The Commutator as Least Fixed Point of a Closure Operator

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Abstract. We present a description of the (non-modular) commutator, inspired by that of Kearnes in [Kea95, p. 930], that provides a simple recipe for computing the commutator.

1. Preliminaries

If A and B are sets and $\alpha \subseteq A \times A$ and $\beta \subseteq B \times B$ are binary relations on A and B, respectively, then we define the *pairwise product* of α and β by

$$\alpha * \beta = \{((a, b), (a', b')) \in (A \times B)^2 \mid a \;\alpha\; a' \text{ and } b \;\beta\; b'\}, \tag{1.1}$$

and we let $\alpha \times \beta$ denote the usual Cartesian product of sets; that is,

$$\alpha \times \beta = \{((a, a'), (b, b')) \in A^2 \times B^2 \mid a \;\alpha\; a' \text{ and } b \;\beta\; b'\}. \tag{1.2}$$

The equivalence class of $\alpha * \beta$ containing the pair (a, b) is denoted and defined by

$$(a, b)/(\alpha * \beta) = a/\alpha \times b/\beta = \{(a', b') \in A \times B \mid a \;\alpha\; a' \text{ and } b \;\beta\; b'\},$$

the Cartesian product of the sets a/α and b/β. The set of all equivalence classes of $\alpha * \beta$ is also a Cartesian product, namely, $(A \times B)/(\alpha * \beta) = A/\alpha \times B/\beta = \{(a, b)/(\alpha * \beta) \mid a \in A \text{ and } b \in B\}$.

For an algebra $\mathbf{A}$ with congruence relations $\alpha, \beta \in \operatorname{Con} \mathbf{A}$, let $\underline{\boldsymbol{\beta}}$ denote the subalgebra of $\mathbf{A} \times \mathbf{A}$ with universe β, and let 0_A denote the least equivalence relation on A. Thus, $0_A = \{(a, a) \mid a \in A\} \leqslant \beta$. Denote by D_α the following subset of $\beta \times \beta$:

$$D_\alpha = (\alpha * \alpha) \cap (0_A \times 0_A) = \{((a, a), (b, b)) \in (0_A \times 0_A) \mid a \;\alpha\; b\}. \tag{1.3}$$

Let $\Delta_{\beta,\alpha} = \operatorname{Cg}^{\underline{\boldsymbol{\beta}}}(D_\alpha)$ denote the congruence relation of $\underline{\boldsymbol{\beta}}$ generated by D_α. The condition $\mathsf{C}(\alpha, \beta; \gamma)$ holds iff for all $a \;\alpha\; b$, for all $u_i \;\beta\; v_i$ $(1 \leqslant i \leqslant n)$, and for all $t \in \operatorname{Pol}_{n+1}(\mathbf{A})$, we have $t(a, \mathbf{u}) \;\gamma\; t(a, \mathbf{v})$ iff $t(b, \mathbf{u}) \;\gamma\; t(b, \mathbf{v})$. There are a number of different ways to define a commutator. See, for example, [Smi76, HH79, Gum80, DG92, KS98, Lip94]. The present note concerns the commutator $[\alpha, \beta]$ defined to be the least congruence γ such that $\mathsf{C}(\alpha, \beta; \gamma)$ holds.

2. Alternate Description of the Commutator

We now describe an alternate way to express the commutator—specifically, it is the least fixed point of a certain closure operator. This description was inspired by the one that is mentioned in passing by Keith Kearnes in [Kea95, p. 930]. Our objective here is to prove that the description we present is correct (i.e., describes the commutator) and to show that it leads to a simple, efficient procedure for computing the commutator.

Let $\operatorname{Tol}(A)$ denote the collection of all tolerances (reflexive symmetric relations) on the set A,[1] and let $\Psi_{\beta,\alpha}\colon \operatorname{Tol}(A) \to \operatorname{Tol}(A)$ be the function defined for each $T \in \operatorname{Tol}(A)$ follows:

$$\Psi_{\beta,\alpha}(T) = \{(x, y) \in A \times A \mid (\exists\, (a, b) \in T)\, (a, b) \;\Delta_{\beta,\alpha}\; (x, y)\}, \tag{2.1}$$

where $\Delta_{\beta,\alpha} = \operatorname{Cg}^{\underline{\boldsymbol{\beta}}}(D_\alpha)$ and $D_\alpha = (\alpha * \alpha) \cap (0_A \times 0_A)$ (as in (1.3)).

