THE FINITE LATTICE REPRESENTATION PROBLEM
AND
INTERVALS IN SUBGROUP LATTICES
PART II

William DeMeo

williamdemeo@gmail.com

joint work with

Ralph Freese, Peter Jipsen, Bill Lampe, J.B. Nation
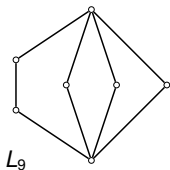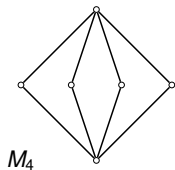
University of South Carolina

Algebra & Logic Seminar

September 28, 2012
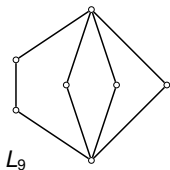
These slides and other resources are available at

http://www.math.sc.edu/~demeow/FLRP.html ⟶

CONTRUCTION OF AN ALGEBRA **A** WITH Con **A** $\cong L_9$.

M_4

L_9

CONTRUCTION OF AN ALGEBRA **A** WITH Con **A** ≅ L_9.

STEP 1 Take a permutational algebra **B** = $\langle B, F \rangle$ with congruence lattice Con **B** ≅ M_4.

Con **B**

L_9

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Example:

- Let $B = \{0, 1, \ldots, 5\}$ index the elements of S_3 and consider the right regular action of S_3 on itself.
- $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- Con $\langle B, \{g_0, g_1\} \rangle \cong M_4$ with congruences

$\alpha = |012|345|, \ \beta = |03|14|25|, \gamma = |04|15|23|, \ \delta = |05|13|24|.$

Con **B**

1_B

α β γ δ

0_B

L_9

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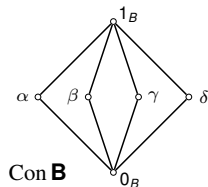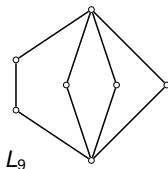
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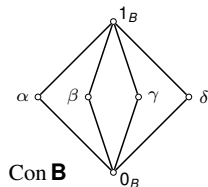- Con $\langle B, \{g_0, g_1\} \rangle \cong M_4$ with congruences

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Goal: *expand* $\mathbf{B}$ *to an algebra* $\mathbf{A}$ *that has* α *"doubled" in* Con $\mathbf{A}$.

Con **B**

Con **A**

CONTRUCTION OF AN ALGEBRA **A** WITH $\operatorname{Con} \mathbf{A} \cong L_9$.

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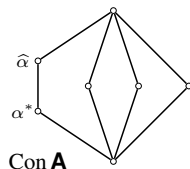
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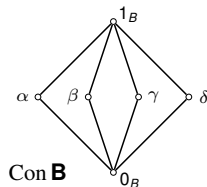$\alpha = |012|345|, \ \beta = |03|14|25|, \gamma = |04|15|23|, \ \delta = |05|13|24|.$

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STEP 2 Since $\alpha = \operatorname{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

$$B_0 = \{0, 1, 2, 3, 4, 5\} = B$$
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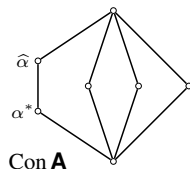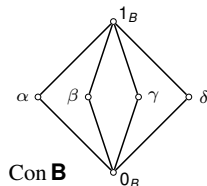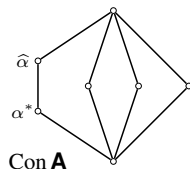
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Con **A**

$\widehat{\alpha}$

α^*

STEP 3 Define unary operations e_0, e_1, e_2, s, $g_0 e_0$, and $g_1 e_0$.

CONTRUCTION OF AN ALGEBRA **A** WITH Con **A** $\cong L_9$.

Con $\langle B, \{g_0, g_1\} \rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

$\beta = |0, 3|1, 4|2, 5|$

$\gamma = |0, 4|1, 5|2, 3|$

$\delta = |0, 5|1, 3|2, 4|$

Con $\langle A, F_A \rangle$

$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$

$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$

$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$

$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$

$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$

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$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$

$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$

$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$

$$\alpha = \alpha^* \cap B^2 = \widehat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \ldots$$
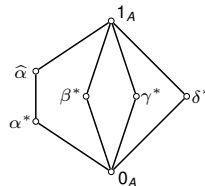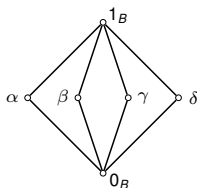
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$\alpha = |0, 1, 2|3, 4, 5|$

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$\gamma = |0, 4|1, 5|2, 3|$

$\delta = |0, 5|1, 3|2, 4|$

0	1	2
3	4	5

B_0

EXTENSION & EXPANSION

Con $\langle B, \{g_0, g_1\} \rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

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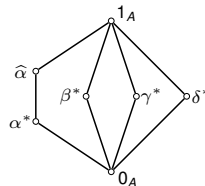
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β

| 0 | 1 | 2 |
| 3 | 4 | 5 |

B_0

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γ

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δ

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$B_0 = \{\ 0\quad 1\quad 2\quad 3\quad 4\quad 5\ \}$

