1. (a) Given two sets X and Y write down the precise definition of X $\subseteq$ Y. That is, give the formula that defines the $\subseteq$ relation using logical symbols like $\forall$, $\in$, $\rightarrow$, among others.

   (b) The empty set is denoted by $\emptyset$; it is all we had at the start of this course. How did we use the empty set and curly braces, { and }, to construct the natural numbers {0, 1, 2, . . . }?
   (*Hint:* the number k is a certain set containing k elements. Construct the first 3 numbers, 0, 1, 2, and then define k in general.)

   (c) Write down the *cardinality* of each of the following sets in the blank space provided, then answer the True/False questions by circling the appropriate choice.

$$A = \emptyset, \quad B = \{0\}, \quad C = \{1\}, \quad D = \{0,1\}, \quad E = \{1,2,3\}, \quad F = \{0,1,2,3\}.$$

   What is the cardinality (size) of each set?

$$|A| = \_\_\_\_, \quad |B| = \_\_\_\_, \quad |C| = \_\_\_\_, \quad |D| = \_\_\_\_, \quad |E| = \_\_\_\_, \quad |F| = \_\_\_\_.$$

   (d) For the sets $A, B, \ldots$ defined above, circle True or False, as appropriate.

   i. $A \in B$ True False

   ii. $A = B$ True False

   iii. $B \in D$ True False

   iv. $D \cap E = C$ True False

   v. $D \cup E = F$ True False

   vi. $E \cap F = E$ True False

2. Consider the following three hypotheses:

- Let A denote "Alan likes kangaroos".
- Let B denote "Betty likes frogs".
- Let C denote "Carl likes hamsters".

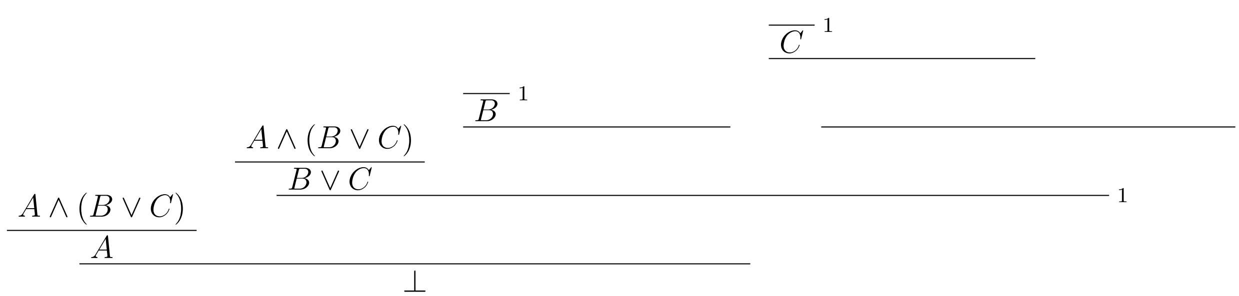
Translate the following statements into symbolic logic, using the letters A, B, C, and the logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$,

(a) Alan likes kangaroos, and either Betty likes frogs or Carl likes hamsters.

(b) If Betty likes frogs, then Alan doesn't like kangaroos.

(c) If Carl likes hamsters, then Betty likes frogs.

(d) Prove that the above hypotheses, (a), (b), (c), are contradictory by completing the derivation tree skeleton below. (You may write an English sentence to explain your reasoning, but your score will be based mainly on the derivation tree.)



3. Give a natural deduction proof of $\neg A \wedge \neg B \rightarrow \neg (A \vee B)$.

4. (a) Complete the truth table below, adding columns for the formulas $(A \vee \neg B) \wedge B$ and $A \wedge B$.

A	B				

(b) On the basis of this truth table, are the formulas $(A \vee \neg B) \wedge B$ and $A \wedge B$ logically equivalent? Explain.

(c) Is the formula $(A \vee \neg B) \wedge B$ a *tautology* or a *contradiction*, or is it neither a tautology nor a contradiction? Explain.

5. (a) Construct a truth table for $A \vee B \rightarrow A \wedge B$.

(b) Is statement $A \vee B \rightarrow A \wedge B$ a *tautology* or a *contradiction*, or is it neither a tautology nor a contradiction? Explain.

6. Construct a proof derivation diagram using basic intro/elim rules of natural deduction to prove the **law of the excluded middle** using the **proof-by-contradiction law**. That is, prove $\forall$ A (A $\vee$ $\neg$ A) from the assumption $\forall$ A ($\neg$ $\neg$ A $\rightarrow$ A).

