

The Commutator as Least Fixed Point of a Closure Operator

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ABSTRACT. We present a description of the (non-modular) commutator, inspired by that of Kearnes in [Kea95, p. 930], that provides a simple recipe for computing the commutator.

1. Preliminaries

If A and B are sets and $\alpha \subseteq A \times A$ and $\beta \subseteq B \times B$ are binary relations on A and B , respectively, then we define the *pairwise product* of α and β by

$$\alpha * \beta = \{((a, b), (a', b')) \in (A \times B)^2 \mid a \alpha a' \text{ and } b \beta b'\}, \quad (1.1)$$

and we let $\alpha \times \beta$ denote the usual Cartesian product of sets; that is,

$$\alpha \times \beta = \{((a, a'), (b, b')) \in A^2 \times B^2 \mid a \alpha a' \text{ and } b \beta b'\}. \quad (1.2)$$

The equivalence class of $\alpha * \beta$ containing the pair (a, b) is denoted and defined by

$$(a, b)/(\alpha * \beta) = a/\alpha \times b/\beta = \{(a', b') \in A \times B \mid a \alpha a' \text{ and } b \beta b'\},$$

the Cartesian product of the sets a/α and b/β . The set of all equivalence classes of $\alpha * \beta$ is also a Cartesian product, namely, $(A \times B)/(\alpha * \beta) = A/\alpha \times B/\beta = \{(a, b)/(\alpha * \beta) \mid a \in A \text{ and } b \in B\}$.

For an algebra \mathbf{A} with congruence relations $\alpha, \beta \in \text{Con } \mathbf{A}$, let $\underline{\beta}$ denote the subalgebra of $\mathbf{A} \times \mathbf{A}$ with universe β , and let 0_A denote the least equivalence relation on A . Thus, $0_A = \{(a, a) \mid a \in A\} \leq \beta$. Denote by D_α the following subset of $\beta \times \beta$:

$$D_\alpha = (\alpha * \alpha) \cap (0_A \times 0_A) = \{((a, a), (b, b)) \in (0_A \times 0_A) \mid a \alpha b\}. \quad (1.3)$$

Let $\Delta_{\beta, \alpha} = \text{Cg}^{\underline{\beta}}(D_\alpha)$ denote the congruence relation of $\underline{\beta}$ generated by D_α . The condition $\text{C}(\alpha, \beta; \gamma)$ holds iff for all $a \alpha b$, for all $u_i \beta v_i$ ($1 \leq i \leq n$), and for all $t \in \text{Pol}_{n+1}(\mathbf{A})$, we have $t(a, \mathbf{u}) \gamma t(a, \mathbf{v})$ iff $t(b, \mathbf{u}) \gamma t(b, \mathbf{v})$. There are a number of different ways to define a commutator. See, for example, [Smi76, HH79, Gum80, DG92, KS98, Lip94]. The present note concerns the commutator $[\alpha, \beta]$ defined to be the least congruence γ such that $\text{C}(\alpha, \beta; \gamma)$ holds.

2. Alternate Description of the Commutator

We now describe an alternate way to express the commutator—specifically, it is the least fixed point of a certain closure operator. This description was inspired by the one that is mentioned in passing by Keith Kearnes in [Kea95, p. 930]. Our objective here is to prove that the description we present is correct (i.e., describes the commutator) and to show that it leads to a simple, efficient procedure for computing the commutator.

Let $\text{Tol}(A)$ denote the collection of all tolerances (reflexive symmetric relations) on the set A ,¹ and let $\Psi_{\beta, \alpha}: \text{Tol}(A) \rightarrow \text{Tol}(A)$ be the function defined for each $T \in \text{Tol}(A)$ follows:

$$\Psi_{\beta, \alpha}(T) = \{(x, y) \in A \times A \mid (\exists (a, b) \in T) (a, b) \Delta_{\beta, \alpha} (x, y)\}, \quad (2.1)$$

where $\Delta_{\beta, \alpha} = \text{Cg}^{\underline{\beta}}(D_\alpha)$ and $D_\alpha = (\alpha * \alpha) \cap (0_A \times 0_A)$ (as in (1.3)).