EXTENSION & EXPANSION
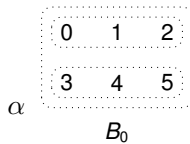
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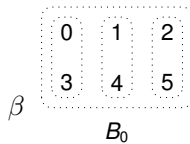$B_2 = \{\; 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15 \;\}$

EXTENSION & EXPANSION
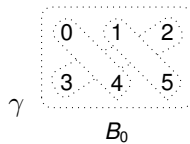
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$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
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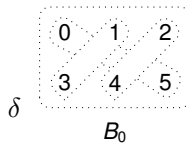$$B_2 = \{\, 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15 \,\}$$

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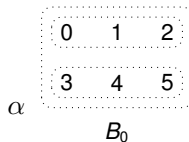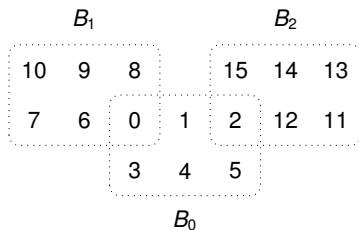
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B_2

B_0

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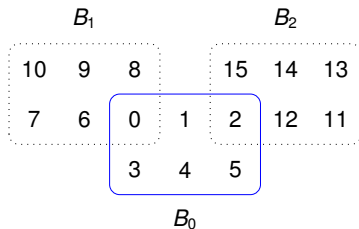$$B_2 = \{\ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15\ \}$$

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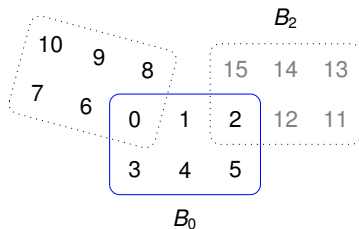
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$$
\begin{array}{ccc}
 & 10 & & & & B_2 \\
7 & & 9 & & & \\
 & 6 & & 8 & 15 \;\; 14 \;\; 13 \\
 & & 0 & 1 & 2 \;\; | \;\; 12 \;\; 11 \\
 & & 3 & 4 & 5 \\
 & & & B_0 &
\end{array}
$$

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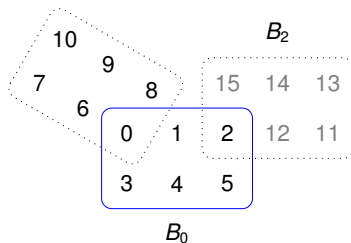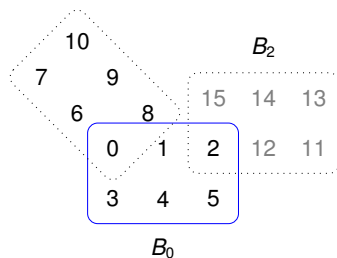
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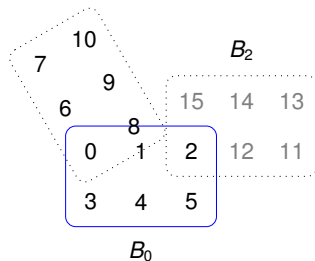$$B_2 = \{ \begin{array}{cccccc} 11 & 12 & 2 & 13 & 14 & 15 \end{array} \}$$

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EXTENSION & EXPANSION
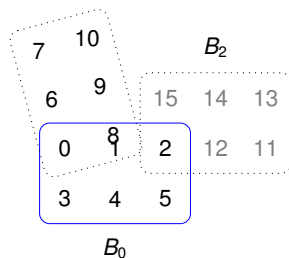
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$\quad\quad\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$

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EXTENSION & EXPANSION
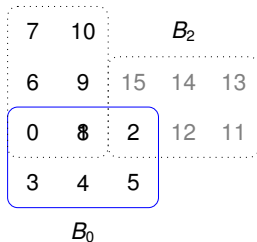
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  $e_1 \colon A \twoheadrightarrow B_1$
  $e_2 \colon A \twoheadrightarrow B_2$
  $s \colon A \twoheadrightarrow B_0$

$ge_0 \colon A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$

$B_1 = \{\ 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10\ \}$

$$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$$
$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$
$$B_2 = \{\ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15\ \}$$

EXTENSION & EXPANSION
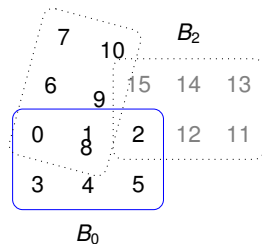
Con $\langle B, \{g_0, g_1\} \rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

$\beta = |0, 3|1, 4|2, 5|$

$\gamma = |0, 4|1, 5|2, 3|$

$\delta = |0, 5|1, 3|2, 4|$

B_0

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

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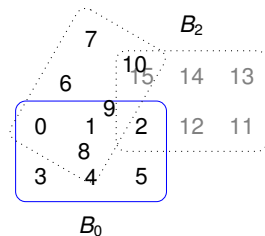$$B_2 = \{\ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15\ \}$$

