THE RECTANGULARITY THEOREM OF LIBOR BARTO AND MARCIN KOZIK

WITH APPLICATIONS TO SMALL CIBS

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joint work with

Cliff Bergman and Josh Thompson

Iowa State University

Workshop on Algebras and Algorithms
University of Colorado, Boulder, May 19–22

slides available at

https://github.com/williamdemeo/Talks

Definition of CSP
(naive version)

Input

- *variables:* $\mathcal{V} = \{v_1, v_2, \ldots\}$
- *domain:* $\mathcal{D}$
- *constraints:* $C_1, C_2, \ldots$

Output

- "yes" if there is a *solution*

  $f : \mathcal{V} \to \mathcal{D}$ (an assignment of values to variables that satisfies all C_i)

- "no" otherwise

Definition of CSP

(jaded version)

$\mathbf{A} = \langle A, \mathcal{F}\rangle$ is a finite idempotent algebra, $\mathsf{Sub}(\mathbf{A})$ is all subuniverses of $\mathbf{A}$.

In this talk $\mathrm{CSP}(\mathbf{A})$ denotes the following decision problem:

An *instance of degree* n of $\mathrm{CSP}(\mathbf{A})$ is the tuple $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R}\rangle$

- *variables* $\mathcal{V} = \{0, 1, \dots, n-1\}$;
- *domains* $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1}\} \subset \mathsf{Sub}(\mathbf{A})$ (one for each variable)
- *scope functions* $\mathcal{S} = (\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{p-1})$ with *constraint arities* $\mathrm{ar}(\mathcal{S}) = (m_0, m_1, \dots, m_{p-1})$;
- *constraint relations* $\mathcal{R} = (\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_{p-1})$, where

$$\mathbf{R}_i \leqslant \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}.$$

A *solution* to $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R}\rangle$ is an assignment $f : \mathcal{V} \to A$ of values to variables that satisfies all constraints. That is,

$$f \in \Pi_{\mathcal{V}} A_j \quad \text{and} \quad \mathrm{Proj}_{\mathbf{s}_i} f \in \mathbf{R}_i, \;\; \text{for each } 0 \leqslant i < p.$$

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Notation: $\underline{n} = \{0, 1, \dots, n-1\}$, so the i-th scope has type $\mathbf{s}_i : \underline{m}_i \to \underline{n}$ and

$$\mathrm{Proj}_{\mathbf{s}_i} f = f \circ \mathbf{s}_i$$

Example 1

...thanks, Ross!

Let $\mathbf{A} = \langle \{0, 1\}, \{f\} \rangle$, where

$$f(x, y, z) = x + y + z \pmod 2.$$

Consider the ternary relations

$$R_0 = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$$

$$R_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$$

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So we have a degree 3 instance of $\mathrm{CSP}(\mathbf{A})$, where

- variables: $\mathcal{V} = \{0, 1, 2\}$
- domains: $A_i = \{0, 1\}, \quad i = 0, 1, 2$
- scope functions: the identity on $\{0, 1, 2\}$
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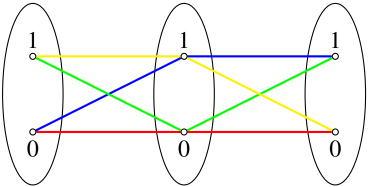
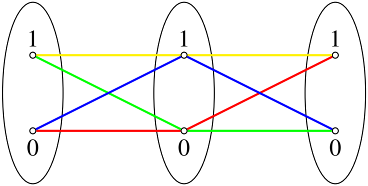
EXAMPLE 1

∩ AND POTATOES



(lines are elements of R_0)



(lines are elements of R_1)

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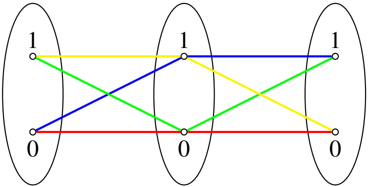

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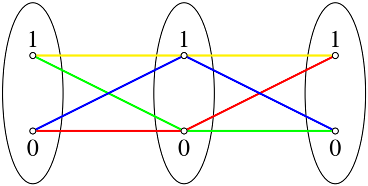

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$$\mathrm{Proj}_{ij} R_0 = \mathrm{Proj}_{ij} R_1$$

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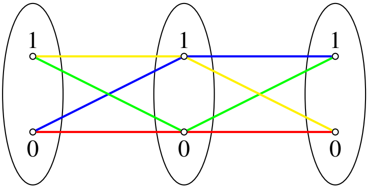

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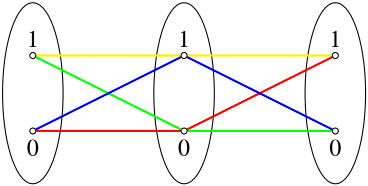

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Notice for all $i, j \in \{0, 1, 2\}$,

$$\mathrm{Proj}_{ij} R_0 = \mathrm{Proj}_{ij} R_1$$

...yet $R_0 \cap R_1 = \emptyset$.

