THE RECTANGULARITY THEOREM
OF LIBOR BARTO AND MARCIN KOZIK
WITH APPLICATIONS TO SMALL CIBS

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joint work with

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Iowa State University

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Input

- *variables:* $\mathcal{V} = \{v_1, v_2, \dots\}$
- *domain:* $\mathcal{D}$
- *constraints:* $C_1, C_2, \dots$

Output

- "yes" if there is a *solution*

    $f : \mathcal{V} \to \mathcal{D}$ (an assignment of values to variables that satisfies all C_i)

- "no" otherwise

DEFINITION OF CSP

$\mathbf{A} = \langle A, \mathcal{F} \rangle$ is a finite idempotent algebra, Sub($\mathbf{A}$) is all subuniverses of $\mathbf{A}$.

In this talk CSP($\mathbf{A}$) denotes the following decision problem:

An *instance of degree* n of CSP($\mathbf{A}$) is the tuple $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$
- *variables* $\mathcal{V} = \{0, 1, \ldots, n-1\}$;
- *domains* $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}\} \subset$ Sub($\mathbf{A}$) (one for each variable)
- *scope functions* $\mathcal{S} = (\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{p-1})$ with *constraint arities* $\text{ar}(\mathcal{S}) = (m_0, m_1, \ldots, m_{p-1})$;
- *constraint relations* $\mathcal{R} = (\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{p-1})$, where

$$\mathbf{R}_i \leqslant \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}.$$

A *solution* to $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$ is an assignment $f : \mathcal{V} \to A$ of values to variables that satisfies all constraints. That is,

$$f \in \Pi_{\mathcal{V}} A_j \quad \text{and} \quad \text{Proj}_{\mathbf{s}_i} f \in \mathbf{R}_i, \text{ for each } 0 \leqslant i < p.$$

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Notation: $\underline{n} = \{0, 1, \ldots, n-1\}$, so the i-th scope has type $\mathbf{s}_i : \underline{m_i} \to \underline{n}$ and

$$\mathrm{Proj}_{\mathbf{s}_i} f = f \circ \mathbf{s}_i$$

EXAMPLE 1
...THANKS, ROSS!

Let $A = \langle \{0, 1\}, \{f\} \rangle$, where

$$f(x, y, z) = x + y + z \pmod{2}.$$

Consider the ternary relations

$$R_0 = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$$

$$R_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$$

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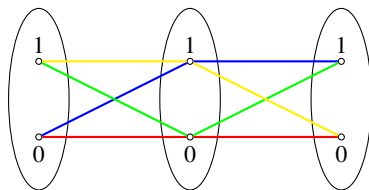
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So we have a degree 3 instance of $\mathrm{CSP}(\mathbf{A})$, where
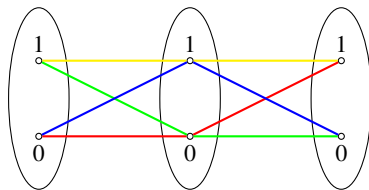
- variables: $\mathcal{V} = \{0, 1, 2\}$
- domains: $A_i = \{0, 1\}, \quad i = 0, 1, 2$
- scope functions: the identity on $\{0, 1, 2\}$
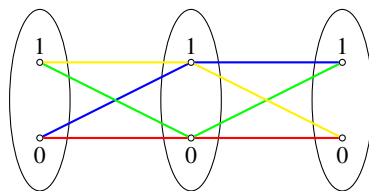- constraint relations: $\mathbf{R}_0$ and $\mathbf{R}_1$

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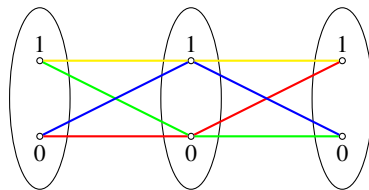
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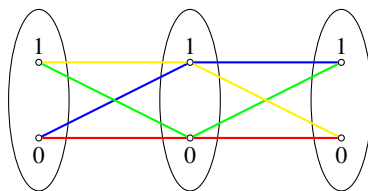∩ AND POTATOES

