OVERALGEBRAS:
EXPANSIONS AND EXTENSIONS OF FINITE ALGEBRAS

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Algebra & Combinatorics Seminar

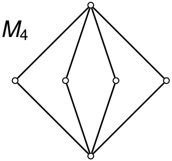
22 Feb 2016

These slides and other resources are available at

https://github.com/williamdemeo/Talks

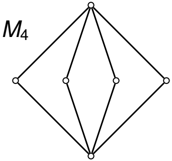
Representing lattices

M_4



Question: Is this a congruence lattice?

Representing lattices



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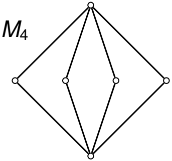

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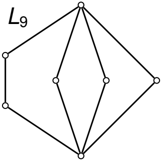
L_9



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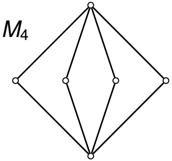

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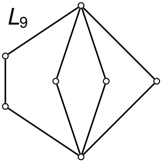
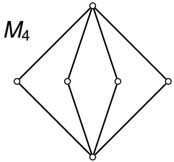
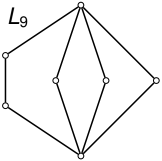


Question: Is this a congruence lattice?

Answer: Yes! ...but this one is harder.

CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\mathrm{Con}\,\mathbf{A} \cong L_9$.





STEP 1 Find a permutational algebra $\mathbf{B} = \langle B, F \rangle$ with congruence lattice $\mathrm{Con}\,\mathbf{B} \cong M_4$.

There are infinitely many but here's an easy one:

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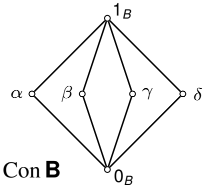

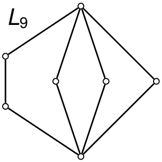

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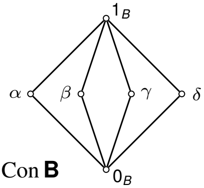
- Let $B = \{0, 1, \ldots, 5\}$ index the elements of S_3.
- the right regular action of S_3 on itself has generators $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$.
- Con $\langle B, \{g_0, g_1\}\rangle \cong M_4$ has elements

$\alpha = |012|345|,\ \beta = |03|14|25|, \gamma = |04|15|23|,\ \delta = |05|13|24|.$

Construction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.



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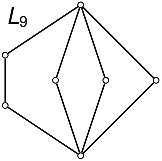

Step 2 Define $A = B_0 \cup B_1 \cup B_2$ where

$$B_0 = \{0, 1, 2, 3, 4, 5\}$$
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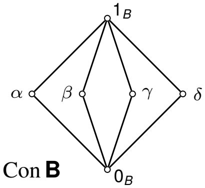

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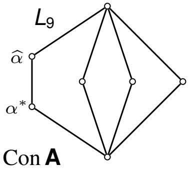

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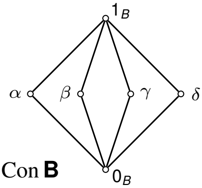

Con **B**

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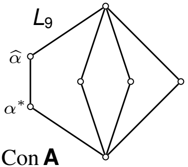

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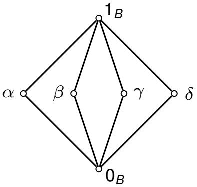
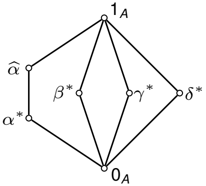


Con $\langle B, \{g_0, g_1\}\rangle$



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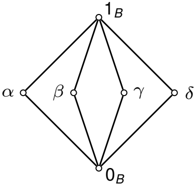

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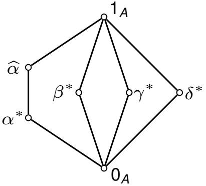

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$$\alpha = \alpha^* \cap B^2 = \widehat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \ldots$$