Example 2

...thanks, Cliff!

Let $\mathbf{A} = \langle \{0, 1\}, \{m\} \rangle$, where $m : A^3 \to A$ is a majority operation,

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$

Let $\mathbf{R}_0, \mathbf{R}_1 \leqslant_{\mathrm{sd}} \mathbf{A}^3$ with universes

$$R_0 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0)\},$$
$$R_1 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}.$$

This describes the instance of $CSP(\mathbf{A})$ with

- variables: $\mathcal{V} = \{0, 1, 2\}$
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Some conveniences

Retrict attention to instances where all constraint relation are subdirect,

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Could visualize $(\mathbf{s}_i, \mathbf{R}_i)$ as specifying a subalgebra of the full product $\Pi_{\mathcal{V}} \mathbf{A}_j$

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(thanks again, Ross!)

Convenient because now solutions are the elements in $\bigcap_{i \in V} [\![\mathbf{s}_i, \mathbf{R}_i]\!]$.

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BUT input size is not a function of these “full” subdirect products!

(Input size could be defined as the length of a string of all tuples in scopes and constraint relations of the instance.)

End of Act I

First Intermission

pause...

...draw more potatoes...

...give audience chance to escape.

ABSORPTION THEORY (FOR MORTALS)

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be a finite algebra in a Taylor variety.

Let $t \in \mathrm{Clo}(\mathbf{A})$ be a k-ary term operation.

A subalgebra $\mathbf{B} \leqslant \mathbf{A}$ is *absorbing in* $\mathbf{A}$ *with respect to* t if

$$a \in A,\ b_i \in B \quad \Longrightarrow \quad t^{\mathbf{A}}(b_0, \dots, b_{j-1}, a, b_{j+1}, \dots, b_{k-1}) \in B \quad \text{(all } j \in \underline{k}\text{)}$$

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Equivalently, $t^{\mathbf{A}}[B^{j-1} \times A \times B^{k-j}] \subseteq B$, for all $0 \leqslant j < k$, that is,

$$(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B^{j-1} \times A \times B^{k-j} \quad \Longrightarrow \quad t^{\mathbf{A}}(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B.$$

Absorption Theory (for mortals)

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Notation:

$\mathbf{B} \triangleleft \mathbf{A}$ means $\mathbf{B}$ is absorbing in $\mathbf{A}$ with respect to some term.

To be explicit about the term, $\mathbf{B} \triangleleft_t \mathbf{A}$.

$\mathbf{B} \triangleleft\!\triangleleft \mathbf{A}$ means $\mathbf{B} \triangleleft \mathbf{A}$ and B is minimal (with respect to inclusion) among absorbing subuniverses of $\mathbf{A}$.

An algebra is *absorption-free* (AF) if it has no proper absorbing subalgebras.

Where are we going with this?

"The Absorption Theorem" of Barto and Kozik (LMCS 2012)

Concerns the special class of "linked" subdirect products.

Identifies some special cases in which a subdirect product is the full product!

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Theorem (Absorption Theorem)

If $\mathcal{V}$ *is an idempotent locally finite variety, then TFAE*

- $\mathcal{V}$ *is a Taylor variety;*
- *if* $\mathbf{A}_0, \mathbf{A}_1 \in \mathcal{V}$ *are finite idempotent absorption-free algebras and* $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ *is linked, then* $\mathbf{R} = \mathbf{A}_0 \times \mathbf{A}_1$*.*

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At Vanderbilt Shanks Workshop (2015), Barto presented more joint work with Kozik generalizing the Absorption Theorem to more than two factors.

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*assuming suitable conditions under which the theorem is true.