[1] Actually, a *tolerance* of an algebra $\mathbf{A} = \langle A, \dots \rangle$ is a reflexive symmetric subalgebra of $\mathbf{A} \times \mathbf{A}$. Therefore, the set of all tolerances of $\mathbf{A}$ forms an algebraic (hence complete) lattice. If we drop the operations and consider only the set A, then a tolerance relation on A is simply a reflexive symmetric binary relation.

Remarks.

(1) It's easy to see that $\Psi_{\beta,\alpha}(T)$ is reflexive and symmetric whenever T has these properties; similarly, $\Psi_{\beta,\alpha}(T)$ is compatible with the operations of $\mathbf{A}$ whenever T is. In other words $\Psi_{\beta,\alpha}$ maps tolerances of A ($\mathbf{A}$, resp.) to tolerances of A ($\mathbf{A}$, resp.).

(2) Since $\Psi_{\beta,\alpha}$ is clearly a monotone increasing function on the complete lattice $\mathrm{Tol}(A)$, it is guaranteed to have a least fixed point—that is, there is a point $\tau \in \mathrm{Tol}(A)$ such that $\Psi_{\beta,\alpha}(\tau) = \tau$ and $\tau \leqslant T$, for every $T \in \mathrm{Tol}(A)$ satisfying $\Psi_{\beta,\alpha}(T) = T$.

(3) Here are two ways the least fixed point of $\Psi_{\beta,\alpha}$ could be computed:

$$\tau = \bigwedge\{T \in \mathrm{Tol}(A) \mid \Psi_{\beta,\alpha}(T) \leqslant T\} \quad \text{and} \quad \tau = \bigvee_{k\geqslant 0} \Psi^k_{\beta,\alpha}(0_A). \tag{2.2}$$

In Lemma 2.1 we will show that the least fixed point of $\Psi_{\beta,\alpha}$ is, in fact, the commutator, $\tau = [\alpha, \beta]$, so either expression in (2.2) could potentially be used to compute it. However, Lemma 2.1 also shows that $\Psi_{\beta,\alpha}$ is a closure operator; in particular, it is idempotent. Therefore, $\Psi^k_{\beta,\alpha}(0_A) = \Psi_{\beta,\alpha}(0_A)$ for all k, so we have the following simple description of the commutator:

$$\begin{aligned}[\alpha, \beta] = \Psi_{\beta,\alpha}(0_A) &= \{(x, y) \in A \times A \mid (\exists\, (a, b) \in 0_A)\, (a, b)\ \Delta_{\beta,\alpha}\ (x, y)\}\\ &= \{(x, y) \in A \times A \mid (\exists a \in A)\, (a, a)\ \Delta_{\beta,\alpha}\ (x, y)\}.\end{aligned}$$

2.1. Fixed Point Lemma.

Lemma 2.1. *If α, $\beta \in \mathrm{Con}(\mathbf{A})$ and if $\Psi_{\beta,\alpha}$ is defined by (2.1), then*

(i) $\Psi_{\beta,\alpha}$ is a closure operator on $\mathrm{Tol}(A)$;

(ii) $[\alpha, \beta]$ is the least fixed point of $\Psi_{\beta,\alpha}$.

Proof.

(i) To prove (i) we verify that $\Psi_{\beta,\alpha}$ has the three properties that define a closure operator—namely for all T, $T' \in \mathrm{Tol}(A)$,

(c.1) $T \leqslant \Psi_{\beta,\alpha}(T)$;

(c.2) $T \leqslant T' \Rightarrow \Psi_{\beta,\alpha}(T) \leqslant \Psi_{\beta,\alpha}(T')$;

(c.3) $\Psi_{\beta,\alpha}(\Psi_{\beta,\alpha}(T)) = \Psi_{\beta,\alpha}(T)$.

Proof of (c.1): $(a, b) \in T$ implies $(a, b) \in \Psi_{\beta,\alpha}(T)$ because $(a, b)\ \Delta_{\beta,\alpha}\ (a, b)$.

Proof of (c.2): $(x, y) \in \Psi_{\beta,\alpha}(T)$ iff there exists $(a, b) \in T \leqslant T'$ such that $(a, b)\ \Delta_{\beta,\alpha}\ (x, y)$; this and $(a, b) \in T'$ implies $(x, y) \in \Psi_{\beta,\alpha}(T')$.

