

CSPs OF FINITE COMMUTATIVE IDEMPOTENT BINARS

William DeMeo

`williamdemeo@gmail.com`

Iowa State University

joint work with

Cliff Bergman

Jiali Li

Shanks Workshop

Vanderbilt University

May 30, 2015

slides available at

<https://github.com/williamdemeo/Talks>

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra \mathbf{A} ...

$\text{CSP}(\mathbf{A})$ is tractable $\iff \mathbf{A}$ has a weak-nu term operation

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The left-to-right direction is known.

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The right-to-left direction is open.

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A **weak near unanimity** (weak-nu) term operation is one that satisfies

$$t(x, x, \dots, x) \approx x \quad (\text{idempotent})$$

$$t(y, x, \dots, x) \approx t(x, y, \dots, x) \approx \dots \approx t(x, x, \dots, y)$$

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

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A *binary* operation $t(x, y)$ is weak-nu if

$$t(x, x) \approx x \quad (\text{idempotent})$$

$$t(y, x) \approx t(x, y) \quad (\text{commutative})$$

So let's try to prove (?) for **commutative idempotent binars**.

COMMUTATIVE IDEMPOTENT BINARS

A **CIB** is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.

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Semilattices are tractable.

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Is every finite commutative idempotent binar tractable?

First Example: a semilattice is an associative CIB.

Semilattices are tractable.

Pause to consider more general case for a minute...

GENERAL CASE

SOME WELL KNOWN FACTS

Let \mathbf{A} be a finite idempotent algebra. Let \mathbf{S}_2 be the 2-elt semilattice.

$$\mathbf{V}(\mathbf{A}) \text{ is CP} \iff \mathbf{A} \text{ has Malcev term}$$

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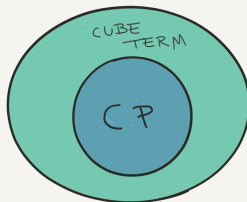


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$$\begin{aligned} \mathbf{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has Malcev term} \\ &\implies \mathbf{A} \text{ has cube term} \end{aligned}$$



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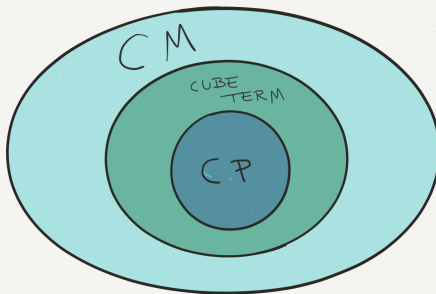
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$\implies V(\mathbf{A})$ is CM



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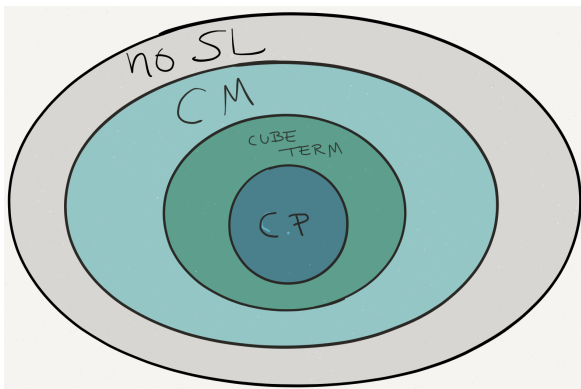
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$\implies \mathbf{S}_2$ is not in $V(\mathbf{A})$



RECENT RESULTS

\mathbf{A} = a finite idempotent algebra

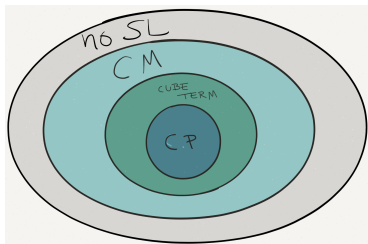
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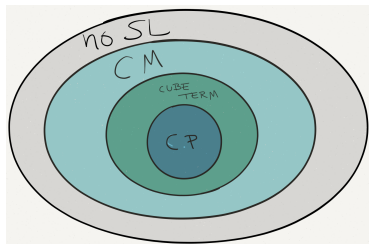
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■ cube term \implies CM

(Berman, Idziak, Marković, McKenzie, Valeriote, Willard 2010)

■ CM \implies \mathbf{S}_2 is not in $V(\mathbf{A})$

Proof: $\mathbf{S}_2 \in V(\mathbf{A}) \Rightarrow \mathbf{S}_2^2 \in V(\mathbf{A})$;

$\text{Con}(\mathbf{S}_2^2)$ is not modular.

