OVERALGEBRAS:
EXPANSIONS AND EXTENSIONS OF FINITE ALGEBRAS

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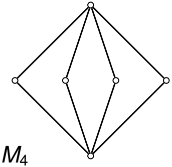
Iowa State University
Algebra & Combinatorics Seminar

22 Feb 2016

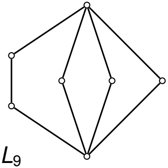
These slides and other resources are available at

https://github.com/williamdemeo/Talks

Contruction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.



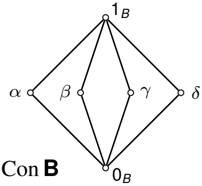
M_4



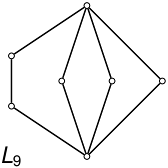
L_9

Contruction of an algebra $\mathbf{A}$ with $\mathrm{Con}\,\mathbf{A} \cong L_9$.

Step 1 Take a permutational algebra $\mathbf{B} = \langle B, F \rangle$ with congruence lattice $\mathrm{Con}\,\mathbf{B} \cong M_4$.

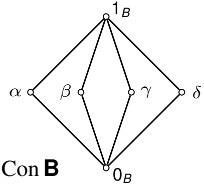


Con **B**

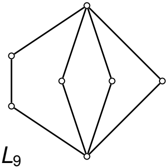


L_9

CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.



$\operatorname{Con}\mathbf{B}$



L_9

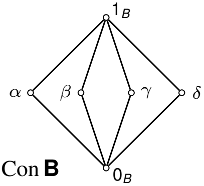
STEP 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\operatorname{Con}\mathbf{B} \cong M_4$.

Example:

- Let $B = \{0, 1, \dots, 5\}$ index the elements of S_3 and consider the right regular action of S_3 on itself.
- $g_0 = (0,4)(1,3)(2,5)$ and $g_1 = (0,1,2)(3,4,5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- $\operatorname{Con}\langle B, \{g_0, g_1\}\rangle \cong M_4$ with congruences

$\alpha = |012|345|,\ \beta = |03|14|25|, \gamma = |04|15|23|,\ \delta = |05|13|24|.$

CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.



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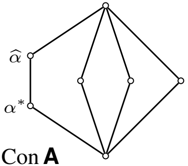
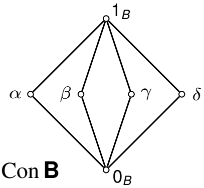
Goal: *expand* **B** *to an algebra* **A** *that has* α *"doubled" in* Con **A**.



Con **A**

Contruction of an algebra $\mathbf{A}$ with $\mathrm{Con}\,\mathbf{A} \cong L_9$.



Con $\mathbf{B}$

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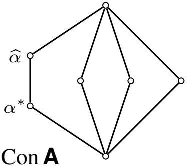

Con $\mathbf{A}$

STEP 2 Since $\alpha = \mathrm{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

$$B_0 = \{0, 1, 2, 3, 4, 5\} = B$$
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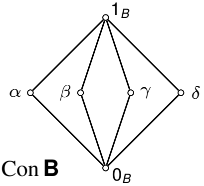
CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\mathrm{Con}\,\mathbf{A} \cong L_9$.



$\mathrm{Con}\,\mathbf{B}$

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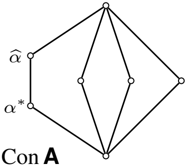

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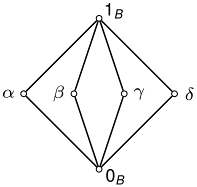
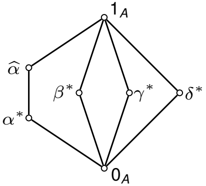
STEP 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0 e_0$, and $g_1 e_0$.

CONTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH Con $\mathbf{A} \cong L_9$.





Con $\langle B, \{g_0, g_1\}\rangle$

Con $\langle A, F_A\rangle$

$$\alpha = |0,1,2|3,4,5|$$

$$\beta = |0,3|1,4|2,5|$$

$$\gamma = |0,4|1,5|2,3|$$

$$\delta = |0,5|1,3|2,4|$$

$$\widehat{\alpha} = |0,1,2,6,7,11,12|3,4,5|8,9,10,13,14,15|$$

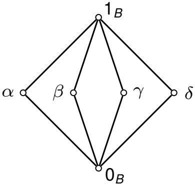
$$\alpha^* = |0,1,2,6,7,11,12|3,4,5|8,9,10|13,14,15|$$

$$\beta^* = |0,3,8|1,4|2,5,15|6,9|7,10|11,13|12,14|$$

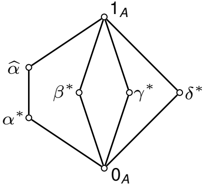
$$\gamma^* = |0,4,9|1,5|2,3,13|6,10|7,8|11,14|12,15|$$

$$\delta^* = |0,5,10|1,3|2,4,14|6,8|7,9,11,15|12,13|$$

Contruction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.



