CS 241: Spring 2022 **HW 5** **Due 2 Mar by 11:59pm**

Name: ______________________________ Student ID: __________________

Instructions. Print out this assignment, fill in your answers in the space provided and upload your work to Gradescope. **Important**. You must write your answers on a printout of this pdf! Please do not upload additional pages or scratch work.

(5pts) 1. Let X = {a, b, c}. For each of the following binary relations on X, check the boxes next to the properties that the relation has. *Select all that apply!*

(a) $\emptyset = \{\ \}$

○ reflexive ○ symmetric ○ antisymmetric ○ transitive

(b) $\mathscr{D} = \{(a, a), (b, b), (c, c)\}$

○ reflexive ○ symmetric ○ antisymmetric ○ transitive

(c) $\mathscr{A} = \{(a, a), (b, a), (b, b), (c, a), (c, b), (c, c)\}$

○ reflexive ○ symmetric ○ antisymmetric ○ transitive

(d) $X \times X = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

○ reflexive ○ symmetric ○ antisymmetric ○ transitive

(e) $(X \times X) \setminus \mathscr{D} = \{(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)\}$

○ reflexive ○ symmetric ○ antisymmetric ○ transitive

(6pts) 2. Let X be a set and let R be a binary relation on X. Recall we sometimes write x R y to express (x, y) $\in$ R. Consider the following statements.

i. $\exists$ x $\forall$ y (x R y)

ii. $\exists$ y $\forall$ x (x R y)

iii. $\forall$ x $\forall$ y (x R y $\land$ x $\neq$ y $\to$ $\exists$ z (x R z $\land$ z R y $\land$ x $\neq$ z $\land$ y $\neq$ z))

For each of the following structures, say whether the above statements are true or false.

Example. Let (X, R) = ($\mathbb{N}$, $\leq$).
(i.e., X is the set of natural numbers and R is the less-than-or-equal-to relation on $\mathbb{N}$)

Answer.

i. $\exists$ x $\forall$ y (x R y) ⊗ True ◯ False (take x = 0)

ii. $\exists$ y $\forall$ x (x R y) ◯ True ⊗ False (take x = y+1)

iii. $\forall$ x $\forall$ y (x R y $\land$ x $\neq$ y $\to$ $\exists$z (x R z $\land$ z R y $\land$ x $\neq$ z $\land$ y $\neq$ z))
◯ True ⊗ False (take y = x + 1)

(a) Let (X, R) = ($\mathbb{Z}$, $\leq$). (Recall, $\mathbb{Z}$ denotes the set of integers.)

i. $\exists$ x $\forall$ y (x R y) ◯ True ◯ False

ii. $\exists$ y $\forall$ x (x R y) ◯ True ◯ False

iii. $\forall$ x $\forall$ y (x R y $\land$ x $\neq$ y $\to$ $\exists$ z (x R z $\land$ z R y $\land$ x $\neq$ z $\land$ y $\neq$ z))
◯ True ◯ False

(b) Let (X, R) = ($\mathbb{Q}$, $\leq$). (Recall, $\mathbb{Q}$ denotes the set of rational numbers.)

i. $\exists$ x $\forall$ y (x R y) ◯ True ◯ False

ii. $\exists$ y $\forall$ x (x R y) ◯ True ◯ False

iii. $\forall$ x $\forall$ y (x R y $\land$ x $\neq$ y $\to$ $\exists$ z (x R z $\land$ z R y $\land$ x $\neq$ z $\land$ y $\neq$ z))
◯ True ◯ False

(c) Let (X, R) = ($\mathbb{N}$, $|$). (Recall, a $|$ b means "a evenly divides b.")

i. $\exists$ x $\forall$ y (x R y) ◯ True ◯ False

ii. $\exists$ y $\forall$ x (x R y) ◯ True ◯ False

iii. $\forall$ x $\forall$ y (x R y $\land$ x $\neq$ y $\to$ $\exists$ z (x R z $\land$ z R y $\land$ x $\neq$ z $\land$ y $\neq$ z))
◯ True ◯ False

(d) Let (X, R) = ($\mathscr{P}(\mathbb{N})$, $\subseteq$). (Each variable in the language denotes a *subset* of $\mathbb{N}$.)

i. $\exists$ x $\forall$ y (x R y) ◯ True ◯ False

ii. $\exists$ y $\forall$ x (x R y) ◯ True ◯ False

iii. $\forall$ x $\forall$ y (x R y $\land$ x $\neq$ y $\to$ $\exists$ z (x R z $\land$ z R y $\land$ x $\neq$ z $\land$ y $\neq$ z))
◯ True ◯ False

(8pts) 3. Let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) Show that if $g \circ f$ is one-to-one (or "injective"), then f is one-to-one.

(b) Give an example of spaces X, Y, Z, and functions f: $X \to Y$ and g: $Y \to Z$, such that that $g \circ f$ is one-to-one, but g is not one-to-one.

Let X = Y = Z =

Define f and g as follows:

(c) Suppose $g \circ f$ is one-to-one and f is onto (surjective). Show that g is one-to-one.

(4pts) 4. Let $\equiv$ be an equivalence relation on a set A. For every element $a \in A$, let [a] denote the equivalence class containing a; that is, $[a] = \{ c \mid c \in A \wedge c \equiv a \}$. Show that for every a and b in A, we have $[a] = [b]$ if and only if $a \equiv b$.
[*Hints.* Proving an iff statement typically requires two separate proof steps, one for each implication direction. [a] and [b] are sets, so $[a] = [b]$ means that $\forall\ c\ (c \in [a] \leftrightarrow c \in [b])$. By definition, an element c is in [a] if and only if $c \equiv a$. In particular, $a \in [a]$.]

(6pts) 5. Let the relation $\sim$ on the natural numbers $\mathbb{N}$ be defined as follows: if n is even, then n $\sim$ n + 1; if n is odd, then n $\sim$ n - 1. Furthermore, for every n, let n $\sim$ n.

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{N}$.

(b) What is the equivalence class of 5?

(c) Describe the set $\{[n] \mid n \in \mathbb{N}\}$ of all equivalence classes of $\sim$.

(4pts) 6. A binary relation $\leq$ on a set X is said to be a **preorder** if it is reflexive and transitive. For example, let X denote the collection of all living earthlings and let $\leq$ be defined as follows: $\forall$ x, y $\in$ X, x $\leq$ y if and only if x is no older than y, where age is measured in whole days. Clearly $\leq$ is reflexive and transitive, but it is not symmetric (since younger doesn't imply older) nor antisymmetric (since different people may be the same age). But prove the following.

Theorem. Let $\leq$ be a preorder on a set X. Define the relation $\equiv$, where x $\equiv$ y holds if and only if x $\leq$ y and y $\leq$ x. Then $\equiv$ is an equivalence relation on X.