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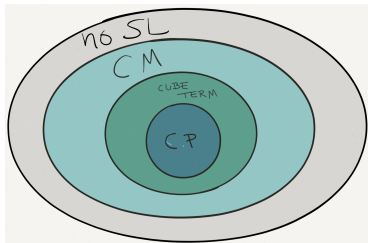
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CIB case

- 1st reduction by cube-term blockers.



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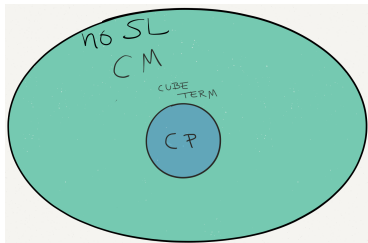
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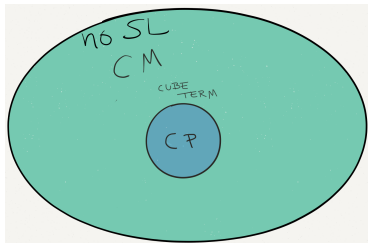
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- 2nd reduction by Kearnes-Tschantz.



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FIRST REDUCTION

BY CUBE-TERM BLOCKERS

Marković, M. Maróti, McKenzie (M^4)

“Finitely related clones and algebras with cube terms” (2012)

A **cube-term blocker** (CTB) is a pair (C, B) of subuniverses satisfying $\emptyset < C < B \leq A$ and for every $t(x_1, \dots, x_n)$ there is an index $i \in [n]$ with

$$(\forall (b_1, \dots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \dots, b_n) \in C).$$

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A finite CIB \mathbf{A} has a CTB if and only if $\mathbf{S}_2 \in \text{HS}(\mathbf{A})$.

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LEMMA

A finite CIB \mathbf{A} has a CTB if and only if $\mathbf{S}_2 \in \text{HS}(\mathbf{A})$.

PROOF.

(C, B) a CTB implies $\theta = C^2 \cup (B - C)^2$ a congruence with $\mathbf{B}/\theta \cong \mathbf{S}_2$.

Conversely, suppose $\mathbf{S}_2 \in \text{HS}(\mathbf{A})$, and \mathbf{B} is a subalgebra of \mathbf{A} with \mathbf{B}/θ a meet-SL for some θ . Let C/θ be the bottom of \mathbf{B}/θ , then (C, B) is a CTB. □

SECOND REDUCTION

Kearnes and Tschantz

“Automorphism groups of squares and of free algebras” (2007)

LEMMA

If V is an idempotent variety that is not congruence permutable, then there are subuniverses U and W of $\mathbf{F} := \mathbf{F}_V\{x, y\}$ satisfying

1. $x \in U \cap W$
2. $y \in U^c \cap W^c$
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For CIB's, either U or W will be an ideal.

This implies a CTB and a semilattice.

REMAINING QUESTIONS FOR FINITE CIBs

CONCLUSION

Let \mathbf{A} be a finite CIB. Then

$\mathbf{S}_2 \notin \mathbf{HS}(\mathbf{A})$ if and only if $\mathbf{V}(\mathbf{A})$ is congruence permutable.

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OPEN QUESTION

Let \mathbf{A} be a finite CIB with S_2 in $HS(\mathbf{A})$. Is $CSP(\mathbf{A})$ tractable?

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OPEN QUESTION

Let \mathbf{A} be a finite CIB with S_2 in $\text{HS}(\mathbf{A})$. Is $\text{CSP}(\mathbf{A})$ tractable?

Recall, if $V(\mathbf{A})$ is SD_\wedge , then $\text{CSP}(\mathbf{A})$ is tractable.

REMAINING QUESTIONS FOR FINITE CIBS

CONCLUSION

Let \mathbf{A} be a finite CIB. Then

$S_2 \notin \text{HS}(\mathbf{A})$ if and only if $V(\mathbf{A})$ is congruence permutable.

