

CONGRUENCE LATTICES OF FINITE ALGEBRAS

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By

William J. DeMeo

Dissertation Committee:

Ralph Freese, Chairperson

William Lampe

J.B. Nation

Peter Jipsen

Nick Kaiser

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ABSTRACT

An important and long-standing open problem in universal algebra asks whether every finite lattice is isomorphic to the congruence lattice of a finite algebra. Until this problem is resolved, our understanding of finite algebras is incomplete, since, given an arbitrary finite algebra, we cannot say whether there are any restrictions on the shape of its congruence lattice. If we find a finite lattice that does not occur as the congruence lattice of a finite algebra (as many suspect we will), then we can finally declare that such restrictions do exist.

By a well known result of Pálffy and Pudlák, the problem would be solved if we could prove the existence of a finite lattice that is not the congruence lattice of a transitive group action or, equivalently, is not an interval in the lattice of subgroups of a finite group. Thus the problem of characterizing congruence lattices of finite algebras is closely related to the problem of characterizing intervals in subgroup lattices.

In this work, we review a number of methods for finding a finite algebra with a given congruence lattice, including searching for intervals in subgroup lattices. We also consider methods for proving that algebras with a given congruence lattice exist without actually constructing them. By combining these well known methods with a new method we have developed, and with much help from computer software like the `UACalc` and `GAP`, we prove that with one possible exception every lattice with at most seven elements is isomorphic to the congruence lattice of a finite algebra. As such, we have identified the unique smallest lattice for which there is no known representation. We examine this exceptional lattice in detail, and prove results that characterize the class of algebras that could possibly represent this lattice.

We conclude with what we feel are the most interesting open questions surrounding this problem and discuss possibilities for future work.

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LIST OF SYMBOLS

2	$\{0, 1\}$, or the two element lattice
3	$\{0, 1, 2\}$, or the three element lattice
n	the set $\{0, 1, \dots, n-1\}$, or the n element chain
ω	the natural numbers, $\{0, 1, 2, \dots\}$
\mathbb{Z}	the integers, $\{\dots, -1, 0, 1, \dots\}$
\mathbb{Q}	the rational numbers
\mathbb{F}	an arbitrary field
A, B, C, ...	universal algebras
A = $\langle A, F \rangle$	an algebra with universe A and operations F
$\text{Clo}(\mathbf{A})$	the clone of term operations of A
$\text{Pol}(\mathbf{A})$	the clone of polynomial operations of A
$\text{Pol}_n(\mathbf{A})$	the set of n -ary members of $\text{Pol}(\mathbf{A})$
$\text{Aut}(\mathbf{A})$	the group of automorphisms of A
$\text{Inn}(\mathbf{A})$	the inner automorphisms of A
$\text{Out}(\mathbf{A})$	the outer of automorphisms of A
$\text{End}(\mathbf{A})$	the monoid of endomorphisms of A
$\text{Hom}(\mathbf{A}, \mathbf{B})$	the set of homomorphisms from A into B
$\text{Con}(\mathbf{A})$	the lattice of congruence relations of A
$\text{Sub}(\mathbf{A})$	the lattice of subalgebras of A
$\text{Sg}^{\mathbf{A}}(X)$	the subuniverse of A generated by the set $X \subseteq A$
$\text{Cg}^{\mathbf{A}}(X)$	the congruence of A generated by the set $X \subseteq A \times A$
$\text{Eq}(X)$	the lattice of equivalence relations on the set X
X^X	the set of unary maps from a set X into itself
$\ker f$	the kernel of f , $\{(x, y) \mid f(x) = f(y)\}$
$\mathcal{ID}(X)$	the idempotent decreasing functions in X^X
\sqsubseteq	the partial order defined on $\mathcal{ID}(X)$ by $f \sqsubseteq g \Leftrightarrow \ker f \leq \ker g$
\mathcal{K}	a class of algebras
$\mathbf{H}(\mathcal{K})$	the class of homomorphic images of algebras in \mathcal{K}
$\mathbf{S}(\mathcal{K})$	the class of subalgebras of algebras in \mathcal{K}
$\mathbf{P}(\mathcal{K})$	the class of direct products of algebras in \mathcal{K}
$\mathbf{P}_{\text{fi}}(\mathcal{K})$	the class of finite direct products of algebras in \mathcal{K}
\mathcal{V}	a variety, or equational class, of algebras
$\mathbf{V}(\mathbf{A})$	the variety generated by A (thus $\mathbf{V}(\mathbf{A}) = \mathbf{HSP}(\mathbf{A})$)
$\mathbf{V}(\mathcal{K})$	the variety generated by the class \mathcal{K}
$\mathbf{F}_{\mathcal{V}}(X)$	the free algebra in the variety \mathcal{V} over the generating set X
\mathcal{L}_0	the class of finite lattices
\mathcal{L}_1	the class of lattices isomorphic to sublattices of finite partition lattices
\mathcal{L}_2	the class of lattices isomorphic to strong congruence lattices of finite partial algebras
\mathcal{L}_3	the class of lattices isomorphic to congruence lattices of finite algebras
\mathcal{L}_4	the class of lattices isomorphic to intervals in subgroup lattices of finite groups
\mathcal{L}_5	the class of lattices isomorphic to subgroup lattices of finite groups

Part I

Background

CHAPTER 1

INTRODUCTION

We begin with an informal overview of some of the basic objects of study. This will help to fix notation and motivate our discussion. (Italicized terms are defined more formally in later sections or in the appendix.) Then we introduce the problem that is the main focus of this dissertation, *the finite lattice representation problem* (FLRP). In subsequent sections, we give further notational and algebraic prerequisites and summarize the well known results surrounding the FLRP. In the final section of this chapter we provide a list of the new results of this thesis.

1.1 Motivation and problem statement

Among the most basic objects of study in all of mathematics are algebras. An *algebra* $\mathbf{A} = \langle A, F \rangle$ consists of a nonempty set A and a collection F of operations; the most important examples are lattices, groups, rings, and modules. To understand a particular algebra, \mathbf{A} , we often study its representations, which are *homomorphisms* from \mathbf{A} into some other algebra \mathbf{B} . A very important feature of such a homomorphism φ is its *kernel*, which we define as the set $\{(x, y) \in A^2 : \varphi(x) = \varphi(y)\}$. This is a *congruence relation* of the algebra \mathbf{A} which tells us how \mathbf{A} is “reduced” when represented by its image under φ in \mathbf{B} .

Thus, every homomorphism gives rise to a congruence relation, and the set $\text{Con } \mathbf{A}$ of all congruence relations of the algebra \mathbf{A} forms a *lattice*. For example, if \mathbf{A} happens to be a group, $\text{Con } \mathbf{A}$ is isomorphic to the lattice of normal subgroups of \mathbf{A} .¹ To each congruence $\theta \in \text{Con } \mathbf{A}$ there corresponds the natural homomorphism of \mathbf{A} onto \mathbf{A}/θ which has θ as its kernel. Thus, there is a one-to-one correspondence between $\text{Con } \mathbf{A}$ and the natural homomorphisms, and the shape of $\text{Con } \mathbf{A}$ provides useful information about the algebra and its representations. For instance, $\text{Con } \mathbf{A}$ tells us whether and how \mathbf{A} can be decomposed as, or embedded in, a product of simpler algebras.

Given an arbitrary algebra, then, we ought to know whether there are, *a priori*, any restrictions on the possible shape of its congruence lattice. A celebrated result of Grätzer and Schmidt says that there are (essentially) no such restrictions. Indeed, in [18] it is proved that every (algebraic) lattice is the congruence lattice of some algebra. Moreover, as Jiří Tůma proves in [45], the Grätzer-Schmidt

¹In this context, by “kernel” of a homomorphism φ one typically means the normal subgroup $\{a \in A : \varphi(a) = e\}$, whereas this is a single congruence class of the kernel as we have defined it.

Theorem still holds if we restrict ourselves to intervals in subgroup lattices. That is, every algebraic lattice is isomorphic to an interval in the subgroup lattice of an (infinite) group.

Now, suppose we restrict our attention to *finite* algebras. Given an arbitrary finite algebra, it is natural to ask whether there are any restrictions (besides finiteness) on the shape of its congruence lattice. If it turns out that, given an arbitrary finite lattice \mathbf{L} , we can always find a finite algebra \mathbf{A} that has \mathbf{L} as its congruence lattice, then apparently there are no such restrictions.

We call a lattice *finitely representable*, or simply *representable*, if it is isomorphic to the congruence lattice of a finite algebra, and deciding whether every finite lattice is representable is known as the *finite lattice representation problem* (FLRP). For the reasons mentioned above, this is a fundamental question of modern algebra, and the fact that it remains unanswered is quite remarkable.

1.2 Universal algebra preliminaries

We now describe in greater detail some of the algebraic objects that are central to our work. A more complete introduction to this material can be found in the books and articles listed in the bibliography. In particular, the following are the main references for this work: [26], [32], [12], [38], and [20]. Two excellent survey articles on the finite lattice representation problem are [29] and [30].

First, a few words about notation. When discussing universal algebras, such as $\mathbf{A} = \langle A, F \rangle$, we denote the algebras using bold symbols, as in $\mathbf{A}, \mathbf{B}, \dots$, and reserve the symbols A, B, \dots for the universes of these algebras. However, this convention becomes tiresome and inconvenient if strictly adhered to for all algebras, and we often find ourselves referring to an algebra by its universe. For example, we frequently use L when referring to the lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$, and we usually refer to “the lattice of congruence relations $\text{Con } \langle A, F \rangle$,” even though it would be more precise to call $\text{Con } \langle A, F \rangle$ the universe (a set) and use $\mathbf{Con } \mathbf{A} = \langle \text{Con } \langle A, F \rangle, \wedge, \vee \rangle$ to denote the lattice (an algebra). Certainly we will feel free to commit this sort of abuse when speaking about groups, preferring to use G when referring to the group $\mathbf{G} = \langle G, \cdot, {}^{-1}, 1 \rangle$. Sometimes we use the more precise notation $\mathbf{Eq}(X)$ to denote the lattice of equivalence relations on the set X , but more frequently we will refer to this lattice by its universe, $\text{Eq}(X)$. This has never been a source of confusion.

An *operation symbol* f is an object that has an associated *arity*, which we denote by $\mathbf{a}(f)$. A set of operation symbols F is called a *similarity type*. An *algebra* of similarity type F is a pair $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ consisting of a set A , which we call the *universe* of \mathbf{A} , and a set $F^{\mathbf{A}} = \{f^{\mathbf{A}} : f \in F\}$ of *operations*

on A , which are functions $f^{\mathbf{A}} : A^{\mathbf{a}(f)} \rightarrow A$ of arity $\mathbf{a}(f)$. Occasionally the set of operations only enters the discussion abstractly, and it becomes unnecessary to refer to specific operation symbols. In such instances, we often denote the algebra by $\langle A, \dots \rangle$.

Note that the symbol f – like the operation symbol $+$ that is used to denote addition in *some* algebras – is an abstract operation symbol which, apart from its arity, has no specific meaning attached to it. We use the notation $f^{\mathbf{A}}$ to signify that we have given the operation symbol a specific interpretation as an operation in the algebra \mathbf{A} . Having said that, when there is only one algebra under consideration, it seems pedantic to attach the superscript \mathbf{A} to every operation. In such cases, when no confusion can arise, we allow the operation symbol f to denote a specific operation interpreted in the algebra. Also, if F is the set of operations (or operation symbols) of \mathbf{A} , we let $F_n \subseteq F$ denote the n -ary operations (or operation symbols) of \mathbf{A} .

Let A and B be sets and let $\varphi : A \rightarrow B$ be any mapping. We say that a pair $(a_0, a_1) \in A^2$ belongs to the *kernel* of φ , and we write $(a_0, a_1) \in \ker \varphi$, provided $\varphi(a_0) = \varphi(a_1)$. It is easily verified that $\ker \varphi$ is an equivalence relation on the set A . If θ is an equivalence relation on a set A , then a/θ denotes the equivalence class containing a ; that is, $a/\theta := \{a' \in A : (a, a') \in \theta\}$. The set of all equivalence classes of θ in A is denoted A/θ . That is, $A/\theta = \{a/\theta : a \in A\}$.

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ and $\mathbf{B} = \langle B, F^{\mathbf{B}} \rangle$ be algebras of the same similarity type. A *homomorphism* from \mathbf{A} to \mathbf{B} is a function $\varphi : A \rightarrow B$ that respects the interpretation of the operation symbols. That is, if $f \in F$ with, say, $n = \mathbf{a}(f)$, and if $a_1, \dots, a_n \in A$, then $\varphi(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(\varphi(a_1), \dots, \varphi(a_n))$. A *congruence relation* of \mathbf{A} is the kernel of a homomorphism defined on \mathbf{A} . We denote the set of all congruence relations of \mathbf{A} by $\text{Con } \mathbf{A}$. Thus, $\theta \in \text{Con } \mathbf{A}$ if and only if $\theta = \ker \varphi$ for some homomorphism $\varphi : \mathbf{A} \rightarrow \mathbf{B}$. It is easy to check that this is equivalent to the following: $\theta \in \text{Con } \mathbf{A}$ if and only if $\theta \in \text{Eq}(A)$ and for all n

$$(a_i, a'_i) \in \theta \quad (0 \leq i < n) \quad \Rightarrow \quad (f(a_0, \dots, a_{n-1}), f(a'_0, \dots, a'_{n-1})) \in \theta, \quad (1.2.1)$$

for all $f \in F_n$ and all $a_0, \dots, a_{n-1}, a'_0, \dots, a'_{n-1} \in A$. Equivalently, $\text{Con } \mathbf{A} = \text{Eq}(A) \cap \text{Sub}(\mathbf{A} \times \mathbf{A})$.

Given a congruence relation $\theta \in \text{Con } \mathbf{A}$, the *quotient algebra* \mathbf{A}/θ is the algebra with universe $A/\theta = \{a/\theta \mid a \in A\}$ and operations $\{f^{\mathbf{A}/\theta} \mid f \in F\}$ defined as follows:

$$f^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = f^{\mathbf{A}}(a_1, \dots, a_n)/\theta, \text{ where } n = \mathbf{a}(f).$$

A *partial algebra* is a set A (the universe) along with a set of *partial operations*, that is, operations which may be defined on only part of the universe. A *strong congruence relation* of a partial algebra \mathbf{A} is an equivalence relation $\theta \in \text{Eq}(A)$ with the following property: for each (partial) operation f of A , if f is k -ary, if $(x_i, y_i) \in \theta$ ($1 \leq i \leq k$), and if $f(x_1, \dots, x_k)$ exists, then $f(y_1, \dots, y_k)$ exists, and $(f(x_1, \dots, x_k), f(y_1, \dots, y_k)) \in \theta$. We will have very little to say about partial algebras, but they appear below in our overview of significant results related to the FLRP.

Let $\mathbf{A} = \langle A, \dots \rangle$ be an algebra with congruence lattice $\text{Con } \langle A, \dots \rangle$. Recall that a *clone* on a non-void set A is a set of operations on A that contains the projection operations and is closed under compositions. The *clone of term operations* of the algebra \mathbf{A} , denoted by $\text{Clo}(\mathbf{A})$, is the smallest clone on A containing the basic operations of \mathbf{A} . The *clone of polynomial operations* of \mathbf{A} , denoted by $\text{Pol}(\mathbf{A})$, is the clone generated by the basic operations of \mathbf{A} and the constant unary maps on A . The set of n -ary members of $\text{Pol}(\mathbf{A})$ is denoted by $\text{Pol}_n(\mathbf{A})$.

By a *unary algebra* we mean an algebra with any number of unary operations.² In our work, as we are primarily concerned with congruence lattices, we may restrict our attention to unary algebras whenever helpful or convenient, as the next result shows (cf. Theorem 4.18 of [26]).

Lemma 1.2.1. *If F is a set of operations on A , then*

$$\text{Con } \langle A, F \rangle = \text{Con } \langle A, F' \rangle,$$

where F' is any of $\text{Pol}(\mathbf{A})$, $\text{Pol}_1(\mathbf{A})$, or the set of basic translations (operations in $\text{Pol}_1(\mathbf{A})$ obtained from F by fixing all but one coordinate).

The lattice formed by all subgroups of a group G , denoted $\text{Sub}(G)$, is called the *subgroup lattice* of G . It is a *complete lattice*: any number of subgroups H_i have a *meet* (greatest lower bound) $\bigwedge H_i$, namely their intersection $\bigcap H_i$, and a *join* (least upper bound) $\bigvee H_i$, namely the subgroup generated by the union of them. We denote the group generated by the subgroups $\{H_i : i \in I\}$ by $\langle H_i : i \in I \rangle$ when I is infinite, and by $\langle H_0, H_1, \dots, H_{n-1} \rangle$, otherwise. Since a complete lattice is algebraic if and only if every element is a join of compact elements, we see that subgroup lattices are always algebraic. We mention these facts because of their general importance, but we remind the reader that all groups in this work are finite.

²Note that some authors reserve this term for algebras with a single unary operation, and use the term *multi-unary algebra* when referring to what we call unary algebra.

1.3 Overview of well known results

Major inroads toward a solution to the FLRP have been made by many prominent researchers, including Michael Aschbacher, Walter Feit, Hans Kurzweil, Adrea Lucchini, Ralph McKenzie, Raimund Netter, Péter Pálffy, Pavel Pudlák, John Snow, and Jiří Tůma, to name a few. We will have occasion to discuss and apply a number of their results in the sequel. Here we merely mention some of the highlights, in roughly chronological order.

In his 1968 book *Universal Algebra* [19], George Grätzer defines the following classes of lattices:

- \mathcal{L}_0 = the class of finite lattices;
- \mathcal{L}_1 = the class of lattices isomorphic to sublattices of finite partition lattices;
- \mathcal{L}_2 = the class of lattices isomorphic to strong congruence lattices of finite partial algebras;
- \mathcal{L}_3 = the class of lattices isomorphic to congruence lattices of finite algebras.

Clearly $\mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \mathcal{L}_2 \supseteq \mathcal{L}_3$. Grätzer asks ([19] prob. 13, p. 116) whether equality holds in each case. Whether $\mathcal{L}_0 = \mathcal{L}_1$ is the finite version of a question Garrett Birkhoff had asked by 1935. In [6] Birkhoff asks whether every lattice is isomorphic to a sublattice of some partition lattice. Whitman [47] answered this affirmatively in 1946, but his proof embeds every finite lattice in a countably infinite partition lattice. Still, the result of Whitman also proves that there is no non-trivial law that holds in the subgroup lattice of every group. That is,

Theorem 1.3.1 (Whitman [47]). *Every lattice is isomorphic to a sublattice of the subgroup lattice of some group.*

Confirmation that $\mathcal{L}_0 = \mathcal{L}_1$ did not come until the late 1970's, when Pavel Pudlák and Jiří Tůma published [35], in which they prove that every finite lattice can be embedded in a finite partition lattice, thus settling this important and long-standing open question. This result also yields the following finite analogue of Whitman's result:

Theorem 1.3.2 (Pudlák-Tůma [35]). *Every finite lattice is isomorphic to a sublattice of the subgroup lattice of some finite group.*

If we confine ourselves to distributive lattices, the analogue of the FLRP is relatively easy. By the 1930's it was already known to Robert Dilworth that every finite distributive lattice is the congruence lattice of a finite lattice.³ (In fact, if we allow representations by infinite algebras –

³This is mentioned in [7] without proof.

which, as a rule in this work, we do not – then the congruence lattices of modular lattices already account for all distributive lattices. This is shown by E.T. Schmidt in [40], and extended by Ralph Freese who shows in [15] that finitely generated modular lattices suffice.)⁴

A lattice L is called *strongly representable* if, whenever L is isomorphic to a *spanning sublattice*⁵ $L_0 \leq \text{Eq}(X)$ for some X , then there is an algebra $\langle X, \dots \rangle$ whose congruence lattice is L_0 .

Theorem 1.3.3 (Berman [5], Quackenbush and Wolk [36]). *Every finite distributive lattice is strongly representable.*

(We give a short proof of this result in Section 3.3.3.) Berman also proves that if \mathbf{A}_p is a finite partial unary algebra with strong congruence lattice $\text{Con}_s \mathbf{A}_p$, then there is a finite unary algebra \mathbf{A} with $\text{Con } \mathbf{A} \cong \text{Con}_s \mathbf{A}_p$. Therefore, by Lemma 1.2.1, $\mathcal{L}_2 = \mathcal{L}_3$. As our focus is mainly on whether $\mathcal{L}_0 = \mathcal{L}_3$, we will not say more about partial algebras except to note that the results of Pudlák, Tůma, and Berman imply that $\mathcal{L}_0 = \mathcal{L}_3$ holds if and only if $\mathcal{L}_1 = \mathcal{L}_2$ holds.

Next, we mention another deep result of Pudlák and Tůma, which proves the existence of congruence lattice representations for a large class of lattices.

Theorem 1.3.4 (Pudlák and Tůma [34]). *Let L be a finite lattice such that both L and its congruence lattice have the same number of join irreducible elements. Then L is representable.*

Notice that finite distributive lattices satisfy the assumption of Theorem 1.3.4, so this provides yet another proof that such lattices are representable.

We now turn to subgroup lattices of finite groups and their connection with the FLRP. The study of subgroup lattices has a long history, starting with Richard Dedekind's work [10] in 1877, including Ada Rottlaender's paper [39] from 1928, and later numerous important contributions by Reinhold Baer, Øystein Ore, Kenkichi Iwasawa, Leonid Efimovich Sadovskii, Michio Suzuki, Giovanni Zacher, Mario Curzio, Federico Menegazzo, Roland Schmidt, Stewart Stonehewer, Giorgio Busetto, and many others. The book [41] by Roland Schmidt gives a comprehensive account of this work.

Suppose H is a subgroup of G (denoted $H \leq G$). By the *interval sublattice* $[H, G]$ we mean the sublattice of $\text{Sub}(G)$ given by:

$$[H, G] := \{K : H \leq K \leq G\},$$

⁴It turns out that the finite distributive lattices are representable as congruence lattices of other restricted classes of algebras. We will say a bit more about this below, but we refer the reader to [28] for more details.

⁵By a *spanning sublattice* of a bounded lattice L_0 , we mean a sublattice $L \leq L_0$ that has the same top and bottom as L_0 . That is $1_L = 1_{L_0}$ and $0_L = 0_{L_0}$.

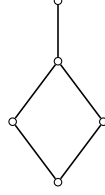
That is $[H, G]$ is the lattice of subgroups of G that contain H .⁶

We define the following classes of lattices:

- \mathcal{L}_4 = the class of lattices isomorphic to intervals in subgroup lattices of finite groups;
- \mathcal{L}_5 = the class of lattices isomorphic to subgroup lattices of finite groups.

Recall that \mathcal{L}_3 , the class of all lattices isomorphic to congruence lattices of finite algebras, is known as the class of *representable* lattices. We adhere to this convention throughout and, moreover, we will call a lattice *group representable* if it belongs to \mathcal{L}_4 .

Clearly, $\mathcal{L}_4 \supseteq \mathcal{L}_5$, since $\text{Sub}(G)$ is itself the interval $[1, G]$. Moreover, it's easy to find a lattice that is in \mathcal{L}_4 but not in \mathcal{L}_5 , so the inclusion is strict. For example, there is no group G for which $\text{Sub}(G)$ is isomorphic to the lattice shown below.



To see this, note that if G has a unique maximal subgroup H , then there exists $g \in G \setminus H$ and we must have $\langle g \rangle = G$. Thus, if $\text{Sub}(G)$ has a unique coatom, then G is cyclic, and subgroup lattices of cyclic groups are self-dual, unlike the lattice shown above. However, this lattice belongs to \mathcal{L}_4 . For example, it is the filter above $H = C_3$ in the subgroup lattice of $G = C_3 \times (C_3 \rtimes C_4)$.

We will have a lot more to say about intervals in subgroup lattices throughout this thesis. Perhaps the most useful fact for our work is the following:

$$\text{Every interval in a subgroup lattice is the congruence lattice of a finite algebra.} \quad (1.3.1)$$

In particular, as we explain below in Chapter 4, if $\langle G/H, G \rangle$ is the algebra consisting of the group G acting on the left (right) cosets of a subgroup $H \leq G$ by left (right) multiplication, then $\text{Con} \langle G/H, G \rangle \cong [H, G]$. Thus, we see that $\mathcal{L}_3 \supseteq \mathcal{L}_4$.

Whether the converse of (1.3.1) holds – and thus whether $\mathcal{L}_3 = \mathcal{L}_4$ – is an open question. In other words, it is not known whether every congruence lattice of a finite algebra is isomorphic to an

⁶The reader may anticipate confusion arising from the conflict between our notation and the well-established notation for the *commutator subgroup*, $[H, G] := \langle \{hgh^{-1}g^{-1} : h \in H, g \in G\} \rangle$, which we will also have occasion to use. However, we have found that context always makes clear which meaning is intended. In any case, we often refer to “the interval $[H, G]$ ” or “the commutator $[H, G]$.”

interval in the subgroup lattice of a finite group. However, a surprising and deep result related to this question was proved in 1980 by Péter Pálffy and Pavel Pudlák. In [32], they prove

Theorem 1.3.5. *The following statements are equivalent:*

- (i) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*
- (ii) *Every finite lattice is isomorphic to the congruence lattice of a finite transitive G -set.*

As we will see later (Theorem 4.1.2), statement (ii) is equivalent to

(ii)' *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

It is important to note that Theorem 1.3.5 does *not* say $\mathcal{L}_3 = \mathcal{L}_4$. Rather, it says that $\mathcal{L}_0 = \mathcal{L}_3$ if and only if $\mathcal{L}_0 = \mathcal{L}_4$. Moreover, this result implies that if we prove the existence of a lattice which is not isomorphic to an interval in a subgroup lattice of a finite group, then we have solved the FLRP.

It is surprising that a problem about general algebras can be reduced to a problem about such a special class of algebras – finite transitive G -sets. Also surprising, in view of all that we know about finite groups and their actions, is that we have yet to determine whether these statements are true or false. To put it another way, given an arbitrary finite lattice L , it is unknown whether there must be a finite group having this lattice as an interval in its lattice of subgroups.

We pause for a moment to consider the $\mathcal{L}_3 = \mathcal{L}_4$ question in the restricted case of finite distributive lattices (which we know are strongly representable). Silcock [42] and Pálffy [28] prove that every finite distributive lattice is an interval in the subgroup lattice of some finite solvable group. The main result is stated below as Theorem 1.3.7, and this can be combined with the following easy lemma to establish the claim.

Lemma 1.3.6. *If $D = \{(g, g) \in G \times G \mid g \in G\}$ then the interval $[D, G \times G]$ is isomorphic to the lattice of normal subgroups of G .*

Theorem 1.3.7. *Every finite distributive lattice is isomorphic to the lattice of normal subgroups of a finite solvable group.*

Beyond those mentioned in this brief introduction, many other results surrounding the FLRP have been proven. Some of these are not as relevant to our work, and others will be discussed in detail in Chapter 2. A more complete overview of the FLRP with an emphasis on group theory can be found in the articles by Pálffy, [29] and [30].

CHAPTER 2

AN OVERVIEW OF FINITE LATTICE REPRESENTATIONS

In this chapter we give a brief overview of various known methods for representing a given lattice as the congruence lattice of a finite algebra or proving that such a representation exists. In later chapters we describe these methods in greater detail and show how to apply them. In particular, in Section 6.2, we use them along with some new methods to show that, with one possible exception, every lattice with no more than seven elements is isomorphic to the congruence lattice of a finite algebra. Throughout this chapter, we continue to use \mathcal{L}_3 to denote the class of finite lattices that are isomorphic to congruence lattices of finite algebras. Again, we call the lattices that belong to \mathcal{L}_3 *representable* lattices.

2.1 Closure properties of the class of representable lattices

This section concerns *closure properties* of the class \mathcal{L}_3 . More precisely, if \mathbf{O} is an operation that can be applied to a lattice or collection of lattices, we say that \mathcal{L}_3 is *closed under* \mathbf{O} provided $\mathbf{O}(\mathcal{K}) \subseteq \mathcal{L}_3$ for all $\mathcal{K} \subseteq \mathcal{L}_3$. For example, if $\mathbf{S}(\mathcal{K}) = \{\text{all sublattices of lattices in } \mathcal{K}\}$, then it is clearly unknown whether \mathcal{L}_3 is closed under \mathbf{S} , for otherwise the FLRP would be solved. (Clearly, $\text{Eq}(X) \in \mathcal{L}_3$ for every finite set X – take the algebra to be the set X with no operations. Then $\text{Con} \langle X, \emptyset \rangle = \text{Eq}(X)$. So, if \mathcal{L}_3 were closed under \mathbf{S} , then \mathcal{L}_3 would contain all finite lattices, by the result of Pudlák and Tůma mentioned above; that is, $\mathcal{L}_0 = \mathcal{L}_1$.)

The following is a list of known closure properties of \mathcal{L}_3 and the names of those who first (or independently) proved them. We discuss some of these results in greater detail later in this section. The class \mathcal{L}_3 of lattices isomorphic to congruence lattices of finite algebras is closed under

1. lattice duals¹ (Hans Kurzweil [23] and Raimund Netter [27], 1986),
2. interval sublattices (follows from Kurzweil-Netter),
3. direct products (Jiří Tůma [45], 1986),
4. ordinal sums (Ralph McKenzie [25], 1984; John Snow [43], 2000),

¹Recall, the *dual of a lattice* is simply the lattice turned on its head, that is, the lattice obtained by reversing the partial order of the original lattice.

5. parallel sums (John Snow [43], 2000),
6. certain sublattices of lattices in \mathcal{L}_3 – namely, those which are obtained as a union of a filter and an ideal of a lattice in \mathcal{L}_3 (John Snow [43], 2000).

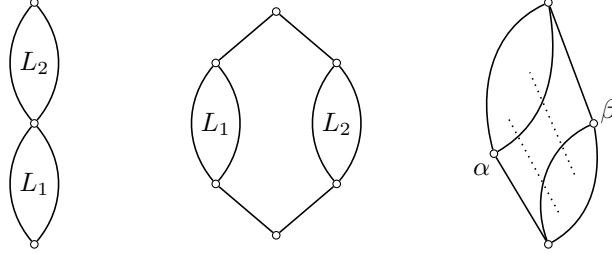


Figure 2.1: The ordinal (left) and parallel (middle) sum of the lattices L_1 and L_2 ; a sublattice obtained as a union of a filter α^\uparrow and an ideal β^\downarrow (right).

Remarks.

1. The first result says that if L is representable then so is the dual of L .
2. It follows from item 1. that any interval sublattice of a representable lattice is representable. For, let $[\alpha, \beta] := \{\theta \in L \mid \alpha \leq \theta \leq \beta\}$ be an interval in the representable lattice $L = \text{Con } \mathbf{A}$. Then $[\alpha, 1_A] \cong \text{Con } \mathbf{A}/\alpha$. By 1., the dual of $\ell := [\alpha, 1_A]$ is representable. Now take the filter above β' in ℓ' (where β' is the image of β under dualization) and we obtain a representation of a lattice isomorphic to the dual of $[\alpha, \beta]$. Apply 1. again and we have the desired representation of $[\alpha, \beta]$.
3. Of course, by direct products we mean *finite* direct products.
- 4.-5. By the ordinal (parallel) sum of two lattices L_1, L_2 , we mean the lattice on the left (middle) of Figure 2.1.
6. The property in item 6. is very useful and we discuss it further in Section 2.3 below, where we present a very short proof of this result. It will come up again in Section 6 when we prove the existence of representations of small lattices.

Whether the class \mathcal{L}_3 is closed under homomorphic images seems to be an open question.

2.2 Lattice duals: the theorem of Kurzweil and Netter

As mentioned above, the class \mathcal{L}_3 – the lattices isomorphic to congruence lattices of finite algebras – is closed under dualization. That is, if L is representable, then so is the dual of L . This was proved in 1986 by Raimund Netter [27], generalizing the idea of his advisor, Hans Kurzweil [23]. Though Kurzweil’s article did appear (in German), it is unclear whether Netter’s article was ever published. In this section we present a proof of their result. The argument requires a fair bit of machinery, but it is a nice idea and well worth the effort.²

If G is a group and X a set, then the set $\{f \mid X \rightarrow G\}$ of functions from X into G is denoted by G^X . This is a group with binary operation $(f, g) \mapsto f \cdot g$, where, for each $x \in X$, $(f \cdot g)(x) = f(x)g(x)$ is simply multiplication in the group G . The identity of the group G^X is of course the constant map $f(x) = 1_G$ for all $x \in X$.

Let X be a finite totally ordered set, with order relation \leq , and consider the set X^X of functions mapping X into itself. The subset of X^X consisting of functions that are both idempotent and decreasing³ will be denoted by $\mathcal{ID}(X)$. That is,

$$\mathcal{ID}(X) = \{f \in X^X \mid f^2 = f \text{ and } \forall x f(x) \leq x\}.$$

Define a partial order \sqsubseteq on the set $\mathcal{ID}(X)$ by

$$f \sqsubseteq g \iff \ker f \leq \ker g, \tag{2.2.1}$$

where $\ker f = \{(x, y) \mid f(x) = f(y)\}$. It is easy to see that $f \sqsubseteq g$ holds if and only if $gf = g$. Moreover, under this partial ordering $\mathcal{ID}(X)$ is a lattice which is isomorphic to $\mathbf{Eq}(X)$ (viz. the map $\Theta : \mathbf{Eq}(X) \rightarrow \mathcal{ID}(X)$ given by $\Theta(\alpha) = f_\alpha$, where $f_\alpha(x) = \min\{y \in X \mid (x, y) \in \alpha\}$.)

Suppose S is a finite nonabelian simple group, and consider S^n , the direct power of n copies of S . An element of S^n may be viewed as a map from the set $n = \{0, 1, \dots, n-1\}$ into S . Thus, if $x = (x_0, x_1, \dots, x_{n-1}) \in S^n$, then by $\ker x$ we mean the relation $(i, j) \in \ker x$ if and only if $x_i = x_j$. The set of constant maps is a subgroup $D < S^n$, sometimes called the *diagonal subgroup*; that is, $D = \{(s, s, \dots, s) \mid s \in S\} \leq S^n$.

²We learned of the main argument used in the proof from slides of a series of three lectures given by Péter Pálffy in 2009 [31]. Pálffy gives credit for the argument to Kurzweil and Netter.

³When we say that the map f is *decreasing* we mean $f(x) \leq x$ for all x . (We do not mean $x \leq y$ implies $f(y) \leq x$.)

For each $f \in \mathcal{ID}(n)$, define

$$K_f = \{(x_{f(0)}, x_{f(1)}, \dots, x_{f(n-1)}) \mid x_{f(i)} \in S, i = 0, 1, \dots, n-1\}.$$

Then $D \leq K_f \leq S^n$, and K_f is the set of maps $K_f = \{xf \in S^n \mid x \in S^n\}$; i.e., compositions of the given map $f \in n^n$, followed by any $x \in S^n$. Thus, $K_f = \{y \in S^n \mid \ker f \leq \ker y\}$. For example, if $f = (0, 0, 2, 3, 2) \in \mathcal{ID}(5)$, then $\ker f = |0, 1|2, 4|3|$ and K_f is the subgroup of all $(y_0, y_1, \dots, y_4) \in S^5$ having $y_0 = y_1$ and $y_2 = y_4$. That is, $K_f = \{(x_0, x_0, x_2, x_3, x_2) \mid x \in S^5\}$.

Lemma 2.2.1. *The map $f \mapsto K_f$ is a dual lattice isomorphism from $\mathbf{Eq}(n)$ onto the interval sublattice $[D, S^n] \leq \text{Sub}(S^n)$.*

Proof. This is clear since $\mathcal{ID}(n)$ is ordered by (2.2.1), and we have $f \sqsubseteq h$ if and only if $K_h = \{y \in S^n \mid \ker h \leq \ker y\} \leq \{y \in S^n \mid \ker f \leq \ker y\} = K_f$. \square

Theorem 2.2.2 (Kurzweil [23], Netter [27]). *If the finite lattice L is representable (as the congruence lattice of a finite algebra), then so is the dual lattice L' .*

Proof. Without loss of generality, we assume that L is concretely represented as $L = \text{Con} \langle n, F \rangle$. By Lemma 1.2.1, we can further assume that F consists of unary operations: $F \subseteq n^n$. As above, let S be a nonabelian simple group and let D be the diagonal subgroup of S^n . Then the unary algebra $\langle S^n/D, S^n \rangle$ is a transitive S^n -set which (by Theorem 4.1.2 below) has congruence lattice isomorphic to the interval $[D, S^n]$. By Lemma 2.2.1, this is the dual of the lattice $\mathbf{Eq}(n)$. That is, $\text{Con} \langle S^n/D, S^n \rangle \cong (\mathbf{Eq}(n))'$.

