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Lambda abstraction algebras: representation theorems[☆]

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Abstract

Lambda abstraction algebras (LAAs) are designed to algebraize the untyped lambda calculus in the same way cylindric and polyadic algebras algebraize the first-order predicate logic. Like combinatory algebras they can be defined by true identities and thus form a variety in the sense of universal algebra, but they differ from combinatory algebras in several important respects. The most natural LAAs are obtained by coordinatizing environment models of the lambda calculus. This gives rise to two classes of LAAs of functions of finite arity: *functional* LAAs (FLA) and *point-relativized functional* LAAs (RFA). It is shown that RFA is a variety and is the smallest variety including FLA.

Dimension-complemented LAAs constitute the widest class of LAAs that can be represented as an algebra of functions and are known to have a natural intrinsic characterization. We prove that every dimension-complemented LAA is isomorphic to RFA. This is the crucial step in showing that RFA is a variety.

0. Introduction

The untyped lambda calculus is formalized as a theory of equations, but it is not an equational theory in the usual algebraic sense because the equations, unlike the associative and commutative laws for example, are not always preserved when arbitrary terms are substituted for variables. Consequently, the general methods that have been developed in universal algebra and category theory, for defining the semantics of an arbitrary algebraic theory for example, are not directly applicable. There have been several attempts to reformulate the lambda calculus as a pure equational theory. The earliest and best known, although apparently not motivated exactly by these considerations, are the *combinatory algebras* (CAs) of Curry [9].

[☆] Most of the results in this paper were announced in [26].

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Combinatory algebras have a simple pure equational characterization. Curry also specified (by a considerably less natural set of axioms) a pure equational subclass of combinatory algebras, the *λ-algebras* (see [3, 5.2.5]), that he viewed as algebraic models of the lambda calculus. It was later discovered that the CAs most closely connected to the natural functional models of the lambda calculus are special lambda algebras called *λ-models* [3, 5.2.7]. They were first axiomatized by Meyer [22] and independently by Scott [33]; the axiomatization while elegant is not equational. It turns out however that the class of λ-algebras is the equational class (variety) generated by the λ-models, in fact every λ-algebra is a homomorphic image of a λ-model [5] and every λ-algebra is a subalgebra of a λ-model [22]; see [3, 5.2.16]. More recently, several pure algebraic theories of the lambda calculus have been developed within the context of category theory: Obtułowicz and Wieger [24] via the *algebraic theories* of Lawvere; Adachi [2] via monads; Curien [8] via categorical combinators.

In [25] we proposed an alternative approach in the context of universal algebra. We introduced the notion of a *lambda abstraction algebra* (LAA), which is intended to provide a pure algebraic theory of the lambda calculus in the same way Boolean algebras constitute an algebraic theory of classical propositional logic and, more to the point, cylindric and polyadic Boolean algebras constitute an algebraic theory of first-order predicate logic.[1] Like combinatory algebras, LAAs can be defined by true identities and thus form a variety in the sense of universal algebra, but they differ from CAs in other important respects. For example, functional abstraction is a fundamental operation in LLAs; in CAs it is defined in terms of the combinators **k** and **s**. The role that variables play in the lambda calculus, as place holders, is not reflected in the structure of CAs, but it is in LAAs as an implicit coordinate system. Because of this, substitution can be abstracted in LAAs by "inverting" (β)-conversion in a natural way. These syntactical differences actually show what is an essential difference between the two algebraic theories. The most natural LAAs are algebras of functions that are obtained by coordinatizing λ-models. The functions of finite arity that arise this way can be encoded as elements of a CA by the well-known method of Schönfinkel and Curry. But there are also functions of infinite arity that cannot be encoded in the same way. A precise algebraic expression to this observation is given in [28]: Each element of an abstract LAA can be assigned a dimension that abstracts the notion of the arity of a function. A LAA is *locally finite-dimensional* (LFA) if each of its elements is finite-dimensional. The set of finite-dimensional elements of a LAA forms a LFA, and its zero-dimensional elements form a CA, in fact a λ-algebra. There is a categorical equivalence between LFAs and λ-algebras that takes each LFA to its zero-dimensional part [28, Theorem 2.15]. For an arbitrary LAA, its locally finite-dimensional subalgebra can be reduced to its zero-dimensional λ-algebra via the categorical equivalence, but the LAA itself cannot be so reduced. This suggests that

[1] Essentially the same notion was discovered independently by Diskin [10], also under the influence of ideas from algebraic logic. See also [11]. The models of the lambda calculus considered by Krivine [20] turn out to be essentially what we call locally finite-dimensional functional FLAs.

there is a broader view of the lambda calculus, which encompasses spaces of functions of infinite arity, for which CAs are not completely adequate.

The situation in algebraic logic is analogous, and it is the algebraic-logic model that mainly motivates the present study. In analogy to the case for LAAs, the most natural cylindric (and polyadic) algebras are algebras of functions that are obtained by coordinatizing models of classical first-order logic. Here again algebras of functions of infinite arity arise in the process. (These functions can be viewed as characteristic functions of infinitary predicates). A certain natural generalization of the class of all cylindric algebras that arise this way turns out to be a variety, and in fact the smallest variety that includes all the functional algebras that are most closely connected with models of first-order logic. This is the class of *representable* cylindric algebras. It is a proper subvariety of the class of all cylindric algebras, so nonrepresentable cylindric algebras exist. Much of the work in algebraic logic has been directed at finding wider classes of representable cylindric algebras with natural intrinsic characterizations. The main references for cylindric algebras are [17, 18]; for polyadic algebras it is [16]; see in particular [15]. We also mention here Németi [23]. It contains an extensive survey of the various algebraic versions of quantifier logics.

In the present paper the main result is a characterization of the smallest variety of LAAs that includes the functional LAAs that are most closely connected with λ-models. In the process we obtain functional representation results for several classes of LAAs that parallel corresponding representation results in the theory of cylindric algebras. These include LFAs, dimension-complemented LAAs, and neat reducts of LAAs. The two key results used in the characterization of the variety generated by the functional LAAs are the functional representability of dimension-complemented LAAs and an analogue for LAAs of the combinatory completeness lemma for CAs.

0.1. Lambda calculus

To keep this article self-contained, we review in this subsection the basic definitions of what is known as the untyped lambda calculus. Originated by Church [6, 7], it is a formalization of an intensional as opposed to extensional theory of functions; that is, a theory of functions viewed as "rules" rather than "sets of ordered pairs". A basic feature of such a theory is the lack of distinction between functions and the elements of the domains on which the functions act. Thus a function can, in theory, take other functions, even itself, as legitimate arguments. There are two primitive notions: *application*, the operation of applying a function to an argument, and *lambda (functional) abstraction*, the process of forming a function from the "rule" that defines it. Application is formalized as a binary operation denoted by $\cdot$, while lambda abstraction as a family of unary operations λx, one for each variable x. Terms of lambda calculus are defined in the usual way: every variable is a term, and if t and s are terms, so are $t \cdot s$ and $\lambda x(t)$ for each variable x. By convention we write ts for $t \cdot s$ and $\lambda x.t$ for $\lambda x(t)$. An occurrence of a variable x is *free* if it is not within the scope of any λx. A term s is *free for* x in a term t if no free occurrence of x in t lies within the scope of a lambda abstraction with respect to a free variable of s. A term without free variables is called *closed*. Terms can also be constructed using constant symbols from some fixed set C. $t[s/x]$ will denote the result of substituting s for all free occurrences of x in t subject to the usual provisions about renaming bound variables in t to avoid capture of free variables in s.

Although lambda calculus is a very basic language, it is sufficient to express all the computable functions. An interesting example suggesting the expressive power of the lambda calculus is the "double composition" functional t defined by the equation $t=\lambda f.(\lambda x.f(fx))$. The value of t at the argument g, for example, is obtainable by applying the lambda term t to g, which entails substituting g for f to obtain $\lambda x.g(gx)$. By the same reasoning, t applied to t equals the "compose 4 times" functional because tt applied to g equals $(tt)g=(\lambda x.t(tx))g=t(tg)=t(\lambda x.g(gx))=\lambda x.[\lambda x.g(gx)]([\lambda x.g(gx)]x)=\lambda x.[\lambda x.g(gx)](g(gx))=\lambda x.g(g(g(gx)))$. The process of application and evaluation reflects the computational behavior of many modern programming languages, which explains the interest in the lambda calculus among computer scientists, and is expressed by the fundamental axioms of (β)-*conversion* and (α)-*conversion*:

- $(\lambda x.t)s=t[s/x]$ for all terms t,s and variable x such that s is free for x in t.
- $\lambda x.t=\lambda y.t[y/x]$ if y does not occur free in t.

α-conversion says that bound variables can be replaced in a term under the obvious condition. The standard axioms for equality complete the inference rules. A λ-*theory* is any set of equations that is closed under (α)- and (β)-conversion and the equality axioms.

Applying a function to itself violates the rules of ordinary set theory which forbid a function from being in its own domain. On the other hand, the naive, intended models of the lambda calculus are sets V that coincide (or at least can be put in one–one correspondence) with the set of functions from V into itself, symbolically, $V=V^V$. This logical difficulty can be overcome by restricting the functions considered to a certain manageable subset of V^V. In this way it is possible to construct domains of *admissible* functions that constitute functional models of the lambda calculus in a very natural sense. The first such models were discovered by Scott [32]. They are called *environment models* in [22]; and they are equivalent to λ-models in a natural sense; see [22, p. 106]. They can be characterized by means of an injective partial mapping $\lambda: V^V \circ\!\!\rightarrow V$ whose domain is the set of admissible functions. λ may be thought of as the process of *encoding* admissible functions as elements of V. With functions encoded this way, application can be viewed as a binary operation on V. Let V be the domain V enriched by the application operation and the encoding mapping, which we denote respectively by $\cdot^V$ and λ^V.

A function in its intensional form is represented by a term $t(x)$ of the lambda calculus with a free variable x. For each $v\in V$, let $t^V(v)$ be the value $t(x)$ takes in V when x is interpreted as v. Then its extensional form is the function $\langle t^V(v)\colon v\in V\rangle\in V^V$, which is encoded as the element $\lambda^V(\langle t^V(v)\colon v\in V\rangle)$ of V and represented by the term $\lambda x.t(x)$. Note that $\langle t^V(v)\colon v\in V\rangle$ and its encoding $\lambda^V(\langle t^V(v)\colon v\in V\rangle)$ both represent the same function, but in environment models only the latter corresponds to an actual element of the universe of the model; this is an essential difference between the models of lambda calculus and LAAs.

The two forms of the function are connected by the operation of application. Intuitively, the valve $t^V(v)$ of the function at a particular argument v is obtained by applying its extensional form to v; symbolically, $\langle t^V(v)\colon v\in V\rangle(v)=t^V(v)$. Expressed in the environment model this becomes

$$\lambda^V(\langle t^V(v)\colon v\in V\rangle)\cdot^V v=t^V(v) \quad \text{for each } v\in V.$$

Thus admissible functions f in V^V and elements of V are related by the equation

$$(\lambda^V f)\cdot^V v=f(v) \quad \text{for each } v\in V. \tag{0.1}$$

The following completeness theorem is a basic result of the lambda calculus; see [22].

Every λ-theory consists of precisely the equations valid in some environment model.

0.2. Applications to the lambda calculus

LAAs have already proved useful in the lambda calculus as a framework for applying the methods of universal algebra. We expect that their use will enrich the theory in the same way Boolean algebras, Heyting algebras, and modal algebras have enriched propositional and modal logic. The connection between models of the lambda calculus and LAAs is explored in detail in [28], and we list some of the results obtained there.

A LAA can be associated with each λ-theory T; its elements are sets of terms that are pairwise equivalent under T. We call this the *term* LAA of T. The fact that each term contains only a finite number of variables is reflected in the fact that the term LAA of every theory is locally finite. With aid of this precise connection between λ-theories and LFAs, the completeness theorem for the lambda calculus can be obtained as a corollary of the functional representation theorem for LFAs (Theorem 3.11); see Diskin [10].

Every λ-theory is uniquely determined by its restriction to closed λ-terms; this is a consequence of the categorical equivalence between λ-algebras and LFAs mentioned previously. More precisely, let $\Lambda(C)$ be the set of terms over a set C of constants. Let T_1 and T_2 be λ-theories on $\Lambda(C_1)$ and $\Lambda(C_2)$, respectively, and let T_1^0 and T_2^0 be the respective *ground theories*, i.e., their restrictions to closed terms. A λ-algebra can be constructed for each λ-theory in much the same way the term LAA is constructed, and we call it the *term* λ-algebra. Then any isomorphism between the term λ-algebras of T_1^0 and T_2^0 can be uniquely extended to an isomorphism between the term LAAs of T_1 and T_2.

A λ-algebra is (isomorphic to) the term λ-algebra of the ground theory of exactly one λ-theory [28, Corollary 2.14, Theorem. 2.15]. A λ-theory T is *rich* if it is closed with respect to the restricted ω-rule: if $(\lambda x.t_1)u=(\lambda x.t_2)u$ is in T for all closed lambda terms u, then $t_1=t_2$ is in T. A λ-model is the term λ-algebra of the ground theory of exactly one rich λ-theory.

Every λ-theory can be uniquely extended in a conservative way to a rich λ-theory [28, Theorem 3.14].

A λ-algebra is a λ-model iff it is *generic* in the variety of λ-algebras, i.e., every other λ-algebra can be constructed from it by products, homomorphic images, and subalgebras. The results of [5, 22] mentioned previously are easy consequences of this result.

Some of the results of the present paper have consequences for the extended lambda calculus over spaces of functions of infinite arity. The notion of a λ-theory can be extended in the following natural way: Let T be a theory on $\Lambda(C)$, where, for the purpose of applying the (α)- and (β)-rules, we assume that, for each $c \in C$, each member of a predetermined and possibly infinite set of variables depending on c is free in c, in a virtual sense. In the general case every variable may be virtually free in a given term. But it is easy to see that, if the dimensions of the constants are so specified that, for each finite set of constants, there is at least one variable that fails to be virtually free in each element of the set, then for every term there will be at least one variable that fails to be virtually free in it. Under these circumstances, the term LAA constructed from T is dimension-complemented. We say that T is *dimension-complemented* in this case.

We have the following completeness theorem for the extended lambda calculus as a corollary of the functional representation theorem for dimension-complemented LAAs (Theorem 3.14):

Every dimension-complemented λ-theory consists precisely of the equations valid (in an extended sense) in some environment model.

0.3. Outline of paper

The formal definition of LAAs is given in Section 1 and the basic theory of abstract substitution developed. This leads to the notion of *dimension*, an abstraction of a variable occurring free in a term of the lambda calculus. The classes of locally finite-dimensional and dimension-complemented LAAs are defined by means of it in Section 2, and the theory of substitution is further refined. Functional LAAs are defined and the basic part of their theory developed in Section 3. We present there the functional representation theorem for locally finite LAAs. We go on to investigate the notion of a point-relativized functional LAAs and prove our second basic representation theorem, that every dimension-complemented LAA is isomorphic to a point-relativized functional LAA.

Section 4 is more technical. Here we develop an algebraic theory of combinatory completeness, a notion that plays an important role in the lambda calculus. It turns out to be equally significant for the theory of LAAs. A new definition of environment model is given in Section 5, and using the combinatory completeness lemma we explore the nature of the functional LAAs that can be defined over a given environment model.