EXTENSION & EXPANSION

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B_0

B_2

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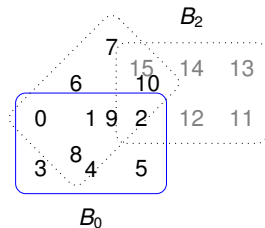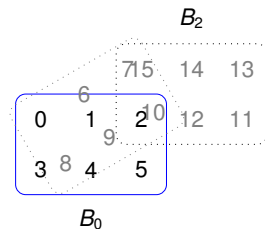
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B_2

B_0

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EXTENSION & EXPANSION
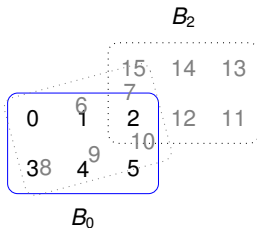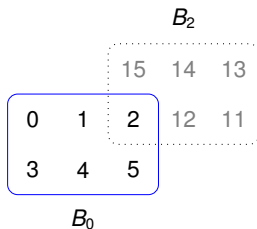
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B_2

	15	14	13	
0	1	2	12	11
3	4	5		

B_0

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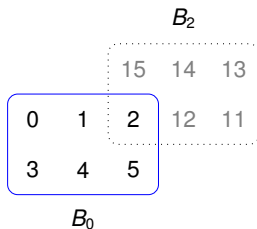
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B_2

15	14	13		

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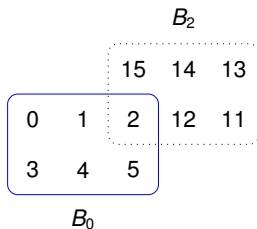$ge_0 \colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\to} B_0$

EXTENSION & EXPANSION

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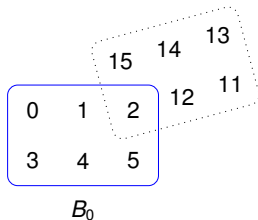$ge_0 \colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\to} B_0$

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B_0

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EXTENSION & EXPANSION
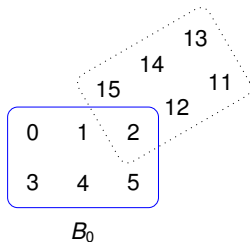
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EXTENSION & EXPANSION
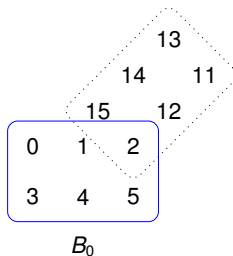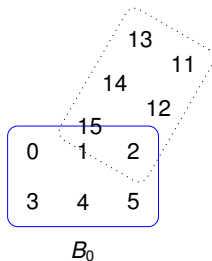
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EXTENSION & EXPANSION
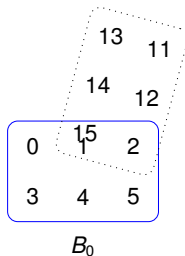
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EXTENSION & EXPANSION

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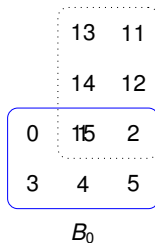$$B_2 = \{\ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15\ \}$$

EXTENSION & EXPANSION

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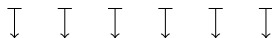
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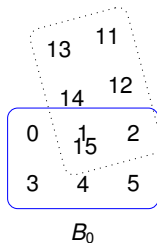
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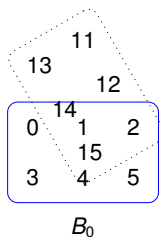
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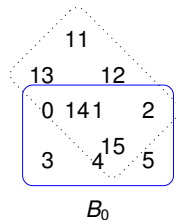$ge_0 : A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$
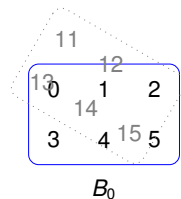
EXTENSION & EXPANSION

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B_0

$$
\begin{array}{ccccccc}
B_1 = \{ & 0 & 6 & 7 & 8 & 9 & 10 \} \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
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\end{array}
$$

EXTENSION & EXPANSION
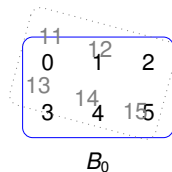
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$$
\begin{array}{ccc}
0 & 1 & 2 \\
3 & 4 & 5
\end{array}
$$

B_0

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

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  $e_2 \colon A \twoheadrightarrow B_2$

  $s \colon A \twoheadrightarrow B_0$

$B_1 = \{\, 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \,\}$

$\quad\quad\ \updownarrow \ \ \updownarrow \ \ \updownarrow \ \ \updownarrow \ \ \updownarrow \ \ \updownarrow$

$B_0 = \{\, 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \,\}$

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$B_2 = \{\, 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15 \,\}$
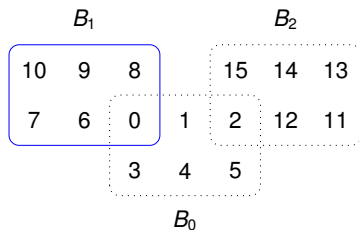
$ge_0 \colon A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$

