Expansions of Finite Algebras
and their congruence lattices

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joint work with

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The Problem

There is essentially no restriction on the shape of a congruence lattice of an arbitrary algebra.

Theorem (Grätzer-Schmidt, 1963)

Every algebraic lattice is isomorphic to the congruence lattice of an algebra.

What if the algebra is finite?

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Definition

We call a finite lattice ***representable*** if it is (isomorphic to) the congruence lattice of a finite algebra.

Theorem (Pálfy and Pudlák, 1980)

The following statements are equivalent:

(i) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*

(ii) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

How to find a representation of a finite lattice

Method 0

Write the given lattice in terms of other representable lattices.

- If L is representable, so is
  - the dual of L
  - any interval sublattice of L
- If L_1 and L_2 are representable, so is
  - the direct product of L_1 and L_2
  - the ordinal sum of L_1 and L_2
  - the parallel sum of L_1 and L_2

How to find a concrete representation of a finite lattice

Method 1 (closure)

Find a "closed" representation of L in $\mathrm{Eq}(X)$.

For $L \leqslant \mathrm{Eq}(X)$ define

$$\lambda(L) = \{f \in X^X : (\forall \theta \in L)\ f(\theta) \subseteq \theta\}$$

For $F \subseteq X^X$ define

$$\rho(F) = \{\theta \in \mathrm{Eq}(X) : (\forall f \in F)\ f(\theta) \subseteq \theta\}$$

For every $L \leqslant \mathrm{Eq}(X)$ we have $L \subseteq \rho\lambda(L)$.

The map $\rho\lambda$ is a *closure operator* on $\mathrm{Sub}[\mathrm{Eq}(X)]$.
(idempotent, extensive, order preserving)

If a lattice $L \leqslant \mathrm{Eq}(X)$ is *closed*, i.e. $\rho\lambda(L) = L$, then

$$L = \mathrm{Con}\,\langle X, \lambda(L)\rangle$$

Method 2 (subgroup lattice interval)

Find L as an interval in a subgroup lattice of a finite group.

If $H < G$ are finite groups, then the interval above H in $Sub(G)$,

$$[H, G] := \{K : H \leqslant K \leqslant G\},$$

is isomorphic to $\mathrm{Con}\,\langle G/H, G\rangle$.

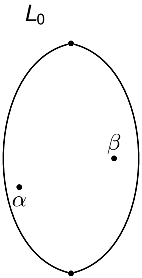
Method 3 (filter+ideal)

Find L as the union of a filter and ideal in a representable lattice.

Method 3

Lemma

Suppose $L_0 \cong \mathrm{Con}\,\langle A, F\rangle$, *and* $\alpha, \beta \in L_0 \setminus \{0, 1\}$.



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Suppose $L_0 \cong \mathrm{Con}\,\langle A, F\rangle$, *and* $\alpha, \beta \in L_0 \setminus \{0, 1\}$. *Consider* $L = \alpha^{\uparrow} \cup \beta^{\downarrow}$.

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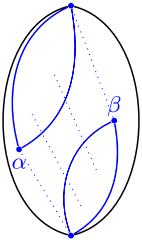

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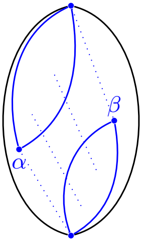

METHOD 3

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There exists a set $F' \subset A^A$ *such that* $L \cong \mathrm{Con}\,\langle A, F \cup F'\rangle$.

Proof:

Fix $\theta \in L_0 \setminus L$. Then $\alpha \nleq \theta \nleq \beta$, so

- $\exists (a, b) \in \alpha \setminus \theta$,
- $\exists (u, v) \in \theta \setminus \beta$.

Define $f_\theta : A \to A$ by

$$f_\theta(x) = \begin{cases} a & x \in u/\beta, \\ b & x \notin u/\beta. \end{cases}$$

Then

- $(f_\theta(u), f_\theta(v)) = (a, b) \notin \theta$, so $f_\theta(\theta) \nsubseteq \theta$,
- $\ker f_\theta \geqslant \beta$, so $f_\theta(\gamma) \subseteq \gamma$ for all $\gamma \leqslant \beta$,
- $f_\theta(A) \subseteq \{a, b\}$, so $f_\theta(\gamma) \subseteq \gamma$ for all $\gamma \geqslant \alpha$.

Let $F' = \{f_\theta : \theta \in L_0 \setminus L\}$.

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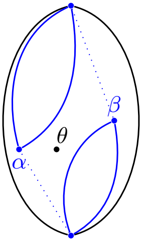
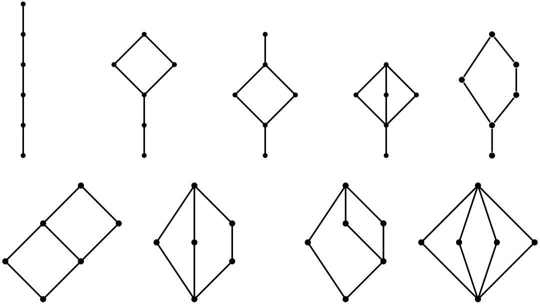


Lattices with at most 6 elements are representable.