The notions of neat reducts and dilations of LAAs are explored in Section 6. The theory is similar to that of the analogous notions in cylindric and polyadic algebras, but in some regards more satisfactory. The algebraic theory of combinatory completeness is the major reason for this. The class of all neat reducts of the LAAs of a fixed dimension form a variety. As a special case of this we obtain a new proof of the fact that the class of λ-algebras is a variety using Birkhoff's characterization of varieties as class of algebras closed under subalgebra, homomorphism, and Cartesian product.

Our main results are in the last section. We prove that the point-relativized functional LAAs coincide (up to isomorphism) with LAAs neatly embeddable in infinitely many more dimensions, and thus form a variety. Thus point-relativized functional LAAs appear to be the LAAs that correspond most closely to representable cylindric algebras. We show that every functional LAA is isomorphic to a point-relativized functional LAA and that every point-relativized functional LAA is a homomorphic image of a (nonrelativized) functional LAA. As a consequence, we have that functional and point-relativized functional LAAs, as well as locally finite-dimensional and dimension-complemented LAAs, all generate the same variety. Some conclusions and open problems are presented in the last section.

1. Lambda abstraction algebras

In this section we define lambda abstraction algebras and develop the basic part of their theory.

In the formulation of the lambda calculus variables play a dual role. They serve to index the arguments of definable functions, i.e., as place holders; they also represent specific functions, namely the projections. This duality is preserved in the theory of lambda abstraction algebras by overloading the notation we use to describe the language. In their role as place holders, the λ-variables are represented as elements of an abstract, nonempty but possibly finite, set I. The language contains an individual constant (i.e., nullary operation) symbol for each element $x \in I$. Although x is a constant, we shall refer to it as a λ-variable and preserve most of the formalism of the lambda calculus in order to keep the connection between the two theories apparent.[2] These constant symbols represent the λ-variables in their role as projections. In addition, the language contains a binary operation symbol $\cdot$, called *application*, and

[2] We depart in this regard from the custom of algebraic logic. In the translation into an algebraic formalism, the individual variables of predicate logic, in their role as place holders, are transformed into indices of a Cartesian coordinate system. The system is indexed by ordinals in the case of cylindric algebras and by elements of an abstract, unordered set I in the polyadic algebra case. Although this facilitates understandability in one regard by clearly separating the two roles variables play, it obscures the connection between the standard and algebraic theories.

a unary operation symbol for each $x \in I$. It is denoted by λx and is called *λ-abstraction*. Note that λx is to be viewed as an indivisible complex symbol; alternatively, $\langle \lambda x\colon x \in I \rangle$ can be viewed as a system of unary operations indexed by elements of I.

We use x, y, z, possibly with subscripts, as metavariables ranging over λ-variables. The real variable symbols of lambda abstraction theory will be referred to as *context variables* and denoted by the greek letters ξ, ν, and μ, possibly with subscripts. The terms of the language of lambda abstraction theory are called *λ-terms*. They are constructed in the usual way and we employ the standard conventions of the lambda calculus in writing them. Formally, they differ from terms of the lambda calculus only to the extent that they may contain context variables. We occasionally revert to universal algebraic notational conventions when the context seems to warrant it. Every λ-variable x and every context variable ξ is a λ-term; if t and s are λ-terms, then so are $t \cdot s$ and $\lambda x(t)$. Instead of $t \cdot s$ and $\lambda x(t)$ we will often write ts and $\lambda x.t$, following the traditional notation. $\lambda xyz.t$ stands for $\lambda x(\lambda y(\lambda z.t))$. When unspecified, application is assumed to be associated to the left and λ-abstraction to the right. For example $\lambda xyz.trs$ means $\lambda x[\lambda y[\lambda z(((t \cdot r) \cdot s))]]$.

We now give the formal definition of lambda abstraction algebras. Readers unfamiliar with the notation of the lambda calculus may find the reformulation of the axioms in terms of the abstract substitution operator given below, following Definition 1.3, easier to grasp.

Definition 1.1. By a *lambda abstraction algebra of dimension I* we mean an algebraic structure of the form

$$\mathbf{A} := \langle A, \cdot^{\mathbf{A}}, \langle \lambda x^{\mathbf{A}}\colon x \in I \rangle \langle x^{\mathbf{A}}\colon x \in I \rangle \rangle$$

satisfying the following quasi-identities for all $x, y, z \in I$ (subject to the indicated conditions) and all $\xi, \mu, \nu \in A$.

(β_1) $(\lambda x.x)\xi = \xi$;
(β_2) $(\lambda x.y)\xi = y$, $x \neq y$;
(β_3) $(\lambda x.\xi)x = \xi$;
(β_4) $(\lambda xx.\xi)\mu = \lambda x.\xi$;
(β_5) $(\lambda x.\xi\mu)\nu = (\lambda x.\xi)\nu((\lambda x.\mu)\nu)$;
(β_6) $(\lambda y.\mu)z = \mu \rightarrow (\lambda xy.\xi)\mu = \lambda y.(\lambda x.\xi)\mu$, $x \neq y$, $z \neq y$;
(α) $(\lambda y.\xi)z = \xi \rightarrow \lambda x.\xi = \lambda y.(\lambda x.\xi)y$, $z \neq y$.

I is called the *dimension set of* $\mathbf{A}$. $\cdot^{\mathbf{A}}$ is called *application* and $\lambda x^{\mathbf{A}}$ is called *λ-abstraction* with respect to x.

A closely related notion, lambda term systems, has recently been introduced by Diskin [10].

Axioms (β_1)–(β_6) constitute a definition of the abstract substitution that corresponds roughly to the axiomatic definition of metalinguistic substitution found in [9]. (α) is a direct algebraic translation of (α)-conversion.

The class of lambda abstraction algebras of dimension I is denoted by LAA_I and the class of all lambda abstraction algebras of any dimension by LAA. We also use LAA_I as shorthand for the phrase "lambda abstraction algebra of dimension I", and similarly for LAA. A LAA_I is of *infinite dimension* if I is infinite.

In the sequel A will be an arbitrary LAA_I, unless otherwise noted. The dimension set I is arbitrary, in particular it can be finite unless otherwise specified. However, many of the results we obtain require some minimal number of variables that varies from result to result. Since our primary interest in this paper are LLA_Is of infinite dimension, we usually do not explicitly indicate the minimal cardinality required of I except in the case it is infinite.

We will omit the superscript A on $\cdot^A$, λx^A, and x^A whenever we are sure we can do so without confusion. This will also apply to defined notions introduced below, such as Δ^A.

We note here one very useful immediately consequence of the axioms: in any LAA A the functions λx are always one–one, i.e.,

$$\lambda x.a = \lambda x.b \text{ iff } a = b \quad \text{for all } a, b \in A.$$

For if $\lambda x.a = \lambda x.b$, then by (β_3), $a = (\lambda x.a)x = (\lambda x.b)x = b$.

A LAA with only one element is said to be *trivial*. λx is onto (i.e., its range is all of A) only if A is trivial. In particular, if A is nontrivial, then x cannot be in the range of λx. For if $\lambda x.b = x$ for same $b \in A$, then for every $a \in A$ we have by (β_1) and (β_4),

$$a = (\lambda x.x)a = (\lambda xx.b)a = \lambda x.b = x.$$

It follows that every nontrivial LAA is infinite.

As this last result indicates, there are no easily described nontrivial LAAs; they do exist however. We know of essentially only two ways of constructing them. (1) Term LAAs of a consistent λ-theory. The Church–Rosser theorem [3, 3.2.8] must be used here to prove the existence of consistent λ-theories. (2) Functional LAAs. Here one requires an environment model (or equivalently a λ-model). As mentioned previously, the first environment model was constructed only in 1976 by Scott [32].

1.1. Substitution and dimension

The metalinguistic substitution operation $t \mapsto t[s/x]$ is abstracted as a system of binary operations $S^x_-(-)$ on A. They are defined in terms of lambda abstraction and application by inverting (β)-conversion. For any set A, A^* is the set of all finite strings of elements of A.

Definition 1.2. Let A be a LAA_I.
(i) $S^x_b(a) = (\lambda x.a)b$ for all $x \in I$ and $a, b \in A$.
(ii) $S^{\boldsymbol{x}}_{\boldsymbol{b}} = S^{x_1}_{b_1}(\ldots S^{x_n}_{b_n}(a)\ldots)$ for all $\boldsymbol{x} = x_1 \cdots x_n \in I^*$, $\boldsymbol{b} = b_1 \cdots b_n \in A^*$, and $a \in A$.

S is called the *(abstract) substitution operator*.

Algebraic dependence abstracts the metalinguistic property of a variable occurring free in a term.

Definition 1.3. *Let A be a* LAA_I. Let $a \in A$ and $x \in I$. a is said to be *algebraically dependent on x (over A)* if $S_z^x(a) \neq a$ for some $z \in I$; otherwise a is *algebraically independent of x (over A)*. The set of all $x \in I$ such that a is algebraically dependent on x over A is called the *dimension set* of a and is denoted by $\Delta^A a$; thus

$$\Delta^A a = \{x \in I \colon S_z^x(a) \neq a \text{ for some } z \in I\}.$$

a is finite- (infinite-) dimensional if Δa is finite (infinite). a is *zero-dimensional* if $\Delta a = \emptyset$.

It is convenient to treat algebraic dependency as a symmetric relation and speak of x being *algebraically dependent on (independent of) a*. We shall prove in Lemma 1.6 that $x \notin \Delta a$ iff $(\lambda x.a)z = a$ for some $z \in I \setminus \{x\}$. From this it is easy to see that the axioms for lambda abstraction algebras can be reformulated in the following way:

(β_1) $S_\xi^x(x) = \xi$;
(β_2) $S_\xi^x(y) = y$, $y \neq x$;
(β_3) $S_x^x(\xi) = \xi$;
(β_4) $S_\mu^x(\lambda x.\xi) = \lambda x.\xi$;
(β_5) $S_\nu^x(\xi\mu) = S_\nu^x(\xi) S_\nu^x(\mu)$;
(β_6) $y \notin \Delta\mu \Rightarrow S_\mu^x(\lambda y.\xi) = \lambda y.S_\mu^x(\xi)$, $x \neq y$;
(α) $y \notin \Delta\xi \Rightarrow \lambda x.\xi = \lambda y.S_y^x(\xi)$.

Note that the two occurrences of x in (β_3) have different meanings corresponding to the different roles of λ-variables as place holders and projections; this is something that is hidden by our overloaded notation. The difference becomes a little clearer when we interpret (β_3) in an actual LAA_I and explicitly relativize all the operations: $(S^A)^x_{x^A}(a) = ((\lambda x)^A . a)x^A = a$ for all $a \in A$ and $x \in I$. We will avoid notation like "$(S^A)^x_{x^A}$" because it is so cumbersome. We leave it to context to determine the particular algebra in which S is being applied.

If x and y are distinct λ-variables, then $x^A \neq y^A$ in any nontrivial LAA A. To see this choose any $a \in A$ such that $a \neq y^A$. Then by (β_2) and (β_3), $S_a^x(x^A) = a \neq y^A = S_a^x(y^A)$. So $x^A \neq y^A$.

We will show in the next proposition that, in the presence of the other axioms, (β_6) and (α) are equivalent to identities.

Lemma 1.4. *Axioms (β_4) and (β_5) imply $S_c^y S_b^y(a) = S^y_{S_c^y(b)}(a)$* for all $x \in I$ *and $a, b, c \in A$.*

Proof. $S_c^y S_b^y(a) = S_c^y((\lambda y.a)b) = S_c^y(\lambda y.a) S_c^y(b) = (\lambda y.a) S_c^y(b) = S^y_{S_c^y(b)}(a)$. □

Proposition 1.5. *In the presence of (β_2), (β_4), and (β_5), the quasi-identities (β_6) and (α) are logically equivalent to the following identities, respectively,*

(β_6') $S^x_{S_z^y(\mu)}(\lambda y.\xi) = \lambda y.S^x_{S_z^y(\mu)}(\xi)$ *if* $x \neq y$, $z \neq y$.
(α') $\lambda x.S_z^y(\xi) = \lambda y.S_y^x S_z^y(\xi)$ *if* $z \neq y$.

Thus LAA_I *is a variety for every dimension set I.*

Proof. Clearly (β_6') implies (β_6). For the opposite implication, substitute $S_z^y(\mu)$ for μ in (β_6) and observe that the antecedent of the resulting quasi-identity, $S_z^y S_z^y(\mu) = S_z^y(\mu)$, is a consequence of the Lemma 1.4 and (β_2). The equivalence of (α) and (α') is established in the same way. □

Since LAA_I is a variety, it is closed under the formation of subalgebras, homomorphic (in particular isomorphic) images, and Cartesian products. In symbols $\mathbb{S}\mathrm{LAA}_I = \mathbb{H}\mathrm{LAA}_I = \mathbb{P}\mathrm{LAA}_I = \mathrm{LAA}_I$.

The following lemma gives two useful alternative characterizations of algebraic dependency and consequently of dimension set.

Lemma 1.6. *Let $A \in \mathrm{LAA}_I$, and let $a \in A$ and $x \in I$. The following are equivalent.*

(i) $S_z^x(a) = a$ *for some* $z \in I \setminus \{x\}$;

(ii) $S_z^x(a) = a$ *for all* $z \in I$, *i.e.*, $x \notin \Delta a$;

(iii) $S_b^x(a) = a$ *for all* $b \in A$.

Proof. It clearly suffices to prove that (i) implies (iii). Assume $S_z^x(a) = a$ for some $z \in I \setminus \{x\}$. Then $S_b^x(a) = S_b^x S_z^x(a) = S_{S_b^x(z)}^x(a)$ (by Lemma 1.4) $= S_z^x(a) = a$. □

Proposition 1.7. *Let $A \in \mathrm{LAA}_I, a, b \in A$, and $x \in I$.*

(i) $\Delta(ab) \subseteq \Delta a \cup \Delta b$.

(ii) $\Delta(\lambda x.a) = \Delta a \setminus \{x\}$.

(iii) $\Delta(S_b^x(a)) \subseteq (\Delta a \setminus \{x\}) \cup \Delta b$.

(iv) $\Delta x \subseteq \{x\}$, *with equality holding if A is nontrivial.*

Proof. (i) follows immediately from (β_5).

(ii) For the inclusion $\Delta(\lambda x.a) \subseteq \Delta a \setminus \{x\}$ of (ii) use (β_4) and (β_6). To get the opposite inclusion, suppose $y \notin \Delta(\lambda x.a)$ and $y \neq x$. Then for any λ-variable $z \neq x, y$, we have $S_z^y(\lambda x.a) = \lambda x.a$. But by (β_2) and (β_6), $S_z^y(\lambda x.a) = \lambda x.S_z^y(a)$. Hence $\lambda x.S_z^y(a) = \lambda x.a$, which implies $S_z^y(a) = a$ since λx is one–one. Thus $y \notin \Delta a$.

(iii) is a direct consequence of (i) and (ii). Finally, the inclusion of (iv) follows from (β_2) and the equality from (β_1). □

Proposition 1.8. *Let $A \in \mathrm{LAA}_I$ and $a \in A$.*

(i) *If $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B} \in \mathrm{LAA}_I$, in symbols $\boldsymbol{A} \subseteq \boldsymbol{B}$, then $\Delta^A a = \Delta^B a$.*

(ii) *If $\boldsymbol{B} \in \mathrm{LAA}_I$, $h: \boldsymbol{B} \to \boldsymbol{A}$ is a homomorphism, and $b \in B$ such that $h(b) = a$, then $\Delta^A a \subseteq \Delta^B b$.*

(iii) *If $\boldsymbol{A} = \prod_{s \in S} \boldsymbol{B}_s$ with $\boldsymbol{B}_s \in \mathrm{LAA}_I$ for all $s \in S$, and if $a = \langle b_s\colon s \in S\rangle$, then $\Delta^A a = \bigcup_{s \in S} \Delta^{B_s} b_s$.*

Proof. All three parts are immediate consequences of the definition of dimension set. □

In the following lemma we give some basic properties of substitution that will be used repeatedly in the sequel.

