

THE ALGEBRAIC APPROACH TO CSP
AND
CSPs OF COMMUTATIVE IDEMPOTENT BINARS

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joint work with

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slides available at

<https://github.com/williamdemeo/Talks>

CONSTRAINT SATISFACTION PROBLEMS

Input

- *variables*: $V = \{v_1, v_2, \dots\}$
- *domain*: D
- *constraints*: C_1, C_2, \dots

Output

- “yes” if there is a *solution*

$\sigma : V \rightarrow D$ (an assignment of values to variables that satisfies all C_i)

- “no” otherwise

CONSTRAINT SATISFACTION PROBLEMS

EXAMPLE: 3-SAT

Input

- *variables*: $V = \{v_1, \dots, v_n\}$
- *domain*: $D = \{0, 1\}$
- *constraints*: a formula, say,

$$f(v_1, \dots, v_n) = (v_1 \vee v_2 \vee \neg v_3) \wedge (\neg v_1 \vee v_3 \vee v_4) \wedge \dots$$

Output

- “yes” if there is a solution: $\sigma : V \rightarrow D$ such that

$$f(\sigma v_1, \dots, \sigma v_n) = 1$$

- “no” otherwise

CONSTRAINT SATISFACTION PROBLEMS

EXAMPLE: NAE-SAT

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- *variables*: $V = \{v_1, \dots, v_n\}$
- *domain*: $D = \{0, 1\}$
- *constraints*: $(s_1, C_1), (s_2, C_2), \dots$ of the form

$$s = (i, j, k) \in \{1, \dots, n\}^3 \quad (\text{scopes})$$

$$C = \neg(v_i = v_j = v_k)$$

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In terms of relational structures...

Let $S := \{(v_i, v_j, v_k) : (i, j, k) \text{ is a scope}\} \subseteq V^3$

$$R := \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0)\} \subseteq D^3$$

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$$\text{that is, } (x, y, z) \in S \implies (\sigma x, \sigma y, \sigma z) \in R$$

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Solutions are homomorphisms!

$$\sigma : \langle V, S \rangle \rightarrow \langle D, R \rangle$$

CSP: RELATIONAL FORMULATION

Let $\mathbb{D} = \langle D, \mathcal{R} \rangle$ be a relational structure.

$\text{CSP}(\mathbb{D})$ (or $\text{CSP}(\mathcal{R})$) is the decision problem with

Input

- A structure $\mathbb{V} = \langle V, \mathcal{C} \rangle$ *similar* to \mathbb{D} .

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- “yes” if there is a homomorphism $\sigma : \mathbb{V} \rightarrow \mathbb{D}$
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Alternatively, let \Rightarrow be the binary relation on similar structures:

$$\mathbb{V} \Rightarrow \mathbb{D} \quad \text{iff there is a homomorphism } \sigma : \mathbb{V} \rightarrow \mathbb{D}$$

Then the CSP of \mathbb{D} is the membership problem for the set

$$\text{CSP}(\mathbb{D}) := \{\mathbb{V} : \mathbb{V} \Rightarrow \mathbb{D}\}$$

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Output

- “yes” if there is a homomorphism $\sigma : \mathbb{V} \rightarrow \mathbb{D}$
- “no” otherwise

We call \mathbb{D} (or \mathcal{R}) “tractable” if there is a polynomial-time algorithm for solving $\text{CSP}(\mathbb{D})$ (or $\text{CSP}(\mathcal{R})$).

CSP: ALGEBRAIC FORMULATION

Let $\mathbb{D} = \langle D, \mathcal{R} \rangle$ be a relational structure.

For $R \subseteq \mathcal{R}$ define the *polymorphisms* of R ,

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that is, $f \in \text{pol}(R)$ iff for every $\rho \in R$

$$(a_1, b_1, \dots, z_1) \in \rho$$

$$\vdots$$

$$(a_k, b_k, \dots, z_k) \in \rho$$

$$(f(a_1, \dots, a_k), \dots, f(z_1, \dots, z_k)) \in \rho$$

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Define the algebra $\mathbf{D} := \langle D, \text{pol}(R) \rangle$.

We call \mathbf{D} “tractable” if the corresponding structure $\langle D, R \rangle$ is tractable.

CSP: ALGEBRAIC FORMULATION

For F a set of operations on D , define the *relational clone* of F ,

$$\text{rel}(F) := \{\rho \subseteq D^n \mid f(\rho) \subseteq \rho \text{ for every } f \in F\}$$

Let $\bar{R} := \text{rel}(\text{pol}(R))$ be the “closure” of R .

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THEOREM

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THEOREM

$$\text{CSP}\langle D, R \rangle \equiv_P \text{CSP}\langle D, \bar{R} \rangle$$

$$\text{Corollary} \quad \text{pol}(R) = \text{pol}(S) \implies \text{CSP}(R) \equiv_P \text{CSP}(S)$$

The algebra $\langle D, \text{pol}(R) \rangle$ determines the complexity of the corresponding CSP!

