This note summarizes the rules of inference for intuitionistic propositional logic (IPL) as I understand them. This is based on the lectures given by Professor Robert Harper in September 2013 at CMU [1]. Notes for Harper's lectures were transcribed by his students and this summary is based on the recorded lectures and the notes.

As advanced by Per Martin-Löf, a modern presentation of IPL distinguishes the notions of *judgment* and *proposition*. A judgment is something that may be known, whereas a proposition is something that sensibly may be the subject of a judgment. For instance, the statement "Every natural number larger than 1 is either prime or can be uniquely factored into a product of primes." is a proposition because it sensibly may be subject to judgment. That the statement is in fact true is a judgment. Only with a proof, however, is it evident that the judgment indeed holds.

Thus, in IPL, the two most basic judgments are $A\,\mathsf{prop}$ and $A\,\mathsf{true}$:

$A\,\mathsf{prop}$	A is a well-formed proposition
$A\,\mathsf{true}$	Proposition A is intuitionistically true, i.e., has a proof.

The inference rules for the prop judgment are called *formation rules*. The inference rules for the true judgment are divided into classes: *introduction rules* and *elimination rules*.

The meaning of a proposition A is given by the introduction rules for the judgment $A\,\mathsf{true}$. The elimination rules are dual and describe what may be deduced from a proof of $A\,\mathsf{true}$. The principle of *internal coherence*, also known as *Gentzen's principle of inversion*, is that the introduction and elimination rules for a proposition A fit together properly. The elimination rules should be strong enough to deduce all information that was used to introduce A (*local completeness*), but not so strong as to deduce information that might not have been used to introduce A (*local soundness*).

CONJUNCTION

(formation) If A and B are well-formed propositions, then so is their *conjunction*, which we write as $A \wedge B$.

$$\frac{A\,\mathsf{prop} \quad B\,\mathsf{prop}}{A \wedge B\,\mathsf{prop}}\ \wedge\mathsf{F}$$

(introduction) To give meaning to conjunction, we must say how to introduce the judgment $A \wedge B\,\mathsf{true}$. A verification of $A \wedge B$ requires a proof of A and a proof of B.

$$\frac{A\,\mathsf{true} \quad B\,\mathsf{true}}{A \wedge B\,\mathsf{true}}\ \wedge\mathsf{I}$$

(elimination) Because every proof of $A \wedge B$ comes from a pair of proofs, one of A and one of B, we are justified in deducing $A\,\mathsf{true}$ and $B\,\mathsf{true}$ from a proof of $A \wedge B$:

$$\frac{A \wedge B\,\mathsf{true}}{A\,\mathsf{true}}\ \wedge\mathsf{E}_1 \qquad \frac{A \wedge B\,\mathsf{true}}{B\,\mathsf{true}}\ \wedge\mathsf{E}_2$$

TRUTH

(formation) The formation rule serves as immediate evidence for the judgment that $\top$ is a well-formed proposition.

$$\frac{}{\top\,\mathsf{prop}}\ \top\mathsf{F}$$

(introduction) Since $\top$ is a trivially true proposition, its introduction rule makes the judgment $\top\,\mathsf{true}$ immediately evident.

$$\frac{}{\top\,\mathsf{true}}\ \top\mathsf{I}$$

(elimination) Since $\top$ is trivially true, an elimination rule should not increase our knowledge—we put in no information when we introduced $\top\,\mathsf{true}$, so, by the principle of conservation of proof, we should get no information out. Thus, there is no elimination rule for $\top$.

ENTAILMENT

Entailment is a judgment and is written as

$$A_1 \ \mathsf{true}, \ldots, A_n \ \mathsf{true} \vdash A \ \mathsf{true}$$

This expresses the judgment that $A \ \mathsf{true}$ follows from $A_1 \ \mathsf{true}, \ldots, A_n \ \mathsf{true}$. One can view $A_1 \ \mathsf{true}, \ldots, A_n \ \mathsf{true}$ as being assumptions from which the conclusion $A \ \mathsf{true}$ may be deduced. We assume that the entailment judgment satisfies several *structural properties*: reflexivity, transitivity, weakening, contraction, and permutation.