Lemma 1.9. *For all $x, y, z \in I$ and $a, b, c \in A$ we have:*
(i) $x \notin \Delta c \;\Rightarrow\; S^y_c S^x_b(a) = S^x_{S^y_c(b)} S^y_c(a)$;
(ii) $y \notin \Delta b \;\Rightarrow\; S^y_b S^x_y(a) = S^y_b S^x_b(a)$;
(iii) $x \notin \Delta c,\, y \notin \Delta b \;\Rightarrow\; S^y_c S^x_b(a) = S^x_b S^y_c(a)$ *if* $x \neq y$;
(iv) $z \notin \Delta a \cup \Delta b \;\Rightarrow\; S^x_b(a) = S^z_b S^x_z(a)$.

Proof. (i) If $x = y$ the equation reduces to Lemma 1.4. Assume $x \neq y$,

$$\begin{aligned}
S^y_c S^x_b(a) &= S^y_c((\lambda x.a)b)\\
&= S^y_c(\lambda x.a)\, S^y_c(b) \quad \text{by } (\beta_4)\\
&= (\lambda x. S^y_c(a))\, S^y_c(b) \quad \text{by } (\beta_6) \text{ since } x \notin \Delta c \text{ and } x \neq y\\
&= S^x_{S^y_c(b)}(S^y_c(a)).
\end{aligned}$$

(ii)
$$\begin{aligned}
S^y_b S^x_y(a) &= S^y_b(\lambda x.a)\, S^y_b(y)\\
&= S^y_b(\lambda x.a)b\\
&= S^y_b(\lambda x.a)\, S^y_b(b) \quad \text{since } y \notin \Delta b\\
&= S^y_b((\lambda x.a)b)\\
&= S^y_b S^x_b(a).
\end{aligned}$$

(iii)
$$\begin{aligned}
S^y_c S^x_b(a) &= S^y_{S^y_c(b)} S^y_c(a) \quad \text{by (i) since } x \notin \Delta c\\
&= S^x_b S^y_c(a) \quad \text{since } y \notin \Delta b.
\end{aligned}$$

(iv)
$$\begin{aligned}
S^z_b S^x_z(a) &= S^z_b S^x_b(a) \quad \text{by (ii) since } z \notin \Delta b\\
&= S^x_b(a) \quad \text{since } z \notin \Delta a \cup \Delta b \supseteq \Delta S^x_b(a). \qquad \square
\end{aligned}$$

For any set A, A^* is the set of all sequences in A^* without repetitions.

Proposition 1.10. *Let A be a* LAA_I, $\boldsymbol{x} = x_1 \cdots x_n \in I^*$, and $\boldsymbol{b} = b_1 \cdots b_n \in A^*$. *If b_i is independent of $x_1, \ldots, x_{i-1}$ for $i = 2, \ldots, n$, in particular, if each b_i is independent of all the x_j, then*

$$S^{\boldsymbol{x}}_{\boldsymbol{b}}(a) = (\lambda x_1 \cdots x_n . a) b_1 \cdots b_n \quad \textit{for all } a \in A.$$

Proof. The conclusion is trivial for $n=1$. Let $x'=x_2\cdots x_n$ and $b'=b_2\cdots b_n$. By hypothesis, x_1 is independent of $b_2,\dots,b_n$. Thus

$$\begin{aligned}(\lambda x_1\cdots x_n.a)b_1\cdots b_n &= S^{x_1}_{b_1}(\lambda x_2\cdots x_n.a)\,b_2\cdots b_n\\ &= S^{x_1}_{b_1}(\lambda x_2\cdots x_n.a)\,S^{x_1}_{b_1}(b_2)\cdots S^{x_1}_{b_1}(b_n)\\ &= S^{x_1}_{b_1}((\lambda x_2\cdots x_n.a)b_2\cdots b_n)\quad \text{by } (\beta_5)\\ &= S^{x_1}_{b_1}S^{x'}_{b'}(a)\quad \text{by induction hypothesis}\\ &= S^{x}_{b}(a).\qquad \square\end{aligned}$$

2. Dimension restricted lambda abstraction algebras

The most natural LAAs, the ones the axioms are intended to characterize, are algebras of functions. They will be discussed in detail in the next section. An abstract LAA is *functionally representable* if it is isomorphic to an algebra of functions; the precise definition can be found in the next section. Not every LAA is functionally representable. It turns out that by restricting the dimension sets of elements of a LAA in different ways we can insure that it is functionally representable.

Definition 2.1. Let $A\in \mathrm{LAA}_I$.
(i) A *is locally finite-dimensional* if it is of infinite dimension (i.e., I is infinite) and every $a\in A$ is of finite dimension (i.e., $|\Delta a|<\omega$).
(ii) A is *dimension-complemented* if it is of infinite dimension and, for all $a\in A$, $\Delta a\neq I$.

The classes of locally finite-dimensional and dimension-complemented LAA_Is are denoted, respectively, by LFA_I and DCA_I. Note that $\mathrm{LFA}_I\subseteq \mathrm{DCA}_I$ for every infinite I.

LFAs correspond most closely to the other algebraic models of the lambda calculus that have appeared in the literature, for instance the *term λ-algebras* [22] and *syntactical models* [3] of combinatory logic and the *Curry theories* of [24]. There seems to be no similar analogue for DCAs.

Lemma 2.2. *Let* $A\in \mathrm{LAA}_I$.
(i) Let $a,b\in A$ *and assume* a *and* b *are both (algebraically) independent of the λ-variable* x. *If* $xa=xb$, *then* $a=b$.
(ii) *Let* $a,b,c,d\in A$ *and assume they are all independent of* x. *Then* $xab=xcd$ *implies* $a=c$ *and* $b=d$.

Proof. (i) Let $\mathbf{i}=\lambda x.x$. Since a is independent of x, $S^x_{\mathbf{i}}(xa)=S^x_{\mathbf{i}}(x)S^x_{\mathbf{i}}(a)=\mathbf{i}a=(\lambda x.x)a=S^x_a(x)=a$. Similarly, $S^x_{\mathbf{i}}(xb)=b$. Thus $xa=xb$ implies $a=S^x_{\mathbf{i}}(xa)=S^x_{\mathbf{i}}(xb)=b$.

(ii) The proof is similar to that of (i). Choose any $y \neq x$ and let $\mathbf{k} = \lambda xy.x$. Then $S^x_{\mathbf{k}}(xab) = \mathbf{k}ab = (\lambda xy.x)ab = S^x_a S^y_b(x)$ (by Proposition 1.10) $= S^x_a(x) = a$. Similarly, $S^x_{\mathbf{k}}(xcd) = c$. Thus $xab = xcd$ implies $a = c$.

Again choose any $y \neq x$ and let $\mathbf{1} = \lambda xy.y$. Then $S^x_{\mathbf{1}}(xab) = \mathbf{1}ab = S^x_a S^y_b(y) = S^x_a(b) = b$. Similarly, $S^x_{\mathbf{1}}(xcd) = d$. Thus $xab = xcd$ implies $b = d$. □

A subset J of I is, respectively, *coinfinite* or *coinfinite* if $I \setminus J$ is finite or infinite.

Proposition 2.3. *Let $A \in \mathrm{LAA}_I$ with I infinite. If A is dimension-complemented, then $\Delta a_1 \cup \cdots \cup \Delta a_n$ is coinfinite for any finite set $a_1, \ldots, a_n$* of elements of A.

Proof. The conclusion is obvious if A is trivial, so we assume otherwise. The proof consists of the following results.

(1) Let $a \in A$ and assume a is independent of x. Then $\Delta(xa) = \Delta a \cup \{x\}$.

It suffices, in view of Proposition 1.7(i) and (iv), to prove $\Delta a \cup \{x\} \subseteq \Delta(xa)$. Suppose xa is independent of x. Then $xa = S^x_b(xa) = S^x_b(x) S^x_b(a) = ba$ for every $b \in A$. Hence $ba = ca$ for all $b, c \in A$. Let y be a λ-variable distinct from x. Then $y = S^x_a(y) = (\lambda x.y)a = (\lambda y.x)a = S^y_a(x) = x$; more precisely, $y^A = x^A$. But this is impossible since A is nontrivial. Thus xa depends on x.

Suppose that a depends on y but xa does not. This implies $y \neq x$ by the first part of the proof. Thus $xa = S^y_z(xa) = xS^y_z(a)$ for any λ-variable $z \neq y$. a is independent of x by hypothesis, and so is $S^y_z(a)$ by Proposition 1.7(iii), provided $z \neq x$. Thus Lemma 2.2(i) applies and we can conclude that $S^y_z(a) = a$ for every $z \neq x, y$. This contradicts the assumption that a depends on y. So xa must also depend on y.

(2) Let $a \in A$. If Δa is cofinite, then there exists a $b \in A$ such that $\Delta b = I$.

This follows directly from (1) by an easy induction.

We can now conclude that, if A is dimension-complemented, then Δa is coinfinite for every $a \in A$.

(3) Let $a, b \in A$ and assume they are both independent of x. Then $\Delta(xab) = \Delta a \cup \Delta b \cup \{x\}$.

It suffices to prove the inclusion from right to left. Suppose xab is independent of x. Then as in the proof of (1) we get $cab = dab$ for all $c, d \in A$. Let y and z be λ-variables such that x, y, and z are pairwise distinct. Then

$$(\lambda zx.y)ab = S^z_a(\lambda x.y)b = (\lambda x S^z_a(y))b = (\lambda x.y)b = S^x_b(y) = y;$$

the second equality holds by (β_6) since $x \notin \Delta a$. Similarly, $(\lambda yx.z)ab = z$. Thus $y^A = z^A$ which contradicts the assumption that A is nontrivial. So $x \in \Delta(xab)$.

Suppose that either a or b depends on y but xab does not. This implies $y \neq x$ by the first part of the proof. Thus $xab = S^y_z(xab) = xS^y_z(a)S^y_z(b)$ for every $z \neq y$. $a, b, S^y_z(a)$, $S^y_z(b)$ are all independent of x, provided $z \neq x$. Thus Lemma 2.2(ii) applies and we get $S^y_z(a) = a$ and $S^y_z(b) = b$ for every $z \neq x, y$. This contradicts the assumption that either a or b depends on y. So xab must depend on y.

(4) Let $a_1, \dots, a_n \in A$ and assume $x_1, \dots, x_n$ are distinct λ-variables that are independent of all the a_i. Define $b_1 = a_1$, $b_2 = x_1 b_1 a_2, \dots, b_n = x_n b_{n-1} a_n$. Then $\Delta b_n = \Delta a_1 \cup \cdots \cup \Delta a_n \cup \{x_1, \dots, x_n\}$.

This is proved by a straightforward induction using (3).

We are now ready to establish the conclusion of the proposition. Assume $\Delta a_1 \cup \cdots \cup \Delta a_n$ is cofinite. By replacing each of the a_i by its λ-abstraction with respect to appropriate λ-variables, if necessary, and using Lemma 1.7(ii), we can assume without loss of generality that there exist n distinct λ-variables $x_1, \dots, x_n$ that are independent of all of the a_i. Let b_n be defined as in (4). Then we can conclude from (4) that Δb_n is cofinite, contradicting a result previously established in the proof. □

2.1. Simultaneous substitution

We abstract the process of simultaneously substituting a finite sequence $t_1, \dots, t_n$ of terms for the variables $x_1, \dots, x_n$ in a term s. Such a substitution can be simulated by sequential substitutions provided the free occurrences of the x_i in s are first replaced by new variables that do not conflict with the free variables of $t_1, \dots, t_n$. This is the basis of our abstraction. Implicit is the assumption that a reservoir of new variables is always available. Consequently simultaneous substitution can only be abstracted under some kind of dimension-restricting assumption.

We introduce some useful notation. Recall that A^* is the set of all finite sequences of elements of A, and $A^\star$ is the set of all finite sequences in A^* without repetitions. Let $A \in \mathrm{LAA}_I$. If $\boldsymbol{a} = a_1 \cdots a_n \in A^*$, then we define $\Delta \boldsymbol{a} = \Delta a_1 \cup \cdots \cup \Delta a_n$. A λ-variable z *is independent* of $\boldsymbol{a}$ if $z \notin \Delta \boldsymbol{a}$. It is *independent* of $\boldsymbol{x} = x_1 \cdots x_n \in I^*$ if $z \neq x_1, \dots, x_n$.

Definition 2.4. Assume A is a DCA_I and let $a \in A$. Let $\boldsymbol{b} = b_1 \cdots b_n \in A^*$ and $\boldsymbol{x} = x_1 \cdots x_n \in I^\star$. Choose $\boldsymbol{z} = z_1 \cdots z_n \in I^\star$ such that

$$\text{each } z_i \text{ is independent of } a,\ \boldsymbol{b}, \text{ and } \boldsymbol{x}. \tag{2.1}$$

We define

$$\hat{S}^{\boldsymbol{x}}_{\boldsymbol{b}}(a) = S^{\boldsymbol{z}}_{\boldsymbol{b}} S^{\boldsymbol{x}}_{\boldsymbol{z}}(a).$$

Lemma 2.5. *The definition of* $\hat{S}^{\boldsymbol{x}}_{\boldsymbol{b}}(a)$ *is independent of the choice of the sequence* $\boldsymbol{z}$ *satisfying* (2.1).

Proof. The proof is by induction on the length n of $\boldsymbol{x}$. □

Thus, if A is dimension-complemented or locally finite-dimensional, then $\hat{S}^{\boldsymbol{x}}_{\boldsymbol{b}}$ is a well-defined mapping from A into itself for all $\boldsymbol{b} \in A^*$ and $\boldsymbol{x} \in I^\star$ such that $\boldsymbol{b}$ and $\boldsymbol{x}$ are of the same length. $\hat{S}$ is called the *simultaneous substitution operator* and $\hat{S}^{\boldsymbol{x}}_{\boldsymbol{b}}(a)$ is the result of *simultaneously substituting* $b_1, \dots, b_n$ for $x_1, \dots, x_n$ in a.

Lemma 2.6. *Let A be a* DCA_I, *and let $\boldsymbol{b}$ and $\boldsymbol{x}$ be as in the Definition* 2.4. *Let $\boldsymbol{x}'$ be any permutation of the sequence $\boldsymbol{x}=x_1 \cdots x_n$ and let $\boldsymbol{b}'$ be the corresponding permutation of $\boldsymbol{b}$. Then $\hat{S}^{\boldsymbol{x}}_{\boldsymbol{b}}(a)=\hat{S}^{\boldsymbol{x}'}_{\boldsymbol{b}'}(a)$ for all $a \in A$.*

Proof. Choose $\boldsymbol{z}=z_1 \cdots z_n$ satisfying (2.1), and let $\boldsymbol{z}'$ be the permutation of $\boldsymbol{z}$ corresponding to the permutation of $\boldsymbol{x}$. Then by repeated application of Lemma 1.9(iii) we have

$$\hat{S}^{\boldsymbol{x}}_{\boldsymbol{b}}(a)=S^{\boldsymbol{z}}_{\boldsymbol{b}}S^{\boldsymbol{x}}_{\boldsymbol{z}}(a)=S^{\boldsymbol{z}'}_{\boldsymbol{b}'}S^{\boldsymbol{x}'}_{\boldsymbol{z}'}(a)=\hat{S}^{\boldsymbol{x}'}_{\boldsymbol{b}'}(a). \qquad \square$$

Lemma 2.7. *Assume A is a* DCA_I. *Let $a \in A$, $x \in I$, $\boldsymbol{b}=b_1 \cdots b_n \in A^*$, and $\boldsymbol{y}=y_1 \cdots y_n \in I^\star$. Let $z \in I$ be distinct from x, disjoint from $\boldsymbol{y}$, and independent of a and $\boldsymbol{b}$.*

(i) *If $x \neq y_1, \ldots, y_n$, then $\hat{S}^{\boldsymbol{y}}_{\boldsymbol{b}}(\lambda x.a)=\lambda z.\hat{S}^{\boldsymbol{y}x}_{\boldsymbol{b}z}(a)$.*

(ii) *If $x=y_i$ for some $i \leqslant n$, then $\hat{S}^{\boldsymbol{y}}_{\boldsymbol{b}}(\lambda x.a)=\lambda z.\hat{S}^{\boldsymbol{y}'x}_{\boldsymbol{b}'z}(a)$,* where $\boldsymbol{y}'=y_1 \cdots y_{i-1}y_{i+1} \cdots y_n$ and $\boldsymbol{b}'=b_1 \cdots b_{i-1}b_{i+1} \cdots b_n$.

