An Open Question About Finite Algebras

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The Problem

A **finite algebra** is an utterly fundamental object in mathematics. The shape of an algebra's **congruence lattice** gives us vital information about the algebra.

However, we still don't know whether there are any restrictions on the possible shapes of congruence lattices. Mathematicians have been on a quest for over 50 years to either find such restrictions or prove that none exist.

Contribution

We have identified large classes of lattices which occur as congruence lattices of finite algebras, and have found ways to build new "representable" lattices out of old ones. Thus, we have made significant progress toward a possible answer to this basic question:

"Is there a finite lattice which cannot be the congruence lattice of a finite algebra?"

What is an algebra?

An **algebra** $\langle A; F\rangle$ is a set A together with a collection F of operations on the set.

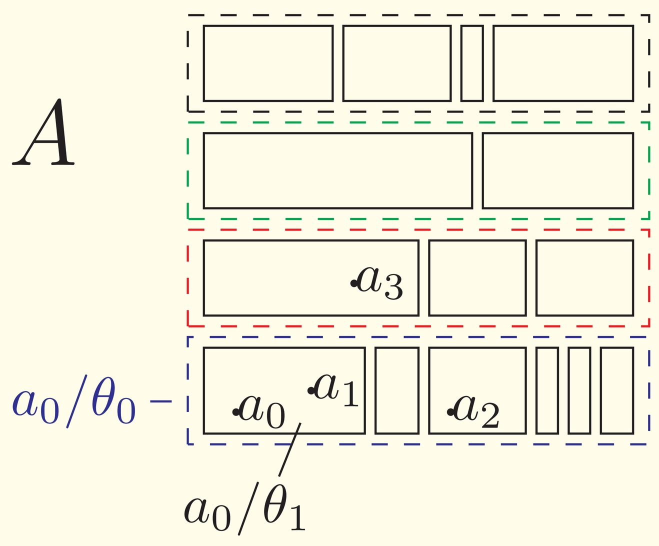
Example: The set of integers

$$A = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

along with the operations $F = \{+, -, \times\}$.
(Note: $\div$ is not an operation on the set A.)

The basic structure of an algebra

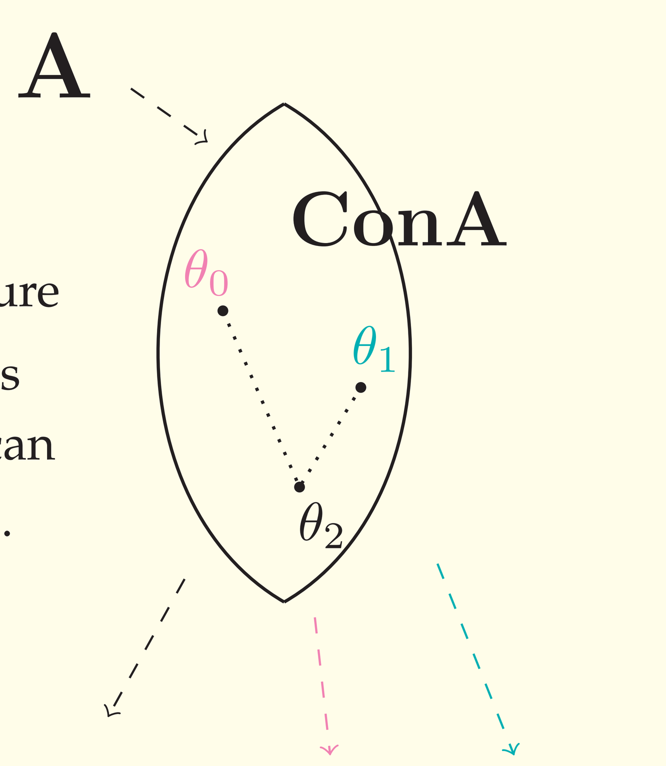
A **congruence** of an algebra $\mathbf{A} = \langle A; F\rangle$ is a partition of the set A into blocks that are preserved by the operations in F.



The set of all congruences of an algebra forms the **congruence lattice** of the algebra, denoted **ConA**. Insight into the structure of **A** is gained by knowing the shape of **ConA**.

Decompositions of algebras

The shape/structure of **ConA** tells us how the algebra can be decomposed.