$\text{Con} \langle B, \{g_0, g_1\} \rangle$

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EXTENSION & EXPANSION
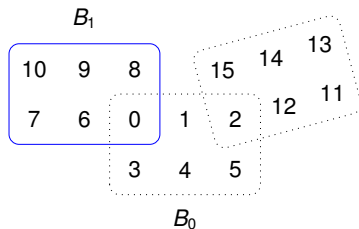
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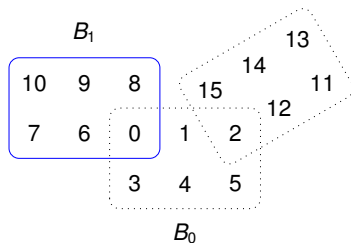$ge_0 \colon A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$

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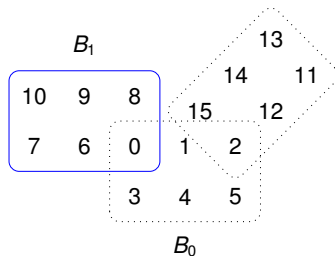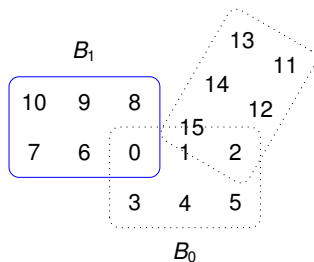
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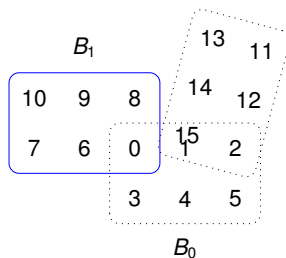
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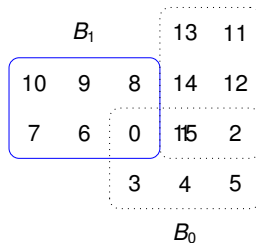
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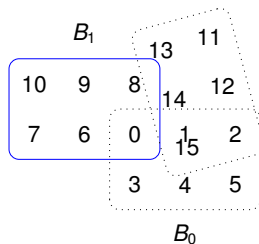$B_2 = \{ \ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15 \ \}$

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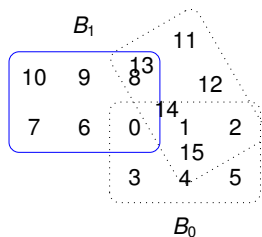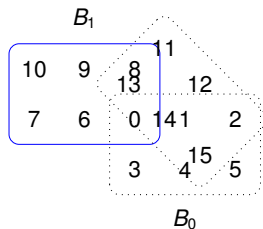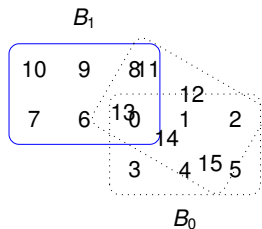$ge_0 \colon A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$

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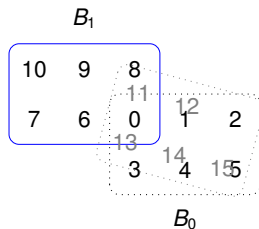
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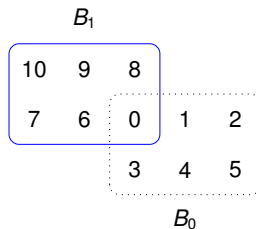$B_2 = \{\ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15\ \}$

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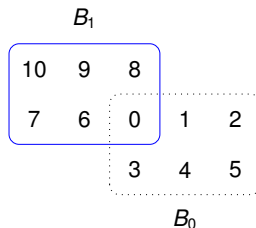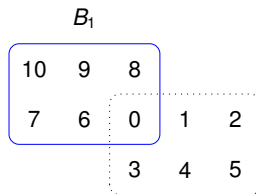
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B_1

10	9	8		
7	6	0	1	2
		3	4	5

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EXTENSION & EXPANSION
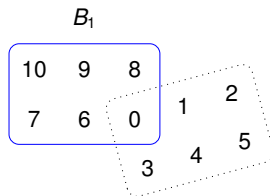
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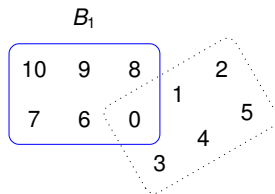$ge_0 \colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\rightarrow} B_0$

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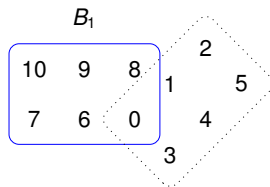
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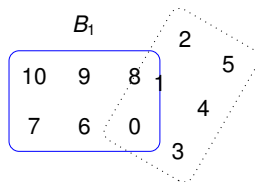
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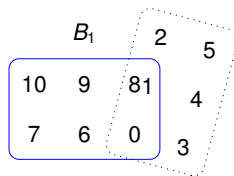
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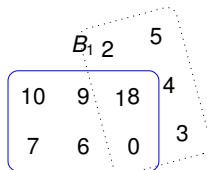$B_2 = \{\ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15\ \}$

EXTENSION & EXPANSION

Con $\langle B, \{g_0, g_1\} \rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