(iii) *If $x \neq y_1, \ldots, y_n$, then $S^z_c\hat{S}^{\boldsymbol{y}x}_{\boldsymbol{b}z}(a)=\hat{S}^{\boldsymbol{y}x}_{\boldsymbol{b}c}(a)$ for every $c \in A$.*

Proof. We prove item (i). The verification of (ii) and (iii) is similar to that of (i) and is omitted.

$$\begin{aligned}\hat{S}^{\boldsymbol{y}}_{\boldsymbol{b}}(\lambda x.a)&=\hat{S}^{\boldsymbol{y}}_{\boldsymbol{b}}(\lambda z.S^x_z(a)) \quad \text{by } (\alpha)\\ &=\lambda z.\hat{S}^{\boldsymbol{y}}_{\boldsymbol{b}}S^x_z(a) \quad \text{by } (\beta_6)\\ &=\lambda z.\hat{S}^{\boldsymbol{y}x}_{\boldsymbol{b}z}(a). \qquad \square\end{aligned}$$

3. Functional lambda abstraction algebras

As previously observed, the most natural LAAs, the algebras that the axioms are intended to characterize, are algebras of functions. Not surprisingly, they are closely related to the environment models of lambda calculus. Indeed, they are obtained by coordinatizing environment models by the λ-variables in a natural way. The exact connection between LAAs and models of the lambda calculus is investigated in detail in [28]. The main results of this section are functional representation theorems for LFA_Is and DCA_Is. Not every LAA_I is functionally representable, but this result will be presented elsewhere.

Recall that $f: A \multimap B$ means that f is a partial function from A to B. Let $\boldsymbol{V}=\langle V, \cdot^V, \lambda^V \rangle$ be an environment model, as described in the Introduction. The application operation is $\cdot^V$, and $\lambda^V: V^V \multimap V$ is the encoding of the admissible functions of $\boldsymbol{V}$ into V; recall that they are related by the equality (0.1). Let I be the set of λ-variables. An element p of V^I, i.e., an assignment of elements of V to λ-variables, is called an *environment*. p_x is the value p assigns to x for each $x \in I$. For any $v \in V$ and $x \in I$, $p(v/x)$ is the new environment defined for each $y \in I$ by $(p(v/x))_y=v$ if $y=x$; p_y, otherwise.

Let $a \in V^{V^I}$, i.e., a function from environments to V. Each λ-variable x and environment p determines a function $a_{x,p}$ in V^V by letting x range over V but fixing the value of each $y \in I$, $y \neq x$, to be p_y; symbolically, $a_{x,p} = \langle a(p(v/x)) : v \in V \rangle \in V^V$. a is *admissible* if each of the functions $a_{x,p}$ is admissible in V, i.e., is in the domain of λ^V. Every functional LAA consists of a set of admissible functions in V^{V^I} for some environment model V. The operations of application and λ-abstraction of V induce operations on the admissible functions in V^{V^I} in a natural way.

We now give a precise definition of a functional LAA. Our definition does not depend on the notion of an environment model, and in fact, in Section 5, we shall see how to define environment models in terms of functional LAAs.

Definition 3.1. Let $V = \langle V, \cdot^V, \lambda^V \rangle$ be a structure where V is a nonempty set, $\cdot^V$ is a binary operation on V, and $\lambda^V : V^V \multimap V$ is a partial function assigning elements of V to certain functions from V into itself. V is called a *functional domain* if for each f in the domain of λ^V,

$$f(v) = (\lambda^V(f)) \cdot^V v \quad \text{for all } v \in V. \tag{3.1}$$

Definition 3.2. Let $V = \langle V, \cdot^V, \lambda^V \rangle$ be a functional domain and let I be a nonempty set. Let $V_I = \{f : f : V^I \multimap V\}$, i.e., the set of all partial functions from V^I to V. By the *I-coordinatization* of V we mean the algebra

$$V_I = \langle V_I, \cdot^{V_I}, \langle \lambda x^{V_I} : x \in I \rangle, \langle x^{V_I} : x \in I \rangle \rangle,$$

where for all $a, b : V^I \multimap V$, $x \in I$, and $p \in V^I$:

- $(a \cdot^{V_I} b)(p) = a(p) \cdot^V b(p)$, provided $a(p)$ and $b(p)$ are both defined; otherwise $(a \cdot^{V_I} b)(p)$ *is undefined.*
- $(\lambda x^{V_I}.a)(p) = \lambda^V(\langle a\,(p(v/x)) : v \in V \rangle)$, provided $\langle a(p(v/x)) : v \in V \rangle$ is in the domain of λ^V (note this implies $a(p(v/x))$ is defined for all $v \in V$); otherwise $(\lambda x^{V_I}.a)(p)$ is undefined.
- $x^{V_I}(p) = p_x$.

Definition 3.3. Let V and I be as in preceding definition. A subalgebra A of total functions of V_I, i.e., a subalgebra such that $(\lambda x^{V_I}.a)(p)$ is defined for all $a \in A$ and $p \in V^I$, is called a *functional lambda abstraction algebra*. I is the *dimension set* of A and V is its *value domain.*

The class of functional lambda abstraction algebras of dimension I is denoted by FLA_I. We shall show in Theorem 3.7 that every FLA_I is a LAA_I. Locally finite-dimensional FLAs are similar to the functional models of the lambda calculus developed in Krivine [20].

In the sequel a subalgebra of V_I of total functions will be called a *total subalgebra* of V_I. It will be shown below that a functional domain V is an environment model of the lambda calculus in the sense of [22] iff V_I has at least one total subalgebra (see Definition 5.1 and the remarks preceding Theorem 5.8).

Definition 3.4. Let A be a FLA_I with value domain V. An element a of A is *functionally independent of* $x \in I$ if, for all $p, q \in V^I$, $p_y = q_y$ for all $y \in I \setminus \{x\}$ implies $a(p) = a(q)$; otherwise a is *functionally dependent on* x.

We treat functional dependency as a symmetric relation and say that x is *functionally independent of (dependent on) a*.

Lemma 3.5. *Let A be a* FLA_I *with value domain V. For all $a, b \in A$, $x \in I$, and $p \in V^I$,*

$$(S_b^x(a))(p) = a(p(b(p)/x)).$$

Proof. $(S_b^x(a))(p) = ((\lambda x^{V_I}.a) \cdot^{V_I} b)(p) = (\lambda x^{V_I}.a)(p) \cdot^V b(p) = \lambda^V(\langle a(p(v/x)) \colon v \in V\rangle) \cdot^V b(p) = a(p(b(p)/x))$ by (3.1). □

Proposition 3.6. *Let A be a* FLA_I *and $a \in A$. Then Δa coincides with the set of all $x \in I$ that functionally depend on a. Thus the notions of algebraic and functional dependence coincide.*

Proof. Assume $x \notin \Delta a$, i.e., $S_z^x(a) = a$ for some $z \neq x$. Then, for every $q \in V^I$, $a(q(q_z/x)) = a(q(z^A(q)/x)) = (S_z^x(a))(q) = a(q)$ by Lemma 3.5. First taking $q = p$ we get

$$a(p(p_z/x)) = a(p). \tag{3.2}$$

Now take $q = p(v/x)$ for arbitrary $v \in V$, and note that $q_z = p_z$ and hence $q(q_z/x) = p(v/x)(p_z/x) = p(p_z/x)$. We thus get

$$a(p(p_z/x)) = a(p(v/x)). \tag{3.3}$$

The combination of (3.2) and (3.3) shows that algebraic dependency implies functional dependency. The proof of the implication in the opposite direction is straightforward and is omitted. □

Theorem 3.7. *Every* FLA_I *is a* LAA_I.

Proof. Let A be a FLA_I. We verify only axiom (β_6). The verification of the other axioms is similar to that of (β_6) and is omitted. Let a, $b \in A$, $x, y \in I$ with $x \neq y$, and $p \in V^I$.

(β_6) Assume $y \notin \Delta b$. For every $v \in V$ set $p^v = p(v/y)$. Then by Proposition 3.6, $b(p) = b(p(v/y)) = b(p^v)$, and hence,

$$p(b(p)/x)(v/y) = p(v/y)(b(p)/x) = p(v/y)(b(p(v/y))/x) = p^v(b(p^v)/x). \tag{3.4}$$

$$\begin{aligned}(S^x_b(\lambda y^A.a))(p) &= (\lambda y^A.a)(p\,(b(p)/x))\\ &= \lambda^V(\langle a(p(b(p)/x)(v/y)) \colon v \in V\rangle)\\ &= \lambda^V(\langle a(p^v(b(p^v)/x)) \colon v \in V\rangle) \quad \text{by (3.4)}\\ &= \lambda^V(\langle S^x_b(a)(p^v) \colon v \in V\rangle)\\ &= \lambda^V(\langle S^x_b(a)(p(v/y)) \colon v \in V\rangle)\\ &= (\lambda y^A.S^x_b(a))(p). \qquad \square\end{aligned}$$

3.1. Functional representation of dimension restricted LAAs

Definition 3.8. Let $A = \langle A, \cdot^A, \lambda x^A, x^A\rangle_{x \in I}$ be an arbitrary LAA_I. The *functional domain* $V = \langle V, \cdot^V, \lambda^V\rangle$ *associated with* A is defined as follows: $V = A$ and $\cdot^V = \cdot^A$. The domain of $\lambda^V \colon V^V \multimap V$ is

$$\mathrm{dom}(\lambda^V) = \{\langle S^x_v(a) \colon v \in V\rangle \colon a \in A \text{ and } x \in I\},$$

and, for each function in this set,

$$\lambda^V(\langle S^x_v(a) \colon v \in V\rangle) := \lambda x^A.a. \tag{3.5}$$

The following lemma shows that λ^A is well defined.

Lemma 3.9. $\langle S^x_v(a) \colon v \in V\rangle = \langle S^y_v(b) \colon v \in V\rangle$ *implies* $\lambda x^A.a = \lambda y^A.b$.

Proof. Assume $S^x_v(a) = S^y_v(b)$ for all $v \in V$. If $x = y$, then taking $v = x$, we get $a = S^x_x(a) = S^x_x(b) = b$, and hence $\lambda x^A.a = \lambda y^A.b$. Suppose now that $x \neq y$. Taking $v = x$, we get $a = S^x_x(a) = S^y_x(b)$. Thus y is independent of a by Proposition 1.7(iii). Now taking $v = y$, we get $b = S^y_y(b) = S^x_y(a)$. Hence, by (α), $\lambda x^A.a = \lambda y^A.S^x_y(a) = \lambda y^A.b$. $\square$

So λ^V and hence the structure V are well defined. Recall that $V = A$.

Lemma 3.10. V *is a functional domain.*

Proof. Let $f = \langle S^x_v(a) \colon v \in V\rangle$ for some $a \in A$, $x \in I$. Then $(\lambda^V(f)) \cdot^V v = (\lambda x^A.a) \cdot^A v = S^x_v(a) = f(v)$. $\square$

The following representation theorem is the main result of [25].[3] It was independently proved by Diskin and Beylin [11]. It is the algebraic analogue of the completeness theorem for lambda calculus which says that every lambda theory consists of precisely the equations valid in some environment model [22].

Theorem 3.11. (Functional representation of LFAs). *Every* LFA_I *is isomorphic to a* FLA_I, *i.e.*, $\mathrm{LFA}_I \subseteq \mathbb{I}\mathrm{FLA}_I$. *More precisely, every* LFA_I A *is isomorphic to a total subalgebra of the I-coordinatization of its associated functional domain.*

Proof (*outline*). Let V be the functional domain associated with A. The mapping Ψ from A into V_I is defined as follows: For each $a \in A$, let $x = x_1 \cdots x_n \in I^\star$ such that $\Delta a \subseteq \{x_1, \ldots, x_n\}$. Define

$$\Psi(a)(p) = \hat{S}^{x}_{p \circ x}(a) \quad \text{for every } a \in A \text{ and } p \in V^I.$$

In [25] it is shown that Ψ is well defined and is an isomorphism from A onto a total subalgebra of V_I. $\square$

This theorem corresponds to the completeness theorem for the lambda calculus [22], which says that every lambda theory consists of precisely the equations valid in some environment model. It is modeled on the functional representation theorem for locally finite-dimensional polyadic Boolean algebras [15, Theorem (10.9)]. It is also closely related to the representation theorem for locally finite-dimensional cylindric algebras [17, Part II, Theorem 3.2.11(i)], which corresponds to the completeness theorem for first-order predicate logic (cf. the Forward of [17], Part I).

LFAs correspond most closely to the other algebraic models of the lambda calculus that have appeared in the literature, for instance the *term λ-algebras* [22] and *syntactical models* [3] of combinatory logic and the *Curry theories* of [24]. On the other hand locally finite-dimensional FLAs correspond to the *environment models* [22] and *λ-models* [3] of combinatory logic and the *functional Curry theories* in [24].

Point-relativized functional LAA's. The functional representation theorem for DCAs requires a more general notion of functional algebra. This more general class of functional LAAs is analogous to the class of *weak cylindric fields of sets* in the theory of cylindric algebras; see [17, Part I, Definition 3.1.2].

Let V be a functional domain and let r be a fixed but arbitrary element of V^I, and let V^I_r be the set of all $p \in V^I$ that differ from r at only finitely many coordinates, i.e.

$$V^I_r = \{p \in V^I : |\{x \in I : p_x \neq r_x\}| < \omega\}.$$

[3] In [25] we prove that every LFA of denumerable dimension is functionally representable. It is a trivial matter, however, to extend the proof to cover arbitrary infinite index sets.

Let $V_{I,r}$ be the set of all partial functions $f: V_r^I \hookrightarrow V$. The (I,r)-*coordinatization* of V,

$$V_{I,r} = \langle V_{I,r}, \cdot^{V_{I,r}}, \langle \lambda x^{V_{I,r}}: x \in I \rangle, \langle x^{V_{I,r}}: x \in I \rangle \rangle,$$

is defined just as V_I except that all functions are restricted to V_r^I.

Definition 3.12. A subalgebra A of $V_{I,r}$ of total functions is called a *point-relativized functional lambda abstraction algebra. I* is the *dimension set* of A and r is its *thread.* V is the *value domain* of A.

The class of point-relativized functional lambda abstraction algebras of dimension I is denoted by RFA_I. Every RFA_I is a LAA_I. The proof is almost identical to that of the corresponding result for FLAs, Theorem 3.7.