GENERAL PROBLEM

Find properties (of algebras) that characterize the complexity of CSPs.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra \mathbf{A} ...

$\text{CSP}(\mathbf{A})$ is tractable $\iff \mathbf{A}$ has a weak-nu term operation

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The left-to-right direction is known.

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The right-to-left direction is open.

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A **weak near unanimity** (weak-nu) term operation is one that satisfies

$$t(x, x, \dots, x) \approx x \quad (\text{idempotent})$$

$$t(y, x, \dots, x) \approx t(x, y, \dots, x) \approx \dots \approx t(x, x, \dots, y)$$

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A *binary* operation $t(x, y)$ is weak-nu if

$$t(x, x) \approx x \quad (\text{idempotent})$$

$$t(y, x) \approx t(x, y) \quad (\text{commutative})$$

So let's try to prove (?) for **commutative idempotent binars**.

COMMUTATIVE IDEMPOTENT BINARS

A **CIB** is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.

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First Example: a semilattice is an associative CIB.

Semilattices are tractable.

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First Example: a semilattice is an associative CIB.

Semilattices are tractable.

Pause to consider more general case for a minute...

GENERAL CASE

SOME WELL KNOWN FACTS

Let \mathbf{A} be a finite idempotent algebra. Let \mathbf{S}_2 be the 2-elt semilattice.

$$\mathbf{V}(\mathbf{A}) \text{ is CP} \iff \mathbf{A} \text{ has Malcev term}$$

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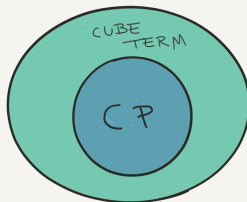


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$$\begin{aligned} \mathbf{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has Malcev term} \\ &\implies \mathbf{A} \text{ has cube term} \end{aligned}$$



GENERAL CASE

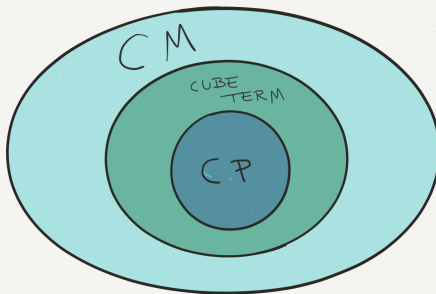
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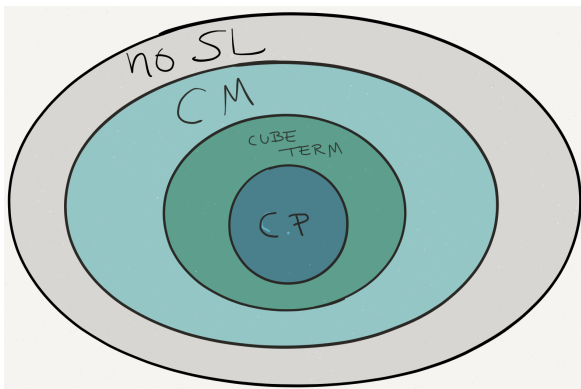
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$\implies \mathbf{S}_2$ is not in $V(\mathbf{A})$



FIRST REDUCTION

BY CUBE-TERM BLOCKERS

Marković, M. Maróti, McKenzie (M^4)

“Finitely related clones and algebras with cube terms” (2012)

A **cube-term blocker** (CTB) is a pair (C, B) of subuniverses satisfying $\emptyset < C < B \leq A$ and for every $t(x_1, \dots, x_n)$ there is an index $i \in [n]$ with

$$(\forall (b_1, \dots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \dots, b_n) \in C)$$

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M^4 prove a finite idempotent algebra has a cube term iff it has no CTB.

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LEMMA

A finite CIB \mathbf{A} has a CTB if and only if $\mathbf{S}_2 \in \text{HS}(\mathbf{A})$.

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LEMMA

A finite CIB \mathbf{A} has a CTB if and only if $\mathbf{S}_2 \in \text{HS}(\mathbf{A})$.

PROOF.

(C, B) a CTB implies $\theta = C^2 \cup (B - C)^2$ a congruence with $\mathbf{B}/\theta \cong \mathbf{S}_2$.

Conversely, suppose $\mathbf{S}_2 \in \text{HS}(\mathbf{A})$, and \mathbf{B} is a subalgebra of \mathbf{A} with \mathbf{B}/θ a meet-SL for some θ . Let C/θ be the bottom of \mathbf{B}/θ , then (C, B) is a CTB. □

SECOND REDUCTION

Kearnes and Tschantz

“Automorphism groups of squares and of free algebras” (2007)

LEMMA

If V is an idempotent variety that is not congruence permutable, then there are subuniverses U and W of $\mathbf{F} := \mathbf{F}_V\{x, y\}$ satisfying

1. $x \in U \cap W$
2. $y \in U^c \cap W^c$
3. $(U \times F) \cup (F \times W) \leq \mathbf{F}^2$

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For CIB's, either U or W will be an ideal.

