

DEDEKIND'S TRANSPOSITION PRINCIPLE

AND

ISOTOPIC ALGEBRAS WITH NONISOMORPHIC CONGRUENCE LATTICES

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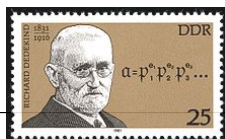
University of South Carolina

AMS Spring Western Sectional Meeting
University of Colorado, Boulder, CO

April 13-14, 2013

These slides and other resources are available at

<http://williamdemeo.wordpress.com>



Notes

DEDEKIND'S TRANSPOSITION PRINCIPLE

FOR MODULAR LATTICES

Notation

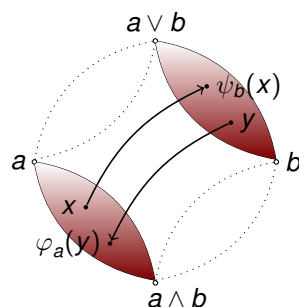
Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice with $a \in L$.

Let φ_a and ψ_a be the *perspectivity maps*

$$\varphi_a(x) = x \wedge a \quad \text{and} \quad \psi_a(x) = x \vee a$$

For $x, y \in L$, let

$$\llbracket x, y \rrbracket_L = \{z \in L \mid x \leq z \leq y\}.$$



THEOREM (DEDEKIND'S TRANSPOSITION PRINCIPLE)

\mathbf{L} is modular iff for all $a, b \in L$ the maps φ_a and ψ_b are inverse lattice isomorphisms of $\llbracket a \wedge b, a \rrbracket$ and $\llbracket b, a \vee b \rrbracket$.

Notes

ANOTHER TRANSPOSITION PRINCIPLE

FOR LATTICES OF EQUIVALENCE RELATIONS

Let X be a set and let $\text{Eq } X$ be the lattice of equivalence relations on X .

If L is a sublattice of $\text{Eq } X$ with $\eta, \theta \in L$, then we define

$$[\eta, \theta]_L = \{\gamma \in L \mid \eta \leq \gamma \leq \theta\}.$$

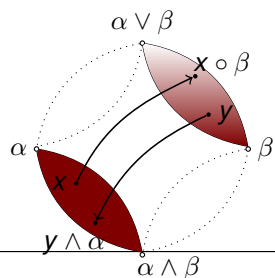
For $\beta \in \text{Eq } X$, let $[\eta, \theta]_L^\beta$ be the set of relations in $[\eta, \theta]_L$ that permute with β ,

$$[\eta, \theta]_L^\beta = \{\gamma \in L \mid \eta \leq \gamma \leq \theta \text{ and } \gamma \circ \beta = \beta \circ \gamma\}.$$

LEMMA

Suppose α and β are permuting relations in $L \leq \text{Eq } X$.

Then $[\beta, \alpha \vee \beta]_L \cong [\alpha \wedge \beta, \alpha]_L^\beta \leq [\alpha \wedge \beta, \alpha]_L$.



Notes

DEDEKIND'S RULE

The proof requires the following version of *Dedekind's Rule*:

LEMMA

Suppose $\alpha, \beta, \gamma \in L \leq \text{Eq } X$ and $\alpha \leq \beta$.

Then the following identities of subsets of X^2 hold:

$$\alpha \circ (\beta \cap \gamma) = \beta \cap (\alpha \circ \gamma)$$

$$(\beta \cap \gamma) \circ \alpha = \beta \cap (\gamma \circ \alpha)$$

Notes

ISOTOPY

BASIC DEFINITIONS

Let \mathbf{A} , \mathbf{B} , \mathbf{C} be algebras of the same type.

\mathbf{A} and \mathbf{B} are *isotopic over* \mathbf{C} , denoted $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$, if there is an isomorphism

$$\varphi : \mathbf{A} \times \mathbf{C} \xrightarrow{\cong} \mathbf{B} \times \mathbf{C} \quad \text{that leaves the second coordinate fixed}$$

i.e. $(\forall a \in A) (\forall c \in C) \quad \varphi(a, c) = (\varphi_1(a, c), c)$

We say that \mathbf{A} and \mathbf{B} are *isotopic*, denoted $\mathbf{A} \sim \mathbf{B}$, if $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ for some \mathbf{C} .

It is easy to verify that \sim is an equivalence relation. $\mathbf{A} \sim_{\mathbf{C}}^{\text{mod}} \mathbf{B}$ and say that \mathbf{A} and \mathbf{B} are *modular isotopic over* \mathbf{C} .

We call \mathbf{A} and \mathbf{B} *modular isotopic in one step*, denoted $\mathbf{A} \sim_1^{\text{mod}} \mathbf{B}$, if they are modular isotopic over some \mathbf{C} .

We call \mathbf{A} and \mathbf{B} are *modular isotopic*, denoted $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$, if (\mathbf{A}, \mathbf{B}) is in the transitive closure of \sim_1^{mod} .