Let A be a FLA_I with value algebra V, and let $r \in V^I$. For each partial function $a: V^I \hookrightarrow V$, let $a|_r$ denote the restriction of a to V_r^I, i.e., $a|_r = a \upharpoonright V_r^I$.

The definitions of functional dependence and independence carry over to RFAs without change, and Lemma 3.5 and Proposition 3.6 continue to hold with the obvious changes. In particular, in any RFA_I A with value domain V and thread r, $\Delta^A a$ is the set of all $x \in I$ that are functionally dependent on a.

Let A be a FLA_I with value domain V and let $r \in V^I$. Let $A|_r = \{a|_r : a \in A\}$ and $A|_r = \langle A|_r, \cdot^{A|_r}, \lambda x^{A|_r}, x^{A|_r} \rangle_{x \in I}$, where

$$a|_r \cdot^{A|_r} b|_r = (a \cdot^A b)|_r, \qquad \lambda x^{A|_r}.a|_r = (\lambda x^A.a)|_r, \qquad x^{A|_r} = x^A|_r.$$

These operations are well defined. It is an easy matter to check that $A|_r$ is a point-relativized FLA_I. Clearly the mapping $a \mapsto a|_r$ is a surjective homomorphism from A onto $A|_r$.

Proposition 3.13. *Every* FLA_I *is isomorphic to a subdirect product of* RFA_I*s with the same value domain. In particular, if* $A \in \mathrm{FLA}_I$ *and* V *is its value domain, then* A *is isomorphic to a subdirect product of the* $A|_r$ *for all* $r \in V^I$.

Proof. If $a, b \in A$ with $a \neq b$, then there is a $r \in V^I$ such that $a(r) \neq b(r)$. Clearly then $a|_r \neq b|_r$. Thus the collection of maps $a \mapsto a|_r$, for all $r \in V^I$, constitute a separating family of surjective homomorphisms from A into $\{A|_r : r \in V^I\}$. □

Later, as a consequence of the functional representation theorem for neat reducts (Theorem 7.4) we shall see that every FLA_I is in fact isomorphic to a RFA_I, but one with a different (and more complex) value domain.

Theorem 3.14 (Functional representation of DCAs). *Every* DCA_I *is isomorphic to a* RFA_I, *i.e.,* $\mathrm{DCA}_I \subseteq \mathbb{I}\mathrm{RFA}_I$. *More precisely, every* DCA_I A *is isomorphic to a total subalgebra of the* (I, ε)*-coordinatization of its associated functional domain where* $\varepsilon = \langle x^A : x \in I \rangle$.

Proof. Let $\boldsymbol{A}=\langle A, \cdot^{A}, \lambda x^{A}, x^{A}\rangle_{x\in I}$ be a DCA_I. (Recall that this implies I is infinite.) Let $\boldsymbol{V}$ be its associated value domain (Definition 3.8). For each $p\in V_\varepsilon^I$ there exist unique (up to ordering) finite sequences $\boldsymbol{y}=y_1\cdots y_n\in I^\star$ and $\boldsymbol{u}=u_1\cdots u_v\in V^*$ such that $p=\varepsilon(\boldsymbol{u}/\boldsymbol{y})(=\varepsilon(u_1/y_1,\dots,u_n/y_n))$. Moreover, every pair of such sequences determines an element of V_ε^I in this way. Recall that $V_{I,\varepsilon}=\{f\colon f\colon V_\varepsilon^I \multimap V\}$. We define a mapping $\Psi: A\to V_{I,\varepsilon}$ as follows for all $a\in A$:

$$\Psi(a)(\varepsilon(\boldsymbol{u}/\boldsymbol{y}))=\hat{S}_{\boldsymbol{u}}^{\boldsymbol{y}}(a) \quad \text{for all } \boldsymbol{y}\in I^*, \quad \boldsymbol{u}\in V^*.$$

$\Psi(a)$ is well defined for every $a\in A$ by Lemma 2.6. Note that $\Psi(a)$ is a total function such that $\Psi(a)(\varepsilon)=a$ for every $a\in A$. Thus Ψ is one–one. We verify that Ψ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{V}_{I,\varepsilon}$.

Let $a,b\in A$, $x\in I$, and $p\in V_\varepsilon^I$; let $p=\varepsilon(\boldsymbol{u}/\boldsymbol{y})$ where $\boldsymbol{u}=u_1\cdots u_n$ and $\boldsymbol{y}=y_1\cdots y_n$,

$$\begin{aligned}
\Psi(a\cdot^{A}b)(p)&=\hat{S}_{\boldsymbol{u}}^{\boldsymbol{y}}(a\cdot^{A}b)\\
&=\hat{S}_{\boldsymbol{u}}^{\boldsymbol{y}}(a)\cdot^{A}\hat{S}_{\boldsymbol{u}}^{\boldsymbol{y}}(b)\\
&=\hat{S}_{\boldsymbol{u}}^{\boldsymbol{y}}(a)\cdot^{V}\hat{S}_{\boldsymbol{u}}^{\boldsymbol{y}}(b) \quad \text{since } \langle V,\cdot^{V}\rangle=\langle A,\cdot^{A}\rangle\\
&=\Psi(a)(p)\cdot^{V}\Psi(b)(p)\\
&=(\Psi(a)\cdot^{V_{I,\varepsilon}}\Psi(b))(p).
\end{aligned}$$

Suppose first of all that $x\neq y_1,\dots,y_n$. Choose z distinct from x and independent of a and $u_1,\dots,u_n$ (such a z exists by Proposition 2.3)

$$\begin{aligned}
\Psi(\lambda x^{A}.a)(p)&=\Psi(\lambda x^{A}.a)(\varepsilon(\boldsymbol{u}/\boldsymbol{y}))\\
&=\hat{S}_{\boldsymbol{u}}^{\boldsymbol{y}}(\lambda x^{A}.a)\\
&=\lambda z^{A}.\hat{S}_{\boldsymbol{u}z}^{\boldsymbol{y}x}(a) \quad \text{by Lemma 2.7(i)}\\
&=\lambda^{V}(\langle S_v^z\hat{S}_{\boldsymbol{u}z}^{\boldsymbol{y}x}(a)\colon v\in V\rangle) \quad \text{by (3.5)}\\
&=\lambda^{V}\langle(\hat{S}_{\boldsymbol{u}v}^{\boldsymbol{y}x}(a)\colon v\in V\rangle) \quad \text{by Lemma 2.7(iii)}\\
&=\lambda^{V}(\langle\Psi(a)(\varepsilon(\boldsymbol{u}v/\boldsymbol{y}x))\colon v\in V\rangle)\\
&=\lambda^{V}\langle\Psi(a)(\varepsilon(\boldsymbol{u}/\boldsymbol{y})(v/x))\colon v\in V\rangle)\\
&=(\lambda x^{V_{I,\varepsilon}}.\Psi(a))(\varepsilon(\boldsymbol{u}/\boldsymbol{y}))\\
&=(\lambda x^{V_{I,\varepsilon}}.\Psi(a))(p).
\end{aligned}$$

Suppose next that $x=y_i$ for some $i\leqslant n$. Again choose z distinct from x and independent of a and $u_1,\dots,u_n$. Without loss of generality we assume $x=y_n$. Let

$y' = y_1 \cdots y_{n-1}$, $u' = u_1 \cdots u_{n-1}$.

$$
\begin{aligned}
\Psi(\lambda x^A . a)(p) &= \Psi(\lambda x^A . a)(\varepsilon(\boldsymbol{u}/\boldsymbol{y}))\\
&= \hat{S}^{\boldsymbol{y}}_{\boldsymbol{u}}(\lambda x^A . a)\\
&= \hat{S}^{\boldsymbol{y}'}_{\boldsymbol{u}'}(\lambda x^A . a)\\
&= \lambda z^A . \hat{S}^{\boldsymbol{y}'x}_{\boldsymbol{u}'z}(a) \quad \text{by Lemma 2.7(ii)}\\
&= \lambda^V(\langle S^z_v \hat{S}^{\boldsymbol{y}'x}_{\boldsymbol{u}'v}(a) \colon v \in V\rangle) \quad \text{by (3.5)}\\
&= \lambda^V(\langle \hat{S}^{\boldsymbol{y}'x}_{\boldsymbol{u}'z}(a) \colon v \in V\rangle) \quad \text{by Lemma 2.7(iii)}\\
&= \lambda^V(\langle \Psi(a)(\varepsilon(\boldsymbol{u}'v/\boldsymbol{y}'x)) \colon v \in V\rangle)\\
&= \lambda^V(\langle \Psi(a)(\varepsilon(\boldsymbol{u}'/\boldsymbol{y}')(v/x)) \colon v \in V\rangle)\\
&= \lambda^V(\langle \Psi(a)(\varepsilon(\boldsymbol{u}/\boldsymbol{y})(v/x)) \colon v \in V\rangle)\\
&= (\lambda x^{V_{I,\varepsilon}} . \Psi(a))(\varepsilon(\boldsymbol{u}/\boldsymbol{y}))\\
&= (\lambda x^{V_{I,\varepsilon}} . \Psi(a))(p).
\end{aligned}
$$

Thus Ψ preserves λ-abstractions. Finally, the verification that the interpretations of the λ-variables are also preserved is left to the reader. □

This theorem can be compared with the *representation theory for dimension-complemented cylindric algebras* that asserts that every such algebra is isomorphic to a representable cylindric algebra; see [17, Part II, Theorem 3.2.11]. The theorem has as a corollary the completeness theorem for the extended lambda calculus; see the Introduction.

4. Combinatory completeness

In this section we define combinatory algebras, then show that DCAs have a combinatory reduct and prove the combinatory completeness lemma for DCAs. We begin with the precise formalization of λ-terms and their basic properties that we deferred in the preceding sections.

Definition 4.1. We assume that there is a fixed, countably infinite set of context variables. Let I be an arbitrary set of λ-variables.

(i) The *λ-terms over I* are defined recursively as follows: every context variable and every λ-variable is a λ-term. It t and s are λ-terms, then so are $(t \cdot s)$ and $(\lambda x t)$ for every λ-variable x.

(ii) Let A be a LAA_I, let $C \subseteq A$, and let $\bar{c}$ be a new constant symbol for each $c \in C$. The *λ-polynomials over I and C* are defined recursively as follows: every context variable, every λ-variable, for and $\bar{c}$ for every $c \in C$ is a λ-polynomial. If t and s are λ-polynomials, then so are $(t \cdot s)$ and (λxt) for every λ-variable x.

By a *λ-expression* we mean either a λ-term or a λ-polynomial. The occurrence of a λ-variable x in a λ-expression is *bound* if it occurs within the scope of an occurrence of λx; otherwise it is *free*. A λ-variable is *bound*, respectively *free*, in a λ-expression if it has at least one bound, respectively free, occurrence. A λ-expression is *pure* if it contains no context variables; it is *closed* if it has no free λ-variables. A λ-term that is both pure and closed is called a *combinator*. The set of all λ-terms over I is denoted by Λ_I; the sets of all pure and of all closed λ-terms over I are denoted, respectively, by Λ_I^P and Λ_I^O. The set of all λ-polynomials over I and C is denoted by $\Lambda_I(C)$.

Pure λ-terms are essentially the same as terms of the lambda calculus (see [3, 2.1.1], and we employ the standard notational conventions of the lambda calculus in writing λ-expressions; see for instance [3, 2.1.3].[4] For example, we use x, y, and z (possibly with subscripts) as metavariables ranging over λ-variables. On the other hand, we use r, s, t, u, v to represent λ-expressions instead of the more common M, N, etc., and ξ, μ, ν represent context variables, which do not appear in terms of the lambda calculus.[5]

λ-expression are evaluated in a LAA_I under a given assignment of values to the context variables in the standard way. Let t be a λ-term and let $\xi_1, \ldots, \xi_n$ be context variables. We write t in the form $t(\xi_1, \ldots, \xi_n)$ to indicate that every context variable that occurs in t is included in that list $\xi_1, \ldots, \xi_n$ (but not every context variable in the list is required to occur in t).

Definition 4.2. Let A be a LAA and $a_1, \ldots, a_n \in A$. We write $t^A(a_1, \ldots, a_n)$ to denote the value t takes in A when $\xi_1, \ldots, \xi_n$ are interpreted, respectively, as $a_1, \ldots, a_n$. The formal recursive definition takes the following form (let $\boldsymbol{a} = a_1, \ldots a_n$): If t is a λ-variable x, then $t^A(\boldsymbol{a}) = x^A$. If t is a context variable ξ_i, then $t^A(\boldsymbol{a}) = a_i$. If t is a constant symbol $\bar{c}$ denoting some element c of A, then $t^A(\boldsymbol{a}) = c$. If $t = rs$, then $t^A(\boldsymbol{a}) = r^A(\boldsymbol{a}) \cdot^A s^A(\boldsymbol{a})$. Finally, if $t = \lambda x.s$, then $t^A(\boldsymbol{a}) = \lambda x^A.s^A(\boldsymbol{a})$.

In the next few results we investigate the connection between abstract substitution and term evaluation.

Lemma 4.3. *Let A be a LAA_I and let $t(\xi_1, \ldots, \xi_n)$ be a λ-expression with free variables $x_1, \ldots, x_m$ and constants $\bar{c}_1, \ldots, \bar{c}_k$ corresponding to elements $c_1, \ldots, c_k$ of A. Then, for all $a_1, \ldots, a_n \in A$,*

$$\Delta(t^A(a_1, \ldots, a_n)) \subseteq \{x_1, \ldots, x_m\} \cup \Delta(a_1) \cup \cdots \cup \Delta(a_n) \cup \Delta(c_1) \cup \cdots \cup \Delta(c_k).$$

---

[4] Note, however, that Λ_I means something different than it does in [3, 2.1.3].

[5] Compare the *contexts* defined in Barendregt [3, Definition 2.1.18]; our context variables correspond roughly to Barendregt's notion of a "hole".

In particular, if t is a pure λ-term, then $\Delta(t^A) \subseteq \{x_1, \ldots, x_n\}$ *and if t is a combinator, then* $\Delta(t^A) = \emptyset$.

Proof. By induction on the structure of terms with the aid of Lemma 1.7. □

Since a combinator evaluates to a zero-dimensional element of A, we will call any zero-dimensional element of a LAA a *combinator.*

We next show that a change of bound variables in a combinator does not affect its value in a LAA provided there are enough λ-variables available. Recall that for any set A, $A^\star$ will denote the set of all $a_1 \cdots a_n \in A^*$ without repetitions (i.e., $a_i \neq a_j$ if $i \neq j$). Let t and s be pure λ-terms and let $\boldsymbol{x} = x_1 \cdots x_n$, $\boldsymbol{y} = y_1 \cdots y_n \in I^\star$. s is said to be obtained from t *by changing variables from* $\boldsymbol{x}$ *to* $\boldsymbol{y}$ if s is obtained from t by replacing every free or bound occurrence of x_i by y_i and every occurrence of λx_i by λy_i. If there is some $\boldsymbol{x}$ and $\boldsymbol{y}$ for which this is true, we say that s is obtained from t *by a change of variables.*

Lemma 4.4. *Let t and s be pure λ-terms without variables in common, and let* $\boldsymbol{x}, \boldsymbol{y} \in I^\star$ *be disjoint. Assume that s is obtained from t by changing variables from* $\boldsymbol{x}$ *to* $\boldsymbol{y}$. *Then for every* LAA_I A, $s^A = S^{\boldsymbol{x}}_{\boldsymbol{y}}(t^A)$.