Reflexivity: An assumption is enough to conclude the same judgment.

$$\frac{}{A \ \mathsf{true} \vdash A \ \mathsf{true}} \ \mathsf{R}$$

Transitivity: If you prove $A \ \mathsf{true}$, then you are justified in using it in a proof.

$$\frac{A \ \mathsf{true} \quad A \ \mathsf{true} \vdash C \ \mathsf{true}}{C \ \mathsf{true}} \ \mathsf{T}$$

Reflexivity and transitivity are undeniable since without them it would be unclear what is meant by an *assumption*. An assumption should be strong enough to prove conclusions (reflexivity), and only as strong as the proofs they stand for (transitivity). The remaining structural properties—weakening, contraction, and permutation—could be denied. Logics that deny any of these properties are called *substructural logics*.

Weakening: We can add assumptions to a proof without invalidating that proof.

$$\frac{A \ \mathsf{true}}{B \ \mathsf{true} \vdash A \ \mathsf{true}} \ \mathsf{W}$$

Contraction: The number of copies of an assumption does not matter.

$$\frac{A \ \mathsf{true}, A \ \mathsf{true} \vdash C \ \mathsf{true}}{A \ \mathsf{true} \vdash C \ \mathsf{true}} \ \mathsf{C}$$

Permutation: aka "exchange;" the order of assumptions does not matter.

$$\frac{\Gamma \vdash C \ \mathsf{true}}{\pi(\Gamma) \vdash C \ \mathsf{true}} \ \mathsf{P}$$

IMPLICATION

(formation)

$$\frac{A \ \mathsf{prop} \quad B \ \mathsf{prop}}{A \supset B \ \mathsf{prop}} \ {\supset}\mathsf{F}$$

(introduction)

$$\frac{A \ \mathsf{true} \vdash B \ \mathsf{true}}{A \supset B \ \mathsf{true}} \ {\supset}\mathsf{I}$$

In this way, implication internalizes the entailment judgment as a proposition, while we nonetheless maintain the distinction between propositions and judgments.

(elimination)

$$\frac{A \supset B \ \mathsf{true} \quad A \ \mathsf{true}}{B \ \mathsf{true}} \ {\supset}\mathsf{E}.$$

This rule is sometimes referred to as *modus ponens*.

DISJUNCTION

(formation)

$$\frac{A\ \mathsf{prop} \quad B\ \mathsf{prop}}{A \vee B\ \mathsf{prop}}\ \vee\mathsf{F}$$

(introduction)

$$\frac{A\ \mathsf{true}}{A \vee B\ \mathsf{true}}\ \vee\mathsf{I}_1 \qquad\qquad \frac{B\ \mathsf{true}}{A \vee B\ \mathsf{true}}\ \vee\mathsf{I}_2$$

(elimination)

$$\frac{A \vee B\ \mathsf{true} \quad A\ \mathsf{true} \vdash C\ \mathsf{true} \quad B\ \mathsf{true} \vdash C\ \mathsf{true}}{C\ \mathsf{true}}\ \vee\mathsf{E}$$

FALSEHOOD

(formation) The unit of disjunction is falsehood, the proposition that is trivially never true, which we write as $\perp$. Its formation rule is immediate evidence that $\perp$ is a well-formed proposition.

$$\frac{}{\perp\ \mathsf{prop}}\ \perp\mathsf{F}$$

(introduction) Because $\perp$ should never be true, it has no introduction rule.
(elimination)

$$\frac{\perp\ \mathsf{true}}{C\ \mathsf{true}}\ \perp\mathsf{E}$$

The elimination rule captures *ex falso quodlibet*: from a proof of $\perp\ \mathsf{true}$, we may deduce that *any* proposition C is true because there is ultimately no way to introduce $\perp\ \mathsf{true}$. Once again, the rules cohere. The elimination rule is very strong, but remains justified due to the absence of any introduction rule for falsehood.

References

[1] Robert Harper. Carnegie Mellon University course: 15-819 Homotopy Type Theory. `http://www.cs.cmu.edu/~rwh/courses/hott/`, Fall 2012.

[2] Frank Pfenning. Lecture notes on harmony. `http://www.cs.cmu.edu/~fp/courses/15317-f09/lectures/03-harmony.pdf`, September 2009.

[3] Frank Pfenning. Lecture notes on natural deduction. `http://www.cs.cmu.edu/~fp/courses/15317-f09/lectures/02-natded.pdf`, August 2009.