Now, each operation $\varphi \in F$ gives rise to an operation on S^n by composition:

$$\hat{\varphi}(\mathbf{s}) = \hat{\varphi}(s_0, s_1, \dots, s_{n-1}) = (s_{\varphi(0)}, s_{\varphi(1)}, \dots, s_{\varphi(n-1)}).$$

Thus, φ induces an operation on S^n/D since, for $\mathbf{d} = (d, d, \dots, d) \in D$ and $\mathbf{s} \in S^n$ we have $\mathbf{s}\mathbf{d} = (s_0d, s_1d, \dots, s_{n-1}d)$ and $\hat{\varphi}(\mathbf{s}\mathbf{d}) = (s_{\varphi(0)}d, s_{\varphi(1)}d, \dots, s_{\varphi(n-1)}d) = \hat{\varphi}(\mathbf{s})\mathbf{d}$, so $\hat{\varphi}(\mathbf{s}D) = \hat{\varphi}(\mathbf{s})D$. Finally, add the set of operations $\hat{F} = \{\hat{\varphi} \mid \varphi \in F\}$ to $\langle S^n/D, S^n \rangle$, yielding the new algebra $\langle S^n/D, S^n \cup \hat{F} \rangle$, and observe that a congruence $\theta \in \text{Con} \langle S^n/D, S^n \rangle$ remains a congruence of $\langle S^n/D, S^n \cup \hat{F} \rangle$ if and only if it corresponds to a partition on n that is invariant under F . \square

2.3 Union of a filter and ideal

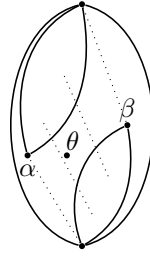
The lemma in this section was originally proved by John Snow using primitive positive formulas. Since it provides such a useful tool for proving that certain finite lattices are representable as congruence lattices, we give our own direct proof of the result below. In Chapter 6 we use this lemma to prove the existence of representations of a number of small lattices.

Before stating the lemma, we need a couple of definitions. (These will be discussed in greater detail in Section 3.2.) Given a relation $\theta \subseteq X \times X$, we say that the map $f : X^n \rightarrow X$ *respects* θ and we write $f(\theta) \subseteq \theta$ provided $(x_i, y_i) \in \theta$ implies $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \theta$. For a set $L \subseteq \text{Eq}(X)$ of equivalence relations we define

$$\lambda(L) = \{f \in X^X : (\forall \theta \in L) f(\theta) \subseteq \theta\},$$

which is the set of all unary maps on X which respect all relations in L .

Lemma 2.3.1. *Let X be a finite set. If $\mathbf{L} \leq \mathbf{Eq}(X)$ is representable and $\mathbf{L}_0 \leq \mathbf{L}$ is a sublattice with universe $\alpha^\uparrow \cup \beta^\downarrow$ where $\alpha^\uparrow = \{x \in L \mid \alpha \leq x\}$ and $\beta^\downarrow = \{x \in L \mid x \leq \beta\}$ for some $\alpha, \beta \in L$, then \mathbf{L}_0 is representable.*



$$L_0 \leq L$$

Proof. Assume $\mathbf{L}_0 \not\cong \mathbf{2}$, otherwise the result holds trivially. Since $\mathbf{L} \leq \mathbf{Eq}(X)$ is representable, we have $\mathbf{L} = \mathbf{Con} \langle X, \lambda(L) \rangle$ (cf. Section 3.2). Take an arbitrary $\theta \in L \setminus L_0$. Since $\theta \notin \alpha^\uparrow$, there is a pair $(a, b) \in \alpha \setminus \theta$. Since $\theta \notin \beta^\downarrow$, there is a pair $(u, v) \in \theta \setminus \beta$. Define $h \in X^X$ as follows:

$$h(x) = \begin{cases} a, & x \in u/\beta, \\ b, & \text{otherwise.} \end{cases}$$

Then, $\beta \leq \ker h = (u/\beta)^2 \cup ((u/\beta)^c)^2$, where $(u/\beta)^c$ denotes the complement of the β class containing u . Therefore, h respects every $\gamma \leq \beta$. Furthermore, $(a, b) \in \gamma$ for all $\gamma \geq \alpha$, so h respects every γ above α . This proves that $h \in \lambda(L_0)$. Now, θ was arbitrary, so we have proved that for every $\theta \in L \setminus L_0$ there exists a function in $\lambda(L_0)$ which respects every $\gamma \in \alpha^\uparrow \cup \beta^\downarrow = L_0$, but violates θ . Finally, since $\mathbf{L}_0 \leq \mathbf{L}$, we have $\lambda(L) \subseteq \lambda(L_0)$. Combining these observations, we see that every $\theta \in \text{Eq}(X) \setminus L_0$ is violated by some function in $\lambda(L_0)$. Therefore, $\mathbf{L}_0 = \mathbf{Con} \langle X, \lambda(L_0) \rangle$. \square

2.4 Ordinal sums

The following theorem is a consequence of McKenzie's shift product construction [25].

Theorem 2.4.1. *If $L_1, \dots, L_n \in \mathcal{L}_3$ is a collection of representable lattices, then the ordinal sum and the adjoined ordinal sum, shown in Figure 2.4, are representable.*

A more direct proof of Theorem 2.4.1 follows the argument given by John Snow in [43]. As noted above, Jiří Tůma proved that the class of finite representable lattices is closed under direct products. Thus, if L_1 and L_2 are representable, then so is $L_1 \times L_2$. Now note that the adjoined ordinal sum of L_1 and L_2 is the union, $\alpha^\uparrow \cup \beta^\downarrow$, of a filter and ideal in the lattice $L_1 \times L_2$, where $\alpha = \beta = 1_{L_1} \times 0_{L_2}$. Therefore, by Lemma 2.3.1, the adjoined ordinal sum is representable. A trivial induction argument proves the result for adjoined ordinal sums of n lattices. The same result for ordinal sums (Figure 2.4 left) follows since the two element lattice is obviously representable.

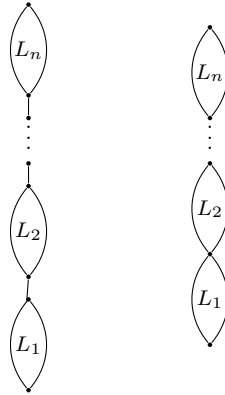


Figure 2.2: The ordinal sum (left) and the adjoined ordinal sum (right) of the lattices L_1, \dots, L_n .

Part II

Finite Lattice Representations

CHAPTER 3

CONCRETE REPRESENTATIONS

In this chapter we introduce a strategy that has proven very useful for showing that a given lattice is representable as a congruence lattice of a finite algebra. We call it the *closure method*, and it has become especially useful with the advent of powerful computers which can search for such representations. Here, as above, $\text{Eq}(X)$ denotes the lattice of equivalence relations on X . Sometimes we abuse notation and take $\text{Eq}(X)$ to mean the lattice of partitions of the set X . This has never caused problems because these two lattices are isomorphic.

3.1 Concrete versus abstract representations

As Bjarni Jónsson explains in [21], there are two types of representation problems for congruence lattices, the concrete and the abstract. The *concrete representation problem* asks whether a specific family of equivalence relations on a set A is equal to $\text{Con } \mathbf{A}$ for some algebra \mathbf{A} with universe A . The *abstract representation problem* asks whether a given lattice is isomorphic to $\text{Con } \mathbf{A}$ for some algebra \mathbf{A} .

These two problems are closely related, and have become even more so since the publication in 1980 of [35], in which Pavel Pudlák and Jiří Tůma prove that every finite lattice can be embedded as a spanning sublattice¹ of the lattice $\text{Eq}(X)$ of equivalence relations on a finite set X . Given this result, we see that even if our goal is to solve the abstract representation problem for some (abstract) lattice L , then we can embed L into $\text{Eq}(X)$ as $L \cong L_0 \leq \text{Eq}(X)$, for some finite set X , and then try to solve the concrete representation problem for L_0 .

A point of clarification is in order here. The term *representation* has become a bit overused in the literature about the finite lattice representation problem. On the one hand, given a finite lattice L , if there is a finite algebra \mathbf{A} such that $L \cong \text{Con } \mathbf{A}$, then L is called a *representable lattice*. On the other hand, given a sublattice $L_0 \leq \text{Eq}(X)$, if $L_0 \cong L$, then L_0 is sometimes called a *concrete representation* of the lattice L (whether or not it is the congruence lattice of an algebra). Below we will define the notion of a *closed concrete representation*, and if we have this special kind of concrete representation of a give lattice, then that lattice is indeed representable in the first sense.

¹Recall, by a *spanning sublattice* of a bounded lattice L_0 , we mean a sublattice $L \leq L_0$ that has the same top and bottom as L_0 . That is $1_L = 1_{L_0}$ and $0_L = 0_{L_0}$.

As we will see below, there are many examples in which a particular concrete representation $L_0 \leq \text{Eq}(X)$ of L is not a congruence lattice of a finite algebra. (In fact, we will describe general situations in which we can guarantee that there are no non-trivial² operations which respect the equivalence relations of L_0 .) This does not imply that $L \notin \mathcal{L}_3$. It may simply mean that L_0 is not the “right” concrete representation of L , and perhaps we can find some other $L \cong L_1 \leq \text{Eq}(X)$ such that $L_1 = \text{Con} \langle X, \lambda(L_1) \rangle$.

3.2 The closure method

The idea described in this section first appeared in *Topics in Universal Algebra* [21], pages 174–175, where Jónsson states, “these or related results were discovered independently by at least three different parties during the summer and fall of 1970: by Stanley Burris, Henry Crapo, Alan Day, Dennis Higgs and Warren Nickols at the University of Waterloo, by R. Quackenbush and B. Wolk at the University of Manitoba, and by B. Jónsson at Vanderbilt University.”

Let X^X denote the set of all (unary) maps from the set X to itself, and let $\text{Eq}(X)$ denote the lattice of equivalence relations on the set X . If $\theta \in \text{Eq}(X)$ and $h \in X^X$, we write $h(\theta) \subseteq \theta$ and say that “ h respects θ ” if and only if for all $(x, y) \in X^2$ $(x, y) \in \theta$ implies $(h(x), h(y)) \in \theta$. If $h(\theta) \not\subseteq \theta$, we sometimes say that “ h violates θ .”

For $L \subseteq \text{Eq}(X)$ define

$$\lambda(L) = \{h \in X^X : (\forall \theta \in L) h(\theta) \subseteq \theta\}.$$

For $H \subseteq X^X$ define

$$\rho(H) = \{\theta \in \text{Eq}(X) : (\forall h \in H) h(\theta) \subseteq \theta\}.$$

The map $\rho\lambda$ is a *closure operator* on $\text{Sub}[\text{Eq}(X)]$. That is, $\rho\lambda$ is

- *idempotent*:³ $\rho\lambda\rho\lambda = \rho\lambda$;
- *extensive*: $L \subseteq \rho\lambda(L)$ for every $L \leq \text{Eq}(X)$;
- *order preserving*: $\rho\lambda(L) \leq \rho\lambda(L_0)$ if $L \leq L_0$.

Given $L \leq \text{Eq}(X)$, if $\rho\lambda(L) = L$, then we say L is a *closed* sublattice of $\text{Eq}(X)$, in which case we

²By a *non-trivial function* we mean a function that is not constant and not the identity.

³In fact, $\rho\lambda\rho = \rho$ and $\lambda\rho\lambda = \lambda$.

clearly have

$$L = \text{Con} \langle X, \lambda(L) \rangle.$$

This suggests the following strategy for solving the representation problem for a given abstract finite lattice L : search for a concrete representation $L \cong L_0 \leq \text{Eq}(X)$, compute $\lambda(L_0)$, compute $\rho\lambda(L_0)$, and determine whether $\rho\lambda(L_0) = L_0$. If so, then we have solved the abstract representation problem for L , by finding a *closed concrete representation*, or simply *closed representation*, of L_0 . We call this strategy the *closure method*.

We now state without proof a well known theorem which shows that the finite lattice representation problem can be formulated in terms of closed concrete representations (cf. [21]).

Theorem 3.2.1. *If $\mathbf{L} \leq \mathbf{Eq}(X)$, then $\mathbf{L} = \mathbf{Con} \mathbf{A}$ for some algebra $\mathbf{A} = \langle X, F \rangle$ if and only if \mathbf{L} is closed.*

In the remaining sections of this chapter, we consider various aspects of the closure method and prove some results about it. Later, in Section 6.2, we apply it to the problem of finding closed representations of all lattices of small order. Before proceeding, however, we introduce a slightly different set-up than the one introduced above that we have found particularly useful for implementing the closure method on a computer. Instead of considering the set of equivalence relations on a finite set, we work with the set of idempotent decreasing maps. These were introduced above in Section 2.2, but we briefly review the definitions here for convenience.

Given a totally ordered set X , let the set $\mathcal{ID}(X) = \{f \in X^X : f^2 = f \text{ and } f(x) \leq x\}$ be partially ordered by \sqsubseteq as follows:

$$f \sqsubseteq g \quad \Leftrightarrow \quad \ker f \leq \ker g.$$

As noted above, this makes $\mathcal{ID}(X)$ into a lattice that is isomorphic to $\mathbf{Eq}(X)$. Define a relation R on $X^X \times \mathcal{ID}(X)$ as follows:

$$(h, f) \in R \quad \Leftrightarrow \quad (\forall (x, y) \in \ker f) (h(x), h(y)) \in \ker f.$$

If $h R f$, we say that h *respects* f .

Let $\mathcal{F} = \mathcal{P}(\mathcal{ID}(X))$ and $\mathcal{H} = \mathcal{P}(X^X)$ be partially ordered by set inclusion, and define the

maps $\lambda : \mathcal{F} \rightarrow \mathcal{H}$ and $\rho : \mathcal{H} \rightarrow \mathcal{F}$ as follows:

$$\lambda(F) = \{h \in X^X : \forall f \in F, h R f\} \quad (F \in \mathcal{F})$$

$$\rho(H) = \{f \in \mathcal{ID}(X) : \forall h \in H, h R f\} \quad (H \in \mathcal{H})$$

The pair (λ, ρ) defines a *Galois correspondence* between $\mathcal{ID}(X)$ and X^X . That is, λ and ρ are antitone maps such that $\lambda\rho \geq \text{id}_{\mathcal{H}}$ and $\rho\lambda \geq \text{id}_{\mathcal{F}}$. In particular, for any set $F \in \mathcal{F}$ we have $F \subseteq \rho\lambda(F)$. These statements are all trivial verifications, and a couple of easy consequences are:

1. $\rho\lambda\rho = \rho$ and $\lambda\rho\lambda = \lambda$,
2. $\rho\lambda$ and $\lambda\rho$ are idempotent.

Since the map $\rho\lambda$ from \mathcal{F} to itself is idempotent, extensive, and order preserving, it is a *closure operator* on \mathcal{F} , and we say a set $F \in \mathcal{F}$ is *closed* if and only if $\rho\lambda(F) = F$. Equivalently, F is closed if and only if $F = \rho(H)$ for some $H \in \mathcal{H}$.

3.3 Superbad representations

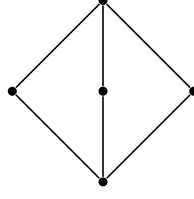
In this section we describe what is in some sense the worst kind of concrete representation. Given an abstract finite lattice \mathbf{L} , it may happen that, upon computing the closure of a particular representation $\mathbf{L} \cong \mathbf{L}_0 \leq \mathbf{Eq}(X)$, we find that $\rho\lambda(L_0)$ is all of $\mathbf{Eq}(X)$. We call such an \mathbf{L}_0 a *dense sublattice* of $\mathbf{Eq}(X)$, or more colloquially, a *superbad representation* of \mathbf{L} .

More generally, if A and B are subsets of $\mathcal{ID}(X)$, we say that A is *dense* in B if and only if $\rho\lambda(A) \supseteq B$. If \mathbf{L} is a finite lattice and there exists an embedding $\mathbf{L} \cong \mathbf{L}_0 \leq \mathbf{Eq}(X)$ such that $\rho\lambda(L_0) = \mathbf{Eq}(X)$, we say that \mathbf{L} can be *densely embedded* in $\mathbf{Eq}(X)$.

3.3.1 Density

One of the first questions we asked concerned the 5-element modular lattice, denoted \mathbf{M}_3 (sometimes called the *diamond*; see Figure 3.3.1). We asked for which sets X does the lattice of equivalence relations on X contain a dense \mathbf{M}_3 sublattice. The answer is given by

Proposition 3.3.1. *The lattice $\mathbf{Eq}(X)$ contains a proper dense \mathbf{M}_3 sublattice if and only if $|X| \geq 5$.*



\mathbf{M}_3

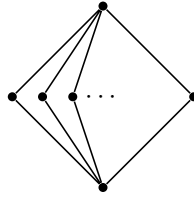
Figure 3.1: The 5-element non-distributive lattice, \mathbf{M}_3 .

This basically says that, when $|X| \geq 5$, the lattice of equivalences on X contains a spanning diamond \mathbf{L} with the property that every non-trivial operation in X^X violates some equivalence relation in the universe L of \mathbf{L} . Thus, the closure $\rho\lambda(L)$ is all of $\mathbf{Eq}(X)$. John Snow proved this for $|X|$ odd. Using the same technique (and some rather tedious calculations), we verified that the result holds for $|X|$ even as well.

Before moving on to the next result, we note that the necessity part of the proposition above is obvious. For, if $|X| \leq 2$, then $\mathbf{Eq}(X)$ has no \mathbf{M}_3 sublattice. If $|X| = 3$, then $\mathbf{Eq}(X)$ is itself \mathbf{M}_3 . It can be checked directly (by computing all possibilities) that, when $|X| = 4$, $\mathbf{Eq}(X)$ has one closed \mathbf{M}_3 sublattice and five \mathbf{M}_3 sublattices that are neither closed nor dense.

For ease of notation, let $\mathbf{Eq}(n)$ denote the set of equivalence relations on an n -element set, and let \mathbf{M}_n denote the $(n+2)$ -element lattice of height two (Figure 3.2).

Figure 3.2: The $(n+2)$ -element lattice of height 2, \mathbf{M}_n .



\mathbf{M}_n

Proposition 3.3.2. *For $n \geq 1$, $\mathbf{Eq}(2n+1)$ contains a dense \mathbf{M}_{n+2} .*

Thus, every \mathbf{M}_n can be densely embedded in $\mathbf{Eq}(X)$ for some finite set X .

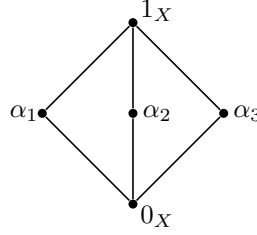
Proof. (sketch) We begin with Snow's example of a dense \mathbf{M}_3 sublattice of $\mathbf{Eq}(X)$, where $X =$

$\{0, 1, 2, 3, 4\}$. Define three partitions of X ,

$$\alpha_1 = |0, 1|2, 3|4|, \quad \alpha_2 = |0|1, 2|3, 4|, \quad \alpha_3 = |0, 2, 4|1, 3|,$$

let $L = \{0_X, \alpha_1, \alpha_2, \alpha_3, 1_X\}$ and let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ denote the sublattice of $\mathbf{Eq}(X)$ generated by the three equivalences $\alpha_1, \alpha_2, \alpha_3$ (Figure 3.3).

Figure 3.3: The lattice $\mathbf{L} = \langle \{0_X, \alpha_1, \alpha_2, \alpha_3, 1_X\}; \wedge, \vee \rangle$.



Obviously $\mathbf{L} \cong \mathbf{M}_3$, and it is not hard to show that the only unary maps which respect all equivalences in \mathbf{L} are the constants and the identity. In other words, the set $\lambda(L) \subseteq X^X$ consists of the six trivial maps in X^X . Therefore, $\rho\lambda(L) = \mathbf{Eq}(X)$.

Now notice that if we adjoin the equivalence $\alpha_4 = |0, 3|1, 4|2|$ to \mathbf{L} we get an \mathbf{M}_4 , which we denote by $\mathbf{L}(\alpha_4)$. Obviously, $\lambda(L) \supseteq \lambda(L(\alpha_4))$, as adding more equivalences only shrinks the set of functions respecting all equivalences. Therefore, $\mathbf{Eq}(X) = \rho\lambda(L) \subseteq \rho\lambda(L(\alpha_4))$, so $\mathbf{L}(\alpha_4)$ is a dense \mathbf{M}_4 sublattice of $\mathbf{Eq}(5)$.

Similarly, letting $X = \{0, 1, \dots, 6\}$ and

$$\alpha_1 = |0, 1|2, 3|4, 5|6|, \quad \alpha_2 = |0|1, 2|3, 4|5, 6|, \quad \alpha_3 = |0, 2, 4, 6|1, 3, 5|,$$

the sublattice $\mathbf{L} = \langle \{0_X, \alpha_1, \alpha_2, \alpha_3, 1_X\}, \wedge, \vee \rangle$ is a dense \mathbf{M}_3 in $\mathbf{Eq}(X)$. Adjoining the partitions

$$\alpha_4 = |0, 3|2, 5|1, 6|4| \quad \text{and} \quad \alpha_5 = |0, 5|1, 4|3, 6|2|$$

results in a dense \mathbf{M}_5 in $\mathbf{Eq}(X)$. Proceeding inductively, when $|X| = 2n+1$ there are $n+1$ partitions of the form $\alpha_i = |x_{i_0}|x_{i_1}, x_{i_2}| \cdots |x_{i_{2n-1}}, x_{i_{2n}}|$, and one of the form $\alpha_{n+2} = |\text{evens}|\text{odds}|$, with the following properties:

1. $\alpha_i \wedge \alpha_j = 0_X$,
2. $\alpha_i \vee \alpha_j = 1_X$,
3. the lattice generated by α_{n+2} and at least two other α_i is dense in $\mathbf{Eq}(X)$.

□

3.3.2 Non-density

The results in this section give sufficient conditions under which a lattice cannot be densely embedded in a lattice of equivalence relations. These results require some standard terminology that we have not yet introduced, so we begin the section with these preliminaries. As always, we will only deal with finite lattices $\mathbf{L} = \langle L, \wedge, \vee \rangle$, and we use $0_L = \bigwedge L$ to denote the bottom of \mathbf{L} and $1_L = \bigvee L$ to denote the top.

If $\mathbf{L} = \langle L, \wedge, \vee \rangle$ is a lattice, a non-empty subset $I \subseteq L$ is called an *ideal* of \mathbf{L} if

- (i) I is a *down-set*: if $\alpha \in I$ and $\beta \leq \alpha$, then $\beta \in I$;
- (ii) I is closed under finite joins: $\alpha, \beta \in I$ implies $\alpha \vee \beta \in I$.

A *filter* of a lattice is defined dually as a non-empty *up-set* that is closed under finite meets. An ideal or filter is said to be *proper* if it is not equal to all of L . The smallest ideal that contains a given element α is a *principal ideal* and α is said to be a *principal element* or *generator* of the ideal in this situation. The *principal ideal generated by α* is defined and denoted by $\alpha^\downarrow = \{\theta \in L \mid \theta \leq \alpha\}$. Similarly, $\alpha^\uparrow = \{\theta \in L \mid \theta \geq \alpha\}$ is the *principal filter generated by α* . An ideal I called a *prime ideal* provided $\alpha \wedge \beta \in I$ implies $\alpha \in I$ or $\beta \in I$ for all $\alpha, \beta \in L$. Equivalently, a *prime ideal* is an ideal whose set-theoretic complement is a filter. Since we require ideals (filters) to be non-empty, every prime filter (ideal) is necessarily proper. An element is called *meet prime* if it is the generator of a principal prime ideal. Equivalently, $\alpha \in L \setminus \{1_L\}$ is meet prime if for all $\beta, \gamma \in L$ we have $\beta \wedge \gamma \leq \alpha$ implies $\beta \leq \alpha$ or $\gamma \leq \alpha$. *Join prime* is defined dually.

Lemma 3.3.3. *Suppose $\mathbf{L} = \langle L, \wedge, \vee \rangle$ is a complete 0,1-lattice. Then the following are equivalent:*

- (i) *There is an element $\alpha \in L \setminus \{0_L\}$ such that $\bigvee \{\gamma \in L : \gamma \not\leq \alpha\} < 1_L$.*
- (ii) *There is an element $\alpha \in L \setminus \{1_L\}$ such that $\bigwedge \{\gamma \in L : \gamma \not\leq \alpha\} > 0_L$.*

(iii) \mathbf{L} is the union of a proper principal ideal and a proper principal filter.

Proof. (i) \Rightarrow (ii): Suppose $\alpha \in L \setminus \{0_L\}$ is such that the element $\alpha' = \bigvee\{\gamma : \gamma \not\geq \alpha\}$ is strictly below 1_L , and consider $\bigwedge\{\gamma : \gamma \not\leq \alpha'\}$. If $\beta \not\leq \alpha'$, then $\beta \notin \{\gamma : \gamma \not\geq \alpha\}$ so $\beta \geq \alpha$. Therefore, $\bigwedge\{\gamma : \gamma \not\leq \alpha'\} \geq \alpha > 0_L$. Thus $\alpha' \in L \setminus \{1_L\}$ is such that $\bigwedge\{\gamma : \gamma \not\leq \alpha'\} > 0_L$ so (ii) holds.

(ii) \Rightarrow (iii): Let $\alpha < 1_L$ be such that $\beta = \bigwedge\{\gamma : \gamma \not\leq \alpha\} > 0_L$. Then, $\mathbf{L} = \alpha^\downarrow \cup \beta^\uparrow$ satisfies (iii).

(iii) \Rightarrow (i): Suppose $\mathbf{L} = \alpha^\uparrow \cup \beta^\downarrow$ for some $\alpha > 0_L$, $\beta < 1_L$. Then $\{\gamma \in L : \gamma \not\geq \alpha\} \subseteq \beta^\downarrow$; i.e. $\gamma \not\geq \alpha \Rightarrow \gamma \leq \beta$. Therefore, $\bigvee\{\gamma : \gamma \not\geq \alpha\} \leq \beta < 1_L$, so (i) holds. \square

Lemma 3.3.4. *If $\mathbf{L} \not\cong \mathbf{2}$ is a sublattice of $\mathbf{Eq}(X)$ satisfying the conditions of Lemma 3.3.3, then $\lambda(L)$ contains a non-trivial unary function.*

Proof. Suppose $\mathbf{L} \not\cong \mathbf{2}$ is a sublattice of $\mathbf{Eq}(X)$ which satisfies condition (i) of the lemma. We must show that there is a non-trivial (i.e. non-constant, non-identity) $h \in X^X$ which respects every $\theta \in L$. By condition (i), there is an element $\alpha \in L \setminus \{0_L\}$ such that $\beta = \bigvee\{\gamma \in L : \gamma \not\geq \alpha\}$ is strictly below 1_L . Since $\alpha > 0_L$, there is a pair (u, v) of distinct elements of X that are α related. Since $\beta < 1_L$, there is a β equivalence class $B \subsetneq X$. Define $h \in X^X$ as follows:

$$h(x) = \begin{cases} u, & x \in B, \\ v, & x \notin B. \end{cases} \quad (3.3.1)$$

Then h is not constant, since $\emptyset \neq B \neq X$; h is not the identity, since $\mathbf{L} \not\cong \mathbf{2}$; h respects everything above α and everything below β , and therefore, $h \in \lambda(\alpha^\uparrow \cup \beta^\downarrow) = \lambda(L)$. \square

Theorem 3.3.5. *If $\mathbf{L} \not\cong \mathbf{2}$ is a lattice satisfying the conditions of Lemma 3.3.3 and X is any set, then \mathbf{L} cannot be densely embedded in $\mathbf{Eq}(X)$.*

Proof. The theorem says that, for any embedding $\mathbf{L} \cong \mathbf{L}_0 \leq \mathbf{Eq}(X)$ of such a lattice, \mathbf{L}_0 is not dense in $\mathbf{Eq}(X)$; i.e. $\rho\lambda(L_0) \subsetneq \mathbf{Eq}(X)$. To prove that this follows from Lemma 3.3.4, we must verify the following statement: If $\mathbf{2} \not\cong \mathbf{L} \leq \mathbf{Eq}(X)$ and if there is a non-trivial unary function $h \in \lambda(L)$, then $\rho\lambda(L) \subsetneq \mathbf{Eq}(X)$.

If $h \in X^X$ is any non-trivial unary function, then there are elements $\{x, y, u, v\}$ of X such that $x \neq y$ and $h(x) = u \neq v = h(y)$. We can assume X has at least three distinct elements since $\mathbf{L} \not\cong \mathbf{2}$. There are two cases to consider. In the first, h simply permutes x and y . In this case, $x = v$ and

$y = u$, and $h(v) = u$, $h(u) = v$. There must be a third element of X , say, $w \notin \{u, v\}$. If $h(w) \neq u$, then h violates any equivalence that puts v, w in the same block and puts u and $h(w)$ in separate blocks. If $h(w) \neq v$, then h violates any equivalence that puts u, w in the same block and v and $h(w)$ in separate blocks. In the second case to consider, $\{x, u, v\}$ are three distinct elements. In this case, h violates every relation that puts x, y in the same block and puts u and v in separate blocks.

We have thus proved that $\rho\lambda(L) \not\subseteq \mathbf{Eq}(X)$ whenever $\lambda(L)$ contains a non-trivial unary function. \square

Corollary 3.3.6. *If $\mathbf{L} \not\cong \mathbf{2}$ is a finite lattice with a meet prime element and X is any set, then \mathbf{L} cannot be densely embedded in $\mathbf{Eq}(X)$.*

Remark. The same result holds if we assume the lattice has a join prime element.

Proof. It is clear by the definition of meet prime that a lattice satisfying the hypotheses of the corollary also satisfies the conditions of Lemma 3.3.3, so the result follows from Theorem 3.3.5. \square

A lattice is called *meet-semidistributive* if it satisfies the *meet-semidistributive law*,

$$\text{SD}_\wedge : \quad \alpha \wedge \beta = \alpha \wedge \gamma \quad \Rightarrow \quad \alpha \wedge (\beta \vee \gamma) = \alpha \wedge \beta.$$

Corollary 3.3.7. *If $\mathbf{L} \not\cong \mathbf{2}$ is a finite meet-semidistributive lattice and X is any set, then \mathbf{L} cannot be densely embedded in $\mathbf{Eq}(X)$.*

Proof. We prove that every finite meet-semidistributive lattice \mathbf{L} contains a meet prime element. The result will then follow by Corollary 3.3.6. Since \mathbf{L} is finite, there exists an atom $\alpha \in L$. If α is the only atom, then α^\uparrow is trivially prime. Suppose $\beta \vee \gamma \in \alpha^\uparrow$. Then $(\beta \vee \gamma) \wedge \alpha = \alpha$, and $\beta \wedge \alpha \leq \alpha$ implies $\beta \wedge \alpha \in \{0_L, \alpha\}$. Similarly for γ . If both $\beta \wedge \alpha = 0_L = \gamma \wedge \alpha$ then SD_\wedge implies $(\beta \vee \gamma) \wedge \alpha = 0_L$, which is a contradiction. \square

The converse of Corollary 3.3.6 is false. That is, there exists a finite lattice $\mathbf{L} \not\cong \mathbf{2}$ with no meet prime element that cannot be densely embedded in some $\mathbf{Eq}(X)$. The lattice $\mathbf{M}_{3,3}$ shown below is an example. It has no meet prime element but it does satisfy the conditions of Lemma 3.3.3. Thus, by Theorem 3.3.5, $\mathbf{M}_{3,3}$ is not densely embeddable.

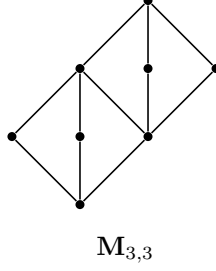


Figure 3.4: The lattice $\mathbf{M}_{3,3}$.

3.3.3 Distributive lattices

A lattice \mathbf{L} is called *strongly representable* as a congruence lattice if whenever $\mathbf{L} \cong \mathbf{L}_0 \leq \mathbf{Eq}(X)$ for some X then there is an algebra based on X whose congruence lattice is \mathbf{L}_0 .

Theorem 3.3.8 (Berman [5], Quackenbush and Wolk [36]). *Every finite distributive lattice is strongly representable.*

Remark: By Theorem 3.2.1 above, the result of Berman, Quackenbush and Wolk says, if \mathbf{L} is a finite distributive lattice then every embedding $\mathbf{L} \cong \mathbf{L}_0 \leq \mathbf{Eq}(X)$ is closed. The following proof is only slightly shorter than to the original in [36], and the methods are similar.

Proof. Without loss of generality, suppose $\mathbf{L} \leq \mathbf{Eq}(X)$. Fix $\theta \in \mathbf{Eq}(X) \setminus L$ and define $\theta^* = \bigwedge \{\gamma \in L : \gamma \geq \theta\}$ and $\theta_* = \bigvee \{\gamma \in L : \gamma \leq \theta\}$. Let α be a join irreducible in L below θ^* and not below θ_* . Note that α is not below θ . Let $\beta = \bigvee \{\gamma \in L : \gamma \not\geq \alpha\}$. If β were above θ , then β would be above θ^* , and so β would be above α . But α is join prime, so β is not above θ .

Choose $(u, v) \in \alpha \setminus \theta$ and note that $u \neq v$. Choose $(x, y) \in \theta \setminus \beta$ and note that $x \neq y$. Let B be the β block of y and define $h \in X^X$ as in (3.3.1). Then it is clear that h violates θ , h respects all elements in the sets $\alpha^\uparrow = \{\gamma \in L : \alpha \leq \gamma\}$ and $\beta^\downarrow = \{\gamma \in L : \gamma \leq \beta\}$, and $L = \alpha^\uparrow \cup \beta^\downarrow$. Since θ was an arbitrary element of $\mathbf{Eq}(X) \setminus L$, we can construct such an $h = h_\theta$ for each $\theta \in \mathbf{Eq}(X) \setminus L$. Let $\mathcal{H} = \{h_\theta : \theta \in \mathbf{Eq}(X) \setminus L\}$ and let \mathbf{A} be the algebra $\langle X, \mathcal{H} \rangle$. Then, $\mathbf{L} = \mathbf{Con}(\mathbf{A})$. \square

3.4 Conclusions and open questions

J.B. Nation has found examples of densely embedded double-winged pentagons none of whose sublattices are densely embedded. John Snow then asked if any of the sublattices are closed embeddings.

In general, we might ask the following: Are there closed sublattices of dense embeddings?

Another question we have not answered is whether the converse of Theorem 3.3.5 is true, but this seems unlikely. Rather, we expect there exists a finite lattice that is neither densely embeddable nor the union of a proper principal ideal and a proper principal filter.

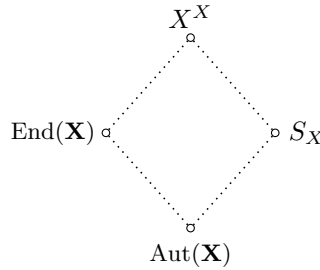
Finally, we mention that even if we restrict ourselves to one of the smaller classes of finite lattices mentioned above – those satisfying the conditions of Lemma 3.3.3 or Corollary 3.3.6, or the finite meet-semidistributive lattices – it is still unknown whether every lattice in this class is representable as the congruence lattice of a finite algebra.

CHAPTER 4

CONGRUENCE LATTICES OF GROUP ACTIONS

Let X be a finite set and consider the set X^X of all maps from X to itself, which, when endowed with composition of maps and the identity mapping, forms a monoid, $\langle X^X, \circ, \text{id}_X \rangle$. The submonoid S_X of all bijective maps in X^X is a group, the *symmetric group on X* . When the underlying set is more complicated, or for emphasis, we denote the symmetric group on X by $\text{Sym}(X)$. When the underlying set isn't important, we usually write S_n to denote the symmetric group on an n -element set.

If we have defined some set F of basic operations on X , so that $\mathbf{X} = \langle X, F \rangle$ is an algebra, then two other important submonoids of X^X are $\text{End}(\mathbf{X})$, the set of maps in X^X which respect all operations in F , and $\text{Aut}(\mathbf{X})$, the set of bijective maps in X^X which respect all operations in F . It is apparent from the definition that $\text{Aut}(\mathbf{X}) = S_X \cap \text{End}(\mathbf{X})$, and $\text{Aut}(\mathbf{X})$ is a submonoid of $\text{End}(\mathbf{X})$ and a subgroup of S_X . These four fundamental monoids associated with the algebra \mathbf{X} , and their relative ordering under inclusion, are shown in the diagram below.



Given a finite group G , and an algebra $\mathbf{X} = \langle X, F \rangle$, a *representation* of G on \mathbf{X} is a group homomorphism from G into $\text{Aut}(\mathbf{X})$. That is, a representation of G is a mapping $\varphi : G \rightarrow \text{Aut}(\mathbf{X})$ which satisfies $\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$, where (as above) \circ denotes composition of maps in $\text{Aut}(\mathbf{X})$.

4.1 Transitive G -sets

From the foregoing, we see that a representation defines an action by G on the set X , as follows: $\bar{g}x = \varphi(g)(x)$. If $\bar{G} = \varphi[G] \leq \text{Aut}(\mathbf{X})$ denotes the image of G under φ , we call the algebra $\langle X, \bar{G} \rangle$ a *G -set*.¹ The action is called *transitive* if for each pair $x, y \in X$ there is some $g \in G$ such that $\bar{g}x = y$.

¹More generally, a G -set is sometimes defined to be a pair (X, φ) , where φ is a homomorphism from a group into the symmetric group S_X , see e.g. [44].

The representation φ is called *faithful* if it is a monomorphism, in which case G is isomorphic to its image under φ , which is a subgroup of $\text{Aut}(\mathbf{X})$. We also say, in this case, that the group acts faithfully, and call it a *permutation group*. A group which acts transitively on some set is called a *transitive group*. Without specifying the set, however, this term is meaningless, since every group acts transitively on some sets and intransitively on others. A representation φ is called *transitive* if the resulting action is transitive. Finally, we define *degree* of a group action on a set X to be the cardinality of X .