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OPEN QUESTION

Let \mathbf{A} be a finite CIB with S_2 in $\text{HS}(\mathbf{A})$. Is $\text{CSP}(\mathbf{A})$ tractable?

Recall, if $V(\mathbf{A})$ is SD_\wedge , then $\text{CSP}(\mathbf{A})$ is tractable.

REVISED QUESTION

Let \mathbf{A} be a finite CIB with S_2 in $\text{HS}(\mathbf{A})$, and $V(\mathbf{A})$ not SD_\wedge .

Is $\text{CSP}(\mathbf{A})$ tractable?

EXAMPLES

\cdot	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

Cliff's idea: replace basic binary operation with a term from $\text{Clo}(\mathbf{A})$, say

$$t(x, y) = (x \cdot (x \cdot y)) \cdot (y \cdot (x \cdot y)).$$

If $\langle A, t \rangle$ tractable, then so is $\mathbf{A} = \langle A, \cdot \rangle$.

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$$\begin{aligned}\{t\} \subseteq \text{Clo}(\mathbf{A}) &\implies \text{Rel}(\text{Clo}(\mathbf{A})) \subseteq \text{Rel}(\{t\}) \\ &\implies \text{CSP}(\mathbf{A}) \leq_P \text{CSP}\langle A, t \rangle\end{aligned}$$

EXAMPLES

\cdot	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

t	0	1	2	3
0	0	0	0	0
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

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$$\langle A, t \rangle \text{ tractable} \implies \mathbf{A} \text{ tractable}$$

EXAMPLES

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Let $t_2(x, y) = x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y))$.

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t_2	0	1	2	3
0	0	0	0	1
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2	0	3	2	1
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$\langle A, t_2 \rangle$ tractable

EXAMPLES

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Let $t_3(x, y) = \dots$?

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0	0	0	2	1
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Let $t_3(x, y) = \dots$?

Let $t_3(x, y, z) = \dots$?

...and about 25 others.

The image shows two overlapping windows. The background window is the UACalculator v1.13 application, which has a menu bar (File, Edit, HSP, Tasks, Maltsev, Idempotent Algs, Equations, Drawing, Help) and a toolbar. The 'Editor' tab is active, showing a form for defining an algebra with fields for Name (CB4-SL-729), Cardinality (4), and Desc (CB with semilattice; V(A) not SD-meet). Below this is a table for operations (g(2)) and a section for the Element Key Table. The foreground window is a web browser displaying the GitHub repository for UACalc/AlgebraFiles. The repository page shows 23 commits, 1 branch, 0 releases, and 2 contributors. A list of commits is visible, including one by 'willanderson' and another by 'Baker'.

UACalculator v1.13 (Feb 28, 2015)

File Edit HSP Tasks Maltsev Idempotent Algs Equations Drawing Help

Editor Algebras Computations Con Sub Drawing

Name: CB4-SL-729 Cardinality: 4 Desc: CB with semilattice; V(A) not SD-meet

Operations: g(2) Del Add Make Into Basic Alg

y	0	1	2	3
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g(3,y)	1	2	1	3

☐ Idempotent Default Element: none

Element Key Table

Index	Elem
0	0
1	1
2	2
3	3

Algebras

Internal	Name	Type	Description
A6	CB4-SL-439	BASIC	CB with semilattice; V(A) not SD-meet
A7	CB4-SL-505	BASIC	CB with semilattice; V(A) not SD-meet
A8	CB4-SL-713	BASIC	CB with semilattice; V(A) not SD-meet
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Msg:

UACalc/AlgebraFiles

GitHub, Inc. [US] <https://github.com/UACalc/AlgebraFiles>

This repository Search Explore Gist Blog Help

UACalc / AlgebraFiles Unwatch

A repository of algebra files for the Universal Algebra Calculator — Edit

23 commits 1 branch 0 releases 2 contributors

branch: master AlgebraFiles / +

minor corrections

willanderson authored 10 days ago latest commit: 926f135

- Baker initial commit 11 months ago
- Bergman minor corrections 10 days ago
- Groups initial commit 11 months ago
- Jipson initial commit 11 months ago

To see them, load UACalc with files from the **Bergman** directory at

<https://github.com/UACalc/AlgebraFiles>

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Thank you for listening!