Expansion & Extension

Expanded Universe

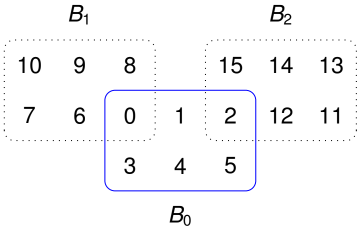
$A = B_0 \cup B_1 \cup B_2$ where

$$
\begin{aligned}
B_0 &= \{0, 1, 2, 3, 4, 5\}\\
B_1 &= \{0, 6, 7, 8, 9, 10\}\\
B_2 &= \{11, 12, 2, 13, 14, 15\}.
\end{aligned}
$$

New Operations

$$
\begin{aligned}
e_0 &: A \twoheadrightarrow B_0\\
e_1 &: A \twoheadrightarrow B_1\\
e_2 &: A \twoheadrightarrow B_2\\
s &: A \twoheadrightarrow B_0,\\
ge_0 &: A \overset{e_0}{\twoheadrightarrow} B_0 \xrightarrow{g} B_0,
\end{aligned}
$$

for each $g \in F^B$.



Why does it work?

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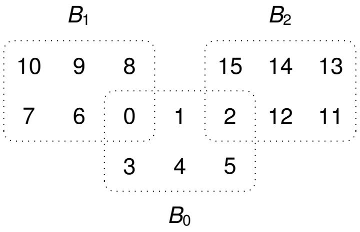

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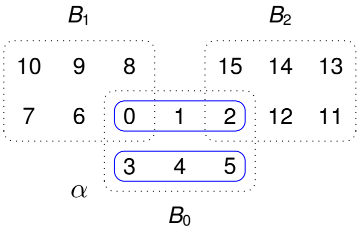

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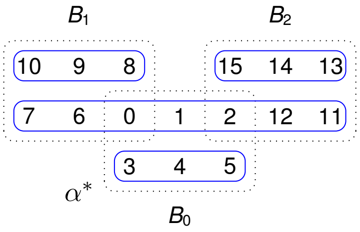

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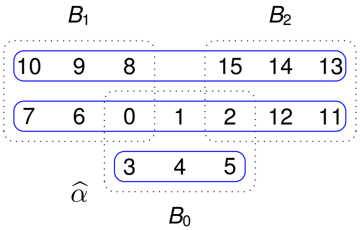

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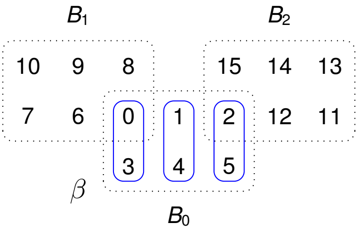

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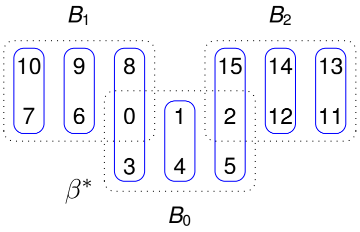

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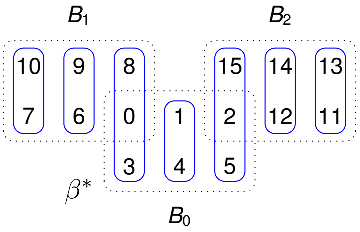

Con $\langle A, F_A\rangle$

Why don't β classes of B_1, B_2 mix?

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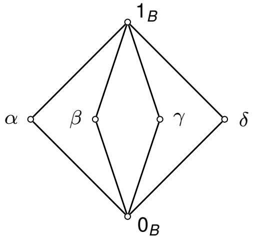
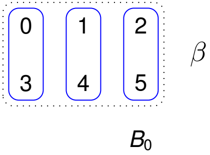
VARIATIONS ON THE SAME EXAMPLE...

- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_{B}^{-1} = [\![\beta^*, \widehat{\beta}]\!]$.





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- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\![\beta^*, \widehat{\beta}]\!]$.
- Select elements 0 and 3 as intersection points:

$$A = B_0 \cup B_1 \cup B_2 \quad \text{where}$$

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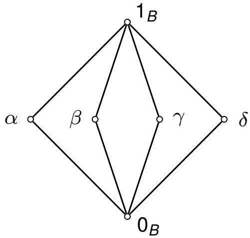

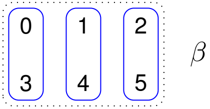

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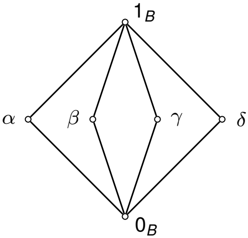

