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The lattice of lambda theories. (English summary)
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The complete lattice λT of lambda theories consists of sets of equations between (untyped) closed lambda terms, closed under deduction in lambda calculus with free variables, in particular under the rule $M = N \Rightarrow \lambda x.M = \lambda x.N$. The set is ordered by inclusion and the intersection of a set of theories forms their meet and the deductive closure of the union their join. This lattice λT has been studied in the signature $\langle L, +, \cdot, 0, 1\rangle$, where $+, \cdot, 0, 1$ stand for join, intersection, the least and largest element, respectively. The relation $\mathcal{S} \subseteq \mathcal{T}$ can be defined by $\mathcal{S} \cdot \mathcal{T} = \mathcal{S}$. Results have been obtained using proof-theory (there is an element $\mathcal{H}$, equating all unsolvables, with a unique maximal extension $\mathcal{H}^*$ properly below the top 1), notions of reduction (there are $2^{\aleph_0}$ elements T with $\mathcal{H}^* \subseteq T \subseteq \mathcal{H}^*$), model-theory ($\mathcal{H}^*$ was discovered as the theory of Scott's D_∞ models of untyped lambda calculus) and even recursion-theory (λT is a dense ordering); these results are due to various authors [see H. P. Barendregt, *The lambda calculus*, Revised edition, North-Holland, Amsterdam, 1984; MR0774952]. The present paper studies λT using methods from universal algebra.

Lambda theories are not usual first-order equational algebraic theories. The reason is that there is an operator 'λx' having a binding effect on the expression following it. For this reason one has studied lambda abstraction algebras [see A. Salibra, Theoret. Comput. Sci. **249** (2000), no. 1, 197–240; MR1791956], in which these 'λx' belong to the algebraic signature and one axiomatizes the binding effect. This ensures that methods of universal algebra are applicable to λT.

The results of the paper are the following. 1. There are quasi-identities, i.e. statements of the form E_1 & ... & $E_n \to E$, with the $E_1, \dots, E_n, E$ all equations, that hold in λT, but not in arbitrary complete lattices. An example is $\mathcal{S} + \mathcal{T} = 1 \,\&\, \mathcal{S}\mathcal{G} = \mathcal{T}\mathcal{G} \to \mathcal{G} \subseteq \mathcal{S}$. 2. If an identity fails in some complete lattice, then it also fails in λT^+, where one has added to the set of lambda terms a finite set of constants. 3. There exists a $\mathcal{J} \in \lambda T$ such that the sublattice $[\mathcal{J}) = \{\mathcal{T} \in \lambda T \mid \mathcal{J} \subseteq \mathcal{T}\}$ satisfies $\mathcal{S}\mathcal{T} = \mathcal{S}\mathcal{G} \to \mathcal{S}\mathcal{T} = \mathcal{S}(\mathcal{T} + \mathcal{G})$. This theory $\mathcal{J}$ is axiomatized by (here $\Omega \equiv (\lambda a.aa)(\lambda a.aa)$ is the standard unsolvable term)

$$\Omega xx = x,\ \Omega xy = \Omega yx,\ \Omega x(\Omega yz) = \Omega(\Omega xy)z,$$

and is proved consistent by constructing a filter model using intersection types [see M. Coppo et al., in *Logic colloquium '82 (Florence, 1982)*, 241–262, North-Holland, Amsterdam, 1984; MR0762113]. *Henk Barendregt*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.