Two special cases are almost always what one means when one speaks of a representation of a finite group. These are the so called

- *linear representations*, where $\mathbf{X} = \langle X, +, \circ, -, 0, 1, \mathbb{F} \rangle$ is a finite dimensional vector space over a field \mathbb{F} , so $\text{Aut}(\mathbf{X})$ is the set of invertible matrices with entries from \mathbb{F} ;
- *permutation representations*, where $\mathbf{X} = X$ is just a set, so $\text{Aut}(\mathbf{X}) = S_X$.

For us the most important representation of a group G is its action on a set of cosets of a subgroup. That is, for any subgroup $H \leq G$, we define a transitive permutation representation of G , which we will denote by $\hat{\lambda}_H$. Specifically, $\hat{\lambda}_H$ is a group homomorphism from G into the symmetric group $\text{Sym}(G/H)$ of permutations on the set $G/H = \{H, x_1H, x_2H, \dots\}$ of *left* cosets of H in G . The action is simply left multiplication by elements of G . That is, $\hat{\lambda}_H(g)(xH) = gxH$. Clearly, $\hat{\lambda}_H(g_1g_2) = \hat{\lambda}_H(g_1)\hat{\lambda}_H(g_2)$ for all $g_1, g_2 \in G$, so $\hat{\lambda}_H$ is a homomorphism. Each xH is a point in the set G/H , and the *point stabilizer* of xH in G is defined by $G_{xH} = \{g \in G \mid gxH = xH\}$. Notice that

$$G_{xH} = \{g \in G \mid x^{-1}gxH = H\} = xG_Hx^{-1} = xHx^{-1} = H^x,$$

where $G_H = \{g \in G \mid gH = H\}$ is the point stabilizer of H in G . Thus, the kernel of the homomorphism $\hat{\lambda}_H$ is

$$\ker \hat{\lambda}_H = \{g \in G \mid \forall x \in G, gxH = xH\} = \bigcap_{x \in G} G_{xH} = \bigcap_{x \in G} xHx^{-1} = \bigcap_{x \in G} H^x.$$

Note that $\ker \hat{\lambda}_H$ is the largest normal subgroup of G contained in H , also known as the *core* of H in G , which we denote by

$$\text{core}_G(H) = \bigcap_{x \in G} H^x.$$

If the subgroup H happens to be *core-free*, that is, $\text{core}_G(H) = 1$, then $\hat{\lambda}_H : G \hookrightarrow \text{Sym}(G/H)$ is an embedding, so $\hat{\lambda}_H$ is a faithful representation; G acts faithfully on G/H . Hence the group G , being isomorphic to a subgroup of $\text{Sym}(G/H)$, is itself a permutation group.

Other definitions relating to G -sets will be introduced as needed and in the appendix, and we assume the reader is already familiar with these. However, we mention one more important concept before proceeding, as it is a potential source of confusion. By a *primitive group* we mean a group that contains a core-free maximal subgroup. This definition is not the typical one found in group theory textbooks, but we feel it is better. (See the appendix Section A.1 for justification.)

4.1.1 G -set isomorphism theorems

We have seen above that the action of a group on cosets of a subgroup H is a transitive permutation representation, and the representation is faithful when H is core-free. The first theorem in this section states that every transitive permutation representation is of this form. (In fact, as we will see in Lemma 4.2.1 below, every permutation representation, whether transitive or not, can be viewed as an action on cosets.)

First, we need some more notation. Given a G -set $\mathbf{A} = \langle A, G \rangle$ and any element $a \in A$, the set $G_a = \{g \in G \mid ga = a\}$ of all elements of G which fix a is a subgroup of G , called the *stabilizer of a in G* .

Theorem 4.1.1 (1st G -set Isomorphism Theorem). *If $\mathbf{A} = \langle A, \bar{G} \rangle$ is a transitive G -set, then \mathbf{A} is isomorphic to the G -set*

$$\Gamma := \langle G/G_a, \{\hat{\lambda}_g : g \in G\} \rangle$$

for any $a \in A$.

Proof. Suppose $\mathbf{A} = \langle A, \bar{G} \rangle$ is a transitive G -set, so $A = \{\bar{g}a \mid g \in G\}$ for any $a \in A$. The operations of the G -set Γ are defined, for each $g \in G$ and each coset $xG_a \in G/G_a$, by $\hat{\lambda}_g(xG_a) = gxG_a$.

Let \mathbf{G}_Λ denote the G -set $\langle G, \{\lambda_g : g \in G\} \rangle$, that is, the group G acting on itself by left multiplication. Fix $a \in A$, and define $\varphi_a : G \rightarrow A$ by $\varphi_a(x) = \bar{x}(a)$ for each $x \in G$. Then φ_a is a

homomorphism from \mathbf{G}_Λ into \mathbf{A} – that is, φ_a respects operations:²

$$\varphi_a(\lambda_g(x)) = \varphi_a(gx) = \overline{g}x(a) = \bar{g} \cdot \bar{x}(a) = \bar{g}\varphi_a(x).$$

Moreover, since \mathbf{A} is transitive, $\varphi_a(G) = \{\bar{g}a \mid g \in G\} = A$, so φ_a is an epimorphism. Therefore, $\mathbf{G}_\Lambda / \ker \varphi_a \cong \mathbf{A}$. To complete the proof, one simply checks that the two algebras $\mathbf{G}_\Lambda / \ker \varphi_a$ and Γ are identical.³ \square

The next theorem shows why intervals of subgroup lattices are so important for our work.

Theorem 4.1.2 (2nd G -set Isomorphism Theorem). *Let $\mathbf{A} = \langle A, G \rangle$ be a transitive G -set and fix $a \in A$. Then the lattice $\text{Con } \mathbf{A}$ is isomorphic to the interval $[G_a, G]$ in the subgroup lattice of G .*

Proof. For each $\theta \in \text{Con } \mathbf{A}$, let $H_\theta = \{g \in G : (g(a), a) \in \theta\}$, and for each $H \in [G_a, G]$, let $(b, c) \in \theta_H$ mean there exist $g \in G$ and $h \in H$ such that $gh(a) = b$ and $g(a) = c$. If $g_1, g_2 \in H_\theta$, then

$$(g_2(a), a) \in \theta \quad \Rightarrow \quad (g_2^{-1}g_2(a), g_2^{-1}(a)) = (a, g_2^{-1}(a)) \in \theta,$$

so $(g_2^{-1}(a), a) \in \theta$, by symmetry. Therefore, $(g_1g_2^{-1}(a), g_1(a)) \in \theta$, so $(g_1g_2^{-1}(a), (a)) \in \theta$, by transitivity. Thus H_θ is a subgroup of G , and clearly $G_a \leq H_\theta$. It is also easy to see that θ_H is a congruence of \mathbf{A} . The equality $H_{\theta_H} = H$ trivially follows from the definitions. On the other hand $(b, c) \in \theta_{H_\theta}$ if and only if there exist $g, h \in G$ for which $(h(a), a) \in \theta$ and $b = gh(a)$, and $c = g(a)$. Since G is transitive, it is equivalent to $(b, c) \in \theta$. Therefore, $\theta_{H_\theta} = \theta$. Finally, $H_\theta \leq H_\varphi$ if and only if $\theta \leq \varphi$, so $\theta \mapsto H_\theta$ is an isomorphism between $\text{Con } \mathbf{A}$ and $[G_a, G]$. \square

Since the foregoing theorem is so central to our work, we provide an alternative statement of it. This is the version typically found in group theory textbooks (e.g., [12]). Keeping these two alternative perspectives in mind can be useful.

²In general, if $\mathbf{A} = \langle A, F \rangle$ and $\mathbf{B} = \langle B, F \rangle$ are two algebras of the same similarity type, then $\varphi : \mathbf{A} \rightarrow \mathbf{B}$ is a homomorphism provided

$$\varphi(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(\varphi(a_1), \dots, \varphi(a_n))$$

whenever $f^{\mathbf{A}}$ is an n -ary operation of \mathbf{A} , $f^{\mathbf{B}}$ is the corresponding n -ary operation of \mathbf{B} , and a_1, \dots, a_n are arbitrary elements of A . (Note that a one-to-one correspondence between the operations of two algebras of the same similarity type is assumed, and required for the definition of homomorphism to make sense.)

³ Indeed, $\ker \varphi_a = \{(x, y) \in G^2 \mid \varphi_a(x) = \varphi_a(y)\}$ and the universe of $\mathbf{G}_\Lambda / \ker \varphi_a$ is $G / \ker \varphi_a = \{x / \ker \varphi_a \mid x \in G\}$, where for each $x \in G$

$$\begin{aligned} x / \ker \varphi_a &= \{y \in G \mid (x, y) \in \ker \varphi_a\} = \{y \in G \mid \varphi_a(x) = \varphi_a(y)\} = \{y \in G \mid \bar{x}(a) = \bar{y}(a)\} \\ &= \{y \in G \mid \text{id}_A(a) = \overline{x^{-1}y}(a)\} = \{y \in G \mid x^{-1}y \in G_a\} = xG_a. \end{aligned}$$

These are precisely the elements of G/G_a , so the universes of $\mathbf{G}_\Lambda / \ker \varphi_a$ and Γ are the same, as are their operations (left multiplication by $g \in G$).

Theorem 4.1.3 (2nd G -set Isomorphism Theorem, version 2). *Let $\mathbf{A} = \langle A, \bar{G} \rangle$ be a transitive G -set and let $a \in A$. Let \mathcal{B} be the set of all blocks B with $a \in B$. Let $[G_a, G] \subseteq \text{Sub}(G)$ denote the set of all subgroups of G containing G_a . Then there is a bijection $\Psi : \mathcal{B} \rightarrow [G_a, G]$ given by $\Psi(B) = G(B)$, with inverse mapping $\Phi : [G_a, G] \rightarrow \mathcal{B}$ given by $\Phi(H) = \bar{H}a = \{\bar{h}a \mid h \in H\}$. The mapping Ψ is order-preserving in the sense that if $B_1, B_2 \in \mathcal{B}$ then $B_1 \subseteq B_2 \Leftrightarrow \Psi(B_1) \leq \Psi(B_2)$.*

Briefly, the poset $\langle \mathcal{B}, \subseteq \rangle$ is order-isomorphic to the poset $\langle [G_a, G], \leq \rangle$.

Corollary 4.1.4. *Let G act transitively on a set with at least two points. Then G is primitive if and only if each stabilizer G_a is a maximal subgroup of G .*

Since the point stabilizers of a transitive group are all conjugate, one stabilizer is maximal only when all of the stabilizers are maximal. In particular, a regular permutation group is primitive if and only if it has prime degree.

Next we describe (up to equivalence) all transitive permutation representations of a given group G . We call two representations (or actions) *equivalent* provided the associated G -sets are isomorphic. The foregoing implies that every transitive permutation representation of G is equivalent to $\hat{\lambda}_H$ for some subgroup $H \leq G$. The following lemma⁴ shows that we need only consider a single representative H from each of the conjugacy classes of subgroups.

Lemma 4.1.5. *Suppose G acts transitively on two sets, A and B . Fix $a \in A$ and let G_a be the stabilizer of a (under the first action). Then the two actions are equivalent if and only if the subgroup G_a is also a stabilizer under the second action of some point $b \in B$.*

The point stabilizers of the action $\hat{\lambda}_H$ described above are the conjugates of H in G . Therefore, the lemma implies that, for any two subgroups $H, K \leq G$, the representations $\hat{\lambda}_H$ and $\hat{\lambda}_K$ are equivalent precisely when $K = xHx^{-1}$ for some $x \in G$. Hence, the transitive permutation representations of G are given, up to equivalence, by $\hat{\lambda}_{K_i}$ as K_i runs over a set of representatives of conjugacy classes of subgroups of G .

4.1.2 An M -set isomorphism theorem

It is natural to ask whether the two theorems of the previous subsection hold more generally for a unary algebra $\langle X, M \rangle$, where M is a monoid (rather than a permutation group). We call such

⁴Lemma 1.6B of [12].

an algebra $\langle X, M \rangle$ an M -set, and although we will see that there is no analogue to the 2nd G -set Isomorphism Theorem, we do have

Theorem 4.1.6 (1st M -set Isomorphism Theorem). *If $\langle X, M \rangle$ is a transitive M -set, then for any fixed $x \in X$, the map $\varphi_x : M \rightarrow X$ defined by $\varphi_x(m) = mx$ is an M -set epimorphism. Moreover, the (transitive) M -set $\langle M/\ker \varphi_x, M \rangle$ is isomorphic to $\langle X, M \rangle$.*

Proof. By transitivity, for each $y \in X$, there is an $m \in M$ such that $\varphi_x(m) = mx = y$, so φ_x is onto. Also, φ_x is a homomorphism of the M -set $\langle M, M \rangle$ onto the M -set $\langle X, M \rangle$, since for all $m, m_1 \in M$,

$$\varphi_x(m \circ m_1) = m(m_1x) = m\varphi_x(m_1).$$

By the usual isomorphism theorem,

$$\langle M/\ker \varphi_x, M \rangle \cong \langle X, M \rangle \tag{4.1.1}$$

where

$$\ker \varphi_x = \{(m_1, m_2) \in M^2 : \varphi_x(m_1) = \varphi_x(m_2)\} = \{(m_1, m_2) \in M^2 : m_1x = m_2x\}.$$

Note that, since $\langle X, M \rangle$ is a transitive M -set, the M -set $\langle M/\ker \varphi_x, M \rangle$ must also be transitive, otherwise (4.1.1) would fail.

Just to be sure, let's verify that $\langle M/\ker \varphi_x, M \rangle$ is indeed transitive. Let $m_1/\ker \varphi_x, m_2/\ker \varphi_x$ be any two $\ker \varphi_x$ -classes of M . We must show there exists $m_3 \in M$ such that $m_3[m_1/\ker \varphi_x] = m_2/\ker \varphi_x$. Let $\varphi_x(m_1) = y_1$ and $\varphi_x(m_2) = y_2$. Let $m_3 \in M$ be a map which takes y_1 to y_2 , (guaranteed to exist by transitivity of $\langle X, M \rangle$). Then for all $m \in m_1/\ker \varphi_x$, we have $m_3mx = m_3y_1 = y_2$, so $m_3m \in m_2/\ker \varphi_x$. Therefore,

$$m_3[m_1/\ker \varphi_x] \subseteq m_2/\ker \varphi_x.$$

By the same argument, there is $m'_3 \in M$ such that

$$m'_3[m_2/\ker \varphi_x] \subseteq m_1/\ker \varphi_x.$$

By cardinality, $m_3[m_1/\ker \varphi_x] = m_2/\ker \varphi_x$. □

An analogue to the 2nd G -set Isomorphism Theorem for monoids would be that $[M_x, M] \cong \text{Con} \langle X, M \rangle$ should hold for a transitive M -set $\langle X, M \rangle$. By the following counter-example, we see that this is false: Consider the monoid M consisting of the identity and constant maps. Of course, $\langle X, M \rangle$ is a transitive M -set, and $\text{Con} \langle X, M \rangle = \text{Eq}(X)$. However, for $x \in X$, the stabilizer is $M_x = \{m \in M : mx = x\}$ which is the set containing the identity map on X and the constant function that maps all points to x . So the lattice $[M_x, M]$ of submonoids of M above M_x is just the lattice of subsets of M which contain the identity and the constant map x . This is a distributive lattice, so it cannot be isomorphic to $\text{Con} \langle X, M \rangle = \text{Eq}(X)$.

4.2 Intransitive G -sets

The problem of characterizing congruence lattices of intransitive G -sets seems open. In this section we prove a couple of results which help determine the shape of congruence lattices of intransitive G -sets. In [11] we use these and other results to show that for many lattices a minimal representation as the congruence lattice of an intransitive G -set is not possible.⁵

In the previous section we considered transitive, or one-generated, G -sets. In Theorem 4.1.1, we presented the well known result that a transitive G -set $\langle \Omega, G \rangle$, with universe Ω , is isomorphic to the G -set $\langle G/H, G \rangle$, where the universe is now the collection of cosets of a subgroup $H = G_\omega$ – the stabilizer of a point $\omega \in \Omega$. Then, Theorem 4.1.2 gave us a precise description of the shape of the congruence lattice: $\text{Con} \langle G/H, G \rangle \cong [H, G]$. It is natural to ask whether results analogous to these hold for intransitive G -sets.

In this section, we first prove that an arbitrary (intransitive) G -set $\langle \Omega, G \rangle$ is isomorphic to a G -set of the form $\langle G_1/H_1 \cup \dots \cup G_r/H_r, G \rangle$, where $H_i \leq G_i \cong G$. This result is well known, and appears as Theorem 3.4 in [26]. Nonetheless we present a short proof and describe the G -set isomorphism explicitly.⁶ Thereafter, we prove lemma which, along with the first, gives a characterization of the congruence lattice of an arbitrary G -set. It is almost certain that this simple result is also well known, but to my knowledge it does not appear in print elsewhere.⁷

⁵In other words, if there exists a representation of such a lattice as the congruence lattice of an algebra (of minimal cardinality), then the algebra must be a *transitive* G -set.

⁶Such an explicit description is useful when we are working with such algebras on the computer, using the Universal Algebra Calculator or **GAP**, for example.

⁷I thank Alexander Hulpke for alerting me to the special case, described below, of the second lemma.

Throughout this section, we adhere to the convention that *groups act on the left*, so we will denote the action of $g \in G$ on an element $\omega \in \Omega$ by $g : \omega \mapsto g\omega$, and we use $G\omega$ to denote the orbit of ω under this action, that is, $G\omega = \{g\omega \mid g \in G\}$. Finally, we remind the reader that *all groups under consideration are finite*.

Our first lemma shows that, even in the intransitive case, we can take the universe of an arbitrary G -set to be a collection cosets of the group G .

Lemma 4.2.1. *Every G -set $\langle \Omega, G \rangle$ is isomorphic to a G -set on a universe of the form $G_1/H_1 \cup \dots \cup G_r/H_r$, where $H_i \leq G_i \cong G$ and G_i/H_i is the set of left cosets of H_i in G_i , for each $1 \leq i \leq r$,*

Proof. Suppose $\Omega = \langle \Omega, G \rangle$ is an arbitrary G -set, and let $\langle \Omega_i, G \rangle$, $1 \leq i \leq r$, be the minimal subalgebras of Ω . That is, each Ω_i is an orbit, say, $\Omega_i = G\omega_i$, and $\Omega = G\omega_1 \cup \dots \cup G\omega_r$ is a disjoint union. For each $1 \leq i \leq r$, let G_i be an isomorphic copy of G , with, say, $\varphi_i : G_i \cong G$ as the isomorphism. Clearly,

$$H_i := \{x \in G_i \mid \varphi_i(x)\omega_i = \omega_i\} \cong \{g \in G \mid g\omega_i = \omega_i\} = G_{\omega_i}.$$

Note that $\langle G_i/H_i, G \rangle \cong \langle G\omega_i, G \rangle$, where G acts on G_i/H_i as one expects: for $g \in G$ and $xH_i \in G_i/H_i$, the action is $g : xH_i \mapsto \varphi_i^{-1}(g)xH_i$.

Define $\psi : G_1/H_1 \cup \dots \cup G_r/H_r \rightarrow \Omega$ by $\psi(xH_i) = \varphi_i(x)\omega_i$. This map is well-defined. For, if $xH_i = x'H_j$, then $i = j$ and $x^{-1}x' \in H_i$, and it is easy to verify that $x^{-1}x' \in H_i$ holds if and only if $\varphi_i(x')\omega_i = \varphi_i(x)\omega_i$. Thus, $\psi(xH_i) = \psi(x'H_j)$.

Now consider the G -set $\langle G_1/H_1 \cup \dots \cup G_r/H_r, G \rangle$ with the same action as above: $g(xH_i) = \varphi_i^{-1}(g)(xH_i)$. We claim that ψ is a G -set isomorphism of $\langle G_1/H_1 \cup \dots \cup G_r/H_r, G \rangle$ onto $\langle \Omega, G \rangle$. It is clearly a bijection.⁸ We check that ψ respects the interpretation of the action of G : Fix $g \in G$ and $x \in G_i$. Then, since φ_i is a homomorphism,

$$\psi(\varphi_i^{-1}(g)(xH_i)) = \varphi_i(\varphi_i^{-1}(g)x)\omega_i = \varphi_i(\varphi_i^{-1}(g))\varphi_i(x)\omega_i = g\psi(xH_i).$$

□

The foregoing lemma shows that we can always take the universe of an intransitive G -set to be a

⁸Define $\zeta : \Omega \rightarrow G_1/H_1 \cup \dots \cup G_r/H_r$ by $\zeta(g\omega_i) = \varphi_i^{-1}(g)H_i$, check that this map is well-defined, and note that $\psi\zeta = \text{id}_\Omega$, and $\zeta\psi$ is the identity on $G_1/H_1 \cup \dots \cup G_r/H_r$.

disjoint union of sets of cosets of stabilizer subgroups. We now use this fact to describe the structure of the congruence lattice of an arbitrary G -set.

As above, let $\mathbf{\Omega} = \langle \Omega, G \rangle$ be a G -set with universe $\Omega = G\omega_1 \cup \dots \cup G\omega_r$, where each $\langle G\omega_i, G \rangle$ is a minimal subalgebra. Consider the partition $\tau \in \text{Eq}(\Omega)$, given by $\tau = |G\omega_1|G\omega_2| \dots |G\omega_r|$. Clearly, this is a congruence relation, since the action of every $g \in G$ fixes each block. We call τ the *intransitivity congruence*. It's clear that we can join two or more blocks of τ and the new larger block will still be preserved by every $g \in G$. Thus, the interval above τ in the congruence lattice $\mathbf{\Omega}$ is isomorphic to the lattice of partitions of a set of size r . That is,

$$[\tau, 1_{\mathbf{\Omega}}] := \{\theta \in \text{Con } \mathbf{\Omega} \mid \tau \leq \theta \leq 1_{\mathbf{\Omega}}\} \cong \text{Eq}(r). \quad (4.2.1)$$

Another obvious fact is that the interval below τ in $\text{Con } \mathbf{\Omega}$ is

$$[0_{\mathbf{\Omega}}, \tau] \cong \prod_{i=1}^r \text{Con}(\langle G\omega_i, G \rangle). \quad (4.2.2)$$

Since each minimal algebra $\langle G\omega_i, G \rangle \cong \langle G_i/H_i, G \rangle$ is transitive, we have $\text{Con}(\langle G\omega_i, G \rangle) \cong [H_i, G_i]$. Thus, the structure of that part of $\text{Con } \mathbf{\Omega}$ that is comparable with the intransitivity congruence is explicitly described by (4.2.1) and (4.2.2).

Our next result describes the congruences that are incomparable with the intransitivity congruence. The description is in terms of the blocks of congruences below the intransitivity congruence. Thus, the lemma does not give a nice abstract characterization of the shape of the $\text{Con } \mathbf{\Omega}$ in terms of the shape of $\text{Sub}(G)$, as we had in the transitive case. However, besides being useful for computing the congruences, this result can be used in certain situations to draw conclusions about the general shape of $\text{Con } \mathbf{\Omega}$, based on the subgroup structure of G (for example, using combinatorial arguments involving the index of subgroups of G). We will say more about this below.

Though the proof of Lemma 4.2.2 is elementary, it gets a bit complicated when presented in full generality. Therefore, we begin by discussing the simplest special case of an intransitive G -set, that is, one which has just two minimal subalgebras. Suppose $\mathbf{\Omega} = \langle \Omega, G \rangle = \langle \Omega_1 \cup \Omega_2, G \rangle$ is a G -set with $\Omega_i = G\omega_i$ for some $\omega_i \in \Omega_i$, $i = 1, 2$. For each subset $\Lambda \subseteq \Omega$, for each $g \in G$, let $g\Lambda := \{g\omega \mid \omega \in \Lambda\}$, and define the *set-wise stabilizer* of Λ in G to be the subgroup

$$\text{Stab}_G(\Lambda) := \{g \in G \mid g\omega \in \Lambda \text{ for all } \omega \in \Lambda\}.$$

As above, we call the congruence $\tau = |\Omega_1|\Omega_2|$ the intransitivity congruence. Fix a congruence τ_0 strictly below τ , and for each $i = 1, 2$ let $\Lambda_i = \omega_i/\tau_0$ denote the block of τ_0 containing ω_i . Then there is a congruence θ above τ_0 with a block $\Lambda_1 \cup \Lambda_2$ if and only if $\text{Stab}_G(\Lambda_1) = \text{Stab}_G(\Lambda_2)$. (We will verify this claim below when we prove it more generally in Lemma 4.2.2.) This characterizes all congruences in $\text{Con } \mathbf{\Omega}$ that are incomparable with the intransitivity congruence, τ , in terms of the congruences below τ .

Let $\mathbf{\Omega} = \langle \Omega_1 \cup \dots \cup \Omega_r, G \rangle$ be a G -set with minimal subalgebras $\Omega_i = G\omega_i$, for some $\omega_i \in \Omega_i$, $1 \leq i \leq r$. Let $\tau = |\Omega_1|\Omega_2| \dots |\Omega_r|$ be the intransitivity congruence and fix $\tau_0 < \tau$ in $\text{Con } \mathbf{\Omega}$. For each $1 \leq i \leq r$, let $\Lambda_i = \omega_i/\tau_0$ denote the block of τ_0 containing ω_i , and let $T_i = \{g_{i,0}=1, g_{i,1}, \dots, g_{i,n_i}\}$ be a transversal of $G/\text{Stab}_G(\Lambda_i)$.⁹

It is important to note that the blocks of τ_0 are $g_{i,k}\Lambda_i$, where $1 \leq i \leq r$ and $0 \leq k \leq n_i$. This is illustrated in the following diagram, where the blocks of τ_0 appear below the blocks of τ to which they belong.

$$\begin{array}{ccccccc} \tau = & | & \Omega_1 & | & \Omega_2 & | & \dots & | & \Omega_r & | \\ & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\ \tau_0 = & | & \Lambda_1|g_{1,1}\Lambda_1| \dots |g_{1,n_1}\Lambda_1 & | & \Lambda_2|g_{2,1}\Lambda_2| \dots |g_{2,n_2}\Lambda_2 & | & \dots & | & \Lambda_r|g_{r,1}\Lambda_r| \dots |g_{r,n_r}\Lambda_r & | \end{array}$$

It should be obvious that the blocks of τ_0 are as given above, but since this plays such an important role in the lemma below, we check it explicitly: If $\Lambda_i \subseteq \Omega_i$ is a block of τ_0 , then so is $g\Lambda_i$ for all $g \in G$, and either $g\Lambda_i \cap \Lambda_i = \emptyset$ or $g\Lambda_i = \Lambda_i$. If $\Lambda' \subseteq \Omega_i$ is also a block of τ_0 , then $\Lambda' = g'\Lambda_i$ for some $g' \in G = \text{Stab}_G(\Lambda_i) \cup g_{i,1}\text{Stab}_G(\Lambda_i) \cup \dots \cup g_{i,n_i}\text{Stab}_G(\Lambda_i)$, say $g' \in g_{i,j}\text{Stab}_G(\Lambda_i)$. Then, $g_{i,j}^{-1}g' \in \text{Stab}_G(\Lambda_i)$, so $g_{i,j}^{-1}g'\Lambda_i = \Lambda_i$. Therefore, $g'\Lambda_i = g_{i,j}\Lambda_i$.

Another obvious but important consequence: If $T_1 = \{g_{1,0}=1, g_{1,1}, \dots, g_{1,n_1}\}$ is a transversal of $G/\text{Stab}_G(\Lambda_1)$, and if $\text{Stab}_G(\Lambda_1) = \text{Stab}_G(\Lambda_j)$, then T_1 is also a transversal of $G/\text{Stab}_G(\Lambda_j)$, so the blocks of τ_0 in Ω_j may be written as $g_{1,k}\Lambda_j$, where $0 \leq k \leq n_1$.

Lemma 4.2.2. *Given a subset $\{i_1, \dots, i_m\} \subseteq \{1, \dots, r\}$, there exists $\theta \in \text{Con } \mathbf{\Omega}$ with block $\Lambda_{i_1} \cup \dots \cup \Lambda_{i_m}$ if and only if $\text{Stab}_G(\Lambda_{i_1}) = \dots = \text{Stab}_G(\Lambda_{i_m})$. For example,*

$$\theta = \tau_0 \cup \bigcup_{k=0}^{n_{i_1}} (g_{i_1 k} \Lambda_{i_1} \cup \dots \cup g_{i_1 k} \Lambda_{i_m})^2. \quad (4.2.3)$$

⁹Here $G/\text{Stab}_G(\Lambda_i)$ denotes the set of right cosets of $\text{Stab}_G(\Lambda_i)$ in G , and a *transversal* is a set containing one element from each coset.

Remarks. The index set $\{i_1, \dots, i_m\}$ identifies the subalgebras from which to choose blocks that will be joined in the new congruence θ . The number of blocks of τ_0 which intersect the subalgebra Ω_{i_j} is n_{i_j} , which is the length of the transversal of $G/\text{Stab}_G(\Lambda_{i_j})$. Therefore, $n_{i_j} = |G : \text{Stab}_G(\Lambda_{i_j})|$.

As noted above, if $\text{Stab}_G(\Lambda_{i_1}) = \text{Stab}_G(\Lambda_{i_m})$, then we can assume the transversals $T_1 = \{g_{i_1 1}, \dots, g_{i_1 n_{i_1}}\}$ and $T_m = \{g_{i_m 1}, \dots, g_{i_m n_{i_m}}\}$ are the same. In the proof below, we will use T to denote this common transversal.

Proof. (\Rightarrow) Assume there is a congruence $\theta \in \text{Con } \mathbf{\Omega}$ with block $\Lambda_{i_1} \cup \dots \cup \Lambda_{i_m}$. Suppose there exists $1 \leq j < k \leq m$ such that $\text{Stab}_G(\Lambda_{i_j}) \neq \text{Stab}_G(\Lambda_{i_k})$. Without loss of generality, assume $g \in \text{Stab}_G(\Lambda_{i_j}) \setminus \text{Stab}_G(\Lambda_{i_k})$, so $g\Lambda_{i_j} = \Lambda_{i_j}$ and there is an $x \in \Lambda_{i_k}$ such that $gx \notin \Lambda_{i_k}$. Of course, $g\Omega_{i_k} = \Omega_{i_k}$, so we must have $gx \notin \Lambda_{i_1} \cup \dots \cup \Lambda_{i_m}$. Thus, choosing any $y \in \Lambda_{i_j}$, we have $(x, y) \in \theta$ while $(gx, gy) \notin \theta$, contradicting $\theta \in \text{Con } \mathbf{\Omega}$. Therefore, it must be the case that $\text{Stab}_G(\Lambda_{i_1}) = \dots = \text{Stab}_G(\Lambda_{i_m})$.

(\Leftarrow) Suppose $\text{Stab}_G(\Lambda_{i_1}) = \dots = \text{Stab}_G(\Lambda_{i_m})$. Let θ be the relation defined in (4.2.3). We will prove $\theta \in \text{Con } \mathbf{\Omega}$. It is easy to see that θ is an equivalence relation, so we just need to check $g\theta \subseteq \theta$; that is, we prove $(\forall (x, y) \in \theta) (\forall g \in G) (gx, gy) \in \theta$.

Fix $(x, y) \in \theta$, say, $x \in g_{i_1 k} \Lambda_{i_j}$ and $y \in g_{i_1 \ell} \Lambda_{i_\ell}$, for some $0 \leq k \leq n_{i_1}$, $1 \leq j < \ell \leq m$. For each $g \in G$ we have $g g_{i_1 k} \Lambda_{i_j} = g_{i_1 s} \Lambda_{i_j}$ for some $g_{i_1 s} \in T$. Thus, $g_{i_1 s}^{-1} g g_{i_1 k} \in \text{Stab}_G(\Lambda_{i_j})$. Similarly, $g g_{i_1 \ell} \Lambda_{i_\ell} = g_{i_1 t} \Lambda_{i_\ell}$ for some $g_{i_1 t} \in T$, so $g_{i_1 t}^{-1} g g_{i_1 \ell} \in \text{Stab}_G(\Lambda_{i_\ell})$. This and the hypothesis $\text{Stab}_G(\Lambda_{i_j}) = \text{Stab}_G(\Lambda_{i_\ell})$ together imply $g_{i_1 s} \text{Stab}_G(\Lambda_{i_j}) = g_{i_1 t} \text{Stab}_G(\Lambda_{i_j})$, so $g_{i_1 s} = g_{i_1 t}$, since they are both elements of the transversal of $\text{Stab}_G(\Lambda_{i_j})$. We have thus shown that the action of $g \in G$ maps pairs of blocks with equal stabilizers to the same block of θ ; that is, $g g_{i_1 k} \Lambda_{i_j} = g_{i_1 s} \Lambda_{i_j} \theta g_{i_1 t} \Lambda_{i_\ell} = g g_{i_1 \ell} \Lambda_{i_\ell}$. \square

CHAPTER 5

INTERVAL SUBLATTICE ENFORCEABLE PROPERTIES

5.1 Introduction

Given a finite lattice L , the expression $L \cong [H, G]$ means “there exist finite groups $H < G$ such that L is isomorphic to the interval $\{K \mid H \leq K \leq G\}$ in the subgroup lattice of G .” A group G is called *almost simple* if G has a normal subgroup $S \triangleleft G$ which is nonabelian, simple, and has trivial centralizer, $C_G(S) = 1$. If $H \leq G$, then the *core* of H in G , denoted $\text{core}_G(H)$, is the largest normal subgroup of G contained in H ; it is given by $\text{core}_G(H) = \bigcap_{g \in G} gHg^{-1}$. A subgroup $H \leq G$ for which $\text{core}_G(H) = 1$ is called *core-free* in G . If every finite lattice can be represented as the congruence lattice of a finite algebra, we say that the FLRP has a positive answer.

If we assume that the FLRP has a positive answer, then for every finite lattice L there is a finite group G having L as an upper interval in $\text{Sub}(G)$. In this chapter we consider the following question: Given a finite lattice L , what can we say about a finite group G that has L as an upper interval in its subgroup lattice? Taking this a step further, we consider certain finite collections of finite lattices ask what sort of properties we can prove about a group G if we assume it has all of these lattices as upper intervals in its subgroup lattice. In this and the next section, we address these questions somewhat informally in order to motivate this approach. In Section 5.3 we introduce a new formalism for *interval sublattice enforceable* properties of groups.

One easy consequence that comes out of this investigation is the following observation:

Proposition 5.1.1. *Let \mathcal{L} be a finite collection of finite lattices. If the FLRP has a positive answer, then there exists a finite group G such that each lattice $L_i \in \mathcal{L}$ is an upper interval $L_i \cong [H_i, G] \leq \text{Sub}(G)$, with H_i core-free in G .*

By the “parachute” construction described in the next section, we will see that the only non-trivial part of this proposition is the conclusion that all the H_i be core-free in G . However, this will follow easily from Lemma 5.2.4 below.

Before proceeding, it might be worth pausing to consider what seems like a striking consequence of the proposition above: If the FLRP has a positive answer, then no matter what we take as our finite collection \mathcal{L} – for example, we might take \mathcal{L} to be *all* finite lattices with at most N elements

for some large $N < \omega$ – we can always find a *single* finite group G such that every lattice in \mathcal{L} is an upper interval in $\text{Sub}(G)$; moreover, (by Lemma 5.2.4) we can assume the subgroup H_i at the bottom of each interval is core-free. As a result, the single finite group G must have so many faithful representations, $G \hookrightarrow \text{Sym}(G/H_i)$ with $\text{Con}\langle G/H_i, G \rangle \cong L_i$, one such representation for each distinct $L_i \in \mathcal{L}$.

5.2 Parachute lattices

As mentioned above, in 1980 Pálffy and Pudlák published the following striking result:

Theorem 5.2.1 (Pálffy-Pudlák [32]). *The following statements are equivalent:*

- (A) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*
- (B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

Also noted in [32] is the important fact that (B) is equivalent to:

- (B') *Every finite lattice is isomorphic to the congruence lattice of a finite transitive G -set.*

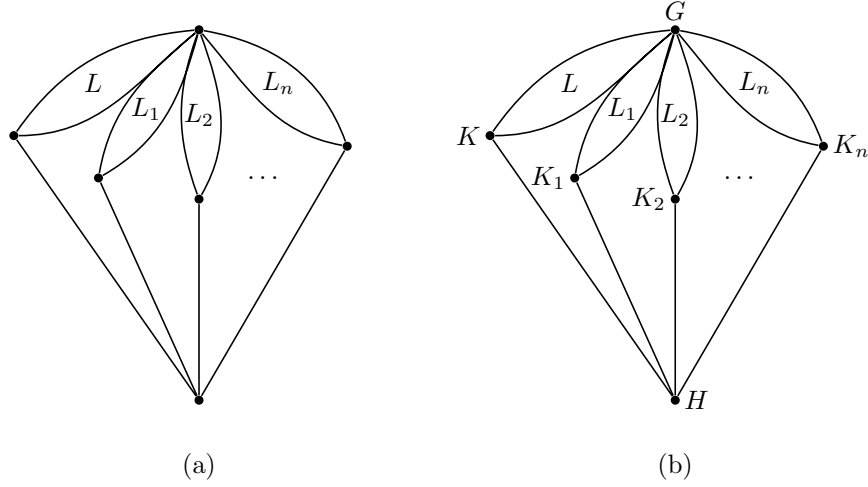
There are a number of examples in the literature of the following situation: a specific finite lattice is considered, and it is shown that if such a lattice is an interval in the subgroup lattice of a finite group, then this group must be of a certain form or have certain properties. As the number of such results grows, it becomes increasingly useful to keep in mind the following simple observation:

Lemma 5.2.2. *Let $\mathcal{G}_1, \dots, \mathcal{G}_n$ be classes of groups and suppose that for each $i \in \{1, \dots, n\}$ there exists a finite lattice L_i such that $L_i \cong [H, G]$ only if $G \in \mathcal{G}_i$. Then (B) is equivalent to*

- (C) *For each finite lattice L , there is a finite group $G \in \bigcap_{i=1}^n \mathcal{G}_i$ such that $L \cong [H, G]$.*

Proof. Obviously, (C) implies (B). Assume (B) holds and let L be any finite lattice. Suppose $\mathcal{G}_1, \dots, \mathcal{G}_n$ and L_1, \dots, L_n satisfy the hypothesis of the lemma. Construct a new lattice $\mathcal{P} = \mathcal{P}(L, L_1, \dots, L_n)$ as shown in Figure 5.1 (a). By (B), there exist finite groups $H \leq G$ with $\mathcal{P} \cong [H, G]$. Let K, K_1, \dots, K_n be the subgroups of G which cover H and satisfy $L \cong [K, G]$, and $L_i \cong [K_i, G]$, $i = 1, \dots, n$ (Figure 5.1 (b)). Thus, L is an interval in the subgroup lattice of G , and, since $L_i \cong [K_i, G]$, we must have $G \in \mathcal{G}_i$, by hypothesis. This is true for all $1 \leq i \leq n$, so $G \in \bigcap_{i=1}^n \mathcal{G}_i$, which proves that (B) implies (C). \square

Figure 5.1: The parachute construction.