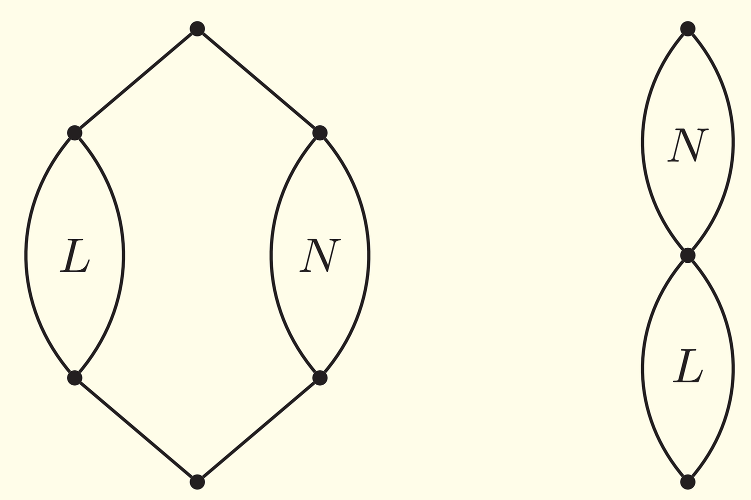
$$\mathbf{A}/\theta_2 \hookrightarrow \mathbf{A}/\theta_0 \times \mathbf{A}/\theta_1$$

Prior Work I

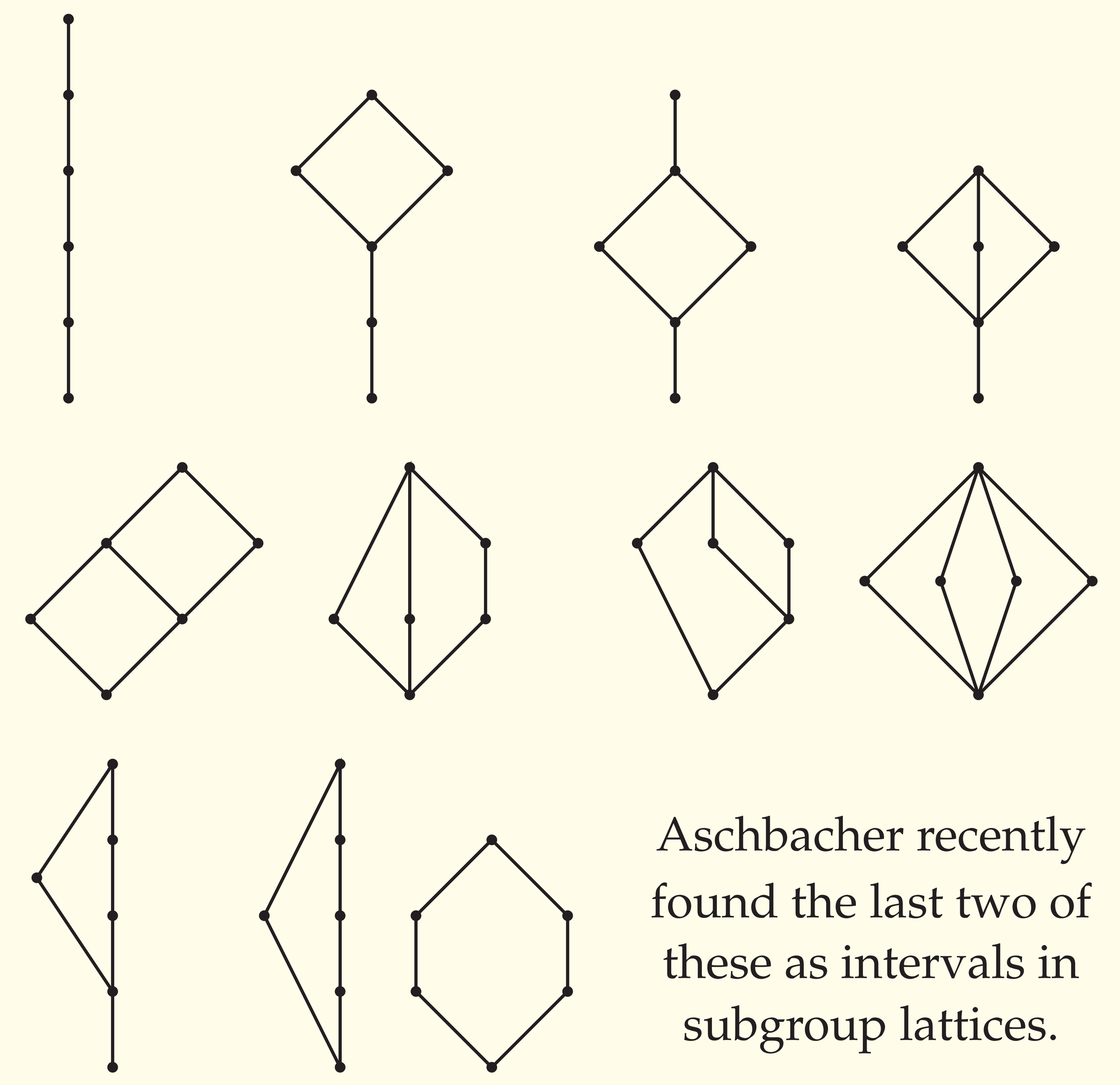
Grätzer & Schmidt [1] proved that there is no restriction on the shape of a congruence lattice of an *infinite* algebra. So, if (as most suspect) there is a lattice which is not the congruence lattice of a *finite* algebra, this will reveal an important distinction between finite vs. infinite.