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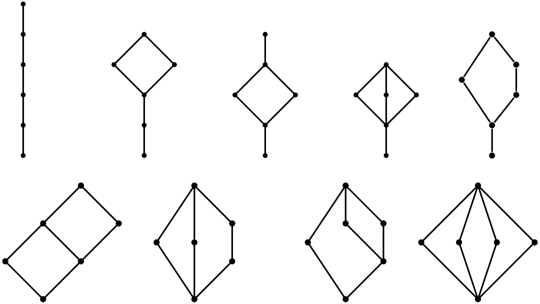

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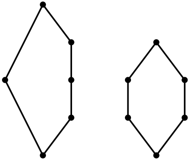

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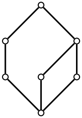
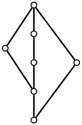
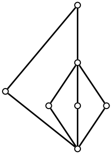


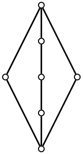
L_{19}



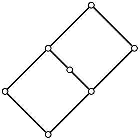
L_{20}



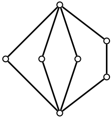
L_{17}



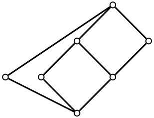
L_{13}



L_{11}



L_{9}



L_{10}

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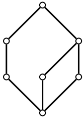

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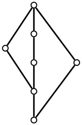

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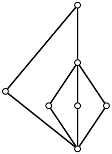

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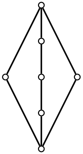

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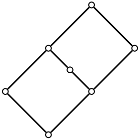

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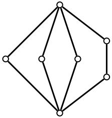

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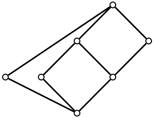

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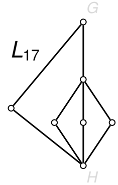
Finding representations…

…as intervals in subgroup lattices



`SmallGroup(288,1025)`

$|G : H| = 48$

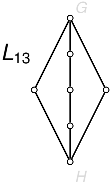
- The group $G = (A_4 \times A_4) \rtimes C_2$ has a subgroup $H \cong S_3$ such that $[H, G] \cong L_{17}$.

- …so the dual L_{16} is also representable.

  L_{16} can be embedded above diagonal of the direct power of a simple group.

  $$L_{16} \hookrightarrow [\Gamma, S^{48}] \cong \mathrm{Eq}(48)^{dual}.$$

  Add the group operations G which closed L_{17}, and L_{16} appears as an upper interval in $S^{48} \rtimes G$.



`SmallGroup(960,11358)`

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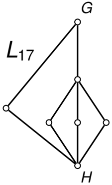

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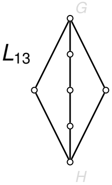

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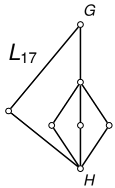

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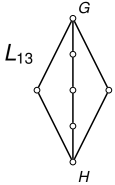

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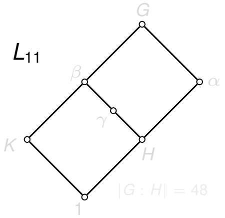

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- G has a subgroup $H \cong C_6$ with $[H, G] \cong N_5$.
- Let $[H, G] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.
- Sub(G) is a congruence lattice, so if there exists a subgroup $K \succ 1$, below β and not below γ, then

$$L_{11} \cong K^{\downarrow} \cup H^{\uparrow}.$$

- Sub(A_4) is a congruence lattice (of A_4 acting regularly on itself).
- Therefore,

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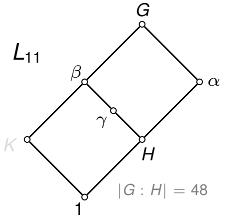

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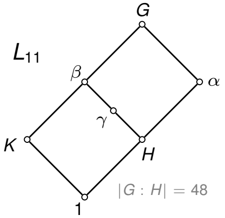

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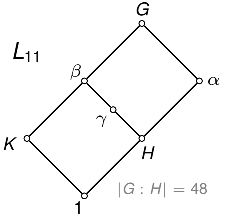

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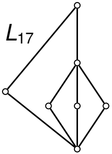

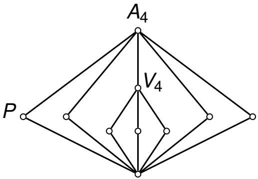

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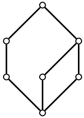
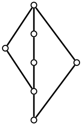
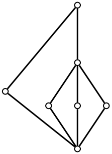


L_{19}



L_{20}



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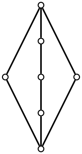
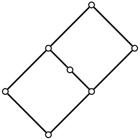


L_{13}



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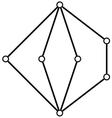
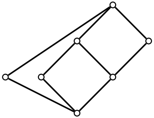


L_9



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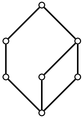
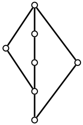
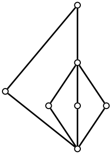


L_{19} ✓



L_{20} ✓



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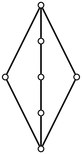
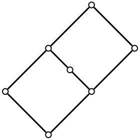
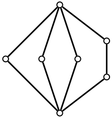


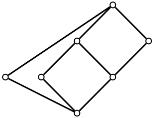
L_{13} ✓



L_{11} ✓

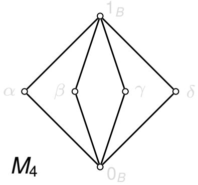


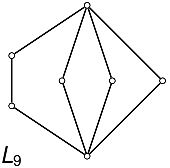
L_9 ✓



L_{10}

Contruction of an algebra $\mathbf{A}$ with $\mathrm{Con}\,\mathbf{A} \cong L_9$.





Step 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\mathrm{Con}\,\mathbf{B} \cong M_4$.

Example:

- Let $B = \{0, 1, \dots, 5\}$ index the elements of S_3 and consider the right regular action of S_3 on itself.
- $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- $\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle \cong M_4$ with congruences

$$\alpha = |012|345|,\ \beta = |03|14|25|,\ \gamma = |04|15|23|,\ \delta = |05|13|24|.$$

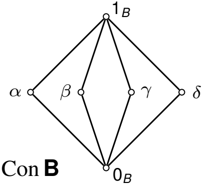
Goal: *expand* $\mathbf{B}$ *to an algebra* $\mathbf{A}$ *that has* α *"doubled" in* $\mathrm{Con}\,\mathbf{A}$.

Step 2 Since $\alpha = \mathrm{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

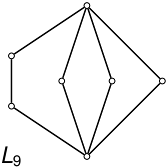
$$B_0 = \{0, 1, 2, 3, 4, 5\} = B$$
$$B_1 = \{0, 6, 7, 8, 9, 10\}$$
$$B_2 = \{11, 12, 2, 13, 14, 15\}.$$

Step 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0e_0$, and g_1e_0.

CONTRUCTION OF AN ALGEBRA **A** WITH Con **A** $\cong L_9$.