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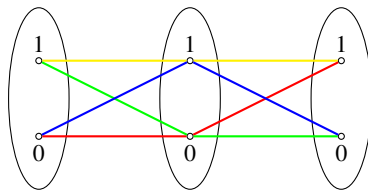Notice for all $i, j \in \{0, 1, 2\}$,

$$\text{Proj}_{ij} R_0 = \text{Proj}_{ij} R_1$$

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Notice for all $i, j \in \{0, 1, 2\}$,

$$\operatorname{Proj}_{ij} R_0 = \operatorname{Proj}_{ij} R_1$$

...yet $R_0 \cap R_1 = \emptyset$.

EXAMPLE 2
...THANKS, CLIFF!

Let $\mathbf{A} = \langle\{0,1\},\{m\}\rangle$, where $m : A^3 \to A$ is a majority operation,

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x.$$

Let $\mathbf{R}_0, \mathbf{R}_1 \leqslant_{sd} \mathbf{A}^3$ with universes

$$R_0 = \{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\},$$
$$R_1 = \{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}.$$

This describes the instance of $CSP(\mathbf{A})$ with

- variables: $\mathcal{V} = \{0,1,2\}$
- domains: $A_i = \{0,1\}, \quad i = 0,1,2$
- scope functions: the identity on $\{0,1,2\}$
- constraint relations: $\mathbf{R}_0$ and $\mathbf{R}_1$

Retrict attention to instances where all constraint relation are subdirect,

$$\mathbf{R}_i \leqslant_{\mathrm{sd}} \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}$$

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(thanks again, Ross!)

Convenient because now solutions are the elements in $\bigcap_{i \in V} [\![\mathbf{s}_i, \mathbf{R}_i]\!]$.

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BUT input size is not a function of these "full" subdirect products!

(Input size could be defined as the length of a string of all tuples in scopes and constraint relations of the instance.)

pause...

...draw more potatoes...

...give audience chance to escape.

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be a finite algebra in a Taylor variety.

Let $t \in \mathrm{Clo}(\mathbf{A})$ be a k-ary term operation.

A subalgebra $\mathbf{B} \leqslant \mathbf{A}$ is *absorbing in* $\mathbf{A}$ *with respect to* t if

$$a \in A, \; b_i \in B \quad \Longrightarrow \quad t^{\mathbf{A}}(b_0, \ldots, b_{j-1}, a, b_{j+1}, \ldots, b_{k-1}) \in B \quad (\text{all } j \in \underline{k})$$

ABSORPTION THEORY (FOR MORTALS)

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Equivalently, $t^{\mathbf{A}}[B^{j-1} \times A \times B^{k-j}] \subseteq B$, for all $0 \leqslant j < k$, that is,

$$(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B^{j-1} \times A \times B^{k-j} \quad \Longrightarrow \quad t^{\mathbf{A}}(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B.$$

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Notation:

$\mathbf{B} \lhd \mathbf{A}$ means $\mathbf{B}$ is absorbing in $\mathbf{A}$ with respect to some term.

To be explicit about the term, $\mathbf{B} \lhd_t \mathbf{A}$.

$\mathbf{B} \lhd\lhd \mathbf{A}$ means $\mathbf{B} \lhd \mathbf{A}$ and B is minimal (with respect to inclusion) among absorbing subuniverses of $\mathbf{A}$.

An algebra is *absorption-free* (AF) if it has no proper absorbing subalgebras.

"The Absorption Theorem" of Barto and Kozik (LMCS 2012)

Concerns the special class of "linked" subdirect products.

Identifies some special cases in which a subdirect product is the full product!

WHERE ARE WE GOING WITH THIS?

