The finite lattice representation problem and intervals in subgroup lattices
Part I

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joint work with

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Algebra & Logic Seminar

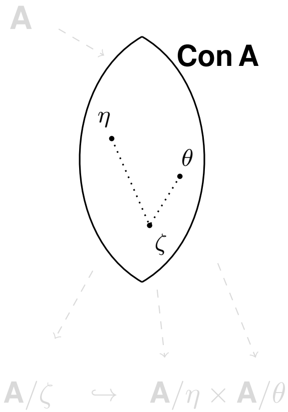
September 21, 2012

These slides and other resources are available at

http://www.math.sc.edu/~demeow/FLRP.html $\longrightarrow$

Congruence Decompositions

Congruence relations reveal structural characteristics.



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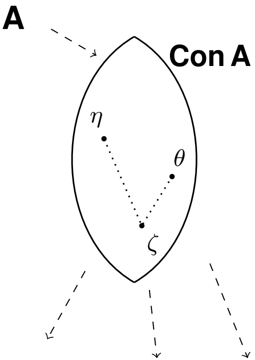

$$\mathbf{A}/\zeta \quad \hookrightarrow \quad \mathbf{A}/\eta \times \mathbf{A}/\theta$$

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THE FINITE LATTICE REPRESENTATION PROBLEM

There is essentially no restriction on the shape of a congruence lattice of an arbitrary algebra.

THEOREM (GRÄTZER-SCHMIDT, 1963)

Every algebraic lattice is isomorphic to the congruence lattice of an algebra.

What if the algebra is finite?

Problem: Given a finite lattice $\mathbf{L}$, does there exist a *finite* algebra $\mathbf{A}$ such that $\mathbf{Con}\,\mathbf{A} \cong \mathbf{L}$?

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We call a finite lattice ***representable*** if it is isomorphic to the congruence lattice of a finite algebra.

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Some important classes of finite lattices

- $\mathscr{L}_0 =$ all finite lattices
- $\mathscr{L}_1 =$ lattices isomorphic to sublattices of finite partition lattices
- $\mathscr{L}_2 =$ ...strong congruence lattices of finite partial algebras
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  This does **not** say $\mathscr{L}_3 = \mathscr{L}_4$. It's possible that $\mathscr{L}_0 \supsetneqq \mathscr{L}_3 \supsetneqq \mathscr{L}_4$.

Recap

Theorem (Pudlák and Tůma, 1980)

Every finite lattice can be embedded in $\mathrm{Eq}(X)$ *with* X *finite.*

In other words, $\mathscr{L}_0 = \mathscr{L}_1$.

Theorem (Pálfy and Pudlák, 1980)

The following statements are equivalent:

(i) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*

(ii) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

In other words, $\mathscr{L}_0 = \mathscr{L}_3$ if and only if $\mathscr{L}_0 = \mathscr{L}_4$.

How to find a representation of a finite lattice

Method 1 (use closure properties)

The class $\mathscr{L}_3$ is closed under the following operations:

- lattice duals (Kurzweil and Netter, 1986)
- interval sublattices (follows from Kurzweil-Netter)
- direct products (Tůma, 1986)
- ordinal sums (McKenzie, 1984; Snow, 2000)
- parallel sums (Snow, 2000)
- certain sublattices of lattices in $\mathscr{L}_3$ (Snow, 2000)
  (namely, those obtained as a union of a filter and ideal)

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Method 2 (use a Galois correspondence)

- Fix $\theta \subseteq X \times X$, $\quad f : X^n \to X$.

  Say that f ***respects*** θ and write $f(\theta) \subseteq \theta$ provided

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- For $L \subseteq \mathrm{Eq}(X)$ define

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- If a lattice $L \leqslant \mathrm{Eq}(X)$ is *closed*, i.e. $\rho\lambda(L) = L$, then

  $$L = \mathrm{Con}\,\langle X, \lambda(L)\rangle$$

Method 3 (subgroup lattice interval)

Find L as an interval in a subgroup lattice of a finite group.