Con **B**



L_9

STEP 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice Con $\mathbf{B} \cong M_4$.

Example:

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$\alpha = |012|345|$, $\beta = |03|14|25|$, $\gamma = |04|15|23|$, $\delta = |05|13|24|$.

Goal: *expand* **B** *to an algebra* **A** *that has* α *"doubled" in* Con **A**.

STEP 2 Since $\alpha = \mathrm{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

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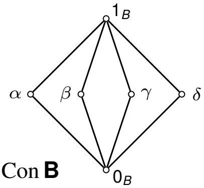
STEP 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0 e_0$, and $g_1 e_0$.

CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\mathrm{Con}\,\mathbf{A} \cong L_9$.



$\mathrm{Con}\,\mathbf{B}$

STEP 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\mathrm{Con}\,\mathbf{B} \cong M_4$.

Example:

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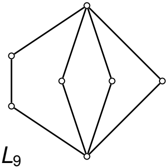

L_9

STEP 2 Since $\alpha = \mathrm{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

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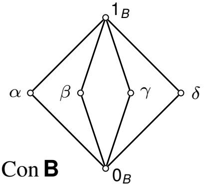
STEP 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0 e_0$, and $g_1 e_0$.

CONTRUCTION OF AN ALGEBRA **A** WITH Con **A** $\cong L_9$.



Con **B**

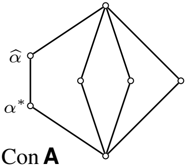
STEP 1 Take a permutational algebra $\mathbf{B} = \langle B, F \rangle$ with congruence lattice Con $\mathbf{B} \cong M_4$.

Example:

- Let $B = \{0, 1, \ldots, 5\}$ index the elements of S_3 and consider the right regular action of S_3 on itself.
- $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- Con $\langle B, \{g_0, g_1\} \rangle \cong M_4$ with congruences

$\alpha = |012|345|,\ \beta = |03|14|25|, \gamma = |04|15|23|,\ \delta = |05|13|24|.$

Goal: *expand* **B** *to an algebra* **A** *that has* α *"doubled" in* Con **A**.



Con **A**

STEP 2 Since $\alpha = \mathrm{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

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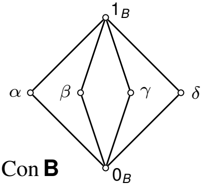
STEP 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0 e_0$, and $g_1 e_0$.

CONTRUCTION OF AN ALGEBRA **A** WITH Con **A** $\cong L_9$.



Con **B**

STEP 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\mathrm{Con}\,\mathbf{B} \cong M_4$.

<u>Example:</u>

- Let $B = \{0, 1, \ldots, 5\}$ index the elements of S_3 and consider the right regular action of S_3 on itself.
- $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
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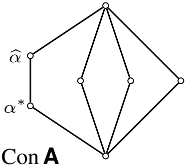
$\alpha = |012|345|,\ \beta = |03|14|25|, \gamma = |04|15|23|,\ \delta = |05|13|24|.$

Goal: *expand* **B** *to an algebra* **A** *that has* α *"doubled" in* Con **A**.



Con **A**

STEP 2 Since $\alpha = \mathrm{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

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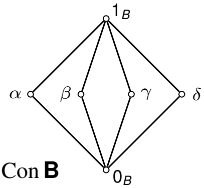
STEP 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0 e_0$, and $g_1 e_0$.

CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.

STEP 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\operatorname{Con}\mathbf{B} \cong M_4$.



Example:

- Let $B = \{0, 1, \ldots, 5\}$ index the elements of S_3 and consider the right regular action of S_3 on itself.
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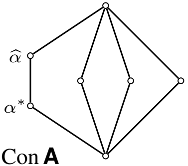
- $\operatorname{Con}\langle B, \{g_0, g_1\}\rangle \cong M_4$ with congruences

$\alpha = |012|345|,\ \beta = |03|14|25|, \gamma = |04|15|23|,\ \delta = |05|13|24|.$

Goal: *expand* $\mathbf{B}$ *to an algebra* $\mathbf{A}$ *that has* α *"doubled" in* $\operatorname{Con}\mathbf{A}$.



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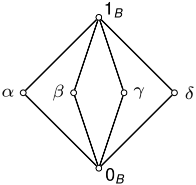
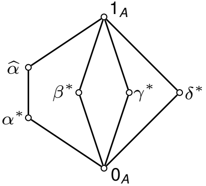
CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\mathrm{Con}\,\mathbf{A} \cong L_9$.



$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$

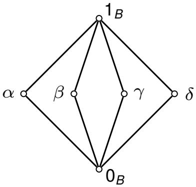


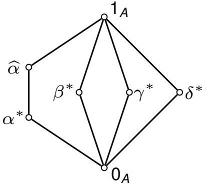
$\mathrm{Con}\,\langle A, F_A\rangle$

$$\alpha = |0, 1, 2|3, 4, 5|$$
$$\beta = |0, 3|1, 4|2, 5|$$
$$\gamma = |0, 4|1, 5|2, 3|$$
$$\delta = |0, 5|1, 3|2, 4|$$

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$
$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$
$$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$$
$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$
$$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$$

CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\mathrm{Con}\,\mathbf{A} \cong L_9$.





$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$

$\mathrm{Con}\,\langle A, F_A\rangle$

$$\alpha = |0, 1, 2|3, 4, 5|$$

$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$

$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$

$$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$$

$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$

$$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$$

$$\alpha = \alpha^* \cap B^2 = \widehat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \ldots$$

WHY DOES IT WORK?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

$$\alpha = |0, 1, 2|3, 4, 5|$$

$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

$$\begin{matrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{matrix}$$

B_0

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

$$e_0\colon A \twoheadrightarrow B_0$$
$$e_1\colon A \twoheadrightarrow B_1$$
$$e_2\colon A \twoheadrightarrow B_2$$
$$s\colon A \twoheadrightarrow B_0$$

$$ge_0\colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\rightarrow} B_0$$

Why does it work?

