Constraint Satisfaction Problems, and Universal Algebra

Cliff Bergman and William DeMeo
Iowa State University
williamdemeo@gmail.com

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What is a CSP?

Informally, a **C**onstraint **S**atisfaction **P**roblem consists of

- a list of variables ranging over a finite domain and
- a set of constraints on those variables.

Problem: can we assign values to all the variables so that all of the constraints are satisfied?

Examples

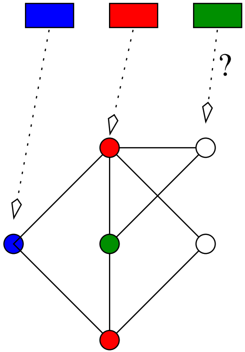
A system of linear equations is a CSP

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Also, a system of nonlinear equations is a CSP

$$
\begin{aligned}
a_{11}x_1^2x_3 + a_{12}x_2x_3x_7 + \cdots + a_{1n}x_4x_n^3 &= b_1 \\
a_{21}x_2x_5 + a_{22}x_2 \qquad\quad + \cdots + a_{2n}x_4^3 \quad &= b_2 \\
&\vdots \\
a_{m1}x_3x_5x_8 + a_{m2}x_2 \qquad + \cdots + a_{mn}x_n \quad &= b_m
\end{aligned}
$$

For a fixed k, determining whether a graph is k-colorable is a CSP



Given a propositional formula $\varphi(x_1, \ldots, x_n)$, determine whether φ is satisfiable

$$\varphi(x, y, z) = (x \vee y \vee z') \wedge (x' \vee y \vee z')$$

then

$$\varphi(0, 0, 1) = 1$$

Algorithms

There is an efficient algorithm (Gaussian elimination) for solving any linear system. That is

There is an algorithm that accepts as input a linear system and decides whether that system has a solution.

The running time of the algorithm is bounded above by $f(s)$ where f is a polynomial and s is the size of the system.

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The particular system is an instance of the problem LINEAR SYSTEM

Similarly

There is an algorithm that accepts as input a graph and decides whether the graph is 2-colorable.

Running time bounded by $f(s)$, a polynomial in size s.

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There is an algorithm that accepts as input a formula, $\varphi = \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k$, each φ_i bijunctive, and decides whether φ is satisfiable.
Running time bounded by $f(k)$, a polynomial in size k.

φ is an instance of the problem 2-SAT.

We say these algorithms run in polynomial time.

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Thus these problems are solvable in nondeterministic polynomial time.

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It is possible for $X \leq_{\mathsf{p}} Y \leq_{\mathsf{p}} X$. In that case, write $X \equiv_{\mathsf{p}} Y$.

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- $\mathbb{P} \subseteq \mathbb{NP}$
- Both $\mathbb{P}$ and $\mathbb{NP}$ are downsets, i.e.,
  $Y \in \mathbb{P} \;\&\; X \leq_{\mathsf{p}} Y \implies X \in \mathbb{P}$

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The maximal members of $\mathbb{NP}$ are called $\mathbb{NP}$-complete.

3-COLORABILITY, NONLINEAR SYSTEM, and 3-SAT are known to be $\mathbb{NP}$-complete.

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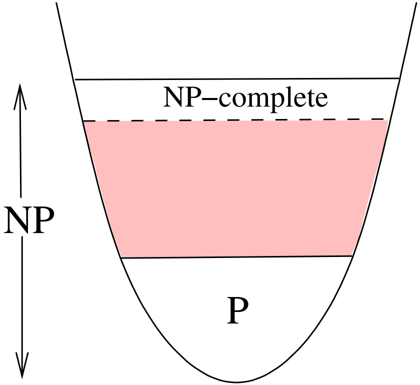
Theorem (Ladner, 1975)

If $\mathbb{P} \neq \mathbb{NP}$ then there are problems in $\mathbb{NP} - \mathbb{P}$ that are not $\mathbb{NP}$-complete.



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Formal Definition of CSP

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An n-ary relation on D is a subset of D^n

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$\mathrm{Rel}_n(D)$ denotes the set of all n-ary relations on D

$$\mathrm{Rel}(D) = \bigcup_{n>0} \mathrm{Rel}_n(D)$$

Let D be a finite set and $\Delta \subseteq \mathrm{Rel}(D)$

$\mathrm{CSP}(\langle D, \Delta \rangle)$ is the following decision problem:

Instance. A finite set $V = \{ v_1, \dots, v_n \}$ of variables and a finite set $\{ C_1, \dots, C_m \}$ of constraints;

each constraint C_i is a pair $(\langle x_{i1}, \dots, x_{i\,p_i} \rangle , \delta_i)$ in which $x_{i1}, \dots, x_{i\,p_i} \in V$ and $\delta_i \in \Delta$

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Question. Does there exist a solution, that is, a “context” $\rho \colon V \to D$, such that for all $i \leq m$, $\langle \rho(x_{i1}), \ldots, \rho(x_{i\,p}) \rangle \in \delta_i$?