Notes

ISOTOPY

MODULAR CASE

Lemma 11. If $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$ then $\text{Con } \mathbf{A} \cong \text{Con } \mathbf{B}$.

The proof is a nice/easy application of Dedekind's Transposition Principle.

Could we use the same strategy with the non-modular version of the transposition principle to show that $\mathbf{A} \sim \mathbf{B}$ implies $\text{Con } \mathbf{A} \cong \text{Con } \mathbf{B}$?

As you have guessed, the answer is no!

The perspectivity map that is so useful when $\text{Con}(\mathbf{A} \times \mathbf{C})$ is modular can fail *miserably* in the non-modular case... *even when* $\mathbf{A} \cong \mathbf{B}$!

But this only shows that the same argument doesn't work...

Notes

COUNTEREXAMPLES

We describe a class of examples in which $\mathbf{A} \sim \mathbf{B}$ and $\text{Con } \mathbf{A} \not\cong \text{Con } \mathbf{B}$.

The examples show that congruence lattices of isotopic algebras can differ arbitrarily in size.

For any group G , let $\text{Sub}(G)$ denote the lattice of subgroups of G .

A group G is called a *Dedekind group* if every subgroup of G is normal.

Let S be any group and let D denote the *diagonal subgroup* of $S \times S$,

$$D = \{(x, x) \mid x \in S\}$$

The interval $[[D, S \times S] \leq \text{Sub}(S \times S)$ is described by the following

LEMMA

The filter above the diagonal subgroup of $S \times S$ is isomorphic to the lattice of normal subgroups of S .

Notes

THE EXAMPLE

Let S be a group, and let $G = S_1 \times S_2$, where $S_1 \cong S_2 \cong S$.

Let

$$D = \{(x_1, x_2) \in G \mid x_1 = x_2\}, \quad T_1 = S_1 \times \langle 1 \rangle, \quad T_2 = \langle 1 \rangle \times S_2.$$

Then $D \cong T_1 \cong T_2$, and these are pair-wise compliments:

$$\langle T_1, T_2 \rangle = \langle T_1, D \rangle = \langle D, T_2 \rangle = G$$

$$T_1 \cap D = D \cap T_2 = T_1 \cap T_2 = \langle (1, 1) \rangle$$

Let $\mathbf{A} = \langle G/T_1, G^{\mathbf{A}} \rangle$ be the algebra with universe the left cosets of T_1 in G , and basic operations the left multiplications by elements of G .

For each $g \in G$ the operation $g^{\mathbf{A}} \in G^{\mathbf{A}}$ is defined by

$$g^{\mathbf{A}}(xT_1) = (gx)T_1 \quad (xT_1 \in G/T_1).$$

Define the algebra $\mathbf{C} = \langle G/T_2, G^{\mathbf{C}} \rangle$ similarly.

Notes

THE EXAMPLE

The algebra \mathbf{B} will have universe $B = G/D$, but we define the action of G on B with a twist.

For each $g = (g_1, g_2) \in G$, for each $(x_1, x_2)D \in G/D$, define

$$g^{\mathbf{B}}((x_1, x_2)D) = (g_2x_1, g_1x_2)D.$$

Let $\mathbf{B} = \langle G/D, G^{\mathbf{B}} \rangle$, where $G^{\mathbf{B}} = \{g^{\mathbf{B}} \mid g \in G\}$.

Consider the binary relation $\varphi \subseteq (A \times C) \times (B \times C)$ that associates to each ordered pair

$$((x_1, x_2)T_1, (y_1, y_2)T_2) \in A \times C$$

the pair

$$((x_2, y_1)D, (y_1, y_2)T_2) \in B \times C$$

It is easy to verify that this relation is a function, and in fact

$$\varphi: \mathbf{A} \times \mathbf{C} \rightarrow \mathbf{B} \times \mathbf{C} \text{ is an isomorphism.}$$

Since φ leaves second coordinates fixed, $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$.

Notes

CONCLUSION

Compare $\text{Con } \mathbf{A}$ and $\text{Con } \mathbf{B}$.

$\text{Con } \mathbf{A} \cong \llbracket T_1, G \rrbracket \leq \text{Sub}(G)$, so $\text{Con } \mathbf{A} \cong \text{Sub}(S)$.

$\text{Con } \mathbf{B}$ is isomorphic to the lattice of normal subgroups of S .

$$\text{Con } \mathbf{B} \cong \text{NSub}(S) \leq \text{Sub}(S) \cong \text{Con } \mathbf{A}$$

So, if S is any non-Dedekind group, $\text{Con } \mathbf{B} \not\cong \text{Con } \mathbf{A}$.

If S is a nonabelian simple group, then $\text{Con } \mathbf{B} \cong \mathbf{2}$, while $\text{Con } \mathbf{A} \cong \text{Sub}(S)$ can be arbitrarily large.

Notes