LINKED SUBDIRECT PRODUCTS

A subdirect product $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked if for all $a, a' \in \operatorname{Proj}_0 R$,

$$\exists\, c_0, c_2, \ldots, c_{2n} \in A_0, \quad \exists\, c_1, c_3, \ldots, c_{2n+1} \in A_1$$

such that

$$a = c_0, \quad (c_{2i}, c_{2i+1}) \in R, \quad (c_{2i+2}, c_{2i+1}) \in R, \quad c_{2n} = a'$$

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[todo: insert potato diagram]

Notation:

For $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$, let η_i denote the kernel of the i-th projection of $\mathbf{R}$. That is,

$$\eta_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid \operatorname{Proj}_i \mathbf{r} = \operatorname{Proj}_i \mathbf{r}'\}$$

Let $R^{-1} = \{(y, x) \in A_1 \times A_0 \mid (x, y) \in R\}$.

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The following are equivalent:

- $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked;
- $\eta_0 \vee \eta_1 = 1_R$;
- if $a, a' \in \operatorname{Proj}_0 R$, then (a, a') is in the transitive closure of $R \circ R^{-1}$.

Properties of Absorption I

Absorption has nice properties...

- (transitivity) $\mathbf{C} \triangleleft \mathbf{B} \triangleleft \mathbf{A} \implies \mathbf{C} \triangleleft \mathbf{A}$
- (closure under nonempty $\cap$ and finite products)

  If $\mathbf{B} \triangleleft_f \mathbf{A}$ and $\mathbf{C} \triangleleft_g \mathbf{A}$ and $B \cap C \neq \emptyset$, then $\mathbf{B} \cap \mathbf{C} \triangleleft \mathbf{A}$.

  If $\mathbf{B}_0 \triangleleft_f \mathbf{A}_0$ and $\mathbf{B}_1 \triangleleft_g \mathbf{A}_1$, then $\mathbf{B}_0 \times \mathbf{B}_1 \triangleleft_t \mathbf{A}_0 \times \mathbf{A}_1$.

  ...with respect to $t = f \star g$ in both cases.

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If $f : A^\ell \to A$ and $g : A^m \to A$, then $f \star g$ is the ℓm-ary operation

$$f(g(a_{11}, \dots, a_{1m}), g(a_{21}, \dots, a_{2m}), \dots, g(a_{\ell 1}, \dots, a_{\ell m}))$$

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More generally, if $\mathbf{B}_i \triangleleft_{t_i} \mathbf{A}_i$ for $0 \leqslant i < n$, then $\Pi \mathbf{B}_i \triangleleft_s \Pi \mathbf{A}_i$.

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A finite product of finite idempotent algebras is AF if each factor is AF.

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Restriction Lemma.
If $\mathbf{B} \lhd_t \mathbf{A}$ and $\mathbf{C} \leqslant \mathbf{A}$ and $D = B \cap C \neq \emptyset$, then $\mathbf{D} \lhd \mathbf{C}$ with respect to the restriction of t to C.

Lemma (LSD 1)

If $\mathbf{B}_i \lhd \mathbf{A}_i$ *and* $\mathbf{R} \leqslant \Pi_i \mathbf{A}_i$ *and* $R' := R \cap \Pi_i B_i \neq \emptyset$*, then* $\mathbf{R}' \lhd \mathbf{R}$*.*

Proof. $\Pi \mathbf{B}_i \lhd_t \Pi \mathbf{A}_i$, so follows Restriction Lemma if we put $C = R$.

Lemma (LSD 2)

Suppose $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ *and* $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi \mathbf{A}_i$*. If* $R' := R \cap \Pi B_i \neq \emptyset$*, then* $\mathbf{R}' \leqslant_{\mathrm{sd}} \Pi \mathbf{B}_i$*.*

Lemma (LSD 2)

If $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ *is linked and* $\mathbf{S} \lhd \mathbf{R}$*, then* $\mathbf{S}$ *is linked.*

In some simple cases we get linking from LSD Lemmas along with the following elementary

Fact. Suppose $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and let $\eta_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i)$.

1. If $\mathbf{A}_0$ is simple, then either $\eta_0 \vee \eta_1 = 1_R$ or $\eta_0 \geqslant \eta_1$.
2. If $\mathbf{A}_0$ and $\mathbf{A}_1$ are both simple, then either $\eta_0 \vee \eta_1 = 1_R$ or $\eta_0 = 0_R = \eta_1$.

...so, if both factors are simple, then $\eta_0 \neq \eta_1$ gives the linking...

Cor 1. Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be simple. If $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$, then $\mathbf{R}$ is linked.

...and if one factor is simple nonabelian and the other abelian, linking is free!

Cor 2. If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ abelian, then every subdirect product of $\mathbf{A}_0 \times \mathbf{A}_1$ is linked.