$\operatorname{Con}\langle B, \{g_0, g_1\}\rangle$



$\operatorname{Con}\langle A, F_A\rangle$

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$$\alpha = \alpha^* \cap B^2 = \widehat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \dots$$

Extension & Expansion

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

$$\alpha = |0,1,2|3,4,5|$$

$$\beta = |0,3|1,4|2,5|$$

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$$\delta = |0,5|1,3|2,4|$$

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \end{array}$$

B_0

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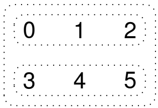

α

B_0

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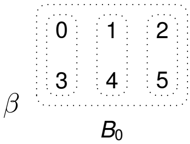

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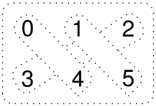
γ 

B_0

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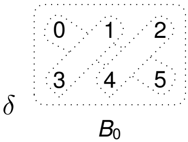

Extension & Expansion

$$\text{Con}\,\langle B, \{g_0, g_1\}\rangle$$

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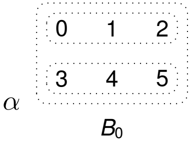

$$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$$

EXTENSION & EXPANSION

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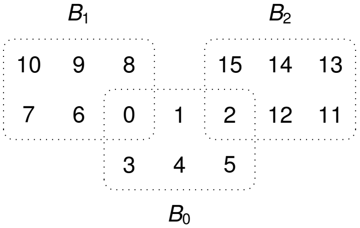

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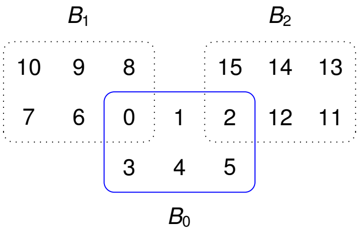

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- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

$$e_0\colon A \twoheadrightarrow B_0$$
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$$ge_0\colon A \overset{e_0}{\twoheadrightarrow} B_0 \xrightarrow{g} B_0$$

Extension & Expansion

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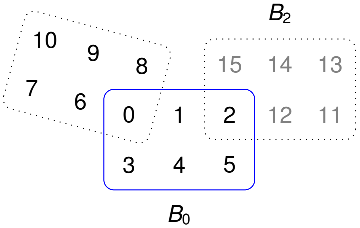

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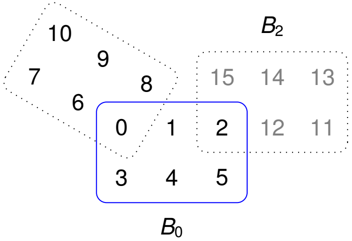

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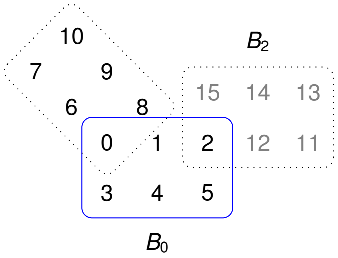

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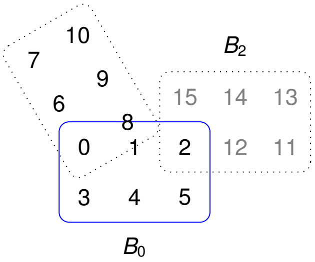

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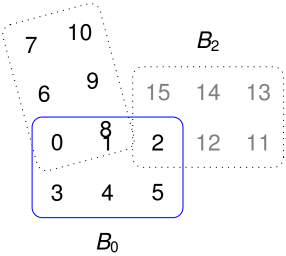

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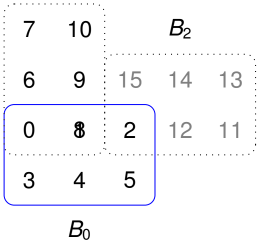
$$\begin{array}{rcccccc} B_1 = \{ & 0 & 6 & 7 & 8 & 9 & 10\ \} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ B_0 = \{ & 0 & 1 & 2 & 3 & 4 & 5\ \} \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ B_2 = \{ & 11 & 12 & 2 & 13 & 14 & 15\ \} \end{array}$$

$$ge_0 \colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\rightarrow} B_0$$

Extension & Expansion

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$



$$\alpha = |0,1,2|3,4,5|$$

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- $A = B_0 \cup B_1 \cup B_2$
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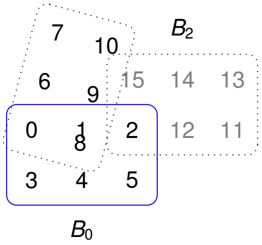

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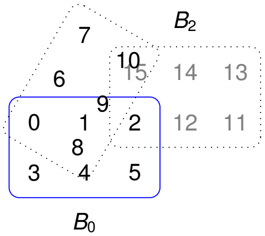

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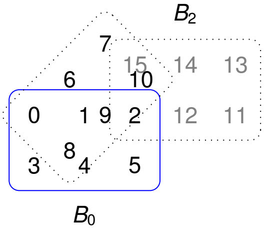

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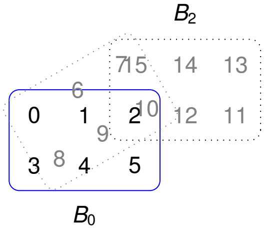

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$$ge_0\colon A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0$$

Extension & Expansion

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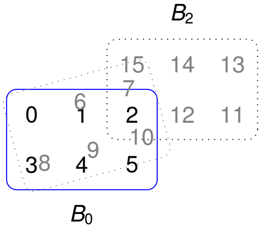

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EXTENSION & EXPANSION

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$$\begin{array}{ccccc} & & & B_2 & \\ & & 15 & 14 & 13 \\ 0 & 1 & 2 & 12 & 11 \\ 3 & 4 & 5 & & \\ & B_0 & & & \end{array}$$

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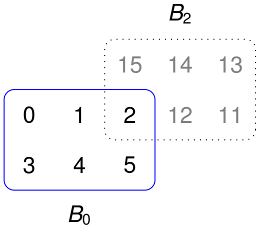