¹Actually, a *tolerance* of an algebra $\mathbf{A} = \langle A, \dots \rangle$ is a reflexive symmetric subalgebra of $\mathbf{A} \times \mathbf{A}$. Therefore, the set of all tolerances of \mathbf{A} forms an algebraic (hence complete) lattice. If we drop the operations and consider only the set A , then a tolerance relation on A is simply a reflexive symmetric binary relation.

Remarks.

- (1) It's easy to see that $\Psi_{\beta,\alpha}(T)$ is reflexive and symmetric whenever T has these properties; similarly, $\Psi_{\beta,\alpha}(T)$ is compatible with the operations of \mathbf{A} whenever T is. In other words $\Psi_{\beta,\alpha}$ maps tolerances of A (\mathbf{A} , resp.) to tolerances of A (\mathbf{A} , resp.).
- (2) Since $\Psi_{\beta,\alpha}$ is clearly a monotone increasing function on the complete lattice $\text{Tol}(A)$, it is guaranteed to have a least fixed point—that is, there is a point $\tau \in \text{Tol}(A)$ such that $\Psi_{\beta,\alpha}(\tau) = \tau$ and $\tau \leq T$, for every $T \in \text{Tol}(A)$ satisfying $\Psi_{\beta,\alpha}(T) = T$.
- (3) Here are two ways the least fixed point of $\Psi_{\beta,\alpha}$ could be computed:

$$\tau = \bigwedge \{T \in \text{Tol}(A) \mid \Psi_{\beta,\alpha}(T) \leq T\} \quad \text{and} \quad \tau = \bigvee_{k \geq 0} \Psi_{\beta,\alpha}^k(0_A). \quad (2.2)$$

In Lemma 2.1 we will show that the least fixed point of $\Psi_{\beta,\alpha}$ is, in fact, the commutator, $\tau = [\alpha, \beta]$, so either expression in (2.2) could potentially be used to compute it. However, Lemma 2.1 also shows that $\Psi_{\beta,\alpha}$ is a closure operator; in particular, it is idempotent. Therefore, $\Psi_{\beta,\alpha}^k(0_A) = \Psi_{\beta,\alpha}(0_A)$ for all k , so we have the following simple description of the commutator:

$$\begin{aligned} [\alpha, \beta] &= \Psi_{\beta,\alpha}(0_A) = \{(x, y) \in A \times A \mid (\exists (a, b) \in 0_A) (a, b) \Delta_{\beta,\alpha} (x, y)\} \\ &= \{(x, y) \in A \times A \mid (\exists a \in A) (a, a) \Delta_{\beta,\alpha} (x, y)\}. \end{aligned}$$

2.1. Fixed Point Lemma.

Lemma 2.1. *If $\alpha, \beta \in \text{Con}(\mathbf{A})$ and if $\Psi_{\beta,\alpha}$ is defined by (2.1), then*

- (i) $\Psi_{\beta,\alpha}$ is a closure operator on $\text{Tol}(A)$;
- (ii) $[\alpha, \beta]$ is the least fixed point of $\Psi_{\beta,\alpha}$.

Proof.

- (i) To prove (i) we verify that $\Psi_{\beta,\alpha}$ has the three properties that define a closure operator—namely for all $T, T' \in \text{Tol}(A)$,
 - (c.1) $T \leq \Psi_{\beta,\alpha}(T)$;
 - (c.2) $T \leq T' \Rightarrow \Psi_{\beta,\alpha}(T) \leq \Psi_{\beta,\alpha}(T')$;
 - (c.3) $\Psi_{\beta,\alpha}(\Psi_{\beta,\alpha}(T)) = \Psi_{\beta,\alpha}(T)$.

Proof of (c.1): $(a, b) \in T$ implies $(a, b) \in \Psi_{\beta,\alpha}(T)$ because $(a, b) \Delta_{\beta,\alpha} (a, b)$.

Proof of (c.2): $(x, y) \in \Psi_{\beta,\alpha}(T)$ iff there exists $(a, b) \in T \leq T'$ such that $(a, b) \Delta_{\beta,\alpha} (x, y)$; this and $(a, b) \in T'$ implies $(x, y) \in \Psi_{\beta,\alpha}(T')$.