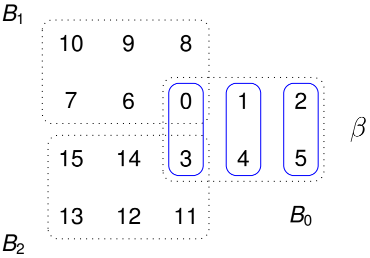

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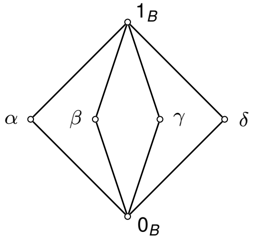

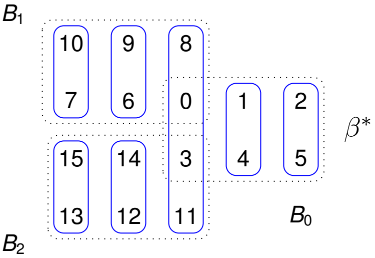

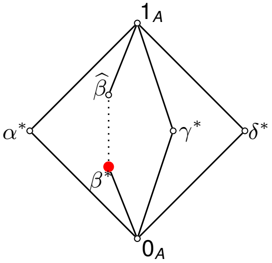

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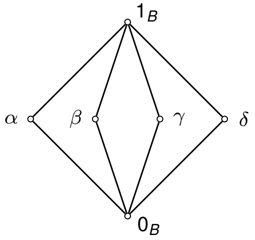

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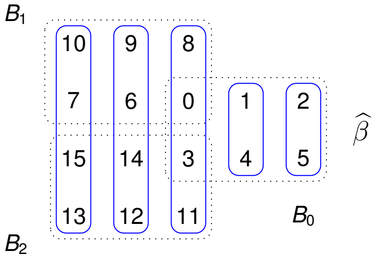

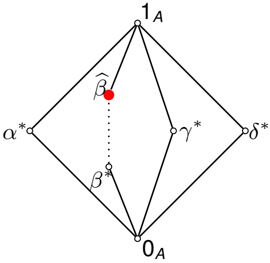

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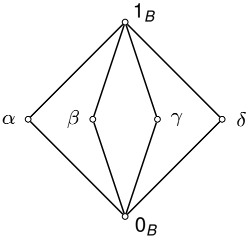

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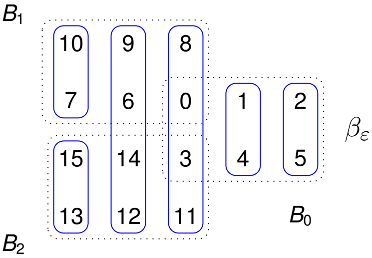

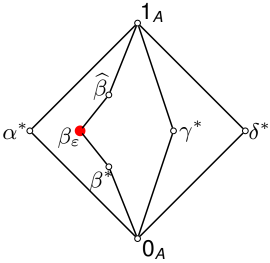

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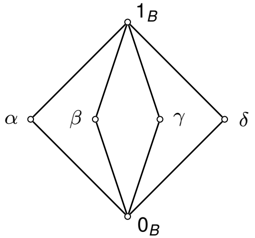

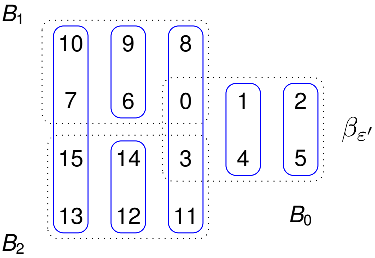

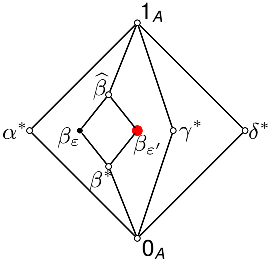

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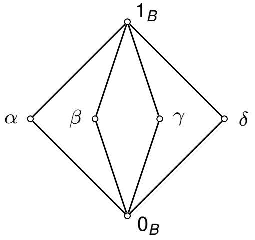
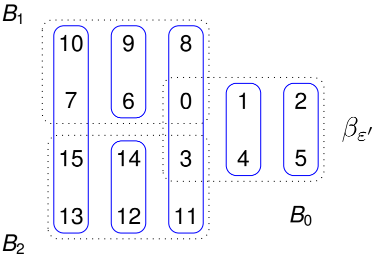
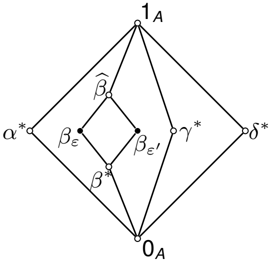






The P^5 Lemma

Lemma (Pálfy-Pudlák, 1980)

Let $\mathbf{A} = \langle A, F \rangle$ *be a unary algebra where* F *is a monoid.*

Suppose $e \in F$ *satisfies* $e \circ e = e$.