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Extension & Expansion

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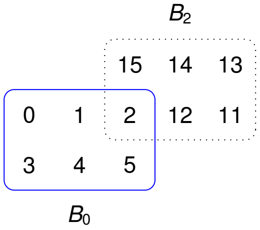

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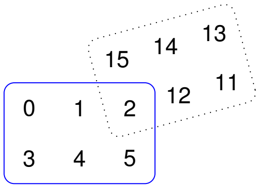

B_0

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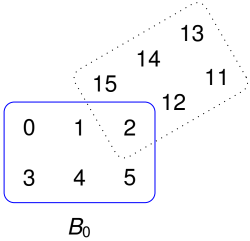

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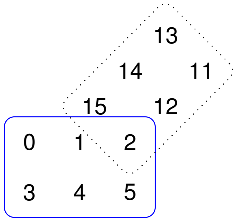

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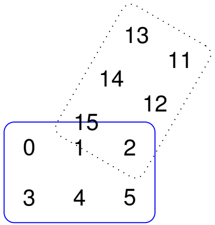

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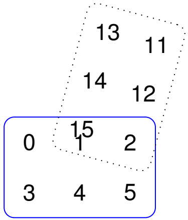

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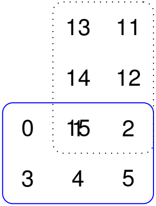

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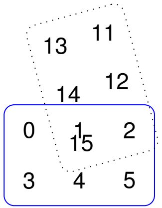

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EXTENSION & EXPANSION

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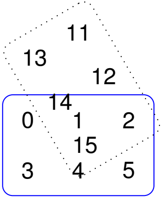

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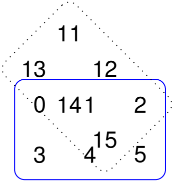

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$$ge_0 \colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\to} B_0$$

EXTENSION & EXPANSION

$$\text{Con}\,\langle B, \{g_0, g_1\}\rangle$$

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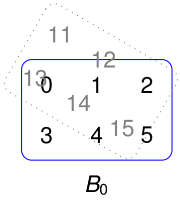

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

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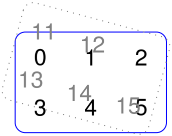

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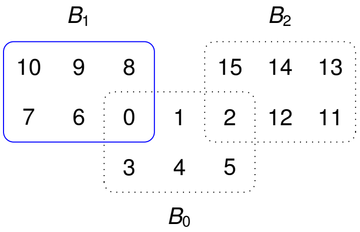

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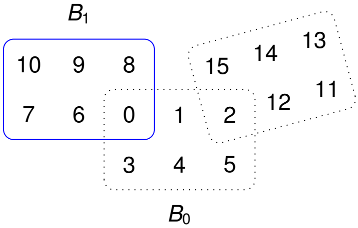

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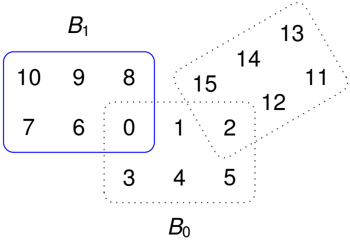

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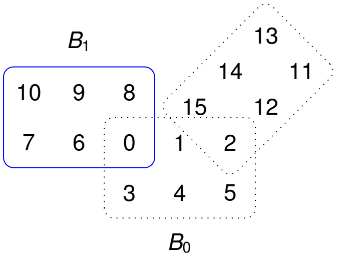

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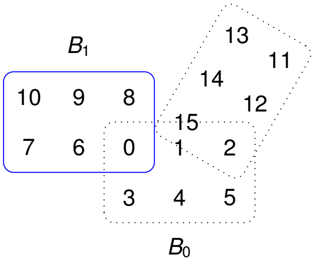

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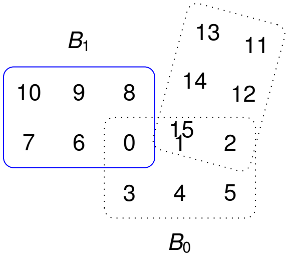

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EXTENSION & EXPANSION

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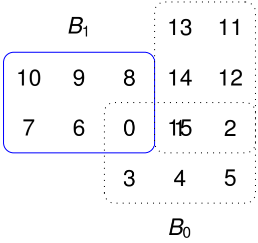

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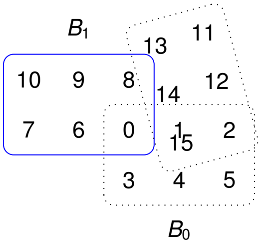

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Extension & Expansion

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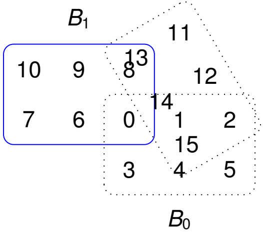

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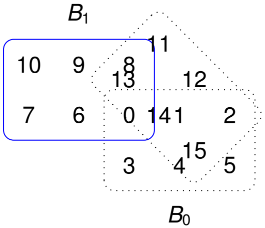

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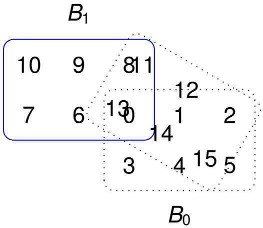

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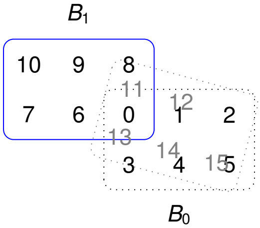