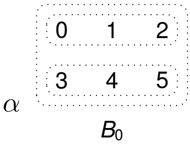
$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

$$\alpha = |0,1,2|3,4,5|$$

$$\beta = |0,3|1,4|2,5|$$

$$\gamma = |0,4|1,5|2,3|$$

$$\delta = |0,5|1,3|2,4|$$



- $A = B_0 \cup B_1 \cup B_2$
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$$e_1 \colon A \twoheadrightarrow B_1$$
$$e_2 \colon A \twoheadrightarrow B_2$$
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WHY DOES IT WORK?

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$$\alpha = |0,1,2|3,4,5|$$

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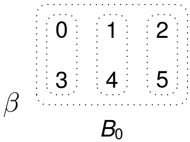

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WHY DOES IT WORK?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

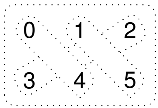
$$\alpha = |0, 1, 2|3, 4, 5|$$

$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

γ



B_0

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WHY DOES IT WORK?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

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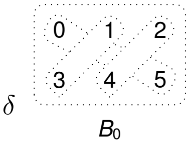
$$\delta = |0,5|1,3|2,4|$$



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Why does it work?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

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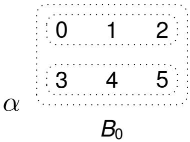

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$$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$$

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Why does it work?

$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$

$\alpha = |0,1,2|3,4,5|$

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$\gamma = |0,4|1,5|2,3|$

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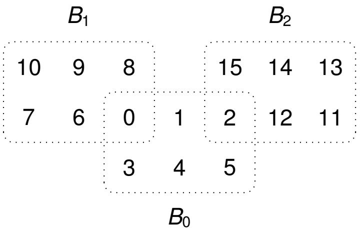

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

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$$e_1\colon A \twoheadrightarrow B_1$$
$$e_2\colon A \twoheadrightarrow B_2$$
$$s\colon A \twoheadrightarrow B_0$$

$$B_1 = \{\ 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10\ \}$$

$$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$$

$$B_2 = \{\ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15\ \}$$

$$ge_0\colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\to} B_0$$

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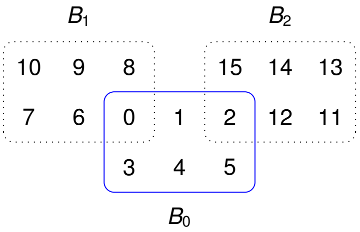

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$$\begin{array}{rcccccccc}
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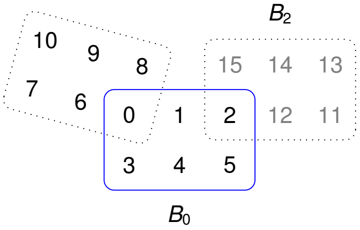

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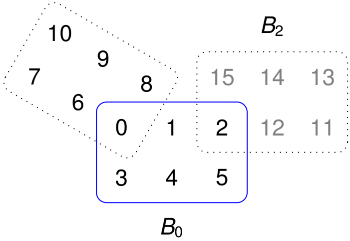

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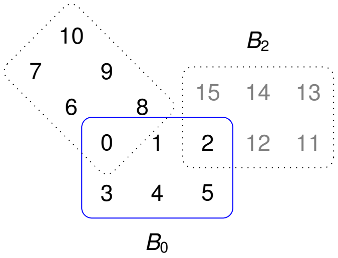

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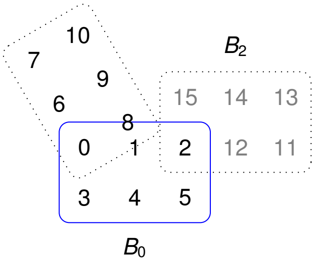

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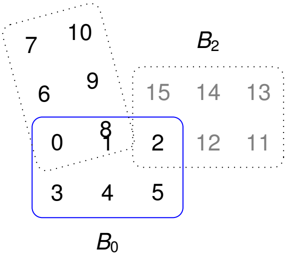

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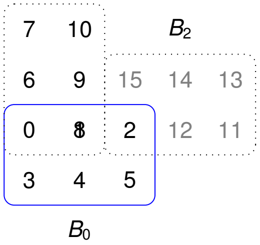

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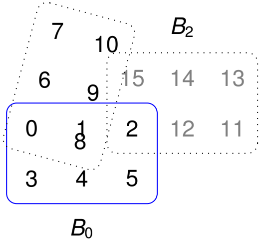

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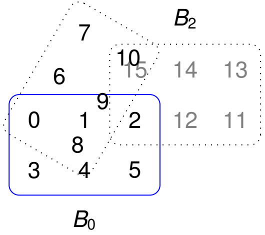

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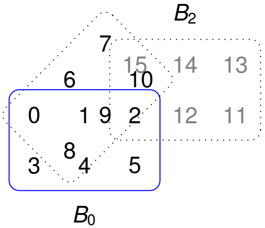

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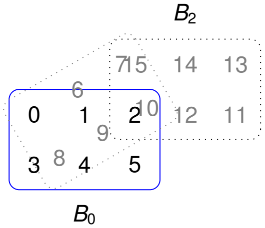

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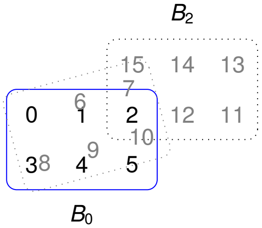

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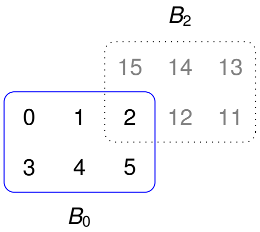

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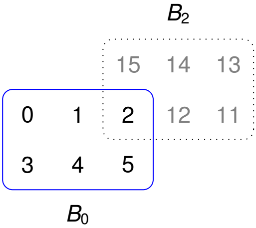

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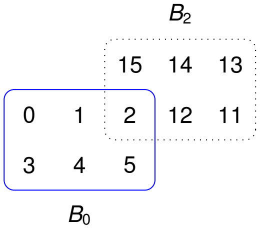