Define $\mathbf{B} = \langle B, G \rangle$

$$B = e(A) \quad \text{and} \quad G = \{e f|_B \mid f \in F\}.$$

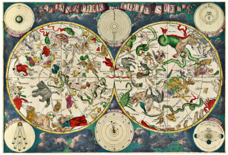
Let $|_B : Con(\mathbf{A}) \to Con(\mathbf{B})$ *be the restriction mapping:*

$$\theta|_B = \theta \cap B^2$$

Then $|_B$ *is a surjective homomorphism (even for arbitrary meets and joins).*

Péter Pál Pálfy and Pavel Pudlák: *Congruence lattices of finite algebras* AU (1980).

The Star Map and Hat Map



star map

hat map

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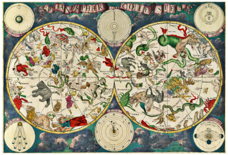

~~star map~~ star chart

~~hat map~~ map hat

THE STAR MAP AND HAT MAP

STAR MAP * : $\operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ is congruence generation:

$$\beta^* = \operatorname{Cg}^{\mathbf{A}}(\beta) \qquad (\forall\, \beta \in \operatorname{Con}\mathbf{B})$$

HAT MAP $\widehat{\ }$: $\operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ is

$$\widehat{\beta} = \{(x,y) \in A^2 \mid (ef(x), ef(y)) \in \beta,\ \forall\, f \in \operatorname{Pol}_1(\mathbf{A})\}.$$

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The hat map appears in McKenzie's "Finite Forbidden Lattices" paper (Puebla, 1982) where he gives an alternative proof of the P^5 Lemma.

Residuation lemma

A lemma relating the three maps *, $|_B$ and $\widehat{\ }$.

Lemma

(I) ${}^* : \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ *is a **residuated mapping** with **residual*** $|_B$.

(II) $|_B : \operatorname{Con} \mathbf{A} \to \operatorname{Con} \mathbf{B}$ *is a **residuated mapping** with **residual*** $\widehat{\ }$.

(III) *For all* $\alpha \in \operatorname{Con} \mathbf{A}$, $\beta \in \operatorname{Con} \mathbf{B}$,

$$\beta = \alpha|_B \quad \Leftrightarrow \quad \beta^* \leqslant \alpha \leqslant \widehat{\beta}.$$

In particular, $\beta^*|_B = \beta = \widehat{\beta}|_B$.

Adjunction lemma

New version of the lemma:

$$^{*} \quad \dashv \quad |_{B} \quad \dashv \quad \widehat{\ }$$

That is,

Lemma

(I) $^{*} : \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ *is **left adjoint** to* $|_{B}$*.*

(II) $|_{B} : \operatorname{Con} \mathbf{A} \to \operatorname{Con} \mathbf{B}$ *is **left adjoint** to* $\widehat{\ }$*.*

(III) *For all* $\alpha \in \operatorname{Con} \mathbf{A}$, $\beta \in \operatorname{Con} \mathbf{B}$*,*

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In particular, $\beta^{*}|_{B} = \beta = \widehat{\beta}|_{B}$*.*

Proof of the P^5 Lemma

Lemma (Pálfy-Pudlák, 1980)

The restriction mapping

$$\operatorname{Con} \mathbf{A} \ni \alpha \mapsto \alpha|_B = \alpha \cap B^2 \in \operatorname{Con} \mathbf{B}$$

is a complete lattice epimorphism.

New Proof

Recall, for $f : X \to Y$ a monotone function on preorders X and Y,
if f has a right (left) adjoint, then f preserves all joins (meets) that exist in X.

By the lemma $|_B$ has both a left and right adjoint.