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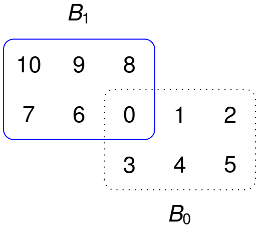

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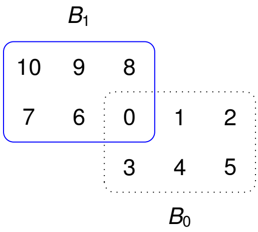

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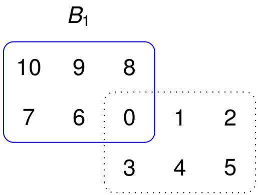

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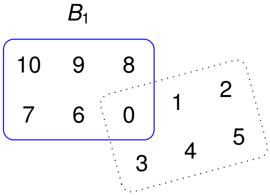

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

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B_1 = \{ & 0 & 6 & 7 & 8 & 9 & 10 & \} \\
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 & & & & & & & \\
B_2 = \{ & 11 & 12 & 2 & 13 & 14 & 15 & \}
\end{array}$$

$$ge_0 \colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\rightarrow} B_0$$

EXTENSION & EXPANSION

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

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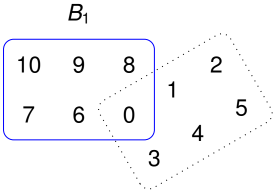

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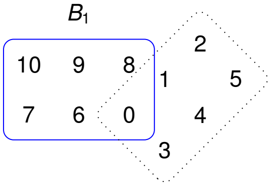

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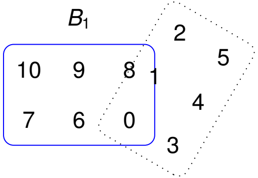

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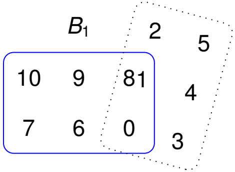

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$$\begin{array}{cccc} & B_1 & 2 & 5 \\ 10 & 9 & 8 & 4 \\ 7 & 6 & 0 & 3 \end{array}$$

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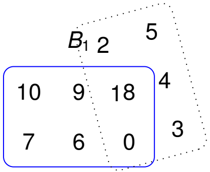

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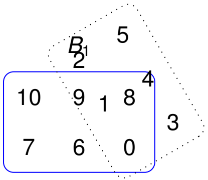

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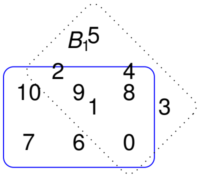

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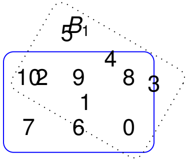

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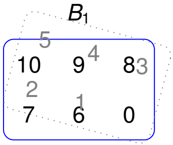

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B_1

10	9	8
7	6	0

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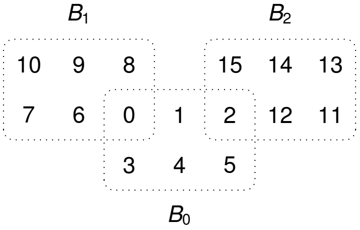

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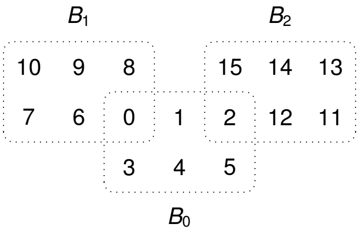

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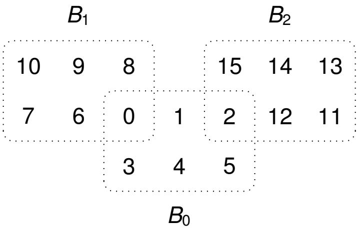

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Why does it work?

Con $\langle B, \{g_0, g_1\}\rangle$

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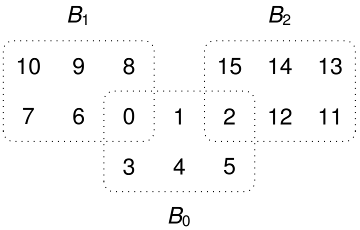

Con $\langle A, F_A\rangle$

$$\widehat{\alpha} = |0,1,2,6,7,11,12|3,4,5|8,9,10,13,14,15|$$

$$\alpha^* = |0,1,2,6,7,11,12|3,4,5|8,9,10|13,14,15|$$

$$\beta^* = |0,3,8|1,4|2,5,15|6,9|7,10|11,13|12,14|$$

$$\gamma^* = |0,4,9|1,5|2,3,13|6,10|7,8|11,14|12,15|$$

$$\delta^* = |0,5,10|1,3|2,4,14|6,8|7,9,11,15|12,13|$$

WHY DOES IT WORK?

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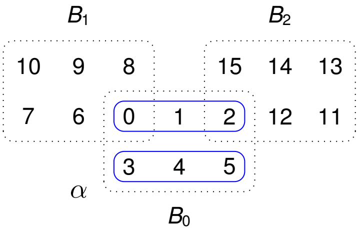

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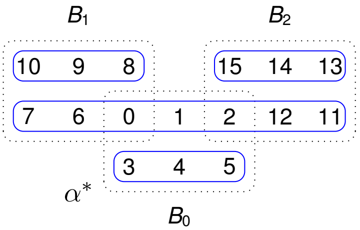

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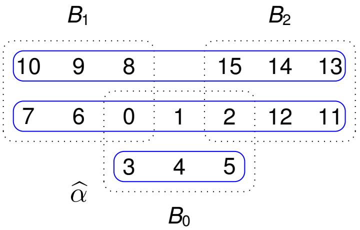

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$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$

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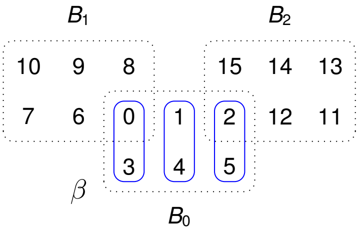

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$$\beta^* = |0,3,8|1,4|2,5,15|6,9|7,10|11,13|12,14|$$

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$$\delta^* = |0,5,10|1,3|2,4,14|6,8|7,9,11,15|12,13|$$

Why does it work?