Kurzweil [2] proved that if L is a congruence lattice of a finite algebra, then so is the *dual* of L (i.e. L "up-side-down").

John Snow [3] proved that if L and N are congruence lattices of finite algebras, so are



Prior Work II



Aschbacher recently found the last two of these as intervals in subgroup lattices.

Theorem: *Every lattice with at most 6 elements is a congruence lattice of a finite algebra.*

Results

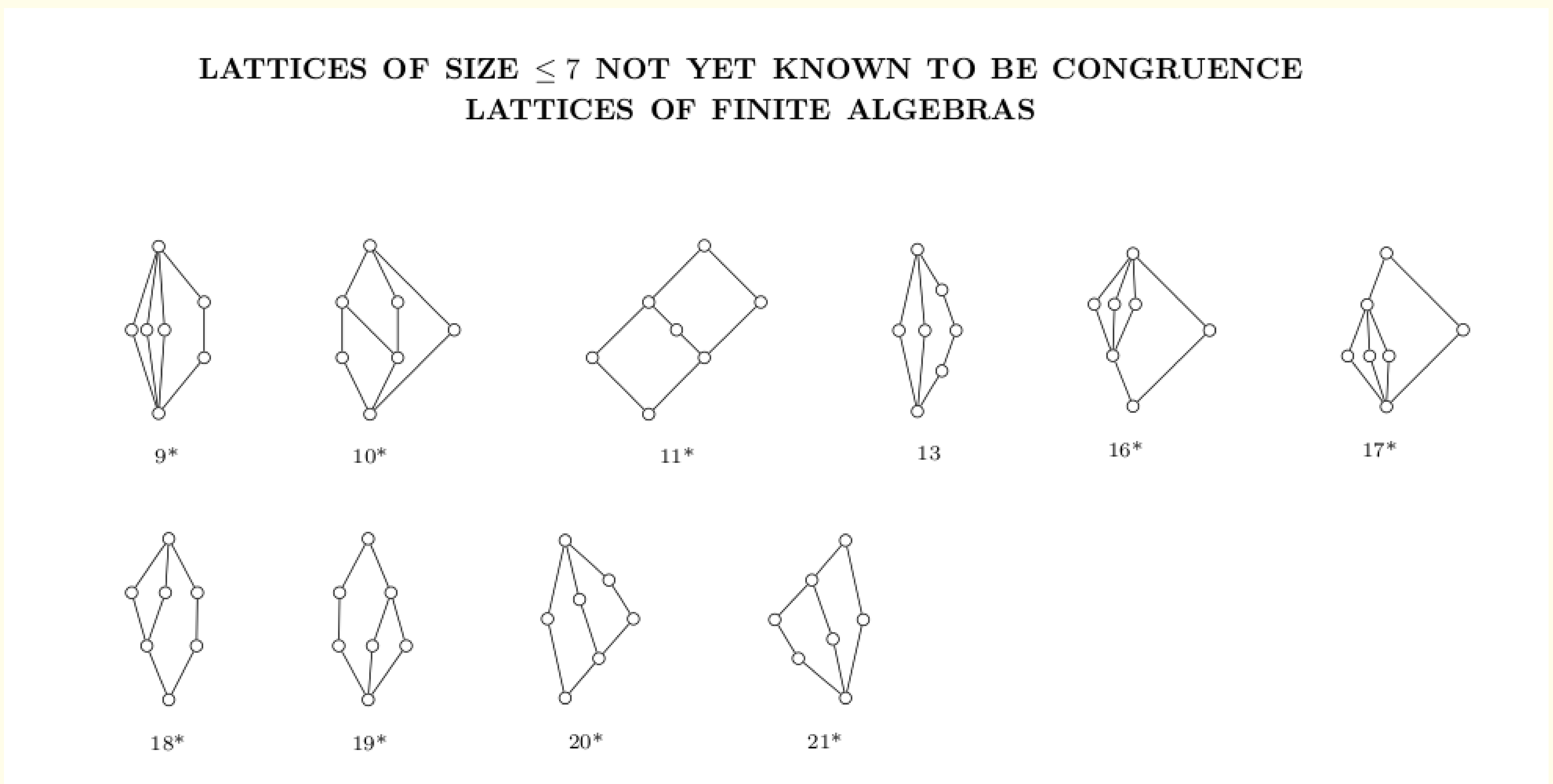
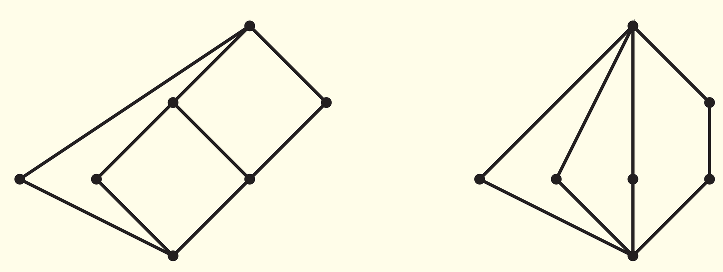