$\beta = |0, 3|1, 4|2, 5|$

$\gamma = |0, 4|1, 5|2, 3|$

$\delta = |0, 5|1, 3|2, 4|$

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

  $e_0 \colon A \twoheadrightarrow B_0$

  $e_1 \colon A \twoheadrightarrow B_1$

  $e_2 \colon A \twoheadrightarrow B_2$

  $s \colon A \twoheadrightarrow B_0$

$ge_0 \colon A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$

$B_1 = \{\ 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10\ \}$
$\quad\quad\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$
$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$
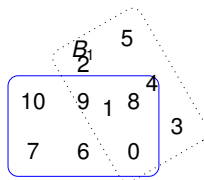
$B_2 = \{\ 11 \quad 12 \quad 2 \quad 13 \quad 14 \quad 15\ \}$

Con $\langle B, \{g_0, g_1\} \rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

$\beta = |0, 3|1, 4|2, 5|$

$\gamma = |0, 4|1, 5|2, 3|$

$\delta = |0, 5|1, 3|2, 4|$

		B_1	2	5
	10	9	8	4
	7	6	0	3

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

  $e_0 \colon A \twoheadrightarrow B_0$
  $e_1 \colon A \twoheadrightarrow B_1$
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$B_1 = \{\, 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \,\}$

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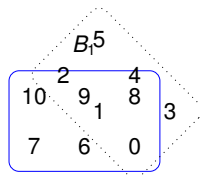
Con $\langle B, \{g_0, g_1\} \rangle$

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EXTENSION & EXPANSION
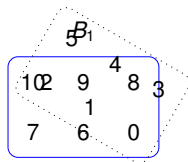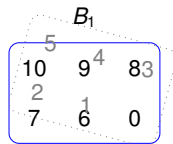
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$ge_0 : A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$

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$\qquad\quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$
$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$

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EXTENSION & EXPANSION

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$e_1 \colon A \twoheadrightarrow B_1$

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$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$
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EXTENSION & EXPANSION
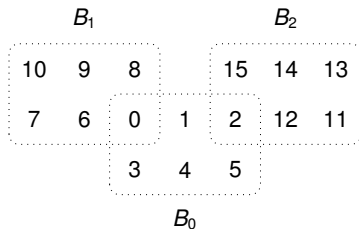
Con $\langle B, \{g_0, g_1\} \rangle$

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$\beta = |0, 3|1, 4|2, 5|$

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- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

$\quad e_0 \colon A \twoheadrightarrow B_0$

$\quad e_1 \colon A \twoheadrightarrow B_1$

$\quad e_2 \colon A \twoheadrightarrow B_2$

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$B_1 = \{\, 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \,\}$

$B_0 = \{\, 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \,\}$

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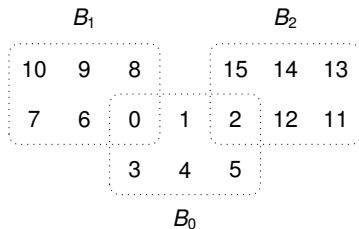$ge_0 \colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\to} B_0$

EXTENSION & EXPANSION

Con $\langle B, \{g_0, g_1\} \rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

$\beta = |0, 3|1, 4|2, 5|$

$\gamma = |0, 4|1, 5|2, 3|$

$\delta = |0, 5|1, 3|2, 4|$

B_1

10	9 ^{4}	8 $_3$
^{2}		
7	6 ^{1}	0

^{5}

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

  $e_0 \colon A \twoheadrightarrow B_0$

  $e_1 \colon A \twoheadrightarrow B_1$

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$\qquad\qquad \updownarrow \ \ \updownarrow \ \ \updownarrow \ \ \updownarrow \ \ \updownarrow \ \ \updownarrow$
$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$

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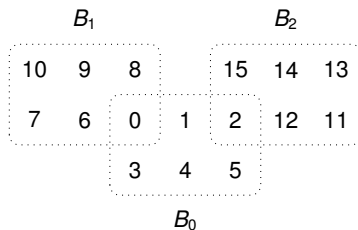$ge_0 \colon A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$

$\text{Con} \langle B, \{g_0, g_1\} \rangle$

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$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

B_1

10	9	8
7	6	0

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

  $e_0 \colon A \twoheadrightarrow B_0$

  $e_1 \colon A \twoheadrightarrow B_1$

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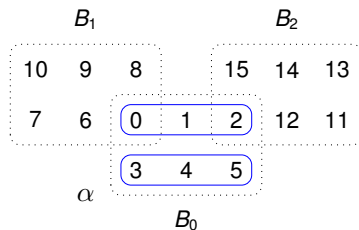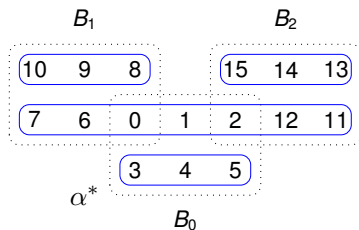
$\text{Con} \langle B, \{g_0, g_1\} \rangle$

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EXTENSION & EXPANSION
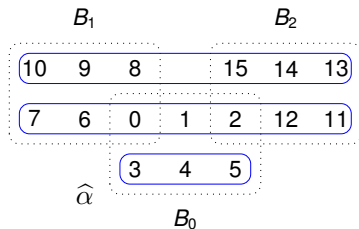
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$ge_0 \colon A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$

WHY DOES IT WORK?