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THEOREM (ABSORPTION THEOREM)

If V is an idempotent locally finite variety, then TFAE

- V is a Taylor variety;
- if $\mathbf{A}_0, \mathbf{A}_1 \in V$ are finite idempotent absorption-free algebras and $\mathbf{R} \leqslant_{sd} \mathbf{A}_0 \times \mathbf{A}_1$ is linked, then $\mathbf{R} = \mathbf{A}_0 \times \mathbf{A}_1$.

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At Vanderbilt Shanks Workshop (2015), Barto presented more joint work with Kozik generalizing the Absorption Theorem to more than two factors.

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[*]assuming suitable conditions under which the theorem is true.

LINKED SUBDIRECT PRODUCTS

A subdirect product $\mathbf{R} \leqslant_{sd} \mathbf{A}_0 \times \mathbf{A}_1$ is linked if for all $a, a' \in \mathrm{Proj}_0 R$,

$$\exists\, c_0, c_2, \ldots, c_{2n} \in A_0, \quad \exists\, c_1, c_3, \ldots, c_{2n+1} \in A_1$$

such that

$$a = c_0, \quad (c_{2i}, c_{2i+1}) \in R, \quad (c_{2i+2}, c_{2i+1}) \in R, \quad c_{2n} = a'$$

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[todo: insert potato diagram]

Notation:

For $\mathbf{R} \leqslant_{sd} \mathbf{A}_0 \times \mathbf{A}_1$, let η_i denote the kernel of the i-th projection of $\mathbf{R}$. That is,

$$\eta_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid \mathrm{Proj}_i \, \mathbf{r} = \mathrm{Proj}_i \, \mathbf{r}'\}$$

Let $R^{-1} = \{(y, x) \in A_1 \times A_0 \mid (x, y) \in R\}$.

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The following are equivalent:

- $\mathbf{R} \leqslant_{sd} \mathbf{A}_0 \times \mathbf{A}_1$ is linked;
- $\eta_0 \vee \eta_1 = 1_R$;
- if $a, a' \in \mathrm{Proj}_0 \, R$, then (a, a') is in the transitive closure of $R \circ R^{-1}$.

Absorption has nice properties...

- (transitivity) $\mathbf{C} \lhd \mathbf{B} \lhd \mathbf{A} \implies \mathbf{C} \lhd \mathbf{A}$
- (closure under nonempty $\cap$ and finite products)

  If $\mathbf{B} \lhd_f \mathbf{A}$ and $\mathbf{C} \lhd_g \mathbf{A}$ and $B \cap C \neq \emptyset$, then $\mathbf{B} \cap \mathbf{C} \lhd \mathbf{A}$.

  If $\mathbf{B}_0 \lhd_f \mathbf{A}_0$ and $\mathbf{B}_1 \lhd_g \mathbf{A}_1$, then $\mathbf{B}_0 \times \mathbf{B}_1 \lhd_t \mathbf{A}_0 \times \mathbf{A}_1$.

  ...with respect to $t = f \star g$ in both cases.

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If $f : A^\ell \to A$ and $g : A^m \to A$, then $f \star g$ is the ℓm-ary operation

$$f(g(a_{11}, \ldots, a_{1m}), g(a_{21}, \ldots, a_{2m}), \ldots, g(a_{\ell 1}, \ldots, a_{\ell m}))$$

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More generally, if $\mathbf{B}_i \lhd_{t_i} \mathbf{A}_i$ for $0 \leqslant i < n$, then $\Pi\mathbf{B}_i \lhd_s \Pi\mathbf{A}_i$.
...with respect to $s = t_0 \star t_1 \star \cdots \star t_{n-1}$.

An obvious but important consequence:

  A finite product of finite idempotent algebras is AF if each factor is AF.

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Restriction Lemma.
If $\mathbf{B} \lhd_t \mathbf{A}$ and $\mathbf{C} \leqslant \mathbf{A}$ and $D = B \cap C \neq \emptyset$, then $\mathbf{D} \lhd \mathbf{C}$ with respect to the restriction of t to C.