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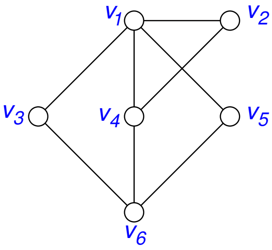
$\mathrm{CSP}(\langle D, \Delta \rangle)$ always lies in $\mathbb{NP}$.

Example: 3-colorability

$D = \{r, g, b\}, \qquad \Delta = \{\kappa_3\}$
$\kappa_3 = \{\, (x, y) \in D : x \neq y \,\}$

Then $\mathrm{CSP}(\langle D, \Delta \rangle)$ is the 3-colorability problem



$$V = \{v_1, \ldots, v_6\}$$
$$\langle v_1, v_2 \rangle \in \kappa$$
$$\langle v_1, v_3 \rangle \in \kappa$$
$$\langle v_1, v_4 \rangle \in \kappa$$
$$\langle v_2, v_4 \rangle \in \kappa$$
$$\vdots$$
$$\langle v_5, v_6 \rangle \in \kappa$$

Two Motivating Questions

1. Dichotomy Conjecture
   Every $\mathrm{CSP}(\langle D, \Delta \rangle)$ either lies in $\mathbb{P}$ or is $\mathbb{NP}$-complete.

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   Characterize those CSPs that lie in $\mathbb{P}$.

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1. Dichotomy Conjecture
   Every $\mathrm{CSP}(\langle D, \Delta \rangle)$ either lies in $\mathbb{P}$ or is $\mathbb{NP}$-complete.
2. Tractability Problem
   Characterize those CSPs that lie in $\mathbb{P}$.

What would a characterization look like? What language could we use?

Polymorphisms

Definition

Let $\delta \in \mathrm{Rel}_k(D)$ and $f\colon D^n \to D$. We say f preserves δ if

$$(a_{11}, \ldots, a_{1k}), \ldots, (a_{n1}, \ldots, a_{nk}) \in \delta \implies \\ \big(f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1k}, \ldots, a_{nk})\big) \in \delta$$

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f is an n-ary operation on D.

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a_{21} & a_{22} & \ldots & a_{2k} & \in & \delta \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nk} & \in & \delta \\
\downarrow f & \downarrow f & & \downarrow f & & \\
\star & \star & \ldots & \star & \in & \delta
\end{array}$$

Definition

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Important point: $\langle D, \mathrm{Pol}(\Delta)\rangle$ is an algebraic structure

Theorem

Let $\Gamma, \Delta \subseteq \mathrm{Rel}(D)$. *Then*

$$\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Delta) \implies \mathrm{CSP}(\Delta) \leq_{\mathsf{p}} \mathrm{CSP}(\Gamma).$$

Theorem

Let $\Gamma, \Delta \subseteq \mathrm{Rel}(D)$. *Then*

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Thus, the richer the algebraic structure, the easier the corresponding CSP

One can go back and forth between relational and algebraic structures

Relational		**Algebraic**
$\langle D, \Delta \rangle$	$\longrightarrow$	$\langle D, \mathrm{Pol}(\Delta) \rangle$
$\langle D, \mathrm{Inv}(F) \rangle$	$\longleftarrow$	$\langle D, F \rangle$

$\mathrm{CSP}\langle D, \Delta \rangle \equiv_{\mathsf{p}} \mathrm{CSP}\langle D, \mathrm{Inv}(\mathrm{Pol}(\Delta)) \rangle$

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Perhaps the expressive power of algebra can be used to classify CSPs.

Algebraic Facts

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras

$\mathbf{B}$ a subalgebra of $\mathbf{A} \implies \mathrm{CSP}(\mathbf{B}) \leq_{\mathsf{p}} \mathrm{CSP}(\mathbf{A})$.

$\mathbf{B}$ a homomorphic image of $\mathbf{A} \implies \mathrm{CSP}(\mathbf{B}) \leq_{\mathsf{p}} \mathrm{CSP}(\mathbf{A})$.

$\mathrm{CSP}(\mathbf{A}^n) \equiv_{\mathsf{p}} \mathrm{CSP}(\mathbf{A})$

Theorem (Bulatov, Jeavons, Krokhin, 2000)

If $\langle D, \Delta \rangle$ is a core and every polymorphism is essentially unary, then $\mathrm{CSP}(\Delta)$ is $\mathbb{NP}$-complete.

f is essentially unary if $f(x_1, \ldots, x_n) = g(x_j)$ for some unary g and some $j \leq n$.

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Corollary

3-COLORABILITY, NONLINEAR SYSTEM, and 3-SAT are $\mathbb{NP}$-complete.

Informal reformulation of the dichotomy conjecture

If $\mathbf{A}$ has some kind of decent algebraic structure then $\mathrm{CSP}(\mathbf{A}) \in \mathbb{P}$ otherwise $\mathrm{CSP}(\mathbf{A})$ is $\mathbb{NP}$-complete.