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A property of the lattice of equational theories

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To the memory of András Huhn

Abstract. It had been conjectured that any algebraic lattice having a compact one could be represented as the lattice of equational theories extending some theory. However, we show that each lattice having such a representation satisfies a nontrivial quasidistributivity condition. In particular, M_3 has no such representation.

If Σ is an equational theory, then $L(\Sigma)$ will denote the lattice of all equational theories which extend (i.e., contain) Σ. As is well known, $L(\Sigma)$ is an algebraic lattice with a compact one. Mod (Σ) will be the class of all algebras satisfying each equation in Σ.

If V is a variety, then $L(V)$ will denote the lattice of subvarieties of V and $Th(V)$ will denote the set of equations true in every member of V. L^∂ will denote the dual of the lattice L. As is also well known, $L(\text{Mod}\,(\Sigma))$ is isomorphic to $L(\Sigma)^\partial$, and $L(Th(V))$ is isomorphic to $L(V)^\partial$.

In [15] Mal'cev asked for a characterization of $L(V)$. Within a few years the conjecture was floating about that any algebraic lattice with a compact one could be represented as $L(V)^\partial$ for some V (i.e., as $L(\Sigma)$ for some theory Σ).

It is part of the folklore that every finite distributive lattice can be represented as $L(\Sigma)$. (This is a fairly easy consequence of Jónsson's Lemma [10] on subdirectly irreducible algebras in congruence distributive varieties.) Pigozzi [19] has shown for every algebraic lattice L that $L+1$ has such a representation. Pigozzi's result extends in some ways earlier work by Burris and Nelson [2], Jezek [7], and McNulty [10]. However Pigozzi's proof requires a similarity type as large as the set of compact elements of L, while, for example, Jezek [7] works with any non trivial similarity type.

At the conference in Szeged in 1983 Kogalovskii announced that for any algebraic lattice L that $L+1$ was isomorphic to $L(\Sigma)$ for some Σ. (This is Pigozzi's Theorem mentioned above.) Kogalovskii also announced that if $\mathbf{A}$ is an algebra with a binary term having a left zero and a left one then Con $\mathbf{A}$ is

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isomorphic to some $L(\Sigma)$. Incidentally, for any algebraic lattice L, it is fairly easy to produce a groupoid $\mathbf{A}$ with left zero and left one so that $L+1 \cong \operatorname{Con} \mathbf{A}$. Thus the Kogalovskii–Pigozzi Theorem is a corollary of the second theorem announced by Kogalovskii.

All the results mentioned above show the universal character of the lattice of equational theories. McKenzie, however, has results showing the special nature of this lattice. In [16] McKenzie showed essentially that the multiplicity type can be recovered from the lattice of all equational theories of a given similarity type. A small part of [17] was devoted to showing that many finite lattices *cannot* be represented as $L(\Sigma)$ for any *locally finite* Σ. (We say that a theory Σ is locally finite just in case $\operatorname{Mod}(\Sigma)$ is locally finite.)

This provides a hint that the old conjecture is false, but this hint is not entirely clear. This is because McKenzie also shows in [17] that many finite lattices are not congruence lattices of finite groupoids. Yet every finite lattice is the congruence lattice of some infinite groupoid. See [12].

However, the old conjecture is false. This follows from our

THEOREM. *The lattice L of all equational theories in a fixed similarity type satisfies the following condition (for each c, $z \in L$, each set I and each family $(a_i : i \in I)$ of elements of L). If*

(i) $a_i \wedge c = z$ *for each* $i \in I$ *and*

(ii) $\bigvee (a_i : i \in I) = 1$,

then $c = z$.

This theorem is an immediate consequence of the two lemmate proved in the next section.

M_κ is the lattice having κ atoms, a zero and one, and no other elements. M_3 is pictured in Figure 1.

COROLLARY. *For every equational theory Σ and for every $\kappa \geq 3$ M_κ is not isomorphic to $L(\Sigma)$.*

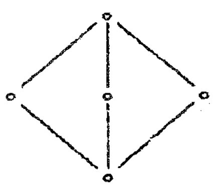

Fig. 1

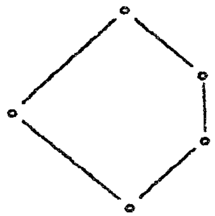

Fig. 2

However N_5 (see Figure 2) is isomorphic to some $L(\Sigma)$. For example one can take Σ to be the theory of Polin's variety, which is locally finite. See [4].