Proof of (c.3): $(x, y) \in \Psi_{\beta,\alpha}(\Psi_{\beta,\alpha}(T))$ if and only if there exists $(a, b) \in \Psi_{\beta,\alpha}(T)$ such that $(a, b)\ \Delta_{\beta,\alpha}\ (x, y)$, and $(a, b) \in \Psi_{\beta,\alpha}(T)$ is in turn equivalent to the existence of $(c, d) \in T$ such that $(c, d)\ \Delta_{\beta,\alpha}\ (a, b)$. By transitivity of $\Delta_{\beta,\alpha}$, we have that $(c, d)\ \Delta_{\beta,\alpha}\ (a, b)\ \Delta_{\beta,\alpha}\ (x, y)$ implies $(c, d)\ \Delta_{\beta,\alpha}\ (x, y)$, proving that there exists $(c, d) \in T$ such that $(c, d)\ \Delta_{\beta,\alpha}\ (x, y)$; equivalently, $(x, y) \in T$.

(ii) As remarked above, from part (i) follows $\Psi^k_{\beta,\alpha}(0_A) = \Psi_{\beta,\alpha}(0_A)$ for all k, so the least fixed point of $\Psi_{\beta,\alpha}$ that appears in the formula on the right in (2.2) reduces to $\tau = \Psi_{\beta,\alpha}(0_A)$. Therefore, to complete the proof it suffices to show $[\alpha, \beta] = \Psi_{\beta,\alpha}(0_A)$.

We first prove $[\alpha, \beta] \leqslant \Psi_{\beta,\alpha}(0_A)$. Since $[\alpha, \beta]$ is the least congruence γ satisfying $\mathsf{C}(\alpha, \beta; \gamma)$, it suffices to prove $\mathsf{C}(\alpha, \beta; \Psi_{\beta,\alpha}(0_A))$ holds. Suppose $a\ \alpha\ a'$ and $b_i\ \beta\ b'_i$ and $t^{\mathbf{A}} \in \mathrm{Pol}_{k+1}(\mathbf{A})$ satisfy $t^{\mathbf{A}}(a, \mathbf{b})\ \Psi_{\beta,\alpha}(0_A)\ t^{\mathbf{A}}(a, \mathbf{b}')$, where $\mathbf{b} = (b_1, \dots, b_k)$ and $\mathbf{b}' = (b'_1, \dots, b'_k)$. We must show $t(a', \mathbf{b})\ \Psi_{\beta,\alpha}(0_A)\ t(a', \mathbf{b}')$. By definition of $\Psi_{\beta,\alpha}$, the antecedent $t^{\mathbf{A}}(a, \mathbf{b})\ \Psi_{\beta,\alpha}(0_A)\ t^{\mathbf{A}}(a, \mathbf{b}')$ is equivalent to the existence of $c \in A$ such that $(c, c)\ \Delta_{\beta,\alpha}\ (t^{\mathbf{A}}(a, \mathbf{b}), t^{\mathbf{A}}(a, \mathbf{b}'))$. Now

$$(t^{\mathbf{A}}(a, \mathbf{b}), t^{\mathbf{A}}(a, \mathbf{b}')) = t^{\underline{\boldsymbol{\beta}}}((a, a), (b_1, b'_1), \dots, (b_k, b'_k)),$$

and since $a\ \alpha\ a'$, we have

$$t^{\underline{\boldsymbol{\beta}}}((a, a), (b_1, b'_1), \dots, (b_k, b'_k))\ \Delta_{\beta,\alpha}\ t^{\underline{\boldsymbol{\beta}}}((a', a'), (b_1, b'_1), \dots, (b_k, b'_k)).$$

The latter is equal to $(t^{\mathbf{A}}(a',\mathbf{b}), t^{\mathbf{A}}(a',\mathbf{b}'))$, and it follows by transitivity of $\Delta_{\beta,\alpha}$ that $(c,c)\ \Delta_{\beta,\alpha}\ (t^{\mathbf{A}}(a',\mathbf{b}), t^{\mathbf{A}}(a',\mathbf{b}'))$. Therefore, $t(a',\mathbf{b})\ \Psi_{\beta,\alpha}(0_A)\ t(a',\mathbf{b}')$, as desired.