Proof. By induction on the structure of t. If $t = x_i$ then $s = y_i$ and the result is obvious. If $t = t_1 t_2$, then $s = s_1 s_2$ where s_i is obtained from t_i by changing variables from $\boldsymbol{x}$ to $\boldsymbol{y}$. Thus by (β_5) and the induction hypothesis, $s^A = s_1^A s_2^A = S^{\boldsymbol{x}}_{\boldsymbol{y}}(t_1^A) S^{\boldsymbol{x}}_{\boldsymbol{y}}(t_2^A) = S^{\boldsymbol{x}}_{\boldsymbol{y}}(t^A)$.

Assume now that $t = \lambda x_i . t'$. Then $s = \lambda y_i . s'$ where s' is obtained from t' by changing variables from $\boldsymbol{x}$ to $\boldsymbol{y}$. Let $\boldsymbol{x}_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$ and let $\boldsymbol{y}_i$ be similarly defined. Note that since t and s have no variables in common, t'^A is independent of all the variables of s. Then

$$
\begin{aligned}
s^A &= \lambda y_i^A . s'^A \\
&= \lambda y_i^A . S^{\boldsymbol{x}}_{\boldsymbol{y}}(t'^A) \quad \text{by the induction hypothesis} \\
&= \lambda y_i^A . S^{\boldsymbol{x}_i}_{\boldsymbol{y}_i} S^{x_i}_{y_i}(t'^A) \quad \text{by Lemmas 1.9(iii) and fact } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are disjoint} \\
&= S^{\boldsymbol{x}_i}_{\boldsymbol{y}_i}(\lambda y_i^A . S^{x_i}_{y_i}(t'^A)) \quad \text{by } (\beta_6) \\
&= S^{\boldsymbol{x}_i}_{\boldsymbol{y}_i}(\lambda x_i^A . t'^A)) \quad \text{by } (\alpha) \\
&= S^{\boldsymbol{x}_i}_{\boldsymbol{y}_i} S^{x_i}_{y_i}(\lambda x_i^A . t'^A)) \quad \text{by Proposition 1.7(ii)} \\
&= S^{\boldsymbol{x}}_{\boldsymbol{y}}(t^A) \quad \text{by Lemma 1.9(iii).} \quad \square
\end{aligned}
$$

Proposition 4.5. *Let t and s be combinators (i.e., pure, closed λ-terms), and assume s is obtained from t by change of variables.*

(i) *If t and s have no variables in common, then* $s^A = t^A$.

(ii) *If* $|I| \geqslant 3n$, *where n is the number of distinct variables of t (in particular, if I is infinite), then* $s^A = t^A$.

Proof. (i) Let s be obtained from t by changing the variables from $\boldsymbol{x}$ to $\boldsymbol{y}$. Then (i) is an immediate consequence of Lemma 4.4 and the fact $S_{\boldsymbol{y}}^{\boldsymbol{x}}(t^A)=t^A$ since t is closed.

(ii) If t and s have variables in common, then by hypothesis, we can find a term r having no variables in common with either t or s and such that s is obtained from r which in turn is obtained from t by a change of variables. Then by (i) $s^A=r^A=t^A$. $\square$

The proof of the combinatory completeness lemma for LAAs below relies heavily on the following lemma that further explores the connection between abstract substitution and term evaluation.

Lemma 4.6. *Let $t(\xi_1,\dots,\xi_n)$ be a λ-term whose context variables occur in the list $\xi_1,\dots,\xi_n$. Let $\boldsymbol{x}=x_1\cdots x_m$ be λ-variables that do not occur in t. Finally, let A be a* LAA_I *and let $\boldsymbol{a}=a_1\cdots a_m\in A^*$ such that each a_i is independent of all $y\in I$ such that λy occurs in t. Then*

$$S_{\boldsymbol{a}}^{\boldsymbol{x}}(t^A(b_1,\dots,b_n))=t^A(S_{\boldsymbol{a}}^{\boldsymbol{x}}(b_1),\dots,S_{\boldsymbol{a}}^{\boldsymbol{x}}(b_n)) \quad \textit{for all } b_1,\dots,b_n\in A.$$

Proof. The proof is an easy induction on the structure of t. $\square$

4.1. Combinatory algebras

Definition 4.7. Let $\boldsymbol{C}=\langle C,\cdot^C,\mathbf{k}^C,\mathbf{s}^C\rangle$ be an algebra where $\cdot^C$ is a binary operation and $\mathbf{k}^C,\mathbf{s}^C$ are constants. $\boldsymbol{C}$ is a *combinatory algebra* (CA) if it satisfies the following identities (as usual the symbol $\cdot$ and the superscript ^{C} are omitted, and association when in doubt, is to the left):

$$\mathbf{k}xy=x; \tag{4.1}$$

$$\mathbf{s}xyz=xz(yz). \tag{4.2}$$

k and **s** are called *combinators* (the connection with pure, closed λ-terms will become apparent later). The derived combinators **i** and **1** are defined as follows: $\mathbf{i}:=\mathbf{skk}$ and $\mathbf{1}:=\mathbf{s}(\mathbf{ki})$. Easy calculations show that the following identities hold in every combinatory algebra.

$$\mathbf{i}x=x \quad \text{and} \quad \mathbf{1}xy=xy. \tag{4.3}$$

In the equational logic of combinatory algebras it is traditional to let λ-variables play the role of context variables. We follow this convention in the next definition. Recall that x,y,z, possibly with subscripts, denote arbitrary distinct λ-variables.

Definition 4.8. Let I be an arbitrary set of λ-variables.

(i) The *combinatory terms over I* are defined recursively: $\mathbf{k},\mathbf{s}$, and every λ-variable is a combinatory term. If t and s are combinatory terms, so is $(t\cdot s)$.

(ii) Let C be a CA and let $\bar{c}$ be a new constant symbol for each $c \in C$. The *combinatory polynomials over I and C* are defined as follows: $\mathbf{k}, \mathbf{s}$, every λ-variable, and $\bar{c}$ for every $c \in C$ is a λ-polynomial. If t and s are λ-polynomials, then so is $(t \cdot s)$.

A *combinatory expression* is either a combinatory term or a combinatory polynomial. A combinatory expression is *closed* if it contains no variables. Closed combinatory terms are also called *combinators.* Note that context variables do not occur in combinatory terms, so every combinatory term is pure in the sense of λ-terms. For this reason we normal speak of *variables* instead of λ-variables in the context of combinatory algebras.

Let C be a combinatory algebra. If $t = t(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ includes all the variables occurring in t, and $v_1, \dots, v_n \in C$, then $t^C(v_1, \dots, v_n)$ will denote the value of t in C when x_i is interpreted as v_i and each new constant $\bar{u}$ as u. The set of all combinatory terms over I is denoted by $\mathscr{C}_I$, and the sets of combinatory polynomials over I and C is denoted by $\mathscr{C}_I(C)$. Where appropriate, we follow the same conventions writing combinatory terms as for λ-terms.

The following result is well known [22, 3, Theorem 5.1.10]; we reproduce its simple proof here for completeness.

Proposition 4.9 (Combinatory completeness lemma). *Let C be a combinatory algebra and let $t(x_1, \dots, x_n)$ be a combinatory polynomial over C whose variables all occur in the list $x_1, \dots, x_n$. Then there exists an element c in C such that, for all $v_1, \dots, v_n \in C$,*

$$t^C(v_1, \dots, v_n) = cv_1 \cdots v_n.$$

The combinatory completeness lemma can best be viewed in terms of the corresponding lemma for LAA$_I$s that we will prove below. Its proof depends on the following definition and lemma that shows that some aspects of lambda abstraction can be simulated in combinatory algebras.

Definition 4.10. Let C be a combinatory algebra. For each variable x the transformation $\lambda^*x : \mathscr{C}_I(C) \to \mathscr{C}_I(C)$ is defined by recursion on the structure of terms as follows: $\lambda^*x(x) = \mathbf{i}$. Let t be a combinatory term different from x. If x does not occur in t, define $\lambda^*x(t) = \mathbf{k}t$. Otherwise, t must be of the form sr where s and r are combinatory terms, at least one of which contains x; in this case define $\lambda^*x(t) = \mathbf{s}\lambda^*x(s)\lambda^*x(r)$. For any finite sequence $x_1, \dots, x_n$ of variables define

$$\lambda^*x_1 \cdots x_n(t) = \lambda^*x_1(\lambda^*x_2(\dots(\lambda^*x_n(t)\dots))).$$

Lemma 4.11. *Let C be a combinatory algebra, t a combinatory polynomial over C, and x a variable.*

(i) *x does not occur in $\lambda^*x(t)$. More precisely, the variables that occur in $\lambda^*(t)$ are exactly the variables except x that occur in t.*

(ii) *Let $y_1, \ldots, y_n$ be any list of variables that includes all variables occurring in t except x, and write $t = t(x, y_1, \ldots, y_n)$* and $\lambda^* x(t) = (\lambda^* x(t))(y_1, \ldots, y_n)$. *Then for all $v, u_1, \ldots, u_n \in C$,*

$$t^C(v, u_1, \ldots, u_n) = ((\lambda^* x(t))^C(u_1, \ldots, u_n))\, v.$$

The proofs of both (i) and (ii) are straightforward inductions on the structure of t; compare [3, Proposition 5.1.9].

The combinatory completeness lemma now follows easily. $\lambda^* x_1, \ldots, x_n(t)$ is closed by Lemma 4.11(i) (i.e., it contains no variables). Hence it defines a unique element $c = (\lambda^* x_1 \cdots x_n(t))^C$ of C. By repeated applications of Lemma 4.11(ii) we get $t^C(v_1, \ldots, v_n) = cv_1 \cdots v_n$.

4.2. Combinatory completeness for LAAs

The main result of this section is an analogue for LAAs of the combinatory completeness lemma for combinatory algebras. The combinatory term lemma and the combinatory completeness lemma (Proposition 4.18 and Theorem 4.19) play fundamental roles in the representation results in Sections 6 and 7.

Definition 4.12. Let A be a LAA_I. By the *combinatory reduct* of A we mean the algebra $\langle A, \cdot^A, \mathbf{k}^A, \mathbf{s}^A \rangle$ where $\mathbf{k}^A = (\lambda xy.xy)^A$ and $\mathbf{s}^A = (\lambda xyz.xz(yz))^A$.

The smallest subalgebra of $\langle A, \cdot^A, \mathbf{k}^A, \mathbf{s}^A \rangle$, i.e., the subalgebra generated by $\mathbf{k}^A$ and $\mathbf{s}^A$, is called the *minimal combinatory subreduct* of A.

The variables x, y, and z are assumed to be distinct. Note that by Proposition 4.5 the definitions of $\mathbf{k}^A$ and $\mathbf{s}^A$ are independent of the choice of x, y, and z, provided I is infinite, or more generally if $|I| \geqslant 9$. In the sequel we will assume this is always the case unless otherwise specified.

Here and in the sequel we omit the superscript on $\mathbf{k}^A$ and $\mathbf{s}^A$, and on other expressions, when no confusion seems likely.

Lemma 4.13. *Let A be a* LAA_I.

(i) $\mathbf{k}^A ab = a$ *for all $a, b \in A$ such that $\Delta^A b \neq I$.*

(ii) $\mathbf{s}^A abc = ac(bc)$ *for all $a, b, c \in A$ such that $|I - (\Delta^A b \cup \Delta^A c)| \geqslant 2$.*

(iii) $\mathbf{i}^A = (\lambda x.x)^A$.

(iv) $\mathbf{1}^A = (\lambda xy.xy)^A$.

Proof. (i) Assume that b is independent of x. Then using Proposition 1.10 we get $\mathbf{k}ab = (\lambda xy.x)ab = S_a^x S_b^y(x) = S_a^x(a) = a$. A similar argument gives (ii). The easy proofs of (iii) and (iv) are omitted. $\square$

Corollary 4.14. *The combinatory reduct of every* DCA_I *is a combinatory algebra.*

We will see in the next section that the combinatory reduct of every FLA_I and every RFA_I is also a combinatory algebra. Consequently, in all of these algebras, combinatory terms can be meaningfully evaluated in them. We can in fact define the evaluation of combinatory terms in an arbitrary LAA by means of its combinatory reduct.

Definition 4.15. Let A be a LAA_I and $t(x_1,\ldots,x_n)$ a combinatory term. Then we take $t^A = t^C(x_1^A,\ldots,x_n^A)$, where C is the combinatory reduct of A. More explicitly, if t is x_i, $\mathbf{k}$, or $\mathbf{s}$, then t^A is, respectively, x_i^A, $(\lambda xy.x)^A$, $(\lambda xyz.xz(yz))^A$; if $t = sr$, then $t^A = s^A \cdot^A r^A$.

Note that for the purposes of evaluating it in a LAA, a combinatory term is treated like a ground term, with each of its λ-variables interpreted as the value of the corresponding constant in A. This illustrates the different role that the λ-variables play in λ-terms and combinatory terms.

At the same time that some LAAs have a combinatory structure, many of the properties of λ-abstraction can be simulated in CAs. We now define a translation of pure λ-terms into combinatory terms that is useful in investigating the connection between LAAs and CAs more closely. Recall that Λ_I^P and $\mathscr{C}_I$ denote, respectively, the sets of pure λ-terms and combinatory terms over I. The translation from Λ_I^P to $\mathscr{C}_I$ makes use of the transformation λ^*x of $\mathscr{C}_I$, for each $x \in I$, given in Definition 4.10.

Definition 4.16. The mapping $t \mapsto t_{\mathrm{com}}$ from Λ_I^P into $\mathscr{C}_I$ is obtained by successively replacing each subterm of t of the form $\lambda x.s$ by $\lambda^*x(s)$, starting with the innermost λ-abstractions. More precisely, $t_{\mathrm{com}} = t$ if t is a λ-variable; $t_{\mathrm{com}} = \lambda^*x(s_{\mathrm{com}})$ if $t = \lambda x.s$; $t_{\mathrm{com}} = s_{\mathrm{com}} r_{\mathrm{com}}$ if $t = sr$.

This translation is well known from the lambda calculus (see [3, 7.1.5]) where it plays an important role. It plays an equally important role here; it is used to show how much of the structure of a DCA, a FLA, and a RFA can be reduced to that of its combinatory reduct. To fully understand how this is done, the relationship between pure λ-terms and combinatory terms must be clearly in mind. Both kinds of terms contain only λ-variables, but in λ-terms the λ-variables represent constant (i.e., nullary operation) symbols, while in combinatory terms they play the role of variables in more-or-less the usual sense. These however are semantical differences. Formally, the only difference between pure λ-terms and combinatory terms is that the former contain λ-abstractions but do not contain the combinators $\mathbf{k}$ and $\mathbf{s}$; in combinatory terms the reverse is true. When we are working exclusively in the context of pure λ-terms and combinatory terms we speak simply of *variables* instead of λ-variables.

λ^*x is a transformation of combinatory terms; it does not define directly a function on combinatory algebras. It can be used, however, to simulate some aspects of λ-abstraction in a sense made precise in the following lemma and proposition. Recall that for every LAA_I A and combinatory term t, the evaluation t^A in A is defined in Definition 4.15.