This implies a CTB and a semilattice.

\mathbf{A} = a finite CIB

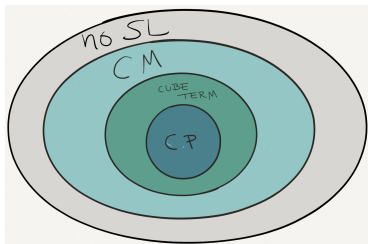
\mathbf{S}_2 = the 2-elt semilattice.

$V(\mathbf{A})$ is CP \iff \mathbf{A} has a Malcev term

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\implies $V(\mathbf{A})$ is CM

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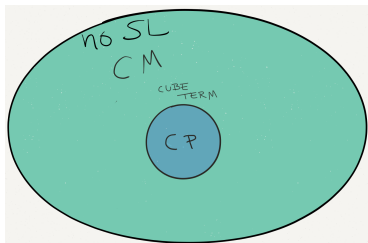
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■ 1st reduction by cube-term blockers.

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■ 1st reduction by cube-term blockers.

■ 2nd reduction by Kearnes-Tschantz.

REMAINING QUESTIONS FOR FINITE CIBs

CONCLUSION

Let \mathbf{A} be a finite CIB. Then

$\mathbf{S}_2 \notin \mathbf{HS}(\mathbf{A})$ if and only if $\mathbf{V}(\mathbf{A})$ is congruence permutable.

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OPEN QUESTION

Let \mathbf{A} be a finite CIB with S_2 in $HS(\mathbf{A})$. Is $CSP(\mathbf{A})$ tractable?

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Recall, if $V(\mathbf{A})$ is SD_\wedge , then $\text{CSP}(\mathbf{A})$ is tractable.

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REVISED QUESTION

Let \mathbf{A} be a finite CIB with S_2 in $\text{HS}(\mathbf{A})$, and $V(\mathbf{A})$ not SD_\wedge .

Is $\text{CSP}(\mathbf{A})$ tractable?

EXAMPLE 1

\cdot	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

EXAMPLE 1

Cliff's trick: replace binary operation with a term from $\text{clo}(\mathbf{A})$, say

$$x * y = (x \cdot (x \cdot y)) \cdot (y \cdot (x \cdot y))$$

If $\langle A, * \rangle$ tractable, then so is $\mathbf{A} = \langle A, \cdot \rangle$.

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$$x * y = (x \cdot (x \cdot y)) \cdot (y \cdot (x \cdot y))$$

If $\langle A, * \rangle$ tractable, then so is $\mathbf{A} = \langle A, \cdot \rangle$.

$$\begin{aligned} \{*\} \subseteq \text{clo}(\mathbf{A}) &\implies \text{rel}(\text{clo}(\mathbf{A})) \subseteq \text{rel}(\{*\}) \\ &\implies \text{CSP}(\mathbf{A}) \leq_P \text{CSP}\langle A, * \rangle \end{aligned}$$

\cdot	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

EXAMPLE 1

Cliff's trick: replace binary operation with a term from $\text{clo}(\mathbf{A})$, say

$$x * y = (x \cdot (x \cdot y)) \cdot (y \cdot (x \cdot y))$$

If $\langle A, * \rangle$ tractable, then so is $\mathbf{A} = \langle A, \cdot \rangle$.

$$\begin{aligned}\{*\} \subseteq \text{clo}(\mathbf{A}) &\implies \text{rel}(\text{clo}(\mathbf{A})) \subseteq \text{rel}(\{*\}) \\ &\implies \text{CSP}(\mathbf{A}) \leq_P \text{CSP}\langle A, * \rangle\end{aligned}$$

$$\langle A, * \rangle \text{ tractable} \implies \mathbf{A} \text{ tractable}$$

\cdot	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

$*$	0	1	2	3
0	0	0	0	0
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

EXAMPLE 2

\cdot	0	1	2	3
0	0	0	1	1
1	0	1	3	2
2	1	3	2	1
3	1	2	1	3

Let $t(x, y) = x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y))$.

EXAMPLE 2

\cdot	0	1	2	3
0	0	0	1	1
1	0	1	3	2
2	1	3	2	1
3	1	2	1	3

t	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

Let $t(x, y) = x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y))$.

$\langle A, t \rangle$ tractable

EXAMPLE 3

\cdot	0	1	2	3
0	0	0	2	1
1	0	1	3	2
2	2	3	2	1
3	1	2	1	3

EXAMPLE 3

\cdot	0	1	2	3
0	0	0	2	1
1	0	1	3	2
2	2	3	2	1
3	1	2	1	3

Let $t_2(x, y) = \dots$?