We now prove $\Psi_{\beta,\alpha}(0_A) \leqslant [\alpha,\beta]$. If $(x,y)\in\Psi_{\beta,\alpha}(0_A)$ then there exists $a\in A$ such that

$$(a,a)\ \Delta_{\beta,\alpha}\ (x,y). \tag{2.3}$$

From the definition of $\Delta_{\beta,\alpha}$ and Mal'tsev's congruence generation theorem, (2.3) holds if and only if for there exist $(z_i,z_i')\in\beta$ $(0\leqslant i\leqslant n+1)$, and $(u_i,v_i)\in\alpha$, $f_i\in\operatorname{Pol}_1(\underline{\boldsymbol{\beta}})$ $(0\leqslant i\leqslant n)$, such that $(a,a)=(z_0,z_0')$ and $(x,y)=(z_{n+1},z_{n+1}')$ hold, and so do the following equations of sets:

$$\{(a,a),(z_1,z_1')\}=\{f_0(u_0,u_0),f_0(v_0,v_0)\}, \tag{2.4}$$

$$\{(z_1,z_1'),(z_2,z_2')\}=\{f_1(u_1,u_1),f_1(v_1,v_1)\}, \tag{2.5}$$

$$\vdots$$

$$\{(z_n,z_n'),(x,y)\}=\{f_n(u_n,u_n),f_n(v_n,v_n)\}.$$

Now $f_i\in\operatorname{Pol}_1(\underline{\boldsymbol{\beta}})$ for all i, so

$$f_i(c,c')=g_i^{\underline{\boldsymbol{\beta}}}((c,c'),(b_1,b_1'),\dots,(b_k,b_k'))=(g_i^{\mathbf{A}}(c,\mathbf{b}),g_i^{\mathbf{A}}(c',\mathbf{b}')),$$

for some k, some $(k{+}1)$-ary term g_i, and some constants $\mathbf{b}=(b_1,\dots,b_k)$ and $\mathbf{b}'=(b_1',\dots,b_k')$ satisfying $b_i\ \beta\ b_i'$ $(1\leqslant i\leqslant k)$. By (2.4), either

$$(a,a)=\big(g_0(u_0,\mathbf{b}),g_0(u_0,\mathbf{b}')\big)\quad\text{and}\quad (z_1,z_1')=\big(g_0(v_0,\mathbf{b}),g_0(v_0,\mathbf{b}')\big),$$

or vice-versa. We assumed $u_0\ \alpha\ v_0$ and $b_i\ \beta\ b_i'$ $(1\leqslant i\leqslant k)$, so the α,β-term condition entails $g_0(u_0,\mathbf{a})\ [\alpha,\beta]\ g_0(u_0,\mathbf{a}')$ iff $g_0(v_0,\mathbf{a})\ [\alpha,\beta]\ g_0(v_0,\mathbf{a}')$. From this and (2.4) we deduce that $(a,a)\in[\alpha,\beta]$ iff $(z_1,z_1')\in[\alpha,\beta]$. Similarly (2.5) and $u_1\ \alpha\ v_1$ imply $(z_1,z_1')\in[\alpha,\beta]$ iff $(z_2,z_2')\in[\alpha,\beta]$. Inductively, and by transitivity of $[\alpha,\beta]$, we conclude $(a,a)\in[\alpha,\beta]$ iff $(x,y)\in[\alpha,\beta]$. Since $(a,a)\in[\alpha,\beta]$, we have $(x,y)\in[\alpha,\beta]$, as desired.

□

3. Computing the Commutator

As a consequence of the description of the commutator given in the last section, we now have the following simple method for computing it.

Input A finite algebra, $\mathbf{A}=\langle A,\dots\rangle$, and two congruence relations α, $\beta\in\operatorname{Con}\mathbf{A}$.

Procedure

- **Step 1** Compute the congruence relation $\Delta_{\beta,\alpha}=\operatorname{Cg}^{\underline{\boldsymbol{\beta}}}\big\{((a,a),(b,b))\mid a\ \alpha\ b\big\}$.
- **Step 2** Compute the commutator

$$[\alpha,\beta]=\big\{(x,y)\in A\times A\mid(\exists a\in A)\,(a,a)\ \Delta_{\beta,\alpha}\ (x,y)\big\}=\bigcup_{a\in A}(a,a)/\Delta_{\beta,\alpha}$$

Note that $\Delta_{\beta,\alpha}$ is a subalgebra of $\mathbf{A}^2\times\mathbf{A}^2$ and such a congruence can be computed in polynomial-time in the size of $\mathbf{A}$. (See [Fre08].)

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