LEMMA (LSD 1)

If $\mathbf{B}_i \lhd \mathbf{A}_i$ and $\mathbf{R} \leqslant \Pi_i \mathbf{A}_i$ and $R' := R \cap \Pi_i B_i \neq \emptyset$, then $\mathbf{R}' \lhd \mathbf{R}$.

Proof. $\Pi \mathbf{B}_i \lhd_t \Pi \mathbf{A}_i$, so follows Restriction Lemma if we put $C = R$.

LEMMA (LSD 2)

Suppose $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$ and $\mathbf{R} \leqslant_{sd} \Pi \mathbf{A}_i$. If $R' := R \cap \Pi B_i \neq \emptyset$, then $\mathbf{R}' \leqslant_{sd} \Pi \mathbf{B}_i$.

LEMMA (LSD 2)

If $\mathbf{R} \leqslant_{sd} \mathbf{A}_0 \times \mathbf{A}_1$ is linked and $\mathbf{S} \lhd \mathbf{R}$, then $\mathbf{S}$ is linked.

In some simple cases we get linking from LSD Lemmas along with the following elementary

Fact. Suppose $\mathbf{R} \leqslant_{sd} \mathbf{A}_0 \times \mathbf{A}_1$ and let $\eta_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i)$.

1. If $\mathbf{A}_0$ is simple, then either $\eta_0 \vee \eta_1 = 1_R$ or $\eta_0 \geqslant \eta_1$.
2. If $\mathbf{A}_0$ and $\mathbf{A}_1$ are both simple, then either $\eta_0 \vee \eta_1 = 1_R$ or $\eta_0 = 0_R = \eta_1$.

...so, if both factors are simple, then $\eta_0 \neq \eta_1$ gives the linking...

Cor 1. Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be simple. If $\mathbf{R} \leqslant_{sd} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$, then $\mathbf{R}$ is linked.

...and if one factor is simple nonabelian and the other abelian, linking is free!

Cor 2. If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ abelian, then every subdirect product of $\mathbf{A}_0 \times \mathbf{A}_1$ is linked.

Suppose we add to the respective contexts of the last three results the hypothesis that the algebras live in an idempotent variety with a Taylor term...

(We will refer to such varieties as "Taylor varieties" and we call the algebras they contain "Taylor algebras.")

...then the Absorption Theorem (in combination with facts above) yields

Lemma: Let A_0 and A_1 be finite Taylor algebras with $B_i \lhdlhd A_i$ ($i = 0, 1$) and suppose $R \leqslant_{sd} A_0 \times A_1$ and $\eta_0 \neq \eta_1$.

(I) If A_0 and A_1 are simple and $R \cap (B_0 \times B_1) \neq \emptyset$, then $B_0 \times B_1 \leqslant R$.

(II) If A_0 is simple nonabelian and A_1 is abelian, then $B_0 \times A_1 \leqslant R$.

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The Rectangularity Theorem.

Let $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ be finite algebras in a Taylor variety, $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$, and

- at most one $\mathbf{A}_i$ is abelian,
- all nonabelian factors are simple,
- $\mathbf{R} \leqslant_{sd} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$.
- $\mathbf{R}' = \mathbf{R} \cap (\mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1})$ is nonempty.

$$\text{Then } \mathbf{R}' = \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}.$$

Let $\underline{n} = \{0, 1, 2, \ldots, n-1\}$.

Let $\sigma' = \underline{n} - \sigma$, when σ is a subset of $\underline{n}$.

For $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi_{\underline{n}} \mathbf{A}_i$ let

$$\eta_\sigma = \ker(R \twoheadrightarrow \Pi_\sigma A_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid \mathrm{Proj}_\sigma \mathbf{r} = \mathrm{Proj}_\sigma \mathbf{r}'\},$$

If $\sigma \subseteq \underline{n}$, then by $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi_\sigma \mathbf{A}_i \times \Pi_{\sigma'} \mathbf{A}_i$ we mean

$$\mathbf{R} \leqslant \Pi_{\underline{n}} \mathbf{A}_i, \quad \mathrm{Proj}_\sigma \mathbf{R} = \Pi_\sigma \mathbf{A}_i, \quad \text{and} \quad \mathrm{Proj}_{\sigma'} \mathbf{R} = \Pi_{\sigma'} \mathbf{A}_i.$$

and we say that $\mathbf{R}$ is a *subdirect product of* $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$ in this case.