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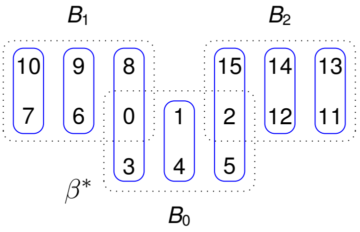

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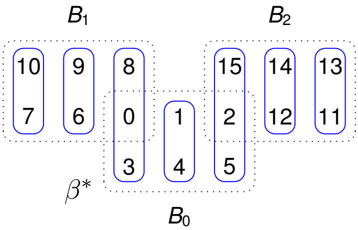

Con $\langle A, F_A\rangle$

Why don't the β classes of B_1 and B_2 mix?

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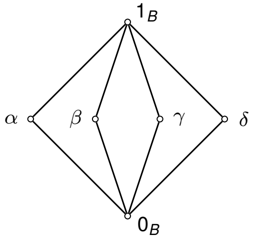

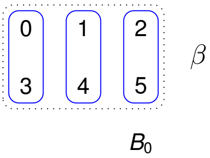

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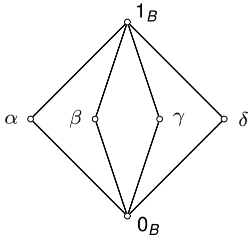

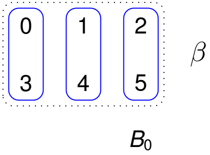

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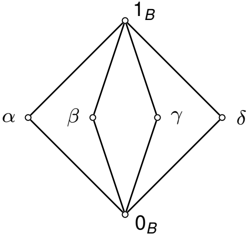

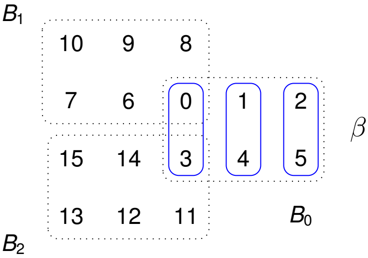

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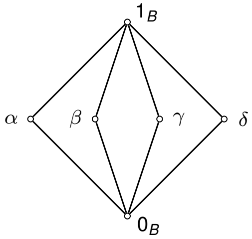

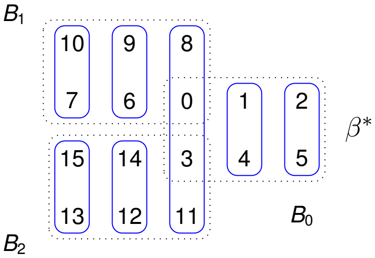

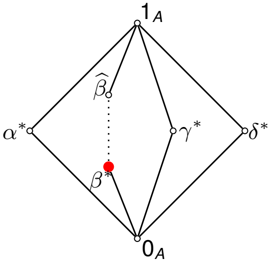

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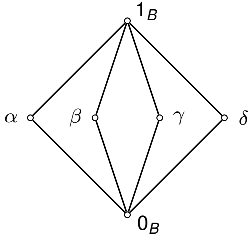

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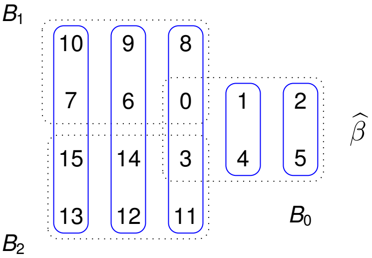

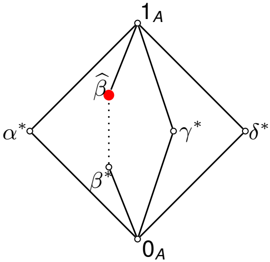

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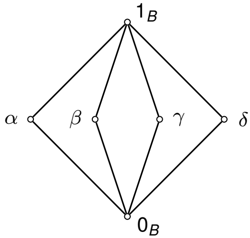

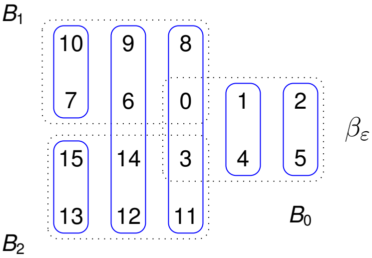

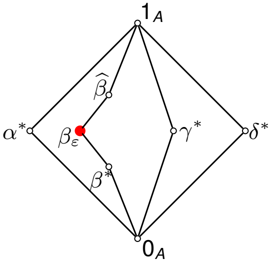

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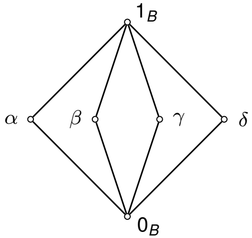