If $H \leqslant G$ are finite groups, then the interval above H in $Sub(G)$,

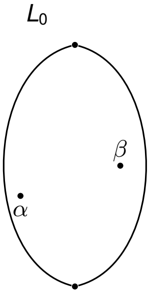
$$[H, G] := \{K \mid H \leqslant K \leqslant G\},$$

is isomorphic to $\mathrm{Con}\,\langle G/H, G\rangle$.

METHOD 4 (FILTER+IDEAL)

Find L as the union of a filter and ideal in a representable lattice.

Suppose $L_0 \cong \mathrm{Con}\,\langle A, F\rangle$, $\alpha, \beta \in L_0 \setminus \{0, 1\}$.



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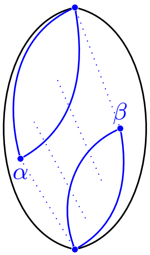

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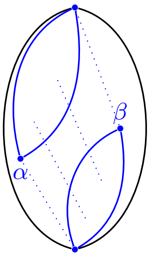
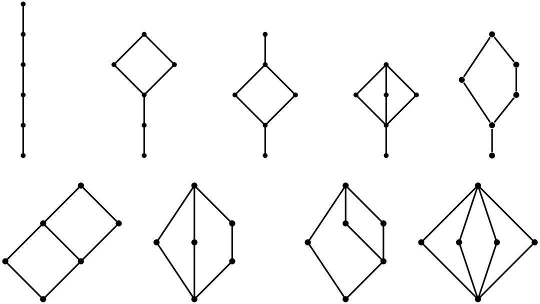
Then there exists a set $F' \subset A^A$ such that

$$L \cong \mathrm{Con}\,\langle A, F \cup F'\rangle.$$

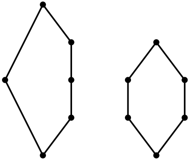
$L \leqslant L_0$



Lattices with at most 6 elements are representable.



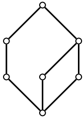
Watatani (1996) *J. Funct. Anal.*



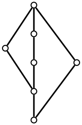
Aschbacher (2008) *JAMS*

Theorem: *Every lattice with at most 6 elements is an interval in the subgroup lattice of a finite group.*

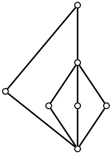
ARE ALL LATTICES WITH AT MOST 7 ELEMENTS REPRESENTABLE?



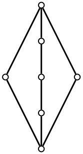
L_{19}



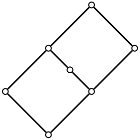
L_{20}



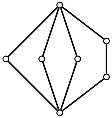
L_{17}



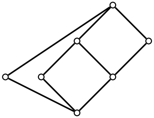
L_{13}



L_{11}



L_9



L_7

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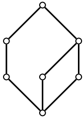

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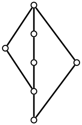
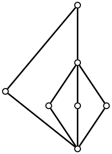
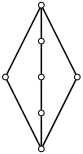


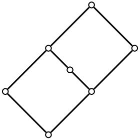
L_{20} ✓



L_{17}



L_{13}



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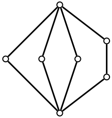

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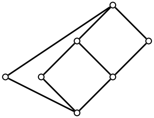

L_7

Finding representations…

…as intervals in subgroup lattices

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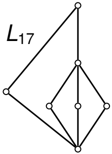

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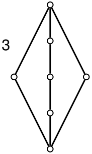
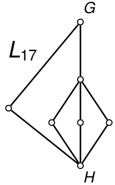


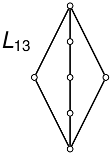
FINDING REPRESENTATIONS…

…AS INTERVALS IN SUBGROUP LATTICES



`SmallGroup(288,1025)`

$|G : H| = 48$



- The group $G = (A_4 \times A_4) \rtimes C_2$ has a subgroup $H \cong S_3$ such that $[H, G] \cong L_{17}$.
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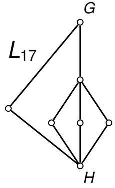

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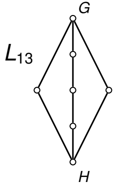


SmallGroup(960,11358)

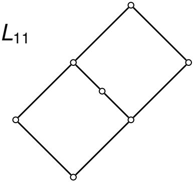
$|G : H| = 80$

- The group $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ has a subgroup $H \cong A_4$ such that $[H, G] \cong L_{13}$.