Figure courtesy of Peter Jipsen.

Claim: *Every lattice with at most 7 elements is a congruence lattice of a finite algebra.*

There are 53 lattices with 7 elements. We have found representations for all but these two:

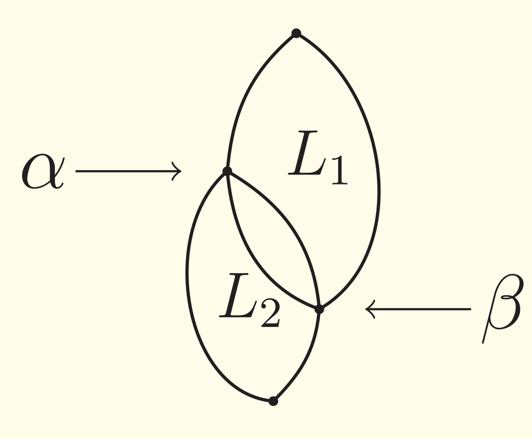


A Galois Correspondence

For any finite lattice L, there is a finite set A such that L can be embedded into the lattices of all partitions of A. We can then consider the set F of all operations that respect the partitions of L. If we're lucky, L is the congruence lattice of the resulting algebra $\langle A; F\rangle$.

Nondensity Lemma: *If L is a union of a filter and an ideal, there will always be nontrivial operations in F.*

Gluing Lemma: *Fix two finite congruence lattices L_1, L_2 with $\alpha \in L_1$ and $\beta \in L_2$. Under certain conditions, we can prove that the following is also a congruence lattice:*



References

[1] G. Grätzer and E. Schmidt. Characterizations of congruence lattices of abstract algebras. *Acta Sci. Math. (Szeged)*, 24:34–59, 1963.

[2] H. Kurzweil. Endliche gruppen mit vielen untergruppen. *J. refine angew. Math.*, 356:140–160, 1985.

[3] J. W. Snow. A constructive approach to the finite congruence lattice representation problem. *Algebra Universalis*, 43:279–293, 2000.