$\text{Con}\, \langle B, \{g_0, g_1\}\rangle$

$\alpha = |0, 1, 2|3, 4, 5|$
$\beta = |0, 3|1, 4|2, 5|$
$\gamma = |0, 4|1, 5|2, 3|$
$\delta = |0, 5|1, 3|2, 4|$

$$B_1 \qquad\qquad B_2$$

10	9	8		15	14	13
7	6	0	1	2	12	11
		3	4	5		

$$B_0$$

$\text{Con}\, \langle A, F_A\rangle$

$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$
$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$
$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$
$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$
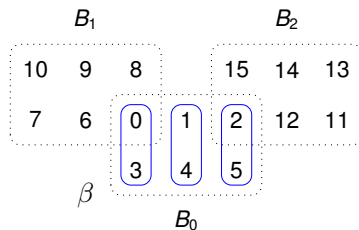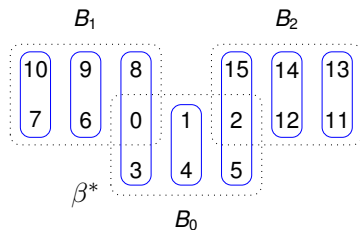$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$

WHY DOES IT WORK?

Con $\langle B, \{g_0, g_1\} \rangle$

$\alpha = |0, 1, 2|3, 4, 5|$
$\beta = |0, 3|1, 4|2, 5|$
$\gamma = |0, 4|1, 5|2, 3|$
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Con $\langle A, F_A \rangle$

$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$
$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$
$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$
$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$
$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$

$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

$\beta = |0, 3|1, 4|2, 5|$

$\gamma = |0, 4|1, 5|2, 3|$

$\delta = |0, 5|1, 3|2, 4|$

$B_1 \qquad B_2$

10	9	8		15	14	13
7	6	0	1	2	12	11
	3	4	5			

α^*

B_0

$\mathrm{Con}\,\langle A, F_A\rangle$

$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$

$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$

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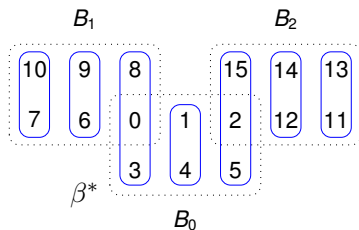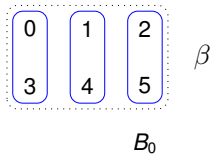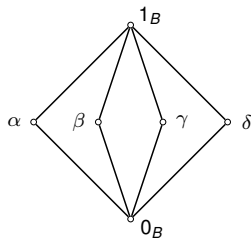$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$

WHY DOES IT WORK?

Con $\langle B, \{g_0, g_1\} \rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

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$\delta = |0, 5|1, 3|2, 4|$

Con $\langle A, F_A \rangle$

$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$

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WHY DOES IT WORK?

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$\alpha = |0, 1, 2|3, 4, 5|$

$\beta = |0, 3|1, 4|2, 5|$

$\gamma = |0, 4|1, 5|2, 3|$

$\delta = |0, 5|1, 3|2, 4|$

B_1 B_2

β^*

B_0

Con $\langle A, F_A \rangle$

*Why don't the β classes
of B_1 and B_2 mix?*

$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$

$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$

$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$

$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$

$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$

- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_{B}^{-1} = [\beta^*, \widehat{\beta}]$.
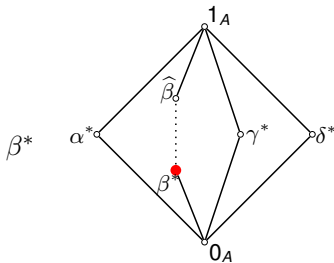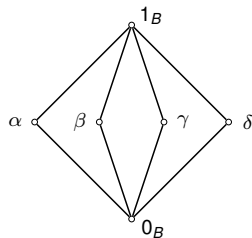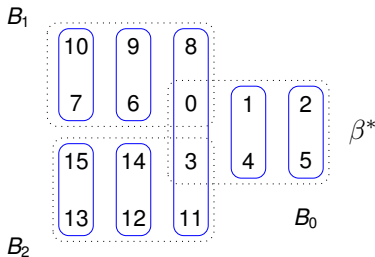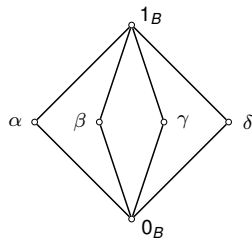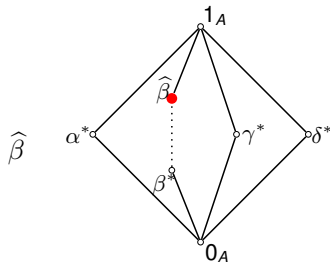
VARIATIONS ON THE SAME EXAMPLE...

- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\beta^*, \widehat{\beta}]$.

- Select elements 0 and 3 as intersection points:

$$A = B_0 \cup B_1 \cup B_2 \quad \text{where}$$

$$B_0 = \{0, 1, 2, 3, 4, 5\}$$
$$B_1 = \{0, 6, 7, 8, 9, 10\}$$
$$B_2 = \{11, 12, 13, 3, 14, 15\}.$$
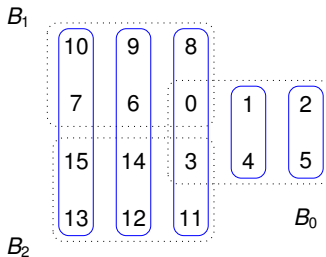

B_0

VARIATIONS ON THE SAME EXAMPLE...

- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_{B}^{-1} = [\beta^*, \widehat{\beta}]$.

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$$A = B_0 \cup B_1 \cup B_2 \quad \text{where}$$

$$B_0 = \{0, 1, 2, 3, 4, 5\}$$
$$B_1 = \{0, 6, 7, 8, 9, 10\}$$
$$B_2 = \{11, 12, 13, 3, 14, 15\}.$$

B_1

10	9	8			
7	6	0	1	2	
15	14	3	4	5	β
13	12	11		B_0	

B_2

VARIATIONS ON THE SAME EXAMPLE...

- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\beta^*, \widehat{\beta}]$.

- Select elements 0 and 3 as intersection points:

$$A = B_0 \cup B_1 \cup B_2 \quad \text{where}$$

$$B_0 = \{0, 1, 2, 3, 4, 5\}$$
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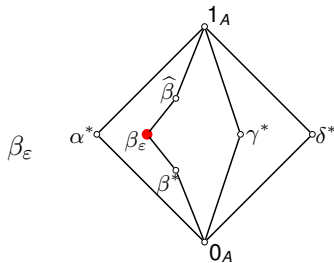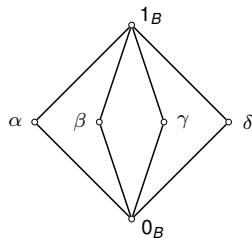$$B_2 = \{11, 12, 13, 3, 14, 15\}.$$
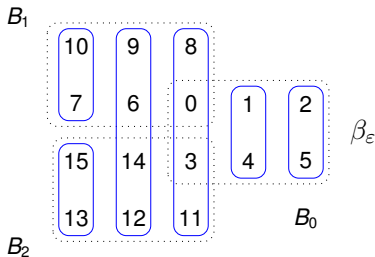
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- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\beta^*, \widehat{\beta}]$.

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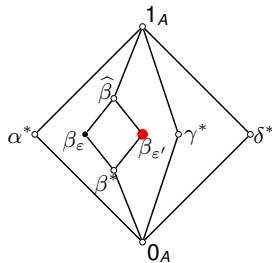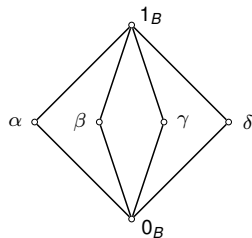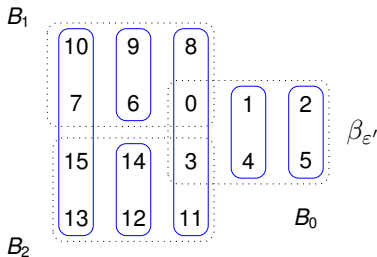
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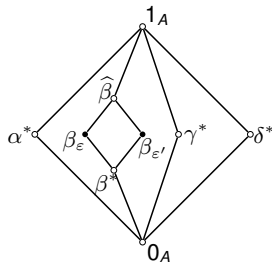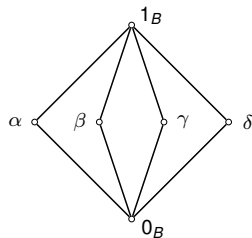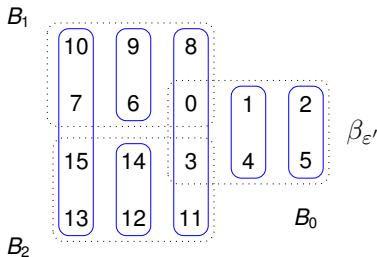
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$$\widehat{\beta} = \{(x, y) \in A^2 : (ef(x), ef(y)) \in \beta \text{ for all } f \in \mathrm{Pol}_1(\mathbf{A})\}.$$

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LEMMA

(I) $^* : \operatorname{Con} \mathbf{B} \to \operatorname{Con} \mathbf{A}$ *is a residuated mapping with residual* $|_{B}$.

(II) $|_{B} : \operatorname{Con} \mathbf{A} \to \operatorname{Con} \mathbf{B}$ *is a residuated mapping with residual* $\widehat{}$.

