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Commutator theory without join-distributivity. (English summary)
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Commutator theory in varieties satisfying weaker properties than congruence modularity is investigated. The commutator $[\alpha,\beta]$ is defined as the smallest γ such that α centralizes β modulo γ, written as $C(\alpha,\beta;\gamma)$. In general, it is not true that $C(\alpha,\beta;\gamma)\wedge\gamma\le\gamma'\Rightarrow C(\alpha,\beta;\gamma')$; rather it is shown that this assumption implies left distributivity, i.e. $[\alpha+\beta,\gamma]=[\alpha,\gamma]+[\beta,\gamma]$. Left semidistributivity of the commutator always holds, i.e., $[\alpha_i,\beta]=\delta\Rightarrow[\bigvee\alpha_i,\beta]=\delta$.

A major tool in modular varieties is a difference term $d(x,y,z)$ satisfying $x\alpha y\Rightarrow d(x,y,y)=x[\alpha,\alpha]d(y,y,x)$. Many nonmodular varieties, amongst them inverse semigroups and all n-permutable varieties, are shown to possess a weak difference term satisfying only $x\alpha y\Rightarrow d(x,y,y)[\alpha,\alpha]x[\alpha,\alpha]d(y,y,x)$. Under the hypothesis of a weak difference term many of the reviewer's permutability results for modular varieties can be obtained, for instance $\alpha+\beta=(\alpha^{(n)}+\beta^{(n)})\circ\alpha\circ\beta\circ(\alpha^{(n)}+\beta^{(n)})$, from which some useful commutator equations, such as $[\alpha+\gamma,\alpha+\gamma]=\alpha+[\gamma,\gamma]$, follow.

As in the classical case, certain sublattices of congruence lattices reveal clues about the commutator. As an example, if M_3, the five-element modular nondistributive lattice, is a sublattice of $\mathrm{Con}(A)$, then the commutator of its top element must be below its bottom. Finally, a list is provided of lattices that cannot be sublattices of $\mathrm{Con}(A)$ if A has a weak difference term.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.