Finding representations…

…using subgroup lattice intervals and the filter+ideal lemma.

L_{11}



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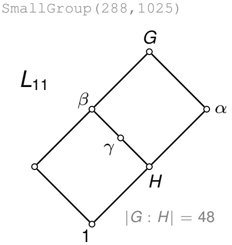

- Let $G = (A_4 \times A_4) \rtimes C_2$.
- G has a subgroup $H \cong C_6$ with $[H, G] \cong N_5$.
- Let $[H, G] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.

Finding representations...

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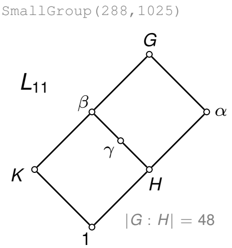

- Let $G = (A_4 \times A_4) \rtimes C_2$.
- G has a subgroup $H \cong C_6$ with $[H, G] \cong N_5$.
- Let $[H, G] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.

- $\mathrm{Sub}(G)$ is a congruence lattice, so if there exists a subgroup $K \succ 1$, below β and not below γ, then

$$L_{11} \cong K^{\downarrow} \cup H^{\uparrow}.$$

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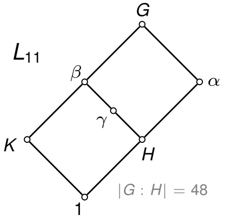
SmallGroup(288,1025)



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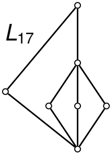
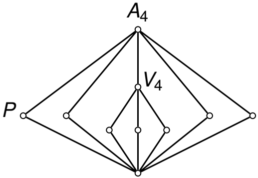




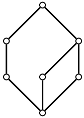
- $\mathrm{Sub}(A_4)$ is a congruence lattice (of A_4 acting regularly on itself).

- Therefore,

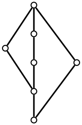
$$L_{17} \cong V_4^{\downarrow} \cup P^{\uparrow}$$

is a congruence lattice.

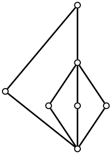
Are all lattices with at most 7 elements representable?



L_{19} ✓



L_{20} ✓



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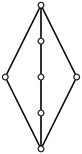
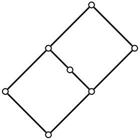
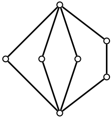


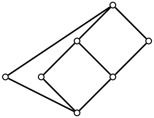
L_{13} ✓



L_{11} ✓

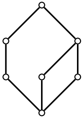


L_9



L_7

Are all lattices with at most 7 elements representable?



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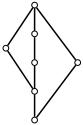

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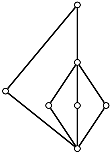

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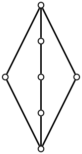

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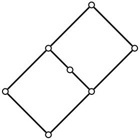

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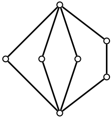

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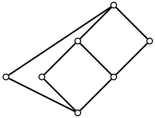
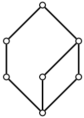


L_7

Seven element lattices: summary



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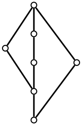

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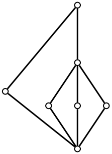

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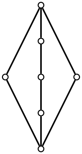

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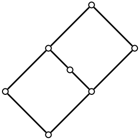

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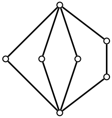

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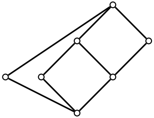

L_7

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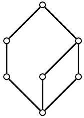

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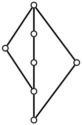

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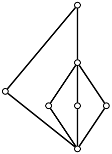

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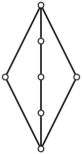

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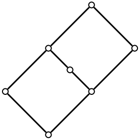

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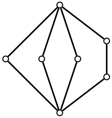

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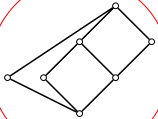

L_7

HAS ANYONE SEEN THIS LATTICE?