(III) *For all* $\alpha \in \operatorname{Con} \mathbf{A}$, $\beta \in \operatorname{Con} \mathbf{B}$,

$$\beta = \alpha|_{B} \quad \Leftrightarrow \quad \beta^* \leqslant \alpha \leqslant \widehat{\beta}.$$
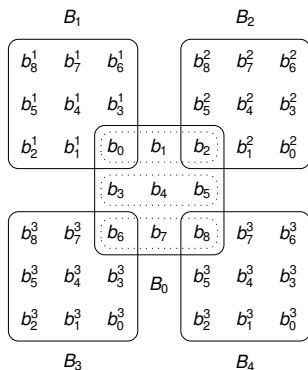
In particular, $\beta^*|_{B} = \beta = \widehat{\beta}|_{B}.$

- If $\beta \in \mathrm{Con}\, \mathbf{B}$ is a coatom of **Con B** with m congruence classes then the interval $[\beta^*, \widehat{\beta}]$ in **Con A** is $\mathbf{2}^{m-1}$.
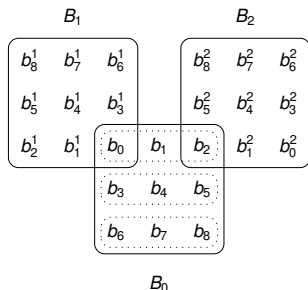
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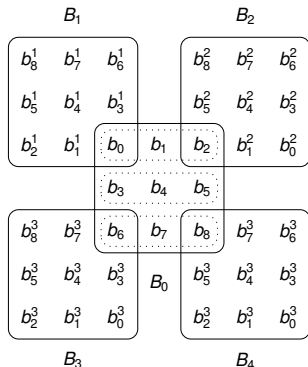

More generally...

- Suppose $\beta \in \mathrm{Con}\,\mathbf{B}$ has transversal $b_{\beta(1)}, \dots, b_{\beta(m)}$.

- Denote by T_r the set of intersection points in the r-th block of β:

$$T_r = \bigcup_{k=1}^{K} B_k \cap b_{\beta(r)}/\beta.$$

THE STRUCTURE OF THE INTERVAL $[\beta^*, \widehat{\beta}] \leqslant$ **Con A**.

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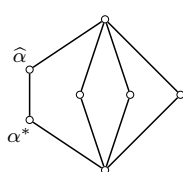$$T_r = \bigcup_{k=1}^{K} B_k \cap b_{\beta(r)}/\beta.$$

Then $\quad [\beta^*, \widehat{\beta}] = \{\theta \in \mathrm{Eq}(A) : \beta^* \subseteq \theta \subseteq \widehat{\beta}\} \cong \prod_{r=1}^{m}(\mathrm{Eq}|\,T_r|)^{m-1}.$
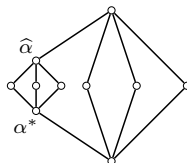
Returning to our original example, the base algebra **B** is the right regular S_3-set, and the nontrivial relations in $\mathrm{Con}\,\mathbf{B}$ are
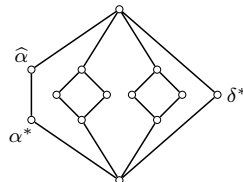
$$\alpha = |0, 1, 2|3, 4, 5| \quad \beta = |0, 3|1, 4|2, 5| \quad \gamma = |0, 4|1, 5|2, 3| \quad \delta = |0, 5|1, 3|2, 4|$$
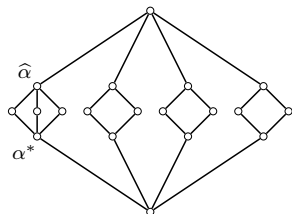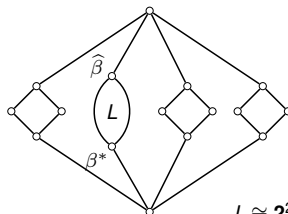

$T = \{0, 1\}$ $T = \{0, 1, 2\}$ $T = \{0, 2, 3\}$

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$L \cong \mathbf{2}^2 \times \mathbf{2}^2$

LIMITATIONS

Two limitations of the foregoing construction:

1. The sizes $|T_r|$ of the partition lattice factors in

$$[\beta^*, \widehat{\beta}] \cong \prod_{r=1}^{m} (\text{Eq}|T_r|)^{m-1}$$

   are limited by the size of the blocks of β.

2. If β is not principal, $[\theta^*, \hat{\theta}]$ may be non-trivial for some $\theta \not\geq \beta$.

THEOREM

Let $\mathbf{B} = \langle B, F \rangle$ be a finite algebra. Suppose

$$\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \ldots, (a_{K-1}, b_{K-1}))$$

has m blocks and fix $N < \infty$.

There exists an overalgebra $\langle A, F_A \rangle$ such that the interval $\beta|_B^{-1} \leqslant \mathrm{Con}\,\mathbf{A}$ is

$$[\beta^*, \widehat{\beta}] \cong (\mathrm{Eq}(N))^{m-1}.$$

Moreover, we can arrange it so that $\theta^* = \widehat{\theta}$ for all $\theta \not\geqslant \beta$ in $\mathrm{Con}\,\mathbf{A}$.

A GENERALIZATION

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