UNIVERSAL ALGEBRAIC METHODS FOR CONSTRAINT SATISFACTION PROBLEMS
with applications to commutative idempotent binars

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After substantial progress over the last 15 years, the "algebraic CSP-dichotomy conjecture" reduces to the following: every local constraint satisfaction problem (CSP) associated with a finite idempotent algebra is tractable if and only if the algebra has a Taylor term operation. Despite the tremendous achievements in this area, there remain examples of small algebras with just a single binary operation whose CSP resists classification as either tractable or NP-complete using known methods. In this paper we present some new methods for approaching this problem, with particular focus on those techniques that help us attack the class of finite algebras known as "commutative idempotent binars" (CIBs). We demonstrate the utility of these methods by using them to prove that every CIB of cardinality at most 4 yields a tractable CSP.

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1. Introduction

The "CSP-dichotomy conjecture" of Feder and Vardi [11] asserts that every constraint satisfaction problem (CSP) over a fixed finite constraint language is either NP-complete or tractable.

A discovery of Jeavons, Cohen and Gyssens in [18]—later refined by Bulatov, Jeavons and Krokhin in [9]—was the ability to transfer the question of the complexity of the CSP over a set of relations to a question of algebra. Specifically, these authors showed that the complexity of any particular CSP depends solely on the *polymorphisms* of the constraint relations, that is, the functions preserving all the constraints. The transfer to universal algebra was made complete by Bulatov, Jeavons, and Krokhin in recognizing that to any set $\mathcal{R}$ of constraint relations one can associate an algebra $\mathbf{A}(\mathcal{R})$ whose operations consist of the polymorphisms of $\mathcal{R}$. Following this, the CSP-dichotomy conjecture of Feder and Vardi was recast as a universal algebra problem once it was recognized that the conjectured sharp dividing line between those CSPs that are NP-complete and those that are tractable was seen to depend upon universal algebraic properties of the associated algebra. One such property is the existence of a Taylor term (defined in Section 2.2.2). Roughly speaking, the "algebraic CSP-dichotomy conjecture" is the following: The CSP associated with a finite idempotent algebra is tractable if and only if the algebra has a Taylor term operation in its clone. We state this more precisely as follows:

Conjecture 1.1. *If* $\mathbf{A}$ *is an algebra with a Taylor term it its clone and if* $\mathcal{R}$ *is a finite set of relations compatible with* $\mathbf{A}$*, then* $\operatorname{CSP}(\mathcal{R})$ *is tractable. Conversely, if* $\mathbf{A}$ *is an idempotent algebra with no Taylor term in its clone, then there exists a finite set* $\mathcal{R}$ *of relations such that* $\operatorname{CSP}(\mathcal{R})$ *is NP-complete.*

The second sentence of the conjecture was already established in [9]. One goal of this paper is to provide further evidence in support of the first sentence.

Algebraists have identified two quite different techniques for proving that a finite idempotent algebra is tractable. One, often called the "local consistency algorithm," works for any finite algebra lying in an idempotent, congruence-meet-semidistributive variety (SD-$\wedge$ for short). See [2] or [4] for details. The other, informally called the "few subpowers technique," applies to any finite idempotent algebra possessing an edge term [17]. Definitions of these terms appear in Section 6.3.

While these two algorithms cover a wide class of interesting algebras, they are not enough to resolve the above conjecture. So the question is, what is the way forward from here? One might hope for an entirely new algorithm, but none seem to be on the horizon. Alternately, we can try to combine the two existing approaches in a way that captures the outstanding cases. Several researchers have investigated this idea. In this paper, we do as well.

For example, suppose that $\mathbf{A}$ is a finite, idempotent algebra possessing a congruence, θ, such that $\mathbf{A}/\theta$ lies in an SD-$\wedge$ variety and every congruence class of θ has an edge term. It is not hard to show that $\mathbf{A}$ has a Taylor term. Can local consistency be combined with few subpowers to prove that $\mathbf{A}$ is tractable?

We can formalize this idea as follows. Let $\mathcal{V}$ and $\mathcal{W}$ be idempotent varieties. The *Mal'tsev product* of $\mathcal{V}$ and $\mathcal{W}$ is the class

$$\mathcal{V} \circ \mathcal{W} = \{\, \mathbf{A} : (\exists\, \theta \in \operatorname{Con}(\mathbf{A}))\ \ \mathbf{A}/\theta \in \mathcal{W} \ \&\ (\forall a \in A)\ a/\theta \in \mathcal{V} \,\}.$$

$\mathcal{V} \circ \mathcal{W}$ is always an idempotent quasivariety, but is generally not closed under homomorphic images. A long term goal would be to prove that if all finite members of $\mathcal{V}$ and $\mathcal{W}$ are tractable, then the same will hold for all members of $\mathcal{V} \circ \mathcal{W}$. Tellingly, Freese and McKenzie show in [13] that a number of important properties are preserved by Mal'tsev product.

Theorem 1.2. *Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be idempotent varieties. For each of the following properties, P, if both* $\mathcal{V}$ *and* $\mathcal{W}$ *have P, then so does* $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$*.*

(1) Being idempotent;
(2) having a Taylor term;
(3) being SD-$\wedge$*;*
(4) having an edge term.

It follows from the theorem that if both $\mathcal{V}$ and $\mathcal{W}$ are SD-$\wedge$, or both have an edge term, then every finite member of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ will be tractable. So the next step would be to consider $\mathcal{V}$ with one of these properties and $\mathcal{W}$ with the other.

But even if we could prove that tractability is preserved by Mal'tsev product, that would not be enough. In particular, simple algebras would remain problematic. Interestingly, Barto and Kozik have recently developed an approach that seems particularly well-suited to simple algebras. In LICS'10 ([3]), these researchers proved a powerful "Absorption Theorem" for products of two absorption-free algebras in a Taylor variety. At a more recent workshop [1], Barto announced further joint work with Kozik on a general "Rectangularity Theorem" that says, roughly, a subdirect product of simple nonabelian algebras contains a full product of minimal absorbing subalgebras

In Section 5 of the present paper we state and prove a version of Barto and Kozik's Rectangularity Theorem. In Section 8, we apply this tool, together with some techniques involving Mal'tsev products and other new decomposition strategies, to prove that every commutative, idempotent binar of cardinality at most 4 is tractable.

2. Definitions and Notations

2.1. *Notation for projections, scopes, and kernels*

Here we give a somewhat informal description of some definitions and notations we use in the sequel. Some of these will be reintroduced more carefully later on, as needed.

An operation $f\colon A^n \to A$ is called *idempotent* provided $f(a, a, \dots, a) = a$ for all $a \in A$. Examples of idempotent operations are the projection functions and these will play an important role in later sections, so we start by introducing a sufficiently general and flexible notation for them.

We define the natural numbers as usual and denote them as follows:

$$\underline{0} := \emptyset, \quad \underline{1} := \{0\}, \quad \underline{2} := \{0, 1\}, \quad \dots \quad \underline{n} := \{0, 1, 2, \dots, n-1\}, \quad \dots$$

Given sets $A_0, A_1, \dots, A_{n-1}$, their Cartesian product is $\prod_{\underline{n}} A_i := A_0 \times \cdots \times A_{n-1}$. An element $\mathbf{a} \in \prod_{\underline{n}} A_i$ is an ordered n-tuple, which may be specified by simply listing its values, as in $\mathbf{a} = (\mathbf{a}(0), \mathbf{a}(1), \dots, \mathbf{a}(n-1))$. Thus, tuples are functions defined on a (finite) index set, and this view may be emphasized symbolically as follows:

$$\mathbf{a}\colon \underline{n} \to \bigcup_{i \in \underline{n}} A_i; \;\; i \mapsto \mathbf{a}(i) \in A_i.$$

If $\sigma\colon \underline{k} \to \underline{n}$ is a k-tuple of numbers in $\underline{n}$, then we can compose an n-tuple $\mathbf{a}$ in $\prod_{i\in\underline{n}} A_i$ with σ yielding the k-tuple $\mathbf{a}\circ\sigma$ in $\prod_{\underline{k}} A_{\sigma(i)}$. Generally speaking, we will try to avoid nonstandard notational conventions, but here are two exceptions: let

$$\underline{A} := \prod_{i\in\underline{n}} A_i \quad \text{and} \quad \underline{A}_\sigma := \prod_{i\in\underline{k}} A_{\sigma(i)}.$$

Now, if the k-tuple $\sigma\colon \underline{k} \to \underline{n}$ happens to be one-to-one, and if we let p_σ denote the map $\mathbf{a} \mapsto \mathbf{a}\circ\sigma$, then p_σ is the usual *projection function* from $\underline{A}$ onto $\underline{A}_\sigma$. Thus, $p_\sigma(\mathbf{a})$ is a k-tuple whose i-th component is $(\mathbf{a}\circ\sigma)(i) = \mathbf{a}(\sigma(i))$. We will make frequent use of such projections, as well as their images under the covariant and contravariant powerset functors $\mathcal{P}$ and $\bar{\mathcal{P}}$. Indeed, we let $\operatorname{Proj}_\sigma\colon \mathcal{P}(\underline{A}) \to \mathcal{P}(\underline{A}_\sigma)$ denote the *projection set function* defined for each $R \subseteq \underline{A}$ by

$$\operatorname{Proj}_\sigma R = \mathcal{P}(p_\sigma)(R) = \{p_\sigma(\mathbf{x}) \mid \mathbf{x}\in R\} = \{\mathbf{x}\circ\sigma \mid \mathbf{x}\in R\},$$

and we let $\operatorname{Inj}_\sigma\colon \bar{\mathcal{P}}(\underline{A}_\sigma) \to \bar{\mathcal{P}}(\underline{A})$ denote the *injection set function* defined for each $S \subseteq \underline{A}_\sigma$ by

$$\operatorname{Inj}_\sigma S = \bar{\mathcal{P}}(p_\sigma)(S) = \{\mathbf{x} \mid p_\sigma(\mathbf{x}) \in S\} = \{\mathbf{x}\in\underline{A} \mid \mathbf{x}\circ\sigma \in S\}. \tag{2.1}$$

Of course, $\operatorname{Inj}_\sigma S$ is nothing more than the inverse image of the set S with respect to the projection function p_σ. We sometimes use the shorthand $R_\sigma = \operatorname{Proj}_\sigma R$ and $S^{\overleftarrow{\sigma}} = \operatorname{Inj}_\sigma S$ for the projection and injection set functions, respectively.

A one-to-one function $\sigma\colon \underline{k} \to \underline{n}$ is sometimes called a *scope* or *scope function.* If $R \subseteq \prod_{j\in\underline{k}} A_{\sigma(j)}$ is a subset of the projection of $\underline{A}$ onto coordinates $(\sigma(0), \sigma(1), \dots, \sigma(k-1))$, then we call R a *relation on* $\underline{A}$ *with scope* σ. The pair (σ, R) is called a *constraint*, and $R^{\overleftarrow{\sigma}}$ is the set of tuples in $\underline{A}$ that *satisfy* (σ, R).

By η_σ we denote the *kernel* of the projection function p_σ, which is the following equivalence relation on $\underline{A}$:

$$\eta_\sigma = \{(\mathbf{a},\mathbf{a}') \in \underline{A}^2 \mid p_\sigma(\mathbf{a}) = p_\sigma(\mathbf{a}')\} = \{(\mathbf{a},\mathbf{a}') \in \underline{A}^2 \mid \mathbf{a}\circ\sigma = \mathbf{a}'\circ\sigma\}. \tag{2.2}$$

More generally, if θ is an equivalence relation on the set $\prod_{j\in\underline{k}} A_{\sigma(j)}$, then we define the equivalence relation θ_σ on the set $\underline{A} = \prod_{\underline{n}} A_i$ as follows:

$$\theta_\sigma = \{(\mathbf{a},\mathbf{a}') \in \underline{A}^2 \mid (\mathbf{a}\circ\sigma)\ \theta\ (\mathbf{a}'\circ\sigma)\}. \tag{2.3}$$

In other words, θ_σ consists of all pairs in $\underline{A}^2$ that land in θ when projected onto the scope σ. Thus, if $0_{\underline{A}}$ denotes the least equivalence relation on $\underline{A}$, then η_σ is shorthand for $(0_{\underline{A}})_\sigma$.

If the domain of σ is a singleton, $\underline{k} = \{0\}$, then of course σ is just a one-element list, say, $\sigma = (j)$. In such cases, we write p_j instead of $p_{(j)}$. Similarly, we write Proj_j instead of $\operatorname{Proj}_{(j)}$, η_j instead of $\eta_{(j)}$, etc. Thus, $p_j(\mathbf{a}) = \mathbf{a}(j)$, and $\eta_j = \{(\mathbf{a},\mathbf{a}') \in \underline{A}^2 \mid \mathbf{a}(j) = \mathbf{a}'(j)\}$, and, if $\theta \in \operatorname{Con}\mathbf{A}_j$, then $\theta_j = \{(\mathbf{a},\mathbf{a}') \in \underline{A}^2 \mid \mathbf{a}(j)\ \theta\ \mathbf{a}'(j)\}$. Here are some obvious consequences of these definitions:

$$\bigvee_{j\in\underline{n}} \eta_j = \underline{A}^2, \qquad \eta_\sigma = \bigwedge_{j\in\sigma} \eta_j, \qquad \eta_{\underline{n}} = \bigwedge_{j\in\underline{n}} \eta_j = 0_{\underline{A}}, \qquad \theta_\sigma = \bigwedge_{j\in\underline{k}} \theta_{\sigma(j)}.$$

2.2. *Notation for algebraic structures*

This section introduces the notation we use for algebras and related concepts. The reader should consult [6] for more details and background on general (universal) algebras and the varieties they inhabit.

2.2.1. *Product algebras*

Fix $n \in \mathbb{N}$, let F be a set of operation symbols, and for each $i \in \underline{n}$ let $\mathbf{A}_i = \langle A_i, F\rangle$ be an algebra of type F. Let $\underline{\mathbf{A}} = \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1} = \prod_{\underline{n}} \mathbf{A}_i$ denote the product algebra. If $k \leqslant n$ and if $\sigma \colon \underline{k} \to \underline{n}$ is a one-to-one function, then we denote by $p_\sigma \colon \underline{\mathbf{A}} \twoheadrightarrow \prod_{\underline{k}} \mathbf{A}_{\sigma(i)}$ the *projection* of $\underline{\mathbf{A}}$ onto the "σ-factors" of $\underline{\mathbf{A}}$, which is an algebra homomorphism; thus the kernel η_σ defined in (2.2) is a congruence relation of $\underline{\mathbf{A}}$.

2.2.2. *Term operations*

For a nonempty set A, we let O_A denote the set of all finitary operations on A. That is, $\mathsf{O}_A = \bigcup_{n\in\mathbb{N}} A^{(A^n)}$. A *clone of operations* on A is a subset of O_A that contains all projection operations and is closed under the (partial) operation of general composition. If $\mathbf{A} = \langle A, F^{\mathbf{A}}\rangle$ denotes the algebra with universe A and set of basic operations F, then $\mathsf{Clo}(\mathbf{A})$ denotes the clone generated by F, which is also known as the *clone of term operations* of $\mathbf{A}$.

Walter Taylor proved in [23] that a variety, $\mathcal{V}$, satisfies some nontrivial idempotent Mal'tsev condition if and only if it satisfies one of the following form: for some n, $\mathcal{V}$ has an idempotent n-ary term t such that for each $i \in \underline{n}$ there is an identity of the form

$$t(*, \cdots, *, x, *, \cdots, *) \approx t(*, \cdots, *, y, *, \cdots, *)$$

true in $\mathcal{V}$ where distinct variables x and y appear in the i-th position on either side of the identity. Such a term t is now commonly called a *Taylor term.*

Throughout this paper we assume all algebras are finite (though some results may apply more generally). Starting in Section 5, we make the additional assumption that the algebras in question come from a single *Taylor variety* $\mathcal{V}$, by which we mean that $\mathcal{V}$ has a Taylor term and every term of $\mathcal{V}$ is idempotent.

2.2.3. *Subdirect products*

If $k, n \in \mathbb{N}$, if $\mathcal{A} = (A_0, A_1, \dots, A_{n-1})$ is a list of sets, and if $\sigma \colon \underline{k} \to \underline{n}$ is a k-tuple, then a relation R over $\mathcal{A}$ with scope σ is a subset of the Cartesian product $A_{\sigma(0)} \times A_{\sigma(1)} \times \cdots \times A_{\sigma(k-1)}$. Let F be a set of operation symbols and for each $i \in \underline{n}$ let $\mathbf{A}_i = \langle A_i, F\rangle$ be an algebra of type F. If $\underline{\mathbf{A}} = \prod_{\underline{n}} \mathbf{A}_i$ is the product of these algebras, then a relation R over $\mathcal{A}$ with scope σ is called *compatible with* $\underline{\mathbf{A}}$ if it is closed under the basic operations in F. In other words, R is compatible if the induced algebra $\mathbf{R} = \langle R, F\rangle$ is a subalgebra of $\prod_{\underline{k}} \mathbf{A}_{\sigma(j)}$. If R is compatible with the product algebra and if the projection of R onto each factor is surjective, then $\mathbf{R}$ is called a *subdirect product* of the algebras in the list $(\mathbf{A}_{\sigma(0)}, \mathbf{A}_{\sigma(1)}, \dots, \mathbf{A}_{\sigma(k-1)})$; we denote this situation by writing $\mathbf{R} \leqslant_{\mathrm{sd}} \prod_{\underline{k}} \mathbf{A}_{\sigma(j)}$.

3. Abelian Algebras

In later sections nonabelian algebras will play the following role: some of the theorems will begin with the assumption that a particular algebra $\mathbf{A}$ is nonabelian and then proceed to show that if the result to be proved were false, then $\mathbf{A}$ would have to be abelian. To prepare the way for such arguments, we review some basic facts about abelian algebras.

3.1. *Definitions*

Let $\mathbf{A} = \langle A, F^{\mathbf{A}}\rangle$ be an algebra. A reflexive, symmetric, compatible binary relation $T \subseteq A^2$ is called a *tolerance of* $\mathbf{A}$. Given a pair $(\mathbf{u}, \mathbf{v}) \in A^m \times A^m$ of m-tuples of A, we write $\mathbf{u}\ \mathbf{T}\ \mathbf{v}$ just in case $\mathbf{u}(i)\ T\ \mathbf{v}(i)$ for all $i \in \underline{m}$. We state a number of definitions in this section using tolerance relations, but the definitions don't change when the tolerance in question happens to be a congruence relation (i.e., a transitive tolerance).

Suppose S and T are tolerances on $\mathbf{A}$. An *S, T-matrix* is a 2×2 array of the form

$$\begin{bmatrix} t(\mathbf{a}, \mathbf{u}) & t(\mathbf{a}, \mathbf{v}) \\ t(\mathbf{b}, \mathbf{u}) & t(\mathbf{b}, \mathbf{v}) \end{bmatrix},$$

where t, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{v}$ have the following properties:

(1) $t \in \mathsf{Clo}_{\ell+m}(\mathbf{A})$,
(2) $(\mathbf{a}, \mathbf{b}) \in A^\ell \times A^\ell$ and $\mathbf{a}\ \mathbf{S}\ \mathbf{b}$,
(3) $(\mathbf{u}, \mathbf{v}) \in A^m \times A^m$ and $\mathbf{u}\ \mathbf{T}\ \mathbf{v}$.

Let δ be a congruence relation of $\mathbf{A}$. If the entries of every S, T-matrix satisfy

$$t(\mathbf{a}, \mathbf{u})\ \delta\ t(\mathbf{a}, \mathbf{v}) \quad \iff \quad t(\mathbf{b}, \mathbf{u})\ \delta\ t(\mathbf{b}, \mathbf{v}), \tag{3.1}$$

then we say that S *centralizes* T *modulo* δ and we write $\mathsf{C}(S, T; \delta)$. That is, $\mathsf{C}(S, T; \delta)$ means that (3.1) holds *for all* ℓ, m, t, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{v}$ satisfying properties (i)–(iii).

The *commutator* of S and T, denoted by $[S, T]$, is the least congruence δ such that $\mathsf{C}(S, T; \delta)$ holds. Note that $\mathsf{C}(S, T; 0_A)$ is equivalent to $[S, T] = 0_A$, and this is sometimes called the *S, T-term condition*; when it holds we say that S *centralizes* T, and write $\mathsf{C}(S, T)$. A tolerance T is called *abelian* if $\mathsf{C}(T, T)$ (i.e., $[T, T] = 0_A$). An algebra $\mathbf{A}$ is called *abelian* if 1_A is abelian (i.e., $\mathsf{C}(1_A, 1_A)$).

3.2. *Facts about centralizers and abelian congruences*

We now collect some useful facts about centralizers of congruence relations that are needed in Section 5. The facts collected in the first lemma are well-known and easy to prove. (For examples, see [16, Prop 3.4] and [19, Thm 2.19].)

Lemma 3.1. *Let* $\mathbf{A}$ *be an algebra and suppose* $\mathbf{B}$ *is a subalgebra of* $\mathbf{A}$. *Let* α, β, γ, δ, α_i β_j, γ_k *be congruences of* $\mathbf{A}$, *for some* $i \in I$, $j \in J$, $k \in K$. *Then the following hold:*

(1) $\mathsf{C}(\alpha, \beta; \alpha \wedge \beta)$;
(2) if $\mathsf{C}(\alpha, \beta; \gamma_k)$ *for all* $k \in K$, *then* $\mathsf{C}(\alpha, \beta; \bigwedge_K \gamma_k)$;
(3) if $\mathsf{C}(\alpha_i, \beta; \gamma)$ *for all* $i \in I$, *then* $\mathsf{C}(\bigvee_I \alpha_i, \beta; \gamma)$;
(4) if $\mathsf{C}(\alpha, \beta; \gamma)$ *and* $\alpha' \leqslant \alpha$, *then* $\mathsf{C}(\alpha', \beta; \gamma)$;
(5) if $\mathsf{C}(\alpha, \beta; \gamma)$ *and* $\beta' \leqslant \beta$, *then* $\mathsf{C}(\alpha, \beta'; \gamma)$;
(6) if $\mathsf{C}(\alpha, \beta; \gamma)$ *in* $\mathbf{A}$, *then* $\mathsf{C}(\alpha \cap B^2, \beta \cap B^2; \gamma \cap B^2)$ *in* $\mathbf{B}$;
(7) if $\gamma \leqslant \delta$, *then* $\mathsf{C}(\alpha, \beta; \delta)$ *in* $\mathbf{A}$ *if and only if* $\mathsf{C}(\alpha/\gamma, \beta/\gamma; \delta/\gamma)$ *in* $\mathbf{A}/\gamma$.

The next two lemmas are essential in a number proofs below. The first lemma identifies special conditions under which certain quotient congruences are abelian. The second gives fairly general conditions under which quotients of abelian congruences are abelian.

Lemma 3.2. *Let* α_0, α_1, β *be congruences of* $\mathbf{A}$ *and suppose* $\alpha_0 \wedge \beta = \delta = \alpha_1 \wedge \beta$. *Then* $\mathsf{C}(\alpha_0 \vee \alpha_1, \beta; \delta)$. *If, in addition,* $\beta \leqslant \alpha_0 \vee \alpha_1$, *then* $\mathsf{C}(\beta, \beta; \delta)$, *so* β/δ *is an abelian congruence of* $\mathbf{A}/\delta$.

Lemma 3.2 is an easy consequence of items (1), (3), (4), and (7) of Lemma 3.1.

Lemma 3.3. *Let* $\mathcal{V}$ *be a locally finite variety with a Taylor term and let* $\mathbf{A} \in \mathcal{V}$. *Then* $\mathsf{C}(\beta, \beta; \gamma)$ *for all* $[\beta, \beta] \leqslant \gamma$.

Lemma 3.3 can be proved by combining the next result, of David Hobby and Ralph McKenzie, with a result of Keith Kearnes and Emil Kiss.

Lemma 3.4 (cf. [16, Thm 7.12]). *A locally finite variety* $\mathcal{V}$ *has a Taylor term if and only if it has a so called weak difference term; that is, a term* $d(x, y, z)$ *satisfying the following conditions for all* $\mathbf{A} \in \mathcal{V}$, *all* $a, b \in A$, *and all* $\beta \in \mathrm{Con}(\mathbf{A})$: $d^{\mathbf{A}}(a, a, b)\ [\beta, \beta]\ b\ [\beta, \beta]\ d^{\mathbf{A}}(b, a, a)$, *where* $\beta = \mathrm{Cg}^{\mathbf{A}}(a, b)$.

Lemma 3.5 ([19, Lem 6.8]). *If* $\mathbf{A}$ *belongs to a variety with a weak difference term and if* β *and* γ *are congruences of* $\mathbf{A}$ *satisfying* $[\beta, \beta] \leqslant \gamma$, *then* $\mathsf{C}(\beta, \beta; \gamma)$.