DEFINITIONS. i) A finite lattice L is 0, 1-simple iff for every congruence θ it is true that $|0/\theta| > 1$ or $|1/\theta| > 1$ implies $\theta = L \times L$.

ii) A lattice L is *tight* iff $2 < |L| < \omega$ and L is 0, 1-simple and for every meet endomorphism f of L if $f(x) > x$ for all $x < 1$ then $f(0) = 1$.

Note that if L is a simple lattice in which the meet of the coatoms is zero and if L satisfies $2 < |L| < \omega$, then L is tight. In a letter McKenzie sent the following

PROPOSITION. *Let a be an atom of the lattice L, and set $u(a) = \bigvee (b : b \not\geq a)$. If L is tight, then $u(a) = 1$.*

The proof is a straightforward computation done in two cases. If $a \leq u(a) < 1$ and $\varphi(x) = u(a)$ for $x \not\geq a$ and $\varphi(x) = 1$ otherwise, then φ is a meet endomorphism violating the definition. If $u(a) \not\geq a$, then L is not 0, 1-simple because $L = [0, u(a)] c \dot{\cup} [a, 1]$.

COROLLARY. *If L is tight, then L is not isomorphic to $L(\Sigma)$ for any Σ.*

In [17] McKenzie had proved that no tight lattice could be $L(\Sigma)$ for any *locally finite* Σ. So our theorem subsumes that part of [17] concerned with lattices of subvarieties.

In the original manuscript [14] the Theorem was stated only for $|I| = 2$. Agnes Szendrei sent an example (see Figure 3) which satisfies the condition for $|I| = 2$ but not for $|I| = 3$. The example is a one element extension of the free distributive lattice on 3 generators. The one new element is an atom which is covered by the median element.

One naturally considers a similar one point extension for the free distributive lattice on k generators. Peter Pálfy pointed out how easy it is to prove that such an extension satisfies the condition for $|I| = k - 1$ but not for k. The main point of

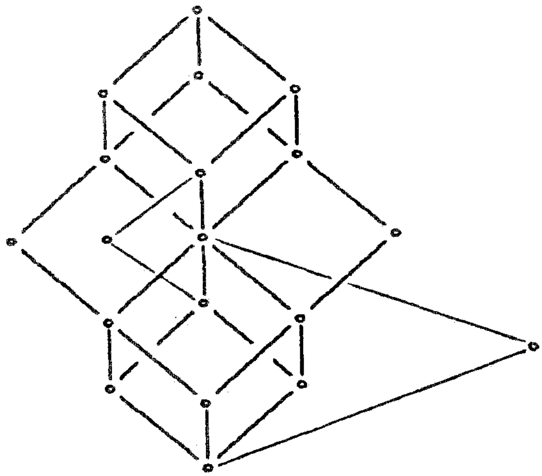

Fig. 3

the argument is that if $\bigvee(a_i : i \in I) = 1$ and $|I| = k - 1$ and no a_i is the "extra" element, then some $a_i \geq$ the join of two generators which is $\geq$ the median element. (That generators are join prime is essential.)

Proof of the Theorem. The first step in the proof is the

MCKENZIE LEMMA. *If Σ is any equational theory, then $L(\Sigma)$ is isomorphic to* Con $\mathbf{A}$ *for some algebra* $\mathbf{A}$ *having a binary term function b which has a left zero and a left one.*

While it is not explicitly stated there, the above lemma can be found in Section 3 of McKenzie's Finite Forbidden Lattices paper [17]. See Def. 3.2 and the proof of Lemma 3.4.

Proof. Let Σ be any equational theory not including "$x \approx y$". Let $V = \operatorname{Mod} \Sigma$. $\mathbf{F}_V(\omega) = \langle F; O \rangle$ is the free algebra on the ω generators $x_0, x_1, x_2, \ldots$. For any $\mathbf{A}$ we let InvCon $\mathbf{A}$ denote the fully invariant congruences of $\mathbf{A}$. On F define the ternary operation sub by $\operatorname{sub}(t, u, v) = \varphi(t)$ where φ is the unique endomorphism of $\mathbf{F}_V(\omega)$ satisfying $\varphi(x_0) = u$ and $\varphi(x_1) = v$ and $\varphi(x_i) = x_i$ for $i \geq 2$. It is easy to see that if Ψ is any fully invariant congruence and $a_i \Psi b_i$, then $\operatorname{sub}(a_0, a_1, a_2) \equiv \operatorname{sub}(b_0, b_1, b_2)$ (Ψ).