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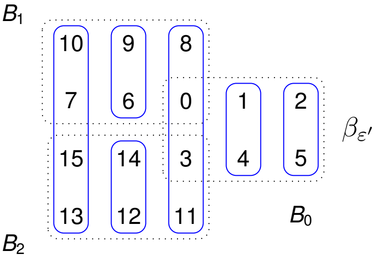

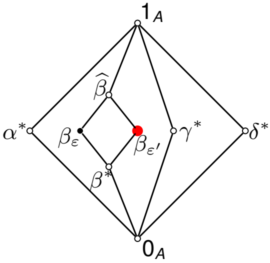

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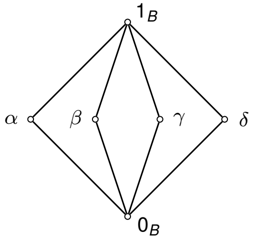

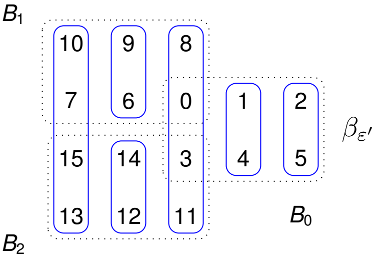

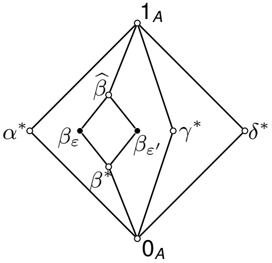

The P^5 Lemma

Lemma (Pálfy and Pudlák)

Let $\mathbf{A} = \langle A, F \rangle$ *be a unary algebra with* $e^2 = e \in F$.

Define $\mathbf{B} = \langle B, G \rangle$ *with*

$$B = e(A) \quad \textit{and} \quad G = \{ef|_B : f \in F\}.$$

Then

$$\operatorname{Con} \mathbf{A} \ni \theta \mapsto \theta \cap B^2 \in \operatorname{Con} \mathbf{B}$$

is a lattice epimorphism.

Residuation Lemma

- Define $\widehat{\ }: \mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ by

$$\widehat{\beta} = \{(x, y) \in A^2 : (ef(x), ef(y)) \in \beta \text{ for all } f \in \mathrm{Pol}_1(\mathbf{A})\}.$$

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Lemma

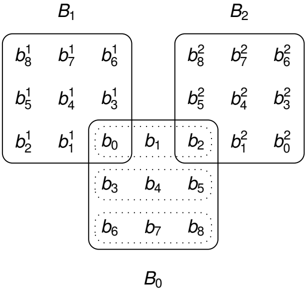
(I) $^* : \operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ *is a residuated mapping with residual* $|_B$.

(II) $|_B : \operatorname{Con}\mathbf{A} \to \operatorname{Con}\mathbf{B}$ *is a residuated mapping with residual* $\widehat{\ }$.

(III) *For all* $\alpha \in \operatorname{Con}\mathbf{A}$, $\beta \in \operatorname{Con}\mathbf{B}$,
$$\beta = \alpha|_B \quad \Leftrightarrow \quad \beta^* \leqslant \alpha \leqslant \widehat{\beta}.$$
In particular, $\beta^*|_B = \beta = \widehat{\beta}|_B$.

THE STRUCTURE OF THE INTERVAL $[\beta^*, \widehat{\beta}] \leqslant$ **Con A**.

- If $\beta \in \operatorname{Con} \mathbf{B}$ is a coatom of **Con B** with m congruence classes then the interval $[\beta^*, \widehat{\beta}]$ in **Con A** is $\mathbf{2}^{m-1}$.



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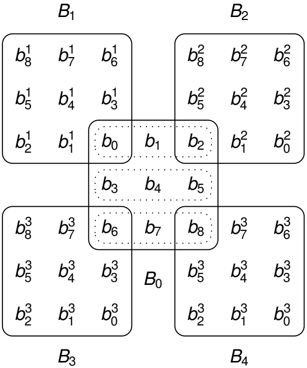

More generally...

- Suppose $\beta \in \mathrm{Con}\,\mathbf{B}$ has transversal $b_{\beta(1)}, \dots, b_{\beta(m)}$.
- Denote by T_r the set of intersection points in the r-th block of β:

$$T_r = \bigcup_{k=1}^{K} B_k \cap b_{\beta(r)}/\beta.$$

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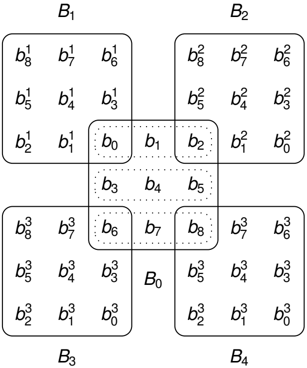

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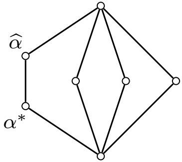
Then $\quad [\beta^*, \widehat{\beta}] = \{\theta \in \mathrm{Eq}(A) : \beta^* \subseteq \theta \subseteq \widehat{\beta}\} \cong \prod_{r=1}^{m} (\mathrm{Eq}|T_r|)^{m-1}.$

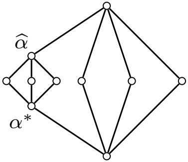
Slightly more general examples...