Remark 3.6. It follows immediately from Lemma 3.3 that in a locally finite Taylor variety, $\mathcal{V}$, quotients of abelian algebras are abelian, so the abelian members of $\mathcal{V}$ form a subvariety. But this can also be derived from Lemma 3.4, since $[\beta, \beta] = 0_A$ implies $d^{\mathbf{A}}$ is a Mal'tsev term operation on the blocks of β, so if $\mathbf{A}$ is abelian—i.e., if $\mathsf{C}(1_A, 1_A; 0_A)$—then Lemma 3.4, implies that $\mathbf{A}$ has a Mal'tsev term operation.

(This will be recorded below in Theorem 6.7.) It then follows that homomorphic images of $\mathbf{A}$ are abelian. (See [6, Cor 7.28] for more details).

4. Absorption Theory

In this section we survey some of the theory related to an important concept called *absorption*, which was invented by Libor Barto and Marcin Kozik. After introducing the concept, we discuss some of the properties that make it so useful. The main results in this section are not new. The only possibly novel contributions are some straightforward observations in Section 4.3 (which have likely been observed by others). Our intention here is merely to collect and present these known results in a way that makes them easy to apply in later sections and future work.

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be a finite algebra in the Taylor variety $\mathcal{V}$ and let $t \in \mathsf{Clo}(\mathbf{A})$ be a k-ary term operation. A subalgebra $\mathbf{B} = \langle B, F^{\mathbf{B}} \rangle \leqslant \mathbf{A}$ is said to be *absorbing in* $\mathbf{A}$ *with respect to* t if for all $1 \leqslant j \leqslant k$ and for all $(b_1, \dots, b_{j-1}, a, b_{j+1}, \dots, b_k) \in B^{j-1} \times A \times B^{k-j}$ we have

$$t^{\mathbf{A}}(b_1, \dots, b_{j-1}, a, b_{j+1}, \dots, b_k) \in B.$$

In other terms, $t^{\mathbf{A}}[B^{j-1} \times A \times B^{k-j}] \subseteq B$, for all $1 \leqslant j \leqslant k$. Here, $t^{\mathbf{A}}[D]$ denotes the set $\{t^{\mathbf{A}}(x) \mid x \in D\}$.

The notation $\mathbf{B} \triangleleft \mathbf{A}$ means that $\mathbf{B}$ is an absorbing subalgebra of $\mathbf{A}$ with respect to some term. If we wish to be explicit about the term, we write $\mathbf{B} \triangleleft_t \mathbf{A}$. We may also call t an "absorbing term" for $\mathbf{B}$ in this case. The notation $\mathbf{B} \triangleleft\!\triangleleft \mathbf{A}$ means that $\mathbf{B}$ is a minimal absorbing subalgebra of $\mathbf{A}$, that is, B is minimal (with respect to set inclusion) among the absorbing subuniverses of $\mathbf{A}$. An algebra is said to be *absorption-free* if it has no proper absorbing subalgebras.

4.1. *Absorption theorem*

In later sections, we will make frequent use of a powerful tool of Libor Barto and Marcin Kozik called the "Absorption Theorem" (Thm 2.3 of [4]). This result concerns the special class of "linked" subdirect products. A subdirect product $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is said to be *linked* if it satisfies the following: for $a, a' \in \operatorname{Proj}_0 R$ there exist elements $c_0, c_2, \dots, c_{2n} \in A_0$ and $c_1, c_3, \dots, c_{2n+1} \in A_1$ such that $c_0 = a$, $c_{2n} = a'$, and for all $0 \leqslant i < n$, $(c_{2i}, c_{2i+1}) \in R$ and $(c_{2i+2}, c_{2i+1}) \in R$. Here is an easily proved fact that provides some equivalent ways to define "linked."

Fact 4.1. Let $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$, let $\tilde{\eta}_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i)$ denote the kernel of the projection of $\mathbf{R}$ onto its i-th coordinate, and let $R^{-1} = \{(y, x) \in A_1 \times A_0 \mid (x, y) \in R\}$. Then the following are equivalent:

(1) $\mathbf{R}$ is linked;
(2) $\tilde{\eta}_0 \vee \tilde{\eta}_1 = 1_R$;
(3) if $a, a' \in \operatorname{Proj}_0 R$, then (a, a') is in the transitive closure of $R \circ R^{-1}$.

Theorem 4.1 (Absorption Theorem [4, Thm 2.3]). *If $\mathcal{V}$ is an idempotent locally finite variety, then the following are equivalent:*

- *$\mathcal{V}$ is a Taylor variety;*
- *if $\mathbf{A}_0, \mathbf{A}_1 \in \mathcal{V}$ are finite absorption-free algebras, and if $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked, then $\mathbf{R} = \mathbf{A}_0 \times \mathbf{A}_1$.*

Before moving on to the next subsection, we need one more definition. If $f\colon A^\ell \to A$ and $g\colon A^m \to A$ are ℓ-ary and m-ary operations on A, then by $f \star g$ we mean the ℓm-ary operation that maps each $\mathbf{a} = (a_{11}, \dots, a_{1m}, a_{21}, \dots, a_{2m}, \dots, a_{\ell m}) \in A^{\ell m}$ to

$$(f \star g)(\mathbf{a}) = f(g(a_{11}, \dots, a_{1m}), g(a_{21}, \dots, a_{2m}), \dots, g(a_{\ell 1}, \dots, a_{\ell m})).$$

4.2. *Properties of absorption*

In this section we prove some easy facts about algebras that hold quite generally. In particular, we don't assume the presence of Taylor terms or other Mal'tsev conditions in this section. However, for the sake of simplicity, we do assume that all algebras are finite and belong to a single idempotent variety, though some of the results of this section hold for infinite algebras as well. Almost all of the results in this section are well known. The only exceptions are mild generalizations of facts that have already appeared in print. We have restated or reworked some of the known results in a way that makes them easier for us to apply.

Lemma 4.2. *Let $\mathbf{A}$ be an idempotent algebra with terms f and g. If $\mathbf{B} \triangleleft \mathbf{A}$ with respect to either f or g, then $\mathbf{B} \triangleleft \mathbf{A}$ with respect to $f \star g$.*

Proof. In this proof, to keep the notation simple, let $ABB\cdots B$ denote the usual Cartesian product $A \times B \times B \times \cdots \times B$, and let $t = f \star g$. We only handle the $j = 1$ case of the definition in Section 4. (The general case is equally simple, but the notation becomes tedious.) That is, we prove the following subset inclusion:

$$\begin{aligned} t[ABB\cdots B] = f\big[g[ABB\cdots B] \times g[BB\cdots B] \times \cdots \\ \cdots \times g[BB\cdots B]\big] \subseteq B. \end{aligned}$$

Suppose first that f is the absorbing term. Since B is a subalgebra, $g[BB\cdots B]$ is contained in B. (This holds for any term.) Hence, $t[ABB\cdots B]$ is contained in $f[ABB\cdots B] \subseteq B$. On the other hand, if g is the absorbing term, then $t[ABB\cdots B] \subseteq f[BB\cdots B] \subseteq B$, again because B is a subalgebra. $\square$

Corollary 4.3. *Fix $p < \omega$ and suppose $\{t_0, t_1, \dots, t_{p-1}\}$ is a set of terms in the variety $\mathcal{V}$. If $\mathbf{A}$ and $\mathbf{B}$ are finite algebras in $\mathcal{V}$ and if $\mathbf{B} \triangleleft_t \mathbf{A}$ for some t in $\{t_0, t_1, \dots, t_{p-1}\}$, then $\mathbf{B} \triangleleft_s \mathbf{A}$, where $s = t_0 \star t_1 \star \cdots \star t_{p-1}$.*

Proof. By induction on p. $\square$

Lemma 4.4 ([4, Prop 2.4]). *Let* $\mathbf{A}$ *be an algebra.*

- *If* $\mathbf{C} \triangleleft \mathbf{B} \triangleleft \mathbf{A}$*, then* $\mathbf{C} \triangleleft \mathbf{A}$*.*
- *If* $\mathbf{B} \triangleleft_f \mathbf{A}$ *and* $\mathbf{C} \triangleleft_g \mathbf{A}$ *and* $B \cap C \neq \emptyset$*, then* $B \cap C$ *is an absorbing subuniverse of* $\mathbf{A}$ *with respect to* $t = f \star g$*.*

Corollary 4.5. *For* $i = 0, 1$*, let* $\mathbf{A}_i$ *and* $\mathbf{B}_i$ *be algebras in the variety* $\mathcal{V}$ *and suppose* $\mathbf{B}_0 \triangleleft_f \mathbf{A}_0$ *and* $\mathbf{B}_1 \triangleleft_g \mathbf{A}_1$*. Then* $\mathbf{B}_0 \times \mathbf{B}_1$ *is absorbing in* $\mathbf{A}_0 \times \mathbf{A}_1$ *with respect to* $f \star g$*.*

Proof. Since $\mathbf{B}_0 \triangleleft_f \mathbf{A}_0$, it follows that $\mathbf{B}_0 \times \mathbf{A}_1 \triangleleft \mathbf{A}_0 \times \mathbf{A}_1$ with respect to the same term f. Similarly $\mathbf{A}_0 \times \mathbf{B}_1 \triangleleft_g \mathbf{A}_0 \times \mathbf{A}_1$. Hence, $\mathbf{B}_0 \times \mathbf{B}_1 = (\mathbf{B}_0 \times \mathbf{A}_1) \cap (\mathbf{A}_0 \times \mathbf{B}_1)$ is absorbing in $\mathbf{A}_0 \times \mathbf{A}_1$ with respect to $f \star g$, by Lemma 4.4. □

Corollary 4.5 generalizes to finite products and *minimal* absorbing subalgebras. We record these observations next. (A proof appears in A.1.)

Lemma 4.6. *Let* $\mathbf{B}_i \leqslant \mathbf{A}_i$ $(0 \leqslant i < n)$ *be algebras in the variety* $\mathcal{V}$*, let* $\mathbf{B} := \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}$*, and let* $\mathbf{A} := \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$*. If* $\mathbf{B}_i \triangleleft_{t_i} \mathbf{A}_i$ *(resp.,* $\mathbf{B}_i \triangleleft\!\triangleleft_{t_i} \mathbf{A}_i$*) for each* $0 \leqslant i < n$*, then* $\mathbf{B} \triangleleft_s \mathbf{A}$ *(resp.,* $\mathbf{B} \triangleleft\!\triangleleft_s \mathbf{A}$*) where* $s := t_0 \star t_1 \star \cdots \star t_{n-1}$*.*

An obvious but important consequence of Lemma 4.6 is that a (finite) product of finite idempotent algebras is absorption-free if each of its factors is absorption-free.

We conclude this section with a few more properties of absorption that will be useful below. The first is a simple observation with a very easy proof.

Lemma 4.7. *Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be subalgebras of a finite idempotent algebra* $\mathbf{A}$*. Suppose* $\mathbf{B} \triangleleft_t \mathbf{A}$*, and suppose* $D = B \cap C \neq \emptyset$*. Then the restriction of* t *to* C *is an absorbing term for* D *in* C*, whence* $D \triangleleft C$*.*

Despite its simplicity, Lemma 4.7 is quite useful for proving that certain algebras are absorption-free. Along with Lemma 4.6, Lemma 4.7 yields the following result (to be applied in Section 5):

Lemma 4.8. *Let* $\mathbf{A}_0$*,* $\mathbf{A}_1$*,* $\ldots$*,* $\mathbf{A}_{n-1}$ *be finite algebras with* $\mathbf{B}_i \triangleleft \mathbf{A}_i$ *for each* i*. Suppose* $\mathbf{R} \leqslant \prod_i \mathbf{A}_i$ *and* $R \cap \prod_i B_i \neq \emptyset$*. Then* $R \cap \prod_i B_i$ *is an absorbing subuniverse of* $\mathbf{R}$*.*

Proof. Let $\mathbf{A} = \prod_i \mathbf{A}_i$ and $\mathbf{B} = \prod_i \mathbf{B}_i$. By Lemma 4.6, $\mathbf{B} \triangleleft_t \mathbf{A}$, so the result follows Lemma 4.7 if we put $C = R$. □

For $0 < j \leqslant k$, we say that a k-ary term operation t of an algebra $\mathbf{A}$ *depends on its j-th argument* provided there exist $a_0, a_1, \ldots, a_{k-1}$ such that $p(x) := t(a_0, \ldots, a_{j-2}, x, a_j, \ldots, a_{k-1})$ is a nonconstant polynomial of $\mathbf{A}$. Let $\mathbf{A}$ be an algebra, and $s \in C \subseteq A$. We call s a *sink for* C provided that for any term $t \in \mathsf{Clo}_k(\mathbf{A})$ and for any $0 < j \leqslant k$, if t depends on its j-th argument, then

$t(c_0, c_1, \dots, c_{j-2}, s, c_j, \dots, c_{k-1}) = s$ for all $c_i \in C$. If C is a subuniverse, then $\{s\}$ is absorbing, in a very strong way. In fact, it is easy to see that if an absorbing subuniverse B intersects nontrivially with a set containing a sink, then B must also contain the sink. We record this as

Lemma 4.9. *If $B \triangleleft A$, if s is a sink for $C \subseteq A$, and if $B \cap C \neq \emptyset$, then $s \in B$.*

Proof of the next lemma appears in A.2 below.

Lemma 4.10. *Let $\mathbf{A}_0$, $\mathbf{A}_1$, $\dots$, $\mathbf{A}_{n-1}$ be finite idempotent algebras of the same type, and suppose $\mathbf{B}_i \triangleleft\!\triangleleft \mathbf{A}_i$ for $0 \leqslant i < n$. Let $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$, and let $R' = R \cap (B_0 \times B_1 \times \cdots \times B_{n-1})$. If $R' \neq \emptyset$, then $\mathbf{R}' \leqslant_{\text{sd}} \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}$.*

For a proof of the next result, see [4, Prop 2.15].

Lemma 4.11 ([4, cf. Prop 2.15]). *Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be finite idempotent algebras of the same type, let $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and assume $\mathbf{R}$ is linked. If $\mathbf{S} \triangleleft \mathbf{R}$, then $\mathbf{S}$ is also linked.*

Finally, we come to a very useful fact about absorption that we exploit heavily in the sections that follow. (This is proved in [5, Lem 4.1], and in Section A.3 below.)

Lemma 4.12 ([5, Lem 4.1]). *Finite idempotent abelian algebras are absorption-free.*

4.3. *Linking is easy, sometimes*

We will apply the Absorption Theorem frequently below, so we pause here to consider one of the hypotheses of the theorem that might seem less familiar to some of our readers. Specifically, one might wonder when we can expect a subdirect product to be *linked*, as is required of $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ in the statement of the Absorption Theorem. Here we consider a few special cases in which this hypothesis comes essentially for free. Most of the proofs in this section, as well as some in later sections, depend on the following elementary observation about subdirect products:

Lemma 4.13. *Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be algebras. Suppose $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and let $\tilde{\eta}_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i)$ denote the kernel of the i-th projection of $\mathbf{R}$ onto $\mathbf{A}_i$.*

(1) If $\mathbf{A}_0$ is simple, then either $\tilde{\eta}_0 \vee \tilde{\eta}_1 = 1_R$ or $\tilde{\eta}_0 \geqslant \tilde{\eta}_1$.
(2) If $\mathbf{A}_0$ and $\mathbf{A}_1$ are both simple, then either $\tilde{\eta}_0 \vee \tilde{\eta}_1 = 1_R$ or $\tilde{\eta}_0 = 0_R = \tilde{\eta}_1$.

Proof. The congruence relation $\tilde{\eta}_0 \in \operatorname{Con} \mathbf{R}$ is maximal, since $\mathbf{R}/\tilde{\eta}_0 \cong \mathbf{A}_0$ and $\mathbf{A}_0$ is simple. Therefore, if $\tilde{\eta}_0 \vee \tilde{\eta}_1 < 1_R$, then $\tilde{\eta}_0 = \tilde{\eta}_0 \vee \tilde{\eta}_1$, so $\tilde{\eta}_1 \leqslant \tilde{\eta}_0$, proving (1). If $\mathbf{A}_1$ is also simple, then the same argument, with the roles of $\tilde{\eta}_0$ and $\tilde{\eta}_1$ reversed, shows that $\tilde{\eta}_0 \leqslant \tilde{\eta}_1$, so $\tilde{\eta}_0 = \tilde{\eta}_1$. Since $\tilde{\eta}_0 \wedge \tilde{\eta}_1 = 0_R$, we see that both projection kernels must be 0_R in this case. $\square$

An immediate corollary is that, in case both factors in the product $\mathbf{A}_0 \times \mathbf{A}_1$ are simple, the linking required in the Absorption Theorem holds under weaker hypotheses.

Corollary 4.14. *Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be simple algebras and suppose $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$. If $\tilde{\eta}_0 \neq \tilde{\eta}_1$ or $\mathbf{A}_0 \ncong \mathbf{A}_1$, then $\mathbf{R}$ is linked.*

Another corollary of 4.13 holds in a locally finite Taylor variety. When one factor of $\mathbf{A}_0 \times \mathbf{A}_1$ is abelian and the other is simple and nonabelian, then we get linking for free. That is, in a locally finite Taylor variety, *every subdirect product of $\mathbf{A}_0 \times \mathbf{A}_1$ is linked*, as we now prove.

Corollary 4.15. *Suppose $\mathbf{A}_0$ and $\mathbf{A}_1$ are algebras in a locally finite Taylor variety. If $\mathbf{A}_0$ is abelian and $\mathbf{A}_1$ is simple and nonabelian, then every subdirect product of $\mathbf{A}_0 \times \mathbf{A}_1$ is linked. Moreover, if $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and if $\mathbf{B}_1 \lhd\!\!\lhd \mathbf{A}_1$, then $\mathbf{R}$ intersects $\mathbf{A}_0 \times \mathbf{B}_1$ nontrivially and this intersection forms a linked subdirect product of $\mathbf{A}_0 \times \mathbf{B}_1$.*

Proof. Suppose $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is not linked. Since $\mathbf{A}_1$ is simple we have $\tilde{\eta}_0 \leqslant \tilde{\eta}_1$ by Lemma 4.13. Therefore, $\tilde{\eta}_0 = \tilde{\eta}_0 \wedge \tilde{\eta}_1 = 0_R$. But then $\mathbf{R} \cong \mathbf{R}/\tilde{\eta}_0 \cong \mathbf{A}_0$ is abelian, while $\mathbf{R}/\tilde{\eta}_1 \cong \mathbf{A}_1$ is nonabelian. This is a contradiction since, by Remark 3.6, $\mathbf{R}/\theta$ is abelian for all $\theta \in \operatorname{Con} \mathbf{R}$. Therefore, $\mathbf{R}$ is linked. For the second part, since $\mathbf{R}$ is a subdirect product, it follows that $R \cap (A_0 \times B_1)$ is nonempty and, by Lemma 4.8, $\mathbf{R} \cap (\mathbf{B}_0 \times \mathbf{A}_1)$ is absorbing in $\mathbf{R}$. Therefore, by Lemma 4.11, the intersection must also be linked. □

We can extend the previous result to multiple abelian factors by collecting them into a single factor. We use the notation $\underline{n} := \{0, 1, \dots, n-1\}$ and $k' := \underline{n} - \{k\}$. For example,

$$0' := \{1, 2, \dots, n-1\} \quad \text{and} \quad \mathbf{R}_{0'} := \operatorname{Proj}_{0'} \mathbf{R}.$$

Corollary 4.16. *Let $\mathbf{A}_0$, $\mathbf{A}_1$, $\dots$, $\mathbf{A}_{n-1}$ be algebras in a locally finite Taylor variety. Suppose $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}$ are abelian. If $\mathbf{R} \leqslant_{\mathrm{sd}} \prod_{\underline{n}} \mathbf{A}_i$, then $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{R}_{0'}$ is linked.*

Proof. Suppose $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{R}_{0'}$ is not linked: $\tilde{\eta}_0 \vee \tilde{\eta}_{0'} < 1_R$. Since $\mathbf{A}_0$ is simple, $\tilde{\eta}_0$ is a coatom, so $\tilde{\eta}_0 \geqslant \tilde{\eta}_{0'}$. Therefore, $\tilde{\eta}_{0'} = \tilde{\eta}_0 \wedge \tilde{\eta}_{0'} = 0_R$, so $\mathbf{R} \cong \mathbf{R}/\tilde{\eta}_{0'} \cong \mathbf{R}_{0'} \leqslant \prod_{0'} \mathbf{A}_i$. This proves that $\mathbf{R}$ is abelian, and yet the quotient $\mathbf{R}/\tilde{\eta}_0 \cong \mathbf{A}_0$ is nonabelian, which contradicts Remark 3.6. □

Suppose we add to the respective contexts of the last three results the hypothesis that the algebras live in an idempotent variety with a Taylor term. As mentioned above, we refer to such varieties as "Taylor varieties" and we call the algebras they contain “Taylor algebras.” In this context, we can apply the Absorption Theorem (in combination with other results from above) to deduce the following:

Lemma 4.17. *Let* $\mathbf{A}_0$ *and* $\mathbf{A}_1$ *be finite algebras in a Taylor variety with minimal absorbing subalgebras* $\mathbf{B}_i \lll \mathbf{A}_i$ $(i = 0, 1)$*, and suppose* $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$*, where* $\tilde{\eta}_0 \neq \tilde{\eta}_1$*.*

(1) If $\mathbf{A}_0$ *and* $\mathbf{A}_1$ *are simple and* $R \cap (B_0 \times B_1) \neq \emptyset$*, then* $\mathbf{B}_0 \times \mathbf{B}_1 \leqslant \mathbf{R}$*.*
(2) If $\mathbf{A}_0$ *is simple nonabelian and* $\mathbf{A}_1$ *is abelian, then* $\mathbf{B}_0 \times \mathbf{A}_1 \leqslant \mathbf{R}$*.*

Proof.

(1) First note that, by Corollary 4.14, $\mathbf{R}$ is linked. Let $\mathbf{R}' := \mathbf{R} \cap (\mathbf{B}_0 \times \mathbf{B}_1)$. Then by Lemma 4.10 $\mathbf{R}' \leqslant_{\mathrm{sd}} \mathbf{B}_0 \times \mathbf{B}_1$, and by Lemma 4.8 $\mathbf{R}' \lhd \mathbf{R}$, so $\mathbf{R}'$ is also linked, by Lemma 4.11. The hypotheses of the Absorption Theorem—with $\mathbf{R}'$ in place of $\mathbf{R}$ and $\mathbf{B}_i$ in place of $\mathbf{A}_i$—are now satisfied. Therefore, $\mathbf{R}' = \mathbf{B}_0 \times \mathbf{B}_1$.
(2) This follows directly from Corollary 4.15 and the Absorption Theorem. □

Once again, by collecting multiple abelian factors into a single factor, we obtain

Corollary 4.18. *Let* $\mathbf{A}_0$*,* $\mathbf{A}_1$*, …,* $\mathbf{A}_{n-1}$ *be finite Taylor algebras, where* $\mathbf{A}_0$ *is simple nonabelian,* $\mathbf{B}_0 \lll \mathbf{A}_0$*, and the remaining* $\mathbf{A}_i$ *are abelian, and suppose* $\mathbf{R} \leqslant_{\mathrm{sd}} \prod_{\underline{n}} \mathbf{A}_i$*. Then* $\mathbf{B}_0 \times \mathbf{R}_{0'} \leqslant \mathbf{R}$*.*

Proof. Obviously, $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{R}_{0'}$ and $R \cap (B_0 \times R_{0'}) \neq \emptyset$. Also, $\mathbf{R}_{0'} := \mathrm{Proj}_{0'} \mathbf{R}$ is abelian, so we can apply Lemma 4.17 (ii), with $\mathbf{R}_{0'}$ in place of $\mathbf{A}_1$. □

5. The Rectangularity Theorem

5.1. *Some recent history*

In late May of 2015 we attended a workshop on “Open Problems in Universal Algebra,” at which Libor Barto announced a new theorem that he and Marcin Kozik proved called the “Rectangularity Theorem.” At this meeting Barto gave a detailed overview of the proof [1]. The authors of the present paper then made a concerted effort over many months to fill in the details and produce a complete proof. Unfortunately, each attempt uncovered a gap that we were unable to fill. Nonetheless, we found a slightly different route to the theorem. Our argument is similar to the one presented by Barto in most of its key aspects. In particular, we make heavy use of the absorption idea and the Absorption Theorem plays a key role. However, we were unable to complete the proof without a new “Linking Lemma” (Lem. 5.4) that we proved using a general result of Kearnes and Kiss.

Thus, our argument is similar in spirit to the original, but provides alternative evidence that Barto and Kozik’s Rectangularity Theorem is correct, and may shed additional light on the result. In the next subsection, we prove the Rectangularity Theorem as well as some corollaries that we use later, in Section 6, to demonstrate how the Rectangularity Theorem can be applied to CSP problems.

