

Name: \_\_\_\_\_

Student ID: \_\_\_\_\_

**Instructions.** Print out this assignment, fill in your answers *on these pages*, scan the document, and then upload your scanned work to Gradescope. **Important.** You must write your answers on a printout of this pdf! Do not upload any other pages or scratch work.

- (6pts) 1. A *perfect number* is a number that equals the sum of its proper divisors, other than itself. For example, 6 is perfect since  $6 = 1 + 2 + 3$ .

Using a language with variables ranging over the natural numbers and suitable functions and predicates, write down first-order sentences asserting the following. Use a predicate **perfect** to express that a number is perfect. (Incidentally, it is not known whether assertions a and d are true. They are open questions.)

**Example 1.** 28 is perfect. **Solution.**  $\text{perfect}(28)$ .

**Example 2.** There are no perfect numbers between 100 and 200. (Let's assume the phrase "between a and b" includes the endpoints a, b).

**Solution a.**  $\forall x (\text{perfect}(x) \rightarrow x < 100 \vee 200 < x)$ .

**Solution b.**  $\forall x (100 \leq x \leq 200 \rightarrow \neg \text{perfect}(x))$ .

Note that a. and b. are equivalent ("contrapositive") statements, so both are acceptable.

- (a) Every perfect number is even. (In your answer, use  $\exists k (x = 2k)$  to express the assertion "x is even.")

- (b) There are (at least) two perfect numbers between 200 and 10,000.

(*Hint.* This is equivalent to the assertion that there are numbers x and y between 200 and 10,000 which are not the same ( $x \neq y$ ) and are perfect.)

- (c) For every number x, there is a perfect number that is larger than x. (This is how we express the claim that there are infinitely many perfect numbers.)

(6pts) 2. Use a language with variables ranging over people, and predicates

- $\text{trusts}(x,y)$  (denoting “x trusts y”)
- $\text{politician}(x)$  (“x is a politician”)
- $\text{crazy}(x)$  (“x is crazy”)
- $\text{knows}(x, y)$  (“x knows y”)
- $\text{related}(x, y)$  (“x is related to y”)
- $\text{rich}(x)$  (“x is rich”)

to write down first-order sentences expressing each of the assertions below.

**Example.** Nobody trusts a politician.

Solution.  $\forall x (\text{politician}(x) \rightarrow \forall y (\neg \text{trusts}(y,x)))$

Note that, in each case, some interpretation of English into logic is required, and that, consequently, using a logical formula forces us to clarify and make precise a statement’s meaning.

(a) Anyone who trusts a politician is crazy.

(b) Everyone knows someone who is related to a politician.

(c) If a person is rich then that person is either a politician or knows a politician.

(6pts) 3. Give a natural deduction proof of  $\forall x B(x)$  from hypotheses

1.  $\forall x (A(x) \vee B(x))$  and
2.  $\forall y (\neg A(y))$

(4pts) 4. Starting with hypotheses

1.  $\forall x (\text{even}(x) \vee \text{odd}(x))$  and
2.  $\forall x (\text{odd}(x) \rightarrow \text{even}(s(x)))$

give a natural deduction proof of  $\forall x (\text{even}(x) \vee \text{even}(s(x)))$

(It might help to think of  $s(x)$  as the “successor” function,  $s(x) = x + 1$ , but your proof shouldn’t depend on this interpretation.)

(6pts) 5. Give a natural deduction proof of  $\exists x A(x) \vee \exists x B(x) \rightarrow \exists x (A(x) \vee B(x))$ .

(4pts) 6. Give a natural deduction proof of  $\forall x, y (x = y \rightarrow y = x)$  using only the following two hypotheses:

- $\forall x (x = x)$
- $\forall u, v, w (u = w \rightarrow (v = w \rightarrow u = v))$

*Hints.* Recall that  $(\forall x, y)$  is shorthand notation for  $(\forall x) (\forall y)$ ; similarly,  $\forall u, v, w$  is short for  $\forall u \forall v \forall w$ . Choose instantiations of  $u, v$ , and  $w$  carefully. You can instantiate all the universal quantifiers in one step.

(6pts) 7. Give a natural deduction proof of  $\neg(\exists x)(A(x) \wedge B(x)) \rightarrow (\forall x)(A(x) \rightarrow \neg B(x))$ .