Returning to our original example, the base algebra **B** is the right regular S_3-set, and the nontrivial relations in $\mathrm{Con}\,\mathbf{B}$ are

$$\alpha = |0,1,2|3,4,5| \quad \beta = |0,3|1,4|2,5| \quad \gamma = |0,4|1,5|2,3| \quad \delta = |0,5|1,3|2,4|$$



$T = \{0,1\}$



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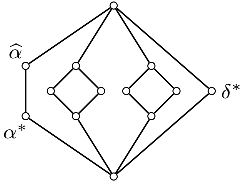
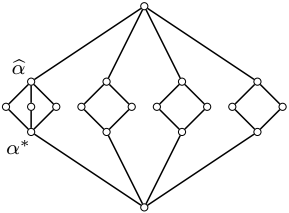


$T = \{0,2,3\}$



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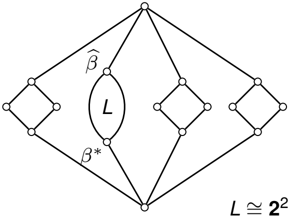

$L \cong \mathbf{2}^2 \times \mathbf{2}^2$

$T = \{0,2,3,5\}$

Limitations

Two limitations of the foregoing construction:

1. The sizes $|T_r|$ of the partition lattice factors in

$$[\beta^*, \widehat{\beta}] \cong \prod_{r=1}^{m} (\mathrm{Eq}|T_r|)^{m-1}$$

   are limited by the size of the blocks of β.

2. If β is not principal, $[\theta^*, \hat{\theta}]$ may be non-trivial for some $\theta \not\geq \beta$.

A Generalization

Theorem

Let $\mathbf{B} = \langle B, F \rangle$ *be a finite algebra. Suppose*

$$\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \dots, (a_{K-1}, b_{K-1}))$$

has m blocks and fix $N < \infty$.

There exists an overalgebra $\langle A, F_A \rangle$ *such that the interval* $\beta|_B^{-1} \leqslant \mathrm{Con}\,\mathbf{A}$ *is*

$$[\beta^*, \widehat{\beta}] \cong (\mathrm{Eq}(N))^{m-1}.$$

Moreover, we can arrange it so that $\theta^* = \widehat{\theta}$ *for all* $\theta \ngeqslant \beta$ *in* $\mathrm{Con}\,\mathbf{A}$.

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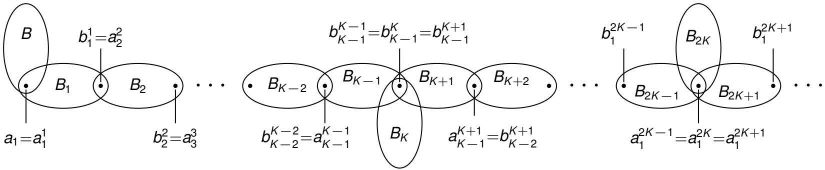

THE P^5 LEMMA

LEMMA (PÁLFY-PUDLÁK, 1980)

Let $\mathbf{A} = \langle A, F\rangle$ *be a unary algebra where* F *is a monoid.*

Suppose $e \in F$ *satisfies* $e \circ e = e$.

Define $\mathbf{B} = \langle B, G\rangle$

$$B = e(A) \quad \textit{and} \quad G = \{e f|_B \mid f \in F\}.$$

Let $|_B : Con(\mathbf{A}) \to Con(\mathbf{B})$ *be the restriction mapping:*

$$\theta|_B = \theta \cap B^2$$

Then $|_B$ *is a surjective homomorphism (even for arbitrary meets and joins).*

Péter Pál Pálfy and Pavel Pudlák: *Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups.*

Algebra Universalis **11**(1), 22–27 (1980).

http://dx.doi.org/10.1007/BF02483080

Star map and hat map

star map $^*: \operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ is the congruence generation operator restricted to the set $\operatorname{Con}\mathbf{B}$:

$$\beta^* = \operatorname{Cg}^{\mathbf{A}}(\beta) \qquad (\forall\, \beta \in \operatorname{Con}\mathbf{B})$$

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HAT MAP $\widehat{\ }: \mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ is

$$\widehat{\beta} = \{(x, y) \in A^2 \mid (ef(x), ef(y)) \in \beta \ \text{ for all } \ f \in \mathrm{Pol}_1(\mathbf{A})\}.$$

(Used by McKenzie (1982) in an alternative proof of the P^5 Lemma.)

Ralph McKenzie: *Finite forbidden lattices.*

In: Universal algebra and lattice theory (Puebla, 1982), *Lecture Notes in Math.*, vol. 1004, pp. 176–205. Springer, Berlin (1983).

`http://dx.doi.org/10.1007/BFb0063438`

Residuation lemma

A little lemma relating the three maps *, $|_B$ and $\widehat{\ }$.

Lemma

(i) $^* : \operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ *is a **residuated mapping** with **residual*** $|_B$.

(ii) $|_B : \operatorname{Con}\mathbf{A} \to \operatorname{Con}\mathbf{B}$ *is a **residuated mapping** with **residual*** $\widehat{\ }$.

(iii) *For all* $\alpha \in \operatorname{Con}\mathbf{A}$, $\beta \in \operatorname{Con}\mathbf{B}$,

$$\beta = \alpha|_B \quad \Leftrightarrow \quad \beta^* \leqslant \alpha \leqslant \widehat{\beta}.$$

In particular, $\beta^*|_B = \beta = \widehat{\beta}|_B$.

Adjunction lemma

New version (of the little lemma):

$$ {}^{*} \quad \dashv \quad |_{B} \quad \dashv \quad \widehat{\ } $$

Lemma

(i) $^{*} : \mathrm{Con}\,\mathbf{B} \to \mathrm{Con}\,\mathbf{A}$ *is* ***left adjoint*** *to* $|_{B}$.