5.2. *Preliminaries*

We are mainly concerned with finite algebras belonging to a single Taylor variety $\mathcal{V}$ (though some results below hold more generally). As mentioned, our main goal in this section is to prove the Rectangularity Theorem of Barto and Kozik, which can be described as follows: Start with finite algebras $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ in a Taylor variety with minimal absorbing subalgebras $\mathbf{B}_i \lll \mathbf{A}_i$. Suppose at most one of these algebras is abelian and suppose the nonabelian algebras are simple. Let $\mathbf{R}$ be a subdirect product of $\prod_{\underline{n}} \mathbf{A}_i$. Then (under a few more assumptions) if R intersects nontrivially with the product $\prod_{\underline{n}} B_i$ of minimal absorbing subuniverses, then R contains the entire product $\prod_{\underline{n}} B_i$.

Before attempting to prove this, we first prove a number of useful lemmas. In doing so, we will make frequent use of the following

Notation:

- $\underline{n} := \{0, 1, 2, \ldots, n-1\}$.
- For $\sigma \subseteq \underline{n}$, $A = A_0 \times A_1 \times \cdots \times A_{n-1}$, and $\mathbf{a} = (a_0, a_1 \ldots, a_{n-1})$, we define

$$\eta_\sigma := \ker(A \twoheadrightarrow \Pi_\sigma A_i) = \{(\mathbf{a}, \mathbf{a}') \in A^2 \mid a_i = a'_i \text{ for all } i \in \sigma\},$$

  the kernel of the projection of A onto coordinates σ.
- For $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$, let

$$\tilde{\eta}_\sigma := \ker(R \twoheadrightarrow \Pi_\sigma A_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid r_i = r'_i \text{ for all } i \in \sigma\},$$

  the kernel of the projection of R onto the coordinates σ. Thus, $\tilde{\eta}_\sigma = \eta_\sigma \cap R^2$.
- For $\sigma \subseteq \underline{n}$, we let $\sigma' := \underline{n} - \sigma$, and by $\mathbf{R} \leqslant_{\text{sd}} \prod_\sigma \mathbf{A}_i \times \prod_{\sigma'} \mathbf{A}_i$ we mean that the following three conditions hold:
  (1) $\mathbf{R}$ is a subalgebra of $\prod_{\underline{n}} \mathbf{A}_i$;
  (2) $\operatorname{Proj}_\sigma \mathbf{R} = \prod_\sigma \mathbf{A}_i$;
  (3) $\operatorname{Proj}_{\sigma'} \mathbf{R} = \prod_{\sigma'} \mathbf{A}_i$;

  we say that $\mathbf{R}$ is a *subdirect product of* $\prod_\sigma \mathbf{A}_i$ *and* $\prod_{\sigma'} \mathbf{A}_i$ in this case.
- The subdirect product $\mathbf{R} \leqslant_{\text{sd}} \prod_\sigma \mathbf{A}_i \times \prod_{\sigma'} \mathbf{A}_i$ is said to be *linked* if $\tilde{\eta}_\sigma \vee \tilde{\eta}_{\sigma'} = 1_R$.
- We sometimes use $\mathbf{R}_\sigma$ as shorthand for $\operatorname{Proj}_\sigma \mathbf{R}$, the projection of $\mathbf{R}$ onto coordinates σ.

5.3. *Rectangularity theorem*

We are almost ready to prove the general Rectangularity Theorem. We need just a few more results that play a crucial role in the proof. The first comes from combining Lemma 4.6, transitivity of absorption, and Lemmas 4.8–4.11.

Lemma 5.1. *Let* $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ *be finite algebras in a Taylor variety, let* $\mathbf{B}_i \lll \mathbf{A}_i$ *for each* $i \in \underline{n}$*, and let* $\underline{n} = \sigma \cup \sigma'$ *be a partition of* $\{0, 1, \ldots, n-1\}$ *into two nonempty disjoint subsets. Assume* $\mathbf{R}$ *is a* linked *subdirect product of* $\prod_\sigma \mathbf{A}_i$ *and* $\prod_{\sigma'} \mathbf{A}_i$*, and suppose* $R' := R \cap \prod_i B_i \neq \emptyset$*. Then* $\prod_i B_i \subseteq R$*, so* $\mathbf{R}' = \prod_i \mathbf{B}_i$*.*

Proof. By Lemma 4.10, $\mathbf{R}' \leqslant_{\text{sd}} \prod_\sigma \mathbf{B}_i \times \prod_{\sigma'} \mathbf{B}_i$, and by Lemma 4.8, $\mathbf{R}' \lhd \mathbf{R}$. Therefore, Lemma 4.11 implies $\mathbf{R}'$ is linked. By Lemma 4.6 and transitivity of absorption, it follows that $\prod_\sigma \mathbf{B}_i$ and $\prod_{\sigma'} \mathbf{B}_i$ are both absorption-free, so the Absorption Theorem implies that $\mathbf{R}' = \prod_\sigma \mathbf{B}_i \times \prod_{\sigma'} \mathbf{B}_i$. □

Next, we recall a theorem of Keith Kearnes and Emil Kiss, proved in [19], about algebras in a variety that satisfies a nontrivial idempotent Mal'tsev condition (equivalently, has a Taylor term). As above, we use the phrase "Taylor variety" to refer to an idempotent variety with a Taylor term, and we call algebras in such varieties "Taylor algebras."

Theorem 5.2 ([19, Thm 3.27]). *Suppose α and β are congruences of a Taylor algebra. Then $\mathsf{C}(\alpha, \alpha; \alpha \wedge \beta)$ if and only if $\mathsf{C}(\alpha \vee \beta, \alpha \vee \beta; \beta)$.*

We use this theorem to prove a "Linking Lemma" that will be central to our proof of the Rectangularity Theorem, and the following corollary of Theorem 5.2 and Lemma 3.2 gives the precise context in which we will apply these results.

Corollary 5.3. *Let α_0, α_1, β, δ be congruences of a Taylor algebra $\mathbf{A}$, and suppose that $\alpha_0 \wedge \beta = \delta = \alpha_1 \wedge \beta$ and $\alpha_0 \vee \alpha_1 = \alpha_0 \vee \beta = \alpha_1 \vee \beta = 1_A$. Then, $\mathsf{C}(1_A, 1_A; \alpha_0)$ and $\mathsf{C}(1_A, 1_A; \alpha_1)$, so $\mathbf{A}/\alpha_0$ and $\mathbf{A}/\alpha_1$ are abelian algebras.*

Proof. The hypotheses of Lemma 3.2 hold, so $\mathsf{C}(\beta, \beta; \delta)$ and β/δ is an abelian congruence of $\mathbf{A}/\delta$. Now, since $\delta = \alpha_i \wedge \beta$, we have $\mathsf{C}(\beta, \beta; \alpha_i \wedge \beta)$, so Theorem 5.2 implies $\mathsf{C}(\alpha_i \vee \beta, \alpha_i \vee \beta; \alpha_i)$. This yields $\mathsf{C}(1_A, 1_A; \alpha_i)$, since $\alpha_i \vee \beta = 1_A$. By Lemma 3.1 (7) then, $\mathsf{C}(1_A/\alpha_i, 1_A/\alpha_i; \alpha_i/\alpha_i)$. Equivalently, $\mathsf{C}(1_{A/\alpha_i}, 1_{A/\alpha_i}; 0_{A/\alpha_i})$. That is, $\mathbf{A}/\alpha_i$ is abelian. □

Before stating the next result, we remind the reader that $k' := \underline{n} - \{k\}$.

Lemma 5.4 (Linking Lemma). *Let $n \geqslant 2$, let $\mathbf{A}_0$, $\mathbf{A}_1$, $\ldots$, $\mathbf{A}_{n-1}$ be finite algebras in a Taylor variety, and let $\mathbf{B}_i \lll \mathbf{A}_i$. Suppose*

- *at most one $\mathbf{A}_i$ is abelian*
- *all nonabelian factors are simple*
- *$\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,*
- *$\tilde{\eta}_i \neq \tilde{\eta}_j$ for all $i \neq j$.*

Then there exists k such that $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_k \times \mathbf{R}_{k'}$ is linked.

Proof. By way of contradiction, suppose $\tilde{\eta}_k \vee \tilde{\eta}_{k'} < 1_R$ for all $0 \leqslant k < n$.

Case 1: There is no abelian factor.
If $n = 2$, the result holds by Corollary 4.14. Assume $n > 2$. Since every factor is simple, each $\tilde{\eta}_k$ is a coatom, so $\tilde{\eta}_k \vee \tilde{\eta}_{k'} < 1_R$ implies $\tilde{\eta}_k \geqslant \tilde{\eta}_{k'}$. Then,

$$\tilde{\eta}_{k'} := \bigwedge_{i \neq k} \tilde{\eta}_i = \Big(\bigwedge_{i \neq k} \tilde{\eta}_i\Big) \wedge \tilde{\eta}_k = \tilde{\eta}_{\underline{n}} = 0_R. \tag{5.1}$$

Note that (5.1) holds for all $k \in \underline{n}$. Now, let $\tau \subseteq \underline{n}$ be a subset that is maximal among those satisfying $\tilde{\eta}_\tau > 0_R$. Then by what we just showed, τ omits at least two indices, say, j and ℓ. Therefore, by maximality of τ,

$$\tilde{\eta}_\tau \wedge \tilde{\eta}_j = 0_R = \tilde{\eta}_\tau \wedge \tilde{\eta}_\ell,$$

and $\tilde{\eta}_\tau \nleqslant \tilde{\eta}_j$, and $\tilde{\eta}_\tau \nleqslant \tilde{\eta}_\ell$. Since all factors are simple, $\tilde{\eta}_j$ and $\tilde{\eta}_\ell$ are coatoms and distinct by assumption, so

$$\tilde{\eta}_\tau \vee \tilde{\eta}_j = \tilde{\eta}_\tau \vee \tilde{\eta}_\ell = \tilde{\eta}_j \vee \tilde{\eta}_\ell = 1_R.$$

But then, by Corollary 5.3, both $\mathbf{A}_j \cong \mathbf{R}/\tilde{\eta}_j$ and $\mathbf{A}_\ell \cong \mathbf{R}/\tilde{\eta}_\ell$ are abelian, which contradicts the assumption that there are no abelian factors.

Case 2: One factor, say $\mathbf{A}_0$, is abelian.
For $k > 0$, $\mathbf{A}_k$ is simple and nonabelian, so $\tilde{\eta}_k$ is a coatom of $\operatorname{Con}\mathbf{R}$. Therefore, $\tilde{\eta}_k \vee \tilde{\eta}_{k'} < 1_R$ implies $\tilde{\eta}_{k'} \leqslant \tilde{\eta}_k$, so

$$\tilde{\eta}_{k'} = \tilde{\eta}_{k'} \wedge \tilde{\eta}_k = \bigwedge_{i \in \underline{n}} \tilde{\eta}_i = 0_R. \tag{5.2}$$

(Note that this holds for every $0 < k < n$.)

Now, let $\theta = \tilde{\eta}_{0'} := \bigwedge_{i \neq 0} \tilde{\eta}_i$. Let τ be a maximal subset of $0' := \{1, 2, \dots, n-1\}$ such that $\tilde{\eta}_\tau > \theta$. Obviously τ is a proper subset of $0'$. Moreover, by (5.2) there exists $j \in 0'$ such that $\tilde{\eta}_\tau \nleqslant \tilde{\eta}_j$. Therefore,

$$\tilde{\eta}_\tau \vee \tilde{\eta}_j = 1_R \quad \text{and} \quad \tilde{\eta}_\tau \wedge \tilde{\eta}_j = \theta. \tag{5.3}$$

The first equality in (5.3) holds since $\tilde{\eta}_j$ is a coatom, while the second holds since τ is a maximal set such that $\tilde{\eta}_\tau > \theta$. By Theorem 5.2, we have $\mathsf{C}(\tilde{\eta}_\tau \vee \tilde{\eta}_j, \tilde{\eta}_\tau \vee \tilde{\eta}_j; \tilde{\eta}_j)$ iff $\mathsf{C}(\tilde{\eta}_\tau, \tilde{\eta}_\tau; \tilde{\eta}_\tau \wedge \tilde{\eta}_j)$. Using (5.3), this means

$$\mathsf{C}(1_R, 1_R; \tilde{\eta}_j) \quad \Longleftrightarrow \quad \mathsf{C}(\tilde{\eta}_\tau, \tilde{\eta}_\tau; \theta).$$

However, $\mathsf{C}(1_R, 1_R; \tilde{\eta}_j)$ implies $\mathbf{A}_j$ is abelian, but $\mathbf{A}_0$ is the only abelian factor, so

$$\mathsf{C}(\tilde{\eta}_\tau, \tilde{\eta}_\tau; \theta) \text{ does not hold.} \tag{5.4}$$

Since $\mathbf{A}_0$ is abelian, $\mathsf{C}(1_R, 1_R; \tilde{\eta}_0)$; *a fortiori*, $\mathsf{C}(\tilde{\eta}_\tau \vee \tilde{\eta}_0, \tilde{\eta}_\tau \vee \tilde{\eta}_0; \tilde{\eta}_0)$. The latter holds iff $\mathsf{C}(\tilde{\eta}_\tau, \tilde{\eta}_\tau; \tilde{\eta}_\tau \wedge \tilde{\eta}_0)$, by Theorem 5.2. Now, if $\tilde{\eta}_0 \wedge \tilde{\eta}_\tau \leqslant \theta$, then by Lemma 3.3 we have $\mathsf{C}(\tilde{\eta}_\tau, \tilde{\eta}_\tau; \theta)$, which is false by (5.4). Therefore, $\tilde{\eta}_0 \wedge \tilde{\eta}_\tau \nleqslant \theta$. It follows that $\tilde{\eta}_0 \wedge \tilde{\eta}_\tau \neq 0_R$. By (5.2), then, there are at least two distinct indices, say, j and ℓ, in $\underline{n}$ that do not belong to τ. By maximality of τ, we have $\tilde{\eta}_\tau \wedge \tilde{\eta}_j = \theta = \tilde{\eta}_\tau \wedge \tilde{\eta}_\ell$. By Corollary 5.3 (with $\alpha_0 = \tilde{\eta}_j$, $\alpha_1 = \tilde{\eta}_\ell$, $\beta = \tilde{\eta}_\tau$, $\delta = \theta$), it follows that $\mathbf{R}/\tilde{\eta}_j \cong \mathbf{A}_j$ and $\mathbf{R}/\tilde{\eta}_\ell \cong \mathbf{A}_\ell$ are both abelian—a contradiction, since, by assumption, the only abelian factor is $\mathbf{A}_0$. □

Finally, we have assembled all the tools we will use to accomplish the main goal of this section, which is to prove the following:

Theorem 5.5 (Rectangularity Theorem). *Let $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1}$ be finite algebras in a Taylor variety with minimal absorbing subalgebras $\mathbf{B}_i \lll \mathbf{A}_i$ and suppose*

- *at most one* $\mathbf{A}_i$ *is abelian,*
- *all nonabelian factors are simple,*
- $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\tilde{\eta}_i \neq \tilde{\eta}_j$ *for all* $i \neq j$,
- $R' := R \cap (B_0 \times B_1 \times \cdots \times B_{n-1}) \neq \emptyset$.

Then, $\mathbf{R}' = \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}$.

Proof. Of course it suffices to prove $B_0 \times B_1 \times \cdots \times B_{n-1} \subseteq R$. We prove this by induction on the number of factors in the product $\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$.

For $n = 2$ the result holds by Lemma 4.17. Fix $n > 2$ and assume that for all $2 \leqslant k < n$ the result holds for subdirect products of k factors. We will prove the result holds for subdirect products of n factors.

Let σ be a nonempty proper subset of $\underline{n}$ (so $1 \leqslant |\sigma| < n$) and let σ' denote the complement of σ in $\underline{n}$. Denote by $\mathbf{R}_\sigma$ the projection of $\mathbf{R}$ onto the σ-factors. That is, $\mathbf{R}_\sigma := \operatorname{Proj}_\sigma \mathbf{R}$ and $\mathbf{R}_{\sigma'} := \operatorname{Proj}_{\sigma'} \mathbf{R}$. Each of these projections satisfies the assumptions of the theorem, since $\mathbf{R}_\sigma \leqslant_{\text{sd}} \prod_\sigma \mathbf{A}_i$ and since for all $i \neq j$ in σ, we have $\ker(R_\sigma \twoheadrightarrow A_i) \neq \ker(R_\sigma \twoheadrightarrow A_j)$. Similarly for $\mathbf{R}_{\sigma'}$. Therefore, the induction hypothesis implies that $\prod_\sigma \mathbf{B}_i \leqslant \mathbf{R}_\sigma$ and $\prod_{\sigma'} \mathbf{B}_i \leqslant \mathbf{R}_{\sigma'}$. By Lemma 4.6, $\prod_\sigma \mathbf{B}_i \lhd\!\lhd \prod_\sigma \mathbf{A}_i$, and since $\prod_\sigma \mathbf{B}_i \leqslant \mathbf{R}_\sigma \leqslant \prod_\sigma \mathbf{A}_i$ it's clear that $\prod_\sigma \mathbf{B}_i \lhd \mathbf{R}_\sigma$. In fact, $\prod_\sigma \mathbf{B}_i \lhd\!\lhd \mathbf{R}_\sigma$ as well, by minimality of $\prod_\sigma \mathbf{B}_i \lhd\!\lhd \prod_\sigma \mathbf{A}_i$, and transitivity of absorption. To summarize, for every $\emptyset \subsetneq \sigma \subsetneq \underline{n}$,

$$\mathbf{R} \leqslant_{\text{sd}} \mathbf{R}_\sigma \times \mathbf{R}_{\sigma'}, \quad \prod_\sigma \mathbf{B}_i \lhd\!\lhd \mathbf{R}_\sigma, \quad \prod_{\sigma'} \mathbf{B}_i \lhd\!\lhd \mathbf{R}_{\sigma'}. \tag{5.5}$$

Finally, by the Linking Lemma (Lem. 5.4) there is a k such that $\mathbf{R} \leqslant_{\text{sd}} \mathbf{R}_k \times \mathbf{R}_{k'}$ is linked. Therefore, by Lemma 5.1 the the proof is complete. □

In case there is more than one abelian factor, we have the following slightly more general result. Recall our notation: $\underline{n} := \{0, 1, \ldots, n\}$, and if $\alpha \subseteq \underline{n}$, then

$$\alpha' := \underline{n} - \alpha \quad \text{and} \quad \mathbf{R}_\alpha := \operatorname{Proj}_\alpha \mathbf{R}.$$

Corollary 5.6. *Let* $\mathbf{A}_0$, $\mathbf{A}_1$, …, $\mathbf{A}_{n-1}$ *be finite algebras in a Taylor variety with* $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ ($i \in \underline{n}$), *and let* $\alpha \subseteq \underline{n}$. *Suppose*

- $\mathbf{A}_i$ *is abelian for each* $i \in \alpha$,
- $\mathbf{A}_i$ *is nonabelian and simple for each* $i \in \alpha'$,
- $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\tilde{\eta}_i \neq \tilde{\eta}_j$ *for all* $i \neq j$,
- $R' := R \cap (B_0 \times B_1 \times \cdots \times B_{n-1}) \neq \emptyset$.

Then $\mathbf{R}' = \mathbf{R}_\alpha \times \prod_{\alpha'} \mathbf{B}_i$.

Proof. Suppose $\alpha' = \{i_0, i_1, \dots, i_{m-1}\}$. Clearly, $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{R}_\alpha \times \mathbf{A}_{i_0} \times \mathbf{A}_{i_1} \times \cdots \times \mathbf{A}_{i_{m-1}}$. If $\alpha \neq \emptyset$, then the product has a single abelian factor $\mathbf{R}_\alpha \leqslant \prod_\alpha \mathbf{A}_i$. Otherwise, $\alpha = \emptyset$ and the product has no abelian factors. In either case, the result follows from Theorem 5.5. □

To conclude this section, we make two more observations that facilitate application of the foregoing results to CSP problems.

Corollary 5.7. *Let $\mathbf{A}_0$, $\mathbf{A}_1$, …, $\mathbf{A}_{n-1}$ be finite algebras in a Taylor variety with $\mathbf{B}_i \lll \mathbf{A}_i$ for each $i \in \underline{n}$ and suppose $\mathbf{R}$ and $\mathbf{S}$ are subdirect products of $\prod_{\underline{n}} \mathbf{A}_i$. Let $\alpha \subseteq \underline{n}$ and assume the following:*

(1) $\mathbf{A}_i$ is abelian for each $i \in \alpha$,
(2) $\mathbf{A}_i$ is nonabelian and simple for each $i \notin \alpha$,
(3) $\tilde{\eta}_i \neq \tilde{\eta}_j$ for all $i \neq j$,
(4) R and S both intersect $\prod_{\underline{n}} B_i$ nontrivially,
(5) there exists $\mathbf{x} \in R_\alpha \cap S_\alpha$.

Then $R \cap S \neq \emptyset$.

Proof. By Corollary 5.6, $\mathbf{R}' = \mathbf{R}_\alpha \times \prod_{\alpha'} \mathbf{B}_i$ and $\mathbf{S}' = \mathbf{S}_\alpha \times \prod_{\alpha'} \mathbf{B}_i$. Therefore, since $\mathbf{x} \in R_\alpha \cap S_\alpha$, we have $\{\mathbf{x}\} \times \prod_{\alpha'} \mathbf{B}_i \subseteq R \cap S$. □

Of course, this can be generalized to more than two subdirect products, as the next result states.

Corollary 5.8. *Let $\mathbf{A}_0$, $\mathbf{A}_1$, …, $\mathbf{A}_{n-1}$ be finite algebras in a Taylor variety with $\mathbf{B}_i \lll \mathbf{A}_i$ ($i \in \underline{n}$). Suppose $\{\mathbf{R}_\ell \mid 0 \leqslant \ell < m\}$ is a set of m subdirect products of $\prod_{\underline{n}} \mathbf{A}_i$. Let $\alpha \subseteq \underline{n}$ and assume the following:*

(1) $\mathbf{A}_i$ is abelian for each $i \in \alpha$,
(2) $\mathbf{A}_i$ is nonabelian and simple for each $i \notin \alpha$,
(3) $\forall \ell \in \underline{m}$, $\forall i \neq j$, $\tilde{\eta}^\ell_i \neq \tilde{\eta}^\ell_j$ (where $\tilde{\eta}^\ell_i := \ker(\mathbf{R}_\ell \twoheadrightarrow \mathbf{A}_i)$),
(4) each R_ℓ intersects $\prod B_i$ nontrivially,
(5) there exists $\mathbf{x} \in \bigcap \operatorname{Proj}_\alpha R_\ell$.

Then $\bigcap R_\ell \neq \emptyset$.

6. CSP Applications

In this section we give a precise definition of what we mean by a "constraint satisfaction problem," and what it means for such a problem to be "tractable." We then give some examples demonstrating how one uses the algebraic tools we have developed to prove tractability.

6.1. *Definition of a constraint satisfaction problem*

We now define a "constraint satisfaction problem" in a way that is convenient for our purposes. This is not the most general definition possible, but for now we postpone consideration of the scope of our setup.

Let $\mathbf{A} = \langle A, \mathcal{F}\rangle$ be a finite idempotent algebra, and let $\mathrm{Sub}(\mathbf{A})$ and $\mathsf{S}(\mathbf{A})$ denote the set of subuniverses and subalgebras of $\mathbf{A}$, respectively.

Definition 6.1. *Let $\mathfrak{A}$ be a collection of algebras of the same similarity type. We define* $\mathrm{CSP}(\mathfrak{A})$ *to be the following decision problem:*

- *An n-variable instance of* $\mathrm{CSP}(\mathfrak{A})$ *is a quadruple $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R}\rangle$ consisting of*
  - *a set $\mathcal{V}$ of n* variables*; often we take $\mathcal{V}$ to be $\underline{n} = \{0, 1, \dots, n-1\}$;*
  - *a list $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1}) \in \mathfrak{A}^n$ of algebras from $\mathfrak{A}$, one for each variable;*
  - *a list $\mathcal{S} = (\sigma_0, \sigma_1, \dots, \sigma_{J-1})$ of* constraint scope functions *with arities $\mathrm{ar}(\sigma_j) = m_j$;*
  - *a list $\mathcal{R} = (R_0, R_1, \dots, R_{J-1})$ of* constraint relations*, where each R_j is the universe of a subdirect product of the algebras in $\mathcal{A}$ with indices in $\mathrm{im}\,\sigma_j$; that is,*

$$\mathbf{R}_j \leqslant_{\mathrm{sd}} \prod_{0 \leqslant i < m_j} \mathbf{A}_{\sigma_j(i)}.$$

- *A solution to the instance $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R}\rangle$ is an assignment $f \colon \mathcal{V} \to \bigcup_{\underline{n}} A_i$ of values to variables that satisfies all constraint relations. More precisely, $f \in \prod_{\underline{n}} A_i$ and $f \circ \sigma_j \in R_j$ holds for all $0 \leqslant j < J$,*

We will denote the set of solutions to the instance $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R}\rangle$ by $\mathrm{Sol}(\mathcal{S}, \mathcal{R}, \underline{n})$.

Remark 6.2.