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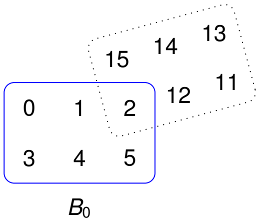

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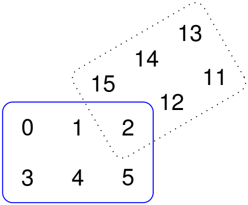

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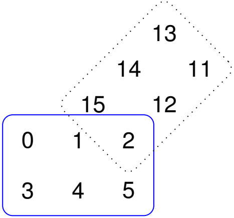

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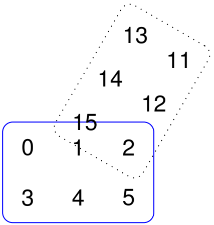

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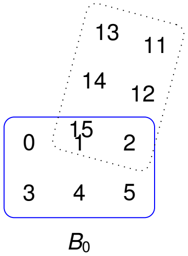

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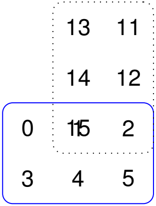

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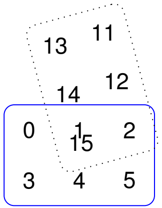

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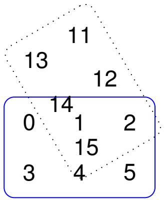

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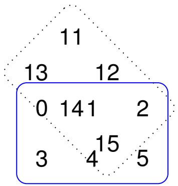

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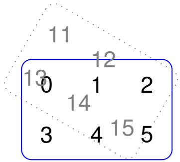

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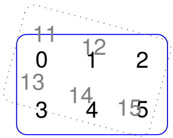

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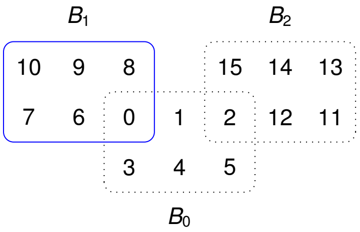

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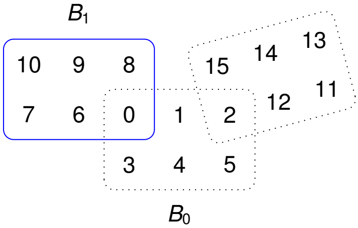

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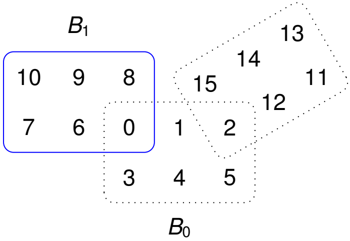

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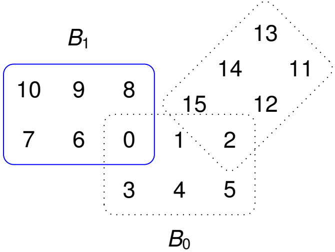

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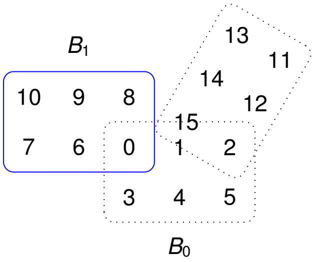

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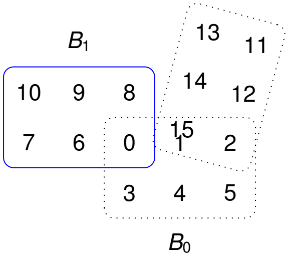

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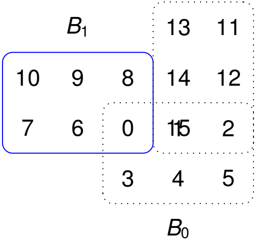

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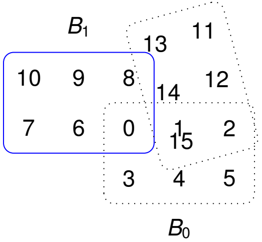

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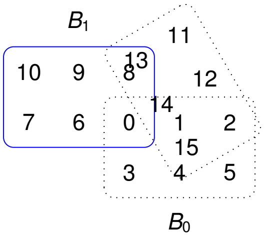

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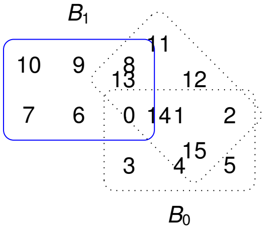

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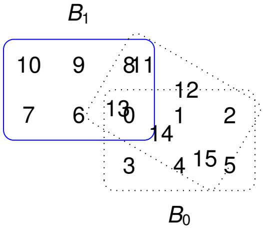

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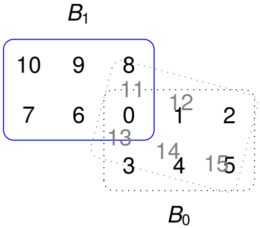

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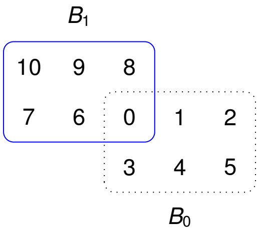

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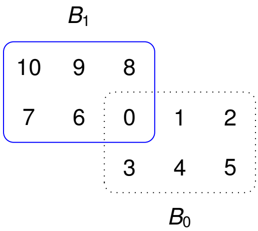

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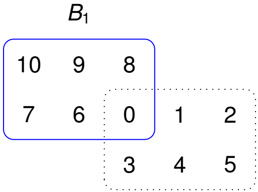

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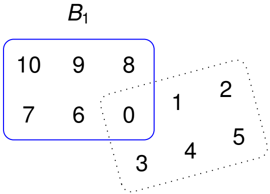

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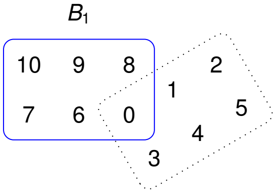