Examples. As usual, we let A_n and S_n denote the alternating and symmetric groups on n letters. In addition, the following notation will be useful:

- \mathfrak{G} = the class of all finite groups;
- \mathfrak{S} = the class of all finite solvable groups;
- $\mathfrak{Gi} = \bigcup_{n < \omega} \{A_n, S_n\}$ = the alternating or symmetric groups, also known as the “giant” groups.

It is easy to find a lattice L with the property that $L \cong [H, G]$ implies $G \notin \mathfrak{S}$. We will see an example of such a lattice in Section 6.3. (For another example, see [29].) In his thesis [4], Alberto Basile proves a result which implies that ${}^1M_6 \cong [H, G]$ only if $G \notin \mathfrak{Gi}$. Given these examples and Lemma 5.2.2, it is clear that (B) holds if and only if for each finite lattice L there exist finite groups $H \leq G$ such that $L \cong [H, G]$ and G is not solvable, not alternating, and not symmetric.

Now, if our goal is to solve the finite lattice representation problem, Lemma 5.2.2 suggests the following path to a negative solution: Find examples of lattices L_i which place restrictions on the G for which $L_i \cong [H, G]$ can hold, say $G \in \mathcal{G}_i$, and eventually reach $\bigcap_i \mathcal{G}_i = \emptyset$ (at which point we are done).

We would like to generalize Lemma 5.2.2 because it is much easier and more common to find a

¹Recall, M_n denotes the $(n + 2)$ -element lattice with n atoms.

class of groups \mathcal{G}_i and a lattice L_i with the following property:

$$\text{If } L_i \cong [H, G] \text{ with } H \text{ core-free in } G, \text{ then } G \in \mathcal{G}_i. \quad (\star)$$

This leads naturally to the following question: Given a class of groups \mathcal{G} and a finite lattice L satisfying (\star) , when can we safely drop the caveat “with H core-free in G ” and get back to the hypothesis of Lemma 5.2.2? There is a very simple sufficient condition involving the class $\mathcal{G}^c := \{G \in \mathfrak{G} \mid G \notin \mathcal{G}\}$. (Recall, if \mathcal{K} is a class of algebras, then $\mathbf{H}(\mathcal{K})$ is the class of homomorphic images of members of \mathcal{K} .)

Lemma 5.2.3. *Let \mathcal{G} be a class of groups and L a finite lattice such that*

$$L \cong [H, G] \text{ with } H \text{ core-free} \Rightarrow G \in \mathcal{G}, \quad (5.2.1)$$

and suppose $\mathbf{H}(\mathcal{G}^c) = \mathcal{G}^c$. Then,

$$L \cong [H, G] \Rightarrow G \in \mathcal{G}. \quad (5.2.2)$$

Proof. Suppose L satisfies (5.2.1) and $\mathbf{H}(\mathcal{G}^c) = \mathcal{G}^c$, that is, \mathcal{G}^c is closed under homomorphic images. (For groups this means if $G \in \mathcal{G}^c$ and $N \trianglelefteq G$, then $G/N \in \mathcal{G}^c$.) If (5.2.2) fails, then there is a finite group $G \in \mathcal{G}^c$ with $L \cong [H, G]$. Let $N = \text{core}_G(H)$. Then $L \cong [H/N, G/N]$ and H/N is core-free in G/N so, by hypothesis (5.2.1), $G/N \in \mathcal{G}$. But $G/N \in \mathcal{G}^c$, since \mathcal{G}^c is closed under homomorphic images. \square

Examples. As mentioned above, there is a lattice L with the property that $L \cong [H, G]$ implies G is not solvable, so let $\mathcal{G} = \mathfrak{S}^c$. Then $\mathcal{G}^c = \mathfrak{S}$ is closed under homomorphic images. For the second example above, we have $\mathcal{G} = \mathfrak{S}^c$, so $\mathcal{G}^c = \bigcup_{n < \omega} \{A_n, S_n\}$. This class is also closed under homomorphic images. It follows from Lemma 5.2.3 that these examples do not require the core-free hypothesis. In contrast, consider the following result of Köhler [22]: If $n - 1$ is not a power of a prime, then²

$$M_n \cong [H, G] \text{ with } H \text{ core-free} \Rightarrow G \text{ is subdirectly irreducible.}$$

²Recall, for groups, subdirectly irreducible is equivalent to having a unique minimal normal subgroup.

Lemma 5.2.3 does not apply in this case since \mathcal{G}^c , the class of subdirectly *reducible* groups, is obviously not closed under homomorphic images.³

Though Lemma 5.2.3 seems like a useful observation, the last example above shows that a generalized version of Lemma 5.2.2 – a version based on hypothesis (\star) – would be more powerful, as it would allow us to impose greater restrictions on G , such as those implied by the results of Köhler and others. Fortunately, the “parachute” construction used in the proof of Lemma 5.2.2 works in the more general case, with only a trivial modification to the hypotheses – namely, the lattices L_i should not be two-element chains (which almost goes without saying in the present context). (Recall, $\mathbf{2}$ denotes the two-element chain.)

Lemma 5.2.4. *Let $\mathcal{G}_1, \dots, \mathcal{G}_n$ be classes of groups and suppose that for each $i \in \{1, \dots, n\}$ there is a finite lattice $L_i \not\cong \mathbf{2}$ which satisfies the following:*

$$\text{If } L_i \cong [H, G] \text{ and } H \text{ is core-free in } G, \text{ then } G \in \mathcal{G}_i. \quad (\star)$$

Then (B) is equivalent to

$$(C) \text{ For every finite lattice } L, \text{ there is a finite group } G \in \bigcap_{i=1}^n \mathcal{G}_i \text{ such that } L \cong [H, G].$$

Proof. Obviously, (C) implies (B). Assume (B) and let L be any finite lattice. Suppose $\mathcal{G}_1, \dots, \mathcal{G}_n$ and L_1, \dots, L_n satisfy (\star) and $L_i \not\cong \mathbf{2}$ for all i . Note that there is no loss of generality in assuming that $n \geq 2$. For if $n = 1$, just throw in one of the examples above to make $n = 2$. Call this additional class of groups \mathcal{G}_2 . Then, at the end of the argument, we’ll have $G \in \mathcal{G}_1 \cap \mathcal{G}_2$, and therefore, $G \in \mathcal{G}_1$, which is the stated conclusion of the theorem in case $n = 1$.

Construct the lattice $\mathcal{P} = \mathcal{P}(L, L_1, \dots, L_n)$ as in the proof of Lemma 5.2.2. By (B) there exist finite groups $H \leq G$ with $\mathcal{P} \cong [H, G]$, and we can assume without loss of generality that H is core-free⁴ in G . Let K, K_1, \dots, K_n be the subgroups of G which cover H and satisfy $L \cong [K, G]$, and $L_i \cong [K_i, G]$, $1 \leq i \leq n$, as in Figure 5.1 (b). Thus, L is an upper interval in the subgroup lattice of G , and it remains to show that $G \in \bigcap_{i=1}^n \mathcal{G}_i$. This will follow from (\star) once we prove that each K_i is core-free in G . We now give an easy direct proof this fact, but we note that it also follows from Lemma 5.4.3 below, as well as from a more general result about *L-P lattices*. (See, e.g., Börner [8].)

³Every algebra, and in particular every group G , has a subdirect decomposition into subdirectly irreducibles, $G \leq G/N_1 \times \dots \times G/N_n$. Thus, there will always be homomorphic images, G/N_i , which are subdirectly irreducible.

⁴This is standard. For, if $\mathcal{P} \cong [H, G]$ with $N := \text{core}_G(H) \neq 1$, then $\mathcal{P} \cong [H/N, G/N]$.

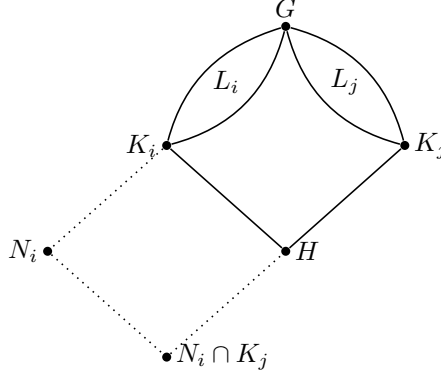


Figure 5.2: The impossibility of a non-trivial core, $N_i = \text{core}_G(K_i)$, in a parachute lattice.

For each $i \in \{1, \dots, n\}$, let $N_i = \text{core}_G(K_i)$. We prove that $N_i = 1$ for all i . Suppose, on the contrary, that $N_i \neq 1$ for some i , and consider any K_j with $j \neq i$.⁵ A sketch of the part of the subgroup lattice under consideration is shown in Figure 5.2. Notice that $N_i K_j = G$. For, N_i is not below H , since H is core-free, so $N_i H = K_i$, so $N_i K_j$ is above both K_i and K_j . Now, clearly, $N_i \cap K_j \trianglelefteq K_j$, and the standard isomorphism theorem implies

$$K_j / (N_i \cap K_j) \cong N_i K_j / N_i = G / N_i.$$

In particular, under this correspondence we have,

$$[N_i \cap K_j, K_j] \ni H \mapsto N_i H = K_i \in [N_i, G],$$

and it follows that the intervals $[K_i, G]$ and $[H, K_j]$ must be isomorphic as lattices. However, by construction, H is a maximal subgroup of K_j , so we have $[H, K_j] \cong \mathbf{2} \not\cong L_i \cong [K_i, G]$. This contradiction proves that $\text{core}_G(K_i) = 1$ for all $1 \leq i \leq n$, as claimed. \square

5.3 ISLE properties of groups

The previous section motivates the study of what we call *interval sublattice enforceable* (ISLE) properties of groups. In this section we formalize this concept, as well as some of the concepts introduced above, and we summarize what we have proved about them. We conclude with some

⁵This is where we use $n \geq 2$; though, if $n = 1$, we could have used K instead of K_j , but then we would need to assume $L \not\cong \mathbf{2}$.

conjectures that will provide the basis for future research.

By a *group theoretical class*, or *class of groups*, we mean a collection \mathcal{G} of groups that is closed under isomorphism: if $G_0 \in \mathcal{G}$ and $G_1 \cong G_0$, then $G_1 \in \mathcal{G}$. A *group theoretical property*, or simply *property of groups*, is a property \mathcal{P} such that if a group G_0 has property \mathcal{P} and $G_1 \cong G_0$, then G_1 has property \mathcal{P} .⁶ Thus if $\mathcal{G}_{\mathcal{P}}$ denotes the collection of groups with group theoretical property \mathcal{P} , then $\mathcal{G}_{\mathcal{P}}$ is a class of groups, and belonging to a class of groups is a group theoretical property. Therefore, we need not distinguish between a property of groups and the class of groups which possess that property. A group in the class \mathcal{G} is called a \mathcal{G} -group, and a group with property \mathcal{P} is called a \mathcal{P} -group. Occasionally we write $G \models \mathcal{P}$ to indicate that G is a \mathcal{P} -group.

We say that a group theoretical property (or class) \mathcal{P} is *interval sublattice enforceable* (ISLE) if there exists a lattice L such that $L \cong [H, G]$ implies G is a \mathcal{P} -group. (By the convention agreed upon at the outset of this chapter, it is implicit in the notation $L \cong [H, G]$ that G is a finite group; thus the class \mathfrak{G} of all finite groups is trivially an ISLE class.) We say that the property (or class) \mathcal{P} is *core-free interval sublattice enforceable* (cf-ISLE) if there exists a lattice L such that if $L \cong [H, G]$ with H core-free in G , then G is a \mathcal{P} -group.

Clearly, if \mathcal{P} is ISLE, then it is also cf-ISLE, and Lemma 5.2.3 above gives a sufficient condition for the converse to hold. We restate this formally as follows:

Lemma 5.2.3'. If \mathcal{P} is cf-ISLE and if $\mathcal{G}_{\mathcal{P}}^c = \{G \in \mathfrak{G} \mid G \not\models \mathcal{P}\}$ is closed under homomorphic images, $\mathbf{H}(\mathcal{G}_{\mathcal{P}}^c) = \mathcal{G}_{\mathcal{P}}^c$, then \mathcal{P} is ISLE.

As we noted in the previous section, two examples of ISLE classes are

- $\mathcal{G}_0 = \mathfrak{S}^c$ = the finite non-solvable groups;
- $\mathcal{G}_1 = (\mathfrak{Gi})^c$ = the finite non-giant groups, $\{G \in \mathfrak{G} \mid (\forall n < \omega) (G \neq A_n \text{ and } G \neq S_n)\}$;

The following classes are at least cf-ISLE:⁷

- \mathcal{G}_2 = the finite subdirectly irreducible groups;
- \mathcal{G}_3 = the finite groups having no nontrivial abelian normal subgroups.
- $\mathcal{G}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for a minimal normal subgroup } M \trianglelefteq G\}$

⁶It seems there is no single standard definition of *group theoretical class*. While some authors (e.g., [13], [3]) use the definition given here, others (e.g. [37], [38]) require that a group theoretical class contain groups of order 1.

⁷The symbols we use to denote these classes are not standard.

Note that $\mathcal{G}_4 \subset \mathcal{G}_2 \cap \mathcal{G}_3 \subset \mathcal{G}_0$.

Given two (group theoretical) properties $\mathcal{P}_1, \mathcal{P}_2$, we write $\mathcal{P}_1 \rightarrow \mathcal{P}_2$ to denote that property \mathcal{P}_1 implies property \mathcal{P}_2 . In other words, $G \models \mathcal{P}_1$ only if $G \models \mathcal{P}_2$. Thus \rightarrow provides a natural partial order on any given set of properties, as follows:

$$\mathcal{P}_1 \leq \mathcal{P}_2 \quad \Leftrightarrow \quad \mathcal{P}_1 \rightarrow \mathcal{P}_2 \quad \Leftrightarrow \quad \mathcal{G}_{\mathcal{P}_1} \subseteq \mathcal{G}_{\mathcal{P}_2},$$

where $\mathcal{G}_{\mathcal{P}_i} = \{G \in \mathfrak{G} \mid G \models \mathcal{P}_i\}$. The following is an obvious corollary of the parachute construction.

Corollary 5.3.1. *If $P = \{\mathcal{P}_i \mid i \in \mathcal{I}\}$ is a collection of (cf-)ISLE properties, then $\bigwedge P$ is (cf-)ISLE.*

Note: the conjunction $\bigwedge \mathcal{P}$ corresponds to the class $\{G \in \mathfrak{G} \mid (\forall i \in \mathcal{I}) G \models \mathcal{P}_i\}$.

It is clear from the foregoing that if solvability were an ISLE property then we would have a solution to the FLRP. But solvability is obviously not ISLE. For, if $L \cong [H, G]$ then for any non-solvable group K we have $L \cong [H \times K, G \times K]$, and of course $G \times K$ is not solvable. Notice, however, that $H \times K$ is not core-free, so a more interesting question to ask might be whether solvability is a cf-ISLE property. The following lemma proves that this is not the case.

Lemma 5.3.2. *Let \mathcal{P} be a cf-ISLE property, and let L be a finite lattice such that $L \cong [H, G]$ with H core-free implies $G \models \mathcal{P}$. Also, suppose there exists a group G witnessing this; that is, G has a core-free subgroup H with $L \cong [H, G]$. Then, for any finite nonabelian simple group S , there exists a wreath product group of the form $W = S \wr \bar{U}$ that is also a \mathcal{P} -group.*

Proof. We apply the idea of Kurzweil twice (cf. Theorem 2.2.2). Fix a finite nonabelian simple group S , and suppose the index of H in G is $|G : H| = n$. Then the action of G on the cosets of H induces an automorphism of the group S^n by permutation of coordinates. Denote this representation by $\varphi : G \rightarrow \text{Aut}(S^n)$, and let the image of G be $\varphi(G) = \bar{G} \leq \text{Aut}(S^n)$. The semidirect product (or wreath product) under this action is the group

$$U := S \wr_{\varphi} G = S^n \rtimes_{\varphi} G = S^n \rtimes \bar{G} = S \wr \bar{G},$$

with multiplication given by

$$(s_1, \dots, s_n, x)(t_1, \dots, t_n, y) = (s_1 t_{x(1)}, \dots, s_n t_{x(n)}, xy),$$

for $s_i, t_i \in S$ and $x, y \in \bar{G}$. An illustration of the subgroup lattice of such a wreath product appears in Figure 5.3. The dual lattice L' is the upper interval of this group, namely, $L' \cong [D \rtimes \bar{G}, U]$. (As

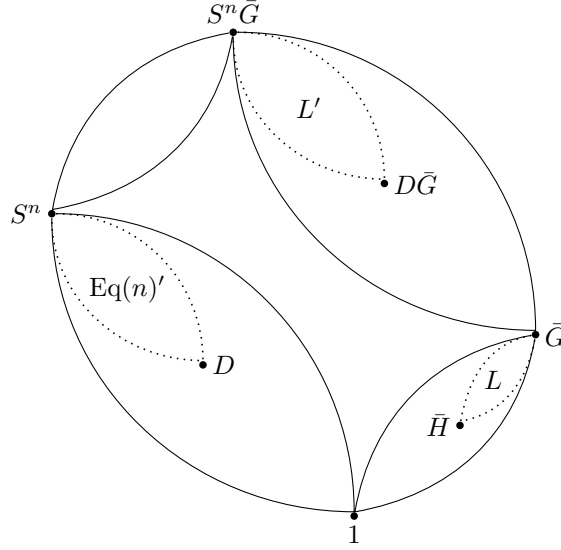


Figure 5.3: Representation of the dual of a group representable lattice.

usual, D denotes the diagonal subgroup of S^n .) It is important to note that if H is core-free in G – equivalently, if $\ker \varphi = 1$ – then the foregoing construction results in the subgroup $D \rtimes \bar{G}$ being core-free in U . (We postpone the proof of this fact.)

Now if we repeat the foregoing procedure, with $H_1 := D \rtimes \bar{G}$ denoting the (core-free) subgroup of U such that $L' \cong [H_1, U]$, then we find that $L = L'' \cong [D_1 \rtimes \bar{U}, S^m \rtimes \bar{U}]$, where $m = |U : H_1|$.⁸ Assuming $D_1 \rtimes \bar{U}$ is core-free in $W = S^m \rtimes \bar{U}$, then, it follows by the original hypothesis that W must be a \mathcal{P} -group.

To complete the proof, we check that starting with a core-free subgroup $H \leq G$ in the Kurzweil construction just described results in a core-free subgroup $D \rtimes \bar{G} \leq U$. Let $N = \text{core}_U(D \rtimes \bar{G})$. Then, for all $n = (d, \dots, d, x) \in N$ and for all $u = (t_1, \dots, t_n, g) \in U$, we have $znz^{-1} \in N$. In particular, we are free to choose $t_1 = t_2$, all other t_k distinct, and $g = 1$. Then

$$znz^{-1} = (t_1, \dots, t_n, 1)(d, \dots, d, x)(t_1^{-1}, \dots, t_n^{-1}, 1) = (t_1 d t_{x(1)}^{-1}, \dots, t_n d t_{x(n)}^{-1}, 1) \in N.$$

Therefore, $t_1 d t_{x(1)}^{-1} = \dots = t_n d t_{x(n)}^{-1}$. With $t_1 = t_2$ and all other t_k distinct, it's clear that x must

⁸Here we use D_1 to denote the diagonal subgroup of S^m to distinguish it from D , the diagonal subgroup of S^n .

stabilize the set $\{1, 2\}$. Of course, the same argument applies in case $t_1 = t_3$ with all other t_k distinct,⁹ so we conclude that x stabilizes the set $\{1, 3\}$ as well. Therefore, $x(i) = i$, for $i = 1, 2, 3$. Since the same argument works for all i , we see that $n = (d, \dots, d, x) \in N$ implies $x \in \ker \varphi = 1$. This puts N below $D \times 1$, and the only normal subgroup of U that lies below $D \times 1$ is the trivial subgroup. \square

The foregoing result enables us to conclude that any class of groups that does not include wreath products of the form $S \wr G$ for all finite simple groups S cannot be a cf-ISLE class.

We conclude this section with the following two equivalent conjectures:

Conjecture 5.1. If \mathcal{P} is a (cf-)ISLE property, then $\neg\mathcal{P}$ is not a (cf-)ISLE property.

Conjecture 5.2. If \mathcal{G} is a (cf-)ISLE class, then \mathcal{G}^c is not a (cf-)ISLE class.

A pair of lattices witnessing the failure of either of these conjectures would solve the FLRP. More precisely, if \mathcal{G} is a class and L_0 and L_1 are lattices such that

$$L_0 \cong [H, G] \Rightarrow G \in \mathcal{G} \quad \text{and} \quad L_1 \cong [H, G] \Rightarrow G \in \mathcal{G}^c$$

Then the parachute lattice $\mathcal{P}(L_0, L_1)$ is not an interval in the subgroup lattice of a finite group.

5.4 Dedekind's rule

We prove a few more lemmas which lead to additional constraints on any group which has a non-trivial parachute lattice as an upper interval in its subgroup lattice. We will need the following standard theorem¹⁰ which we refer to as *Dedekind's rule*:

Theorem 5.4.1 (Dedekind's rule). *Let G be a group and let A, B and C be subgroups of G with $A \leq B$. Then,*

$$A(C \cap B) = AC \cap B, \quad \text{and} \quad (5.4.1)$$

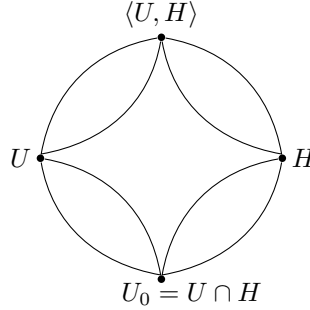
$$(C \cap B)A = CA \cap B. \quad (5.4.2)$$

⁹Note that we can be sure $|G : H| = n > 2$, since $|G : H| = 2$ would imply $H \trianglelefteq G$, which contradicts that H is core-free in G .

¹⁰See, for example, page 122 of Rose, *A Course on Group Theory* [38].

Our next lemma (Lemma 5.4.2) is a slight variation on a standard result that we find very useful. The standard result is essentially part (ii) of Lemma 5.4.2. Surely part (i) of the lemma is also well known, though we have not seen it elsewhere. We will see that the standard result is powerful enough to answer all of our questions about parachute lattices, but later, in Section 6.3, we make use of (i) in a situation where (ii) does not apply.

To state Lemma 5.4.2, we need some new notation. Let U and H be subgroups of a group, let $U_0 := U \cap H$, and consider the interval $[U_0, U] := \{V \mid U_0 \leq V \leq U\}$. In general, when we write UH we mean the set $\{uh \mid u \in U, h \in H\}$, and we write $U \vee H$ or $\langle U, H \rangle$ to mean the group generated by U and H . Clearly $UH \subseteq \langle U, H \rangle$. Equality holds if and only if U and H permute, that is, $UH = HU$. In any case, it is often helpful to visualize part of the subgroup lattice of $\langle U, H \rangle$, as shown below.



Recall that the usual isomorphism theorem for groups implies that if H is a normal subgroup of $\langle U, H \rangle$, then the interval $[H, \langle U, H \rangle]$ is isomorphic to the interval $[U \cap H, U]$. The purpose of the next lemma is to relate these two intervals in cases where we drop the assumption $H \trianglelefteq \langle U, H \rangle$ and add the assumption $UH = \langle U, H \rangle$.

If the two subgroups U and H permute, then we define

$$[U_0, U]^H := \{V \in [U_0, U] \mid VH = HV\}, \quad (5.4.3)$$

which consists of those subgroups V in $[U_0, U]$ that permute with H .

If H normalizes U (which implies $UH = HU$), then we define

$$[U_0, U]_H := \{V \in [U_0, U] \mid H \leq N_G(V)\}, \quad (5.4.4)$$

where $G := UH$. This is the set consisting of those subgroups V in $[U_0, U]$ that are normalized by H . The latter are sometimes called *H-invariant subgroups*. Notice that to even define $[U_0, U]_H$

we must have $H \leq N_G(U)$, and in this case, as we will see below, the two sublattices coincide: $[U_0, U]_H = [U_0, U]^H$.

We are finally ready to state the main result relating the sets defined in (5.4.1) and (5.4.2) (when they exist) to the interval $[H, UH]$.

Lemma 5.4.2. *Suppose U and H are permuting subgroups of a group. Let $U_0 := U \cap H$. Then*

$$(i) \quad [H, UH] \cong [U_0, U]^H \leq [U_0, U].$$

$$(ii) \quad \text{If } U \trianglelefteq UH, \text{ then } [U_0, U]_H = [U_0, U]^H \leq [U_0, U].$$

$$(iii) \quad \text{If } H \trianglelefteq UH, \text{ then } [U_0, U]_H = [U_0, U]^H = [U_0, U].$$

Remarks. Since $G = UH$ is a group, the hypothesis of (ii) is equivalent to $H \leq N_G(U)$, and the hypothesis of (iii) is equivalent to $U \leq N_G(H)$. Part (i) of the lemma says that when two subgroups permute, we can identify the interval above either one of them with the sublattice of subgroups below the other that permute with the first. Part (ii) is similar except we identify the interval above H with the sublattice of H -invariant subgroups below U . Once we have proved (i), the proof of (iii) follows trivially from the standard isomorphism theorem for groups, so we omit the details.

Proof. To prove (i), we show that the following maps are inverse order isomorphisms:

$$\begin{aligned} \varphi : [H, UH] \ni X &\mapsto U \cap X \in [U_0, U]^H \\ \psi : [U_0, U]^H \ni V &\mapsto VH \in [H, UH]. \end{aligned} \tag{5.4.5}$$

Then we show that $[U_0, U]^H$ is a sublattice of $[U_0, U]$, that is, $[U_0, U]^H \leq [U_0, U]$.

Fix $X \in [H, UH]$. We claim that $U \cap X \in [U_0, U]^H$. Indeed,

$$(U \cap X)H = UH \cap X = HU \cap X = H(U \cap X).$$

The first equality holds by (5.4.2) since $H \leq X$, the second holds by assumption, and the third by (5.4.1). This proves $U \cap X \in [U_0, U]^H$. Moreover, by the first equality, $\psi \circ \varphi(X) = (U \cap X)H = UH \cap X = X$, so $\psi \circ \varphi$ is the identity on $[H, UH]$.

If $V \in [U_0, U]^H$, then $VH = HV$ implies $VH \in [H, UH]$. Also, $\varphi \circ \psi$ is the identity on $[U_0, U]^H$, since $\varphi \circ \psi(V) = VH \cap U = V(H \cap U) = VU_0 = V$, by (5.4.1). This proves that φ and ψ are inverses of each other on the sets indicated, and it's easy to see that they are order preserving: $X \leq Y$

implies $U \cap X \leq U \cap Y$, and $V \leq W$ implies $VH \leq WH$. Therefore, φ and ψ are inverse order isomorphisms.

To complete the proof of (i), we show that $[U_0, U]^H$ is a sublattice of $[U_0, U]$. Suppose V_1 and V_2 are subgroups in $[U_0, U]$ which permute with H . It is easy to see that their join $V_1 \vee V_2 = \langle V_1, V_2 \rangle$ also permutes with H , so we just check that their intersection permutes with H . Fix $x \in V_1 \cap V_2$ and $h \in H$. We show $xh = h'x'$ for some $h' \in H$, $x' \in V_1 \cap V_2$. Since V_1 and V_2 permute with H , we have $xh = h_1v_1$ and $xh = h_2v_2$ for some $h_1, h_2 \in H$, $v_1 \in V_1$, $v_2 \in V_2$. Therefore, $h_1v_1 = h_2v_2$, which implies $v_1 = h_1^{-1}h_2v_2 \in HV_2$, so v_1 belongs to $V_1 \cap HV_2$. Note that $V_1 \cap HV_2$ is below both V_1 and $U \cap HV_2 = \varphi\psi(V_2) = V_2$. Therefore, $v_1 \in V_1 \cap HV_2 \leq V_1 \cap V_2$, and we have proved that $xh = h_1v_1$ for $h_1 \in H$ and $v_1 \in V_1 \cap V_2$, as desired.

To prove (ii), assuming $U \triangleleft G$, we show that if $U_0 \leq V \leq U$, then $VH = HV$ if and only if $H \leq N_G(V)$. If $H \leq N_G(V)$, then $VH = HV$ (even when $U \not\triangleleft G$). Suppose $VH = HV$. We must show $(\forall v \in V)(\forall h \in H) hvh^{-1} \in V$. Fix $v \in V$, $h \in H$. Then, $hv = v'h'$ for some $v' \in V$, $h' \in H$, since $VH = HV$. Therefore, $v'h'h^{-1} = hvh^{-1} = u$ for some $u \in U$, since $H \leq N_G(U)$. This proves that $hvh^{-1} \in VH \cap U = V(H \cap U) = VU_0 = V$, as desired. \square

Next we prove that any group which has a nontrivial parachute lattice as an upper interval in its subgroup lattice must have some rather special properties.

Lemma 5.4.3. *Let $\mathcal{P} = \mathcal{P}(L_1, \dots, L_n)$ with $n \geq 2$ and $|L_i| > 2$ for all i , and suppose $\mathcal{P} \cong [H, G]$, with H core-free in G .*

- (i) *If $1 \neq N \triangleleft G$, then $NH = G$.*
- (ii) *If M is a minimal normal subgroup of G , then $C_G(M) = 1$.*
- (iii) *G is subdirectly irreducible.*
- (iv) *G is not solvable.*

Remark. If a subgroup $M \leq G$ is abelian, then $M \leq C_G(M)$, so (ii) implies that a minimal normal subgroup (hence, every normal subgroup) of G must be nonabelian.

Proof. (i) Let $1 \neq N \triangleleft G$. Then $N \not\leq H$, since H is core-free in G . Therefore, $H < NH$. As in Section 5.2, we let K_i denote the subgroups of G corresponding to the atoms of \mathcal{P} . Then H is covered by each K_i , so $K_j \leq NH$ for some $1 \leq j \leq n$. Suppose, by way of contradiction, that

$NH < G$. By assumption, $n \geq 2$ and $|L_i| > 2$. Thus for any $i \neq j$ we have $K_i \leq Y < Z < G$ for some subgroups Y and Z which satisfy $(NH) \cap Z = H$ and $(NH) \vee Y = G$. Also, $(NH)Y = NY$ is a group, so $(NH)Y = NH \vee Y = G$. But then, by Dedekind's rule, we have

$$Y = HY = ((NH) \cap Z)Y = (NH)Y \cap Z = G \cap Z = Z,$$

contrary to $Y < Z$. This contradiction proves that $NH = G$.

(ii) If $C_G(M) \neq 1$, then (i) implies $C_G(M)H = G$, since $C_G(M) \leq N_G(M) = G$. Consider any $H < K < G$. Then $1 < M \cap K < M$ (strictly, by Lemma 5.4.2). Now $M \cap K$ is normalized by H and centralized (hence normalized) by $C_G(M)$. (Indeed, $C_G(M)$ centralizes every subgroup of M .) Therefore, $M \cap K \leq C_G(M)H = G$, contradicting the minimality of M .

(iii) We prove that G has a *unique* minimal normal subgroup. Let M be a minimal¹¹ normal subgroup of G and let $N \leq G$ be any normal subgroup not containing M . We show that $N = 1$. Since both subgroups are normal, the *commutator*¹² of M and N lies in the intersection $M \cap N$, which is trivial by the minimality of M . Thus, M and N centralize each other. In particular, $N \leq C_G(M) = 1$, by (ii).

(iv) Let M' denote the commutator of M . As remarked above, M is nonabelian, so $M' \neq 1$. Also, $M' \leq M \leq G$, and M' is a *characteristic* subgroup of M (i.e., M' invariant under $\text{Aut}(M)$). Therefore, $M' \leq G$, and, as M is a *minimal* normal subgroup of G , we have $M' = M$. Thus, M is not solvable, so G is not solvable. \square

Remark. It follows from (i) that, if \mathcal{P} is a nontrivial parachute lattice with $\mathcal{P} \cong [H, G]$, where H is core-free, then $\text{core}_G(X) = 1$ for every $H \leq X < G$. This gives a second way to complete the proof of Lemma 5.2.4.

To summarize what we have thus far, the lemmas above imply that (B) holds if and only if every finite lattice is an interval $[H, G]$, with H core-free in G , where

(i) G is not solvable, not alternating, and not symmetric;

(ii) G has a unique minimal normal subgroup M which satisfies $MH = G$ and $C_G(M) = 1$; in

¹¹If G is simple, then $M = G$; “minimal” assumes nontrivial.

¹²The *commutator* of M and N is the subgroup generated by the set $\{mnmm^{-1}n^{-1} : m \in M, n \in N\}$. The *commutator* of M is the subgroup generated by $\{aba^{-1}b^{-1} : a, b \in M\}$. The n^{th} *degree commutator* of M , denoted $M^{(n)}$, is defined recursively as the commutator of $M^{(n-1)}$. A group M is *solvable* if $M^{(n)} = 1$ for some $n \in \mathbb{N}$.

particular, M is nonabelian and $\text{core}_G(X) = 1$ for all $H \leq X < G$.

Finally, we note that Theorem 4.3.A of Dixon and Mortimer [12] describes the structure of the unique minimal normal subgroup as follows:

- (iii) $M = T_0 \times \cdots \times T_{r-1}$, where T_i are simple minimal normal subgroups of M which are conjugate (under conjugation by elements of G). Thus, M is a direct power of a simple group T .

In fact, when $C_G(M) = 1$, as in our application, we can specify these conjugates more precisely. Let T be any minimal normal subgroup of M . Note that T is simple. Let $N = N_H(T) = \{h \in H : T^h = T\}$ be the normalizer of T in H . Then the proof of the following lemma is routine, so we omit it.

Lemma 5.4.4. *If $H/N = \{N, h_1N, \dots, h_{k-1}N\}$ is a full set of left cosets of N in H , then $k = r$ and $M = T_0 \times \cdots \times T_{r-1} = T \times T^{h_1} \times \cdots \times T^{h_{r-1}}$.*

We conclude this chapter by noting that other researchers, such as Baddeley, Börner, and Lucchini, have proved similar results for the more general case of *quasiprimitive permutation groups*. In particular, our proof of Lemma 5.4.3 (i) uses the same argument as the one in [8], where it is used to prove Lemma 2.4: if $L \cong [H, G]$ is an *LP-lattice*,¹³ then G must be a quasiprimitive permutation group. We remark that parachute lattices, in which each panel L_i has $|L_i| > 2$, are LP-lattices, so Lemma 5.4.3 follows from theorems of Baddeley, Börner, Lucchini, et al. (cf. [2], [8]).

However, the main purpose of the parachute construction, besides providing a quick route to Lemma 5.4.3, is to demonstrate a natural way to insert arbitrary finite lattices L_i as upper intervals $[K_i, G]$ in $\text{Sub}[G]$, with K_i core-free in G . Then, once we prove special properties of groups G for which $L_i = [K_i, G]$ (K_i core-free), it follows that *every* finite lattice L must be an upper interval $L = [K, G]$ for some G satisfying all of these properties, assuming the FLRP has a positive answer. This forms the basis and motivation for the idea of (cf-)ISLE properties, as discussed in Section 5.3.

¹³An LP-lattice is one in which every element except 0 and 1 is a non-modular element.

CHAPTER 6

LATTICES WITH AT MOST SEVEN ELEMENTS

6.1 Introduction

In the spring of 2011, our research seminar was fortunate enough to have as a visitor Peter Jipsen, who initiated the project of cataloging every small finite lattice L for which there is a known finite algebra \mathbf{A} with $\text{Con } \mathbf{A} \cong L$. It is well known that all lattices with at most six elements are representable. In fact, these can be found as intervals in subgroup lattices of finite groups, but this fact was not known until recently.

By 1996, Yasuo Watatani had found each six-element lattice, except for the two lattices appearing below, as intervals in subgroup lattices of finite groups. See [46].