The subdirect product $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi_\sigma \mathbf{A}_i \times \Pi_{\sigma'} \mathbf{A}_i$ is said to be *linked* if $\eta_\sigma \vee \eta_{\sigma'} = 1_R$.

We may use $\mathbf{R}_\sigma$ for $\mathrm{Proj}_\sigma \mathbf{R}$, the projection of $\mathbf{R}$ onto coordinates in σ.

From now on, *all algebras are finite and belong to the same Taylor variety*.

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Lemma 1.

Let $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$ for each $i \in \underline{n}$, and let $\underline{n} = \sigma \cup \sigma'$ be a disjoint union.

Assume $\mathbf{R}$ is a *linked* subdirect product of $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$.

Suppose $R' = R \cap \Pi_i B_i \neq \emptyset$. Then $\mathbf{R}' = \Pi_i \mathbf{B}_i$.

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Lemma 2. [Kearnes-Kiss, Thm 3.27]

Suppose α and β are congruences of a Taylor algebra. Then

$$\mathsf{C}(\alpha, \alpha; \alpha \wedge \beta) \quad \Longleftrightarrow \quad \mathsf{C}(\alpha \vee \beta, \alpha \vee \beta; \beta).$$

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Lemma 3. [Linking Lemma]
Let $n \geqslant 2$ and $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$ for all $i \in \underline{n}$. Suppose

- at most one $\mathbf{A}_i$ is abelian
- all nonabelian factors are simple
- $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$.

Then there exists k such that $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_k \times \mathbf{R}_{k'}$ is linked.

RECTANGULARITY THEOREM
PROOF SKETCH

The Theorem. Assume $\mathbf{A}_i$ are finite Taylor algebras with $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$, and

- at most one $\mathbf{A}_i$ abelian, all nonabelian factors simple,
- $\mathbf{R} \leqslant_{sd} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$, with $\eta_i \neq \eta_j$ for $i \neq j$,
- $\mathbf{R}' = \mathbf{R} \cap (\mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1})$ nonempty.

    Then $\mathbf{R}' = \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}$.

Proof sketch.

Induct on the number of factors in the product $\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$.

For $n = 2$ the result holds by an earlier Lemma (slide 16).

Fix $n > 2$ and assume for all $2 \leqslant k < n$ the result holds for k factors. We prove it for subdirect products of n factors.

Fix $\emptyset \subsetneq \sigma \subsetneq \underline{n}$.

Then $\mathbf{R}_\sigma = \mathrm{Proj}_\sigma \mathbf{R}$ and $\mathbf{R}_{\sigma'} = \mathrm{Proj}_{\sigma'} \mathbf{R}$ satisfy assumptions of RT.

Induction hypothesis implies $\Pi_\sigma \mathbf{B}_i \leqslant \mathbf{R}_\sigma$ and $\Pi_{\sigma'} \mathbf{B}_i \leqslant \mathbf{R}_{\sigma'}$.

A few more easy steps gives, for all $\emptyset \subsetneq \sigma \subsetneq \underline{n}$,

$$\mathbf{R} \leqslant_{sd} \mathbf{R}_\sigma \times \mathbf{R}_{\sigma'}, \quad \Pi_\sigma \mathbf{B}_i \lhd\lhd \mathbf{R}_\sigma, \quad \Pi_{\sigma'} \mathbf{B}_i \lhd\lhd \mathbf{R}_{\sigma'}.$$

By Linking Lemma and Absorption Theorem, the proof is complete.

What if there is more than one abelian factor?