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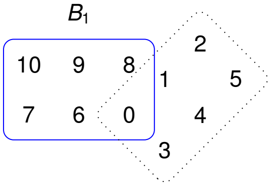

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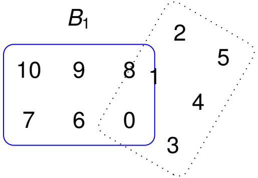

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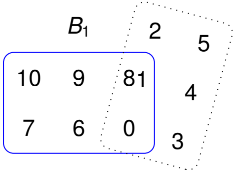

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	B_1	2	5
10	9	8	4
7	6	0	3

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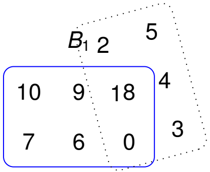

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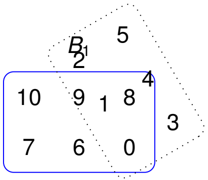

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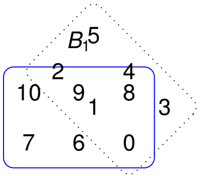

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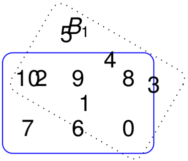

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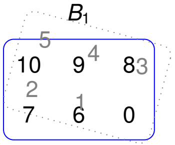

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B_1

$$\boxed{\begin{matrix} 10 & 9 & 8 \\ 7 & 6 & 0 \end{matrix}}$$

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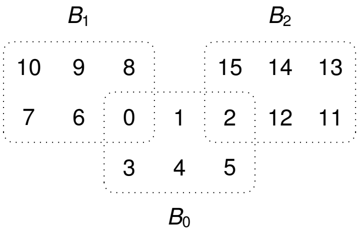

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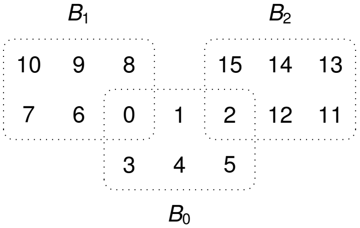

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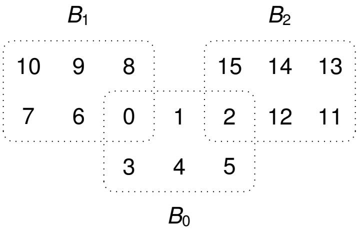

Con $\langle A, F_A\rangle$

$$\widehat{\alpha} = |0,1,2,6,7,11,12|3,4,5|8,9,10,13,14,15|$$
$$\alpha^* = |0,1,2,6,7,11,12|3,4,5|8,9,10|13,14,15|$$
$$\beta^* = |0,3,8|1,4|2,5,15|6,9|7,10|11,13|12,14|$$
$$\gamma^* = |0,4,9|1,5|2,3,13|6,10|7,8|11,14|12,15|$$
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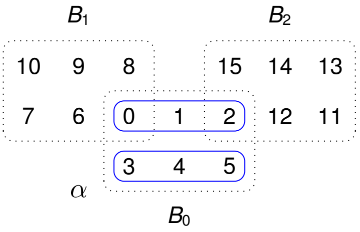

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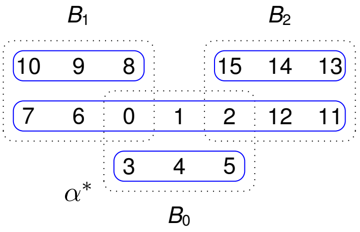

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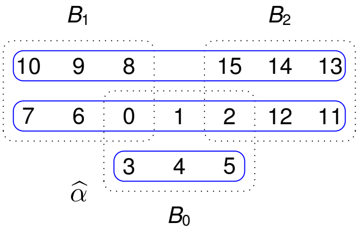

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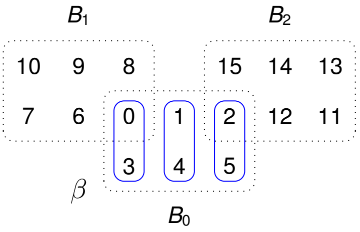

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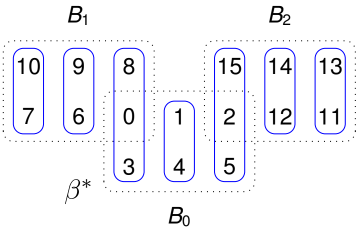

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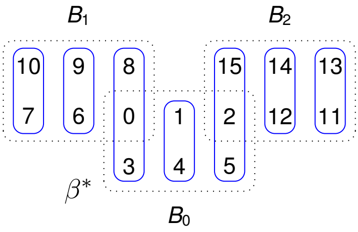

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Why don't the β classes of B_1 and B_2 mix?

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- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\beta^*, \widehat{\beta}]$.
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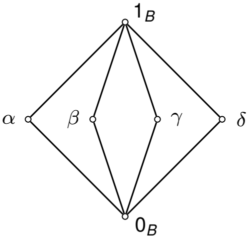

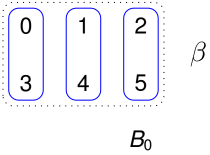

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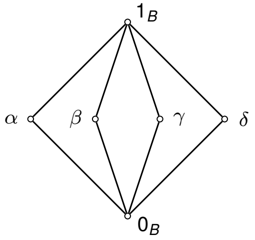

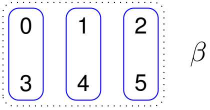

B_0

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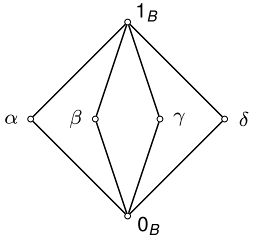

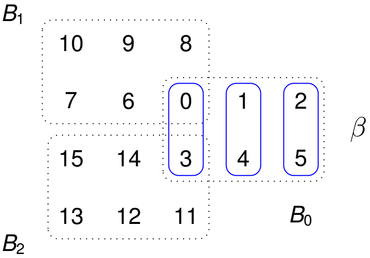

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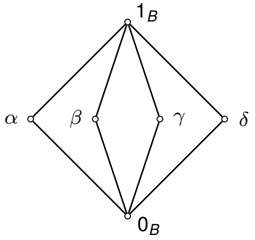