Then, in 2008, Michael Aschbacher showed in [1] how to construct some (very large) twisted wreath product groups that have the lattices above as intervals in their subgroup lattices. Note that, although it was apparently quite difficult to find *group* representations of the lattices shown above, it is quite easy to represent them concretely as the lattices of congruences of very small finite algebras. Take, for example, the set $X = \{0, 1, \dots, 6\}$ and consider the lattice $L \leq \text{Eq}(X)$ generated by the partitions

$$|0, 3, 4|1, 6|2, 5| \text{ and } |0, 6|1, 5|2|3|4| \leq |0, 6|1, 4, 5|2|3| \leq |0, 6|1, 4, 5|2, 3|.$$

This concrete representation of the lattice on the left above happens to be closed: $\rho\lambda(L) = L$, so it is equal to the congruence lattice $\text{Con } \langle X, \lambda(L) \rangle$.

We prove two main results in this chapter. The first is

Theorem 6.1.1. *Every finite lattice with at most seven elements, with one possible exception, is representable as the congruence lattice of a finite algebra.*

The second result concerns the one possible exception of this theorem, a seven element lattice, which we call L_7 . It is the focus of Section 6.3. As we explain below, if L_7 is representable as the congruence lattice of a finite algebra, then it must appear as an interval in the subgroup lattice of a finite group.¹ Our main result, Theorem 6.3.1, places some fairly strong restrictions on such a group. Our motivation is to apply this new theorem, along with some well known theorems classifying finite groups, to eventually either find such a group or prove that none exists. This application will be the focus of future research.

6.2 Seven element lattices

In this section we show that, with one possible exception (discussed in the next section), every lattice with at most seven elements is representable as a congruence lattice of a finite algebra. There are 53 lattices with at most seven elements.² Representations for most of these lattices can be found quite easily by applying the methods described in previous chapters. The easiest, of course, are the distributive lattices, which we know are representable by Theorem 1.3.3. Some others are found to be representable by searching (with a computer) for closed concrete representations $L \leq \text{Eq}(X)$ over some small set X , say $|X| < 8$. Still others are found by checking that they are obtained by applying operations under which \mathcal{L}_3 is closed (§ 2.1). For example, the lattice on the left in Figure 6.1 is the ordinal sum of two copies of the distributive lattice $\mathbf{2} \times \mathbf{2}$. On the right of the same figure is the parallel sum of the distributive lattices $\mathbf{2}$ and $\mathbf{3}$.



Figure 6.1: The ordinal sum of $\mathbf{2} \times \mathbf{2}$ with itself (left) and the parallel sum of $\mathbf{2}$ and $\mathbf{3}$ (right).

Using these methods, it was not hard to find, or at least prove the existence of, congruence lattice representations of all seven element lattices except for the seven lattices appearing in Figure 6.2,

¹Note that the result of Pálffy and Pudlák does *not* say that every representable lattice is isomorphic to an interval in a subgroup lattice of a finite group. Rather, it is a statement about the whole class of representable lattices. However, for certain lattices, such as the one described in Section 6.3, we can prove that it belongs to \mathcal{L}_3 if and only if it belongs to \mathcal{L}_4 .

²The Hasse diagrams of all lattices with at most seven elements are shown here <http://db.tt/2qJUkoaG> or alternatively here <http://math.chapman.edu/~jipsen/mathposters/lattices7.pdf> (courtesy of Peter Jipsen).

plus their duals. Four of these seven are self-dual, so there are ten lattices in total for which a representation is not relatively easy to find.³

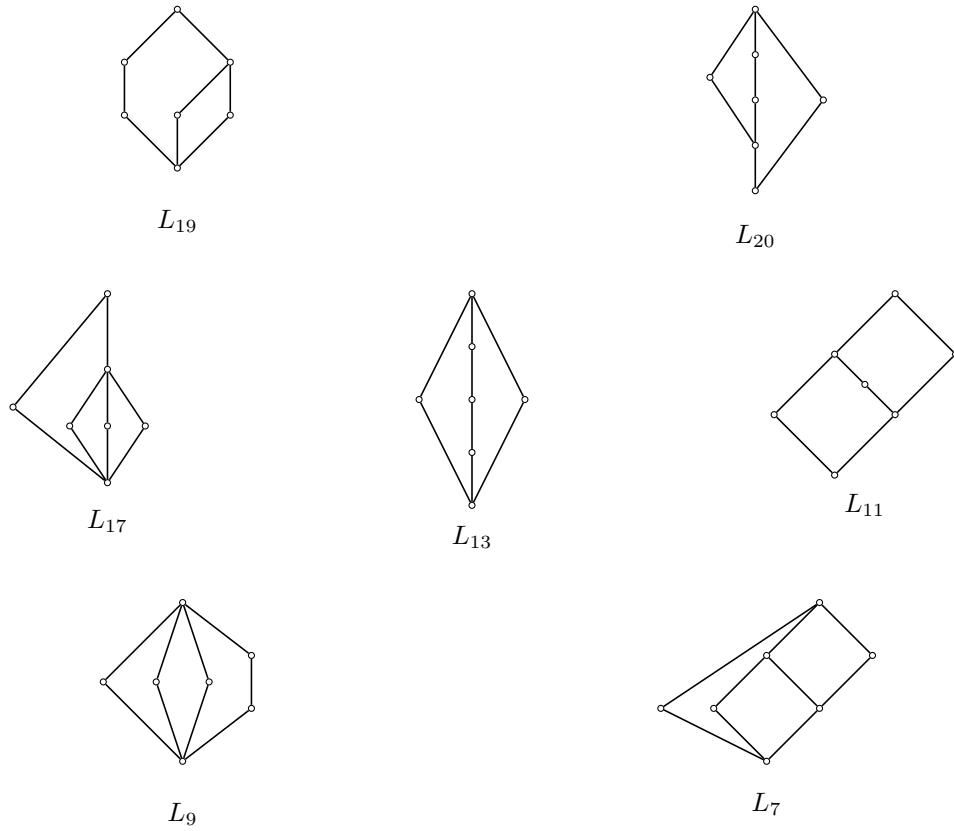
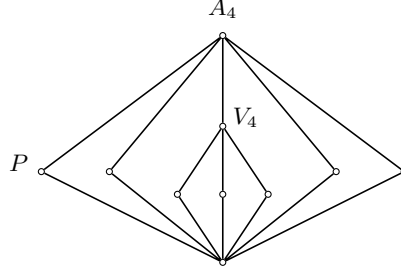


Figure 6.2: Seven element lattices with no obvious congruence lattice representation.

We now prove the existence of congruence lattice representations for all but the last of these. The first two, L_{19} and L_{20} were found using the closure method with the help of Sage by searching for closed concrete representations in the partition lattice $\text{Eq}(8)$. As for L_{17} , recall that the lattice $\text{Sub}(A_4)$ of subgroups of the group A_4 (the group of all even permutations of a four element set) is the lattice shown below.

³The names of these lattices do not conform to any well established naming convention.



Here V_4 denotes the Klein four subgroup and P marks one of the four Sylow 3 subgroups of A_4 . Of course, $\text{Sub}(A_4)$ is the congruence lattice of the permutational algebra consisting of A_4 acting regularly on itself by multiplication. Now note that $L_{17} \cong P^\uparrow \cup V_4^\downarrow$, the union of a filter and ideal of a representable lattice. Therefore, L_{17} is representable.

The question of whether the existence of such a “filter-idea representation” implies that the lattice in question is also an interval in a subgroup lattice seems open. Although, in the present case, we have found that L_{17} has a group representation. Indeed, the group $G = (A_4 \times A_4) \rtimes C_2$ has a subgroup $H \cong S_3$ such that $[H, G] \cong L_{17}$.

Now, by the Kurzweil-Netter result, the dual of L_{17} is also representable. Explicitly, since L_{17} is representable on a 12-element set (the elements of A_4) via the filter-ideal method,⁴ the dual of L_{17} can be embedded above diagonal subgroup of the 12-th power of a simple group: $L'_{17} \hookrightarrow [D, S^{12}] \cong (\text{Eq}(12))'$. Then, adding the operations from the original representation of L_{17} as described in the proof of Theorem 2.2.2, we have an algebra with universe S^{12}/D and congruence lattice isomorphic to L'_{17} .⁵

The lattice L_{13} is an interval in a subgroup lattice. Specifically, a **GAP** search reveals that the group⁶ $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ has a subgroup $H \cong A_4$ such that $[H, G] \cong L_{13}$. The index is $|G : H| = 80$, so the action of G on the cosets G/H is an algebra on an 80 element universe.

Though we have not found L_{11} as an interval in a subgroup lattice, we have found that the pentagon N_5 is an upper interval in the subgroup lattice of the groups $G = ((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$ and $G = (A_4 \times A_4) \rtimes C_2$.⁷ In each of these groups, there exists a subgroup $H < G$ (of index 36) with $[H, G] \cong N_5$. Let $[H, G] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$. (See Figure 6.3.) Of course, $\text{Sub}(G)$ is a congruence

⁴Note that the filter plus ideal method only adds operations to the algebra of which the original lattice was the congruence lattice, leaving the universe fixed. Thus, the filter-ideal sublattice is the congruence lattice of an algebra with the same number of elements as the original algebra.

⁵Incidentally, since L_{17} is also representable as an interval above a subgroup (of index 48), we could apply the Kurzweil-Netter method using this representation instead. Then we would obtain a *group* representation of the dual (namely, an upper interval in a group of the form $S^{48} \rtimes G$, where $G = (A_4 \times A_4) \rtimes C_2$).

⁶In **GAP** this is `SmallGroup(960, 11358)`.

⁷ Q_8 denotes the eight element quaternion group.

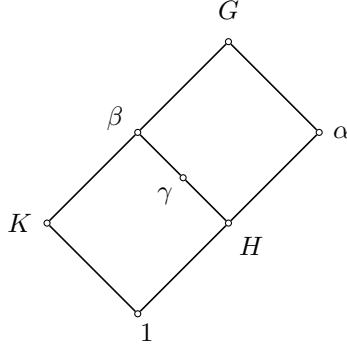


Figure 6.3: The lattice L_{11} represented as the union of a filter and ideal in the subgroup lattice of the group G . Two choices for G that work are $\text{SmallGroup}(216, 153) = ((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$ and $\text{SmallGroup}(288, 1025) = (A_4 \times A_4) \rtimes C_2$.

lattice, so if there exists a subgroup $K \succ 1$, below β and not below γ , then $L_{11} \cong K^\downarrow \cup H^\uparrow$. Indeed, there is such a subgroup K .

Apart from the easy cases, which we only briefly covered at the start of this section, there remain just two seven element lattices for which we have not yet described a representation. These are the lattices at the bottom of Figure 6.2. Finding a representation of L_9 , dubbed the “triple-wing pentagon,” was quite challenging. It sparked the idea of expanding finite algebras, which we describe at length in the next chapter (Ch. 7). Here we only mention the basic idea as it applies to this particular lattice. As the goal is to find an algebra with congruence lattice L_9 , we start with an algebra having an M_4 congruence lattice – that is, a six element lattice of height two with four atoms (which are also coatoms). Then we expand the algebra by adding elements to the universe and adding certain operations so that the newly expanded algebra has almost the same congruence lattice as the original, except one of the atoms has been doubled. That is, the resulting congruence lattice is isomorphic to L_9 . This example and the powerful techniques that grew out of it are described in Chapter 7.

It is still unknown whether the final lattice appearing in Figure 6.2 is representable as the congruence lattice of a finite algebra. Thus, L_7 is the unique smallest lattice for which there is no known representation. It is the subject of the next section.

6.3 The exceptional seven element lattice

In this section we consider L_7 , the last seven element lattice appearing in Figure 6.2. As yet, we are unable to find a finite algebra which has a congruence lattice isomorphic to L_7 , and this is the smallest lattice for which we have not found such a representation.

Suppose \mathbf{A} is a finite algebra with $\text{Con } \mathbf{A} \cong L_7$, and suppose \mathbf{A} is of minimal cardinality among those algebras having a congruence lattice isomorphic to L_7 . Then \mathbf{A} must be isomorphic to a transitive G -set. (This fact is proved in a forthcoming article, [11].) Therefore, if L_7 is representable, we can assume there is a finite group G with a core-free⁸ subgroup $H < G$ such that L_7 is isomorphic to the interval sublattice $[H, G] \leq \text{Sub}(G)$. In this section we present some restrictions on the possible groups for which this can occur.

The first restriction, which is the easiest to observe, is that G must act primitively on the cosets of one of its maximal subgroups. This suggests the possibility of describing G in terms of the O’Nan-Scott Theorem which characterizes primitive permutation groups. The goal is to eventually find enough restrictions on G so as to rule out all finite groups. As yet, we have not achieved this goal. However, the new results in this section reduce the possibilities to very special subclasses of the O’Nan-Scott classification theorem. This paves the way for future studies to focus on these subclasses when searching for a group representation of L_7 , or proving that none exists.

The main result of this section is the following:

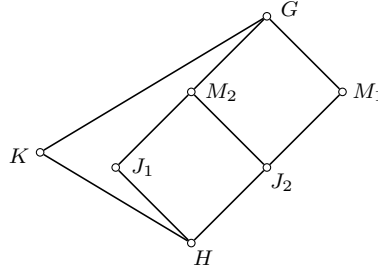
Theorem 6.3.1. *Suppose $H < G$ are finite groups with $\text{core}_G(H) = 1$ and suppose $L_7 \cong [H, G]$. Then the following hold.*

- (i) G is a primitive permutation group.
- (ii) If $N \triangleleft G$, then $C_G(N) = 1$.
- (iii) G contains no non-trivial abelian normal subgroup.
- (iv) G is not solvable.
- (v) G is subdirectly irreducible.
- (vi) With the possible exception of at most one maximal subgroup, all proper subgroups in the interval $[H, G]$ are core-free.

⁸Recall that the core of a subgroup X in G is the largest normal subgroup of G contained in X . This is denoted by $\text{core}_G(X)$. We say the X is *core-free* in G provided $\text{core}_G(X) = 1$.

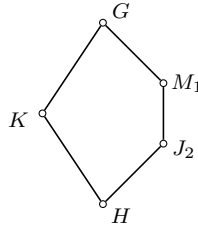
Remark. It is obvious that (ii) \Rightarrow (iii) \Rightarrow (iv), and (ii) \Rightarrow (v), but we include these easy consequences in the statement of the result for emphasis; for, although the hard work will be in proving (ii) and (vi), our main goal is the pair of restrictions (iii) and (v), which allow us to rule out a number of the O’Nan-Scott types describing primitive permutation groups. (Section A.2.1 includes a detailed description of these types.)

Assume the hypotheses of the theorem above. In particular, throughout this section *all groups are finite, H is a core-free subgroup of G , and $[H, G] \cong L_7$.* Label the seven subgroups of G in the interval $[H, G]$ as in the following diagram:



The labels are chosen with the intention of helping us remember to which subgroups they refer: the maximal subgroup M_2 covers two subgroups in the interval $[H, G]$, while J_2 is covered by two subgroups of G .

We now prove the foregoing theorem through a series of claims. The first thing to notice about the interval $[H, G]$ is that K is a *non-modular element* of the interval. This means that there is a spanning pentagonal (N_5) sublattice of the interval with K as the incomparable proper element. (See the diagram below, for example.)



Using this non-modularity property of K , it is easy to prove the following

Claim 6.1. K is a core-free subgroup of G .

Proof. Let $N := \text{core}_G(K)$. If $N \leq X$ for some $X \in \{M_1, M_2, J_1, J_2\}$, then $N < X \cap K = H$, so $N = 1$ (since H is core-free). If $N \not\leq X$ for all $X \in \{M_1, M_2, J_1, J_2\}$, then $NJ_2 = G$. But then

Dedekind's rule leads to the following contradiction:

$$J_2 \leq M_1 \quad \Rightarrow \quad J_2 = J_2(N \cap M_1) = J_2N \cap M_1 = G \cap M_1 = M_1.$$

Therefore, $N = 1$. □

Note that (i) of the theorem follows from Claim 6.1. Since K is core-free, G acts faithfully on the cosets G/K by right multiplication. Since K is a maximal subgroup, the action is primitive.

The next claim is only slightly harder than the previous one as it requires the more general consequence of Dedekind's rule that we established above in Lemma 5.4.2 (i).

Claim 6.2. J_1 and J_2 are core-free subgroups of G .

Proof. First note that if $N \triangleleft G$ then the subgroup NH permutes⁹ with any subgroup containing H . To see this, let $H \leq X \leq G$ and note that

$$NHX = NX = XN = XHN = XNH,$$

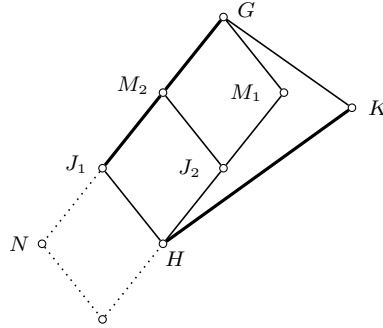
since $H \leq X$ and $N \triangleleft G$.

Suppose $1 \neq N \leq J_1$ for some $N \triangleleft G$. Then $NH = J_1$, so J_1 and K are permuting subgroups. Since $J_1K = G$ and $J_1 \cap K = H$, Lemma 5.4.2 yields

$$[J_1, G] \cong [H, K]^{J_1} := \{X \in [H, K] \mid J_1X = XJ_1\}.$$

But this is impossible since $[H, K]^{J_1} \leq [H, K] \cong \mathbf{2}$, while $[J_1, G] \cong \mathbf{3}$. This proves that $\text{core}_G(J_1) = 1$.

The intervals involved in the argument are drawn with bold lines in the following diagram.



⁹Recall, for subgroups X and Y of a group G , we define the sets $XY = \{xy \mid x \in X, y \in Y\}$, and $YX = \{yx \mid x \in X, y \in Y\}$, and we say that X and Y are *permuting subgroups* (or that X and Y permute, or that X permutes with Y) provided the two sets XY and YX coincide, in which case the set forms a group: $XY = \langle X, Y \rangle = YX$.

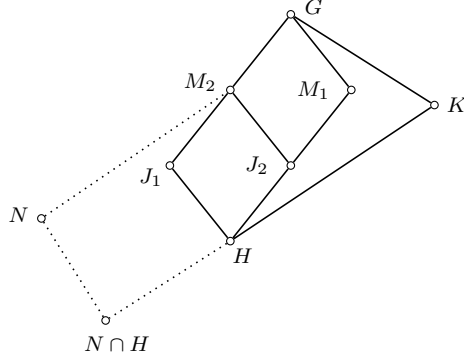


Figure 6.4: Hasse diagram illustrating the cases in which M_2 has non-trivial core: $1 \neq N \leq M_2$ for some $N \triangleleft G$.

The proof that J_2 is core-free is similar. Suppose $1 \neq N \leq J_2$ where $N \triangleleft G$. Then $NH = J_2$ and the subgroups J_2 and K permute. Therefore, $[H, K]^{J_2} \cong [J_2, G]$, by Lemma 5.4.2, which is a contradiction since $[H, K]^{J_2} \leq [H, K] \cong \mathbf{2}$, while $[J_2, G] \cong \mathbf{2} \times \mathbf{2}$. \square

Now that we know K, J_1, J_2 are each core-free in G , we use this information to prove that at least one of the other maximal subgroups, M_1 or M_2 , is core-free in G , thereby establishing (vi) of the theorem. We will also see that G is subdirectly irreducible, proving (v). The proof of (ii) will then follow from the same argument used to prove Lemma 5.4.2 (ii), which we repeat below.

Claim 6.3. Either M_1 or M_2 is core-free in G . If M_2 has non-trivial core and $N \triangleleft G$ is contained in M_2 , then $C_G(N) = 1$ and G is subdirectly irreducible.

Proof. Suppose M_2 has non-trivial core. Then there is a minimal normal subgroup $1 \neq N \triangleleft G$ contained in M_2 . Since H, J_1, J_2 are core-free, $NH = M_2$. Consider the centralizer, $C_G(N)$, of N in G . Of course, this is a normal subgroup of G .¹⁰ If $C_G(N) = 1$, then, since minimal normal subgroups centralize each other, N must be the unique minimal normal subgroup of G . Furthermore, M_1 must be core-free in this case. Otherwise $N \leq M_1 \cap M_2 = J_2$, contradicting $\text{core}_G(J_2) = 1$. Therefore, in case $C_G(N) = 1$ we conclude that G is subdirectly irreducible and M_1 is core-free.

We now prove that the alternative, $C_G(N) \neq 1$, does not occur. This case is a bit more challenging and must be split up into further subcases, each of which leads to a contradiction. Throughout, the assumption $1 \neq N \leq M_2$ is in force, and it helps to keep in mind the diagram in Figure 6.4.

¹⁰The centralizer of a normal subgroup $N \trianglelefteq G$ is itself normal in G . For, it is the kernel of the conjugation action of G on N . Thus, $C_G(N) \trianglelefteq N_G(N) = G$.

Suppose $C_G(N) \neq 1$. Then, since $C_G(N) \trianglelefteq G$, and since H, J_1, J_2, K are core-free, it's clear that $C_G(N)H \in \{G, M_1, M_2\}$. We consider each case separately.

Case 1: Suppose $C_G(N)H = G$. Note that $N \cap H < N \cap J_1 < N$ (strictly). The subgroup $N \cap J_1$ is normalized by J_1 and by $C_G(N)$, and so it is normal in $C_G(N)J_1 \geq C_G(N)H = G$, contradicting the minimality of N . Thus, the case $C_G(N)H = G$ does not occur.

Case 2: Suppose $C_G(N)H = M_1$. The subgroup $N \cap J_1$ is normalized by both H and $C_G(N)$. For, $C_G(N)$ centralizes, hence normalizes, every subgroup of N . Therefore, $N \cap J_1$ is normalized by $C_G(N)H = M_1$. Of course, it's also normalized by J_1 , so $N \cap J_1$ is normalized by the set M_1J_1 , so it's normalized by the group generated by that set, which is $\langle M_1, J_1 \rangle = G$.¹¹ The conclusion is that $N \cap J_1 \triangleleft G$. Since J_1 is core-free, $N \cap J_1 = 1$. But this contradicts the (by now familiar) consequence of Dedekind's rule:

$$H < J_1 < M_2 \quad \Rightarrow \quad N \cap H < N \cap J_1 < N \cap M_2.$$

Therefore, $C_G(N)H = M_1$ does not occur.

Case 3: Suppose $C_G(N)H = M_2$. The subgroup $N \cap M_1$ is normalized by both H and $C_G(N)$. Therefore, $N \cap M_1$ is normalized by $C_G(N)H = M_2$. Of course, it's also normalized by M_1 , so $N \cap M_1$ is normalized by $\langle M_1, M_2 \rangle = G$. The conclusion is that $N \cap M_1 \triangleleft G$. By minimality of the normal subgroup N , we must have either $N \cap M_1 = 1$ or $N \cap M_1 = N$. The former equality implies $N \cap J_2 = 1$, which contradicts the strict inequalities of Dedekind's rule,

$$H < J_2 < M_2 \quad \Rightarrow \quad N \cap H < N \cap J_2 < N \cap M_2, \quad (6.3.1)$$

while the latter equality ($N \cap M_1 = N$) implies that $N \leq M_1 \cap M_2 = J_2$ which contradicts $\text{core}_G(J_2) = 1$.

□

We have proved that either M_1 or M_2 is core-free in G , and we have shown that, if M_2 has non-trivial core, then G is subdirectly irreducible. In fact, we proved that $C_G(N) = 1$ for the unique minimal normal subgroup N in this case. It remains to prove that G is subdirectly irreducible in

¹¹Actually, the set is already a group in this case since $M_1J_1 = C_G(N)HJ_1 = J_1C_G(N)H = J_1M_1$.

case M_1 has non-trivial core. The argument is similar to the foregoing, and we omit some of the details that can be checked exactly as above.

Claim 6.4. If M_1 has non-trivial core and $N \triangleleft G$ is contained in M_1 , then $C_G(N) = 1$ and G is subdirectly irreducible.

Proof. If M_1 has non-trivial core, then there is a minimal normal subgroup $N \triangleleft G$ contained in M_1 . We proved above that M_2 must be core-free in this case, so either $C_G(N)H = G$, $C_G(N)H = M_1$, or $C_G(N) = 1$. The first case is easily ruled out exactly as in Case 1 above. The second case is handled by the argument we used in Case 3. Indeed, if we suppose $C_G(N)H = M_1$, then $N \cap M_2$ is normalized by both H and $C_G(N)$, hence by M_1 . It is also normalized by M_2 , so $N \cap M_2 \triangleleft G$. Thus, by minimality of N , and since M_2 is core-free, $N \cap M_2 = 1$. But then $N \cap J_2 = 1$, leading to a contradiction similar to (6.3.1) but with M_1 replacing M_2 . Therefore, the case $C_G(N)H = M_1$ does not occur, and we have proved $C_G(N) = 1$. \square

So far we have proved that all intermediate proper subgroups in the interval $[H, G]$ are core-free except possibly at most one of M_1 or M_2 . Moreover, we proved that if one of the maximal subgroups has non-trivial core, then there is a unique minimal normal subgroup $N \triangleleft G$ with trivial centralizer, $C_G(N) = 1$. As explained above, G is subdirectly irreducible in this case, since minimal normal subgroups centralize each other.

In order to prove (ii), there remains only one case left to check, and the argument is by now very familiar.

Claim 6.5. If each $H \leq X < G$ is core-free and N is a minimal normal subgroup of G , then $C_G(N) = 1$.

Proof. Let N be a minimal normal subgroup of G . Then, by the core-free hypothesis we have $NH = G$. Fix a subgroup $H < X < G$. Then $N \cap H < N \cap X < N$. The subgroup $N \cap X$ is normalized by H and by $C_G(N)$. If $C_G(N) \neq 1$, then $C_G(N)H = G$, by the core-free hypothesis, so $N \cap X \triangleleft G$, contradicting the minimality of N . Therefore, $C_G(N) \neq 1$. \square

Finally, we note that the claims above taken together prove (ii), and thereby complete the proof of the theorem. For if G is subdirectly irreducible with unique minimal normal subgroup N , and if $C_G(N) = 1$, then all normal subgroups (which necessarily lie above N) must have trivial centralizers.

6.4 Conclusion

We conclude this chapter with a final observation which helps us describe the O’Nan-Scott type of a group which has L_7 as an interval in its subgroup lattice. We end with a conjecture that should be the subject of future research.

By what we have proved above, G acts primitively on the cosets of K , and it also acts primitively on the cosets of at least one of M_1 or M_2 . Suppose M_1 is core-free so that G is a primitive permutation group in its action on cosets of M_1 and let N be the minimal normal subgroup of G . As we have seen, N has trivial centralizer, so it is nonabelian and is the unique minimal normal subgroup of G . Now, we have seen that $NH \geq M_2$ in this case, so $H < J_2 < NH$ implies that $N \cap M_1 \neq 1$. Similarly, if we had started out by assuming that M_2 is core-free, then $NH \geq M_1$, and $H < J_2 < NH$ would imply that $N \cap M_2 \neq 1$.

By the following elementary result (see, e.g., [20]) we see that the action of N on the cosets of the core-free maximal subgroup M_i is not regular.¹² Consequently, G is characterized by case 2 of the version of the O’Nan-Scott Theorem given in the appendix, Section A.2.

Lemma 6.4.1. *If G acts transitively on a set Ω with stabilizer G_ω , then a subgroup $N \leq G$ acts transitively on Ω if and only if $NG_\omega = G$. Also, N is regular if and only if in addition $N \cap G_\omega = 1$.*

¹²Recall, a transitive permutation group N is *acts regularly* on a set Ω provided the stabilizer subgroup of N is trivial. Equivalently, every non-identity element of N is fixed-point-free. Equivalently, N is regular on Ω if and only if for each $\omega_1, \omega_2 \in \Omega$ there is a unique $n \in N$ such that $n\omega_1 = \omega_2$. In particular, $|N| = |\Omega|$.

CHAPTER 7

EXPANSIONS OF FINITE ALGEBRAS

7.1 Background and motivation

In this chapter we present a novel approach to the construction of new finite algebras and describe the congruence lattices of these algebras. Given a finite algebra $\langle B, \dots \rangle$, let B_1, B_2, \dots, B_K be sets which intersect B at specific points. We construct an *overalgebra* $\langle A, F_A \rangle$, by which we mean an expansion of $\langle B, \dots \rangle$ with universe $A := B \cup B_1 \cup \dots \cup B_K$, and a certain set F_A of unary operations which include idempotent mappings e and e_i satisfying $e(A) = B$ and $e_i(A) = B_i$. We explore a number of such constructions and prove results about the shape of the new congruence lattices $\text{Con } \langle A, F_A \rangle$ that result. Thus, descriptions of some new classes of finitely representable lattices is one of our primary contributions. Another, perhaps more significant contribution is the announcement of a novel approach to the discovery of new classes of representable lattices.

Our main contribution is the description and analysis of a new procedure for generating finite lattices which are, by construction, finitely representable. Roughly speaking, we start with an arbitrary finite algebra $\mathbf{B} := \langle B, \dots \rangle$, with known congruence lattice $\text{Con } \mathbf{B}$, and we let B_1, B_2, \dots, B_K be sets which intersect B at certain points. The choice of intersection points plays an important rôle which we describe in detail later. We then construct an *overalgebra* $\mathbf{A} := \langle A, F_A \rangle$, by which we mean an expansion of \mathbf{B} with universe $A = B \cup B_1 \cup \dots \cup B_K$, and a certain set F_A of unary operations which include idempotent mappings e and e_i satisfying $e(A) = B$ and $e_i(A) = B_i$.

Given our interest in the problem mentioned above, the important consequence of this procedure is the new (finitely representable) lattice $\text{Con } \mathbf{A}$ that it produces. The shape of this lattice is, of course, determined by the shape of $\text{Con } \mathbf{B}$, the choice of intersection points of the B_i , and the unary operations chosen for inclusion in F_A . In this chapter, we describe a number of constructions of this type and prove some results about the shape of the congruence lattices of the resulting overalgebras.

Before giving an overview of this chapter, we give a bit of background about the original example which provided the impetus for this work. In the spring of 2011, our research seminar was fortunate enough to have as a visitor Peter Jipsen, who initiated the ambitious project of cataloging every small finite lattice L for which there is a known finite algebra \mathbf{A} with $\text{Con } \mathbf{A} \cong L$. Before long, we had identified such finite representations for all lattices of order seven or less, except for the two lattices

appearing in Figure 7.1. (Section 6.2 describes some of the methods we used to find representations of the other seven-element lattices.) Ralph Freese then discovered a way to construct an algebra which

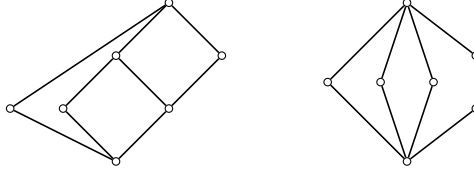


Figure 7.1: Lattices of order 7 with no obvious finite algebraic representation.

has the second of these as its congruence lattice. The idea is to start with an algebra $\mathbf{B} = \langle B, \dots \rangle$ having congruence lattice $\text{Con } \mathbf{B} \cong M_4$, expand the universe to the larger set $A = B \cup B_1 \cup B_2$, and then define the right set F_A of operations on A so that the congruence lattice of $\mathbf{A} = \langle A, F_F \rangle$ will be an M_4 with one atom “doubled” – that is, $\text{Con } \mathbf{A}$ will be the second lattice in figure 7.1.

In this chapter we formalize this approach and extend it in four ways. The first is a straightforward generalization of the original overalgebra construction, and the second is a further expansion of these overalgebras. The third is a construction based on one suggested by Bill Lampe which addresses a basic limitation of the original procedure. Finally, we give a generalization of the third construction. For each of these constructions we prove results which allow us to describe the congruence lattices of the resulting overalgebras.

Here is a brief outline of the remaining sections of this chapter: In Section 7.2 we prove a lemma which greatly simplifies the analysis of the structure of the newly enlarged congruence lattice and its relation to the original congruence lattice. In Section 7.3 we define *overalgebra* and in Section 7.3.1 we give a formal description of the original construction mentioned above. We then describe the original example in detail before proving some general results about the congruence lattices of such overalgebras. At the end of Section 7.3.1 we describe a further expansion of the set of operations defined in the first construction, and we conclude the section with an example demonstrating the utility of these additional operations. Section 7.3.2 presents a second overalgebra construction which overcomes a basic limitation of the first. We then prove a result about the structure of the congruence lattices of these overalgebras, and close the section with some further examples which illustrate the procedure and demonstrate its utility. In Section 7.3.3 we describe a construction that further generalizes the one in Section 7.3.2. The last section discusses the impact that our results have on the main problem – the finite congruence lattice representation problem – as well as the inherent limitations of this approach, and concludes with some open questions and suggestions for further

research.

7.2 A residuation lemma

Let $e^2 = e \in \text{Pol}_1(\mathbf{A})$ be an idempotent unary polynomial, define $B := e(A)$ and $F_B := \{ef|_B : f \in \text{Pol}_1(\mathbf{A})\}$, and consider the unary¹ algebra $\mathbf{B} := \langle B, F_B \rangle$. Pálffy and Pudlák prove in Lemma 1 of [32] that the restriction mapping $|_B$, defined on $\text{Con } \mathbf{A}$ by $\alpha|_B = \alpha \cap B^2$, is a lattice epimorphism of $\text{Con } \mathbf{A}$ onto $\text{Con } \mathbf{B}$. In [24], McKenzie, taking Lemma 1 as a starting point, develops the foundations of what would become tame congruence theory. In reproving the Pálffy-Pudlák congruence lattice epimorphism lemma, McKenzie introduces the mapping $\hat{}$ defined on $\text{Con } \mathbf{B}$ by

$$\hat{\beta} = \{(x, y) \in A^2 : (ef(x), ef(y)) \in \beta \text{ for all } f \in \text{Pol}_1(\mathbf{A})\}.$$

It is not hard to see that $\hat{}$ maps $\text{Con } \mathbf{B}$ into $\text{Con } \mathbf{A}$. For example, if $(x, y) \in \hat{\beta}$ and $g \in \text{Pol}_1(\mathbf{A})$, then for all $f \in \text{Pol}_1(\mathbf{A})$ we have $(efg(x), efg(y)) \in \beta$, so $(g(x), g(y)) \in \hat{\beta}$.

For each $\beta \in \text{Con } \mathbf{B}$, let $\beta^* = \text{Cg}^{\mathbf{A}}(\beta)$. That is, $*$: $\text{Con } \mathbf{B} \rightarrow \text{Con } \mathbf{A}$ is the congruence generation operator restricted to the set $\text{Con } \mathbf{B}$. The following lemma concerns the three mappings, $|_B$, $\hat{}$, and $*$. The third statement of the lemma, which follows from the first two, will be useful in the later sections of this chapter.

Lemma 7.2.1.

- (i) $*$: $\text{Con } \mathbf{B} \rightarrow \text{Con } \mathbf{A}$ is a residuated mapping with residual $|_B$.
- (ii) $|_B$: $\text{Con } \mathbf{A} \rightarrow \text{Con } \mathbf{B}$ is a residuated mapping with residual $\hat{}$.
- (iii) For all $\alpha \in \text{Con } \mathbf{A}$, $\beta \in \text{Con } \mathbf{B}$,

$$\beta = \alpha|_B \quad \Leftrightarrow \quad \beta^* \leq \alpha \leq \hat{\beta}.$$

In particular, $\beta^*|_B = \beta = \hat{\beta}|_B$.

Proof. We first recall the definition of *residuated mapping*. If X and Y are partially ordered sets, and if $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are order preserving maps, then the following are equivalent:

¹In the definition of F_B , we could have used $\text{Pol}(\mathbf{A})$ instead of $\text{Pol}_1(\mathbf{A})$, and then our discussion would not be limited to *unary* algebras. However, as we are mainly concerned with congruence lattices, we lose nothing by restricting the scope in this way. Also, later sections of this chapter will be solely concerned with unary algebras, so for consistency we define \mathbf{B} to be unary in this section as well.

(a) $f : X \rightarrow Y$ is a *residuated mapping* with *residual* $g : Y \rightarrow X$;

(b) for all $x \in X, y \in Y, f(x) \leq y$ iff $x \leq g(y)$;

(c) $g \circ f \geq \text{id}_X$ and $f \circ g \leq \text{id}_Y$.

The definition says that for each $y \in Y$ there is a unique $x \in X$ that is maximal with respect to the property $f(x) \leq y$, and the maximum x is given by $g(y)$. Thus, (i) is equivalent to

$$\beta^* \leq \alpha \Leftrightarrow \beta \leq \alpha|_B \quad (\forall \alpha \in \text{Con } \mathbf{A}, \forall \beta \in \text{Con } \mathbf{B}). \quad (7.2.1)$$

This is easily verified, as follows: If $\beta^* \leq \alpha$ and $(x, y) \in \beta$, then $(x, y) \in \beta^* \leq \alpha$ and $(x, y) \in B^2$, so $(x, y) \in \alpha|_B$. If $\beta \leq \alpha|_B$ then $\beta^* \leq (\alpha|_B)^* \leq \text{Cg}^{\mathbf{A}}(\alpha) = \alpha$.