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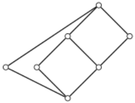
Given a lattice L with n elements, are there finite groups H < G such that L ≅ the lattice of subgroups between H and G?

13

6

If there is no restriction on n, this is a famous open problem. I'm wondering if any recent work has been done for small $n > 6$. I believe the question is answered (positively) for $n = 6$ by Watatani (1996) MR1409040 and Aschbacher (2008) MR2393428. I also believe we can answer it for $n = 7$, with one possible exception. The exceptional case is shown below.



So my two questions are these:

1) Does anyone know of recent work on this special case of the problem (specifically for $n = 7$ or $n = 8$)?

2) Has anyone found a finite group G with a subgroup H such that the interval

$$[H, G] = \{K : H \leq K \leq G\}$$

is the lattice shown above?

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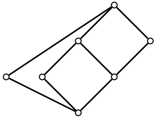
Interval sublattice enforceable properties

- L_7 cannot be obtained using the overalgebra construction.



L_7

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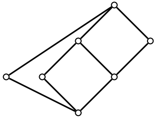
- A minimal representation of L_7 must come from a transitive G-set.



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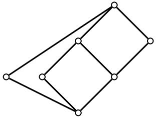

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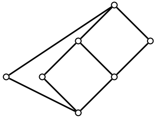

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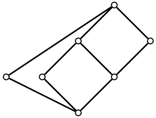

L_7

THEOREM

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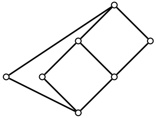

L_7

Theorem

Suppose $H < G$, $\mathrm{core}_G(H) = 1$, $L_7 \cong [H, G]$.

(I) *G is a primitive permutation group.*
(II) *If $N \lhd G$, then $C_G(N) = 1$.*
(III) *G contains no non-trivial abelian normal subgroup.*
(IV) *G is not solvable.*
(V) *G is subdirectly irreducible.*
(VI) *With the possible exception of at most one maximal subgroup, all proper subgroups in the interval $[H, G]$ are core-free.*

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- By UH we mean the *set* $\{uh \mid u \in U, h \in H\}$.

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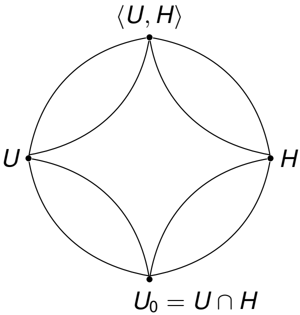

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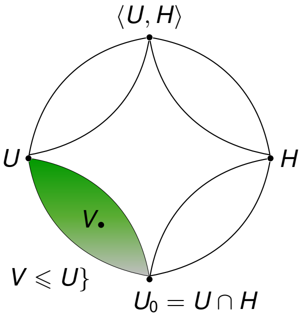
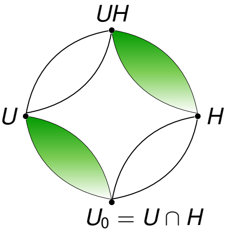


$[U_0, U] := \{V \mid U_0 \leqslant V \leqslant U\}$

Interval Isomorphisms



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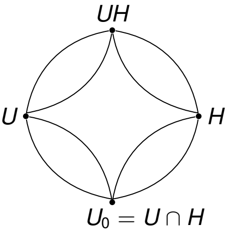

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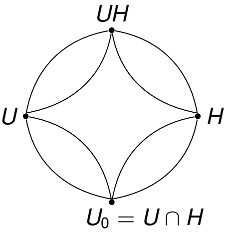