Lemma 4.17. *Let A be a LAA_I and let t be combinatory term. Then for any $x \in I$, $\lambda x^A . t^A = (\lambda^* x(t))^A$.*

Proof. By induction on the structure of t. If $t = x$,

$$(\lambda^* x(t))^A = \mathbf{i}^A = (\lambda x . x)^A = \lambda x A . t^A.$$

If x does not occur in t,

$$\begin{aligned}(\lambda^* x(t))^A &= (\mathbf{k} t)^A = (\lambda y x . y)^A t^A \\ &= S^y_{t^A}(\lambda x . y)^A \\ &= \lambda x^A . t^A \quad \text{since } x \notin \Delta^A t^A.\end{aligned}$$

Assume that $t = sr$ and at least one of s and r contains x. Let y and z be two λ-variables independent of $\lambda x^A . s^A$ and $\lambda x^A . r^A$,

$$\begin{aligned}(\lambda^* x(sr))^A &= \mathbf{s}^A(\lambda x^*(s))^A(\lambda^* x(r))^A \\ &= \mathbf{s}^A(\lambda x^A . s^A)(\lambda x^A . r^A) \\ &= (\lambda y z x . y x (z x))^A(\lambda x^A . s^A)(\lambda x^A . r^A) \\ &= S^y_{\lambda x^A . s^A} S^z_{\lambda x^A . r^A}(\lambda x . y x)(z x))^A \quad \text{by Proposition 1.10} \\ &= \lambda x^A . (\lambda x^A . s^A) x^A((\lambda x^A . r^A) x^A) \\ &\qquad \text{by } (\beta_6) \text{ since } x \notin \Delta(\lambda x^A . s^A) \cup \Delta(\lambda x A . r^A) \\ &= \lambda x^A . S^x_{x^A}(s^A) S^x_{x^A}(r^A). \\ &= \lambda x^A . s^A r^A \\ &= \lambda x^A . t^A. \quad \square\end{aligned}$$

The next proposition is the direct analogue of the combinatory term lemma for the lambda calculus. See [22].

Proposition 4.18 (Combinatory term lemma). *Let A be a LAA_I. Then for every pure λ-term t we have $t^A = t^A_{\mathrm{com}}$.* If t is closed, so is t_{com}.

Proof. A straightforward induction on the structure of t; apply Lemma 4.17 for t of the form $\lambda x . s$. By Lemma 4.11(i), the variables of t_{com} are exactly the free variables of t. So t_{com} is closed if t is closed. $\square$

Theorem 4.19 (Combinatory completeness lemma for LAAs). *Let $t(\xi_1, \ldots, \xi_n)$ be a λ-term whose context variables occur in the list $\xi_1, \ldots, \xi_n$. There exists a combinatory term $\tilde{t}$ with the following property: for every* DCA_I *A, and for all $a_1, \ldots, a_n \in A$ that are independent of each $y \in I$ such that λy occurs in t, we have*

$$t^A(a_1, \ldots, a_n) = \tilde{t}^A a_1 \cdots a_n.$$

If in addition t is closed, then $\tilde{t}$ can also be taken to be closed.

Proof. Let $x_1, \ldots, x_n$ be λ-variables that do not occur in t and that are independent of all the a_i. Then

$$\begin{aligned} t^A(a_1, \ldots, a_n) &= t^A(S_a^x(x_1^A), \ldots, S_a^x(x_n^A)) \\ &= S_a^x(t^A(x_1^A, \ldots, x_n^A)) \quad \text{by Lemma 4.6} \\ &= (\lambda \boldsymbol{x}.t(\boldsymbol{x}))^A a_1 a_2 \cdots a_n \quad \text{by Proposition 1.10.} \end{aligned}$$

$\lambda \boldsymbol{x}.t(\boldsymbol{x})$ is a pure λ-term. Set $\tilde{t} = (\lambda \boldsymbol{x}.t(\boldsymbol{x}))_{\mathrm{com}}$. Then by the combinatory term lemma, $\tilde{t}$ is a combinatory term such that $(\lambda \boldsymbol{x}.t(\boldsymbol{x}))^A = \tilde{t}^A$. So

$$t^A(a_1, \ldots, a_n) = \tilde{t}^A a_1 \cdots a_n,$$

By Proposition 4.18, $\tilde{t}$ is closed if t is closed. $\square$

The following technical corollary will be used in Section 6.

Corollary 4.20. *Let A be a* DCA_I *and let M be its minimal combinatory subreduct.*

(i) *Let $x_1, \ldots, x_n, y_1, \ldots, y_m \in I$. There exists a $k \in M$ such that, for all $a, b \in A$,*

$$(a x_1^A \cdots x_n^A)(b y_1^A \cdots y_n^A) = k a b x_1^A \cdots x_n^A y_1^A \cdots y_m^A.$$

(ii) *Let $x_1, \ldots, x_n \in I$. There exists an $l \in M$ such that, for all $a, b \in A$,*

$$(a x_1^A \cdots x_n^A)(b x_1^A \cdots x_n^A) = l a b x_1^A \cdots x_n^A.$$

5. Environment models

Only recently has a general consensus developed as to what the models of the lambda calculus should be. (A brief but illuminating history of the process can be found in [22].) The notion of an *environment model* (the name is due to Meyer [22]) originated with Hindley and Longo [19]. Meyer describes them as "the natural, most general formulation of what might be meant by mathematical models of the untyped lambda calculus". We give a definition of environment model in terms of LAAs that is equivalent to the one given in [22]. The connection between LAAs and models of the lambda calculus is studied in detailed in [28].

Definition 5.1. An *environment model* is a functional domain V with the property that there exists at least one FLA_I with value domain V and infinite dimension set I, i.e., the I-coordinatization $\mathbf{V}_I$ of V has at least one total algebra.

Let V be an arbitrary functional domain. Any subalgebra of $\mathbf{V}_I$ must contain the projection function x^{V_I} for each $x \in I$. Hence the intersection of subalgebras is nonempty and hence a subalgebra.

Definition 5.2. For any functional domain V, the intersection of all subalgebras of $\mathbf{V}_I$ is denoted by $\mathbf{V}_I^{\perp}$.

The universe of $\mathbf{V}_I^{\perp}$ is denoted by $V_I^{\perp}$. It is easy to see that $V_I^{\perp} = \{t^{V_I}: t \in \Lambda_I^P\}$. (Recall that Λ_I^P is the set of all pure λ-terms over I.) Clearly, V is an environment model iff $\mathbf{V}_I^{\perp}$ is total subalgebra of $\mathbf{V}_I$ and hence a LAA_I.

Analogously, let $\mathbf{V}_I^{\perp}(V)$ be the intersection of all subalgebras of $\mathbf{V}_I$ that includes the constant function $\bar{v}^{V_I} = \langle v\colon p \in V^I \rangle$, for every $v \in V$. It is also a subalgebra of $\mathbf{V}_I$; its universe $V_I^{\perp}(V) = \{t^{V_I}: t \in \Lambda_I(V)\}$. Meyer [22] defines a functional domain V to be an environment model if $\mathbf{V}_I(V)$ is a total subalgebra, i.e., a LAA_I. We shall see below that $\mathbf{V}_I^{\perp}(V)$ is total iff $\mathbf{V}_I^{\perp}$ is total. Hence our notion of environment model is the same as Meyer's.

In our view the most natural models of the lambda calculus are FLAs (and RFAs), which correspond via coordinatization exactly to environment models. This highlights the main difference between our approach and the traditional one to models of the lambda calculus: the latter focuces attention on functional domains while we focus on their coordinatization.

Lemma 5.3. *The combinatory reduct of every* FLA_I *and* RFA_I *is a combinatory algebra.*

Proof. We only outline the proof for FLA_Is; the proof for RFA_Is is similar. Let $\mathbf{A}$ be a FLA_I and let V be its value domain. For every $p \in V^I$ we have $(\lambda xy.x)^{\mathbf{A}}(p) = \lambda^V \langle \lambda^V \langle v\colon u \in V \rangle\colon v \in V \rangle$. Thus, by the defining condition of a functional domain, (3.1), we have for all $a, b \in V^I$ and each $p \in V^I$ $(\mathbf{k}^V ab)(p) = a(p)$, and hence $\mathbf{k}^V ab = a$. A similar calculation works for $\mathbf{s}^V$. $\square$

Let V be an environment model and let $\mathbf{A} = \mathbf{V}_I^{\perp}$. For every combinator (i.e., pure, closed λ-term) t we have $\Delta t^{\mathbf{A}} = \emptyset$ by Lemma 4.3. Thus, since $\mathbf{A}$ is minimal, we can apply Proposition 3.6 to conclude that $t^{\mathbf{A}}$ is a constant function from V^I to V. Define $\mathbf{k}^V$, $\mathbf{s}^V$, $\mathbf{i}^V$, and $\mathbf{1}^V$ to be the constant values in V of the combinators $\mathbf{k}^{\mathbf{A}}$, $\mathbf{s}^{\mathbf{A}}$, $\mathbf{i}^{\mathbf{A}}$, and $\mathbf{1}^{\mathbf{A}}$, respectively.

Corollary 5.4. *Let V be an environment model and let $\mathbf{k}^V$ and $\mathbf{s}^V$ be the elements of V defined above. Then $\langle V, \cdot^V, \mathbf{k}^V, \mathbf{s}^V \rangle$ is a combinatory algebra.*

The next lemma characterizes the lambda abstraction operation of an environment model V in terms of the combinatory reduct of $V_I^{\perp}$.

Lemma 5.5. *Let V be an environment model. Then* $\mathrm{dom}(\lambda^V)=\{\langle uv\colon v\in V\rangle\colon u\in V\}$, *and*

$$\lambda^V\langle uv\colon v\in V\rangle=\mathbf{1}^V u.$$

Proof. Let $A=V_I^{\perp}$. Consider an arbitrary function of the form $\langle uv\colon v\in V\rangle$, and choose any $p\in V^I$ such that $p_x=u$. Then $(\lambda y.xy)^A(p)$ is defined and

$$\begin{aligned}\mathbf{1}^V u=\mathbf{1}^A(p)\cdot^V x^A(p)=(\lambda xy.xy)^A(p)\cdot^V u\\ &=(\lambda x^A.(\lambda y.xy)^A)(p)\cdot^V u\\ &=\lambda^V\langle(\lambda y.xy)^A(p(v/x))\colon v\in V\rangle\cdot^V u\\ &=(\lambda y.xy)^A(p(u/x))\\ &=\lambda^V\langle(xy)^A(p(u/x)(v/y))\colon v\in V\rangle\\ &=\lambda^V\langle uv\colon v\in V\rangle.\end{aligned}$$

Thus $\lambda^V\langle uv\colon v\in V\rangle$ is defined and equal to $\mathbf{1}^V u$. Conversely, let f be in the domain of λ^V. Then $f(v)=(\lambda^V f)v$ by the defining condition of a functional domain, and hence $f=\langle(\lambda^V f)v\colon v\in V\rangle$. □

This lemma shows that every environment model satisfies the Meyer–Scott axiom [3, 5, 27]; cf. [22, p. 100].

Proposition 5.6. *Let V be an environment model and let V_I be its I-coordinatization. Let U be the set of all functions $a\in V_I$ such that*

for every $p\in V^I$ and $\boldsymbol{x}=x_1\cdots x_n\in I^\star$, there exists an element $u_{a,p,\boldsymbol{x}}\in V$ such that, for all $v_1\cdots v_n\in V^*$,

$$a(p(v_1/x_1,\ldots,v_n/x_n))=u_{a,p,\boldsymbol{x}}v_1\cdots v_n. \tag{5.1}$$

Then U is the universe of a total subalgebra of the I-coordinatization V_I of V that contains the constant function $\bar{v}^{V_I}$ for each $v\in V$. Moreover, U is the largest total subuniverse of V_I in the sense that a total function a is contained in some total subalgebra of V_I iff $a\in U$.

Proof. U is clearly a set of total functions. That it is a subuniverse of V_I is a straightforward consequence of Corollary 5.4 and the combinatory completeness lemma (Proposition 4.9), as we now show. We also denote the combinatory algebra $\langle V,\cdot^V,\mathbf{k}^V,\mathbf{s}^V\rangle$ by V.

Consider any $y\in I$. We verify y^{V_I} has property (5.1). Let $p\in V^I$ and $\boldsymbol{x}=x_1\cdots x_n\in I^\star$. For all $\boldsymbol{v}=v_1\cdots v_n\in V^*$, we write $p(\boldsymbol{v}/\boldsymbol{x})$ as shorthand for $p(v_1/x_1,\ldots,v_n/x_n)$. Take the combinatory polynomial t in Proposition 4.9 to be x_i if $y=x_i$ for some $i=1,\dots,n$; if $y\neq x_i$ for all i, take t to be the constant $\bar{p}_y$. Let $u_{y,p,x}$ be the element c of V associated with t in Proposition 4.9. Then for all $v_1,\dots,v_n\in V$,

$$y^{V_I}(p(\boldsymbol{v}/\boldsymbol{x}))=t^V(v_1,\dots,v_n)=u_{y,p,x}v_1\cdots v_n.$$

Thus $y^{V_I}\in U$ for all $y\in I$. In a similar way we can show that every constant function $\bar{v}^{V_I}=\langle v\colon p\in V^I\rangle$ has the property (5.1): take the combinatory polynomial t in Proposition 4.9 to be $\bar{v}$ for every $p\in V^I$ and $\boldsymbol{x}\in I^\star$.

Suppose $a,b\in U$. Then

$$(a\cdot^{V_I}b)(p(\boldsymbol{v}/\boldsymbol{x}))=a(p(\boldsymbol{v}/\boldsymbol{x}))\cdot^V b(p(\boldsymbol{v}/\boldsymbol{x}))=(u_{a,p,x}\boldsymbol{v})\cdot^V(u_{b,p,x}\boldsymbol{v}).$$

Let t be the combinatory polynomial $(\overline{u_{a,p,x}}\,\boldsymbol{x})(\overline{u_{b,p,x}}\,\boldsymbol{x})$. By the combinatory completeness lemma there is a $c\in V$ such that, for all $v_1,\dots,v_n\in V$,

$$(u_{a,p,x}v_1\cdots v_n)(u_{b,p,x}v_1\cdots v_n)=cv_1\cdots v_n.$$

Take $u_{a\cdot^{V_I}b,p,x}=c$.

Suppose $a\in U$ and $y\in I$. If $y\neq x_1,\dots,x_n$, let $\boldsymbol{x}y=x_1\cdots x_n y$ and $\boldsymbol{v}w=v_1\cdots v_n w$ for every $w\in V$.

$$\begin{aligned}(\lambda y^{V_I}.a)(p(\boldsymbol{v}/\boldsymbol{x}))&=\lambda^V\langle a(p(\boldsymbol{v}/\boldsymbol{x})(w/y))\colon w\in V\rangle\\&=\lambda^V\langle a(p(\boldsymbol{v}w/\boldsymbol{x}y))\colon w\in V\rangle\\&=\lambda^V\langle u_{a,p,xy}\,\boldsymbol{v}w\colon w\in V\rangle\quad\text{by assumption } a\in U\\&=\lambda^V\langle (u_{a,p,xy}\,\boldsymbol{v})w\colon w\in V\rangle\\&=\mathbf{1}^V(u_{a,p,xy}\,\boldsymbol{v})\quad\text{by Lemma 5.5.}\end{aligned}$$

By the combinatory completeness lemma there is a $c\in V$ such that

$$\mathbf{1}^V(u_{a,p,xy}v_1\cdots v_n)=cv_1\cdots v_n\quad\text{for all } v_1\cdots v_n\in V^n.$$

Take $u_{(\lambda y^{V_I}.a),p,x}=c$.