Cor. 1 Let $\mathbf{A}_i$ be finite Taylor algebras with $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$ ($i \in \underline{n}$).
Let $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$ ($i \in \underline{n}$) and $\alpha \subseteq \underline{n}$. Suppose

- $\mathbf{A}_i$ is abelian for each $i \in \alpha$,
- $\mathbf{A}_i$ is nonabelian and simple for each $i \in \alpha'$,
- $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$,
- $R' := R \cap (B_0 \times B_1 \times \cdots \times B_{n-1}) \neq \emptyset$.

Then $\mathbf{R}' = \mathbf{R}_\alpha \times \Pi_{\alpha'} \mathbf{B}_i$.

Proof.
Suppose $\alpha' = \{i_0, i_1, \ldots, i_{m-1}\}$. Clearly, $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{R}_\alpha \times \mathbf{A}_{i_0} \times \mathbf{A}_{i_1} \times \cdots \times \mathbf{A}_{i_{m-1}}$. If $\alpha \neq \emptyset$, then the product has a single abelian factor $\mathbf{R}_\alpha \leqslant \Pi_\alpha \mathbf{A}_i$. If $\alpha = \emptyset$, then the product has no abelian factors. In either case, the result follows from the RT Theorem.

Two more observations facilitate application to CSP problems.

Cor. 2 Let $\mathbf{A}_i$ be finite Taylor algebras with $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$ ($i \in \underline{n}$).
Let $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$ for each $i \in \underline{n}$ and suppose $\mathbf{R}$ and $\mathbf{S}$ are subdirect products of $\Pi_{\underline{n}}\mathbf{A}_i$. Let $\alpha \subseteq \underline{n}$ and assume

- $\mathbf{A}_i$ is abelian for each $i \in \alpha$,
- $\mathbf{A}_i$ is nonabelian and simple for each $i \notin \alpha$,
- $\eta_i \neq \eta_j$ for all $i \neq j$,
- R and S both intersect $\Pi_{\underline{n}}B_i$ nontrivially,
- there exists $\mathbf{x} \in R_\alpha \cap S_\alpha$.

Then $R \cap S \neq \emptyset$.
Proof.
By Cor 1, $\mathbf{R}' = \mathbf{R}_\alpha \times \Pi_{\alpha'}\mathbf{B}_i$ and $\mathbf{S}' = \mathbf{S}_\alpha \times \Pi_{\alpha'}\mathbf{B}_i$. Therefore, since $\mathbf{x} \in R_\alpha \cap S_\alpha$, we have $\{\mathbf{x}\} \times \Pi_{\alpha'}\mathbf{B}_i \subseteq R \cap S$.

Generalizing to more than two relations is easy...

Cor. 3 Let $\mathbf{A}_i$ be finite Taylor algebras with $\mathbf{B}_i \lhd\lhd \mathbf{A}_i$ ($i \in \underline{n}$).
Suppose $\{\mathbf{R}_\ell : 0 \leqslant \ell < m\}$ are subdirect products of $\Pi_{\underline{n}}\mathbf{A}_i$.
Let $\alpha \subseteq \underline{n}$, and assume

- $\mathbf{A}_i$ is abelian for $i \in \alpha$ and nonabelian simple for $i \notin \alpha$,
- $\forall \ell \in \underline{m}$, $\forall i \neq j$, $\eta_i^\ell \neq \eta_j^\ell$ (where $\eta_i^\ell := \ker(\mathbf{R}_\ell \twoheadrightarrow \mathbf{A}_i)$),
- each R_ℓ intersects ΠB_i nontrivially,
- there exists $\mathbf{x} \in \bigcap \mathrm{Proj}_\alpha R_\ell$.

Then $\bigcap R_\ell \neq \emptyset$.

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3. **Intersecting mass products.** RT and corollaries assume that the universe R of the subdirect product in question intersects nontrivially with a product ΠB_i of minimal absorbing subuniverses (or "mass product").

In a CSP instance, there are typically many constraint relations. To apply Rectangularity, we have to be sure they all instersect a single mass product.