CSPs of Finite Commutative Idempotent Binars

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joint work with

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CONSTRAINT SATISFACTION PROBLEMS

Input

- *variables:* $V = \{v_1, v_2, \ldots\}$
- *domain:* D
- *contraints:* $C_1, C_2, \ldots$

Output

- "yes" if there is a *solution*

  $\sigma : V \to D$ (an assignment of values to variables that satisfies all C_i)

- "no" otherwise

Input

- *variables:* $V = \{v_1, \dots, v_n\}$
- *domain:* $D = \{0, 1\}$
- *constraints:* a formula, say,

$$f(v_1, \dots, v_n) = (v_1 \vee v_2 \vee \neg v_3) \wedge (\neg v_1 \vee v_3 \vee v_4) \wedge \cdots$$

Output

- "yes" if there is a solution: $\sigma : V \to D$ such that

$$f(\sigma v_1, \dots, \sigma v_n) = 1$$

- "no" otherwise

Constraint Satisfaction Problems

Example: NAE-SAT

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- *variables:* $V = \{v_1, \ldots, v_n\}$
- *domain:* $D = \{0, 1\}$
- *constraints:* $(s_1, C_1), (s_2, C_2), \ldots$ of the form

$$s = (i, j, k) \in \{1, \ldots, n\}^3 \qquad \text{(scopes)}$$
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In terms of relational structures...

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Solutions are homomorphisms!

$$\sigma : \langle V, S\rangle \to \langle D, R\rangle$$

CSP: RELATIONAL FORMULATION

Let $\mathbb{D} = \langle D, \mathcal{R} \rangle$ be a relational structure.

$\mathrm{CSP}(\mathbb{D})$ is the decision problem with

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- A structure $\mathbb{V} = \langle V, \mathcal{C} \rangle$ *similar* to $\mathbb{D}$.

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Alternatively, let $\Rightarrow$ be the binary relation on similar structures

$$\mathbb{V} \Rightarrow \mathbb{D} \quad \text{iff there is a homomorphism } \sigma : \mathbb{V} \to \mathbb{D}$$

Then the CSP is the membership problem for the set

$$\mathrm{CSP}(\mathbb{D}) := \{\mathbb{V} : \mathbb{V} \Rightarrow \mathbb{D}\}$$

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We call $\mathbb{D}$ "tractable" if there is a polynomial-time algorithm for $\mathrm{CSP}(\mathbb{D})$.

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that is, $f \in \mathrm{pol}(R)$ iff for every $\rho \in R$ (say, n-ary)

$$\begin{array}{rcc}
(a_1, b_1, \ldots, z_1) & \in & \rho \\
 & \vdots & \\
(a_k, b_k, \ldots, z_k) & \in & \rho \\
\hline
(f(a_1, \ldots, a_k), \ldots, f(z_1, \ldots, z_k)) & \in & \rho
\end{array}$$

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$$\textbf{Theorem:} \quad \text{CSP}\langle D, \bar{R} \rangle \leqslant_P \text{CSP}\langle D, R \rangle$$