Star Map and Hat Map

hat map?

star map?

The Star Map and Hat Map

Star map ${}^{*} : \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ is congruence generation:

$$\beta^{*} = \operatorname{Cg}^{\mathbf{A}}(\beta) \qquad (\forall\, \beta \in \operatorname{Con} \mathbf{B})$$

Hat map $\widehat{\ } : \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ is

$$\widehat{\beta} = \{(x, y) \in A^2 \mid (ef(x), ef(y)) \in \beta,\ \forall\, f \in \operatorname{Pol}_1(\mathbf{A})\}.$$

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The hat map appears in McKenzie's "Finite Forbidden Lattices" paper (Puebla, 1982) where he gives an alternative proof of the P^5 Lemma.

RESIDUATION LEMMA

A lemma relating the three maps *, $|_B$, and $\widehat{\ }$.

LEMMA

(I) * : $\operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ *is a* ***residuated mapping*** *with* ***residual*** $|_B$.

(II) $|_B$: $\operatorname{Con}\mathbf{A} \to \operatorname{Con}\mathbf{B}$ *is a* ***residuated mapping*** *with* ***residual*** $\widehat{\ }$.

(III) *For all* $\alpha \in \operatorname{Con}\mathbf{A}$, $\beta \in \operatorname{Con}\mathbf{B}$,

$$\beta = \alpha|_B \quad \Leftrightarrow \quad \beta^* \leqslant \alpha \leqslant \widehat{\beta}.$$

In particular, $\beta^*|_B = \beta = \widehat{\beta}|_B$.

Residuation/adjunction lemma

New version...

Lemma

$$ {}^{*} \qquad \dashv \qquad \big|_{B} \qquad \dashv \qquad \widehat{} $$

Residuation/adjunction lemma

New version...

Lemma

$$^* \quad \dashv \quad |_B \quad \dashv \quad \widehat{\ }$$

...that is...

(I) $^* : \mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ is ***left adjoint*** to $|_B$;

(II) $|_B : \mathrm{Con}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{B}$ is ***left adjoint*** to $\widehat{\ }$;

(III) For all $\alpha \in \mathrm{Con}\,\mathbf{A}$, $\beta \in \mathrm{Con}\,\mathbf{B}$,

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New Proof of the P^5 Lemma

Lemma (Pálfy-Pudlák, 1980)

The restriction mapping

$$\mathrm{Con}\,\mathbf{A} \ni \alpha \mapsto \alpha|_B = \alpha \cap B^2 \in \mathrm{Con}\,\mathbf{B}$$

is a complete lattice epimorphism.

Proof.

Recall, for $f : X \to Y$ a monotone function on preorders X and Y,
if f has a right (left) adjoint, then f preserves all joins (meets) that exist in X.

By the lemma $|_B$ has both a left and right adjoint. $\square$