Let $\mathbf{A} = \langle F; O \cup \operatorname{End} \mathbf{F}_V(\omega) \cup \{\operatorname{sub}\} \rangle$. Thus we have that $\operatorname{Con} \mathbf{A} =$

InvCon $\mathbf{F}_V(\omega)$. Now we define b on F by $b(x, y) = \mathrm{sub}(x, x, y)$, and we compute a bit.

$$b(x_0, y) = \mathrm{sub}\,(x_0, x_0, y) = x_0 \text{ for any } y \in F.$$

We also have $b(x_1, y) = \mathrm{sub}\,(x_1, x_1, y) = y$ for any $y \in F$.

ZIPPER LEMMA. *Let* $\mathbf{A}$ *be any algebra having a binary term function* b *and elements* 0 *and* 1 *satisfying* $b(0, y) = 0$ *and* $b(1, y) = y$, *for all* $y \in A$. *Suppose* $L \simeq \mathrm{Con}\,\mathbf{A}$ *and* c, $z \in L$ *and* $(a_i\colon i \in I)$ *is a family of elements of* L. *Then,* $\bigvee (a_i\colon i \in I) = 1$ *and* $a_i \wedge c = z$ *for each* $i \in I$ *imply* $c = z$.

Proof. We may suppose without loss of generality that $z = 0$. For notational simplicity we prove the case when $|I| = 2$ and leave the general case to the reader.

Now we suppose $\mathbf{A}$ is any algebra having a binary term function b and elements 0 and 1 such that $b(0, y) = 0$ and $b(1, y) = y$, for all $y \in A$. We use Δ_A to denote the equality relation on A. Suppose α, β, γ are congruence relations such that $\alpha \vee \beta = A \times A$ and $\alpha \wedge \gamma = \beta \wedge \gamma = \Delta_A$. So there is a sequence $0 = s_0, s_1, \ldots, s_n = 1$ such that $s_0 \equiv s_1(\alpha)$ and $s_1 \equiv s_2(\beta)$ and $s_2 \equiv s_3(\alpha)$, etc. Now let $\langle x, y\rangle \in \gamma$. We set $u_i = b(s_i, x)$ and $v_i = b(s_i, y)$. Note that $u_0 = b(s_0, x) = b(0, x) = 0 = b(0, y) = v_0$. Since $x\gamma y$, we have $u_i = b(s_i, x)\ \gamma\ b(s_i, y) = v_i$ for each i. We also have $u_0 = b(s_0, x)\ \alpha\ b(s_1, x) = u_1\ \beta\ b(s_2, x) = u_2\ \alpha \cdots u_n$. Similarly we obtain $v_0\ \alpha\ v_1\ \beta\ v_2\ \alpha\ v_3 \cdots$. All these relations are depicted in Figure 4 below.

Now we have $u_1\ \alpha\ u_0 = v_0\ \alpha\ v_1$. Thus $u_1 \alpha v_1$. But also $u_1 \gamma v_1$. Hence $u_1 \equiv v_1\ (\alpha \wedge \gamma)$. Since $\alpha \wedge \gamma = \Delta_A$, this means that $u_1 = v_1$. Continuing we see that $u_2 \equiv v_2\ (\beta \wedge \gamma)$, and so $u_2 = v_2$. Inductively, $u_i = v_i$ for each i. In particular, we have $x = b(1, x) = b(s_n, x) = u_n = v_n = b(1, y) = y$. We may conclude that $\gamma = \Delta_A$, which finishes the proof.

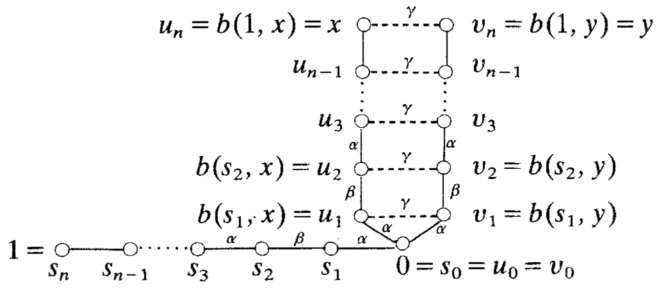

Fig. 4. The algebra $\mathbf{A}$.

The reader will, no doubt, have observed the similarity between the Zipper Lemma and Lemma 1 of [5].

The Theorem of this note is an obvious consequence of these two lemmate.

Problems

Of course we still have Mal'cev's problem

1. Characterize $L(\Sigma)$.

2. Is the converse of the Theorem true? That is, suppose L is an algebraic lattice having a compact one, and suppose L satisfies the conclusion of the Theorem. Is there always an equational theory Σ with $L(\Sigma)$ isomorphic to L?