Absorption Theorem: Application

Suppose we add to the respective contexts of the last three results the hypothesis that the algebras live in an idempotent variety with a Taylor term...

(We will refer to such varieties as "Taylor varieties" and we call the algebras they contain "Taylor algebras.")

...then the Absorption Theorem (in combination with facts above) yields

Lemma: Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be finite Taylor algebras with $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ $(i = 0, 1)$ and suppose $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$.

(I) If $\mathbf{A}_0$ and $\mathbf{A}_1$ are simple and $R \cap (B_0 \times B_1) \neq \emptyset$, then $\mathbf{B}_0 \times \mathbf{B}_1 \leqslant \mathbf{R}$.

(II) If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ is abelian, then $\mathbf{B}_0 \times \mathbf{A}_1 \leqslant \mathbf{R}$.

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...simple nonabelian potatoes cannot.

Barto and Kozik generalized the Absorption Theorem to multiple simple nonabelian factors.

The Rectangularity Theorem.

Let $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ be finite algebras in a Taylor variety, $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$, and suppose

- at most one $\mathbf{A}_i$ is abelian,
- all nonabelian factors are simple,
- $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$,

If $\mathbf{R}' = \mathbf{R} \cap (\mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1})$ is nonempty, then

$$\mathbf{R}' = \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}.$$

Notation:

Let $\underline{n} = \{0, 1, 2, \ldots, n-1\}$.

Let $\sigma' = \underline{n} - \sigma$, when σ is a subset of $\underline{n}$.

For $\mathbf{R} \leqslant_{\text{sd}} \Pi_{\underline{n}} \mathbf{A}_i$ let

$$\eta_\sigma = \ker(R \twoheadrightarrow \Pi_\sigma A_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid \operatorname{Proj}_\sigma \mathbf{r} = \operatorname{Proj}_\sigma \mathbf{r}'\},$$

If $\sigma \subseteq \underline{n}$, then by $\mathbf{R} \leqslant_{\text{sd}} \Pi_\sigma \mathbf{A}_i \times \Pi_{\sigma'} \mathbf{A}_i$ we mean

$$\mathbf{R} \leqslant \Pi_{\underline{n}} \mathbf{A}_i, \quad \operatorname{Proj}_\sigma \mathbf{R} = \Pi_\sigma \mathbf{A}_i, \quad \text{and} \quad \operatorname{Proj}_{\sigma'} \mathbf{R} = \Pi_{\sigma'} \mathbf{A}_i.$$

and we say that $\mathbf{R}$ is a *subdirect product of* $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$ in this case.

The subdirect product $\mathbf{R} \leqslant_{\text{sd}} \Pi_\sigma \mathbf{A}_i \times \Pi_{\sigma'} \mathbf{A}_i$ is said to be *linked* if $\eta_\sigma \vee \eta_{\sigma'} = 1_R$.

We may use $\mathbf{R}_\sigma$ for $\operatorname{Proj}_\sigma \mathbf{R}$, the projection of $\mathbf{R}$ onto coordinates in σ.

From now on, *all algebras are finite and belong to the same Taylor variety*.

Lemma 1.
Let $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ for each $i \in \underline{n}$, and let $\underline{n} = \sigma \cup \sigma'$ be a disjoint union.
Assume $\mathbf{R}$ is a *linked* subdirect product of $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$.
Suppose $R' = R \cap \Pi_i B_i \neq \emptyset$. Then $\mathbf{R}' = \Pi_i \mathbf{B}_i$.

Lemma 2. [Kearnes-Kiss, Th. 3.27]
Suppose α and β are congruences of a Taylor algebra. Then $\mathrm{C}(\alpha, \alpha; \alpha \wedge \beta)$ if and only if $\mathrm{C}(\alpha \vee \beta, \alpha \vee \beta; \beta)$.

The Kearnes and Kiss theorem can be used to prove

Lemma 3. [Linking Lemma]
Let $n \geqslant 2$, let $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ be finite algebras in a Taylor variety, and let $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$. Suppose

- at most one $\mathbf{A}_i$ is abelian
- all nonabelian factors are simple
- $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$.

Then there exists k such that $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_k \times \mathbf{R}_{k'}$ is linked.

Theorem of Kearnes and Kiss

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3. **Intersecting mass products.** RT and corollaries assume that the universe R of the subdirect product in question intersects nontrivially with a product ΠB_i of minimal absorbing subuniverses.