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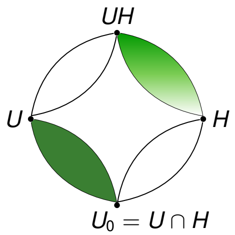

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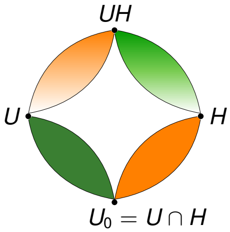

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$$G = \langle a, b, c \mid a^2, b^3, c^3, [b, a], [c, b], c^{-1}a^{-1}a^c \rangle$$

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- The subgroups

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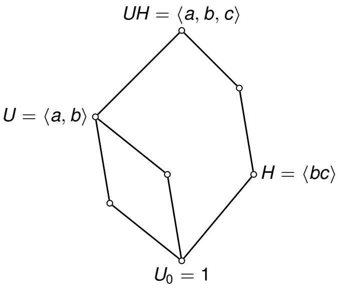

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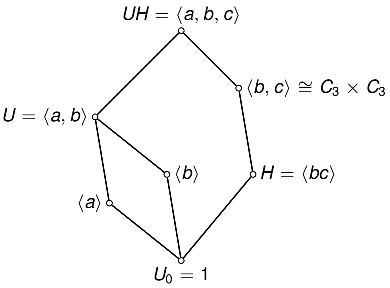

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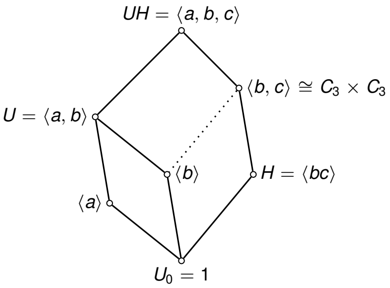

- Three of the four subgroups of U permute with H.

  As the lemma predicts, $U \cap \langle b, c \rangle = \langle b \rangle$.

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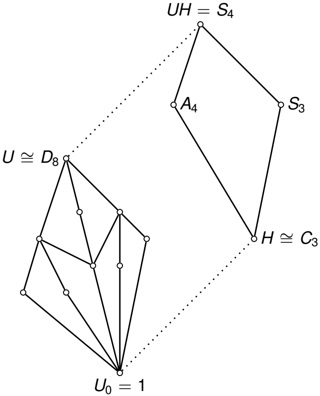

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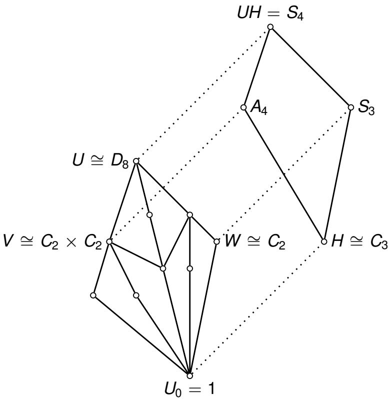

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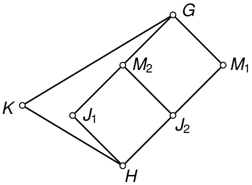
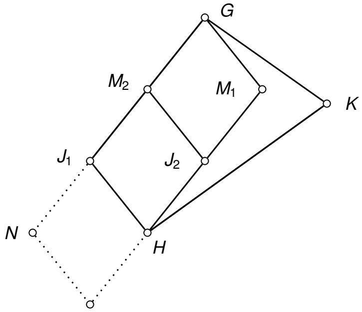


THEOREM

Suppose $H < G$, $\mathrm{core}_G(H) = 1$, and $L_7 \cong [H, G]$. Then

(I) *G is a primitive permutation group.*
(II) *If $N \lhd G$, then $C_G(N) = 1$.*
(III) *G contains no non-trivial abelian normal subgroup.*
(IV) *G is not solvable.*
(V) *G is subdirectly irreducible.*
(VI) *With the possible exception of at most one maximal subgroup, M_1 or M_2, all proper subgroups in the interval $[H, G]$ are core-free.*

Idea of the proof



Claim: J_1 and J_2 are core-free subgroups of G.