Statement (ii) is equivalent to

$$\alpha|_B \leq \beta \Leftrightarrow \alpha \leq \widehat{\beta} \quad (\forall \alpha \in \text{Con } \mathbf{A}, \forall \beta \in \text{Con } \mathbf{B}). \quad (7.2.2)$$

This is also easy to check. For, suppose $\alpha|_B \leq \beta$ and $(x, y) \in \alpha$. Then $(ef(x), ef(y)) \in \alpha$ for all $f \in \text{Pol}_1(\mathbf{A})$ and $(ef(x), ef(y)) \in B^2$, therefore, $(ef(x), ef(y)) \in \alpha|_B \leq \beta$, so $(x, y) \in \widehat{\beta}$. Suppose $\alpha \leq \widehat{\beta}$ and $(x, y) \in \alpha|_B$. Then $(x, y) \in \alpha \leq \widehat{\beta}$, so $(ef(x), ef(y)) \in \beta$ for all $f \in \text{Pol}_1(\mathbf{A})$, including $f = \text{id}_A$, so $(e(x), e(y)) \in \beta$. But $(x, y) \in B^2$, so $(x, y) = (e(x), e(y)) \in \beta$.

Combining (7.2.1) and (7.2.2), we obtain statement (iii) of the lemma. \square

The lemma above was inspired by the two approaches to proving Lemma 1 of [32]. In the original paper $*$ is used, while McKenzie uses the $\widehat{}$ operator. Both β^* and $\widehat{\beta}$ are mapped onto β by the restriction map $|_B$, so the restriction map is indeed onto $\text{Con } \mathbf{B}$. However, our lemma emphasizes the fact that the interval

$$[\beta^*, \widehat{\beta}] = \{\alpha \in \text{Con } \mathbf{A} : \beta^* \leq \alpha \leq \widehat{\beta}\}$$

is precisely the set of congruences for which $\alpha|_B = \beta$. In other words, the inverse image of β under $|_B$ is $\beta|_B^{-1} = [\beta^*, \widehat{\beta}]$. This fact plays a central rôle in the theory developed below. Nonetheless, for the sake of completeness, we conclude this section by verifying that Lemma 1 of [32] can be obtained from the lemma above.

Corollary 7.2.2. $|_B : \text{Con } \mathbf{A} \rightarrow \text{Con } \mathbf{B}$ is onto and preserves meets and joins.

Proof. Given $\beta \in \text{Con } \mathbf{B}$, each $\theta \in \text{Con } \mathbf{A}$ in the interval $[\beta^*, \widehat{\beta}]$ is mapped to $\theta|_B = \beta$, so $|_B$ is clearly onto. That $|_B$ preserves meets is obvious. To see that $|_B$ is join preserving, note that for all $\eta, \theta \in \text{Con } \mathbf{A}$, we have

$$\eta|_B \vee \theta|_B \leq (\eta \vee \theta)|_B$$

since $|_B$ is order preserving. The opposite inequality follows from (7.2.2) above. For,

$$(\eta \vee \theta)|_B \leq \eta|_B \vee \theta|_B \quad \Leftrightarrow \quad \eta \vee \theta \leq \widehat{\eta|_B \vee \theta|_B},$$

and the second inequality holds since, by (7.2.2) again,

$$\eta \leq \widehat{\eta|_B \vee \theta|_B} \quad \Leftrightarrow \quad \eta|_B \leq \eta|_B \vee \theta|_B$$

and

$$\theta \leq \widehat{\eta|_B \vee \theta|_B} \quad \Leftrightarrow \quad \theta|_B \leq \eta|_B \vee \theta|_B.$$

□

Remark. This approach to proving Lemma 1 of [32], which is similar to the proof given in [24], does not reveal any information about the permutability of the congruences of \mathbf{A} , unlike the more direct proof given in [32].

7.3 Overalgebras

In the previous section, we started with an algebra \mathbf{A} and considered a subreduct \mathbf{B} with universe $B = e(A)$, the image of an idempotent unary polynomial of \mathbf{A} . In this section, we start with a fixed finite algebra $\mathbf{B} = \langle B, \dots \rangle$ and consider various ways to construct an *overalgebra*, that is, an algebra $\mathbf{A} = \langle A, F_A \rangle$ having \mathbf{B} as a subreduct where $B = e(A)$ for some idempotent $e \in F_A$. Beginning with a specific finite algebra \mathbf{B} , our goal is to understand what (finitely representable) congruence lattices $\text{Con } \mathbf{A}$ can be built up from $\text{Con } \mathbf{B}$ by expanding the algebra \mathbf{B} in this way.

7.3.1 Overalgebras I

Let B be a finite set, say, $B = \{b_1, b_2, \dots, b_n\}$, let $F \subseteq B^B$ be a set of unary maps taking B into itself, and consider the unary algebra $\mathbf{B} = \langle B, F \rangle$, with universe B and basic operations F . When clarity demands it, we call this collection of operations F_B . Let B_1, B_2, \dots, B_K be sets of the same cardinality as B , which intersect B at exactly one point, as follows:

$$\begin{aligned}
 B &= \{b_1, b_2, b_3, \dots, b_n\} \\
 B_1 &= \{b_1, b_2^1, b_3^1, \dots, b_n^1\} \\
 B_2 &= \{b_1^2, b_2, b_3^2, \dots, b_n^2\} \\
 B_3 &= \{b_1^3, b_2^3, b_3, \dots, b_n^3\} \\
 &\vdots \\
 B_K &= \{b_1^K, \dots, b_{K-1}^K, b_K, b_{K+1}^K, \dots, b_n^K\}.
 \end{aligned} \tag{7.3.1}$$

That is, for all $1 \leq i < j \leq K$, we have

$$|B_i| = n \geq K, \quad B \cap B_i = \{b_i\}, \quad \text{and} \quad B_i \cap B_j = \emptyset.$$

Sometimes it is notationally convenient to use the label $B_0 := B$.

Let $\pi_i : B \rightarrow B_i$ be given by $\pi_i(b_j) = b_j^i$, for $i = 0, 1, 2, \dots, n$ and $j = 1, 2, \dots, K$. (It is convenient to include $i = 0$ in this definition, in which case we let $\pi_0(b_j) = b_j^0 := b_j$.) The map π_i and the operations F induce a set F_i of unary operations on B_i , as follows: to each $f \in F$ corresponds the operation $f^{\pi_i} : B_i \rightarrow B_i$ defined by $f^{\pi_i} = \pi_i f \pi_i^{-1}$. Thus, for each i , $\mathbf{B}_i := \langle B_i, F_i \rangle$ and $\mathbf{B} = \langle B, F \rangle$ are isomorphic algebras. That is, for all $i = 1, \dots, K$, we have

$$\pi_i : \langle B, F \rangle \cong \langle B_i, F_i \rangle$$

$$B \ni b \mapsto b^i \in B_i$$

$$F \ni f \mapsto f^{\pi_i} \in F_i$$

To say that π_i is an isomorphism of two non-indexed algebras is to say that π_i is a bijection of the universes which respects the interpretation of the basic operations; that is, $\pi_i f(b) = f^{\pi_i}(\pi_i b)$. In

the present case, this holds by construction:² $\pi_i f(b) = \pi_i f(\pi_i^{-1} \pi_i b) = f^{\pi_i}(\pi_i b)$.

Let $A = \bigcup_{i=0}^K B_i$ and define the following unary maps on A :

- $e_k : A \rightarrow A$ is $e_k(b_i^j) = b_i^k$ ($1 \leq i \leq n$; $0 \leq j, k \leq K$);
- $s : A \rightarrow A$ is

$$s(x) = \begin{cases} x, & \text{if } x \in B_0, \\ b_i, & \text{if } x \in B_i. \end{cases}$$

Let

$$F_A := \{f e_0 : f \in F\} \cup \{e_k : 0 \leq k \leq K\} \cup \{s\},$$

and define the unary algebra $\mathbf{A} := \langle A, F_A \rangle$.

Throughout, the map $\hat{}$ is defined in essentially the same way as it is in McKenzie's paper [24]. That is, given two algebras $\mathbf{A} = \langle A, \dots \rangle$ and $\mathbf{B} = \langle B, \dots \rangle$ with $B = e(A)$ for some idempotent $e \in \text{Pol}_1(\mathbf{A})$, we define $\hat{} : \text{Con } \mathbf{B} \rightarrow \text{Con } \mathbf{A}$ by

$$\hat{\beta} = \{(x, y) \in A^2 : (ef(x), ef(y)) \in \beta, \forall f \in \text{Pol}_1(\mathbf{A})\} \quad (\beta \in \text{Con } \mathbf{B}).$$

Example 7.3.1. Before proving some results about the basic structure of the congruence lattice of an overalgebra, we present the original example, discovered by Ralph Freese, of a finite algebra with a congruence lattice isomorphic to the second lattice in Figure 7.1. Consider a finite permutational algebra $\mathbf{B} = \langle B, F \rangle$ with congruence lattice $\text{Con } \mathbf{B} \cong M_4$. (Figure 7.2) There are only a few small algebras to choose from.³ We consider the right regular S_3 -set – i.e. the algebra S_3 acting on itself by right multiplication. In GAP,⁴

```
gap> G:=Group([(1,2), (1,2,3)]);;
gap> G:=Action(G,G,OnRight);
Group([ (1,5)(2,4)(3,6), (1,2,3)(4,5,6) ])
```

² This generalizes to k -ary operations if we adopt the following convention: $f^{\pi_i}(a_1, \dots, a_k) = \pi_i f(\pi_i^{-1}(a_1), \dots, \pi_i^{-1}(a_k))$.

³In fact, there are infinitely many, but apart from those involving S_3 , $C_3 \times C_3$, and $(C_3 \times C_3) \rtimes C_3$, they are quite large. The next smallest G-set with M_4 congruence lattice that we know of comes from the group $G = [(C_3 \times C_3) \rtimes C_2] \times ((C_3 \times C_3) \rtimes C_2) \rtimes C_2$ acting on right cosets of $H = D_8$. The index in this case is $|G : H| = 81$. (In GAP, $G := \text{SmallGroup}(648, 725)$, and H is found to be the fourth maximal subgroup class representative of the fourth maximal subgroup class representative of G .)

⁴ All of the computational experiments we describe in this chapter rely on two open source programs, GAP [17] and the Universal Algebra Calculator [16] (UACalc). To conduct our experiments, we have written a small collection of GAP functions; these are available at <http://math.hawaii.edu/~williamdemeco/Overalgebras.html>.

We prefer to use “0-offset” notation, and define the universe of the S_3 -set described above to be $\{0, 1, \dots, 5\}$ instead of $\{1, 2, \dots, 6\}$. As such, the nontrivial congruence relations of this algebra are,

```
gap> for b in AllBlocks(G) do Print(Orbit(G,b,OnSets)-1, "\n"); od;
[ [ 0, 1, 2 ], [ 3, 4, 5 ] ]
[ [ 0, 3 ], [ 2, 5 ], [ 1, 4 ] ]
[ [ 0, 4 ], [ 2, 3 ], [ 1, 5 ] ]
[ [ 0, 5 ], [ 2, 4 ], [ 1, 3 ] ]
```

Next, we create an algebra in **UACalc** format using the two generators of the group as basic operations.⁵

```
gap> Read("gap2uacalc.g");
gap> gset2uacalc([G,"S3action"]);
```

This creates a **UACalc** file specifying an algebra with universe $B = \{0, 1, \dots, 5\}$ and two basic unary operations $g_0 = (4\ 3\ 5\ 1\ 0\ 2)$ and $g_1 = (1\ 2\ 0\ 4\ 5\ 3)$. These operations are the permutations $(0, 4)(1, 3)(2, 5)$ and $(0, 1, 2)(3, 4, 5)$, which, in “1-offset” notation, are the generators $(1, 5)(2, 4)(3, 6)$ and $(1, 2, 3)(4, 5, 6)$ of the S_3 -set appearing in the **GAP** output above. Figure 7.2 displays the congruence lattice of this algebra.

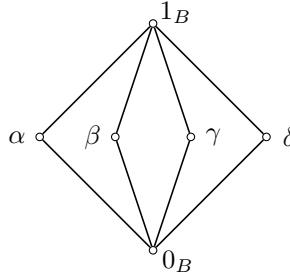


Figure 7.2: Congruence lattice of the right regular S_3 -set, where $\alpha = |0, 1, 2|3, 4, 5|$, $\beta = |0, 3|2, 5|1, 4|$, $\gamma = |0, 4|2, 3|1, 5|$, $\delta = |0, 5|2, 4|1, 3|$.

We now construct an overalgebra which “doubles” the congruence $\alpha = \text{Cg}^{\mathbf{B}}(0, 2) = |0, 1, 2|3, 4, 5|$ by choosing intersection points 0 and 2. The **GAP** function **Overalgebra** carries out the construction, and is invoked as follows:⁶

```
gap> Read("Overalgebras.g");
gap> Overalgebra([G, [0,2]]);
```

⁵The **GAP** routine `gap2uacalc.g` is available at www.uacalc.org.

⁶The **GAP** file `Overalgebras.g` is available at <http://dl.dropbox.com/u/17739547/diss/Overalgebras.g>.

This gives an overalgebra with universe $A = B_0 \cup B_1 \cup B_2 = \{0, 1, 2, 3, 4, 5\} \cup \{0, 6, 7, 8, 9, 10\} \cup \{11, 12, 13, 14, 15\}$, and the following operations:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
e_0	0	1	2	3	4	5	1	2	3	4	5	0	1	3	4	5
e_1	0	6	7	8	9	10	6	7	8	9	10	0	6	8	9	10
e_2	11	12	2	13	14	15	12	2	13	14	15	11	12	13	14	15
s	0	1	2	3	4	5	0	0	0	0	0	2	2	2	2	2
g_0e_0	4	3	5	1	0	2	3	5	1	0	2	4	3	1	0	2
g_1e_0	1	2	0	4	5	3	2	0	4	5	3	1	2	4	5	3

If $F_A = \{e_0, e_1, e_2, s, g_0e_0, g_1e_0\}$, then the algebra $\langle A, F_A \rangle$ has the congruence lattice shown in Figure 7.3.

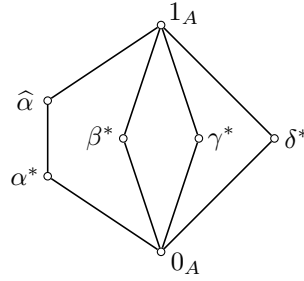


Figure 7.3: Congruence lattice of the overalgebra of the S_3 -set with intersection points 0 and 2.

The congruence relations in Figure 7.3 are as follows:

$$\hat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$

$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$

$$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$$

$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$

$$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|.$$

It is important to note that the resulting congruence lattice depends on our choice of which congruence to “expand,” which is controlled by our specification of the intersection points of the overalgebra. For example, suppose we want one of the congruences having three blocks, say, $\beta = \text{Cg}^{\mathbf{B}}(0, 3) = |0, 3|2, 5|1, 4|$, to have a non-trivial inverse image $\beta|_B^{-1} = [\beta^*, \hat{\beta}]$. Then we would select

the elements 0 and 3, (or 2 and 5, or 1 and 4) as the intersection points of the overalgebra. To select 0 and 3, we invoke the command

```
gap> Overalgebra([G, [0,3]]);
```

This produces an overalgebra with universe $A = B_0 \cup B_1 \cup B_2 = \{0, 1, 2, 3, 4, 5\} \cup \{0, 6, 7, 8, 9, 10\} \cup \{11, 12, 13, 3, 14, 15\}$ and congruence lattice shown in figure 7.4.

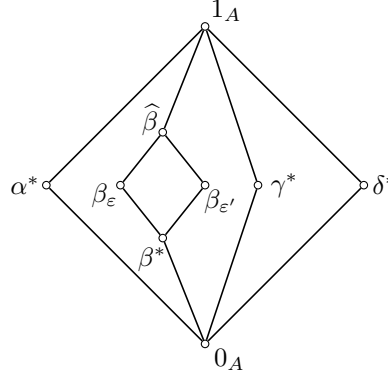


Figure 7.4: Congruence lattice of the overalgebra of the S_3 -set with intersection points 0 and 3.

where

$$\alpha^* = |0, 1, 2, 6, 7|3, 4, 5, 14, 15|8, 9, 10|11, 12, 13|$$

$$\widehat{\beta} = |0, 3, 8, 11|1, 4|2, 5|6, 9, 12, 14|7, 10, 13, 15|$$

$$\beta_\varepsilon = |0, 3, 8, 11|1, 4|2, 5|6, 9, 12, 14|7, 10|13, 15|$$

$$\beta_{\varepsilon'} = |0, 3, 8, 11|1, 4|2, 5|6, 9|7, 10, 13, 15|12, 14|$$

$$\beta^* = |0, 3, 8, 11|1, 4|2, 5|6, 9|7, 10|12, 14|13, 15|$$

$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$

$$\delta^* = |0, 5, 10|1, 3, 12|2, 4|6, 8|7, 9|11, 15|13, 14|.$$

We now prove two theorems which describe the basic structure of the congruence of an overalgebra constructed as described at the outset of this section. In particular, the theorems explain why the interval $[\alpha^*, \widehat{\alpha}] \cong \mathbf{2}$ appears in the first example above, while $[\beta^*, \widehat{\beta}] \cong \mathbf{2} \times \mathbf{2}$ appears in the second.

Given a congruence relation $\beta \in \text{Con } \mathbf{B}$, let $\{b_{\beta(1)}, \dots, b_{\beta(m)}\}$ denote a *transversal* of β ; i.e. a full set of β -class representatives. Thus, as a partition of the set B , β has m classes, or blocks. (Using the notation $\beta(r)$ for the indices of the representatives helps us to remember that $b_{\beta(r)}$ is

a representative of the r -th block of the congruence β .) By the isomorphisms π_i defined above, to each $\beta \in \text{Con } \mathbf{B}$ there corresponds a congruence relation $\beta^{\mathbf{B}_i} \in \text{Con } \mathbf{B}_i$, and if $\{b_{\beta(1)}, \dots, b_{\beta(m)}\}$ is a transversal of β , then the map π_i also gives a transversal of $\beta^{\mathbf{B}_i}$, namely $\{\pi_i(b_{\beta(1)}), \dots, \pi_i(b_{\beta(m)})\} = \{b_{\beta(1)}^i, \dots, b_{\beta(m)}^i\}$. Thus, the r -th block of $\beta^{\mathbf{B}_i}$ is $b_{\beta(r)}^i / \beta^{\mathbf{B}_i}$.

Let $T = \{b_1, b_2, \dots, b_K\}$ be the set of *tie-points*, that is, the points at which the sets B_i ($1 \leq i \leq K$) intersect the set B . Let $T_r = \{b \in T : (b, b_{\beta(r)}) \in \beta\}$ be the set of those tie-points that are in the r -th congruence class of β .

Theorem 7.3.2. *For each $\beta \in \text{Con } \mathbf{B}$,*

$$\text{Cg}^{\mathbf{A}}(\beta) = \bigcup_{k=0}^K \beta^{\mathbf{B}_k} \cup \bigcup_{r=1}^m \left(b_{\beta(r)} / \beta \cup \bigcup_{b_j \in T_r} b_j / \beta^{\mathbf{B}_j} \right)^2. \quad (7.3.2)$$

Remark. Before proceeding to the proof, we advise the reader to consider the small example illustrated in Figures 7.5 and 7.6. Identifying the objects on the right of equation (7.3.2) in these figures will make the proof of the theorem easier to follow. In particular, as the figures make clear, transitivity requires that $\beta^{\mathbf{B}_j}$ classes which are linked together by tie-points must end up in the same class of $\text{Cg}^{\mathbf{A}}(\beta)$. This is the purpose of the $\bigcup_{r=1}^m (\cdot)^2$ term.

Proof. Let β^* denote the right-hand side of (7.3.2). We first check that $\beta^* \in \text{Con } \mathbf{A}$. It is easy to see that β^* is an equivalence relation, so we need only show $f(\beta^*) \subseteq \beta^*$ for all⁷ $f \in F_A$, where

$$F_A := \{f e_0 : f \in F\} \cup \{e_k : 0 \leq k \leq K\} \cup \{s\}.$$

In other words, we prove: if $(x, y) \in \beta^*$ and $f \in F_A$, then $(f(x), f(y)) \in \beta^*$.

Case 1: $(x, y) \in \beta^{\mathbf{B}_k}$ for some $0 \leq k \leq K$.

Then, $(e_i(x), e_i(y)) \in \beta^{\mathbf{B}_i} \subseteq \beta^*$ for all $0 \leq i \leq K$, and $(f e_0(x), f e_0(y)) \in \beta \subseteq \beta^*$ for all $f \in F_B$.

Also,

$$(s(x), s(y)) = \begin{cases} (x, y), & \text{if } k = 0 \\ (b_k, b_k), & \text{if } k \neq 0 \end{cases}$$

belongs to β^* . Thus, $(f(x), f(y)) \in \beta^*$ for all $f \in F_A$.

⁷Note that $\beta^{\mathbf{B}_0} = \beta$.

Case 2: $(x, y) \in \left(b_{\beta(r)}/\beta \cup \bigcup_{b_j \in T_r} b_j/\beta^{\mathbf{B}_j} \right)^2$ for some $1 \leq r \leq m$.

Assume $x \in b_j/\beta^{\mathbf{B}_j}$ and $y \in b_k/\beta^{\mathbf{B}_k}$ for some $b_j, b_k \in T_r$. Then $(e_0(x), b_j) \in \beta$, $(e_0(y), b_k) \in \beta$, and $b_j \beta b_{\beta(r)} \beta b_k$ so

$$(e_0(x), e_0(y)) \in \beta. \quad (7.3.3)$$

Thus, for all $0 \leq \ell \leq K$ we have $(e_\ell e_0(x), e_\ell e_0(y)) \in \beta^{\mathbf{B}_\ell}$. But note that $e_\ell e_0 = e_\ell$. It also follows from (7.3.3) that $(f e_0(x), f e_0(y)) \in \beta$ for all $f \in F_B$. Finally, $(s(x), s(y)) = (b_j, b_k) \in \beta$.

The only remaining possibility for case 2 is $x \in b_{\beta(r)}/\beta$ and $y \in b_j/\beta^{\mathbf{B}_j}$ for some $b_j \in T_r$. Since $b_j \in T_r$, we have $(b_j, b_{\beta(r)}) \in \beta$, so $(e_0(y), b_j) \in \beta$, so $(e_0(y), b_{\beta(r)}) \in \beta$, so $(e_0(x), x) = (e_0(y), e_0(x)) \in \beta$. Therefore, $(e_\ell(y), e_\ell(x)) \in \beta^{\mathbf{B}_\ell}$ for all $0 \leq \ell \leq K$ and $(f e_0(y), f e_0(x)) \in \beta$ for all $f \in F_B$. Finally, $s(x) = x \beta b_{\beta(r)} \beta b_j = s(y)$, so $(s(x), s(y)) \in \beta$.

We have established that $f(\beta^*) \subseteq \beta^*$ for all $f \in F_A$. To complete the proof of Theorem 7.3.2, we must show that $\beta \subseteq \eta \in \text{Con } \mathbf{A}$ implies $\beta^* \leq \eta$. If $\beta \subseteq \eta \in \text{Con } \mathbf{A}$, then $\bigcup \beta^{\mathbf{B}_k} \subseteq \eta$, since $(x, y) \in \beta$ implies $(e_k(x), e_k(y)) \in \beta^{\mathbf{B}_k}$ for all $0 \leq k \leq K$. To see that the second term of (7.3.2) belongs to η , let (x, y) be an arbitrary element of that term, say, $(x, b_i) \in \beta^{\mathbf{B}_i}$ and $(y, b_j) \in \beta^{\mathbf{B}_j}$. As we just observed, β , $\beta^{\mathbf{B}_i}$, and $\beta^{\mathbf{B}_j}$ are subsets of η , and $(b_i, b_j) \in \beta$, so $x \beta^{\mathbf{B}_i} b_i \beta b_j \beta^{\mathbf{B}_j} y$, so $(x, y) \in \eta$. \square

As above, for a given $\beta \in \text{Con } \mathbf{B}$ with transversal $\{b_{\beta(1)}, \dots, b_{\beta(m)}\}$, we denote the set of tie-points contained in the r -th block of β by T_r ; that is,

$$T_r = \{b \in T : (b, b_{\beta(r)}) \in \beta\} = \bigcup_{k=1}^K B_k \cap b_{\beta(r)}/\beta.$$

Suppose this set is $T_r = \{b_{i_1}, b_{i_2}, \dots, b_{i_{|T_r|}}\}$ and let $\mathcal{I}_r = \{i_1, i_2, \dots, i_{|T_r|}\}$ be the indices of these tie-points. Also, we define $\beta^* = \text{Cg}^{\mathbf{A}}(\beta)$, for $\beta \in \text{Con } \mathbf{B}$.

Figures 7.5 and 7.6 illustrate these objects for a simple example in which $B_0 = \{b_0, b_1, \dots, b_8\}$, $\beta = |b_0, b_1, b_2 | b_3, b_4, b_5 | b_6, b_7, b_8|$, and two blocks of β contain two tie-points each. In particular, the set of tie-points in the first block of β is $T_1 = \{b_0, b_2\}$. For the second and third blocks, $T_2 = \emptyset$ and $T_3 = \{b_6, b_8\}$.

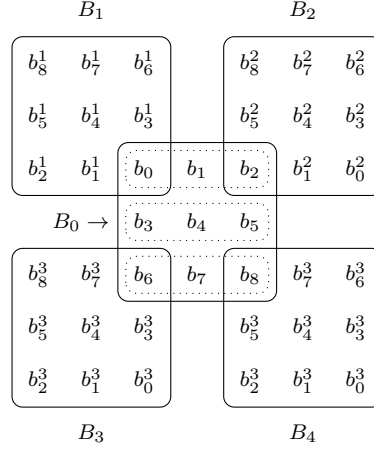


Figure 7.5: The universe $A = B_0 \cup \dots \cup B_4$ for a simple example; dotted lines surround each congruence class of β .

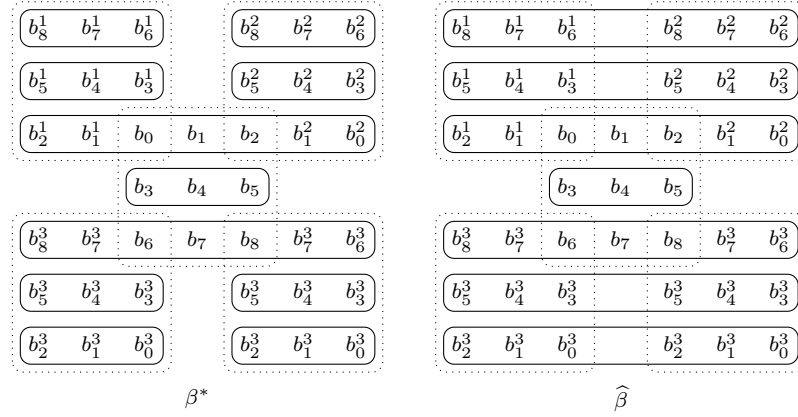


Figure 7.6: Solid lines show the congruence classes of β^* (left) and $\widehat{\beta}$ (right); dotted lines delineate the sets B_i .

Theorem 7.3.3. *For each $\beta \in \text{Con } \mathbf{B}$,*

$$\widehat{\beta} = \beta^* \cup \bigcup_{r=1}^m \bigcup_{\substack{\ell=1 \\ \ell \neq r}}^m \bigcup_{(j,k) \in \mathcal{J}_r^2} \left(b_{\beta(\ell)}^j / \beta^{\mathbf{B}_j} \cup b_{\beta(\ell)}^k / \beta^{\mathbf{B}_k} \right)^2. \quad (7.3.4)$$

Moreover, the interval $[\beta^*, \widehat{\beta}]$ of $\text{Con } \mathbf{A}$ contains every equivalence relation of A between β^* and $\widehat{\beta}$, and is isomorphic to $\prod (\text{Eq}|T_r|)^{m-1}$; that is,

$$[\beta^*, \widehat{\beta}] = \{ \theta \in \text{Eq}(A) : \beta^* \subseteq \theta \subseteq \widehat{\beta} \} \cong \prod_{r=1}^m (\text{Eq}|T_r|)^{m-1}. \quad (7.3.5)$$

Remark. Blocks containing only one tie-point, i.e. those for which $|T_r| = 1$, contribute nothing to the direct product in (7.3.5). Also, for some $1 \leq r \leq m$ we may have $T_r = \emptyset$, in which case we agree to let $\text{Eq}|T_r| = \text{Eq}(0) := \mathbf{1}$.

Proof. Let $\tilde{\beta}$ denote the right-hand side of (7.3.4). It is easy to see that $\tilde{\beta}$ is an equivalence relation on A . To see that it is also a congruence relation, we will prove $f(\tilde{\beta}) \subseteq \tilde{\beta}$ for all $f \in F_A$. Fix $(x, y) \in \tilde{\beta}$. If $(x, y) \in \beta^*$, then $(f(x), f(y)) \in \beta^*$ holds for all $f \in F_A$, as in Theorem 7.3.2. Suppose $(x, y) \notin \beta^*$, say, $x \in b_{\beta(\ell)}^j / \beta^{\mathbf{B}_j}$ and $y \in b_{\beta(\ell)}^k / \beta^{\mathbf{B}_k}$ for some $j, k \in \mathcal{J}_r$, $1 \leq r \leq m$, and $\ell \neq r$. Then x and y are in the ℓ -th blocks of their respective subreduct universes, B_j and B_k , so for each $0 \leq i \leq K$, $(e_i(x), e_i(y)) \in \beta^{\mathbf{B}_i}$. In particular, $(e_0(x), e_0(y)) \in \beta$, so $(ge_0(x), ge_0(y)) \in \beta$ for all $g \in F_B$. Also, $(s(x), s(y)) = (b_j, b_k) \in T_r^2 \subseteq \beta$. This proves that for each $f \in F_A$ we have $(f(x), f(y)) \in \tilde{\beta}$. (In fact, $(f(x), f(y)) \in \beta^*$.) Whence $\tilde{\beta} \in \text{Con } \mathbf{A}$.

Now notice that $\tilde{\beta}|_B = \beta$. Therefore, by the residuation lemma of Section 7.2, we have $\tilde{\beta} \leq \hat{\beta}$. To prove the reverse inclusion, we suppose $(x, y) \notin \tilde{\beta}$ and show $(x, y) \notin \hat{\beta}$. Without loss of generality, assume $x \in b_{\beta(p)}^j / \beta^{\mathbf{B}_j}$ and $y \in b_{\beta(q)}^k / \beta^{\mathbf{B}_k}$, for some $1 \leq p, q \leq m$ and $1 \leq j, k \leq K + 1$. If $p = q$, then $(j, k) \notin \mathcal{J}_r^2$ for all $1 \leq r \leq m$ (otherwise $(x, y) \in \tilde{\beta}$), so $(e_0s(x), e_0s(y)) = (e_0(b_j), e_0(b_k)) = (b_j, b_k) \notin \beta$, so $(x, y) \notin \hat{\beta}$. If $p \neq q$, then $e_0(x) \in b_{\beta(p)} / \beta$ and $e_0(y) \in b_{\beta(q)} / \beta$ – distinct β classes – so $(e_0(x), e_0(y)) \notin \beta$, so $(x, y) \notin \hat{\beta}$.

To prove (7.3.5), we first note that every *equivalence* relation θ on A with $\beta^* \subseteq \theta \subseteq \hat{\beta}$ satisfies $f(\theta) \subseteq \theta$ for all $f \in F_A$, and is therefore a congruence of \mathbf{A} . Indeed, in proving $\tilde{\beta} = \hat{\beta}$ above, we saw that $f(\tilde{\beta}) \subseteq \beta^*$ for all $f \in F_A$, so, *a fortiori*, $f(\theta) \subseteq \beta^*$ for all equivalence relations $\theta \subseteq \hat{\beta}$. Therefore,

$$[\beta^*, \hat{\beta}] = \{\theta \in \text{Eq}(A) : \beta^* \subseteq \theta \subseteq \hat{\beta}\}.$$

To complete the proof, we must show that this interval is isomorphic to the lattice $\prod_{r=1}^m (\text{Eq}|T_r|)^{m-1}$. Consider,

$$\hat{\beta} / \beta^* = \{(x / \beta^*, y / \beta^*) \in (A / \beta^*)^2 : (x, y) \in \hat{\beta}\}.$$

Let N be the number of blocks of $\hat{\beta} / \beta^*$ (which, of course, is the same as the number of blocks of $\hat{\beta}$). For $1 \leq k \leq N$, let x_k / β^* be a representative of the k -th block of $\hat{\beta} / \beta^*$. Let $\mathcal{B}_k = (x_k / \beta^*) / (\hat{\beta} / \beta^*)$ denote this block; that is,

$$\mathcal{B}_k = \{y / \beta^* \in A / \beta^* : (x_k / \beta^*, y / \beta^*) \in \hat{\beta} / \beta^*\}.$$

Then,

$$\prod_{k=1}^N \text{Eq}(\mathcal{B}_k) \cong \{\theta \in \text{Eq}(A) : \beta^* \subseteq \theta \subseteq \widehat{\beta}\} = [\beta^*, \widehat{\beta}].$$

The isomorphism is given by the maps,

$$\begin{aligned} \prod_{k=1}^N \text{Eq}(\mathcal{B}_k) \ni \eta &\mapsto \bigcup_{k=1}^N \eta_k \in [\beta^*, \widehat{\beta}] \\ [\beta^*, \widehat{\beta}] \ni \theta &\mapsto \prod_{k=1}^N \theta \cap \mathcal{B}_k^2 \in \prod_{k=1}^N \text{Eq}(\mathcal{B}_k), \end{aligned}$$

where η_k denotes the projection of η onto its k -th coordinate.

Now, the r -th β -class of B_0 , denoted $b_{\beta(r)}/\beta$, has $|T_r|$ tie-points, so there are $|T_r|$ sets, $B_{i_1}, B_{i_2}, \dots, B_{i_{|T_r|}}$, each of which intersects B_0 at a distinct tie-point in $b_{\beta(r)}/\beta$; that is,

$$B_{i_j} \cap b_{\beta(r)}/\beta = \{b_{i_j}\} \quad (b_{i_j} \in T_r).$$

(See Figure 7.6.) A block \mathcal{B}_k of $\widehat{\beta}/\beta^*$ has a single element when it contains $b_{\beta(r)}/\beta$. Otherwise, it has $|T_r|$ elements, namely,

$$b_{\beta(\ell)}^{i_1}/\beta^{\mathbf{B}_{i_2}}, b_{\beta(\ell)}^{i_2}/\beta^{\mathbf{B}_{i_2}}, \dots, b_{\beta(\ell)}^{i_{|T_r|}}/\beta^{\mathbf{B}_{i_{|T_r|}}},$$

for some $1 \leq \ell \leq m$; $\ell \neq r$. Thus, for each $1 \leq r \leq m$, we have $m-1$ such $|T_r|$ -element blocks, so

$$\prod_{k=1}^N \text{Eq}(\mathcal{B}_k) \cong \prod_{r=1}^m (\text{Eq}|T_r|)^{m-1}.$$

□

We now describe the situation in which the foregoing construction is most useful. Here and in the sequel, instead of $\text{Eq}(2)$, we usually write $\mathbf{2}$ to denote the two element lattice. Given a finite congruence lattice $\text{Con } \mathbf{B}$ and a pair $(x, y) \in B^2$, let $\beta \in \text{Con } \mathbf{B}$ be the unique smallest congruence containing (x, y) . Then $\beta = \text{Cg}^{\mathbf{B}}(x, y)$, and if we build an overalgebra as described above using $\{x, y\}$ as tie-points, then, by Theorem 7.3.3, the interval of all $\theta \in \text{Con } \mathbf{A}$ for which $\theta|_B = \beta$ will be $[\beta^*, \widehat{\beta}] \cong \text{Eq}(2)^{m-1} = \mathbf{2}^{m-1}$, where m is the number of congruence classes in β . Also, since β is the smallest congruence containing (x, y) we can be sure that, for all $\theta \not\geq \beta$, the interval $[\theta^*, \widehat{\theta}]$ is trivial; that is, $\theta^* = \widehat{\theta}$. Finally, for each $\theta > \beta$, we will have $[\theta^*, \widehat{\theta}] \cong \mathbf{2}^{r-1}$, where r is the number of

congruence classes of θ .

Example 7.3.4. With the theorems above, we can explain the shapes of the congruence lattices of Example 7.3.1. Returning to that example, with base algebra \mathbf{B} equal to the right regular S_3 -set, we now show some other congruence lattices that result by simply changing the set of tie-points, T . Recall, the relations in $\text{Con } \mathbf{B}$ are $\alpha = |0, 1, 2|3, 4, 5|$, $\beta = |0, 3|2, 5|1, 4|$, $\gamma = |0, 4|2, 3|1, 5|$, and $\delta = |0, 5|2, 4|1, 3|$.

As Theorems 7.3.2 and 7.3.3 make clear, choosing T to be $\{0, 1\}$, $\{0, 1, 2\}$, or $\{0, 2, 3\}$ yields the congruence lattices appearing in Figure 7.7. Figure 7.8 shows the congruence lattices resulting from the choices $T = \{0, 1, 2, 3\}$ and $T = \{0, 2, 3, 5\}$.

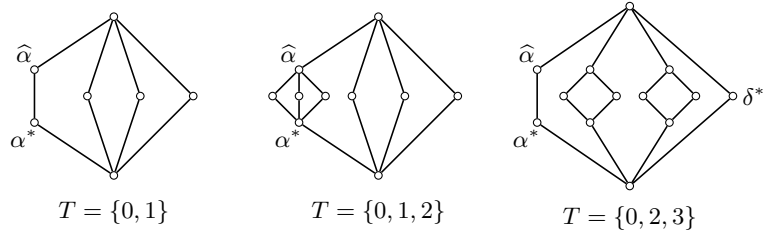


Figure 7.7: Congruence lattices of overalgebras of the S_3 -set for various choices of T , the set of tie-points.