EXAMPLE 3

\cdot	0	1	2	3
0	0	0	2	1
1	0	1	3	2
2	2	3	2	1
3	1	2	1	3

Let $t_2(x, y) = \dots$?

Let $t_3(x, y, z) = \dots$?

...and about 25 others.

The image shows two overlapping windows. The background window is the UACalculator v1.13 application, which has a menu bar (File, Edit, HSP, Tasks, Maltsev, Idempotent Algs, Equations, Drawing, Help) and a toolbar. The 'Editor' tab is active, showing a form for defining an algebra with fields for Name (CB4-SL-729), Cardinality (4), and Desc (CB with semilattice; V(A) not SD-meet). Below this is a table for operations (g(2)) and a section for the Element Key Table. The foreground window is a web browser showing the GitHub repository for UACalc/AlgebraFiles. The repository page includes a search bar, repository name, description ('A repository of algebra files for the Universal Algebra Calculator'), statistics (23 commits, 1 branch, 0 releases, 2 contributors), and a list of contributors with their commit history.

UACalculator v1.13 (Feb 28, 2015)

File Edit HSP Tasks Maltsev Idempotent Algs Equations Drawing Help

Editor Algebras Computations Con Sub Drawing

Name: CB4-SL-729 Cardinality: 4 Desc: CB with semilattice; V(A) not SD-meet

Operations: g(2) Del Add Make Into Basic Alg

y	0	1	2	3
g(0,y)	0	0	2	1
g(1,y)	0	1	3	2
g(2,y)	2	3	2	1
g(3,y)	1	2	1	3

☐ Idempotent Default Element: none

Element Key Table

Index	Elem
0	0
1	1
2	2
3	3

Algebras

Internal	Name	Type	Description
A6	CB4-SL-439	BASIC	CB with semilattice; V(A) not SD-meet
A7	CB4-SL-505	BASIC	CB with semilattice; V(A) not SD-meet
A8	CB4-SL-713	BASIC	CB with semilattice; V(A) not SD-meet
A9	CB4-SL-729	BASIC	CB with semilattice; V(A) not SD-meet

Msg:

UACalc/AlgebraFiles

GitHub, Inc. [US] <https://github.com/UACalc/AlgebraFiles>

This repository Search Explore Gist Blog Help

UACalc / AlgebraFiles Unwatch

A repository of algebra files for the Universal Algebra Calculator — Edit

23 commits 1 branch 0 releases 2 contributors

branch: master AlgebraFiles / +

minor corrections

willanderson authored 10 days ago latest commit: 926f135

Contributor	Commit	Time
Baker	initial commit	11 months
Bergman	minor corrections	10 days
Groups	initial commit	11 months
Jipson	initial commit	11 months

To see them, load UACalc with files from the Bergman directory at

<https://github.com/UACalc/AlgebraFiles>

...and about 25 others.

The image shows two overlapping windows. The background window is the UACalc application, version 1.13 (Feb 28, 2015). It has a menu bar (File, Edit, HSP, Tasks, Maltsev, Idempotent Algs, Equations, Drawing, Help) and a toolbar. The 'Editor' tab is active, showing a form for defining an algebra. The 'Name' field contains 'CB4-SL-729', 'Cardinality' is '4', and 'Desc' is 'CIB with semilattice; V(A) not SD-meet'. Below this is a table for 'Operations' with columns 'y' and 'x' (0, 1, 2, 3). The table contains values for $g(0,y)$, $g(1,y)$, $g(2,y)$, and $g(3,y)$. There are also checkboxes for 'Idempotent' and a 'Default Element' dropdown set to 'none'. At the bottom is an 'Algebras' table with columns 'Internal', 'Name', 'Type', and 'Description'. It lists four entries: AB (CB4-SL-439, BASIC, CIB with semilattice; V(A) not SD-meet), A7 (CB4-SL-505, BASIC, CIB with semilattice; V(A) not SD-meet), AB (CB4-SL-713, BASIC, CIB with semilattice; V(A) not SD-meet), and AB (CB4-SL-729, BASIC, CIB with semilattice; V(A) not SD-meet). The foreground window is a web browser showing the GitHub repository for 'UACalc / AlgebraFiles'. The repository has 23 commits, 1 branch, 0 releases, and 2 contributors. The 'master' branch is selected, showing a commit by 'williamdewo' 10 days ago. A list of contributors is shown: Baker (initial commit, 11 months ago), Bergman (minor corrections, 10 days ago), Groups (initial commit, 11 months ago), and Jipson (initial commit, 11 months ago).

To see them, load UACalc with files from the **Bergman** directory at

<https://github.com/UACalc/AlgebraFiles>

Thank you for listening!