Dedekind's transposition principle
and
isotopic algebras with nonisomorphic congruence lattices

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These slides and other resources are available at

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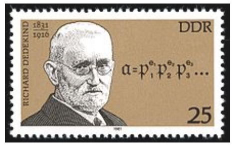


Dedekind's Transposition Principle
for modular lattices

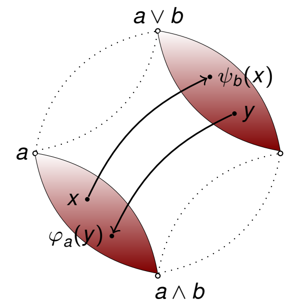
Notation

Let $\mathbf{L} = \langle L, \wedge, \vee\rangle$ be a lattice with $a \in L$.

Let φ_a and ψ_a be the *perspectivity maps*

$$\varphi_a(x) = x \wedge a \quad \text{and} \quad \psi_a(x) = x \vee a$$

For $x, y \in L$, let $[\![x, y]\!]_L = \{z \in L \mid x \leqslant z \leqslant y\}$.



Theorem (Dedekind's Transposition Principle)

$\mathbf{L}$ is modular iff for all $a, b \in L$ the maps φ_a and ψ_b are inverse lattice isomorphisms of $[\![a \wedge b, a]\!]$ and $[\![b, a \vee b]\!]$.

Another transposition principle
for lattices of equivalence relations

Let X be a set and let $\operatorname{Eq} X$ be the lattice of equivalence relations on X.

If L is a sublattice of $\operatorname{Eq} X$ with $\eta, \theta \in L$, then we define

$$[\![\eta, \theta]\!]_L = \{\gamma \in L \mid \eta \leqslant \gamma \leqslant \theta\}.$$

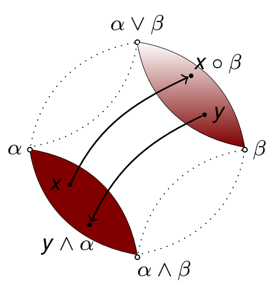
For $\beta \in \operatorname{Eq} X$, let $[\![\eta, \theta]\!]_L^\beta$ be the set of relations in $[\![\eta, \theta]\!]_L$ that permute with β,

$$[\![\eta, \theta]\!]_L^\beta = \{\gamma \in L \mid \eta \leqslant \gamma \leqslant \theta \text{ and } \gamma \circ \beta = \beta \circ \gamma\}.$$

Lemma

Suppose α and β are permuting relations in $L \leqslant \operatorname{Eq} X$.

Then $[\![\beta, \alpha \vee \beta]\!]_L \cong [\![\alpha \wedge \beta, \alpha]\!]_L^\beta \leqslant [\![\alpha \wedge \beta, \alpha]\!]_L$.



Dedekind's Rule

The proof requires the following version of *Dedekind's Rule:*

Lemma

Suppose $\alpha, \beta, \gamma \in L \leqslant \operatorname{Eq} X$ and $\alpha \leqslant \beta$.

Then the following identities of subsets of X^2 hold:

$$\alpha \circ (\beta \cap \gamma) = \beta \cap (\alpha \circ \gamma)$$

$$(\beta \cap \gamma) \circ \alpha = \beta \cap (\gamma \circ \alpha)$$

Isotopy

Basic definitions

Let **A**, **B**, **C** be algebras of the same type.

A and **B** are *isotopic over* **C**, denoted $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$, if there is an isomorphism

$$\varphi : \mathbf{A} \times \mathbf{C} \xrightarrow{\cong} \mathbf{B} \times \mathbf{C} \quad \text{that leaves the second coordinate fixed}$$

$$\text{i.e. } (\forall a \in A)\,(\forall c \in C) \quad \varphi(a, c) = (\varphi_1(a, c), c)$$

We say that **A** and **B** are *isotopic*, denoted $\mathbf{A} \sim \mathbf{B}$, if $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ for some **C**.

It is easy to verify that $\sim$ is an equivalence relation. $\mathbf{A} \sim_{\mathbf{C}}^{\text{mod}} \mathbf{B}$ and say that **A** and **B** are *modular isotopic over* **C**.

We call **A** and **B** *modular isotopic in one step*, denoted $\mathbf{A} \sim_1^{\text{mod}} \mathbf{B}$, if they are modular isotopic over some **C**.

We call **A** and **B** are *modular isotopic*, denoted $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$, if $(\mathbf{A}, \mathbf{B})$ is in the transitive closure of $\sim_1^{\text{mod}}$.