At the moment, such a result would surprise me. Perhaps there is more to uncover about the structure of $L(\Sigma)$.

3. Is there some "quasi-distributivity" condition which is stronger than the Theorem and true in $L(\Sigma)$ for every equational theory Σ?

4. (Jónsson) Suppose L is an algebraic lattice and Σ is any equational theory. Let M be the lattice obtained from the ordinal sum $L + L(\Sigma)$ by collapsing 1_L and $0_{L(\Sigma)}$. (M is depicted by Figure 5.) Is there always an equational theory T with M isomorphic to $L(T)$?

This would generalize Pigozzi's theorem nicely and would suggest that we are close to characterizing $L(\Sigma)$.

5. (Jónsson) What about the modular case?

Incidentally the lattice pictured in Figure 6 is not isomorphic to $L(\Sigma)$ for any Σ as the Theorem shows.

6. Is there some (finite) lattice L such that L is isomorphic to $L(\Sigma)$ for some Σ but L is not isomorphic to $L(T)$ for any *locally finite* equational theory T?

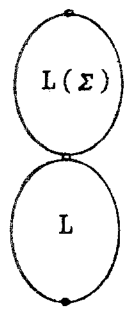

Fig. 5

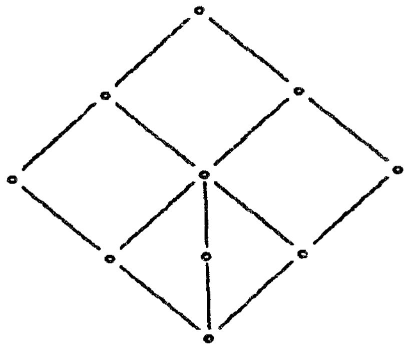

Fig. 6

The corollary about tight lattices suggests the answer could be no. However, Emil Kiss and Péter Pröhle have concocted a candidate. Their lattice L is depicted in Figure 7. Using McKenzie's unpublished tame congruence theory they can prove easily that L is not $L(\Sigma)$ for any locally finite Σ. However, their lattice L satisfies the necessary condition given in the Theorem.

The author of this note has not seen a proof of the second theorem announced by Kogalovskii in Szeged in 1983. There seems to be no proof available in the West at this time. It would be very useful to have such a proof.

7. Find and publish a proof of the second theorem announced by Kogalovskii; i.e., prove the converse to the McKenzie Lemma.

This would provide an indirect answer to Mal'cev's original question.

8. Is every distributive algebraic lattice with a compact one isomorphic to $L(\Sigma)$ for some equational theory Σ?

9. In [12] we see that the ideal lattice of every distributive lattice is

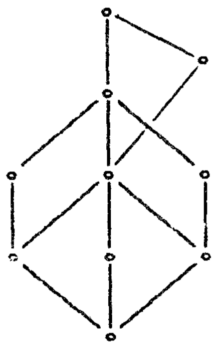

Fig. 7

isomorphic to the congruence lattice of a groupoid with left zero. Is the ideal lattice of every distributive lattice with zero and one isomorphic to the congruence lattice of a groupoid having a left zero and a left one?

10. What about the lattice of subquasivarieties of a quasivariety, or dually, what about the lattice of universal Horn theories extending a fixed universal Horn theory?

The lattice of quasivarieties seems to be very different from the lattice of varieties. For example, the former lattice is semidistributive, while for lattices of varieties, the results of Jezek [7] and Pigozzi [9] rule out any such condition. For further information, the reader should start with the three papers by Gorbunov, Tropin and Tumanov in Number 2 of V. 22 of Algebra and Logic ([6], [20], [21]). The bibliographies there are especially useful.

Acknowledgements

The inspiration for this note came from Pigozzi's paper [19] and a remark by George McNulty. My debt to McKenzie's Finite Forbidden Lattices paper [17] is clear. Jiri Sichler's enthusiasm caused me to read Pigozzi's paper.

Because of Pigozzi's paper, I was trying to prove the old conjecture. My hoped-for first step was to represent each L as Con $\mathbf{A}$ so that the implication $(0=1 \rightarrow x=y)$ was true in $V(\mathbf{A})$. I had a scheme in mind. Because of Kogalovskii's results, McNulty suggested that one could perhaps take $\mathbf{A}$ to be a groupoid with left zero and left one. My experience with congruence lattice representations suggested that this could not be general enough. Ironically, we both seem to be more or less right.

I also wish to thank George McNulty for a very thorough reading of the original version of this paper and for his many helpful comments.

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