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In particular, $\beta^{*}|_{B} = \beta = \widehat{\beta}|_{B}$.

Proof of the P^5 Lemma

Lemma (Pálfy-Pudlák, 1980)

The restriction mapping

$$\operatorname{Con} \mathbf{A} \ni \alpha \mapsto \alpha|_B = \alpha \cap B^2 \in \operatorname{Con} \mathbf{B}$$

is a complete lattice epimorphism.

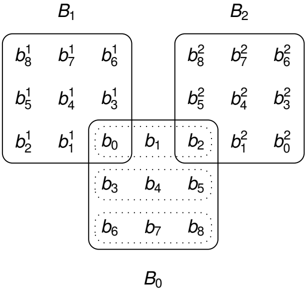
Proof.

Recall, for $f : X \to Y$ a monotone function on preorders X, Y, if f has a right (left) adjoint, then f preserves all joins (meets) existing in X.

By the little lemma $|_B$ has both a left and right adjoint. $\square$

THE STRUCTURE OF THE INTERVAL $[\beta^*, \widehat{\beta}] \leqslant$ **Con A**.

- If $\beta \in \mathrm{Con}\,\mathbf{B}$ is a coatom of **Con B** with m congruence classes then the interval $[\beta^*, \widehat{\beta}]$ in **Con A** is $\mathbf{2}^{m-1}$.



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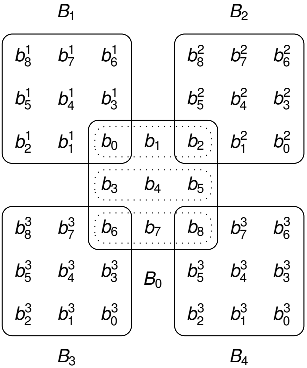

More generally...

- Suppose $\beta \in \mathrm{Con}\,\mathbf{B}$ has transversal $b_{\beta(1)}, \dots, b_{\beta(m)}$.
- Denote by T_r the set of intersection points in the r-th block of β:

$$T_r = \bigcup_{k=1}^{K} B_k \cap b_{\beta(r)}/\beta.$$

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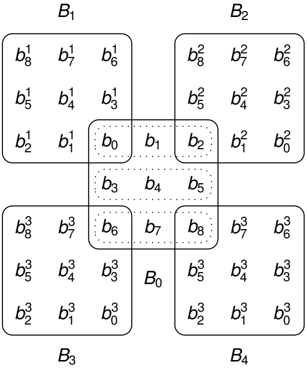

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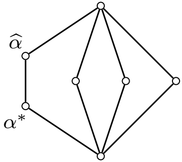
Then $\quad [\beta^*, \widehat{\beta}] = \{\theta \in \mathrm{Eq}(A) : \beta^* \subseteq \theta \subseteq \widehat{\beta}\} \cong \prod_{r=1}^{m} (\mathrm{Eq}|T_r|)^{m-1}.$

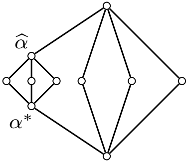
SLIGHTLY MORE GENERAL EXAMPLES...

Returning to our original example, the base algebra **B** is the right regular S_3-set, and the nontrivial relations in $\operatorname{Con} \mathbf{B}$ are

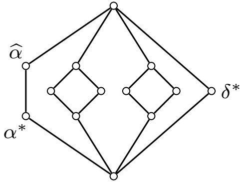
$$\alpha = |0,1,2|3,4,5| \quad \beta = |0,3|1,4|2,5| \quad \gamma = |0,4|1,5|2,3| \quad \delta = |0,5|1,3|2,4|$$



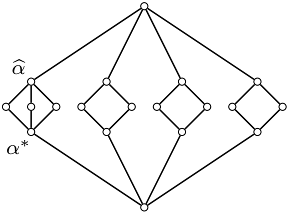
$T = \{0, 1\}$



$T = \{0, 1, 2\}$



$T = \{0, 2, 3\}$



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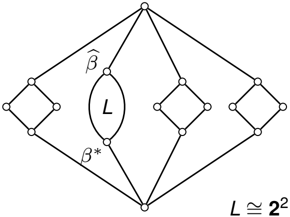

$L \cong \mathbf{2}^2 \times \mathbf{2}^2$

$T = \{0, 2, 3, 5\}$

Limitations

Two limitations of the foregoing construction:

1. The sizes $|T_r|$ of the partition lattice factors in

$$[\beta^*, \widehat{\beta}] \cong \prod_{r=1}^{m} (\mathrm{Eq}|T_r|)^{m-1}$$

are limited by the size of the blocks of β.

2. If β is not principal, $[\theta^*, \hat{\theta}]$ may be non-trivial for some $\theta \ngeq \beta$.

A Generalization

Theorem

Let $\mathbf{B} = \langle B, F \rangle$ *be a finite algebra. Suppose*

$$\beta = \mathrm{Cg}^{\mathbf{B}}((a_1, b_1), \ldots, (a_{K-1}, b_{K-1}))$$

has m blocks and fix $N < \infty$*.*

There exists an overalgebra $\langle A, F_A \rangle$ *such that the interval* $\beta|_B^{-1} \leqslant \mathrm{Con}\,\mathbf{A}$ *is*

$$[\beta^*, \widehat{\beta}] \cong (\mathrm{Eq}(N))^{m-1}.$$

Moreover, we can arrange it so that $\theta^* = \widehat{\theta}$ *for all* $\theta \ngeqslant \beta$ *in* $\mathrm{Con}\,\mathbf{A}$*.*

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