(1) The i-th scope function $\sigma_i \colon \underline{m}_i \to \mathcal{V}$ picks out the m_i variables from $\mathcal{V}$ involved in the constraint relation R_i. Thus, the list $\mathcal{S}$ of scope functions belongs to $\prod_{i<J} \mathcal{V}^{m_i}$.

(2) Frequently we require that the arities of the scope functions be bounded above, i.e., $\mathrm{ar}(\sigma_j) \leqslant m$, for all $j < J$. This gives rise to the *local constraint satisfaction problem* $\mathrm{CSP}(\mathfrak{A}, m)$ consisting of instances of this restricted type.

(3) If $(\sigma, R) \in \mathcal{C}$ is a constraint of an n-variable instance then we denote by $\mathrm{Sol}((\sigma, R), \underline{n})$ the set of all tuples in $\prod_{\underline{n}} A_i$ that satisfy (σ, R). In fact, we already introduced the notation $R^{\overleftarrow{\sigma}}$ for this set in (2.1) of Section 2.1. Recall, $\mathbf{x} \in R^{\overleftarrow{\sigma}}$ iff $\mathbf{x} \circ \sigma \in R$. Thus,

$$\mathrm{Sol}((\sigma, R), \underline{n}) = R^{\overleftarrow{\sigma}} := \{\mathbf{x} \in \prod_{i \in \underline{n}} A_i \mid \mathbf{x} \circ \sigma \in R\}.$$

Therefore, the set of solutions to the instance $\langle \mathcal{V}, \mathcal{A}, \mathcal{C} \rangle$ is

$$\mathrm{Sol}(\mathcal{C}, \underline{n}) = \bigcap_{(\sigma,R)\in\mathcal{C}} R^{\overleftarrow{\sigma}}.$$

(4) If $\mathfrak{A}$ contains a single algebra, we write $\mathrm{CSP}(\mathbf{A})$ instead of $\mathrm{CSP}(\{\mathbf{A}\})$. It is important to note that, in our definition of an instance of $\mathrm{CSP}(\mathfrak{A})$, a constraint relation is a subdirect product of algebras in $\mathfrak{A}$. This means that the constraint relations of an instance of $\mathrm{CSP}(\mathbf{A})$ are subdirect powers of $\mathbf{A}$. In the literature it is conventional to allow constraint relations of $\mathrm{CSP}(\mathbf{A})$ to be subpowers of $\mathbf{A}$. Such interpretations would correspond to instances of $\mathrm{CSP}(\mathsf{S}(\mathbf{A}))$ in our notation.

6.2. *Instance size and tractability*

We measure the computational complexity of an algorithm for solving instances of $\mathrm{CSP}(\mathfrak{A})$ as a function of input size. In order to do this, and to say what it means for $\mathrm{CSP}(\mathfrak{A})$ to be "computationally tractable," we first need a definition of input size. In our case, this amounts to determining the number of bits required to completely specify an instance of the problem. In practice, an upper bound on the size is usually sufficient.

Using the notation in Definition 6.1 as a guide, we bound the size of an instance $\mathcal{I} = \langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$ of $\mathrm{CSP}(\mathfrak{A})$. Let $q = \max(|A_0|, |A_1|, \dots, |A_{n-1}|)$, let r be the maximum rank of an operation symbol in the similarity type, and p the number of operation symbols. Then each member of the list $\mathcal{A}$ requires at most $pq^r \log q$ bits to specify. Thus

$$\mathrm{size}(\mathcal{A}) \leqslant npq^r \log q.$$

Similarly, each constraint scope $\sigma_j \colon \underline{m}_j \to \underline{n}$ can be encoded using $m_j \log n$ bits. Taking $m = \max(m_1, \dots, m_{J-1})$ we have

$$\mathrm{size}(\mathcal{S}) \leqslant Jm \log n.$$

Finally, the constraint relation R_j requires at most $q^{m_j} \cdot m_j \cdot \log q$ bits. Thus

$$\mathrm{size}(\mathcal{R}) \leqslant Jq^m \cdot m \log q.$$

Combining these encodings and using the fact that $\log q \leqslant q$, we deduce that

$$\mathrm{size}(\mathcal{I}) \leqslant npq^{r+1} + Jmq^{m+1} + Jmn. \tag{6.1}$$

In particular, for the problem $\mathrm{CSP}(\mathfrak{A}, m)$, the parameter m is considered fixed, as is r. In this case, we can assume $J \leqslant n^m$. Consequently $\mathrm{size}(\mathcal{I}) \in O((nq)^{m+1})$ which yields a polynomial bound (in nq) for the size of the instance.

A problem is called *tractable* if there exists a deterministic polynomial-time algorithm solving all instances of that problem. We can use Definition 6.1 above to classify the complexity of an algebra $\mathbf{A}$, or collection of algebras $\mathfrak{A}$, according to the complexity of their corresponding constraint satisfaction problems.

An algorithm A is called a *polynomial-time algorithm* for CSP($\mathfrak{A}$) if there exist constants c and d such that, given an instance $\mathfrak{I}$ of CSP($\mathfrak{A}$) of size $S = \text{size}(\mathfrak{I})$, A halts in at most cS^d steps and outputs whether or not $\mathfrak{I}$ has a solution. In this case, we say A "solves the decision problem CSP($\mathfrak{A}$) in polynomial time" and we call the algebras in $\mathfrak{A}$ "jointly tractable." Some authors say that an algebra $\mathbf{A}$ as tractable when $\mathfrak{A} = \mathsf{S}(\mathbf{A})$ is jointly tractable, or when $\mathfrak{A} = \mathsf{SP}_{\text{fin}}(\mathbf{A})$ is jointly tractable. We say that $\mathfrak{A}$ is *jointly locally tractable* if, for every natural number, m, there is a polynomial-time algorithm A_m that solves CSP($\mathfrak{A}, m$).

We wish to emphasize that, as is typical in computational complexity, the problem CSP($\mathfrak{A}$) is a *decision problem,* that is, the algorithm is only required to respond "yes" or "no" to the question of whether a particular instance has a solution, it does not have to actually produce a solution. However, it is a surprising fact that if CSP($\mathfrak{A}$) is tractable then the corresponding *search problem* is also tractable, in other words, one can design the algorithm to find a solution in polynomial time, if a solution exists, see [9, Cor 4.9].

6.3. *Sufficient conditions for tractability*

A lattice is called *meet semidistributive* if it satisfies the quasiidentity

$$x \wedge y \approx x \wedge z \to x \wedge y \approx x \wedge (y \vee z).$$

A variety is SD-$\wedge$ if every member algebra has a meet semidistributive congruence lattice. Idempotent SD-$\wedge$ varieties are known to be Taylor [16]. In [4], Barto and Kozik proved the following.

Theorem 6.3. *Let $\mathbf{A}$ be a finite idempotent algebra lying in an SD-$\wedge$ variety. Then $\mathbf{A}$ is tractable.*

A second significant technique for establishing tractability is the "few subpowers algorithm," which, according to its discoverers, is a broad generalization of Gaussian elimination.

Definition 6.4. *Let $\mathcal{V}$ be a variety and k an integer, $k > 1$. A $(k+1)$-ary term t is called a k-*edge term *for $\mathcal{V}$ if the following k identities hold in $\mathcal{V}$:*

$$\begin{aligned} t(y, y, x, x, x, \dots, x) &\approx x \\ t(y, x, y, x, x, \dots, x) &\approx x \\ t(x, x, x, y, x, \dots, x) &\approx x \\ &\vdots \\ t(x, x, x, x, x, \dots, y) &\approx x. \end{aligned}$$

Clearly every edge term is idempotent and Taylor. It is not hard to see that every Mal'tsev term and every near unanimity term is an edge term. Combining the main results of [8] and [17] yields the following theorem.

Theorem 6.5. *Let* $\mathbf{A}$ *be a finite idempotent algebra with an edge term. Then* $\mathbf{A}$ *is tractable.*

Finally, we comment that tractability is largely preserved by familiar algebraic constructions.

Theorem 6.6 ([9]). *Let* $\mathbf{A}$ *be a finite, idempotent, tractable algebra. Every subalgebra and finite power of* $\mathbf{A}$ *is tractable. Every homomorphic image of* $\mathbf{A}$ *is locally tractable.*

6.4. *Rectangularity Theorem: obstacles and applications*

The goal of this section is to consider aspects of the Rectangularity Theorem that seem to limit its utility as a tool for proving tractability of CSPs. We first give a brief overview of the potential obstacles, and then consider each one in more detail in the following subsections.

(1) **Abelian factors must have easy partial solutions.** One potential limitation concerns the abelian factors in the product algebra associated with a CSP instance. Indeed, Corollary 5.8 assumes that when the given constraint relations are projected onto abelian factors, we can efficiently find a solution to this restricted instance—that is, an element satisfying all constraint relations after projecting these relations onto the abelian factors of the product. Section 6.4.1 shows that this concern is easily dispensed with and is not a real limitation of Corollary 5.8.

(2) **Intersecting products of minimal absorbing subalgebras.** Another potential obstacle concerns the nonabelian simple factors. As we saw in Section 5.3, the Rectangularity Theorem (and its corollaries) assumes that the universes of the subdirect products in question all intersect nontrivially with a single product $\prod B_i$ of minimal absorbing subuniverses. (We refer to "minimal absorbing subuniverses" quite frequently, so from now on we call them *masses*; that is, a *mass* is a **m**inimal **a**bsorbing **s**ubunivers**e**, and the product of masses will be called a *mass product*.) Moreover, assumption (4) of Corollary 5.8 requires that all constraint relations intersect nontrivially with a single mass product. This is a real limitation, as we demonstrate in Section 6.4.2 below.

(3) **Nonabelian factors must be simple.** This is the most obvious limitation of the theorem and at this point we don't have a completely general means of overcoming it. However, some methods that work in particular cases are described below.

In the next two subsections we address potential limitations (1) and (2). In Section 7 we develop some alternative methods for proving tractability of nonsimple algebras, and then in Section 8 we apply these methods in the special setting of "commutative idempotent binars."

6.4.1. *Tractability of abelian algebras*

To address concern (1) of the previous subsection, we observe that finite abelian algebras yield tractable CSPs. We will show how to use this fact and the Rectangularity Theorem to find solutions to a given CSP instance (or determine that none exists). To begin, recall the fundamental result of tame congruence theory [16, Thm 7.12] that we reformulated as Lemma 3.4. As noted in Remark 3.6 above, this result has the following important corollary. (A proof that avoids tame congruence theory appears in [5, Thm 5.1].)

Theorem 6.7. *Let $\mathcal{V}$ be a locally finite variety with a Taylor term. Every finite abelian member of $\mathcal{V}$ generates a congruence-permutable variety. Consequently, every finite abelian member of $\mathcal{V}$ is tractable.*

Let $\mathbf{A}$ be a finite algebra in a Taylor variety and fix an instance $\mathcal{I} = \langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$ of CSP($\mathsf{S}(\mathbf{A})$) with $n = |\mathcal{V}|$ variables. Suppose all nonabelian algebras in the list $\mathcal{A}$ are simple. Let $\alpha \subseteq \underline{n}$ denote the indices of the abelian algebras in $\mathcal{A}$, and assume without loss of generality that $\alpha = \{0, 1, \dots, q-1\}$. That is, $\mathbf{A}_0$, $\mathbf{A}_1$, …, $\mathbf{A}_{q-1}$ are finite idempotent abelian algebras. Consider now the restricted instance $\mathcal{I}_\alpha$ obtained by dropping all constraint relations with scopes that don't intersect α, and by restricting the remaining constraint relations to the abelian factors. Since the only algebras involved in $\mathcal{I}_\alpha$ are abelian, this is an instance of a tractable CSP. Therefore, we can check in polynomial-time whether or not $\mathcal{I}_\alpha$ has a solution. If there is no solution, then the original instance $\mathcal{I}$ has no solution. On the other hand, suppose $f_\alpha \in \prod_{j\in\alpha} A_j$ is a solution to $\mathcal{I}_\alpha$. In Corollary 5.8, to reach the conclusion that the full instance has a solution, we required a partial solution $\mathbf{x} \in \bigcap \operatorname{Proj}_\alpha R_\ell$. This is precisely what f_α provides.

To summarize, we try to find a partial solution by restricting the instance to abelian factors and, if such a partial solution exists, we use it for $\mathbf{x}$ in Corollary 5.8. Then, assuming the remaining hypotheses of Corollary 5.8 hold, we conclude that a solution to the original instance exists. If no solution to the restricted instance exists, then the original instance has no solution. Thus we see that assumption (5) of Corollary 5.8 does not limit the application scope of this result.

6.4.2. *Mass products*

This section concerns products of minimal absorbing subalgebras, or "mass products." Hypothesis (4) of Corollary 5.8 assumes that all constraint relations intersect nontrivially with a single mass product. However, it's easy to contrive instances where this hypothesis does not hold. For example, take the algebra $\mathbf{A} = \langle \{0, 1\}, m \rangle$, where $m \colon A^3 \to A$ is the ternary idempotent majority operation—that is, $m(x, x, x) \approx x$ and $m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x$. Consider subdirect products $\mathbf{R} = \langle R, m\rangle$ and $\mathbf{S} = \langle S, m\rangle$ of $\mathbf{A}^3$ with universes

$$\begin{aligned} R &= \{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\},\\ S &= \{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}. \end{aligned}$$

Then there are mass products that intersect nontrivially with either R or S, but no mass product intersects nontrivially with both R and S. As the Rectangularity Theorem demands, each of R and S fully contains every mass product that it intersects, but there is no single mass product intersecting nontrivially with both R and S. Hence, hypothesis (4) of Corollary 5.8 is not satisfied so it would seem that we cannot use rectangularity to determine whether a solution exists in such instances.

The example described in the last paragraph is very special. For one thing, there is no solution to the instance with constraint relations R and S. We might hope that when an instance *does* have a solution, then there should be a solution that passes through a mass product. As we now demonstrate, this is not always the case. In fact, Example 6.9 describes a case in which two subdirect powers intersect nontrivially, yet each intersects trivially with every mass product.

Proposition 6.8. *There exists an algebra* $\mathbf{A}$ *with subdirect powers* $\mathbf{R}$ *and* $\mathbf{S}$ *such that* $R \cap S \neq \emptyset$ *and, for every collection* $\{B_i\}$ *of masses,* $R \cap \prod B_i = \emptyset = S \cap \prod B_i$.

We prove Proposition 6.8 by simply producing an example that meets the stated conditions.

Example 6.9. Let $\mathbf{A} = \langle \{0,1,2\}, \circ\rangle$ be an algebra with binary operation $\circ$ given by

$\circ$	0	1	2
0	0	1	2
1	1	1	0
2	2	0	2

The proper nonempty subuniverses of $\mathbf{A}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{0,1\}$, and $\{0,2\}$. Note that $\{1\}$ and $\{2\}$ are both minimal absorbing subuniverses with respect to the term $t(x,y,z,w) = (x \circ y) \circ (z \circ w)$. Note also that $\{0\}$ is not an absorbing subuniverse of $\mathbf{A}$. For if $\{0\} \triangleleft \mathbf{A}$, then $\{0\} \triangleleft \{0,1\}$ which is false, since $\{0,1\}$ is a semilattice with absorbing element 1.

Let $\mathbf{A}_0 \cong \mathbf{A} \cong \mathbf{A}_1$, $R = \{(0,0),(1,1),(2,2)\}$, and $S = \{(0,0),(1,2),(2,1)\}$. Then $\mathbf{R}$ and $\mathbf{S}$ are subdirect products of $\mathbf{A}_0 \times \mathbf{A}_1$ and $R \cap S = \{(0,0)\}$. There are four minimal absorbing subuniverses of $\mathbf{A}_0 \times \mathbf{A}_1$. They are

$$B_{11} = \{1\} \times \{1\} = \{(1,1)\},\ B_{12} = \{(1,2)\},\ B_{21} = \{(2,1)\},\ B_{22} = \{(2,2)\}.$$

Finally, observe

$$R \cap B_{ij} = \begin{cases} \{(i,j)\}, & i = j, \\ \emptyset, & i \neq j. \end{cases} \qquad S \cap B_{ij} = \begin{cases} \emptyset, & i = j \\ \{(i,j)\}, & i \neq j. \end{cases}$$

This proves Proposition 6.8.

6.5. *Algorithm synthesis for heterogeneous problems*

We now present a new result (Theorem 6.10) that, like the Rectangularity Theorem, aims to describe some of the elements that must belong to certain subdirect products. The conclusions we draw are weaker than those of Theorem 5.5. However, the hypotheses required here are simpler, and the motivation is different.

We have in mind subdirect products of "heterogeneous" families of algebras. What we mean by this is described and motivated as follows: let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be classes of algebras, all of the same signature. Perhaps we are lucky enough to possess a single algorithm that proves the algebras in $\bigcup \mathcal{C}_i$ are jointly tractable (defined in Section 6.2). Suppose instead that we are a little less fortunate and, although we have no such single algorithm at hand, at least we do happen to know that the classes $\mathcal{C}_i$ are *separately tractable*. By this we mean that for each i we have a proof (or algorithm) P_i witnessing the tractability of $\mathrm{CSP}(\mathcal{C}_i)$. It is natural to ask whether and under what conditions it might be possible to derive from $\{\mathrm{P}_i \mid 1 \leqslant i \leqslant m\}$ a single proof (algorithm) establishing that the algebras in $\bigcup \mathcal{C}_i$ are *jointly tractable*, so that $\mathrm{CSP}(\bigcup \mathcal{C}_i)$ is tractable.

The results in this section take a small step in this direction by considering two special classes of algebras that are known to be separately tractable, and demonstrating that they are, in fact, jointly tractable. We apply this tool in Section 7.2.2 where we prove tractability of a CSP involving algebras that were already known to be tractable, but not previously known to be jointly tractable.

The results here involve special term operations called cube terms and transitive terms. A k-ary idempotent term t is a *cube term* if for every coordinate $i \leqslant k$, t satisfies an identity of the form $t(x_1, \ldots, x_k) \approx y$, where $x_1, \ldots, x_k \in \{x, y\}$ and $x_i = x$. A k-ary operation f on a set A is called *transitive in the i-th coordinate* if for every $u, v \in A$, there exist $a_1, \ldots, a_k \in A$ such that $a_i = u$ and $f(a_1, \ldots, a_n) = v$. Operations that are transitive in every coordinate are called *transitive*.

Fact 6.1. Let $\mathbf{A}$ be a finite idempotent algebra and suppose t is a cube term operation on $\mathbf{A}$. Then t is a transitive term operation on $\mathbf{A}$.

Proof. Assume t is a k-ary cube term operation on $\mathbf{A}$ and fix $0 \leqslant i < k$. We will prove that t is transitive in its i-th argument. Fix $u, v \in A$. We want to show there exist $a_0, \ldots, a_{k-1} \in A$ such that $a_{i-1} = u$ and $t(a_0, \ldots, a_{k-1}) = v$. Since t is a cube term, it satisfies an identity of the form $t(x_1, \ldots, x_k) \approx y$, where $(\forall j)(x_j \in \{x, y\})$ and $x_i = x$. So we simply substitute u for x and v for y in the argument list in of this identity. Denoting this substitution by $t[u/x, v/y]$, we have $t[u/x, v/y] = v$, proving that t is transitive. $\square$

Fact 6.2. The class

$$\mathcal{T} = \{\mathbf{A} \mid \mathbf{A} \text{ finite and every subalgebra of } \mathbf{A} \text{ has a transitive term op}\} \tag{6.2}$$

is closed under the taking of homomorphic images, subalgebras, and finite products. That is, $\mathcal{T}$ is a *pseudovariety.*

We also require the following obvious fact about nontrivial algebras in a Taylor variety. (We call an algebra *nontrivial* if it has more than one element.)

Fact 6.3. If $\mathbf{A}$ and $\mathbf{B}$ are nontrivial algebras in a Taylor variety $\mathcal{V}$, then for some $k > 1$ there is a k-ary term t in $\mathcal{V}$ such that $t^{\mathbf{A}}$ and $t^{\mathbf{B}}$ each depends on at least two of its arguments.

Finally, we are ready for the main result of this section.

Theorem 6.10. *Let $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1}$ be finite idempotent algebras in a Taylor variety and assume there exists a proper nonempty subset $\alpha \subset \underline{n}$ such that*

- *$\mathbf{A}_i$ has a cube term for all $i \in \alpha$,*
- *$\mathbf{A}_j$ has a sink $s_j \in A_j$ for all $j \in \alpha'$; let $\mathbf{s} \in \prod_{\alpha'} A_j$ be a tuple of sinks.*

If $\mathbf{R} \leqslant_{\mathrm{sd}} \prod_{\underline{n}} \mathbf{A}_i$, then the set $X := R_\alpha \times \{\mathbf{s}\}$ is a subuniverse of $\mathbf{R}$.

Remark 6.11. To foreshadow applications of Theorem 6.10, imagine we have algebras of the type described, and a collection $\mathcal{R}$ of subdirect products of these algebras. Suppose also that we have somehow determined that the intersection of the α-projections of these subdirect products is nonempty, say,

$$\mathbf{x}_\alpha \in \bigcap_{R \in \mathcal{R}} R_\alpha.$$

Then the full intersection $\bigcap \mathcal{R}$ will also be nonempty, since according to the theorem it must contain the tuple that is $\mathbf{x}_\alpha$ on α and $\mathbf{s}$ off α.

Proof. Fix $\mathbf{x} \in X$, so $\mathbf{x} \circ \alpha' = \mathbf{s}$ and $\mathbf{x} \circ \alpha \in R_\alpha$. We will prove that $\mathbf{x} \in R$. Define $\mathbf{A}_\alpha = \prod_\alpha \mathbf{A}_i$ and $\mathbf{A}_{\alpha'} = \prod_{\alpha'} \mathbf{A}_i$, so R_α is a subuniverse of $\mathbf{A}_\alpha$ and R is a subuniverse of $\mathbf{A}_\alpha \times \mathbf{A}_{\alpha'}$.

By Fact 6.1, for each $i \in \alpha$ every subalgebra of $\mathbf{A}_i$ has a transitive term operation. Moreover, the class $\mathcal{T}$ defined in (6.2) is a pseudovariety, so every subalgebra of $\mathbf{A}_\alpha$ has a transitive term operation. Suppose there are J subalgebras of $\mathbf{A}_\alpha$ and let $\{t_j \mid 0 \leqslant j < J\}$ denote the collection of transitive term operations, one for each subalgebra. Then it is not hard to prove that $t := t_0 \star t_1 \star \cdots \star t_{J-1}$ is a transitive term for every subalgebra of $\mathbf{A}_\alpha$. (Recall, $\star$ was defined at the end of Section 4.1.) In particular t is transitive for the subalgebra with universe R_α. Assume t is N-ary.

Fix $j \in \alpha'$. As usual, $t^{\mathbf{A}_j}$ denotes the interpretation of t in $\mathbf{A}_j$. We may assume without loss of generality that $t^{\mathbf{A}_j}$ depends on its first argument, at position 0. (It must depend on at least one of its arguments by idempotence.) Now, since $\mathbf{R}$ is a subdirect product of $\prod_{\underline{n}} \mathbf{A}_i$, there exists $\mathbf{r}^{(j)} \in R$ such that $\mathbf{r}^{(j)}(j) = s_j$, the sink in $\mathbf{A}_j$. Since $\mathbf{x} \circ \alpha \in R_\alpha$ and since $t^{\mathbf{A}_\alpha}$ is transitive over $\mathbf{R}_\alpha$, there exist $\mathbf{r}_1, \dots, \mathbf{r}_{N-1}$ in R such that

$$\begin{aligned}\mathbf{y}^{(j)} &:= t^{\mathbf{A}_\alpha \times \mathbf{A}_{\alpha'}}(\mathbf{r}^{(j)}, \mathbf{r}_1, \dots, \mathbf{r}_{N-1}) \\ &= (t^{\mathbf{A}_\alpha}(\mathbf{r}^{(j)} \circ \alpha, \mathbf{r}_1 \circ \alpha, \dots, \mathbf{r}_{N-1} \circ \alpha), t^{\mathbf{A}_{\alpha'}}(\mathbf{r}^{(j)} \circ \alpha', \mathbf{r}_1 \circ \alpha', \dots, \mathbf{r}_{N-1} \circ \alpha'))\end{aligned}$$

belongs to R and satisfies $\mathbf{y}^{(j)} \circ \alpha = \mathbf{x} \circ \alpha$ and $\mathbf{y}^{(j)}(j) = s_j$.

(To make the role played by transitivity here more transparent, note that we asserted the existence of elements in R whose "α-segments," $\mathbf{r}_1 \circ \alpha, \dots, \mathbf{r}_{N-1} \circ \alpha$ could be plugged in for all but one of the arguments of $t^{\mathbf{A}_\alpha}$, resulting in a map (unary polynomial) taking $\mathbf{r}^{(j)} \circ \alpha$ to $\mathbf{x} \circ \alpha$. It is the transitivity of $t^{\mathbf{A}_\alpha}$ over $\mathbf{R}_\alpha$ that justifies this assertion.)