7. Prove that the **law of the excluded middle** is irrefutable. That is, prove $\forall$ A ($\neg\neg$ (A $\vee$ $\neg$ A)).

8. Prove the following claim using "proof by contraposition."
(For credit you must at least write down the correct contrapositive statement.)

Claim: If $n^2 + 1$ is odd, then n is even.

9. Let X and Y be sets. Give *precise and complete definitions* of the italicized terms below by completing the given sentence. You may use words like *reflexive*, *symmetric*, *transitive*, etc., without providing the formulas that define these terms.

   Example. An *antisymmetric relation* on a set A is...

   Answer. ...a binary relation R on A satisfying $\forall x, y \; (x \; R \; y \; \bigwedge \; y \; R \; x \; \longrightarrow \; x = y)$.

   (a) A *equivalence relation* on a set A is...

   (b) An *partial order relation* on a set A is...

   (c) Let A be a set and let $\leq$ be a partial order on A. An element x of A is called a *minimum element* of A if $x \leq y$ for every y in A. Show with an ordinary mathematical proof (using complete English sentences) that a minimum element is unique. (In other words, any two minimum elements have to be equal.)

10. Let $f\colon S \to T$ and $g\colon T \to U$ be functions.

(a) Say precisely what it means for the function $f\colon S \to T$ to be *one-to-one*.

(b) Say precisely what it means for the function $g \circ f\colon S \to U$ to be *one-to-one*.

(c) Write an ordinary mathematical proof (using complete English sentences) to show that if $g \circ f$ is one-to-one, then f is one-to-one.

11. Give a natural deduction proof of $\exists x A(x)$ from $\exists x(A(x) \vee B(x))$ and $\forall x(B(x) \to A(x))$. That is, *assume* that both $\exists x(A(x) \vee B(x))$ and $\forall x(B(x) \to A(x))$ hold, and use that to *derive* $\exists x A(x)$.

12. In a first-order language with a binary relation $R(x, y)$ on a set U, consider the following sentences:

    A. $\exists x\, \forall y\, R(x, y)$
    B. $\exists y\, \forall x\, R(x, y)$
    C. $\forall x, y\, (R(x, y) \land x \neq y \to \exists z\, (R(x, z) \land R(z, y) \land x \neq z \land y \neq z))$

    For each of the following models, determine whether each sentence above is true or false, and fill in the blanks below with T if the sentence is true and F if the sentence is false. (No justification is required.)

    Example. If $(U, R) = (\mathbb{N}, \leq)$, then A is ___T___, B is ___F___, C is___F___.

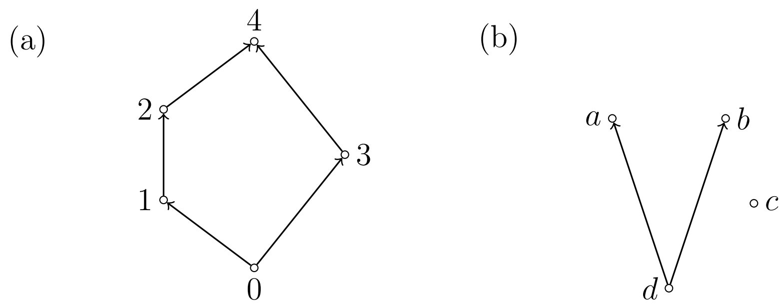
    (a) If $(U, R) = (\mathbb{Z}, \leq)$, then A is _______, B is _______, C is _______.

    (b) If $(U, R) = (\mathbb{Q}, \leq)$, then A is _______, B is _______, C is _______.

    (c) If $(U, R) = (\mathbb{N} \setminus \{0\}, |)$, then A is _______, B is _______, C is _______.
    (Here the interpretation is the set $U = \{1, 2, \ldots\}$ of natural numbers *without zero* and the relation $R = |$, the "divides" relation.)

    (d) If $(U, R) = (\mathcal{P}(\mathbb{N}), \subseteq)$, then A is _______, B is _______, C is _______.
    [*Hint.* $\mathcal{P}(\mathbb{N})$ denotes the *power set* of the natural numbers (i.e., all subsets of $\mathbb{N}$) and the relation $\subseteq$ is the subset inclusion relation.]

13. Each Hasse diagram in the figure below represents a partial ordering R on a set S. Find R in each case. That is, write down the set of ordered pairs in R.



(a) $S = \{0, 1, 2, 3, 4\}$

$R =$

(b) $S = \{a, b, c, d\}$

$R =$