Proof of (c.3): $(x, y) \in \Psi_{\beta,\alpha}(\Psi_{\beta,\alpha}(T))$ if and only if there exists $(a, b) \in \Psi_{\beta,\alpha}(T)$ such that $(a, b) \Delta_{\beta,\alpha} (x, y)$, and $(a, b) \in \Psi_{\beta,\alpha}(T)$ is in turn equivalent to the existence of $(c, d) \in T$ such that $(c, d) \Delta_{\beta,\alpha} (a, b)$. By transitivity of $\Delta_{\beta,\alpha}$, we have that $(c, d) \Delta_{\beta,\alpha} (a, b) \Delta_{\beta,\alpha} (x, y)$ implies $(c, d) \Delta_{\beta,\alpha} (x, y)$, proving that there exists $(c, d) \in T$ such that $(c, d) \Delta_{\beta,\alpha} (x, y)$; equivalently, $(x, y) \in T$.

- (ii) As remarked above, from part (i) follows $\Psi_{\beta,\alpha}^k(0_A) = \Psi_{\beta,\alpha}(0_A)$ for all k , so the least fixed point of $\Psi_{\beta,\alpha}$ that appears in the formula on the right in (2.2) reduces to $\tau = \Psi_{\beta,\alpha}(0_A)$. Therefore, to complete the proof it suffices to show $[\alpha, \beta] = \Psi_{\beta,\alpha}(0_A)$.

We first prove $[\alpha, \beta] \leq \Psi_{\beta,\alpha}(0_A)$. Since $[\alpha, \beta]$ is the least congruence γ satisfying $\mathbf{C}(\alpha, \beta; \gamma)$, it suffices to prove $\mathbf{C}(\alpha, \beta; \Psi_{\beta,\alpha}(0_A))$ holds. Suppose $a \alpha a'$ and $b_i \beta b'_i$ and $t^{\mathbf{A}} \in \text{Pol}_{k+1}(\mathbf{A})$ satisfy $t^{\mathbf{A}}(a, \mathbf{b}) \Psi_{\beta,\alpha}(0_A) t^{\mathbf{A}}(a', \mathbf{b}')$, where $\mathbf{b} = (b_1, \dots, b_k)$ and $\mathbf{b}' = (b'_1, \dots, b'_k)$. We must show $t^{\mathbf{A}}(a', \mathbf{b}) \Psi_{\beta,\alpha}(0_A) t^{\mathbf{A}}(a', \mathbf{b}')$. By definition of $\Psi_{\beta,\alpha}$, the antecedent $t^{\mathbf{A}}(a, \mathbf{b}) \Psi_{\beta,\alpha}(0_A) t^{\mathbf{A}}(a, \mathbf{b}')$ is equivalent to the existence of $c \in A$ such that $(c, c) \Delta_{\beta,\alpha} (t^{\mathbf{A}}(a, \mathbf{b}), t^{\mathbf{A}}(a, \mathbf{b}'))$. Now

$$(t^{\mathbf{A}}(a, \mathbf{b}), t^{\mathbf{A}}(a, \mathbf{b}')) = t^{\beta}((a, a), (b_1, b'_1), \dots, (b_k, b'_k)),$$

and since $a \alpha a'$, we have

$$t^{\beta}((a, a), (b_1, b'_1), \dots, (b_k, b'_k)) \Delta_{\beta,\alpha} t^{\beta}((a', a'), (b_1, b'_1), \dots, (b_k, b'_k)).$$

The latter is equal to $(t^{\mathbf{A}}(a', \mathbf{b}), t^{\mathbf{A}}(a', \mathbf{b}'))$, and it follows by transitivity of $\Delta_{\beta, \alpha}$ that $(c, c) \Delta_{\beta, \alpha} (t^{\mathbf{A}}(a', \mathbf{b}), t^{\mathbf{A}}(a', \mathbf{b}'))$. Therefore, $t(a', \mathbf{b}) \Psi_{\beta, \alpha}(0_A) t(a', \mathbf{b}')$, as desired.