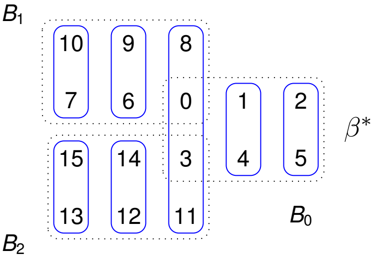

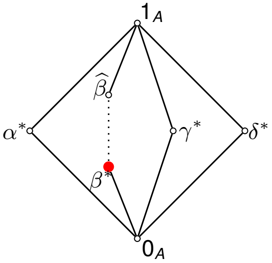

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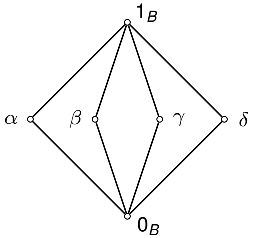

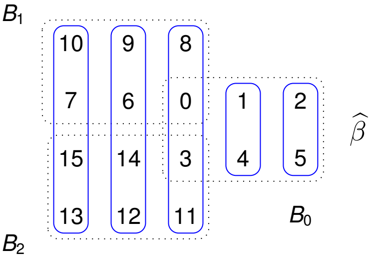

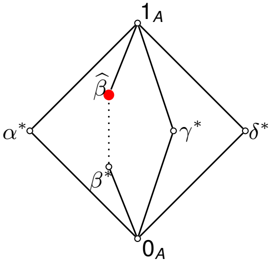

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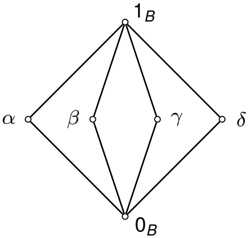

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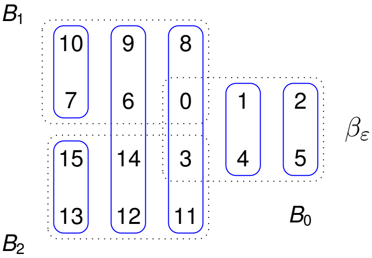

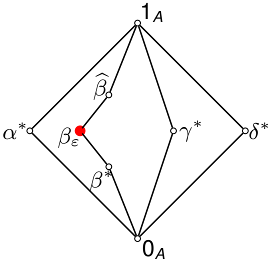

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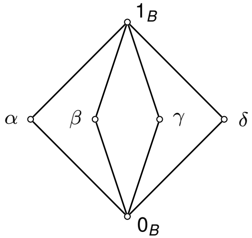

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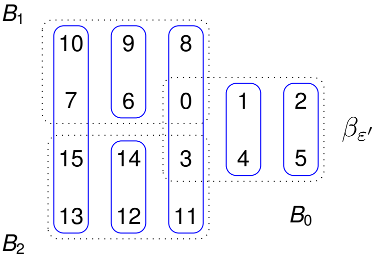

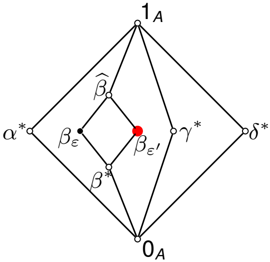

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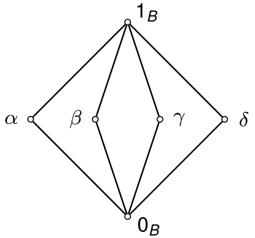

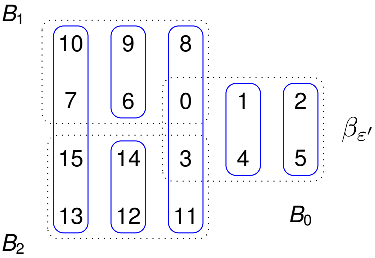

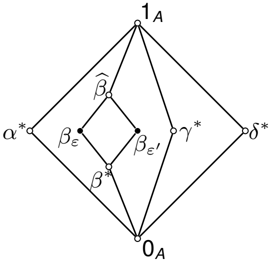

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is the congruence generation operator restricted to $\mathrm{Con}\,\mathbf{B}$.

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(i) $^*: \mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ *is a residuated mapping with residual* $|_B$.

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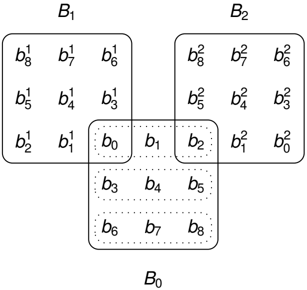

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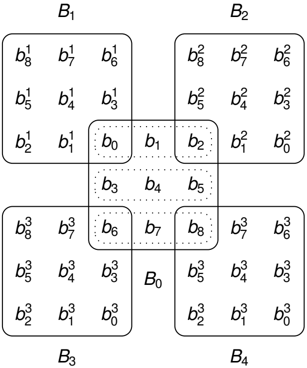

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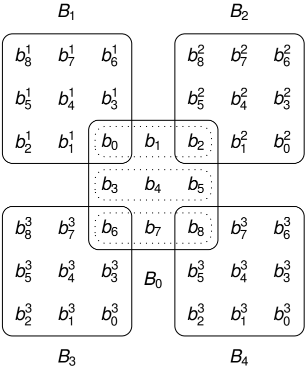

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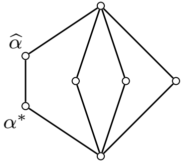
Then $$[\beta^*, \widehat{\beta}] = \{\theta \in \mathrm{Eq}(A) : \beta^* \subseteq \theta \subseteq \widehat{\beta}\} \cong \prod_{r=1}^{m} (\mathrm{Eq}|T_r|)^{m-1}.$$

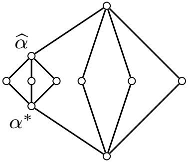
SLIGHTLY MORE GENERAL EXAMPLES...