Isotopy

Modular case

Lemma 11. If $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$ then $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$.

The proof is a nice/easy application of Dedekind's Transposition Principle.

Could we use the same strategy with the non-modular version of the transposition principle to show that $\mathbf{A} \sim \mathbf{B}$ implies $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$?

As you have guessed, the answer is no!

The perspectivity map that is so useful when $\operatorname{Con}(\mathbf{A} \times \mathbf{C})$ is modular can fail *miserably* in the non-modular case... *even when* $\mathbf{A} \cong \mathbf{B}$!

But this only shows that the same argument doesn't work...

Counterexamples

We describe a class of examples in which $\mathbf{A} \sim \mathbf{B}$ and $\operatorname{Con} \mathbf{A} \ncong \operatorname{Con} \mathbf{B}$.

The examples show that congruence lattices of isotopic algebras can differ arbitrarily in size.

For any group G, let $\operatorname{Sub}(G)$ denote the lattice of subgroups of G.

A group G is called a *Dedekind group* if every subgroup of G is normal.

Let S be any group and let D denote the *diagonal subgroup* of $S \times S$,

$$D = \{(x, x) \mid x \in S\}$$

The interval $[\![D, S \times S]\!] \leqslant \operatorname{Sub}(S \times S)$ is described by the following

Lemma

The filter above the diagonal subgroup of $S \times S$ is isomorphic to the lattice of normal subgroups of S.

The example

Let S be a group, and let $G = S_1 \times S_2$, where $S_1 \cong S_2 \cong S$.

Let $D = \{(x_1, x_2) \in G \mid x_1 = x_2\}$, $\quad T_1 = S_1 \times \langle 1 \rangle$, $\quad T_2 = \langle 1 \rangle \times S_2$.

Then $D \cong T_1 \cong T_2$, and these are pair-wise compliments:

$$\langle T_1, T_2 \rangle = \langle T_1, D \rangle = \langle D, T_2 \rangle = G$$

$$T_1 \cap D = D \cap T_2 = T_1 \cap T_2 = \langle (1, 1) \rangle$$

Let $\mathbf{A} = \langle G/T_1, G^{\mathbf{A}} \rangle =$ the algebra with universe the left cosets of T_1 in G, and basic operations the left multiplications by elements of G.

For each $g \in G$ the operation $g^{\mathbf{A}} \in G^{\mathbf{A}}$ is defined by

$$g^{\mathbf{A}}(xT_1) = (gx)T_1 \qquad (xT_1 \in G/T_1).$$

Define the algebra $\mathbf{C} = \langle G/T_2, G^{\mathbf{C}} \rangle$ similarly.

The example

The algebra **B** will have universe $B = G/D$, but we define the action of G on B with a twist.

For each $g = (g_1, g_2) \in G$, for each $(x_1, x_2)D \in G/D$, define

$$g^{\mathbf{B}}((x_1, x_2)D) = (g_2 x_1, g_1 x_2)D.$$

Let $\mathbf{B} = \langle G/D, G^{\mathbf{B}} \rangle$, where $G^{\mathbf{B}} = \{g^{\mathbf{B}} \mid g \in G\}$.

Consider the binary relation $\varphi \subseteq (A \times C) \times (B \times C)$ that associates to each ordered pair

$$((x_1, x_2)T_1, (y_1, y_2)T_2) \in A \times C$$

the pair

$$((x_2, y_1)D, (y_1, y_2)T_2) \in B \times C$$

It is easy to verify that this relation is a function, and in fact

$$\varphi \colon \mathbf{A} \times \mathbf{C} \to \mathbf{B} \times \mathbf{C} \text{ is an isomorphism.}$$

Since φ leaves second coordinates fixed, $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$.

Conclusion

Compare Con **A** and Con **B**.

Con $\mathbf{A} \cong [\![T_1, G]\!] \leqslant \mathrm{Sub}(G)$, so Con $\mathbf{A} \cong \mathrm{Sub}(S)$.

Con **B** is isomorphic to the lattice of normal subgroups of S.

$$\mathrm{Con}\,\mathbf{B} \cong \mathrm{NSub}(S) \leqslant \mathrm{Sub}(S) \cong \mathrm{Con}\,\mathbf{A}$$

So, if S is any non-Dedekind group, Con $\mathbf{B} \ncong$ Con $\mathbf{A}$.

If S is a nonabelian simple group, then Con $\mathbf{B} \cong \mathbf{2}$, while Con $\mathbf{A} \cong \mathrm{Sub}(S)$ can be arbitrarily large.