If $\alpha' = \{j\}$, we are done, since $\mathbf{y}^{(j)} = \mathbf{x}$ in that case. If $|\alpha'| > 1$, then we repeat the foregoing procedure for each $j \in \alpha'$ and obtain a subset $\{\mathbf{y}^{(j)} \mid j \in \alpha'\}$ of R, each member of which agrees with $\mathbf{x}$ on α and has a sink in some position $j \in \alpha'$.

Next, choose distinct $j, k \in \alpha'$. Suppose w is a Taylor term for $\mathcal{V}$. Then by Fact 6.3 we may assume without loss of generality that $w^{\mathbf{A}_j}$ depends on its p-th argument and $w^{\mathbf{A}_k}$ depends on its q-th argument, for some $p \neq q$. Consider

$$\mathbf{z} := w^{\Pi \mathbf{A}_i}(\mathbf{y}^{(j)}, \dots, \mathbf{y}^{(j)}, \underbrace{\mathbf{y}^{(k)}}_{q\text{-th argument}}, \mathbf{y}^{(j)}, \dots, \mathbf{y}^{(j)})$$

Evidently, $\mathbf{z}(j) = s_j$, $\mathbf{z}(k) = s_k$, and $\mathbf{z} \circ \alpha = \mathbf{x} \circ \alpha$ by idempotence, since, when restricted to indices in α, all the input arguments agree and are equal to $\mathbf{x} \circ \alpha$. If $\alpha' = \{j, k\}$, we are done. Otherwise, choose $\ell \in \alpha' - \{j, k\}$, and again $w^{\mathbf{A}_\ell}$ depends on at least one of its arguments, say, the u-th. Let

$$\mathbf{z}' := w^{\Pi \mathbf{A}_i}(\mathbf{z}, \dots, \mathbf{z}, \underbrace{\mathbf{y}^{(\ell)}}_{u\text{-th argument}}, \mathbf{z} \dots, \mathbf{z}).$$

Then $\mathbf{z}'$ belongs to R, agrees with $\mathbf{x}$ on α, and satisfies $\mathbf{z}'(j) = s_j$, $\mathbf{z}'(k) = s_k$, and $\mathbf{z}'(\ell) = s_\ell$. Continuing in this way until the set α' is exhausted produces an element in R that agrees with $\mathbf{x}$ everywhere. In other words, $\mathbf{x}$ itself belongs to R. □

In Section 7.2.2 we apply Theorem 6.10 in the special case where "has a cube term" in the first hypothesis is replaced with "is abelian." Let us be explicit.

Corollary 6.12. *Let $\mathbf{A}_1, \dots, \mathbf{A}_{n-1}$ be finite idempotent algebras in a locally finite Taylor variety. Suppose there exists $0 < k < n-1$ such that*

- $\mathbf{A}_i$ *is abelian for all $i < k$;*
- $\mathbf{A}_i$ *has a sink $s_i \in A_i$ for all $i \geqslant k$.*

If $\mathbf{R} \leqslant_{\text{sd}} \prod \mathbf{A}_i$, *then* $R_{\underline{k}} \times \{s_k\} \times \{s_{k+1}\} \times \cdots \times \{s_{n-1}\} \subseteq R$.

Proof. Since $\mathbf{A}_{\underline{k}} := \prod_{i<k} \mathbf{A}_i$ is abelian and lives in a locally finite Taylor variety, there exists a term m such that $m^{\mathbf{A}_{\underline{k}}}$ is a Mal'tsev operation on $\mathbf{A}_{\underline{k}}$ (Theorem 6.7). Since a Mal'tsev term is a cube term, the result follows from Theorem 6.10. $\square$

As an alternative, a direct proof of Corollary 6.12 appears in Section A.4 below.

7. Problem Instance Reductions

In this section we develop some useful notation for taking an instance of a CSP and restricting or reducing it in various ways, either by removing variables or by reducing modulo a sequence of congruence relations. The utility of these tools will be demonstrated in Section 8.

Throughout this section, $\mathbf{A}$ will denote a finite idempotent algebra. The problem we will focus on is $\text{CSP}(\mathfrak{A})$, defined in Section 6.1, and we will be particularly interested in the special case in which $\mathfrak{A} = \mathsf{S}(\mathbf{A})$.

Recall, we denote an n-variable instance of $\text{CSP}(\mathsf{S}(\mathbf{A}))$ by $\mathcal{I} = \langle \underline{n}, \mathcal{A}, \mathcal{C}\rangle$, where $\underline{n} = \{0, 1, \dots, n-1\}$ represents the set of variables, $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1})$ is a list of n subalgebras of $\mathbf{A}$, and $\mathcal{C} = ((\sigma_0, R_0), (\sigma_1, R_1), \dots, (\sigma_{J-1}, R_{J-1}))$ is a list of constraints with respective arities $\text{ar}(\sigma_j) = m_j$. Thus, $R_j \subseteq \prod_{i<m_j} A_{\sigma(i)}$. Much of the discussion below refers to an arbitrary constraint in $\mathcal{C}$. In such cases it will simplify notation to drop subscripts and denote the constraint by (σ, R).

7.1. *Variable reductions*

7.1.1. *Partial scopes and partial constraints*

Consider the restriction of an n-variable CSP instance $\mathcal{I}$ to the first k of its variables, for some $k \leqslant n$. To start, we restrict an arbitrary scope σ to the first k variables. This results in a new *partial scope* given by the function $\sigma\big|_{\sigma^{-1}(\underline{k}\,\cap\,\text{im}\,\sigma)}$. Call this the *$k$-partial scope of* σ and, to simplify the notation, let

$$\sigma\big|_{\overleftarrow{\underline{k}}} = \sigma\big|_{\sigma^{-1}(\underline{k}\,\cap\,\text{im}\,\sigma)}.$$

If $\underline{k} \cap \text{im}\,\sigma = \emptyset$ then the k-partial scope of σ is the empty function. To make the notation easier to digest, we give a small example below, but first let's consider how to restrict a constraint to the first k variables. To obtain the *k-partial constraint of* (σ, R), we take the k-partial scope of σ as the new scope; for the constraint relation we take the restriction of each tuple in R to its first $p = |\underline{k} \cap \text{im}\,\sigma|$ coordinates. If we let

$$R\big|_{\overleftarrow{\underline{k}}} = R\big|_{\sigma^{-1}(\underline{k}\,\cap\,\text{im}\,\sigma)},$$

then the *k-partial constraint* of (σ, R) is given by $(\sigma\big|_{\overleftarrow{\underline{k}}}, R\big|_{\overleftarrow{\underline{k}}})$. The constraint relation $R\big|_{\overleftarrow{\underline{k}}}$ consists of all p-element initial segments of the tuples in R. For example, suppose σ is a scope consisting of the variables 2, 4, and 7; that is, σ corresponds to the list $(\sigma(0), \sigma(1), \sigma(2)) = (2, 4, 7)$. To find, say, the 5-partial constraint of (σ, R), restrict (σ, R) to the first $k = 5$ variables of the instance. We have $\underline{k} = \{0, 1, 2, 3, 4\}$ and

$$\sigma^{-1}(\underline{k} \cap \operatorname{im} \sigma) = \sigma^{-1}(\{0, 1, 2, 3, 4\} \cap \{2, 4, 7\}) = \sigma^{-1}\{2, 4\} = \{0, 1\}.$$

Therefore, $\sigma\big|_{\overleftarrow{\underline{k}}} = (\sigma(0), \sigma(1)) = (2, 4)$, and $R\big|_{\overleftarrow{\underline{k}}}$ consists of the initial pairs of the triples in R, that is, $\{(x, y) \mid (x, y, z) \in R\}$.

7.1.2. *Partial instances*

The *k-partial instance* of $\mathcal{I}$ is the restriction of $\mathcal{I}$ to its first k variables. We will denote this partial instance by $\mathcal{I}_{\underline{k}}$. Thus, $\mathcal{I}_{\underline{k}}$ is the instance with constraint set $\mathcal{C}_{\underline{k}}$ equal to the set of all k-partial constraints of $\mathcal{I}$. If we let $\mathrm{Sol}(\mathcal{I}, \underline{k})$ denote the set of solutions to $\mathcal{I}_{\underline{k}}$, then $f \in \mathrm{Sol}(\mathcal{I}, \underline{k})$ means that for each $j \in J$,

$$\sigma_j\big|_{\overleftarrow{\underline{k}}} \;\in\; R_j\big|_{\overleftarrow{\underline{k}}}.$$

We might be tempted to call $\mathrm{Sol}(\mathcal{I}, \underline{k})$ a set of "partial solutions," but that's a bit misleading since an $f \in \mathrm{Sol}(\mathcal{I}, \underline{k})$ may or may not extend to a solution to the full instance $\mathcal{I}$.

7.2. *Quotient reductions*

As usual, let $\mathbf{A}$ be a finite idempotent algebra, let $\underline{n} = \{0, 1, \dots, n-1\}$, and let $\mathcal{I} = \langle \underline{n}, \mathcal{A}, \mathcal{C} \rangle$ denote an n-variable instance of $\mathrm{CSP}(\mathsf{S}(\mathbf{A}))$. As above, we assume $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1}) \in \mathsf{S}(\mathbf{A})^n$, and $\mathcal{C} = ((\sigma_0, R_0), (\sigma_1, R_1), \dots, (\sigma_{J-1}, R_{J-1}))$, and $\mathbf{R}_j \leqslant_{\mathrm{sd}} \prod_{\underline{m}_j} A_{\sigma(i)}$, for $0 \leqslant j < J$.

7.2.1. *Quotient instances*

Suppose $\theta_i \in \mathrm{Con}(\mathbf{A}_i)$ for each $0 \leqslant i < n$ and define

$$\boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_{n-1}) \in \mathrm{Con}(\mathbf{A}_0) \times \mathrm{Con}(\mathbf{A}_1) \times \cdots \times \mathrm{Con}(\mathbf{A}_{n-1}).$$

If $\mathbf{x} = (x_0, x_1, \dots, x_{n-1}) \in \prod_{\underline{n}} A_i$, then denote by $\mathbf{x}/\boldsymbol{\theta}$ the tuple whose i-th component is the θ_i-class of $\mathbf{A}_i$ that contains x_i, so

$$\mathbf{x}/\boldsymbol{\theta} = (x_0/\theta_0, x_1/\theta_1, \dots, x_{n-1}/\theta_{n-1}) \in \prod_{i \in \underline{n}} A_i/\theta_i.$$

We need slightly more general notation than this since our tuples will often come from $\prod_{\underline{m}} A_{\sigma(i)}$ for some scope function $\sigma \colon \underline{m} \to \underline{n}$. If we view a general tuple $\mathbf{x} \in \prod_{\underline{n}} A_i$ as a function from $\underline{n}$ to $\bigcup A_i$, and $\boldsymbol{\theta}$ as a function from $\underline{n}$ to $\bigcup \mathrm{Con}(\mathbf{A}_i)$, then we write $\mathbf{x} \circ \sigma \in \prod_{\underline{m}} A_{\sigma(i)}$ and $\boldsymbol{\theta} \circ \sigma \in \prod_{\underline{m}} \mathrm{Con}(\mathbf{A}_{\sigma(i)})$ for the corresponding scope-restricted tuples, and define

$$(\mathbf{x} \circ \sigma)/(\boldsymbol{\theta} \circ \sigma) = (x_{\sigma(0)}/\theta_{\sigma(0)}, x_{\sigma(1)}/\theta_{\sigma(1)}, \dots, x_{\sigma(m-1)}/\theta_{\sigma(m-1)}) \in \prod_{i \in \underline{m}} A_{\sigma(i)}/\theta_{\sigma(i)}.$$

Given $\boldsymbol{\theta} \in \prod_{\underline{n}} \operatorname{Con}(\mathbf{A}_i)$ and a constraint $(\sigma, R) \in \mathcal{C}$ of $\mathcal{I}$, define the *quotient constraint* $(\sigma, R/\boldsymbol{\theta})$, where

$$R/\boldsymbol{\theta} := \bigl\{\mathbf{r}/(\boldsymbol{\theta} \circ \sigma) \in \prod_{i \in \underline{m}} A_{\sigma(i)}/\theta_{\sigma(i)} \mid \mathbf{r} \in R\bigr\}. \tag{7.1}$$

Define the *quotient instance* $\mathcal{I}/\boldsymbol{\theta}$ to be the n-variable instance of $\operatorname{CSP}(\mathsf{HS}(\mathbf{A}))$ with constraint set

$$\mathcal{C}/\boldsymbol{\theta} := \{(\sigma, R/\boldsymbol{\theta}) \mid (\sigma, R) \in \mathcal{C}\}. \tag{7.2}$$

Here are a few easily proved facts about quotient instances.

Fact 7.1. If $\boldsymbol{\theta} \in \prod_{\underline{n}} \operatorname{Con}(\mathbf{A}_i)$ and if $\mathbf{R} \leqslant_{\mathrm{sd}} \prod_{\underline{m}} \mathbf{A}_{\sigma(i)}$, then $R/\boldsymbol{\theta}$ defined in (7.1) is a subuniverse of $\prod_{\underline{m}} \mathbf{A}_{\sigma(i)}/\theta_{\sigma(i)}$ and the corresponding subalgebra is subdirect.

Fact 7.2. If $\boldsymbol{\theta} \in \prod_{\underline{n}} \operatorname{Con}(\mathbf{A}_i)$ and if $\mathcal{I}$ is an n-variable instance of $\operatorname{CSP}(\mathbf{A})$, then the constraint set $\mathcal{C}/\boldsymbol{\theta}$ defined in (7.2) defines an n-variable instance of $\operatorname{CSP}(\mathsf{HS}(\mathbf{A}))$.

Fact 7.3. If $\mathbf{x}$ is a solution to $\mathcal{I}$, then $\mathbf{x}/\boldsymbol{\theta}$ is a solution to $\mathcal{I}/\boldsymbol{\theta}$.

By Fact 7.3, if there is a quotient instance with no solution, then $\mathcal{I}$ has no solution. However, the converse is false; that is, there may be solutions to every proper quotient instance but no solution to the original instance.

7.2.2. *Block instances*

Let $\mathfrak{A}$ be a collection of finite idempotent algebras. Let $\mathcal{I} = \langle \underline{n}, \mathcal{A}, \mathcal{C}\rangle$ be an n-variable instance of $\operatorname{CSP}(\mathfrak{A})$, where $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1})$ and $\mathcal{C}$ is a finite set of constraints. If $\boldsymbol{\theta} \in \prod_{\underline{n}} \operatorname{Con}(\mathbf{A}_i)$ and $\mathbf{x} = (x_0, x_1, \dots, x_{n-1}) \in \prod_{\underline{n}} A_i$, then by idempotence, the list of blocks $\mathbf{x}/\boldsymbol{\theta} = (x_0/\theta_0, x_1/\theta_1, \dots, x_{n-1}/\theta_{n-1})$ is actually a list of algebras. For each constraint $(\sigma, R) \in \mathcal{C}$, consider restricting the relation R to the x_i/θ_i-classes in its scope σ. In other words, replace the constraint (σ, R) with the *block constraint* $(\sigma, R \cap \Pi_\sigma \mathbf{x}/\boldsymbol{\theta})$, where we have defined

$$\Pi_\sigma \mathbf{x}/\boldsymbol{\theta} := \prod_{i \in \underline{m}} x_{\sigma(i)}/\theta_{\sigma(i)}. \tag{7.3}$$

Finally, let $\mathcal{I}_{\mathbf{x}/\boldsymbol{\theta}} = \langle \underline{n}, \mathbf{x}/\boldsymbol{\theta}, \mathcal{C}_{\mathbf{x}/\boldsymbol{\theta}}\rangle$ denote the problem instance of $\operatorname{CSP}(\mathsf{S}(\mathcal{A}))$ specified by the constraint set

$$\mathcal{C}_{\mathbf{x}/\boldsymbol{\theta}} := \bigl\{(\sigma, R \cap \Pi_\sigma \mathbf{x}/\boldsymbol{\theta}) \mid (\sigma, R) \in \mathcal{C}\bigr\}.$$

We call $\mathcal{I}_{\mathbf{x}/\boldsymbol{\theta}}$ the $\mathbf{x}/\boldsymbol{\theta}$*-block instance of* $\mathcal{I}$. It is obvious that a solution to $\mathcal{I}_{\mathbf{x}/\boldsymbol{\theta}}$ is also a solution to $\mathcal{I}$.

The notions "quotient instance" and "block instance" suggest a strategy for solving CSPs that works in certain special cases (one of which we will see in Example 7.1). First search for a solution $\mathbf{x}/\boldsymbol{\theta}$ to the quotient instance $\mathcal{I}/\boldsymbol{\theta}$ for some conveniently chosen $\boldsymbol{\theta}$. If no quotient solution exists, then the original instance $\mathcal{I}$

$\cdot$	0	1	2	3
0	0	0	3	2
1	0	1	3	2
2	3	3	2	1
3	2	2	1	3

$._{\mathbf{Sq}_3}$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

t	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	2	2	2	2
3	3	3	3	3

Fig. 1. Operation tables of Example 7.1: for the algebra $\mathbf{A}$ (left); for the quotient $\mathbf{A}/\Theta \cong \mathbf{Sq}_3$ (middle); for $t(x,y) = x \cdot (y \cdot (x \cdot y))$ (right).

has no solution. Otherwise, if $\mathbf{x}/\boldsymbol{\theta}$ is a solution to $\mathcal{I}/\boldsymbol{\theta}$, then we try to solve the $\mathbf{x}/\boldsymbol{\theta}$-block instance of $\mathcal{I}$. If we are successful, then the instance $\mathcal{I}$ has a solution. Otherwise, the strategy is inconclusive and we try again with a different solution to $\mathcal{I}/\boldsymbol{\theta}$. After we have exhausted all solutions to the quotient instance, if we have still not found a solution to any of the corresponding block instances, then $\mathcal{I}$ has no solution.

This approach is effective as long as we can find a quotient instance $\mathcal{I}/\theta$ for which there is a polynomial bound on the number of quotient solutions. Although this is not always possible (consider instances involving only simple algebras!), there are situations in which an appropriate choice of congruences makes it easy to check that every instance has quotient instance with a very small number of solutions. We present an example.

Example 7.1. Let $\mathbf{A} = \langle\{0,1,2,3\}, \cdot\rangle$, be an algebra with a single binary operation, '$\cdot$', given by the table on the left in Figure 1. The proper nonempty subuniverses are $\{0,1\}$, $\{1,2,3\}$, and the singletons. The algebra $\mathbf{A}$ has a single proper nontrivial congruence relation, Θ, with partition $|01|2|3|$. The quotient algebra $\mathbf{A}/\Theta$ is a 3-element Steiner quasigroup, which happens to be abelian. We denote the latter by $\mathbf{Sq}_3$ and give its binary operation in the middle table in Figure 1. Note that the subalgebra $\{1,2,3\}$ of $\mathbf{A}$ is also isomorphic to $\mathbf{Sq}_3$.

As Theorem 6.7 predicts, the algebra $\mathbf{A}/\Theta$ has a Mal'tsev term $q(x,y,z) = y \cdot (x \cdot z)$. Let $s(x,y) = q(x,y,y) = y \cdot (x \cdot y)$. Then $\mathbf{A}/\Theta \models s(x,y) \approx x$. By iterating the term s on its second variable, we arrive at a term $t(x,y)$ such that, in $\mathbf{A}$, $t(x, t(x,y)) = t(x,y)$. In fact, $t(x,y) = x \cdot (y \cdot (x \cdot y))$. To summarize

$$\mathbf{A} \models t(x, t(x,y)) \approx t(x,y)$$
$$\mathbf{A}/\Theta \models t(x,y) \approx x.$$

The table for t appears on the right in Figure 1.

For the next result, we denote the two-element semilattice by $\mathbf{S}_2 = \langle\{0,1\}, \wedge\rangle$.

Lemma 7.2. *Let* $\mathfrak{A} = \{\mathbf{S}_2, \mathbf{Sq}_3\}$. *Then* $\mathrm{CSP}(\mathfrak{A})$ *is tractable.*

Proof. Let $\mathcal{I} = \langle \underline{n}, \mathcal{A}, \mathcal{C}\rangle$ be an instance of $\mathrm{CSP}(\mathfrak{A})$. Recall, the set of solutions to $\mathcal{I}$ is $\mathrm{Sol}(\mathcal{C}, \underline{n}) = \bigcap_{\mathcal{C}} R^{\overleftarrow{\sigma}}$. We shall apply Corollary 6.12 to establish tractability. We have $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1}) \in \mathfrak{A}^n$. Let $\alpha = \{\, i : \mathbf{A}_i \cong \mathbf{Sq}_3 \,\}$ and $\alpha' = \{\, i : \mathbf{A}_i \cong \mathbf{S}_2 \,\}$, so $\underline{n} = \alpha \cup \alpha'$. We may assume without loss of generality that, for each constraint (σ, R) appearing in $\mathcal{C}$, the associated algebra $\mathbf{R}$ is a subdirect product of $\prod_{\underline{m}} \mathbf{A}_{\sigma(i)}$, where $\mathbf{A}_{\sigma(i)} \cong \mathbf{Sq}_3$ for all $\sigma(i) \in \alpha$, and $\mathbf{A}_{\sigma(i)} \cong \mathbf{S}_2$ for all $\sigma(i) \in \alpha'$.

Let 0 denote the bottom element of each semilattice. For each constraint (σ, R), let $R\big|_{\overleftarrow{\alpha}}$ be an abbreviation for $R\big|_{\sigma^{-1}(\alpha \cap \operatorname{im} \sigma)}$, which is the projection of R onto the factors in its scope whose indices lie in α. Let $\mathbf{0}$ denote a tuple of 0's of length $|\alpha' \cap \operatorname{im} \sigma|$. Then Corollary 6.12 implies that for each constraint $(\sigma, R) \in \mathcal{C}$, we have

$$R\big|_{\overleftarrow{\alpha}} \times \{\mathbf{0}\} \subseteq R. \tag{7.4}$$

Obviously, if $\mathcal{I}$ has a solution, then the α-partial instance $\mathcal{I}_\alpha$ (i.e., the restriction of $\mathcal{I}$ to the abelian factors) also has a solution. Conversely, if $f \in \prod_\alpha A_i$ is a solution to $\mathcal{I}_\alpha$, then (7.4) implies that $g \in \prod_{\underline{n}} A_i$ defined by

$$g(i) = \begin{cases} f(i), & \text{if } i \in \alpha, \\ 0, & \text{if } i \in \alpha', \end{cases}$$

is a solution to $\mathcal{I}$. We conclude that $\mathcal{I}$ has a solution if and only if the α-partial instance has a solution. Since abelian algebras yield tractable CSPs, the proof is complete. ☐

Now we address the tractability of the example at hand.

Proposition 7.3. *If $\langle \mathbf{A}, \cdot \rangle$ is the four-element CIB with operation table given in Figure 1, then* $\mathrm{CSP}(\mathsf{S}(\mathbf{A}))$ *is tractable.*

Proof. Let $\mathcal{I} = \langle \underline{n}, \mathcal{A}, \mathcal{C} \rangle$ be an n-variable instance of $\mathrm{CSP}(\mathsf{S}(\mathbf{A}))$ specified by $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1}) \in \mathsf{S}(\mathbf{A})^n$ and $\mathcal{C} = ((\sigma_0, R_0), (\sigma_1, R_1), \dots, (\sigma_{J-1}, R_{J-1}))$. For each $i \in \underline{n}$, let

$$\theta_i = \begin{cases} \Theta, & \text{if } A_i = A, \\ 0_{A_i}, & \text{if } A_i \neq A, \end{cases} \tag{7.5}$$

and define $\boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_{n-1})$. The universes A_i/θ_i of the quotient algebras defined above satisfy

$$A_i/\theta_i = \begin{cases} \{\{0,1\}, \{2\}, \{3\}\}, & \text{if } A_i = \{0,1,2,3\}, \\ \{\{1\}, \{2\}, \{3\}\}, & \text{if } A_i = \{1,2,3\}, \\ \{\{0\}, \{1\}\}, & \text{if } A_i = \{0,1\}. \end{cases}$$

In the first two cases $\mathbf{A}_i/\theta_i$ is isomorphic to $\mathbf{Sq}_3$ and in the third case $\mathbf{A}_i/\theta_i$ is a 2-element meet semilattice. (Singleton factors, $\mathbf{A}_i/\theta_i = \{a\}$, are ignored because all solutions must obviously assign the associated variable to the value a.)

Consider the quotient instance $\mathcal{I}/\boldsymbol{\theta}$. From the observations in the previous paragraph we see that $\mathcal{I}/\boldsymbol{\theta}$ is an instance of $\mathrm{CSP}(\mathfrak{A})$, where $\mathfrak{A} = \{\mathbf{Sq}_3, \mathbf{S}_2\}$. By Lemma 7.2, $\mathcal{I}/\boldsymbol{\theta}$ can be solved in polynomial-time. If $\mathcal{I}/\boldsymbol{\theta}$ has no solution, then neither does $\mathcal{I}$. Otherwise, let $f \in \mathrm{Sol}(\mathcal{I}/\boldsymbol{\theta}, \underline{n})$ be a solution to the quotient instance.