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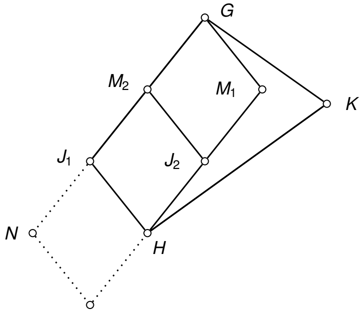

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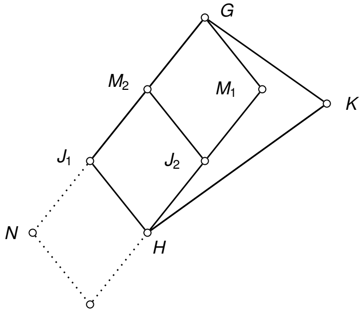

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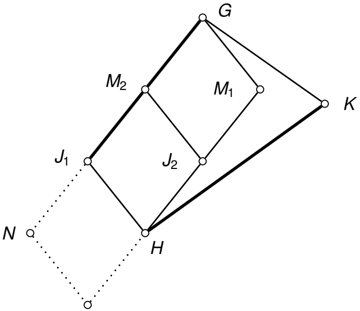

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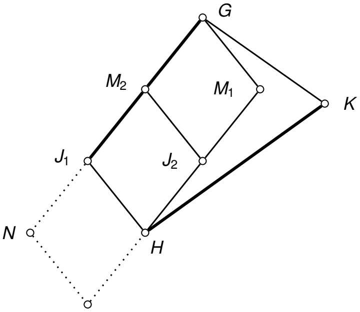

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Impossible!


JUL 9 2003

Aschbacher-O'Nan-Scott Theorem

Let G be a primitive permutation group of degree d, and let $N := \mathrm{Soc}(G) \cong T^m$ with $m \geqslant 1$. Then one of the following holds.

1. N is regular and
   - ***(Affine type)*** T is cyclic of order p, so $|N| = p^m$. Then $d = p^m$ and G is permutation isomorphic to a subgroup of the affine general linear group $\mathrm{AGL}(m, p)$.
   - ***(Twisted wreath product type)*** $m \geqslant 6$, the group T is nonabelian and G is a group of *twisted wreath product type*, with $d = |T|^m$.
2. N is non-regular, non-abelian, and
   - ***(Almost simple type)*** $m = 1$ and $T \leqslant G \leqslant \mathrm{Aut}(T)$.
   - ***(Product action type)*** $m \geqslant 2$ and G is permutation isomorphic to a subgroup of the product action wreath product $P \wr S_{m/l}$ of degree $d = nm/l$. The group P is primitive of type 2.(a) or 2.(c), P has degree n and $\mathrm{Soc}(P) \cong T^l$, where $l \geqslant 1$ divides m.
   - ***(Diagonal type)*** $m \geqslant 2$ and $T^m \leqslant G \leqslant T^m.(\mathrm{Out}(T) \times S_m)$, with the diagonal action. The degree $d = |T|^{m-1}$.

Aschbacher-O'Nan-Scott Theorem

For some interesting history, see Peter Cameron's blog at

`http://cameroncounts.wordpress.com/tag/onan-scott-theorem/`

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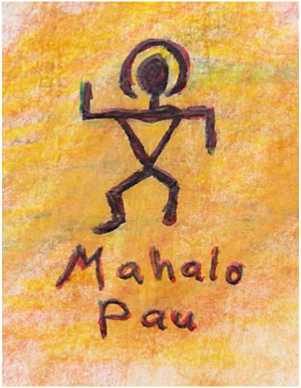
- Future work: Explore "interval enforceable properties of finite groups" and try to restrict to almost simple groups. Then solve the problem using the CFSG Theorem.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