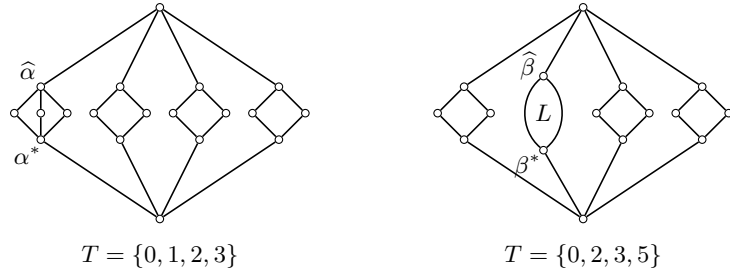


Figure 7.8: Congruence lattices of overalgebras of the S_3 -set for various choices of T ; $L \cong \mathbf{2}^2 \times \mathbf{2}^2$.

Since $\beta = |0, 3|2, 5|1, 4|$, when $T = \{0, 2, 3, 5\}$, the interval $[\beta^*, \widehat{\beta}]$ is $\mathbf{2}^2 \times \mathbf{2}^2$. In Figure 7.8, we denote this abstractly by L , instead of drawing all 16 points of this interval.

Next, consider the situation depicted in the last congruence lattice of Figure 7.8, where $L \cong \mathbf{2}^2 \times \mathbf{2}^2$, and suppose we prefer that all the other $|_B$ -inverse images be trivial: $[\beta^*, \widehat{\beta}] \cong \mathbf{2}^2 \times \mathbf{2}^2$; $\alpha^* = \widehat{\alpha}$; $\gamma^* = \widehat{\gamma}$; $\delta^* = \widehat{\delta}$. In other words, we seek a finite algebraic representation of the lattice in Figure 7.9. This is easy to achieve by adding more operations in the overalgebra construction described above. In fact, it is possible to introduce additional operations so that, if $\beta = \text{Cg}^{\mathbf{B}}(x, y)$, then $\theta^* = \widehat{\theta}$ for

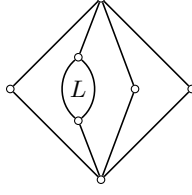


Figure 7.9: A lattice which motivates further expansion of the set of basic operations in the overalgebra.

all $\theta \in \text{Con } \mathbf{B}$ with $\theta \not\geq \beta$. We now describe these operations and state this claim more formally as Proposition 7.3.5 below.

We start with the overalgebra construction described above. Suppose $\beta = \text{Cg}^{\mathbf{B}}(x, y)$ has transversal $\{b_{\beta(1)}, \dots, b_{\beta(m)}\}$, and for each $1 \leq r \leq m$, let

$$T_r = \{b \in T : (b, b_{\beta(r)}) \in \beta\} = \{b_{i_1}, b_{i_2}, \dots, b_{i_{|T_r|}}\}$$

be the tie-points contained in the r -th block of β , as above. Let $\mathcal{I}_r = \{i_1, i_2, \dots, i_{|T_r|}\}$ be the indices of these tie-points. Then $\{B_i : i \in \mathcal{I}_r\}$ is the collection of subreduct universes which intersect the r -th β block of B . For each $1 \leq r \leq m$, define the operation $s_r : A \rightarrow A$ as follows:

$$s_r(x) = \begin{cases} b_i & \text{if } x \in B_i \text{ for some } i \in \mathcal{I}_r, \\ x & \text{otherwise.} \end{cases}$$

Define all other operations as above and let

$$F_A := \{fe_0 : f \in F\} \cup \{e_k : 0 \leq k \leq K\} \cup \{s_r : 0 \leq r \leq m\},$$

where $s_0 := s$ was defined earlier. Finally, let $\mathbf{A} := \langle A, F_A \rangle$, and define θ^* and $\widehat{\theta}$ as above.

Proposition 7.3.5. *For each $\theta \in \text{Con } \mathbf{B}$,*

1. *if $\theta \wedge \beta = 0_B$, then $\theta^* = \widehat{\theta}$;*
2. *if $\theta \geq \beta$, then $[\theta^*, \widehat{\theta}] \cong \prod_{r=1}^n (\text{Eq}[T \cap b_{\theta(r)}/\theta])^{n-1}$, where $n \leq m$ is the number of congruence classes of θ .*

The first part of the proposition is easy to prove, given the additional operations s_r , $1 \leq r \leq m$. The second part follows from Theorem 7.3.3.

Note that T_r was defined above to be $T \cap b_{\beta(r)}/\beta$, so $T = \bigcup_{r=1}^m T_r$ is a partition of the tie-points, and it is on this partition that our definition of the additional operations s_r is based. A modified version of the GAP function used above to construct overalgebras allows the user to specify an arbitrary partition of the tie-points, and the extra operations will be defined accordingly. For example, to base the selection and partition of the tie-points on the congruence β in the example above, we invoke the following command:

```
gap> OveralgebraX0([ G, [[0,3], [2,5]] ]);
```

The resulting overalgebra has congruence lattice isomorphic to the lattice in Figure 7.9, with $L \cong 2^2 \times 2^2$. Similarly,

```
gap> OveralgebraX0([ G, [[0,1,2], [3,4,5]] ]);
```

produces an overalgebra with congruence lattice isomorphic to the one in Figure 7.9, but with $L \cong \text{Eq}(3) \times \text{Eq}(3)$.

Incidentally, with the additional operations s_r , we are not limited with respect to how many terms appear in the direct product. For example,

```
gap> OveralgebraX0([ G, [[0,1,2], [0,1,2], [3,4,5]] ]);
```

produces an overalgebra with a 130 element congruence lattice like the one in Figure 7.9, with $L \cong \text{Eq}(3) \times \text{Eq}(3) \times \text{Eq}(3)$, while

```
gap> OveralgebraX0([ G, [[0,3], [0,3], [0,3], [0,3]] ]);
```

gives a 261 element congruence lattice with $L \cong 2^{16}$.

We close this subsection with a result which describes one way to add even more operations to the overalgebra in case we wish to eliminate some of the congruences in $[\beta^*, \hat{\beta}]$ without affecting congruences outside that interval. In the following claim we assume the base algebra $\mathbf{B} = \langle B, G \rangle$ is a transitive G -set.

Claim 7.1. Consider the collection of maps $\hat{g} : A \rightarrow A$ defined for each $g \in \text{Stab}_G T := \{g \in G : gb = b \forall b \in T\}$ by the rules

$$\hat{g}|_{B_i} = e_{g(b_i)}ge_0 \quad (i = 1, \dots, n).$$

Then, for each $\theta \in \text{Con } \mathbf{A}$,

$$\hat{g}(\theta) \not\subseteq \theta \quad \text{only if} \quad \beta^* < \theta < \hat{\beta}. \quad (7.3.6)$$

Of course, these \widehat{g} maps may not be the only functions in A^A which have the property stated in (7.3.6). Also, in general, even with the whole collection of maps \widehat{g} defined above, we may not be able to eliminate every $\beta^* < \theta < \widehat{\beta}$. In fact, it's easy to construct examples in which there exist $\beta^* < \theta < \widehat{\beta}$ such that $g(\theta) \subseteq \theta$ for every $g \in A^A$.

7.3.2 Overalgebras II

In the previous section we described a procedure for building an overalgebra \mathbf{A} of \mathbf{B} such that for some *principal* congruence $\beta \in \text{Con } \mathbf{B}$ and for all $\beta \leq \theta < 1_B$, the inverse image $\theta|_B^{-1} = [\theta^*, \widehat{\theta}] \leq \text{Con } \mathbf{A}$ is non-trivial. In this section, we start with a non-principal congruence $\beta \in \text{Con } \mathbf{B}$ and ask if it is possible to construct an overalgebra \mathbf{A} such that $\theta|_B^{-1} \leq \text{Con } \mathbf{A}$ is non-trivial if and only if $\beta \leq \theta < 1_B$. To answer this question, we now describe an overalgebra construction that is based on a construction proposed by Bill Lampe.

Let $\mathbf{B} = \langle B; F \rangle$ be a finite algebra, and suppose

$$\beta = \text{Cg}^{\mathbf{B}}((a_1, b_1), \dots, (a_K, b_K))$$

for some $a_1, \dots, a_K, b_1, \dots, b_K \in B$. Let $B = B_0, B_1, B_2, \dots, B_{K+1}$ be sets of cardinality $|B| = n$ which intersect as follows:

$$\begin{aligned} B_0 \cap B_1 &= \{a_1\} = \{a_1^1\}, \\ B_i \cap B_{i+1} &= \{b_i^i\} = \{a_{i+1}^{i+1}\} \text{ for } 1 \leq i < K, \\ B_K \cap B_{K+1} &= \{b_K^K\} = \{a_1^{K+1}\}. \end{aligned}$$

All other intersections are empty. (See Figure 7.10.)

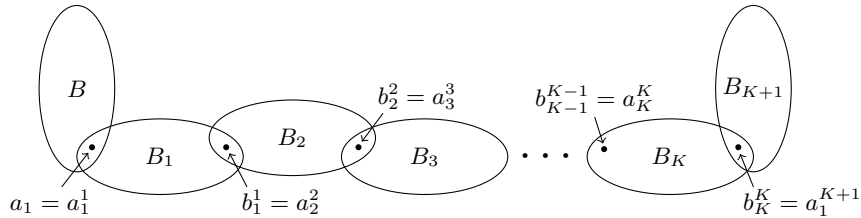


Figure 7.10: The universe of the overalgebra.

For $0 \leq i, j \leq K+1$, let $S_{i,j} : B_i \rightarrow B_j$ be the bijection $S_{i,j}(x^i) = x^j$. Put $A := B_0 \cup \dots \cup B_{K+1}$,

and define the following functions in A^A :

$$e_0(x) = \begin{cases} x, & x \in B_0, \\ a_1, & x \in B_j, 1 \leq j \leq K, \\ S_{K+1,0}(x), & x \in B_{K+1}; \end{cases}$$

$$e_i(x) = \begin{cases} a_i^i, & x \in B_j, j < i, \\ x, & x \in B_i, \\ b_i^i, & x \in B_j, j > i; \end{cases} \quad (1 \leq i \leq K),$$

$$e_{K+1}(x) = \begin{cases} S_{0,K+1}(x), & x \in B_0, \\ a_1^{K+1}, & x \in B_j, 1 \leq j \leq K, \\ x, & x \in B_{K+1}. \end{cases}$$

Using these maps we define the set F_A of operations on A as follows: let $q_{i,j} = S_{i,j} \circ e_i$ for $0 \leq i, j \leq K+1$ and define⁸

$$F_A := \{fe_0 : f \in F\} \cup \{q_{i,0} : 0 \leq i \leq K+1\} \cup \{q_{0,j} : 1 \leq j \leq K+1\}.$$

The overalgebra in this section is defined to be the unary algebra $\mathbf{A} := \langle A, F_A \rangle$.

Theorem 7.3.6. *Suppose $\mathbf{A} = \langle A, F_A \rangle$ is the overalgebra based on the congruence relation $\beta = \text{Cg}^{\mathbf{B}}((a_1, b_1), \dots, (a_K, b_K))$, as described above, and define*

$$\beta^* = \bigcup_{j=0}^{K+1} \beta^{\mathbf{B}^j} \cup (a_1/\beta \cup a_1^1/\beta^{\mathbf{B}^1} \cup a_2^2/\beta^{\mathbf{B}^2} \cup \dots \cup a_K^K/\beta^{\mathbf{B}^K} \cup a_1^{K+1}/\beta^{\mathbf{B}^{K+1}})^2.$$

Then, $\beta^* = \text{Cg}^{\mathbf{A}}(\beta)$.

If β has transversal $\{a_1, c_1, c_2, \dots, c_{m-1}\}$, then

$$\widehat{\beta} = \beta^* \cup \bigcup_{i=1}^{m-1} (c_i/\beta \cup c_i^{K+1}/\beta^{\mathbf{B}^{K+1}})^2. \quad (7.3.7)$$

⁸If we were to include $q_{i,j}$ for all $0 \leq i, j \leq K+1$, the resulting overalgebra would have the same congruence lattice as $\langle A, F_A \rangle$, but using a reduced set of operations simplifies our proofs.

Moreover, $[\beta^*, \widehat{\beta}] \cong \mathbf{2}^{m-1}$.

Proof. It is clear that β^* is an equivalence relation on A , so we first check that $f(\beta^*) \subseteq \beta^*$ for all $f \in F_A$. This will establish that $\beta^* \in \text{Con } \mathbf{A}$. Thereafter we show that $\beta \subseteq \eta \in \text{Con } \mathbf{A}$ implies $\beta^* \leq \eta$, which will prove that β^* is the smallest congruence of \mathbf{A} containing β , as claimed in the first part of the theorem.

Fix $(x, y) \in \beta^*$. To show $(f(x), f(y)) \in \beta^*$ we consider two possible cases.

Case 1: $(x, y) \in \beta^{\mathbf{B}_j}$ for some $0 \leq j \leq K+1$.

In this case it is easy to verify that $(q_{i,0}(x), q_{i,0}(y)) \in \beta$ and $(q_{0,i}(x), q_{0,i}(y)) \in \beta^{\mathbf{B}_i}$ for all $0 \leq i \leq K+1$. For example, if $(x, y) \in \beta^{\mathbf{B}_j}$ with $1 \leq j \leq K$, then $(q_{0,i}(x), q_{0,i}(y)) = (a_1^i, a_1^i)$ and $(q_{i,0}(x), q_{i,0}(y))$ is either (b_i, b_i) or (a_i, a_i) depending on whether i is below or above j , respectively. If $i = j$, then $(q_{i,0}(x), q_{i,0}(y))$ is the pair in B^2 corresponding to $(x, y) \in \beta^{\mathbf{B}_j}$, so $(q_{i,0}(x), q_{i,0}(y)) \in \beta$. A special case is $(q_{0,0}(x), q_{0,0}(y)) \in \beta$. Now, since $q_{0,0} = e_0$, we have $(fe_0(x), fe_0(y)) \in \beta$ for all $f \in F_B$. Altogether, the foregoing implies that $(f(x), f(y)) \in \beta^*$ for all $f \in F_A$.

Case 2: $(x, y) \in \mathcal{B}^2$ where $\mathcal{B} := a_1/\beta \cup a_1^1/\beta^{\mathbf{B}_1} \cup \dots \cup a_K^K/\beta^{\mathbf{B}_K} \cup a_1^{K+1}/\beta^{\mathbf{B}_{K+1}}$.

Note that $e_0(\mathcal{B}) = a_1/\beta$. Therefore, $(e_0(x), e_0(y)) \in \beta$, so $(fe_0(x), fe_0(y)) \in \beta$ for all $f \in F_B$. Also,

$$q_{0,k}(\mathcal{B}) = S_{0,k}e_0(\mathcal{B}) = S_{0,k}(a_1/\beta) = a_1^k/\beta^{\mathbf{B}_k},$$

which is a single block of β^* . Similarly, $e_k(\mathcal{B}) = a_k^k/\beta^{\mathbf{B}_k}$, so

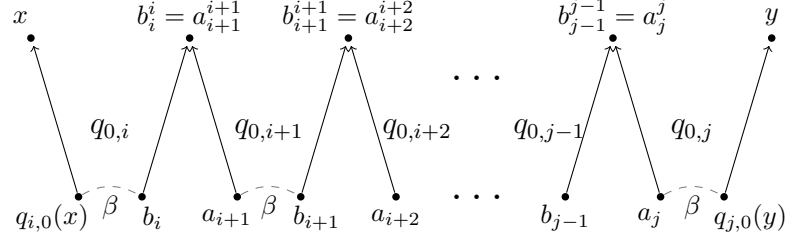
$$q_{k,0}(\mathcal{B}) = S_{k,0}e_k(\mathcal{B}) = S_{k,0}(a_k^k/\beta^{\mathbf{B}_k}) = a_k/\beta.$$

Whence, $(x, y) \in \mathcal{B}^2$ implies $(f(x), f(y)) \in \beta^*$ for all $f \in F_A$.

We have thus established that β^* is a congruence of \mathbf{A} which contains β . We now show that it is the smallest such congruence. Indeed, suppose $\beta \subseteq \eta \in \text{Con } \mathbf{A}$, and fix $(x, y) \in \beta^*$. If $(x, y) \in \beta^{\mathbf{B}_j}$ for some $0 \leq j \leq K+1$, then $(q_{j,0}(x), q_{j,0}(y)) \in \beta \subseteq \eta$, so $(x, y) = (q_{0,j}q_{j,0}(x), q_{0,j}q_{j,0}(y)) \in \eta$.

If, instead of $(x, y) \in \beta^{\mathbf{B}_j}$, we have $(x, y) \in \mathcal{B}^2$, then without loss of generality $x \in a_i^i/\beta^{\mathbf{B}_i}$ and $y \in a_j^j/\beta^{\mathbf{B}_j}$ for some $0 \leq i < j \leq K+1$. We only discuss the case $1 \leq i < j \leq K$, as the other cases can be handled similarly. Since $x \in a_i^i/\beta^{\mathbf{B}_i} = b_i^i/\beta^{\mathbf{B}_i}$, we have $(q_{i,0}(x), b_i) \in \beta$. Similarly, $(a_j, q_{j,0}(y)) \in \beta$. Therefore, we obtain the following diagram⁹

⁹ The diagram illustrates the case $1 \leq i < j \leq K$ where $i+1 < j$. In case $j = i+1$, the diagram is even simpler.



Since $\beta \subseteq \eta \in \text{Con } \mathbf{A}$, and since $q_{0,k} \in F_A$ for each k , the diagram makes it clear that (x, y) must belong to η .

To prove (7.3.7), let $\tilde{\beta}$ denote the right-hand side. That is,

$$\tilde{\beta} := \beta^* \cup \bigcup_{i=1}^{m-1} (c_i/\beta \cup c_i^{K+1}/\beta^{\mathbf{B}_{K+1}})^2.$$

It is clear that $\tilde{\beta} \in \text{Eq}(A)$, so we verify $\tilde{\beta} \in \text{Con } \mathbf{A}$ by proving that $f(\tilde{\beta}) \subseteq \tilde{\beta}$ for all $f \in F_A$. Fix $(x, y) \in \tilde{\beta}$. If $(x, y) \in \beta^*$, then $(f(x), f(y)) \in \beta^*$ for all $f \in F_A$, by the first part of the theorem. So suppose $(x, y) \in (c_i/\beta \cup c_i^{K+1}/\beta^{\mathbf{B}_{K+1}})^2$, for some $1 \leq i \leq m-1$. For ease of notation, define

$$\mathcal{C}_i := c_i/\beta \cup c_i^{K+1}/\beta^{\mathbf{B}_{K+1}}.$$

Then, since $e_0(\mathcal{C}_i) = c_i/\beta$, we have $(e_0(x), e_0(y)) \in \beta$, so $(fe_0(x), fe_0(y)) \in \beta$ for all $f \in F_B$. Also, for $0 \leq k \leq K+1$, we have¹⁰

$$q_{0,k}(\mathcal{C}_i) = S_{0,k}(c_i/\beta) = c_i^k/\beta^{\mathbf{B}_k}.$$

Therefore, $q_{0,k}(\mathcal{C}_i)$ is in a single block of β^* , so $(q_{0,k}(x), q_{0,k}(y)) \in \beta^*$. Also, for $1 \leq k \leq K$, we have $e_k(c_i/\beta) = \{a_k^k\}$ and $e_k(c_i^{K+1}/\beta^{\mathbf{B}_{K+1}}) = \{b_k^k\}$, so

$$q_{k,0}(\mathcal{C}_i) = S_{k,0}(\{a_k^k, b_k^k\}) = \{a_k, b_k\} \subseteq a_k/\beta,$$

while, for $k = K+1$, we have $e_{K+1}(\mathcal{C}_i) = c_i^{K+1}/\beta^{\mathbf{B}_{K+1}}$, so

$$q_{K+1,0}(\mathcal{C}_i) = S_{K+1,0}(c_i^{K+1}/\beta^{\mathbf{B}_{K+1}}) = c_i/\beta.$$

Also, the cases involving $i = 0$ and/or $j = K+1$ can be handled similarly.

¹⁰By $c_i^0/\beta^{\mathbf{B}_0}$ we mean, of course, c_i/β .

Thus, for all $0 \leq k \leq K+1$, we have $(q_{k,0}(x), q_{k,0}(y)) \in \beta^*$. This proves that $(f(x), f(y)) \in \beta^* \subseteq \tilde{\beta}$ holds for all $f \in F_A$, so $\tilde{\beta} \in \text{Con } \mathbf{A}$.

Next, note that $\tilde{\beta}|_B = \beta$, so by the residuation lemma of Section 7.2, $\tilde{\beta} \leq \hat{\beta}$. Thus, to prove (7.3.7), it suffices to show that $(x, y) \notin \tilde{\beta}$ implies $(x, y) \notin \hat{\beta}$. This is straight-forward, and similar to the argument we used to check the analogous fact in the proof of Theorem 7.3.3. Nonetheless, we verify most of the cases, and omit only a few special cases which are easy to check.

Suppose $(x, y) \notin \tilde{\beta}$, and suppose $x \in c_p^j / \beta^{\mathbf{B}_j}$ and $y \in c_q^k / \beta^{\mathbf{B}_k}$ for some $0 \leq j \leq k \leq K+1$ and $1 \leq p, q \leq m-1$. If $j = 0$ and $k = K+1$, then $p \neq q$ (otherwise, $(x, y) \in \tilde{\beta}$). Therefore, $e_0(x) \in c_p / \beta$ and $e_0(y) \in c_q / \beta$, so $(e_0(x), e_0(y)) \notin \beta$, so $(x, y) \notin \hat{\beta}$. If $p = q$, then $j \neq k$ (otherwise, $(x, y) \in \tilde{\beta}$). Thus,

$$\begin{aligned} (e_j(x), e_j(y)) = (x, b_j^j) &\Rightarrow (q_{j,0}(x), q_{j,0}(y)) = (q_{j,0}(x), b_j); \\ (e_k(x), e_k(y)) = (a_k^k, y) &\Rightarrow (q_{k,0}(x), q_{k,0}(y)) = (a_k, q_{k,0}(y)). \end{aligned}$$

One of the pairs on the right is not in β . For if both are in β , then

$$\begin{aligned} x = q_{0,j} q_{j,0}(x) \beta^* q_{0,j}(b_j) &= b_j^j = a_{j+1}^{j+1} \beta^* \cdots \\ \cdots \beta^* a_k^k &= q_{0,k}(a_k) \beta^* q_{0,k} q_{k,0}(y) = y, \end{aligned}$$

which contradicts $(x, y) \notin \tilde{\beta}$, so we must have either $(q_{j,0}(x), q_{j,0}(y)) \notin \beta$ or $(q_{k,0}(x), q_{k,0}(y)) \notin \beta$. Therefore, since $e_0 q_{i,0} = q_{i,0}$, we see that $(x, y) \notin \hat{\beta}$. The other cases, e.g. $x \in a_1 / \beta$, $y \in c_q^k / \beta^{\mathbf{B}_k}$, can be checked similarly.

It remains to prove that $[\beta^*, \hat{\beta}] \cong 2^{m-1}$, but this follows easily from the first part of the proof, where we saw that $(f(x), f(y)) \in \beta^*$ for all $f \in F_A$ and for all $(x, y) \in \hat{\beta}$. This implies that all equivalence relations on A that are above β^* and below $\hat{\beta}$ are, in fact, congruence relations of \mathbf{A} . The shape of this interval of equivalence relations is even simpler than the shape of the analogous interval we found in Theorem 7.3.3. In the present case, we have

$$[\beta^*, \hat{\beta}] = \{\theta \in \text{Eq}(A) : \beta^* \subseteq \theta \subseteq \hat{\beta}\} \cong 2^{m-1}.$$

□

Before stating the next result, we remind the reader that $\theta^* = \text{Cg}^{\mathbf{A}}(\theta)$ for each $\theta \in \text{Con } \mathbf{B}$.

Lemma 7.3.7. *If $\eta \in \text{Con } \mathbf{A}$ satisfies $\eta|_B = \theta$, and if $(x, y) \in \eta \setminus \theta^*$ for some $x \in B_i$, $y \in B_j$, then $i = 0$, $j = K + 1$, and $\theta \geq \beta$.*

In other words, unless $i = 0$ and $j = K + 1$, the congruence η doesn't join blocks of B_i with blocks of B_j (except for those already joined by θ^*).

Proof. We rule out all $0 \leq i \leq j \leq K + 1$ except for $i = 0$ and $j = K + 1$ by showing that, in each of the following cases, we arrive at the contradiction $(x, y) \in \theta^* := \text{Cg}^{\mathbf{A}}(\theta)$.

Case 1: $i = j$.

If $(x, y) \in B_i^2$ for some $0 \leq i \leq K + 1$, then $(q_{i,0}(x), q_{i,0}(y)) \in \eta|_B = \theta \leq \theta^*$, so $(x, y) = (q_{0,i}q_{i,0}(x), q_{0,i}q_{i,0}(y)) \in \theta^*$.

Case 2: $1 \leq i < j \leq K$.

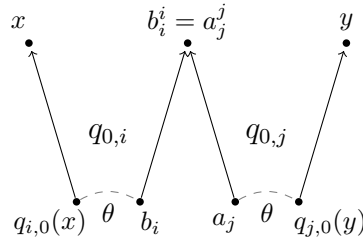
In this case,

$$(q_{i,0}(x), q_{i,0}(y)) = (q_{i,0}(x), b_i) \in \theta, \quad (q_{j,0}(x), q_{j,0}(y)) = (a_j, q_{j,0}(y)) \in \theta,$$

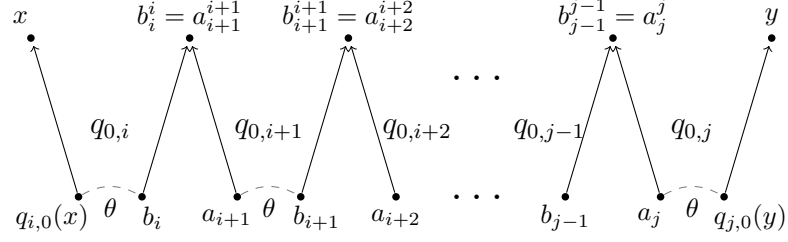
When $j = i + 1$, we obtain

$$x = q_{0,i}q_{i,0}(x) \theta^* q_{0,i}(b_i) = b_i^i = a_j^j = q_{0,j}(a_j) \theta^* q_{0,j}q_{j,0}(y) = y, \quad (7.3.8)$$

so $(x, y) \in \theta^*$. This can be seen more transparently in a diagram.



If $j > i + 1$, then $(q_{k,0}(x), q_{k,0}(y)) = (a_k, b_k) \in \theta$ for all $i < k < j$, and we have the following diagram:



Here too we could write out a line analogous to (7.3.8), but it is obvious from the diagram that $(x, y) \in \theta^*$.

The case $i = 0$; $1 \leq j \leq K$, as well as the case $1 \leq i \leq K$; $j = K + 1$, can be handled with diagrams similar to those used above, and the proofs are almost identical, so we omit them.

The only remaining possibility is $x \in B_0$ and $y \in B_{K+1}$. In this case we have $(q_{k,0}(x), q_{k,0}(y)) = (a_k, b_k) \in \theta$, for all $1 \leq k \leq K$. Therefore, $\theta \geq \beta = \text{Cg}^{\mathbf{A}}((a_1, b_1), \dots, (a_K, b_K))$. \square

Theorem 7.3.8. *Suppose $\mathbf{A} = \langle A, F_A \rangle$ is the overalgebra based on the congruence relation $\beta = \text{Cg}^{\mathbf{B}}((a_1, b_1), \dots, (a_K, b_K))$, as described above. Then, $\theta^* < \hat{\theta}$ if and only if $\beta \leq \theta < 1_B$, in which case $[\theta^*, \hat{\theta}] \cong \mathbf{2}^{r-1}$, where r is the number of congruence classes of θ .*

Consequently, if $\theta \not\geq \beta$, then $\hat{\theta} = \theta^*$.

Proof. Lemma 7.3.7 implies that $\theta^* < \hat{\theta}$ only if $\beta \leq \theta < 1_B$. On the other hand, if $\beta \leq \theta < 1_B$, then we obtain $[\theta^*, \hat{\theta}] \cong \mathbf{2}^{r-1}$ by the same argument used to prove $[\beta^*, \hat{\beta}] \cong \mathbf{2}^{m-1}$ in Theorem 7.3.6. \square

We now consider an example of a congruence lattice having a coatom β that is not principal, and we use the method described in this section to construct an overalgebra \mathbf{A} for which $\beta^* < \hat{\beta}$ in $\text{Con } \mathbf{A}$, and $\theta^* = \hat{\theta}$ for all $\theta \not\geq \beta$ in $\text{Con } \mathbf{B}$.

Example 7.3.9. Let G be the group $C_2 \times A_4$ defined in GAP as follows:¹¹

```
gap> G:=Group([ (9,10)(11,12)(5,6)(7,8),
>              (3,7,12)(9,1,6)(11,4,8)(5,10,2),
>              (3,2)(9,11)(5,7)(1,4)(10,12)(6,8) ]);;
```

This is a group of order 24 which acts transitively on the set $\{1, 2, \dots, 12\}$. (If we let H denote the stabilizer of a point, say $H := G_1 \cong C_2$, then the group acts transitively by right multiplication on the set G/H of right cosets. These two G -sets are of course isomorphic.) The congruence lattice of

¹¹The GAP command `TransitiveGroup(12,7)` also gives a group isomorphic to $C_2 \times A_4$, but by defining it explicitly in terms of certain generators, we obtain more attractive partitions in the congruence lattice.

this algebra (which is isomorphic to the interval from H up to G in the subgroup lattice of G) is shown in Figure 7.11. After relabeling the elements to conform to our 0-offset notation, the universe is $B := \{0, 1, \dots, 11\}$, and the non-trivial congruences are as follows:

$$\alpha = |0, 1, 4, 5, 8, 9|2, 3, 6, 7, 10, 11|$$

$$\beta = |0, 1, 2, 3|4, 5, 6, 7|8, 9, 10, 11|$$

$$\gamma_1 = |0, 1|2, 3|4, 5|6, 7|8, 9|10, 11|$$

$$\gamma_2 = |0, 2|1, 3|4, 7|5, 6|8, 11|9, 10|$$

$$\gamma_3 = |0, 3|1, 2|4, 6|5, 7|8, 10|9, 11|.$$

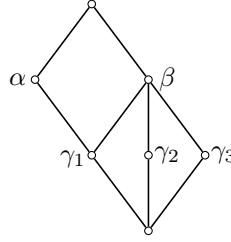


Figure 7.11: The congruence lattice of the permutational algebra $\langle B, G \rangle$, where $B = \{0, 1, \dots, 11\}$ and $G \cong C_2 \times A_4$.

Clearly, the coatom β is not principal. It is generated by $\{(0, 3), (8, 11)\}$, for example. If our goal is to construct an overalgebra which has $\widehat{\beta} > \beta^*$ in $\text{Con } \mathbf{A}$, and $\theta^* = \widehat{\theta}$ for all $\theta \not\geq \beta$ in $\text{Con } \mathbf{B}$, it is clear that the method described in the Section 7.3.1 will not work. For, if we base the overalgebra on tie-points $\{0, 3\}$, then the universe is $A = B \cup B_1 \cup B_2$, where $B \cap B_1 = \{0\}$, $B \cap B_2 = \{3\}$, and $B_1 \cap B_2 = \emptyset$, and the operations are $F_A := \{ge_0 : g \in G\} \cup \{e_0, e_1, e_2, s\}$. Since β has three congruence classes, by Theorem 7.3.3 the interval of all $\theta \in \text{Con } \mathbf{A}$ for which $\theta|_B = \beta$ is $[\beta^*, \widehat{\beta}] \cong \mathbf{2}^2$. However, we also have $\gamma_3 = \text{Cg}^{\mathbf{B}}(0, 3)$, a congruence with 6 classes, so again by Theorem 7.3.3, $[\gamma_3^*, \widehat{\gamma}_3] \cong \mathbf{2}^5$. Thus, using this method it is not possible to obtain a non-trivial interval $[\beta^*, \widehat{\beta}]$ while preserving the original congruence lattice structure below β . This is true no matter which pair $(x, y) \in \beta$ we choose as tie-points, since, in every case, the pair will belong to a congruence below β .

The procedure described in this subsection does not have the same limitation. Indeed, if we set $(a_1, b_1) = (0, 3)$ and $(a_2, b_2) = (8, 11)$ in this construction, then the universe of the overalgebra is $A = \bigcup_{i=0}^3 B_i$ where $B_0 = \{0, 1, \dots, 11\}$, $B_1 = \{0, 12, 13, \dots, 22\}$, $B_2 = \{23, 24, \dots, 29, 30, 14, 31, 32, 33\}$, and $B_3 = \{33, 34, \dots, 44\}$. (See Figure 7.12.)

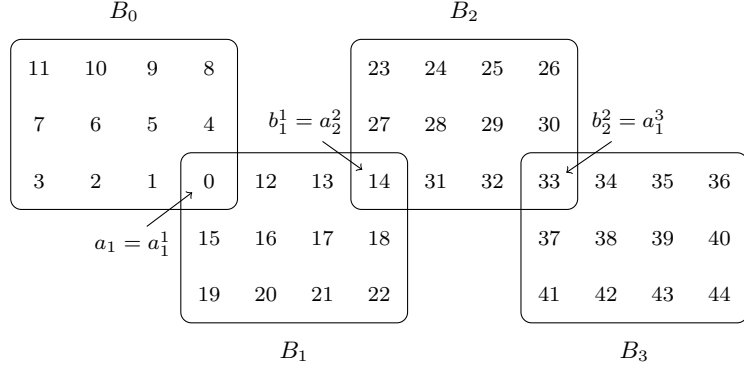


Figure 7.12: The universe of the overalgebra of the $(C_2 \times A_4)$ -set, arranged to reveal the congruences above β^* .

Arranging the subreduct universes as in Figure 7.12 reveals the congruences above β^* . In fact, the four congruences in the interval $[\beta^*, \widehat{\beta}]$ can be read off directly from the diagram. For example, the congruence classes of β^* are shown in Figure 7.13, while the congruence $\widehat{\beta}$, in addition to these relations, joins blocks $|4, 5, 6, 7|$ and $|37, 38, 39, 40|$, as well as blocks $|8, 9, 10, 11|$ and $|41, 42, 43, 44|$. As for the congruences $\beta_\varepsilon, \beta_{\varepsilon'}$, one joins $|4, 5, 6, 7|$ and $|37, 38, 39, 40|$, while the other joins $|8, 9, 10, 11|$ and $|41, 42, 43, 44|$. The full congruence lattice, $\text{Con } \mathbf{A}$, appears in Figure 7.14.

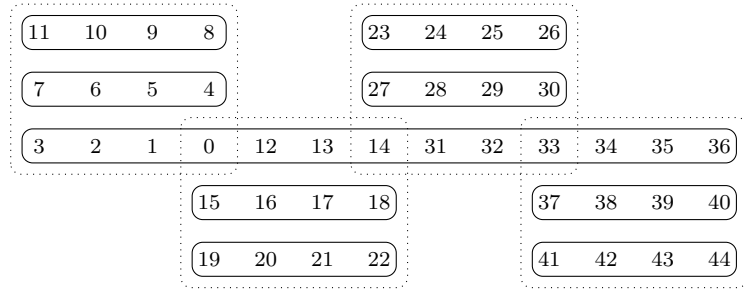


Figure 7.13: The universe of the overalgebra; solid lines delineate the congruence classes of β^* .

7.3.3 Overalgebras III

In Section 7.3.1 we constructed an algebra \mathbf{A} with a congruence lattice $\text{Con } \mathbf{A}$ having interval sublattices $[\beta^*, \widehat{\beta}]$ that are isomorphic to products of powers of partition lattices. We saw that the construction has two main limitations. First, the size of the partition lattices is limited by the size of the congruence classes of $\beta \in \text{Con } \mathbf{B}$. Second, when β is non-principal, it is impossible with this construction to obtain a nontrivial inverse image $[\beta^*, \widehat{\beta}]$ without also having nontrivial inverse

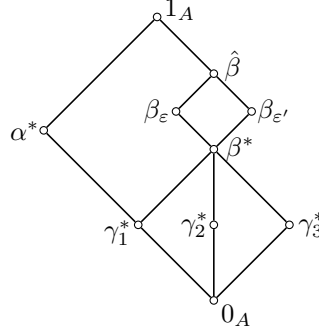


Figure 7.14: The congruence lattice of the overalgebra $\langle A, F_A \rangle$ of $\langle B, G \rangle$, where $B = \{0, 1, \dots, 11\}$ and $G \cong C_2 \times A_4$.

images $[\theta^*, \hat{\theta}]$ for some $\theta \not\geq \beta$. In Section 7.3.2, we presented a construction which resolves the second limitation. However, the first limitation is even more severe in that the resulting intervals $[\beta^*, \hat{\beta}]$ are simply powers of $\mathbf{2}$ – i.e., Boolean algebras. In this section, we present a generalization of the previous constructions which overcomes both of the limitations mentioned above.