Returning to our original example, the base algebra **B** is the right regular S_3-set, and the nontrivial relations in $\operatorname{Con}\mathbf{B}$ are

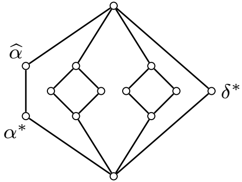
$$\alpha = |0,1,2|3,4,5| \quad \beta = |0,3|1,4|2,5| \quad \gamma = |0,4|1,5|2,3| \quad \delta = |0,5|1,3|2,4|$$



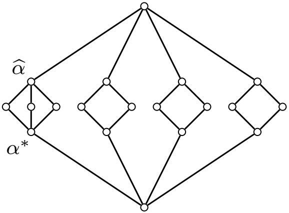
$T = \{0,1\}$



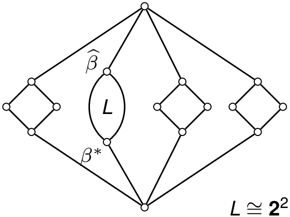
$T = \{0,1,2\}$



$T = \{0,2,3\}$



$T = \{0,1,2,3\}$



$L \cong \mathbf{2}^2 \times \mathbf{2}^2$

$T = \{0,2,3,5\}$

LIMITATIONS

Two limitations of the foregoing construction:

1. The sizes $|T_r|$ of the partition lattice factors in

$$[\beta^*, \widehat{\beta}] \cong \prod_{r=1}^{m} (\mathrm{Eq}|T_r|)^{m-1}$$

   are limited by the size of the blocks of β.

2. If β is not principal, $[\theta^*, \widehat{\theta}]$ may be non-trivial for some $\theta \ngeq \beta$.

A Generalization

Theorem

Let $\mathbf{B} = \langle B, F\rangle$ *be a finite algebra. Suppose*

$$\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \dots, (a_{K-1}, b_{K-1}))$$

has m blocks and fix $N < \infty$*.*

There exists an overalgebra $\langle A, F_A\rangle$ *such that the interval* $\beta|_B^{-1} \leqslant \mathrm{Con}\,\mathbf{A}$ *is*

$$[\beta^*, \widehat{\beta}] \cong (Eq(N))^{m-1}.$$

Moreover, we can arrange it so that $\theta^* = \widehat{\theta}$ *for all* $\theta \ngeq \beta$ *in* $\mathrm{Con}\,\mathbf{A}$*.*

A GENERALIZATION

THEOREM

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$$\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \ldots, (a_{K-1}, b_{K-1}))$$

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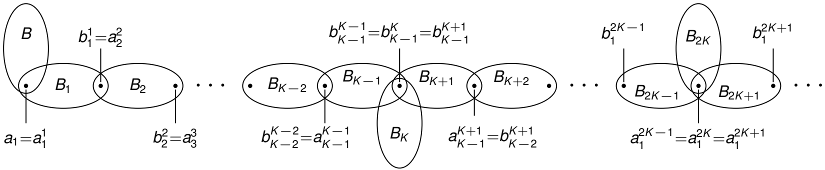
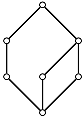
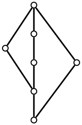
Moreover, we can arrange it so that $\theta^* = \widehat{\theta}$ for all $\theta \ngeq \beta$ in $\mathrm{Con}\,\mathbf{A}$.



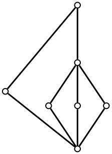
Concluding Remarks



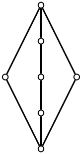
L_{19}



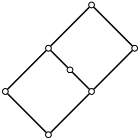
L_{20}



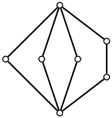
L_{17}



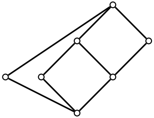
L_{13}



L_{11}

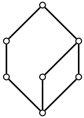


L_9

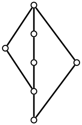


L_{10}

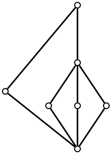
Concluding Remarks



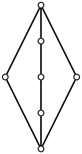
L_{19}



L_{20}



L_{17}



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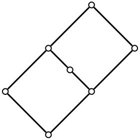

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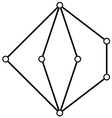

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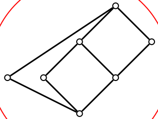


L_{10}

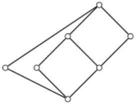
HAS ANYONE SEEN THIS LATTICE?

Given a lattice L with n elements, are there finite groups H < G such that L ≅ the lattice of subgroups between H and G?

13

4

If there is no restriction on n, this is a famous open problem. I'm wondering if any recent work has been done for small $n > 6$. I believe the question is answered (positively) for $n = 6$ by Watatani (1996) MR1409040 and Aschbacher (2008) MR2393428. I also believe we can answer it for $n = 7$, with one possible exception. The exceptional case is shown below.



So my two questions are these:

1) Does anyone know of recent work on this special case of the problem (specifically for $n = 7$ or $n = 8$)?

2) Has anyone found a finite group G with a subgroup H such that the interval

$[H, G] = \{K : H \leq K \leq G\}$

is the lattice shown above?

tagged

finite-groups × 262

open-problem × 190

lattices × 123

universal-algebra × 51

congruences × 5

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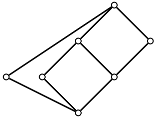
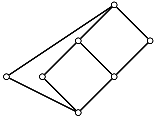
CONCLUDING REMARKS



L_{10}

- L_{10} cannot be obtained using the overalgebra construction.
- A minimal representation of L_{10} must come from a transitive G-set.
- If $[H, G] \cong L_{10}$ with H core-free in G then
  - G is a non-solvable primitive permutation group.
  - If N is a minimal normal subgroup of G, then N is nonabelian.

Concluding Remarks



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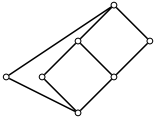

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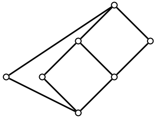
CONCLUDING REMARKS



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