**CS 241: Spring 2022** **HW 03** **Due 18 Feb 11:59pm**

Name: ______________________________ Student ID: __________________

**Instructions**. Print out this assignment, fill in your answers in the space provided and upload your work to Gradescope. **Important**. You must write your answers on a printout of this pdf! Please do not upload additional pages or scratch work.

1. (6 points) A *perfect number* is a number that equals the sum of its proper divisors, other than itself. For example, 6 is perfect since $6 = 1 + 2 + 3$.

   Using a language with variables ranging over the natural numbers and suitable functions and predicates, write down first-order sentences asserting the following. Use a predicate `perfect` to express that a number is perfect. (Incidentally, it is not known whether assertions a and d are true. They are open questions.)

   **Example 1**. 28 is perfect. **Solution.** $\texttt{perfect}(28)$.

   **Example 2**. There are no perfect numbers between 100 and 200. (Let's assume the phrase "between a and b" includes the endpoints a, b).

   **Solution a**. $\forall$ x ($\texttt{perfect}$(x) $\rightarrow$ x $<$ 100 $\vee$ 200 $<$ x).

   **Solution b**. $\forall$ x (100 $\leq$ x $\leq$ 200 $\rightarrow \neg$ $\texttt{perfect}$(x)).
   Note that a. and b. are equivalent ("contrapositive") statements, so both are acceptable.

   (a) Every perfect number is even. (In your answer, use $\exists$ k (x = 2k) to express the assertion "x is even.")

   (b) There are (at least) two perfect numbers between 200 and 10,000.

   (*Hint*. This is equivalent to the assertion that there are numbers x and y between 200 and 10,000 which are not the same (x $\neq$ y) and are perfect.)

   (c) For every number x, there is a perfect number that is larger than x. (This is one way to express that there are infinitely many perfect numbers.)

2. (6 points) Use a language with variables ranging over people, and predicates

   - `trusts`(x,y) (denoting "x trusts y")
   - `politician`(x) ("x is a politician")
   - `crazy`(x) ("x is crazy")
   - `knows`(x, y) ("x knows y")
   - `related`(x, y) ("x is related to y")
   - `rich`(x) ("x is rich")

   to write down first-order sentences expressing each of the assertions below.

   **Example**. Nobody trusts a politician.

   Solution. $\forall$ x (`politician`(x) $\rightarrow$ $\forall$ y ($\neg$ `trusts`(y,x)))

   Note that, in each case, some interpretation of English into logic is required, and that, consequently, using a logical formula forces us to clarify and make precise a statement's meaning.

   (a) Anyone who trusts a politician is crazy.

   (b) Everyone knows someone who is related to a politician.

   (c) If a person is rich then that person is either a politician or knows a politician.

3. (6 points) Give a natural deduction proof of $\forall$ x B(x) from hypotheses

   1. $\forall$ x (A(x) $\vee$ B(x)) and
   2. $\forall$ y ($\neg$ A(y))

4. (0 points) Starting with hypotheses

    1. $\forall$ x ($\texttt{even}$(x) $\vee$ $\texttt{odd}$(x)) and
    2. $\forall$ x ($\texttt{odd}$(x) $\rightarrow$ $\texttt{even}$(s(x)))

    give a natural deduction proof of $\forall$ x ($\texttt{even}$(x) $\vee$ $\texttt{even}$(s(x)))

    (It might help to think of s(x) as the "successor" function, s(x) = x + 1, but your proof shouldn't depend on this interpretation.)

5. (4 points) Give a natural deduction proof of $\exists$ x A(x) $\vee$ $\exists$ x B(x) $\rightarrow$ $\exists$ x (A(x) $\vee$ B(x)).

6. (0 points) Give a natural deduction proof of $\forall$ x, y (x = y $\rightarrow$ y = x) using only the following two hypotheses:

   - $\forall$ x (x = x)
   - $\forall$ u, v, w (u = w $\rightarrow$ (v = w $\rightarrow$ u = v))

   *Hints.* Recall that ($\forall$ x, y) is shorthand notation for ($\forall$ x) ($\forall$ y); similarly, $\forall$ u, v, w is short for $\forall$ u $\forall$ v $\forall$ w. Choose instantiations of u, v, and w carefully. You can instantiate all the universal quantifiers in one step.

7. (4 points) Give a natural deduction proof of
$\neg(\exists\, x)(A(x) \wedge B(x)) \rightarrow (\forall\, x)(A(x) \rightarrow \neg\, B(x))$.