Let $\mathbf{B} = \langle B, F \rangle$ be a finite algebra, and suppose

$$\beta = \text{Cg}^{\mathbf{B}}((a_1, b_1), \dots, (a_{K-1}, b_{K-1}))$$

for some $a_1, \dots, a_{K-1}, b_1, \dots, b_{K-1} \in B$. Define $B_0 = B$ and, for some fixed $Q \geq 0$, let $B_1, B_2, \dots, B_{(2Q+1)K}$ be sets of cardinality $|B| = n$. As above, we use the label x^i to denote the element of B_i which corresponds to $x \in B$ under the bijection. For ease of notation, let $M := (2Q+1)$. We arrange the sets so that they intersect as follows:

$$\begin{aligned} B_0 \cap B_1 &= \{a_1\} = \{a_1^1\}, \\ B_1 \cap B_2 &= \{b_1^1\} = \{a_2^2\}, \\ B_2 \cap B_3 &= \{b_2^2\} = \{a_3^3\}, \\ &\vdots \\ B_{K-2} \cap B_{K-1} &= \{b_{K-2}^{K-2}\} = \{a_{K-1}^{K-1}\}, \\ B_{K-1} \cap B_K &= B_K \cap B_{K+1} = \{b_{K-1}^{K-1}\} = \{b_{K-1}^K\} = \{b_{K-1}^{K+1}\}, \\ B_{K+1} \cap B_{K+2} &= \{a_{K-1}^{K+1}\} = \{b_{K-2}^{K+2}\}, \\ B_{K+2} \cap B_{K+3} &= \{a_{K-2}^{K+2}\} = \{b_{K-3}^{K+3}\}, \dots \end{aligned}$$

$$\begin{aligned}
& \dots, B_{2K-2} \cap B_{2K-1} = \{a_2^{2K-2}\} = \{b_1^{2K-1}\}, \\
& B_{2K-1} \cap B_{2K} = B_{2K} \cap B_{2K+1} = \{a_1^{2K-1}\} = \{a_1^{2K}\} = \{a_1^{2K+1}\}, \\
& B_{2K+1} \cap B_{2K+2} = \{b_1^{2K+1}\} = \{b_2^{2K+2}\}, \\
& B_{2K+2} \cap B_{2K+3} = \{b_2^{2K+2}\} = \{b_3^{2K+3}\}, \\
& \vdots \\
& B_{MK-2} \cap B_{MK-1} = \{b_{MK-2}^{K-2}\} = \{a_{MK-1}^{K-1}\}, \\
& B_{MK-1} \cap B_{MK} = \{b_{K-1}^{MK-1}\} = \{b_{K-1}^{MK}\}.
\end{aligned}$$

All other intersections are empty. (See Figure 7.15.)

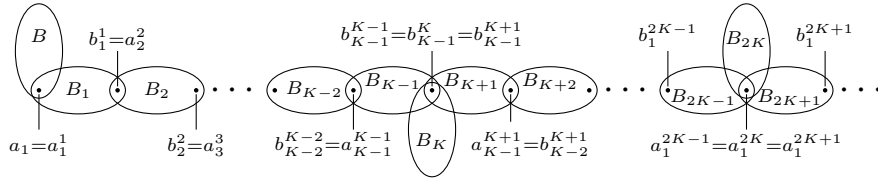


Figure 7.15: The universe of the overalgebra.

As usual, we put $A := B_0 \cup \dots \cup B_{MK}$, and we proceed to define some unary operations on A .

First, for $0 \leq i, j \leq MK$, let $S_{i,j} : B_i \rightarrow B_j$ be the bijection $S_{i,j}(x^i) = x^j$, and note that $S_{i,i} = \text{id}_{B_i}$. Define the following subsets of even and odd multiples of K , respectively: $\mathcal{E} = \{2qK : q = 0, 1, \dots, Q\}$ and $\mathcal{O} = \{(2q+1)K : q = 0, 1, \dots, Q\}$. For each $\ell \in \mathcal{E}$, let

$$e_\ell(x) = \begin{cases} S_{j,\ell}(x), & \text{if } x \in B_j \text{ for some } j \in \mathcal{E}, \\ a_1^\ell, & \text{otherwise.} \end{cases}$$

and, for $0 < i < K$,

$$e_{\ell+i}(x) = \begin{cases} a_i^{\ell+i}, & \text{if } x \in B_j \text{ for some } j < \ell+i, \\ x, & \text{if } x \in B_{\ell+i}, \\ b_i^{\ell+i}, & \text{if } x \in B_j \text{ for some } j > \ell+i. \end{cases}$$

For each $\ell \in \mathcal{O}$, let

$$e_\ell(x) = \begin{cases} S_{j,\ell}(x), & \text{if } x \in B_j \text{ for some } j \in \mathcal{O}, \\ b_{K-1}^\ell, & \text{otherwise.} \end{cases}$$

and, for $0 < i < K$,

$$e_{\ell+i}(x) = \begin{cases} b_{K-i}^{\ell+i}, & \text{if } x \in B_j \text{ for some } j < \ell+i, \\ x, & \text{if } x \in B_{\ell+i}, \\ a_{K-i}^{\ell+i}, & \text{if } x \in B_j \text{ for some } j > \ell+i, \end{cases}$$

In other words, if $\ell \in \mathcal{E}$, then e_ℓ maps each up-pointing set in Figure 7.15 bijectively onto the up-pointing set B_ℓ , and maps all other points of A to the tie-point $a_1^\ell \in B_\ell$; if $\ell \in \mathcal{O}$, then e_ℓ maps each down-pointing set in the figure onto the down-pointing set B_ℓ , and maps all other points to the tie-point b_{K-1}^ℓ . For each set $B_{\ell+i}$ in between – represented in the figure by an ellipse with horizontal major axis – there corresponds a map $e_{\ell+i}$ which act as the identity on $B_{\ell+i}$ and maps all points in A left of $B_{\ell+i}$ to the left tie-point of $B_{\ell+i}$ and all points to the right of $B_{\ell+i}$ to the right tie-point of $B_{\ell+i}$.

Finally, for $0 \leq i, j \leq MK$, we define $q_{i,j} = S_{i,j} \circ e_i$ and take the set of basic operations on A to be

$$F_A := \{fe_0 : f \in F\} \cup \{q_{i,0} : 0 \leq i \leq MK\} \cup \{q_{0,j} : 1 \leq j \leq MK\}.$$

We then consider the overalgebra $\mathbf{A} := \langle A, F_A \rangle$. This overalgebra is, once again, based on the specific congruence $\beta = \text{Cg}^{\mathbf{B}}((a_1, b_1), \dots, (a_{K-1}, b_{K-1})) \in \text{Con } \mathbf{B}$, and the following theorem describes the inverse image of β under $|_B$ – that is, the interval $[\beta^*, \widehat{\beta}]$ in $\text{Con } \mathbf{A}$.

Theorem 7.3.10. *Let $\mathbf{A} = \langle A, F_A \rangle$ be the overalgebra described above, and, for each $0 \leq i \leq MK$, let t_i denote a tie-point of the set B_i . Define*

$$\beta^* = \bigcup_{j=0}^{MK} \beta^{\mathbf{B}_j} \cup \left(\bigcup_{i=0}^{MK} t_i / \beta^{\mathbf{B}_i} \right)^2.$$

Then, $\beta^ = \text{Cg}^{\mathbf{A}}(\beta)$.*

If β has transversal $\{a_1, c_1, c_2, \dots, c_{m-1}\}$, then

$$\widehat{\beta} = \beta^* \cup \bigcup_{i=1}^{m-1} \left(\bigcup_{\ell \in \mathcal{E}} c_i^\ell / \beta^{\mathbf{B}_\ell} \right)^2 \cup \bigcup_{i=1}^{m-1} \left(\bigcup_{\ell \in \mathcal{O}} c_i^\ell / \beta^{\mathbf{B}_\ell} \right)^2. \quad (7.3.9)$$

Moreover, $[\beta^*, \widehat{\beta}] \cong (\text{Eq}|\mathcal{E}|)^{m-1} \times (\text{Eq}|\mathcal{O}|)^{m-1}$.

Remark. Recall that m is the number of congruence classes in β . The number of up-pointing sets in Figure 7.15 is $|\mathcal{E}|$, while $|\mathcal{O}|$ counts the number of down-pointing sets. In our construction, we took $|\mathcal{E}| = |\mathcal{O}| = Q + 1$, but, apart from being notationally convenient, this choice was arbitrary; in fact, there's no reason \mathcal{E} and \mathcal{O} should be equal in number, and they could even be empty. Choosing $\mathcal{O} = \emptyset$, for example, would result in the interval $[\beta^*, \widehat{\beta}] \cong (\text{Eq}|\mathcal{E}|)^{m-1}$. Thus, for any N , we can construct an algebra \mathbf{A} that has $(\text{Eq}N)^{m-1} \cong [\beta^*, \widehat{\beta}] < \text{Con } \mathbf{A}$.

Proof of Theorem 7.3.10. It is easy to check that β^* is an equivalence relation on A , so we first check that $f(\beta^*) \subseteq \beta^*$ for all $f \in F_A$. This will establish that $\beta^* \in \text{Con } \mathbf{A}$. Thereafter we show that $\beta \subseteq \eta \in \text{Con } \mathbf{A}$ implies $\beta^* \leq \eta$, which will prove that β^* is the smallest congruence of \mathbf{A} containing β , as claimed in the first part of the theorem.

Fix $(x, y) \in \beta^*$. To show $(f(x), f(y)) \in \beta^*$ we consider two possible cases.

Case 1: $(x, y) \in \beta^{\mathbf{B}^j}$ for some $0 \leq j \leq (2q + 1)K$.

In this case it is easy to verify that $(q_{i,0}(x), q_{i,0}(y)) \in \beta$ and $(q_{0,i}(x), q_{0,i}(y)) \in \beta^{\mathbf{B}^i}$ for all $0 \leq i \leq K + 1$. For example, if $(x, y) \in \beta^{\mathbf{B}^j}$ with $1 \leq j \leq K$, then $(q_{0,i}(x), q_{0,i}(y)) = (a_1^i, a_1^i)$ and $(q_{i,0}(x), q_{i,0}(y))$ is either (b_i, b_i) or (a_i, a_i) depending on whether i is below or above j , respectively. If $i = j$, then $(q_{i,0}(x), q_{i,0}(y))$ is the pair in B^2 corresponding to $(x, y) \in \beta^{\mathbf{B}^j}$, so $(q_{i,0}(x), q_{i,0}(y)) \in \beta$. A special case is $(q_{0,0}(x), q_{0,0}(y)) \in \beta$. Therefore, $q_{0,0} = e_0$, implies $(fe_0(x), fe_0(y)) \in \beta$ for all $f \in F_B$. Altogether, we have proved that $(f(x), f(y)) \in \beta^*$ for all $f \in F_A$.

Case 2: $(x, y) \in \mathcal{B}^2$ where $\mathcal{B} := \bigcup_{i=0}^{MK} t_i / \beta^{\mathbf{B}^i}$.

Note that $e_0(\mathcal{B}) = a_1 / \beta$. Therefore, $(e_0(x), e_0(y)) \in \beta$, so $(fe_0(x), fe_0(y)) \in \beta$ for all $f \in F_B$. Also,

$$q_{0,k}(\mathcal{B}) = S_{0,k}e_0(\mathcal{B}) = S_{0,k}(a_1 / \beta) = a_1^k / \beta^{\mathbf{B}^k},$$

which is a single block of β^* . Similarly, $e_k(\mathcal{B}) = t_k / \beta^{\mathbf{B}^k}$, so

$$q_{k,0}(\mathcal{B}) = S_{k,0}e_k(\mathcal{B}) = S_{k,0}(t_k / \beta^{\mathbf{B}^k}) = S_{k,0}(t_k) / \beta,$$

a single block of β^* . Whence, $(x, y) \in \mathcal{B}^2$ implies $(f(x), f(y)) \in \beta^*$ for all $f \in F_A$.

We have thus established that β^* is a congruence of \mathbf{A} which contains β . We now show that β^*

is the smallest such congruence. Indeed, suppose $\beta \subseteq \eta \in \text{Con } \mathbf{A}$, and fix $(x, y) \in \beta^*$. If $(x, y) \in \beta^{\mathbf{B}_j}$ for some $0 \leq j \leq MK$, then $(q_{j,0}(x), q_{j,0}(y)) = (S_{j,0}e_j(x), S_{j,0}e_j(y)) = (S_{j,0}(x), S_{j,0}(y)) \in \beta \subseteq \eta$, so $(x, y) = (q_{0,j}q_{j,0}(x), q_{0,j}q_{j,0}(y)) \in \eta$.

If, instead of $(x, y) \in \beta^{\mathbf{B}_j}$, we have $(x, y) \in \mathcal{B}^2$, then without loss of generality $x \in a_i^i/\beta^{\mathbf{B}_i}$ and $y \in a_j^j/\beta^{\mathbf{B}_j}$ for some $0 \leq i < j \leq K+1$. Then, $(q_{i,0}(x), q_{i,0}(t_i)) \in \beta$ and $(q_{j,0}(t_j), q_{j,0}(y)) \in \beta$ and, since $i < j$, there is a sequence of tie points $c_i^i, d_{i+1}^{i+1}, c_{i+1}^{i+1}, d_{i+2}^{i+2}, c_{i+2}^{i+2}, \dots, c_j^j$ (where $\{c, d\} = \{a, b\}$) such that

$$t_i \beta^{\mathbf{B}_i} c_i^i = d_{i+1}^{i+1} \beta^{\mathbf{B}_{i+1}} c_{i+1}^{i+1} = d_{i+2}^{i+2} \beta^{\mathbf{B}_{i+2}} c_{i+2}^{i+2} = \dots = c_j^j \beta^{\mathbf{B}_j} t_j. \quad (7.3.10)$$

We could sketch a diagram similar to the one given in the proof of Theorem 7.3.6, but it should be obvious by now that the relations (7.3.10) imply $(t_i, t_j) \in \eta$. Therefore, $\beta^* = \text{Cg}^{\mathbf{A}}(\beta)$.

Next we prove equation (7.3.9). Let $\tilde{\beta}$ denote the right hand side of (7.3.9). We first show $\tilde{\beta} \in \text{Con } \mathbf{A}$.

Let

$$\mathcal{C}_i^{\mathcal{E}} := \bigcup_{\ell \in \mathcal{E}} c_i^\ell / \beta^{\mathbf{B}_\ell} \quad \text{and} \quad \mathcal{C}_i^{\mathcal{O}} := \bigcup_{\ell \in \mathcal{O}} c_i^\ell / \beta^{\mathbf{B}_\ell}.$$

Note that $\mathcal{C}_i^{\mathcal{E}}$ is the join of the corresponding (i -th) β blocks in the up-pointing sets in Figure 7.15. Thus, $\mathcal{C}_i^{\mathcal{E}}$ can be visualized as a single slice through all of the up-pointing sets. Similarly, $\mathcal{C}_i^{\mathcal{O}}$ is the join of corresponding blocks in the down-pointing sets in Figure 7.15. If $0 < i < K$ and $\ell \in \mathcal{E}$, then $e_{\ell+i}(\mathcal{C}_i^{\mathcal{E}}) = e_{\ell+i}(\mathcal{C}_i^{\mathcal{O}}) = \{a_i^{\ell+i}, b_i^{\ell+i}\}$. Thus, for each such $k = \ell + i$ we have

$$q_{k0}(\mathcal{C}_i^{\mathcal{E}}) = S_{k0} e_k(\mathcal{C}_i^{\mathcal{E}}) = S_{k0} e_k(\mathcal{C}_i^{\mathcal{O}}) = q_{k0}(\mathcal{C}_i^{\mathcal{O}}) = \{a_i, b_i\},$$

a single block of β . Similarly, if $0 < i < K$ and $\ell \in \mathcal{O}$, then $e_{\ell+i}(\mathcal{C}_i^{\mathcal{E}}) = e_{\ell+i}(\mathcal{C}_i^{\mathcal{O}}) = \{a_{K-i}^{\ell+i}, b_{K-i}^{\ell+i}\}$. Thus, for each such $k = \ell + i$ we have

$$q_{k0}(\mathcal{C}_i^{\mathcal{E}}) = S_{k0} e_k(\mathcal{C}_i^{\mathcal{E}}) = S_{k0} e_k(\mathcal{C}_i^{\mathcal{O}}) = q_{k0}(\mathcal{C}_i^{\mathcal{O}}) = \{a_{K-i}, b_{K-i}\},$$

which is also a single block of β . It follows that $q_{k0}(\tilde{\beta}) \subseteq \tilde{\beta}$ for all $k \notin \mathcal{E} \cup \mathcal{O}$. If $k \in \mathcal{E}$, then $e_k(\mathcal{C}_i^{\mathcal{E}}) = c_i^k / \beta^{\mathbf{B}_k}$ and $e_k(\mathcal{C}_i^{\mathcal{O}}) = a_1^k$, so $q_{k0}(\mathcal{C}_i^{\mathcal{E}}) = c_i / \beta$ and $q_{k0}(\mathcal{C}_i^{\mathcal{O}}) = a_1$. Thus, $q_{k0}(\tilde{\beta}) \subseteq \tilde{\beta}$. If $k \in \mathcal{O}$, then $e_k(\mathcal{C}_i^{\mathcal{E}}) = b_{K-1}^k$ and $e_k(\mathcal{C}_i^{\mathcal{O}}) = c_i^k / \beta^{\mathbf{B}_k}$, so $q_{k0}(\mathcal{C}_i^{\mathcal{E}}) = b_{K-1}$ and $q_{k0}(\mathcal{C}_i^{\mathcal{O}}) = c_i / \beta$. Thus, $q_{k0}(\tilde{\beta}) \subseteq \tilde{\beta}$. Finally, $e_0(\mathcal{C}_i^{\mathcal{E}}) = c_i / \beta$ and $e_0(\mathcal{C}_i^{\mathcal{O}}) = a_1$, so, for each $f \in F_B$, the operation $f e_0$ takes all of $\mathcal{C}_i^{\mathcal{E}}$ to a single β class, and all of $\mathcal{C}_i^{\mathcal{O}}$ to a single beta class. That is, $f e_0(\tilde{\beta}) \subseteq \tilde{\beta}$ for all $f \in F_B$.

This completes the proof that $f(\tilde{\beta}) \subseteq \tilde{\beta}$ for all $f \in F_A$.

Since the restriction of $\tilde{\beta}$ to B is clearly $\tilde{\beta}|_B = \beta$, the residuation lemma yields $\tilde{\beta} \leq \hat{\beta}$, and we now prove $\tilde{\beta} \geq \hat{\beta}$. Indeed, it is easy to see that, for each $(x, y) \notin \tilde{\beta}$, there is an operation $f \in \text{Pol}_1(\mathbf{A})$ such that $(e_0 f(x), e_0 f(y)) \notin \beta$, and thus $(x, y) \notin \hat{\beta}$. Verification of this statement is trivial. For example, if $x \in c_i^\ell / \beta^{\mathbf{B}^\ell}$ for some $1 \leq i < m$, $\ell \in \mathcal{E}$ and $y \notin \mathcal{C}_i^\mathcal{E}$, then $e_0(x) \in c_i / \beta$ and $e_0(y) \notin c_i / \beta$, so $(e_0(x), e_0(y)) \notin \beta$. To take a slightly less trivial case, suppose $x \in c_i^\ell / \beta^{\mathbf{B}^\ell}$ for some $1 \leq i < m$, $\ell \in \mathcal{O}$ and $y \notin \mathcal{C}_i^\mathcal{O}$. Then $(e_\ell(x), e_\ell(y)) \notin \beta^{\mathbf{B}^\ell}$, so $(e_0 q_{\ell 0}(x), e_0 q_{\ell 0}(y)) = (q_{\ell 0}(x), q_{\ell 0}(y)) \notin \beta$. The few remaining cases are even easier to verify, so we omit them. This completes the proof of (7.3.9).

It remains to prove $[\beta^*, \hat{\beta}] \cong (\text{Eq}|\mathcal{E}|)^{m-1} \times (\text{Eq}|\mathcal{O}|)^{m-1}$. This follows trivially from what we have proved above. For, in proving that $\tilde{\beta}$ is a congruence, we showed that, in fact, each operation $f \in F_A$ maps blocks of $\tilde{\beta} (= \hat{\beta})$ into blocks of β^* . That is, each operation collapses the interval $[\beta^*, \hat{\beta}]$. Therefore, every equivalence relation on the set A that lies between β^* and $\hat{\beta}$ is respected by every operation of \mathbf{A} . In other words,

$$[\beta^*, \hat{\beta}] = \{\theta \in \text{Eq}(A) : \beta^* \leq \theta \leq \hat{\beta}\}.$$

In view of the configuration of the universe of \mathbf{A} , as shown in Figure 7.15, it is clear that the interval sublattice $\{\theta \in \text{Eq}(A) : \beta^* \leq \theta \leq \hat{\beta}\}$ is isomorphic to $(\text{Eq}|\mathcal{E}|)^{m-1} \times (\text{Eq}|\mathcal{O}|)^{m-1}$. \square

7.4 Conclusions

We have described an approach to building new finite algebras out of old which is useful in the following situation: given an algebra \mathbf{B} with a congruence lattice $\text{Con } \mathbf{B}$ of a particular shape, we seek an algebra \mathbf{A} with congruence lattice $\text{Con } \mathbf{A}$ which has $\text{Con } \mathbf{B}$ as a (non-trivial) homomorphic image; specifically, we construct \mathbf{A} so that $|_B : \text{Con } \mathbf{A} \rightarrow \text{Con } \mathbf{B}$ is a lattice epimorphism. We described the original example – the “triple-winged pentagon” shown on the right of Figure 7.1 – found by Ralph Freese, which motivated us to develop a general procedure for finding such finite algebraic representations.

We mainly focused on a few specific overalgebra constructions. In each case, the congruence lattice that results has the same basic shape as the one with which we started, except that some congruences are replaced with intervals that are direct products of powers of partition lattices. Thus we have identified a broad new class of finitely representable lattices. However, the fact that the

new intervals in these lattices must be products of partition lattices seems quite limiting, and this is the first limitation that we think future research might aim to overcome.

We envision potential variations on the constructions described herein, which might bring us closer toward the goal of replacing certain congruences $\beta \in \text{Con } \mathbf{B}$ with an more general finite lattices, $L \cong [\beta^*, \widehat{\beta}] \leq \text{Con } \mathbf{A}$. Using the constructions described above, we have found examples of overalgebras for which it is not possible to simply add operations in order to eliminate *all* relations strictly contained in the interval $(\beta^*, \widehat{\beta})$. Nonetheless, we remain encouraged by the success of a very modest example in this direction, which we now describe.

Example 7.4.1. Suppose $\langle C, \dots \rangle$ is an arbitrary finite algebra with congruence lattice $L_C := \text{Con } \langle C, \dots \rangle$. Relabel the elements so that $C = \{1, 2, \dots, N\}$. We show how to use the overalgebra construction described in Section 7.3.1 to obtain a finite algebra with congruence lattice appearing in Figure 7.16.¹²

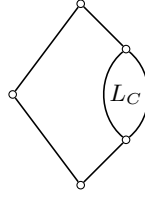


Figure 7.16: L_C an arbitrary finitely representable lattice.

Let $\mathbf{B} = \langle B, F_B \rangle$ be a unary algebra with universe

$$B = \{a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N\},$$

and congruence lattice $\text{Con } \mathbf{B} = \{0_B, \alpha, \beta, 1_B\} \cong \mathbf{2} \times \mathbf{2}$, where

$$\alpha = |a_1, b_1|a_2, b_2| \cdots |a_N, b_N| \quad \text{and} \quad \beta = |a_1, a_2, \dots, a_N|b_1, b_2, \dots, b_N|.$$

Such an algebra exists by the theorem of Berman[5], and Quackenbush and Wolk [36]. Let B_1, B_2, \dots, B_N be sets of size $2N$ which intersect B as follows: for all $1 \leq i < j \leq N$,

$$B_0 \cap B_i = \{b_i\}, \quad \text{and} \quad B_i \cap B_j = \emptyset.$$

¹²John Snow has already proved that “parallel sums” of finitely representable lattices are finitely representable (See Lemmas 3.9 and 3.10 of [43]).

If $\mathbf{A} = \langle A, F_A \rangle$ is the overalgebra constructed as in Section 7.3.1, then $\text{Con } \mathbf{A}$ is isomorphic to the lattice in Figure 7.16, but with L_C replaced with $\text{Eq}(C)$. Now expand the set F_A of operations on A as follows: for each $f \in F_C$, define $f_0 : B \rightarrow B$ by $f_0(a_i) = a_{f(i)}$ and $f_0(b_i) = b_{f(i)}$, and define $\hat{f} : A \rightarrow A$ by $\hat{f}(x) = f_0(s(x))$. Defining $F_A^+ = F_A \cup \{\hat{f} : f \in F_C\}$, we claim that the congruence lattice of the algebra $\langle A, F_A^+ \rangle$ is (isomorphic to) the lattice appearing in Figure 7.16.

As a final remark, we call attention to another obvious limitation of the methods describe in this chapter – they cannot be used to find an algebra with congruence lattice isomorphic to the lattice L_7 , which is the subject of Section 6.3. This lattice is simple, so it is certainly not the inverse image under $|_B$ of some smaller lattice.

CHAPTER 8

OPEN QUESTIONS

We conclude this thesis by listing some open questions, the answers to which will help us better understand finite algebras in general and finite groups in particular. It is the author's view that such progress will undoubtedly lead to a solution to the FLRP in the very near future.

Let $\mathbf{H}(\mathcal{K})$ denote the class of homomorphic images of a class \mathcal{K} of algebras. Let \mathcal{L}_3 denote the class of *representable* lattices; that is, $L \in \mathcal{L}_3$ if and only if $L \cong \text{Con } \mathbf{A}$ for some finite algebra \mathbf{A} . Let \mathcal{L}_4 denote the class of *group representable* lattices; that is, $L \in \mathcal{L}_4$ iff $L \cong [H, G]$ for some finite groups $H \leq G$. As we know, $\mathcal{L}_3 \supseteq \mathcal{L}_4$.

1. Is \mathcal{L}_4 closed under homomorphic images, $\mathbf{H}(\mathcal{L}_4) = \mathcal{L}_4$?
2. Is $\mathbf{H}(\mathcal{L}_4) \subseteq \mathcal{L}_3$ true?
3. Is $\mathbf{H}(\mathcal{L}_3) = \mathcal{L}_3$ true?
4. Is $\mathcal{L}_3 = \mathcal{L}_4$ true? In other words, if L is the congruence lattice of a finite algebra, is L (isomorphic to) the congruence lattice of a transitive G-set? Equivalently, is every congruence lattice of a finite algebra (isomorphic to) an interval in the subgroup lattice of a finite group?
5. Suppose $L \in \mathcal{L}_4$. It is true that, $L_0 = \{x \in L : x \leq \alpha \text{ or } \beta \leq x\} \in \mathcal{L}_4$ for all $\alpha, \beta \in L$? Note that, by the result of John Snow (Lemma 2.3.1) this is true if we replace \mathcal{L}_4 with \mathcal{L}_3 .
6. What other properties of groups, in addition to those described in Chapter 5, are interval sublattice enforceable (ISLE) properties?
7. If a group property is ISLE, is it true that the negation of that property cannot be ISLE? (This is Conjecture 5.1.)
8. Is the lattice M_7 the congruence lattice of an algebra of cardinality less than $30!/10$?
(In [14], Walter Feit finds $M_7 \cong [H, A_{31}]$, where $|H| = 31 \cdot 5$, so M_7 is the congruence lattice of a transitive G -set on $|A_{31} : H| = 30!/10$ elements.)
9. Is there a general characterization of the class of finite lattices that occur as congruence lattices of overalgebras? As we pointed out in Section 7.4.1, a simple lattice is not the congruence lattice of a (non-trivial) expansion of the type described in Chapter 7. Are there other such

properties, besides simplicity, describing lattices that cannot be the congruence lattice of an overalgebra?

10. Is the seven element lattice L_{11} group representable?

(Recall, we proved that L_{11} is representable in Section 6.2 using the filter+ideal method which necessarily results in a non-permutational algebra.)

11. Is every lattice with at most seven elements group representable?

(In Section 6.2 we described the seven element lattices which are the most challenging to represent. These appear in Figure 7.1. We saw that both L_{13} and L_{17} are group representable. Though we did not mention it above, we have also found the lattice L_9 (which motivated the invention of overalgebras) as an interval in the subgroup lattice of A_{10} . At the bottom of this interval is a subgroup of index 25,400. So the smallest G -set we have found with congruence lattice isomorphic to L_9 is on 25,400 elements. Clearly this is not the minimal representation of L_9 . Indeed, in Example 7.3.1 we constructed an overalgebra with 16 elements that has a congruence lattice isomorphic to L_9 . We suspect it will not be very difficult to prove that the lattices L_{19} and L_{20} are group representable. Of the lattices appearing in Figure 7.1 then, L_7 may not be representable, and L_{11} , though representable, seems difficult to find as an interval in a subgroup lattice of a finite group.)

Part III

Appendix

APPENDIX A

GROUP THEORY BACKGROUND

In this section we review some aspects of group theory that are relevant to our problem of representing a finite lattice as the congruence lattice of a finite algebra.

A.1 Group actions and permutation groups

Let G be a group, $\mathbf{A} = \langle A, \bar{G} \rangle$ a G -set, and let $\text{Sym}(A)$ denote the group of permutations of A . For $a \in A$, the one-generated subalgebra $\langle a \rangle \in \text{Sub}(\mathbf{A})$ is called the *orbit* of a in \mathbf{A} . It is easily verified that $\langle a \rangle$ is the set $\bar{G}a := \{\bar{g}a \mid g \in G\}$, and we often use the more suggestive $\bar{G}a$ when referring to this orbit.

The orbits of the G -set \mathbf{A} partition the set A into disjoint equivalence classes. The equivalence relation \sim is defined on A^2 as follows: $x \sim y$ if and only if $\bar{g}x = y$ for some $g \in G$. In fact, \sim is a congruence relation of the algebra \mathbf{A} since, $x \sim y$ implies $\bar{g}x \sim \bar{g}y$. Thus, as mentioned above, each orbit is indeed a *subalgebra* of \mathbf{A} .

Keep in mind that A is the disjoint union of the orbits. That is, if $\{a_1, \dots, a_r\}$ is a full set of \sim -class representatives, then $A = \bigcup_{i=1}^r \bar{G}a_i$ is a disjoint union.

A G -set with only one orbit is called *transitive*. Equivalently, $\langle A, \bar{G} \rangle$ is a transitive G -set if and only if $(\forall a, b \in A)(\exists g \in G)(\bar{g}a = b)$. In this case, we say that G *acts transitively* on A , and occasionally we refer to the group G itself as a *transitive group* of *degree* $|A|$.

For $a \in A$, the set $\text{Stab}_G(a) := \{g \in G \mid \bar{g}a = a\}$ is called the *stabilizer* of a . It is easy to verify that $\text{Stab}_G(a)$ is a subgroup of G . An alternative notation for the stabilizer is $G_a := \text{Stab}_G(a)$.

Let $\lambda : G \rightarrow \bar{G} \leq \text{Sym}(A)$ denote the permutation representation of G ; that is, $\lambda(g) = \bar{g}$. Then

$$\ker \lambda = \{g \in G \mid \bar{g}a = a \text{ for all } a \in A\} = \bigcap_{a \in A} \text{Stab}_G(a) = \bigcap_{a \in A} G_a. \quad (\text{A.1.1})$$

Therefore, $G/\ker \lambda \cong \lambda[G] \leq \text{Sym}(A)$. We say that the representation λ of G is *faithful*, or that G *acts faithfully* on A , just in case $\ker \lambda = 1$. In this case $\lambda : G \hookrightarrow \text{Sym}(A)$, so G itself is isomorphic to a subgroup of $\text{Sym}(A)$, and we call G a *permutation group*.

If $H \leq G$ are groups, the *core* of H in G , denoted $\text{core}_G(H)$, is the largest normal subgroup of

G that is contained in H . It is easy to see that

$$\text{core}_G(H) = \bigcap_{g \in G} gHg^{-1}.$$

A subgroup H is called *core-free* provided $\text{core}_G(H) = 1$.

Elements in the same orbit of a G -set have conjugate stabilizers. Specifically, if $a, b \in A$ and $g \in G$ are such that $\bar{g}a = b$, then $G_b = G_{\bar{g}a} = gG_ag^{-1}$. If the G -set happens to be transitive, then it is faithful if and only if the stabilizer G_a is core-free in G . For,

$$\ker \lambda = \bigcap_{a \in A} G_a = \bigcap_{g \in G} G_{\bar{g}a} = \bigcap_{g \in G} gG_ag^{-1}.$$

Thus G_a is core-free if and only if $\ker \lambda = 1$ if and only if G acts faithfully on A .

In case G is a transitive permutation group, we say that G is *regular* (or that G *acts regularly* on A , or that $\lambda : G \rightarrow \bar{G}$ is a *regular representation*) provided $G_a = 1$ for each $a \in A$; i.e., every non-identity element of G is fixed-point-free.¹ Equivalently, G is regular on A if and only if for each $a, b \in A$ there is a unique $g \in G$ such that $\bar{g}a = b$. In particular, $|G| = |A|$.

A *block system* for G is a partition of A that is preserved by the action of G . In other words, a block system is a congruence relation of the algebra $\mathbf{A} = \langle A, \bar{G} \rangle$. The *trivial block systems* are $0_A = |a_1|a_2|\cdots|a_i|\cdots$ and $1_A = |a_1a_2\cdots a_i\cdots|$. The non-trivial block systems are called *systems of imprimitivity*.

A nonempty subset $B \subseteq A$ is a *block* for \mathbf{A} if for each $g \in G$ either $\bar{g}B = B$ or $\bar{g}B \cap B = \emptyset$.

Let $\mathbf{A} = \langle A, \bar{G} \rangle$ be a transitive G -set. In most group theory textbooks one finds the following definition: a group G is called *primitive* if \mathbf{A} has no systems of imprimitivity; otherwise G is called *imprimitive*. In other words, G is primitive if and only if the transitive G -set $\langle A, \bar{G} \rangle$ is a *simple algebra* – that is, $\text{Con} \langle A, \bar{G} \rangle \cong \mathbf{2}$. In the author's view, this definition of primitive is meaningless and is the source of unnecessary confusion. Clearly *every* finite group acts transitively on the cosets of a maximal subgroup H and the resulting G -set has $\text{Con} \langle G/H, \bar{G} \rangle \cong [H, G] \cong \mathbf{2}$. This means that, according to the usual definition, every finite group is primitive. To make the definition more meaningful, we should require that a primitive group be isomorphic to a permutation group. That is, we call a transitive permutation group *primitive* if the induced algebra is simple. To see the distinction, take an arbitrary group G acting on the cosets of a subgroup H . This action is faithful,

¹The action of a regular permutation group is sometimes called a “free” action.

and G is a permutation group, if and only if H is core-free. If, in addition, H is a maximal subgroup, then the induced algebra $\langle G/H, \bar{G} \rangle$ is simple. For these reasons, we will call a group *primitive* if and only if it has a core-free maximal subgroup. (Note that the terms “primitive” and “imprimitive” are used only with reference to *transitive* G -sets.)

A.2 Classifying permutation groups

A permutation group is either transitive or is a subdirect product of transitive groups, while a transitive group is either primitive or is a subgroup of an iterated wreath product of primitive groups. (See, e.g., Praeger [33].) Hence primitive groups can be viewed as the building blocks of all permutations groups and their classification helps us to better understand the structure of permutation groups in general.

The *socle* of a group G is the subgroup generated by the minimal normal subgroups of G and is denoted by $\text{Soc}(G)$. By [12], Corollary 4.3B, the socle of a finite primitive group is isomorphic to the direct product of one or more copies of a simple group T . The O’Nan-Scott Theorem classifies the primitive permutation groups according to the structure of their socles. The following version of the theorem seems to be among the most useful, and it appears for example in the Ph.D. thesis of Hannah Coutts [9].

A.2.1 The O’Nan-Scott Theorem

Theorem A.2.1 (O’Nan-Scott Theorem). *Let G be a primitive permutation group of degree d , and let $N := \text{Soc}(G) \cong T^m$ with $m \geq 1$. Then one of the following holds.*

1. N is regular and
 - (a) *Affine type* T is cyclic of order p , so $|N| = p^m$. Then $d = p^m$ and G is permutation isomorphic to a subgroup of the affine general linear group $\text{AGL}(m, p)$. We call G a group of *affine type*.
 - (b) *Twisted wreath product type* $m \geq 6$, the group T is nonabelian and G is a group of *twisted wreath product type*, with $d = |T|^m$.
2. N is non-regular and non-abelian and
 - (a) *Almost simple* $m = 1$ and $T \leq G \leq \text{Aut}(T)$.

- (b) *Product action* $m \geq 2$ and G is permutation isomorphic to a subgroup of the product action wreath product $P \wr S_{m/l}$ of degree $d = nm/l$. The group P is primitive of type 2.(a) or 2.(c), P has degree n and $\text{Soc}(P) \cong T^l$, where $l \geq 1$ divides m .
- (c) *Diagonal type* $m \geq 2$ and $T^m \leq G \leq T^m \cdot (\text{Out}(T) \times S_m)$, with the diagonal action. The degree $d = |T|^{m-1}$.

We can see immediately that there are no twisted wreath product type groups of degree less than 60^6 ($= 46.656$ billion). Note that this definition of product action groups is more restrictive than that given by some authors. This is in order to make the O’Nan-Scott classes disjoint.

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