We will use f to construct a solution $g \in \mathrm{Sol}(\mathcal{I}, \underline{n})$ to the original instance. In doing so, we will occasionally map a singleton set to the element it contains; observe that set union is such a map: $\bigcup\{x\} = x$.

Assume $\underline{n} = \alpha \cup \beta \cup \gamma$ is a disjoint union where

$$A_i = \begin{cases} \{0,1,2,3\}, & \text{if } i \in \alpha, \\ \{1,2,3\}, & \text{if } i \in \beta, \\ \{0,1\}, & \text{if } i \in \gamma. \end{cases}$$

In other words, $\prod_{\underline{n}} A_i = \{0,1,2,3\}^{\alpha} \times \{1,2,3\}^{\beta} \times \{0,1\}^{\gamma}$. Then, according to definition (7.5), θ_i is Θ for $i \in \alpha$ and 0_{A_i} for $i \notin \alpha$. Thus, the i-th entry of $\mathcal{A}/\boldsymbol{\theta}$ is

$$A_i/\theta_i = \begin{cases} \{\{0,1\},\{2\},\{3\}\}, & \text{if } i \in \alpha, \\ \{\{1\},\{2\},\{3\}\}, & \text{if } i \in \beta, \\ \{\{0\},\{1\}\}, & \text{if } i \in \gamma. \end{cases}$$

The i-th value $f(i)$ of the quotient solution must belong to A_i/θ_i. Let $\alpha = \alpha_0 \cup \alpha_1$ be such that $f(i) = \{0,1\}$ when $i \in \alpha_0$ and $f(i) \in \{\{2\},\{3\}\}$ when $i \in \alpha_1$. Then,

$$f \in \{\{0,1\}\}^{\alpha_0} \times \{\{2\},\{3\}\}^{\alpha_1} \times \{\{1\},\{2\},\{3\}\}^{\beta} \times \{\{0\},\{1\}\}^{\gamma} \tag{7.6}$$

Note that $f(i)$ is a singleton for all $i \notin \alpha_0$, and recall that $\bigcup\{a\} = a$. Thus, the following defines an element of $\prod_{\underline{n}} A_i$:

$$g(i) = \begin{cases} 0, & \text{if } i \in \alpha_0 \cup \gamma, \\ \bigcup f(i), & \text{if } i \in \alpha_1 \cup \beta. \end{cases}$$

We will prove that g is a solution to the instance $\mathcal{I}$; that is, $g \in \mathrm{Sol}(\mathcal{I}, \underline{n})$. Recall, this holds if and only if $g \circ \sigma \in R$ for each constraint $(\sigma, R) \in \mathcal{C}$.

Fix an arbitrary constraint $(\sigma, R) \in \mathcal{C}$, let $S := \operatorname{im} \sigma$ be the set of indices in the scope of R, and let $S^c := \underline{n} - S$ denote the complement of S with respect to $\underline{n}$. Since we assumed f solves $\mathcal{I}/\boldsymbol{\theta}$ and satisfies (7.6) there must exist $\mathbf{r} \in \prod_{\underline{n}} A_i$ satisfying not only (σ, R) but also the following: $\mathbf{r}(i) \in \{0,1\}$ for $i \in \alpha_0$ and $\mathbf{r}(i) = \bigcup f(i)$ for $i \notin \alpha_0$. (Otherwise f wouldn't satisfy the quotient constraint $(\sigma, R/\boldsymbol{\theta})$.)

We now describe other elements satisfying the (arbitrary) constraint (σ, R) that we will use to prove g is a solution. By subdirectness of R, for each $\ell \in \alpha_0 \cup \gamma$ there exists $\mathbf{x}^{\ell} \in \prod_{\underline{n}} A_i$ satisfying (σ, R) and $\mathbf{x}^{\ell}(\ell) = 0$. Since $\mathbf{r}$ and $\mathbf{x}^{\ell}$ satisfy (σ, R)—that is $\mathbf{r} \circ \sigma \in R$ and $\mathbf{x}^{\ell} \circ \sigma \in R$—we also have $t(\mathbf{r}, \mathbf{x}^{\ell}) \circ \sigma \in R$. That is, $t(\mathbf{r}, \mathbf{x}^{\ell})$ satisfies the constraint (σ, R). Let's compute the entries of $t(\mathbf{r}, \mathbf{x}^{\ell})$ using the partition $\underline{n} = \alpha_0 \cup \alpha_1 \cup \beta \cup \gamma$ defined above. First, since $\mathbf{x}^{\ell}(\ell) = 0$, we have $t(\mathbf{r}, \mathbf{x}^{\ell})(\ell) = 0$. For $i \neq \ell$, we observe the following:

- If $i \in \alpha_0 - \{\ell\}$, then $\mathbf{r}(i) \in \{0,1\}$, so $t(\mathbf{r}, \mathbf{x}^{\ell})(i) \in \{0,1\}$.
- If $i \in \alpha_1$, then $\mathbf{r}(i) = \bigcup f(i) \in \{2,3\}$, so $t(\mathbf{r}, \mathbf{x}^{\ell})(i) = \bigcup f(i)$.
- If $i \in \beta$, then $\mathbf{r}(i) = \bigcup f(i) \in \{1,2,3\}$, so $t(\mathbf{r}, \mathbf{x}^{\ell})(i) = \bigcup f(i)$.
- If $i \in \gamma - \{\ell\}$, then $\mathbf{r}(i) = \bigcup f(i) \in \{0,1\}$, so $t(\mathbf{r}, \mathbf{x}^{\ell})(i) \in \{0,1\}$.

To summarize, for each $\ell \in \alpha_0 \cup \gamma$, there exists $t(\mathbf{r}, \mathbf{x}^\ell) \in \prod_{\underline{n}} A_i$ satisfying (σ, R) and having values $t(\mathbf{r}, \mathbf{x}^\ell)(i) \in \{0, 1\}$ for $i \in \alpha_0 \cup \gamma$ and

$$t(\mathbf{r}, \mathbf{x}^\ell)(i) = \begin{cases} 0, & i = \ell, \\ \bigcup f(i), & i \in \alpha_1 \cup \beta. \end{cases}$$

Finally, if we take the product of all members of $\{t(\mathbf{r}, \mathbf{x}^\ell) \mid \ell \in \alpha_0 \cup \gamma\}$ with respect to the binary operation of the original algebra $\langle A, \cdot \rangle$, then we find

$$t(\mathbf{r}, \mathbf{x}^{\ell_1}) \cdot t(\mathbf{r}, \mathbf{x}^{\ell_2}) \cdot \dots \cdot t(\mathbf{r}, \mathbf{x}^{\ell_L}) = g,$$

where $L = |\alpha_0 \cup \gamma|$. Therefore, $g \in \operatorname{Sol}((\sigma, R), \underline{n})$. Since (σ, R) was an arbitrary constraint, we have proved $g \in \operatorname{Sol}(\mathcal{I}, \underline{n})$, as desired. □

7.3. *The least block algorithm*

This section describes an algorithm that was apparently discovered by "some subset of" Petar Marković and Ralph McKenzie. We first learned about this by reading Marković's slides [22] from a 2011 talk in Krakow which gives a one-slide description of the algorithm. Since no proof (or even formal statement of the assumptions) from the originators seems to be forthcoming, we include our own version and proof in this section.

Let $\mathbf{A}$ be a finite idempotent algebra and let θ be a congruence on $\mathbf{A}$. Recall that by idempotence, every θ-class of A will be a subalgebra of $\mathbf{A}$. Suppose that, as an algebra, each θ-class comes from a variety, E, possessing a $(k+1)$-ary edge term, e. Define $s(x, y) = e(y, y, x, x, \dots, x)$. Then according to Definition 6.4,

$$E \models s(x, y) \approx x.$$

By iterating s on its second variable, we obtain a binary term t such that

$$\mathbf{A} \models t(x, t(x, y)) \approx t(x, y) \tag{7.7}$$

$$E \models t(x, y) \approx x. \tag{7.8}$$

Assume, finally, that the induced algebra $\langle A/\theta, t^{\mathbf{A}/\theta} \rangle$ is a linearly ordered meet semilattice. When this occurs, let us call $\mathbf{A}$ an *Edge-by-Chain* (EC) algebra. Our objective in this subsection is to prove the following theorem.

Theorem 7.4. *Every finite, idempotent EC algebra is tractable.*

Proof. Recall that to call an algebra, $\mathbf{A}$, tractable is to assert that the problem $\operatorname{CSP}(\mathsf{S}(\mathbf{A}))$ is solvable in polynomial time. Let $\mathbf{A}$ be finite, idempotent and EC. We continue to use the congruence θ and term t developed above. Let $\mathcal{I}$ be an instance of the problem $\operatorname{CSP}(\mathsf{S}(\mathbf{A}))$. Thus $\mathcal{I}$ is a triple $\langle \mathcal{V}, \mathcal{A}, \mathcal{C} \rangle$ in which $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1})$ is a list of subalgebras of $\mathbf{A}$, see Definition 6.1. The congruence θ induces a congruence (which we will continue to call θ) on each $\mathbf{A}_i$, and the term

t will induce a linearly ordered semilattice structure on A_i/θ. In other words, each $\mathbf{A}_i$ is also an EC algebra.

Therefore, for each $i < n$, the structure of $\langle A_i/\theta, t\rangle$ will be of the form $C_{i,0} < C_{i,1} < \cdots < C_{i,q_i}$ in which each $C_{i,j}$ is a θ-class of $\mathbf{A}_i$. We have the following useful relationship.

$$i < n,\ j \leqslant k,\ u \in C_{i,j},\ v \in C_{i,k} \implies t(u,v) = u. \tag{7.9}$$

To see this, let $w = t(u,v)$. Since the quotient structure is a semilattice under t, $w \in C_{i,j}$. Therefore, by (7.7), $t(u,w) = t(u,t(u,v)) = t(u,v) = w$. On the other hand, since $u, w \in C_{i,j} \in E$ we have $t(u,w) = u$ by (7.8). Thus $u = w$.

Recall that for an index $k \leqslant n$, the kth-partial instance of $\mathcal{I}$ is the instance, $\mathcal{I}_{\underline{k}}$, obtained by restricting $\mathcal{I}$ to the first k variables (Section 7). The set of solutions to $\mathcal{I}_{\underline{k}}$ is denoted $\mathrm{Sol}(\mathcal{I}, \underline{k})$.

For every $k < n$ we recursively define the index j_k by

$$j_k = \text{least } j \leqslant q_k \text{ such that } \mathrm{Sol}(\mathcal{I}, \underline{k+1}) \cap \prod_{i=1}^{k} C_{i,j_i} \neq \emptyset.$$

If, for some k, no such j exists, then $j_k,\ j_{k+1}, \ldots$ are undefined. The proof of the theorem follows from the following two assertions.

Claim 7.5.

(1) $\mathcal{I}$ has a solution if and only if j_{n-1} is defined.
(2) j_i can be computed in polynomial time for all $i < n$.

One direction of the first claim is immediate: if j_{n-1} is defined then $\mathrm{Sol}(\mathcal{I}) = \mathrm{Sol}(\mathcal{I}, \underline{n})$ is nonempty. We address the converse. Assume that $\mathcal{I}$ has solutions. We argue, by induction on k, that every j_k is defined. For the base step, choose any $g \in \mathrm{Sol}(\mathcal{I})$. Since $g(0) \in A_0$, there is some ℓ such that $g(0) \in C_{0,\ell}$. Then j_0 is defined and is at most ℓ.

Now assume that $j_0, j_1, \ldots, j_{k-1}$ are defined, but j_k is undefined. Let $f \in \mathrm{Sol}(\mathcal{I}, \underline{k}) \cap \prod_{i=0}^{k-1} C_{i,j_i}$. Since j_k is undefined, f has no extension to a member of $\mathrm{Sol}(\mathcal{I}, \underline{k+1})$. We shall derive a contradiction. Since $\mathcal{I}$ has a solution, the restricted instance $\mathcal{I}_{k+1}$ certainly has solutions. Choose any $g \in \mathrm{Sol}(\mathcal{I}, k+1)$ with $g(k) \in C_{k,\ell}$ for the smallest possible ℓ. Let $\mathcal{C} = \big((\sigma_0, R_0), (\sigma_1, R_1), \ldots, (\sigma_{J-1}, R_{J-1})\big)$ be the list of constraints of $\mathcal{I}$. By assumption, for each $m < J$, f is a solution to $(\sigma_m, R_m)\big|_{\underline{k}}$ (the restriction of R_m to its first k variables). Therefore, there is an extension, f_m, of f, to $k+1$ variables such that f_m is a solution to $(\sigma_m, R_m)\big|_{\underline{k+1}}$.

Let $m < J$ and set $h_m = t(g, f_m)$. Since both g and f_m are solutions to $(\sigma_m, R_m)\big|_{\underline{k+1}}$, and t is a term, h_m is also a solution to $(\sigma_m, R_m)\big|_{\underline{k+1}}$. For every $i < k$ we have $f_m(i) = f(i) = f_0(i)$. Thus

$$h_m(i) = t\big(g(i), f_m(i)\big) = t\big(g(i), f_0(i)\big) = h_0(i), \text{ for } i < k \text{ and}$$
$$h_m(k) = t\big(g(k), f_m(k)\big) = g(k).$$

The latter relation holds by (7.9) and the choice of g. It follows that every h_m coincides with h_0. Since h_m satisfies R_m, we have $h_0 \in \mathrm{Sol}(\mathcal{I}, k+1)$. This contradicts our assumption that f has no extension to a member of $\mathrm{Sol}(\mathcal{I}, k+1)$.

Finally, we must verify the second claim. Assume we have computed j_i, for $i < k$. To compute j_k we proceed as follows.

for $j = 0$ to q_k **do**
  $\mathcal{B} \leftarrow (\mathbf{C}_{0,j_0}, \mathbf{C}_{1,j_1}, \ldots, \mathbf{C}_{k-1,j_{k-1}}, \mathbf{C}_{k,j})$
  if $\mathcal{I}_{\mathcal{B}}$ has a solution in CSP($\mathcal{B}$) **then return** j
return FAILURE

In this algorithm, $\mathcal{I}_{\mathcal{B}}$ is the instance of CSP($\mathcal{B}$) obtained by first restricting $\mathcal{I}$ to its first $k+1$ variables, and then restricting each constraint relation to $\prod \mathcal{B}$. Since the members of $\mathcal{B}$ all lie in the edge-term variety E, it follows from Theorems 6.5 and 6.6 that CSP($\mathcal{B}$) runs in polynomial-time. □

8. CSPs of Commutative Idempotent Binars

Taylor terms were defined in Section 2.2.2. It is not hard to see that a binary term is Taylor if and only if it is idempotent and commutative. This suggests, in light of the algebraic CSP-dichotomy conjecture, that we study commutative, idempotent binars (CIB's for short), that is, algebras with a single basic binary operation that is commutative and idempotent. If the conjecture is true, then every finite CIB should be tractable.

The associative CIBs are precisely the semilattices. Finite semilattices have long been known to be tractable, [18]. The variety of semilattices is SD-$\wedge$ and is equationally complete. Every nontrivial semilattice must contain a subalgebra isomorphic to the unique two-element semilattice, $\mathbf{S}_2 = \langle\{0,1\}, \wedge\rangle$, and conversely, $\mathbf{S}_2$ generates the variety of all semilattices.

As we shall see, the algebra $\mathbf{S}_2$ plays a central role in the structure of CIBs. In particular, the omission of $\mathbf{S}_2$ implies tractability.

Theorem 8.1. *If* $\mathbf{A}$ *is a finite* CIB *then the following are equivalent:*

(1) $\mathbf{A}$ *has an edge term.*
(2) $\mathcal{V}(\mathbf{A})$ *is congruence modular.*
(3) $\mathbf{S}_2 \notin \mathsf{HS}(\mathbf{A})$

The proof that (1) implies (2) appears in [8] and holds for general finite idempotent algebras. The contrapositive of (2) implies (3) is easy: if $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$, then $\mathbf{S}_2^2 \in \mathcal{V}(\mathbf{A})$, and the congruence lattice of $\mathbf{S}_2^2$ is not modular. So it remains to show that (3) implies (1). For this we use the idea of a "cube-term blocker" ([20]). A *cube term blocker* (CTB) for an algebra $\mathbf{A}$ is a pair (D, S) of subuniverses with the following properties: $\emptyset < D < S \leqslant A$ and for every term $t(x_0, x_1, \ldots, x_{n-1})$ of $\mathbf{A}$ there is an index $i \in \underline{n}$ such that, for all $\mathbf{s} = (s_0, s_1, \ldots, s_{n-1}) \in S^n$, if $s_i \in D$ then

$t(\mathbf{s}) \in D$.

Theorem 8.2 ([20, Thm 2.1]). *Let* $\mathbf{A}$ *be a finite idempotent algebra. Then* $\mathbf{A}$ *has an edge term iff it possesses no cube-term blockers.*

(The notions of cube term and edge term both originate in [8]. In that paper it is proved that a finite algebra has a cube term if and only if it has an edge term.)

Lemma 8.3. *A finite* CIB $\mathbf{A}$ *has a* CTB *if and only if* $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$.

Proof. Assume (D, S) is a CTB for $\mathbf{A}$. Then there exists $s \in S - D$. Consider $D^+ := D \cup \{s\}$. Evidently D^+ is a subuniverse of $\mathbf{A}$, and if $\mathbf{D}^+$ denotes the corresponding subalgebra, then $\theta_D := D^2 \cup \{(s, s)\}$ is a congruence of $\mathbf{D}^+$. It's easy to see that $\mathbf{D}^+/\theta_D \cong \mathbf{S}_2$, so $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$.

Conversely, if $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$, then there exists $\mathbf{B} \leqslant \mathbf{A}$ and a surjective homomorphism $h\colon \mathbf{B} \to \mathbf{S}_2$. Let $B_0 = h^{-1}(0)$. Then $\emptyset \neq B_0 < B \leqslant A$, and (B_0, B) is a CTB for $\mathbf{A}$. □

Lemma 8.3, along with Theorem 8.2, completes the proof of Theorem 8.1 by showing (1) is false if and only if (3) is false. In fact, something stronger is true. Kearnes has shown that if $\mathcal{V}$ is any variety of CIBs that omits $\mathbf{S}_2$, then $\mathcal{V}$ is congruence permutable.

Corollary 8.4. *Let* $\mathbf{A}_0$, $\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ *be finite* CIBs *satisfying the equivalent conditions in Theorem 8.1. Then both* $\mathbf{A}_0 \times \cdots \times \mathbf{A}_{n-1}$ *and* $\mathbf{A}_0 \times \cdots \times \mathbf{A}_{n-1} \times \mathbf{S}_2$ *are tractable.*

Proof. By the theorem, $\mathcal{V}(\mathbf{A}_0)$ and $\mathcal{V}(\mathbf{A}_1)$ each have an edge term. By Theorem 1.2, $\mathsf{H}\big(\mathcal{V}(\mathbf{A}_0) \circ \mathcal{V}(\mathbf{A}_1)\big)$ has an edge term as well. Since $\mathbf{A}_0 \times \mathbf{A}_1$ lies in this variety, it possesses that same edge term. Iterating this process, the algebra $\mathbf{B} = \mathbf{A}_0 \times \cdots \times \mathbf{A}_{n-1}$ has an edge term, hence is tractable by Theorem 6.5.

Let $t(x_1, \ldots, x_{k+1})$ be an edge term for $\mathbf{B}$. Then according to the first identity in Definition 6.4, $\mathcal{V}(\mathbf{B})$ is a strongly irregular variety. Consequently, the algebra $\mathbf{B} \times \mathbf{S}_2$ is a Płonka sum of two copies of $\mathbf{B}$. As a Płonka sum of tractable algebras, Theorem 4.1 of [7] implies that $\mathbf{B} \times \mathbf{S}_2$ is tractable as well. □

It follows from the corollary and Theorem 6.6 that every finite member of $\mathcal{V}(\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1} \times \mathbf{S}_2)$ is at least locally tractable.

As a counterpoint to Theorem 8.1, we mention the following. The proof requires some basic knowledge of tame congruence theory.

Theorem 8.5. *Let* $\mathbf{A}$ *be a finite* CIB *and suppose that* $\mathsf{HS}(\mathbf{A})$ *contains no nontrivial abelian algebras. Then* $\mathcal{V}(\mathbf{A})$ *is* SD-$\wedge$. *Consequently,* $\mathbf{A}$ *is a tractable algebra.*

Proof. Since every CIB has a Taylor term, and since $\mathsf{HS}(\mathbf{A})$ has no abelian algebras, $\mathsf{S}(\mathbf{A})$ omits types $\{1, 2\}$. Then by [15, Cor 2.2], $\mathcal{V}(\mathbf{A})$ omits types $\{1, 2\}$. But then, by [16, Thm 9.10], $\mathcal{V}(\mathbf{A})$ is SD-$\wedge$. □

In a congruence-permutable variety, an algebra is abelian if and only if it is polynomially equivalent to a faithful, unital module over a ring. See [6, Thm 7.35] for a full discussion. For example, consider the ring $\mathbb{Z}_3$ as a module over itself. Define the binary polynomial $x \cdot y = 2x + 2y$ on this module. The table for this operation is given at the left of Figure 2. Conversely, we can retrieve the module operations from '$\cdot$' by $x + y = 0 \cdot (x \cdot y)$ and $2x = 0 \cdot x$. Consequently, the commutative idempotent binar $\mathbf{Sq}_3$ is abelian.

In light of Theorem 6.7, every finite, abelian CIB will be of this form. We make the following observation.

Proposition 8.6. *A finite, abelian CIB has odd order.*

Proof. Let $\mathbf{A}$ be a finite, abelian CIB. By Theorem 6.7 (since every CIB is Taylor), $\mathbf{A}$ is polynomially equivalent to a faithful, unital module M over a ring R. The basic operation of $\mathbf{A}$ must be a polynomial of M. Thus there are $r, s \in R$ and $b \in M$ such that $x \cdot y = rx + sy + b$. Let 0 denote the zero element of M. Then $0 = 0 \cdot 0 = r0 + s0 + b = b$. The condition $x \cdot y = y \cdot x$ implies $rx + sy = ry + sx$, so (by taking $y = 0$ and since M is faithful), $r = s$. Finally by idempotence, $2r = 1$.

Now, if A, hence M, has even cardinality, there is an element $a \in A$, $a \neq 0$, of additive order 2. Thus

$$a = a \cdot a = 2ra = 0$$

which is a contradiction. □

In an effort to demonstrate the utility of the techniques developed in this paper, we shall now show that every CIB of cardinality at most 4 is tractable. Of course, it is known that the algebraic dichotomy conjecture holds for every idempotent algebra of cardinality at most 3 [10,21]. However, our arguments are relatively short and may indicate why CIBs may be more manageable than arbitrary algebras.

For the remainder of this section, let $\mathbf{A}$ be a CIB with universe $\underline{n}$. We shall consider the various possibilities for n and $\mathbf{A}$ (up to isomorphism), and in each case show that it is tractable. In order to make it clear that our inventory is complete, a few of the arguments and computations are postponed until the end.

$n = 2$. It is easy to see that there is a unique 2-element CIB, namely $\mathbf{S}_2$. Idempotence determines the diagonal entries in the table, and by commutativity, the remaining two entries must be equal. The choices 0 and 1 for the off-diagonal yield a meet-semilattice and a join-semilattice respectively. As we remarked above, every finite semilattice is tractable, and, in fact, Sl, the variety of semilattices, is SD-$\wedge$.

$n = 3$, $\mathbf{A}$ not simple. There is a congruence, θ, on $\mathbf{A}$ with $0_A < \theta < 1_A$. It is important to note that by idempotence, every congruence class is a subalgebra. Based purely on cardinality concerns, $\mathbf{A}/\theta$ has cardinality 2, one θ class has 2 elements, the other has 1. From the uniqueness of the 2-element algebra, $\mathbf{A}/\theta$ and

$\cdot$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

$\mathbf{Sq}_3$

$\cdot$	0	1	2
0	0	0	1
1	0	1	2
2	1	2	2

$\mathbf{T}_1$

$\cdot$	0	1	2
0	0	0	2
1	0	1	1
2	2	1	2

$\mathbf{T}_2$

Fig. 2. The simple CIBs of cardinality 3

each of the θ-classes are semilattices. Consequently, $\mathbf{A} \in Sl \circ Sl$. By Theorem 1.2, $Sl \circ Sl$ is SD-$\wedge$, so by Theorem 6.3, $\mathbf{A}$ is tractable.

***n* = 3, A simple**. On the other hand, suppose that $\mathbf{A}$ is simple. If $\mathbf{A}$ has no proper nontrivial subalgebras, then in $\mathbf{A}$, $x \neq y \implies x \cdot y \notin \{x, y\}$. It follows that $\mathbf{A}$ must be the 3-element Steiner quasigroup, $\mathbf{Sq}_3$, of Figure 2. By Corollary 8.4, $\mathbf{A}$ is tractable.