We now prove $\Psi_{\beta, \alpha}(0_A) \leq [\alpha, \beta]$. If $(x, y) \in \Psi_{\beta, \alpha}(0_A)$ then there exists $a \in A$ such that

$$(a, a) \Delta_{\beta, \alpha} (x, y). \quad (2.3)$$

From the definition of $\Delta_{\beta, \alpha}$ and Mal'tsev's congruence generation theorem, (2.3) holds if and only if for there exist $(z_i, z'_i) \in \beta$ ($0 \leq i \leq n+1$), and $(u_i, v_i) \in \alpha$, $f_i \in \text{Pol}_1(\underline{\beta})$ ($0 \leq i \leq n$), such that $(a, a) = (z_0, z'_0)$ and $(x, y) = (z_{n+1}, z'_{n+1})$ hold, and so do the following equations of sets:

$$\{(a, a), (z_1, z'_1)\} = \{f_0(u_0, u_0), f_0(v_0, v_0)\}, \quad (2.4)$$

$$\{(z_1, z'_1), (z_2, z'_2)\} = \{f_1(u_1, u_1), f_1(v_1, v_1)\}, \quad (2.5)$$

\vdots

$$\{(z_n, z'_n), (x, y)\} = \{f_n(u_n, u_n), f_n(v_n, v_n)\}.$$

Now $f_i \in \text{Pol}_1(\underline{\beta})$ for all i , so

$$f_i(c, c') = g_i^{\beta}((c, c'), (b_1, b'_1), \dots, (b_k, b'_k)) = (g_i^{\mathbf{A}}(c, \mathbf{b}), g_i^{\mathbf{A}}(c', \mathbf{b}')),$$

for some k , some $(k+1)$ -ary term g_i , and some constants $\mathbf{b} = (b_1, \dots, b_k)$ and $\mathbf{b}' = (b'_1, \dots, b'_k)$ satisfying $b_i \beta b'_i$ ($1 \leq i \leq k$). By (2.4), either

$$(a, a) = (g_0(u_0, \mathbf{b}), g_0(u_0, \mathbf{b}')) \quad \text{and} \quad (z_1, z'_1) = (g_0(v_0, \mathbf{b}), g_0(v_0, \mathbf{b}')),$$

or vice-versa. We assumed $u_0 \alpha v_0$ and $b_i \beta b'_i$ ($1 \leq i \leq k$), so the α, β -term condition entails $g_0(u_0, \mathbf{a}) [\alpha, \beta] g_0(u_0, \mathbf{a}')$ iff $g_0(v_0, \mathbf{a}) [\alpha, \beta] g_0(v_0, \mathbf{a}')$. From this and (2.4) we deduce that $(a, a) \in [\alpha, \beta]$ iff $(z_1, z'_1) \in [\alpha, \beta]$. Similarly (2.5) and $u_1 \alpha v_1$ imply $(z_1, z'_1) \in [\alpha, \beta]$ iff $(z_2, z'_2) \in [\alpha, \beta]$. Inductively, and by transitivity of $[\alpha, \beta]$, we conclude $(a, a) \in [\alpha, \beta]$ iff $(x, y) \in [\alpha, \beta]$. Since $(a, a) \in [\alpha, \beta]$, we have $(x, y) \in [\alpha, \beta]$, as desired. □

3. Computing the Commutator

As a consequence of the description of the commutator given in the last section, we now have the following simple method for computing it.

Input A finite algebra, $\mathbf{A} = \langle A, \dots \rangle$, and two congruence relations $\alpha, \beta \in \text{Con } \mathbf{A}$.

Procedure

- **Step 1** Compute the congruence relation $\Delta_{\beta, \alpha} = \text{Cg}^{\beta}\{((a, a), (b, b)) \mid a \alpha b\}$.
- **Step 2** Compute the commutator

$$[\alpha, \beta] = \{(x, y) \in A \times A \mid (\exists a \in A) (a, a) \Delta_{\beta, \alpha} (x, y)\} = \bigcup_{a \in A} (a, a) / \Delta_{\beta, \alpha}$$

Note that $\Delta_{\beta, \alpha}$ is a subalgebra of $\mathbf{A}^2 \times \mathbf{A}^2$ and such a congruence can be computed in polynomial-time in the size of \mathbf{A} . (See [Fre08].)

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