Finally, if $\mathbf{A}$ has a proper nontrivial subalgebra, that subalgebra must be isomorphic to $\mathbf{S}_2$. Thus no nontrivial subalgebra of $\mathbf{A}$ is abelian. By Theorem 8.5, $\mathbf{A}$ generates an SD-$\wedge$ variety, so is tractable by Theorem 6.3. For future reference, there are 2 algebras meeting the description in this paragraph, $\mathbf{T}_1$ and $\mathbf{T}_2$. (This is easy to check by hand.) Their tables are given in Figure 2.

***n* = 4, A not simple**. Let θ be a maximal congruence of $\mathbf{A}$. Then $\mathbf{A}/\theta$ is simple, so from the previous few paragraphs, $\mathbf{A}/\theta$ is isomorphic to one of $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{Sq}_3$, or $\mathbf{S}_2$. If $\mathbf{A}/\theta \cong \mathbf{T}_i$, then the θ-classes must have size $1, 1, 2$. Consequently, each θ-class is a semilattice, so $\mathbf{A} \in Sl \circ \mathcal{V}(\mathbf{T}_i)$ which is SD-$\wedge$ by the computations above and Theorem 1.2. Thus $\mathbf{A}$ is tractable.

The next case to consider is $\mathbf{A}/\theta \cong \mathbf{Sq}_3$. Without loss of generality, assume that θ partitions the universe as $|01|2|3|$, and further that $0 \cdot 1 = 0$. From this data we deduce that the operation table for $\mathbf{A}$ must be

$\cdot$	0	1	2	3
0	0	0	3	2
1	0	1	3	2
2	3	3	2	a
3	2	2	a	3

with $a \in \{0, 1\}$.

If $a = 0$, then the only way to obtain 1 as a product is $1 \cdot 1$. In that case there is a homomorphism onto $\mathbf{S}_2$ with kernel $\psi = |023|1|$. We have $\theta \cap \psi = 0_A$, so $\mathbf{A}$ is a subdirect product of $\mathbf{Sq}_3 \times \mathbf{S}_2$. Therefore, by 8.4, $\mathbf{A}$ is tractable.

More problematic is the case $a = 1$. We established tractability of that algebra in Example 7.1.

Finally, suppose that $\mathbf{A}/\theta \cong \mathbf{S}_2$. The possible sizes of the θ-classes are 3,1 or 2,2. If neither class is isomorphic to $\mathbf{Sq}_3$ then the classes both lie in an SD-$\wedge$ variety, so by Theorem 1.2, $\mathbf{A}$ too lies in an SD-$\wedge$ variety, and is tractable. On the other hand, suppose that one of the classes is isomorphic to $\mathbf{Sq}_3$, (there are 7 such algebras).

	u_2	u_3	Proper Nontrivial Subalgebras
$\mathbf{A}_0$	0	1	$\{0,1\}$, $\{0,2\}$, $\{1,2,3\}$
$\mathbf{A}_1$	1	1	$\{0,1\}$, $\{1,2,3\}$
$\mathbf{A}_2$	1	2	$\{0,1\}$, $\{1,2,3\}$
$\mathbf{A}_3$	0	3	$\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,2,3\}$
$\mathbf{A}_4$	1	3	$\{0,1\}$, $\{0,3\}$, $\{1,2,3\}$
$\mathbf{A}_5$	2	2	$\{0,1\}$, $\{0,2\}$, $\{0,3\}$ $\{1,2,3\}$
$\mathbf{A}_6$	2	3	$\{0,1\}$, $\{0,2\}$, $\{0,3\}$ $\{1,2,3\}$

Fig. 3. 7 simple CIBs of size 4.

Then $\mathbf{A}$ is an EC algebra (with $t(x,y) = y \cdot (x \cdot y)$) and we can apply Theorem 7.4 to establish that $\mathbf{A}$ is tractable.

***n* = 4, A simple**. By Theorem 8.5, if $\mathsf{HS}(\mathbf{A})$ contains no nontrivial abelian algebra, then $\mathbf{A}$ is tractable. Thus, we may as well assume that $\mathsf{HS}(\mathbf{A})$ contains an abelian algebra. By Proposition 8.6, $\mathbf{A}$ itself is nonabelian. Consequently (since $\mathbf{A}$ is simple) $\mathbf{A}$ must have a subalgebra isomorphic to $\mathbf{Sq}_3$. Without loss of generality, let us assume that $\{1,2,3\}$ forms this subalgebra.

Similarly, by Corollary 8.4, we can assume that $\mathsf{HS}(\mathbf{A})$ contains a copy of $\mathbf{S}_2$. Examining the 2- and 3-element CIBs, it is easy to see that if $\mathbf{S}_2 \in \mathsf{HS}(\mathbf{A})$, then already $\mathbf{S}_2 \in \mathsf{S}(\mathbf{A})$. By the symmetry of $\mathbf{Sq}_3$, and the fact that it contains no semilattice, we can assume that $\{0,1\}$ forms the semilattice.

Thus, the table for $\mathbf{A}$ must have one of the following two forms.

$\cdot$	0	1	2	3
0	0	0	u_2	u_3
1	0	1	3	2
2	u_2	3	2	1
3	u_3	2	1	3

$\cdot$	0	1	2	3
0	0	1	v_2	v_3
1	1	1	3	2
2	v_2	3	2	1
3	v_3	2	1	3

By checking the 32 possible tables, either via the universal algebra calculator [14] or directly using Freese's algorithm [12], one determines that there are 7 pairwise nonisomorphic algebras of one of these two forms. Interestingly, every simple algebra of the second form is isomorphic to one of the first form. Thus we will use the first form for all the candidates. These 7 algebras are indicated in Figure 3.

Our intent is to apply the rectangularity theorem to establish the tractability of these 7 algebras. For this, we need to determine the minimal absorbing subuniverses of each subalgebra of $\mathbf{A}$. First observe that if $\{a,b\}$ forms a copy of $\mathbf{S}_2$ with $a \cdot b = a$, then $\{a\} \triangleleft_t \{a,b\}$, with $t(x,y) = x \cdot y$, while $\{b\}$ is not absorbing since a is a sink (Lemma 4.9). Second, by Lemma 4.12, $\mathbf{Sq}_3$ is absorption-free, which is to say, it is its own minimal absorbing subalgebra.

Now let B be a proper minimal absorbing subalgebras of $\mathbf{A}_i$, for $i < 7$. Then $B \cap \{1,2,3\}$ is absorbing in $\{1,2,3\}$ (Lemma 4.7). As $\{1,2,3\} \cong \mathbf{Sq}_3$ is absorption free and B is a proper subalgebra of $\mathbf{A}$, we must have $B = \{1,2,3\}$ or $B = \{0\}$. However the first of these is impossible because $\{1,2,3\} \cap \{0,1\} = \{1\}$, which is not absorbing in $\{0,1\}$ since 0 is a sink. Thus the only possible absorbing subalgebra of $\mathbf{A}$ is $\{0\}$.

For $i \leqslant 2$, $\{0\}$ is indeed absorbing in $\mathbf{A}_i$ with binary absorbing term $t(x,y) = \big(x(xy)\big)\big(y(xy)\big)$. However, observe that if $i > 2$ then according to Figure 3 either $u_2 = 2$ or $u_3 = 3$. If $u_2 = 2$ then 2 is a sink for the subuniverse $\{0,2\}$. In that case $\{0\} = B \cap \{0,2\}$ is not absorbing in $\{0,2\}$ contradicting the fact that B is absorbing in $\mathbf{A}$. A similar argument works for $u_3 = 3$ and $\{0,3\}$. Thus $\mathbf{A}_i$ is absorption free for $i > 2$.

We summarize these computations in the following table.

Algebra	Minimal Absorbing Subalgebra
$\mathbf{S}_2 = \{0,1\}$	$\{0\}$
$\{1,2,3\}$	$\{1,2,3\}$
$\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2$	$\{0\}$
$\mathbf{A}_3, \dots, \mathbf{A}_6$	A

The crucial point here is that every member of $\mathsf{S}(\mathbf{A}_i)$ has a unique minimal absorbing subalgebra.

We now proceed to argue that each algebra from Figure 3 is tractable. Following Definition 6.1, let $\mathcal{I} = \langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$ be an instance of CSP($\mathbf{A}_i$), for some $i < 7$. We have $\mathcal{A} = (\mathbf{C}_0, \mathbf{C}_1, \dots, \mathbf{C}_{n-1})$, in which each $\mathbf{C}_i$ is a subalgebra of $\mathbf{A}_i$, and $\mathcal{R} = (R_0, \dots, R_{J-1})$ is a list of subdirect products of the various $\mathbf{C}_k$'s. We must show that there is an algorithm that determines, in polynomial time, whether this instance has a solution. For this we will use Corollary 5.8.

For each $j < J$, let the relation $\bar{R}_j$ be obtained from R_j as follows. First, for any distinct variables $\ell, k \in \operatorname{im}(\sigma_j)$, if $\tilde{\eta}^j_\ell = \tilde{\eta}^j_k$, remove the k-th variable from the scope of R_j. Then, define $\bar{R}_j = R_j \times \prod_{\ell \notin \operatorname{im} \sigma_j} C_i$. Let $\bar{\mathcal{I}}$ denote the modified instance with $(\bar{R}_j : j < J)$ replacing $\mathcal{R}$ and every scope equal to $\mathcal{V}$. Then the solution set of $\bar{\mathcal{I}}$ is identical to the solution set of the original instance $\mathcal{I}$, and, in fact, is equal to $\bigcap_j \bar{R}_j$.

Dropping variable k does not affect the solution set: if $\tilde{\eta}^\ell = \tilde{\eta}^k$ then there is an isomorphism $h\colon \mathbf{C}_\ell \to \mathbf{C}_k$ such that $h \circ \operatorname{Proj}_\ell = \operatorname{Proj}_k$. Thus, the k-th coordinate of any tuple in R can be recovered by applying h to the ℓ-th coordinate. The second step in the transformation clearly does not lose any of the original solutions. Furthermore, it is easy to see that it does not create any new pairs ℓ, k with $\tilde{\eta}_\ell = \tilde{\eta}_k$. We stress that the creation of $\bar{R}$ from R is not part of the algorithm, so it is not necessary to execute in polynomial-time. It exists only for the purpose of this proof.

We now consider the 5 conditions in Corollary 5.8, applied to $\{\, \bar{R}_j : j < J \,\}$. Set $\alpha = \{\, \ell < n : \mathbf{C}_\ell \cong \mathbf{Sq}_3 \,\}$. Since $\mathbf{Sq}_3$ is the only possible abelian subalgebra of $\mathbf{A}_i$, the first two conditions will be satisfied. Our preprocessing ensures that condition 3 holds as well. We argue momentarily that condition 4 holds. Assuming for the moment that it does, then according to the corollary and our computations above, the following statements are equivalent.

- $\mathcal{I}$ has a solution;
- $\bar{\mathcal{I}}$ has a solution;
- $\bigcap_j \bar{R}_j \neq \emptyset$;
- $\bigcap_j \operatorname{Proj}_\alpha \bar{R}_j \neq \emptyset$;
- $\mathcal{I}_\alpha$ has a solution.

However, $\mathcal{I}_\alpha$ is an instance of CSP($\mathbf{Sq}_3$), which, as an abelian algebra, is known to be tractable. Thus by running the algorithm on this partial instance, we can determine the existence of a solution to the original instance, $\mathcal{I}$.

Thus, we are left with the task of verifying condition 4, that is, show that every $\bar{R}_j$ intersects $\prod B_\ell$. Recall that B_ℓ is a minimal absorbing subalgebra of $\mathbf{C}_\ell$. We have two cases to consider. First, suppose that $i < 3$. In that case the minimal absorbing subalgebra of $\mathbf{A}_i$ is $\{0\}$, which is, in fact, a sink of $\mathbf{A}_i$. Therefore, for every $\ell \in \alpha$, $B_\ell = \{1, 2, 3\} = C_\ell$, while for $\ell \notin \alpha$, $B_\ell = \{0\}$. Let $\bar{R}$ denote any $\bar{R}_j$. Since $\bar{R}$ is subdirect, for every $\ell \in \alpha'$ there is a tuple $\mathbf{r}^\ell \in R$ with $r^\ell_\ell = 0$. Let $\mathbf{r}$ be the product of all the $\mathbf{r}^\ell$'s (in any order). Then, since 0 is a sink, $\mathbf{r}_\ell = 0 \in B_\ell$ for every $\ell \in \alpha'$, while for $\ell \in \alpha$, $\mathbf{r}_\ell \in \mathbf{C}_\ell = B_\ell$. Thus the condition is satisfied.

The argument when $i \geqslant 3$ is not very different. In this case, let $\beta = \{\ell < n : \mathbf{C}_\ell = \mathbf{S}_2\}$. Once again, for $\ell \in \beta$ let $\mathbf{r}^\ell$ be a tuple with $\mathbf{r}^\ell_\ell = 0$, and let $\mathbf{r}$ be the product. Then for $\ell \in \beta$, $\mathbf{r}_\ell = 0 \in B_\ell$, while, for $\ell \notin \beta$, $\mathbf{r}_\ell \in B_\ell$ since $B_\ell = \{1, 2, 3\} = \mathbf{C}_\ell$ (if $\ell \in \alpha$) and $B_\ell = A_i$ otherwise.

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Appendix A. Miscellaneous Proofs

A.1. *Proof of Lemma 4.6*

We prove the following: Let $\mathbf{B}_i \leqslant \mathbf{A}_i$ $(0 \leqslant i < n)$ be algebras in the variety $\mathcal{V}$, let $\mathbf{B} := \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}$, and let $\mathbf{A} := \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$. If $\mathbf{B}_i \vartriangleleft_{t_i} \mathbf{A}_i$ (resp., $\mathbf{B}_i \vartriangleleft\!\!\vartriangleleft_{t_i} \mathbf{A}_i$) for each $0 \leqslant i < n$, then $\mathbf{B} \vartriangleleft_s \mathbf{A}$ (resp., $\mathbf{B} \vartriangleleft\!\!\vartriangleleft_s \mathbf{A}$) where $s := t_0 \star t_1 \star \cdots \star t_{n-1}$.

Proof. In case $n = 2$, the fact that $\mathbf{B} \vartriangleleft_s \mathbf{A}$ follows directly from Corollary 4.5. We first extend this result to *minimal* absorbing subalgebras, still for $n = 2$, and then an easy induction argument will complete the proof for arbitrary finite n.

Assume $\mathbf{B}_0 \vartriangleleft\!\!\vartriangleleft_{t_0} \mathbf{A}_0$ and $\mathbf{B}_1 \vartriangleleft\!\!\vartriangleleft_{t_1} \mathbf{A}_1$. Then, as mentioned, Corollary 4.5 implies $\mathbf{B} \vartriangleleft_s \mathbf{A}$, where $\mathbf{B} := \mathbf{B}_0 \times \mathbf{B}_1$, $\mathbf{A} := \mathbf{A}_0 \times \mathbf{A}_1$, and $s = t_0 \star t_1$. To show that $\mathbf{B}_0 \times \mathbf{B}_1$

is minimal absorbing, we let $\mathbf{S}$ be a proper subalgebra of $\mathbf{B}_0 \times \mathbf{B}_1$ and prove that $\mathbf{S}$ is not absorbing in $\mathbf{A}_0 \times \mathbf{A}_1$. By transitivity of absorption, it suffices to prove that $\mathbf{S}$ is not absorbing in $\mathbf{B}_0 \times \mathbf{B}_1$.

Let t be an arbitrary term of arity q. For each $b \in B_1$, the set

$$S^{-1}b := \{b_0 \in B_0 \mid (b_0, b) \in S\}$$

is easily seen to be a subuniverse of $\mathbf{B}_0$. Indeed, if v is a term of arity k and if $x_0, \ldots, x_{k-1}$ belong to $S^{-1}b$, then $(x_i, b) \in S$ and, since $\mathbf{S} \leqslant \mathbf{A}$, the element

$$\begin{aligned} v^{\mathbf{A}}((x_0, b), (x_1, b), \ldots, (x_{k-1}, b)) &= (v^{\mathbf{A}_0}(x_0, x_1, \ldots, x_{k-1}), v^{\mathbf{A}_1}(b, \ldots, b)) \\ &= (v^{\mathbf{A}_0}(x_0, x_1, \ldots, x_{k-1}), b) \end{aligned}$$

belongs to S. Therefore, $v^{\mathbf{A}_0}(x_0, x_1, \ldots, x_{k-1}) \in S^{-1}b$.

Since $\mathbf{S}$ is a proper subalgebra, there exists $b^* \in B_1$ such that $S^{-1}b^*$ is not all of B_0. Therefore, $S^{-1}b^*$ is a proper subuniverse of $\mathbf{B}_0$, so $S^{-1}b^*$ is not absorbing in $\mathbf{A}_0$ (by minimality of $\mathbf{B}_0$). Consequently,

$$\begin{gathered} \exists x_i \in S^{-1}b^*, \quad \exists b \in B_0, \quad \exists j < q, \quad \exists b' \notin S^{-1}b^* \\ \text{such that } \; t^{\mathbf{A}_0}(x_0, x_1, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_{q-1}) = b'. \end{gathered}$$

Therefore,

$$\begin{aligned} & t^{\mathbf{A}_0 \times \mathbf{A}_1}((x_0, b^*), \ldots, (x_{j-1}, b^*), (b, b^*), (x_{j+1}, b^*), \ldots, (x_{q-1}, b^*)) \\ & = (t^{\mathbf{A}_0}(x_0, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_{q-1}), t^{\mathbf{A}_1}(b^*, \ldots, b^*)) = (b', b^*) \end{aligned}$$

and $(b', b^*) \notin S$ since $b' \notin S^{-1}b^*$. Finally, because $(x_i, b^*) \in S$ for all i, and since t was an arbitrary term, it follows that $\mathbf{S}$ is not absorbing in $\mathbf{B}_0 \times \mathbf{B}_1$.

Now fix $n > 2$ and assume the result holds when there are at most $n-1$ factors. Let $\mathbf{B}' := \mathbf{B}_0 \times \cdots \times \mathbf{B}_{n-2}$ and $\mathbf{A}' := \mathbf{A}_0 \times \cdots \times \mathbf{A}_{n-2}$. By the induction hypothesis, $\mathbf{B}' \lhd\!\!\lhd_{s'} \mathbf{A}'$, and since $\mathbf{B}_{n-1} \lhd\!\!\lhd_{t_{n-1}} \mathbf{A}_{n-1}$ we have (by the $n = 2$ case) $\mathbf{B}' \times \mathbf{B}_{n-1} \lhd\!\!\lhd_s \mathbf{A}' \times \mathbf{A}_{n-1}$, where $s = s' \star t_{n-1}$. $\square$

A.2. *Proof of Lemma 4.10*

We prove the following: Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be finite idempotent algebras of the same type, and suppose $\mathbf{B}_i \lhd\!\!\lhd \mathbf{A}_i$ for $i = 1, \ldots, n$. Let $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, and let $R' = R \cap (B_1 \times \cdots \times B_n)$. If $R' \neq \emptyset$, then $\mathbf{R}' \leqslant_{\text{sd}} \mathbf{B}_1 \times \cdots \times \mathbf{B}_n$.

Proof. If $R' \neq \emptyset$, then for $1 \leqslant i \leqslant n$ the projection $S_i := \operatorname{Proj}_i R'$ is also nonempty. We want to show $S_i = B_i$. By minimality of $\mathbf{B}_i \lhd\!\!\lhd \mathbf{A}_i$ and by transitivity of absorption, it suffices to prove $\mathbf{S}_i \lhd \mathbf{B}_i$. Assume $\mathbf{B}_i \lhd\!\!\lhd \mathbf{A}_i$ with respect to t, say, $k = \operatorname{ar}(t)$. Fix $s_1, \ldots, s_k \in S_i$, $b \in B_1$, and $j \leqslant k$. Then

$$\tilde{b_i} := t^{\mathbf{A}_i}(s_1, \ldots, s_{j-1}, b, s_{j+1}, \ldots, s_k) \in B_i,$$

and we must show $\tilde{b_i} \in S_i$. (For this will prove $\mathbf{S}_i \lhd \mathbf{B}_i$.) Since $\mathbf{R}$ is subdirect, there exist $a_i \in A_i$ (for all $i \neq j$) such that $\mathbf{a}^* := (a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) \in R$. Also, for each $1 \leqslant j \leqslant n$ there exist $s_1^{(j)}, \ldots, s_k^{(j)}$ in S_j such that for all $1 \leqslant \ell \leqslant k$ we have

$$\mathbf{s}_\ell := (s_\ell^{(1)}, \ldots, s_\ell^{(i-1)}, s_\ell, s_\ell^{(i+1)}, \ldots, s_\ell^{(n)}) \in R'.$$

Since all these n-tuples belong to R, the following expression is also in R:

$$\begin{aligned}
&t^{\mathbf{A}_1 \times \cdots \times \mathbf{A}_n}(\mathbf{s}_1, \ldots, \mathbf{s}_{j-1}, \mathbf{a}^*, \mathbf{s}_{j+1}, \ldots, \mathbf{s}_k) \\
&= t^{\mathbf{A}_1 \times \cdots \times \mathbf{A}_n}((s_1^{(1)}, \ldots, s_1^{(i-1)}, s_1, s_1^{(i+1)}, \ldots, s_1^{(n)}), \ldots \\
&\qquad \ldots, (s_{j-1}^{(1)}, \ldots, s_{j-1}^{(i-1)}, s_{j-1}, s_{j-1}^{(i+1)}, \ldots, s_{j-1}^{(n)}), \\
&\qquad\qquad (a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n), \\
&\qquad\qquad\quad (s_{j+1}^{(1)}, \ldots, s_{j+1}^{(i-1)}, s_{j+1}, s_{j+1}^{(i+1)}, \ldots, s_{j+1}^{(n)}), \ldots \\
&\qquad\qquad\qquad \ldots, (s_k^{(1)}, \ldots, s_k^{(i-1)}, s_k, s_k^{(i+1)}, \ldots, s_k^{(n)})).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\big(t^{\mathbf{A}_1}(s_1^{(1)}, \ldots, s_{j-1}^{(1)}, a_1, s_{j+1}^{(1)}, \ldots, s_k^{(1)}), \ldots \\
&\qquad \ldots, t^{\mathbf{A}_{i-1}}(s_1^{(i-1)}, \ldots, s_{j-1}^{(i-1)}, a_{j-1}, s_{j+1}^{(i-1)}, \ldots, s_k^{(i-1)}), \\
&\qquad\qquad t^{\mathbf{A}_i}(s_1, \ldots, s_{j-1}, b, s_{j+1}, \ldots, s_k), \\
&\qquad\qquad\quad t^{\mathbf{A}_{i+1}}(s_1^{(i+1)}, \ldots, s_{j-1}^{(i+1)}, a_{j+1}, s_{j+1}^{(i+1)}, \ldots, s_k^{(i+1)}), \ldots \\
&\qquad\qquad\qquad \ldots t^{\mathbf{A}_n}(s_1^{(n)}, \ldots, s_{j-1}^{(n)}, a_n, s_{j+1}^{(n)}, \ldots, s_k^{(n)})\big),
\end{aligned}$$

which reduces to $(\tilde{b_1}, \ldots, \tilde{b_n})$. Since $\mathbf{B}_i$ is absorbing in $\mathbf{A}_i$, we see that $(\tilde{b_1}, \ldots, \tilde{b_n}) \in (B_1 \times \cdots \times B_n)$. Therefore, $(\tilde{b_1}, \ldots, \tilde{b_n}) \in R \cap B_1 \times \cdots \times B_n$, which means $\tilde{b_i} \in S_i$, as desired. Of course, the same argument works for all $1 \leqslant i \leqslant n$. □

A.3. *Proof of Lemma 4.12*

An very useful property of abelian algebras is that they are absorption-free. A proof of this appears in [5, Lem 4.1], but we include a proof in this section for easy reference and to keep the paper somewhat self-contained. First we require an elementary fact about functions on finite sets.

Fact A.1. If $f\colon X \to X$ is a (unary) function on a finite set X, then there is a natural number $k \geqslant 1$ such that the k-fold composition of f with itself is the same function as the $2k$-fold composition. That is, for all $x \in X$, $f^{2k}(x) = f^k(x)$.

Lemma 4.12. Finite idempotent abelian algebras are absorption-free.

Proof. Suppose $\mathbf{A}$ is a finite idempotent abelian algebra with $\mathbf{B} \lhd_t \mathbf{A}$. We show $\mathbf{B} = \mathbf{A}$. If t is unary, then by idempotence t is the identity function and absorption in this case means $t[A] \subseteq B$. It follows that $A = B$ and we're done. So assume t has arity $k > 1$. We will show that there must also be a $(k-1)$-ary term operation

$s \in \mathsf{Clo}(\mathbf{A})$ such that $\mathbf{B} \lhd_s \mathbf{A}$. It follows inductively that there must also be a unary absorbing term operation. Since a unary idempotent operation is the identity function, this will complete the proof.

Define a sequence of terms $t_0, t_1, \ldots$ as follows: for each $\mathbf{x} = (x_1, \ldots, x_{k-1}) \in A^{k-1}$ and $y \in A$,

$$\begin{aligned}
t_0(\mathbf{x}, y) &= t(\mathbf{x}, y),\\
t_1(\mathbf{x}, y) &= t(\mathbf{x}, t_0(\mathbf{x}, y)) = t(\mathbf{x}, t(\mathbf{x}, y)),\\
t_2(\mathbf{x}, y) &= t(\mathbf{x}, t_1(\mathbf{x}, y)) = t(\mathbf{x}, t(\mathbf{x}, t(\mathbf{x}, y))),\\
&\vdots\\
t_m(\mathbf{x}, y) &= t(\mathbf{x}, t_{m-1}(\mathbf{x}, y)) = t(\mathbf{x}, \ldots, t(\mathbf{x}, t(\mathbf{x}, t(\mathbf{x}, y))) \cdots)).
\end{aligned}$$

It is easy to see that $\mathbf{B}$ is absorbing in $\mathbf{A}$ with respect to t_m, that is, $\mathbf{B} \lhd_{t_m} \mathbf{A}$.

For each $\mathbf{x}_i \in A^{k-1}$, define $p_i : A \to A$ by $p_i(y) = t(\mathbf{x}_i, y)$. Then, $p_i^m(y) = t_m(\mathbf{x}_i, y)$, so by Fact A.1 there exists an $m_i \geqslant 1$ such that $p_i^{2m_i} = p_i^{m_i}$. That is, $t_{m_i}(\mathbf{x}_i, t_{m_i}(\mathbf{x}_i, y)) = t_{m_i}(\mathbf{x}_i, y)$. Let m be the product of all the m_i as $\mathbf{x}_i$ varies over A^{k-1}. Then, for all $\mathbf{x}_i \in A^{k-1}$, we have $p_i^{2m} = p_i^m$. Therefore, for all $\mathbf{x} \in A^{k-1}$, we have $t_m(\mathbf{x}, t_m(\mathbf{x}, y)) = t_m(\mathbf{x}, y)$.

We now show that the $(k-1)$-ary term operation s, defined for all $x_1, \ldots, x_{k-1} \in A$ by

$$s(x_1, \ldots, x_{k-2}, x_{k-1}) = t(x_1, \ldots, x_{k-2}, x_{k-1}, x_{k-1})$$

is absorbing for $\mathbf{B}$, that is, $\mathbf{B} \lhd_s \mathbf{A}$. It suffices to prove that $s[B \times \cdots \times B \times A] \subseteq B$. (For if the factor involving A occurs earlier, we appeal to absorption with respect to t.) So, for $\mathbf{b} \in B^{k-2}$ and $a \in A$, we will show $s(\mathbf{b}, a) = t_m(\mathbf{b}, a, a) \in B$. For all $b \in B$, we have

$$t_m(\mathbf{b}, b, a) = t_m(\mathbf{b}, b, t_m(\mathbf{b}, b, a)).$$

Therefore, if we apply (at the $(k-1)$-st coordinate) the fact that $\mathbf{A}$ is abelian, then we have

$$t_m(\mathbf{b}, a, a) = t_m(\mathbf{b}, a, t_m(\mathbf{b}, b, a)). \tag{A.1}$$

By absorption, $t_m(\mathbf{b}, b, a)$ belongs to B, thus so does the entire expression on the right of (A.1). This proves that $s(\mathbf{b}, a) = t_m(\mathbf{b}, a, a) \in B$, as desired. □

A.4. *Direct Proof of Corollary 6.12*

We prove the following: Let $\mathbf{A}_0, \ldots, \mathbf{A}_{n-1}$ be finite idempotent algebras in a Taylor variety and suppose $\mathbf{R} \leqslant_{\text{sd}} \prod \mathbf{A}_i$. For some $0 < k < n-1$, assume the following:

- if $0 \leqslant i < k$ then $\mathbf{A}_i$ is abelian;
- if $k \leqslant i < n$ then $\mathbf{A}_i$ has a sink $s_i \in A_i$.

Then $Z := R_{\underline{k}} \times \{s_k\} \times \{s_{k+1}\} \times \cdots \times \{s_{n-1}\} \subseteq R$, where $R_{\underline{k}} = \operatorname{Proj}_{\underline{k}} R$.

Proof. Since $\mathbf{A}_{\underline{k}} := \prod_{i<k} \mathbf{A}_i$ is abelian and lives in a Taylor variety, there exists a term m such that $m^{\mathbf{A}_{\underline{k}}}$ is a Mal'tsev term operation on $\mathbf{A}_{\underline{k}}$ (Theorem 6.7). Since we are working with idempotent terms, we can be sure that for each $i \in \underline{n}$ the term operation $m^{\mathbf{A}_i}$ is not constant (so depends on at least one of its arguments).

Fix $\mathbf{z} := (r_0, r_1, \ldots, r_{k-1}, s_k, s_{k+1}, \ldots, s_{n-1}) \in Z$. We will show that $\mathbf{z} \in R$. Since $\mathbf{z}_{\underline{k}} \in R_{\underline{k}}$, there exists $\mathbf{r} \in R$ whose first k elements agree with those of $\mathbf{z}$. That is, $\mathbf{r}_{\underline{k}} = (r_0, r_1, \ldots, r_{k-1}) = \mathbf{z}_{\underline{k}}$.

Now, since $\mathbf{R}$ is subdirect, there exists $\mathbf{x}^{(0)} \in R$ such that $\mathbf{x}^{(0)}(k) = s_k$, the sink in $\mathbf{A}_k$. If the term operation $m^{\mathbf{A}_k}$ depends on its second or third argument, consider $\mathbf{y}^{(0)} = m(\mathbf{r}, \mathbf{x}^{(0)}, \mathbf{x}^{(0)}) \in R$. (Otherwise, $m^{\mathbf{A}_k}$ depends on its first argument, so consider $\mathbf{y}^{(0)} = m(\mathbf{x}^{(0)}, \mathbf{x}^{(0)}, \mathbf{r})$.) For each $0 \leqslant i < k$ we have $\mathbf{y}^{(0)}(i) = m^{\mathbf{A}_i}(r_i, \mathbf{x}^{(0)}(i), \mathbf{x}^{(0)}(i)) = r_i$, since $m^{\mathbf{A}_i}$ is Mal'tsev. Thus, $\mathbf{y}^{(0)}_{\underline{k}} = \mathbf{z}_{\underline{k}}$. At index $i = k$, we have $\mathbf{y}^{(0)}(k) = m^{\mathbf{A}_k}(r_k, s_k, s_k) = s_k$, since s_k is a sink in $\mathbf{A}_k$. By the same argument, but starting with $\mathbf{x}^{(1)} \in R$ such that $\mathbf{x}^{(1)}(k+1) = s_{k+1}$, there exists $\mathbf{y}^{(1)} \in R$ such that $\mathbf{y}^{(1)}_{\underline{k}} = \mathbf{z}_{\underline{k}}$ and $\mathbf{y}^{(1)}(k+1) = s_{k+1}$.

Let t be any term of arity $\ell \geqslant 2$ that depends on at least two of its arguments, say, arguments p and q, and consider $t(\mathbf{y}^{(0)}, \ldots, \mathbf{y}^{(0)}, \mathbf{y}^{(1)}, \mathbf{y}^{(0)}, \ldots, \mathbf{y}^{(0)})$, where $\mathbf{y}^{(1)}$ appears as argument p (or q) and $\mathbf{y}^{(0)}$ appears elsewhere. By idempotence, and by the fact that s_k and s_{k+1} are sinks, we have

$$t\begin{pmatrix} (r_0, \ldots, r_{k-1}, & s_k, & *, & *, \ldots, *) \\ \vdots & & & \vdots \\ (r_0, \ldots, r_{k-1}, & s_k, & *, & *, \ldots, *) \\ (r_0, \ldots, r_{k-1}, & *, & s_{k+1}, & *, \ldots, *) \\ (r_0, \ldots, r_{k-1}, & s_k, & *, & *, \ldots, *) \\ \vdots & & & \vdots \\ (r_0, \ldots, r_{k-1}, & s_k, & *, & *, \ldots, *) \end{pmatrix} = (r_0, \ldots, r_{k-1}, s_k, s_{k+1}, *, \ldots, *),$$

where the wildcard $*$ represents unknown elements. Denote this element of R by $\mathbf{r}^{(1)} = (r_0, \ldots, r_{k-1}, s_k, s_{k+1}, *, \ldots, *)$. Continuing as above, we find $\mathbf{y}^{(2)} = (r_0, \ldots, r_{k-1}, *, *, s_{k+2}, *, \ldots, *) \in R$, and compute

$$\mathbf{r}^{(2)} := t(\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(1)}, \underset{\uparrow\, p\text{-th argument}}{\mathbf{y}^{(2)}}, \mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(1)}) = (r_0, \ldots, r_{k-1}, s_k, s_{k+1}, s_{k+2}, *, \ldots, *),$$

which also belongs to R. In general, once we have

$$\begin{aligned} \mathbf{r}^{(j)} &:= (r_0, \ldots, r_{k-1}, s_k, \ldots, s_{k+j}, *, \ldots, *) \in R, \text{ and} \\ \mathbf{y}^{(j+1)} &:= (r_0, \ldots, r_{k-1}, *, \ldots, *, s_{k+j+1}, *, \ldots, *) \in R, \end{aligned}$$

we compute

$$\begin{aligned}\mathbf{r}^{(j+1)} &= t(\mathbf{r}^{(j)}, \dots, \mathbf{r}^{(j)}, \mathbf{y}^{(j+1)}, \mathbf{r}^{(j)}, \dots, \mathbf{r}^{(j)}) \\ &= (r_0, \dots, r_{k-1}, s_k, \dots, s_{k+j+1}, *, \dots, *) \in R.\end{aligned}$$

Proceeding inductively in this way yields $\mathbf{z} = (r_0, \dots, r_{k-1}, s_k, \dots, s_{n-1}) \in R$, as desired. □

A.5. *Other elementary facts*

The remainder of this section collects some observations that can be useful when trying to prove that an algebra is abelian. We have moved the statements and proofs of these facts to the appendix since we didn't end up using any of them in the paper.

Denote the diagonal of A by $D(A) := \{(a, a) \mid a \in A\}$.

Lemma Appendix A.1. *An algebra* $\mathbf{A}$ *is abelian if and only if there is some* $\theta \in \mathrm{Con}(\mathbf{A}^2)$ *that has the diagonal set* $D(A)$ *as a congruence class.*

Proof. (⇐) Assume Θ is such a congruence. Fix $k < \omega$, $t^{\mathbf{A}} \in \mathsf{Clo}_{k+1}(\mathbf{A})$, $u, v \in A$, and $\mathbf{x}, \mathbf{y} \in A^k$. We will prove the implication (3.1), which in the present context is

$$t^{\mathbf{A}}(\mathbf{x}, u) = t^{\mathbf{A}}(\mathbf{y}, u) \quad \Longrightarrow \quad t^{\mathbf{A}}(\mathbf{x}, v) = t^{\mathbf{A}}(\mathbf{y}, v).$$

Since $D(A)$ is a class of Θ, we have $(u, u)\ \Theta\ (v, v)$, and since Θ is a reflexive relation, we have $(x_i, y_i)\ \Theta\ (x_i, y_i)$ for all i. Therefore,

$$t^{\mathbf{A}\times\mathbf{A}}((x_1, y_1), \dots, (x_k, y_k), (u, u))\ \Theta\ t^{\mathbf{A}\times\mathbf{A}}((x_1, y_1), \dots, (x_k, y_k), (v, v)). \tag{A.2}$$

since $t^{\mathbf{A}\times\mathbf{A}}$ is a term operation of $\mathbf{A} \times \mathbf{A}$. Note that (A.2) is equivalent to

$$(t^{\mathbf{A}}(\mathbf{x}, u), t^{\mathbf{A}}(\mathbf{y}, u))\ \Theta\ (t^{\mathbf{A}}(\mathbf{x}, v), t^{\mathbf{A}}(\mathbf{y}, v)). \tag{A.3}$$

If $t^{\mathbf{A}}(\mathbf{x}, u) = t^{\mathbf{A}}(\mathbf{y}, u)$ then the first pair in (A.3) belongs to the Θ-class $D(A)$, so the second pair must also belong this Θ-class. That is, $t^{\mathbf{A}}(\mathbf{x}, v) = t^{\mathbf{A}}(\mathbf{y}, v)$, as desired.

(⇒) Assume $\mathbf{A}$ is abelian. We show $\mathrm{Cg}^{\mathbf{A}^2}(D(A)^2)$ has $D(A)$ as a block. Assume

$$((x, x), (c, c')) \in \mathrm{Cg}^{\mathbf{A}^2}(D(A)^2). \tag{A.4}$$

It suffices to prove that $c = c'$. Recall, Mal'tsev's congruence generation theorem states that (A.4) holds iff

$$\begin{aligned}&\exists\, (z_0, z_0'), (z_1, z_1'), \dots, (z_n, z_n') \in A^2 \\ &\exists\, ((x_0, x_0'), (y_0, y_0')), ((x_1, x_1'), (y_1, y_1')), \dots, ((x_{n-1}, x_{n-1}'), (y_{n-1}, y_{n-1}')) \in D(A)^2 \\ &\exists\, f_0, f_1, \dots, f_{n-1} \in F^*_{\mathbf{A}^2}\end{aligned}$$

such that

$$\{(x,x),(z_1,z_1')\} = \{f_0(x_0,x_0'), f_0(y_0,y_0')\} \tag{A.5}$$
$$\{(z_1,z_1'),(z_2,z_2')\} = \{f_1(x_1,x_1'), f_1(y_1,y_1')\}$$
$$\vdots$$
$$\{(z_{n-1},z_{n-1}'),(c,c')\} = \{f_{n-1}(x_{n-1},x_{n-1}'), f_{n-1}(y_{n-1},y_{n-1}')\} \tag{A.6}$$

The notation $f_i \in F^*_{\mathbf{A}^2}$ means

$$\begin{aligned} f_i(x,x') &= g_i^{\mathbf{A}^2}((a_1,a_1'),(a_2,a_2'),\dots,(a_k,a_k'),(x,x')) \\ &= (g_i^{\mathbf{A}}(a_1,a_2,\dots,a_k,x), g_i^{\mathbf{A}}(a_1',a_2',\dots,a_k',x')), \end{aligned}$$

for some $g_i^{\mathbf{A}} \in \mathsf{Clo}_{k+1}(\mathbf{A})$ and some constants $\mathbf{a} = (a_1,\dots,a_k)$ and $\mathbf{a}' = (a_1',\dots,a_k')$ in A^k. Now, $((x_i,x_i'),(y_i,y_i')) \in D(A)^2$ implies $x_i = x_i'$, and $y_i = y_i'$, so in fact we have

$$\{(z_i,z_i'),(z_{i+1},z_{i+1}')\} = \{f_i(x_i,x_i), f_i(y_i,y_i)\} \quad (0 \leqslant i < n).$$

Therefore, by Equation (A.5) we have either

$$(x,x) = (g_i^{\mathbf{A}}(\mathbf{a},x_0), g_i^{\mathbf{A}}(\mathbf{a}',x_0)) \quad \text{or} \quad (x,x) = (g_i^{\mathbf{A}}(\mathbf{a},y_0), g_i^{\mathbf{A}}(\mathbf{a}',y_0)).$$

Thus, either $g_i^{\mathbf{A}}(\mathbf{a},x_0) = g_i^{\mathbf{A}}(\mathbf{a}',x_0)$ or $g_i^{\mathbf{A}}(\mathbf{a},y_0) = g_i^{\mathbf{A}}(\mathbf{a}',y_0)$. By the abelian assumption, if one of these equations holds, then so does the other. This and and Equation (A.5) imply $z_1 = z_1'$. Applying the same argument inductively, we find that $z_i = z_i'$ for all $1 \leqslant i < n$ and so, by (A.6) and the abelian property, we have $c = c'$. □

Lemma Appendix A.1 can be used to prove the next result which states that if there is a congruence of $\mathbf{A}_1 \times \mathbf{A}_2$ that has the graph of a bijection between A_1 and A_2 as a block, then both $\mathbf{A}_1$ and $\mathbf{A}_2$ are abelian algebras.

Lemma Appendix A.2. *Suppose $\rho\colon A_0 \to A_1$ is a bijection and suppose the graph $\{(x,\rho x) \mid x \in A_0\}$ is a block of some congruence $\beta \in \mathrm{Con}(A_0 \times A_1)$. Then both $\mathbf{A}_0$ and $\mathbf{A}_1$ are abelian.*

Proof. Define the relation $\alpha \subseteq (A_1 \times A_1)^2$ as follows: for $((a,a'),(b,b')) \in (A_1 \times A_1)^2$,

$$(a,a')\ \alpha\ (b,b') \quad \iff \quad (a,\rho a')\ \beta\ (b,\rho b')$$

We prove that the diagonal $D(A_1)$ is a block of α by showing that $(a,a)\ \alpha\ (b,b')$ implies $b = b'$. Indeed, if $(a,a)\ \alpha\ (b,b')$, then $(a,\rho a)\ \beta\ (b,\rho b')$, which means that $(b,\rho b')$ belongs to the block and $(a,\rho a)/\beta = \{(x,\rho x) \mid x \in A_1\}$. Therefore, $\rho b = \rho b'$, so $b = b'$ since ρ is injective. This proves that $\mathbf{A}_1$ is abelian.

To prove $\mathbf{A}_2$ is abelian, we reverse the roles of A_1 and A_2 in the foregoing argument. If $\{(x,\rho x) \mid x \in A_1\}$ is a block of β, then $\{(\rho^{-1}(\rho x),\rho x) \mid \rho x \in A_2\}$

is a block of β; that is, $\{(\rho^{-1}y, y) \mid y \in A_2\}$ is a block of β. Define the relation $\alpha \subseteq (A_2 \times A_2)^2$ as follows: for $((a, a'), (b, b')) \in (A_2 \times A_2)^2$,

$$(a, a') \; \alpha \; (b, b') \quad \iff \quad (\rho^{-1}a, \rho a') \; \beta \; (\rho^{-1}b, \rho b').$$

As above, we can prove that the diagonal $D(A_2)$ is a block of α by using the injectivity of ρ^{-1} to show that $(a, a) \; \alpha \; (b, b')$ implies $b = b'$. □

Lemma Appendix A.3. *If* $\mathsf{Clo}(\mathbf{A})$ *is trivial (i.e., generated by the projections), then* $\mathbf{A}$ *is abelian.*

(In fact, it can be shown that $\mathbf{A}$ is *strongly abelian* in this case. We won't need this stronger result, and the proof that $\mathbf{A}$ is abelian is elementary.)

Proof. We want to show $\mathsf{C}(1_A, 1_A)$. Equivalently, we must show that for all $t \in \mathsf{Clo}(\mathbf{A})$ (say, $(\ell + m)$-ary) and all $a, b \in A^\ell$, we have $\ker t(a, \cdot) = \ker t(b, \cdot)$. We prove this by induction on the height of the term t. Height-one terms are projections and the result is obvious for these. Let $n > 1$ and assume the result holds for all terms of height less than n. Let t be a term of height n, say, k-ary. Then for some terms $g_1, \dots, q_k$ of height less than n and for some $j \leqslant k$, we have $t = p_j^k[q_1, q_2, \dots, q_k] = q_j$ and since q_j has height less than n, we have

$$\ker t(a, \cdot) = \ker g_j(a, \cdot) = \ker g_j(b, \cdot) = \ker t(b, \cdot).$$

□

Appendix B. Nonrectangularity of abelian algebras

The example in this section reveals why the rectangularity theorem cannot be generalized to products of abelian algebras. Before examining the example, we state a lemma that will be useful when discussing the example. The proof is straightforward.

Lemma Appendix B.1. *Let* $\mathbf{A}_1, \dots, \mathbf{A}_n$ *be finite simple algebras in a Taylor variety and suppose*

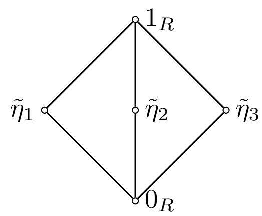
- *each* $\mathbf{A}_i$ *is absorption-free,*
- $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$*,*
- $\tilde{\eta}_i \neq \tilde{\eta}_j$ *for all* $i \neq j$*, and*
- $\mu \subseteq \underline{n}$ *is minimal among the sets in* $\{\sigma \subseteq \underline{n} \mid \bigwedge_\sigma \tilde{\eta}_i = 0_R\}$*.*

Then, $|\mu| > 1$ *and* $\mathrm{Proj}_\tau \mathbf{R} = \Pi_\tau \mathbf{A}_i$ *for every set* $\tau \subseteq \underline{n}$ *with* $1 < |\tau| \leqslant |\mu|$*.*

Example Appendix B.2. Let $\mathbf{A}$ be the algebra $\langle \{0, 1\}, \{f\} \rangle$, with universe $\{0, 1\}$, and a single basic operation given by the ternary function $f(x, y, z) = x + y + z$ where addition is modulo 2. This algebra is clearly simple and has two proper subuniverses, $\{0\}$ and $\{1\}$, neither of which is absorbing, so $\mathbf{A}$ is absorption-free. Let $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A} \times \mathbf{A} \times \mathbf{A}$ be the subdirect power of $\mathbf{A}$ with universe

$$R = \{(x, y, z) \in A^3 \mid x + y + z = 0 \pmod 2\} = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}.$$

It's convenient to give short names to the elements of R. Let us use the integers that they (as binary tuples) represent. That is, $0 = (0,0,0)$, $3 = (1,1,0)$, $5 = (1,0,1)$, and $6 = (0,1,1)$, so $R = \{0,3,5,6\}$. Let $\tilde{\eta}_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i)$, and identify each congruence with the associated partition of the set R. Then, $\tilde{\eta}_1 = |0,6|3,5|$; $\tilde{\eta}_2 = |0,5|3,6|$; $\tilde{\eta}_3 = |0,3|5,6|$, so $\tilde{\eta}_i \wedge \tilde{\eta}_j = 0_R$ and $\tilde{\eta}_i \vee \tilde{\eta}_j = 1_R$. In fact, the three projection kernels are the only nontrivial congruence relations of $\mathbf{R}$.

$\mathrm{Con}(\mathbf{R}) =$ 

Each projection of $\mathbf{R}$ onto 2 coordinates of $\mathbf{A}^3$ is "linked;" these binary projections are all readily seen to be $\{(0,0),(0,1),(1,0),(1,1)\} = A \times A$ in this case. Lemma Appendix B.1 tells us that this must be so. For the set $\{S \subseteq \underline{n} \mid \bigwedge_{i \in S} \tilde{\eta}_i = 0_R\}$ that appears in the lemma is, in this example, $\mathcal{S} = \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$, and the projection of R onto the coordinates in each minimal set in $\mathcal{S}$ must equal $A \times A$ by the lemma. Now let $\mathbf{S} \leqslant_{\mathrm{sd}} \mathbf{A} \times \mathbf{A} \times \mathbf{A}$ be the subdirect power of $\mathbf{A}$ with universe

$$\begin{aligned} S &= \{(x,y,z) \in A^3 \mid x + y + z = 1 \pmod 2\} \\ &= \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\} \\ &= \{1,2,4,7\}. \end{aligned}$$

All of the facts observed above about $\mathbf{R}$ are also true of $\mathbf{S}$. In particular, $\mathrm{Proj}_{\{i,j\}} S = A_i \times A_j$ for each pair $i \neq j$ in $\{1,2,3\}$. If both $\mathbf{R}$ and $\mathbf{S}$ belong to the set of relations (or "constraints") of a single $\mathrm{CSP}(\mathsf{S}(\mathbf{A}))$ instance, then the instance has no solution since $R \cap S = \emptyset$, despite the fact that $\mathrm{Proj}_{\{i,j\}} \mathbf{R} = \mathrm{Proj}_{\{i,j\}} \mathbf{S}$ for each pair $i \neq j$ in $\{1,2,3\}$. To put it another way, $\mathbf{R}$ and $\mathbf{S}$ witness a failure of the *2-intersection property*. The "potato diagram" of $\mathbf{A}^3$ below depicts the elements of R and S as colored lines (with colors chosen to emphasize equality of the projections onto $\{1,2\}$).

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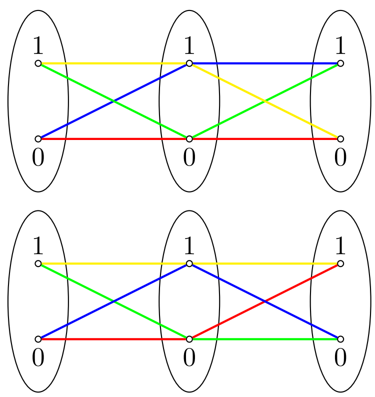
Fig. 4. potatoes



(lines are elements of R)

(lines are elements of S)

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