Finally, assume $y=x_i$ for some $i=1,\dots,n$.

$$\begin{aligned}(\lambda y^{V_I}.a)(p(\boldsymbol{v}/\boldsymbol{x}))&=\lambda^V\langle a(p(\boldsymbol{v}/\boldsymbol{x})(w/x_i))\colon w\in V\rangle\\&=\lambda^V\langle u_{a,p,x}v_1\cdots v_{i-1}wv_{i+1}\cdots v_n\colon w\in V\rangle.\end{aligned}$$

Let $t(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,x_i)$ be the combinatory polynomial $\bar{u}_{a,p,x}x_1\cdots x_n$. By the combinatory completeness lemma there is a $c\in V$ such that

$$u_{a,p,x}v_1\cdots v_{i-1}wv_{i+1}\cdots v_n=cv_1\cdots v_{i-1}v_{i+1}\cdots v_nw\quad\text{for all } v_1,\dots,v_n,\ w\in V.$$

Thus continuing the above string of equalities we have

$$\begin{aligned}(\lambda y^{V_I}.a)(p(\boldsymbol{v}/\boldsymbol{x}))&=\lambda^V\langle (cv_1\cdots v_{i-1}v_{i+1}\cdots v_n)w\colon w\in V\rangle\\&=\mathbf{1}^V(cv_1\cdots v_{i-1}v_{i+1}\cdots v_n),\end{aligned}$$

which, by another application of the combinatory completeness lemma,

$$=dv_1 \cdots v_n \quad \text{for some } d \in V.$$

Take $u_{(\lambda y^{V_I}.a),p,x} = d$.

This completes the proof that U is a total subuniverse of V_I. To show that it is the largest such universe it suffices to show that, if a is contained in any total subuniverse of V_I, then a satisfies (5.1). In particular we show that

$$u_{a,p,x} = (\lambda x_1 \cdots x_n{}^{V_I}.a)(p) \quad \text{for all } a \in V_I, \quad p \in V^I \text{ and } x \in I^\star.$$

Note first that $\Delta(\lambda x_1 \cdots x_n{}^{V_I}.a) \cap \{x_1, \ldots, x_n\} = \emptyset$ by Proposition 1.7(ii). Thus by Proposition 3.6, $\lambda x_1 \cdots x_n{}^{V_I}.a$ is functionally independent of $x_1, \ldots, x_n$. So

$$(\lambda x_1 \cdots x_n{}^{V_I}.a)(p) = (\lambda x_1 \cdots x_n{}^{V_I}.a)(p(v_1/x_1, \ldots, v_n/x_n)) \quad \text{for all } p \in V^I.$$

We now calculate: for all $\boldsymbol{v} = v_1 \cdots v_n \in V^*$,

$$\begin{aligned}
&((\lambda x_1 \cdots x_n{}^{V_I}.a)(p))v_1 \cdots v_n \\
&\quad = (\lambda x_1 \cdots x_n{}^{V_I}.a)(p(\boldsymbol{v}/\boldsymbol{x}))x_1^{V_I}(p(\boldsymbol{v}/\boldsymbol{x})) \ldots x_n^{V_I}(p(\boldsymbol{v}/\boldsymbol{x})) \\
&\quad = ((\lambda x_1 \cdots x_n{}^{V_I}.a)(x_1 \cdots x_n)^{V_I})(p(\boldsymbol{v}/\boldsymbol{x})) \\
&\quad = S^{x_1 \cdots x_n}_{x_1 \cdots x_n}(a)(p(\boldsymbol{v}/\boldsymbol{x})) \quad \text{by Proposition 1.10.} \\
&\quad = a(p(\boldsymbol{v}/\boldsymbol{x})). \quad \square
\end{aligned}$$

5.1. *Full functional* LAAs

An immediate consequence of Proposition 5.6 is that, if V is an environment model, then V_I must have a largest total subalgebra.

Definition 5.7. Let V be an environment model and V_I its I-coordinatization. We denote the largest total subuniverse of V_I (the set of elements of V_I satisfying (5.1)) by $V_I^\top$ and the corresponding largest total subalgebra by $V_I^\top$. FLA$_I$s of the form $V_I^\top$ for some environment model V_I are called *full* FLA$_I$s.

Every FLA$_I$ over V is subalgebra of $V_I^\top$. Moreover, $V_I^\top$ contains all constant functions. Thus $V_I^\top$ includes $V_I^\perp(V)$ and hence the latter is total. This verifies that our definition of environment model coincides with Meyer's [22].

We reformulate the basic properties of full FLA$_I$s as a theorem.

Theorem 5.8. *Let V be an environment model. Then V_I has a largest total subalgebra $V_I^\top$. For any $a: V^I \to V$ we have $a \in V_I^\top$ iff there exists a $(V^I \times I^\star)$-indexed system u_a of elements of V such that, for all $p \in V^I$, $\boldsymbol{x} = x_1 \cdots x_n \in I^\star$, and $\boldsymbol{v} = v_1 \cdots v_n \in V^*$*

$$a(p(v_1/x_1, \ldots, v_n/x_n)) = u_{a,p,x}\, v_1 \cdots v_n.$$

Theorem 5.8 can be useful for determining various properties of $V_I^{\top}$. For example, we use it to show that every full FLA_I fails to be dimension-complemented. This result is a corollary of the following interesting property of full FLA_Is.

Recall that the elements of $V_I^{\top}$ are total functions with domain V^I. Let $\equiv_\omega$ be the equivalence relation on V^I defined by $p \equiv_\omega q$ iff $|\{x \in I: p_x \neq q_x\}| < \omega$. The equivalence class of $r \in V^I$ is just the set V_r^I considered previously in connection with the definition of RFA_Is. Note that $p \in V_r^I$ iff p and r differ at only finitely many λ-variables. One can always find an element b of $V_I^{\top}$ with the property that, for every equivalence class S of $\equiv_\omega$, the restriction of b to S agrees with the restriction of a to S, where a is some arbitrarily chosen element of $V_I^{\top}$ that depends only on S. This property is formalized in the following proposition.

Proposition 5.9. *Let V be an environment model. Let $a = \langle a_r : r \in V^I \rangle$ be any V^I-indexed system of elements of $V_I^{\top}$ such that $r \equiv_\omega s$ implies $a_r = a_s$. Then there exists a $b \in V_I^{\top}$ such that $b(p) = a_p(p)$, for all $p \in V^I$.*

Proof. For all $p \in V^I$, $\boldsymbol{x} = x_1 \cdots x_n \in I^\star$, and $\boldsymbol{v} = v_1 \cdots v_n \in V^\star$ we have

$$\begin{aligned} b(p(v_1/x_1, \ldots, v_n/x_n)) &= a_{p(\boldsymbol{v}/\boldsymbol{x})}(p(v_1/x_1, \ldots, v_n/x_n)) \quad \text{by definition of } b \\ &= a_p(p(v_1/x_1, \ldots, v_n/x_n)) \quad \text{since } p \equiv_\omega p(v_1/x_1, \ldots, v_n/x_n) \\ &= u_{a_p, p, \boldsymbol{x}}\, v_1 \cdots v_n \quad \text{by Theorem 5.8 and the premiss } a_p \in V_I^{\top}. \end{aligned}$$

Taking $u_{b,p,\boldsymbol{x}} = u_{a_p,p,\boldsymbol{x}}$ for all $p \in V^I$ and $x_1 \cdots x_n \in I^*$, we conclude by Theorem 5.8 that $b \in V_I^{\top}$. □

Corollary 5.10. *Every full* FLA_I *fails to be dimension-complemented.*

Proof. Since every finite dimensional LAA_I fails to dimension-complemented by definition, we assume without loss of generality that I is infinite. Let $q: I \to V^I$ be a mapping such that $q(x) \not\equiv_\omega q(y)$ if $x \neq y$. Such a function exists since $\equiv_\omega$ clearly has more that $|I|$ equivalence classes. Let $c \in V_I^{\top}$ be fixed but arbitrary, and define for all $p \in V^I$

$$a_p := \begin{cases} x^{V_I} & \text{if } p \equiv_\omega q(x) \\ c & \text{otherwise.} \end{cases}$$

Let b be as in the proposition. Then $b(q(x)(v/x)) = v$ for each $x \in I$. So b functionally depends on every $x \in I$ and hence $\Delta b = I$ by Proposition 3.6. □

The natural analogue of Theorem 5.8 for RFAs holds and has essentially the same proof.

Theorem 5.11. *Let V be an environment model and $r \in V^I$. Then $V_{I,r}$ has a largest total subalgebra $V_{I,r}^{\top}$. For any $a: V_r^I \to V$, we have $a \in V_{I,r}^{\top}$ iff there exists an $I^\star$-indexed system*

w_a *of elements of* V *such that, for all* $\boldsymbol{x}=x_1\cdots x_n$ *and* $\boldsymbol{v}=v_1\cdots v_n\in V^*$, $a(r(v_1/x_1,\ldots,v_n/x_n))=w_{a,\boldsymbol{x}}v_1\cdots v_n$. $\square$

Proposition 5.9 can be reformulated in a more natural form.

Proposition 5.12. *Let* V *be an environment model. Let* R *be a set of unique representatives of the equivalence classes of* $\equiv_\omega$, *and for each* $p\in V^I$, *let* $\bar{p}$ *be the unique element of* R *such that* $p\equiv_\omega\bar{p}$. *Let* $a\in\prod_{r\in R}\boldsymbol{V}_{I,r}^{\top}$, *i.e., a function that assigns an element of* $\boldsymbol{V}_{I,r}^{\top}$ *to each* $r\in R$. *Then there exists a* $b\in\boldsymbol{V}_I^{\top}$ such that

$$b(p)=a_{\bar{p}}(p) \quad \textit{for all } p\in V^I.$$

It follows from this proposition that, given an R-indexed system $\boldsymbol{B}$ of RFA_Is such that $\boldsymbol{B}_r$ is a subalgebra of $\boldsymbol{V}_{I,r}^{\top}$ for each $r\in R$, then the Cartesian product $\prod\boldsymbol{B}_r$ is isomorphic to a FLA_I, in fact to a subalgebra of $\boldsymbol{V}_I^{\top}$. Taking $\boldsymbol{B}_r=\boldsymbol{V}_{I,r}^{\top}$, we have

$$\boldsymbol{V}_I^{\top}\cong\prod_{r\in R}\boldsymbol{V}_{I,r}^{\top}.$$

This gives the following proposition.

Proposition 5.13. *Every* RFA_I *is a homomorphic image of a* FLA_I *with the same value domain.*

6. Neat reducts and dilations

We consider a more general notion of reduct of a LAA_I $\boldsymbol{A}$ in which the λ-abstraction operations $\lambda x^{\boldsymbol{A}}$ are discarded for only some of the variables of I. The process corresponds exactly to that of forming the *compression* of a polyadic algebra [16, p. 137] and the *neat reduct* of a cylindric algebra [17, Part I, p. 401]; we shall appropriate the latter terminology. The theory of neat reducts of LAA_Is proves to more regular than that of cylindric algebras, mainly because of the combinatory completeness lemma for LAAs. For example, it turns out that the class of all neat J-reducts of LAA_Is forms a variety for certain $J\subseteq I$; for cylindric algebras it is the subalgebras of neat reducts that form a variety. In the main result of the section we show that a LAA_I is a neat reduct of an LAA_{I^+}, with $|I^+\setminus I|\geqslant\omega$, iff it is isomorphic to a RFA_I. Thus the class of RFA_Is and their isomorphic images forms a variety. This should be compared with the fact that the class of α-dimensional cylindric algebras that can be neatly embedded in some $(\alpha+\omega)$-dimensional cylindric algebra coincides (up to isomorphism) with the class of generalized cylindric set algebras of dimension α (i.e., the representable α-dimensional, cylindric algebras); see [17, Part II, Theorem 3.2.10]. In view of this result it is tempting to define the representable LAA_Is to be RFA_Is (and their isomorphic images), rather than the FLA_Is. We do not know if the class of RFA_Is coincides with FLA_I. We do show, however, that they generate the same variety.

Let A be a LAA_I and $J \subseteq I$. By the *J-reduct* of A we mean the algebra

$$\langle A, \cdot^A, \lambda x^A, x^A \rangle_{x \in J}.$$

Clearly this is a LAA_J. Define $\mathrm{Nr}_J A = \{a \in A\colon \Delta a \subseteq J\}$. By Proposition 1.7 this set is closed under the operations of the J-reduct of A and forms the universe of a subalgebra.

Definition 6.1. Let A be an LAA_I and $J \subseteq I$. By the *J-neat reduct* of A we mean the algebra

$$\mathbf{Nr}_J A := \langle \mathrm{Nr}_J A, \cdot^{\mathrm{Nr}_J A}, \lambda x^{\mathrm{Nr}_J A}, x^{\mathrm{Nr}_J A} \rangle_{x \in J},$$

whose operations are the corresponding operations of A restricted to $\mathrm{Nr}_J A$. For a class K of LAA_Is and $J \subseteq I$ we define $\mathrm{Nr}_J \mathrm{K} := \{\mathbf{Nr}_J A\colon A \in \mathrm{K}\}$.

$\mathbf{Nr}_J A$ is obviously a LAA_J. If $|J| \geqslant 3$, then the combinators $\mathbf{k}^A$ and $\mathbf{s}^A$ are term definable in $\mathbf{Nr}_J A$. It is convenient to assume that $\mathbf{k}^A$ and $\mathbf{s}^A$ are term-definable in $\mathbf{Nr}_J A$ for all $J \subseteq I$. Thus, if $|J| \leqslant 2$, we will assume $\mathbf{k}^A$ and $\mathbf{s}^A$ have been adjoined to $\mathbf{Nr}_J A$ as nullary operations. The case $J = \emptyset$ is especially important, we introduce a special term and notation for it.

Definition 6.2. Let $A \in \mathrm{LAA}_I$. The *zero-dimensional subreduct* of A is the algebra

$$\mathbf{Zd}\, A = \langle \mathrm{Zd}\, A, \cdot^A, \mathbf{k}^A, \mathbf{s}^A \rangle,$$

where $\mathrm{Zd}\, A = \{a \in A\colon \Delta^A a = \emptyset\}$, the set of zero-dimensional elements of A.

$\mathbf{Zd}\, A$ is a subalgebra of the combinatory reduct of A and is always a combinatory algebra, even if the full combinatory reduct of A is not. Note that by our convention, $\mathbf{Nr}_\emptyset A = \mathbf{Zd}\, A$.

Theorem 6.3. *Assume* $J \subseteq I$. *Then* $\mathrm{Nr}_J \mathrm{LAA}_I$ *is closed under formation of direct products, i.e.,* $\mathbb{P}\mathrm{Nr}_J \mathrm{LAA}_I = \mathrm{Nr}_J \mathrm{LAA}_I$.

Proof. Let $B_k \in \mathrm{Nr}_J \mathrm{LAA}_J$ for each $k \in K$. Let $B_k = \mathbf{Nr}_J A_k$ for $A_k \in \mathrm{LAA}_I$. Let $b = \langle b_k\colon k \in K \rangle \in \prod_k A_k$. Then $\Delta^{\Pi_k A_k} b = \bigcup_k \Delta^{A_k} b_k$. So $\Delta^{\Pi_k A_k} b \subseteq J$ iff $\Delta^{A_k} b_k \subseteq J$ for all $k \in K$ iff $b \in \prod_k \mathrm{Nr}_J A_k$. Thus

$$\prod_k B_k = \prod_k \mathbf{Nr}_J A_k = \mathbf{Nr}_J \left(\prod_k A_k \right). \qquad \square$$

Definition 6.4. A LAA_I A is said to be *locally finite relative* to a set $K \subseteq I$ if $|\Delta a \cap K| < \omega$ for all $a \in A$.

Thus A is locally finite-dimensional (in the absolute sense) iff it is locally finite relative to I. The set $\{a\in A: |\Delta a \cap K| < \omega\}$ is a universe of a subalgebra of A, so every LAA_I has a largest subalgebra locally finite relative to any fixed $K \subseteq I$. Note that if A is locally finite relative to some infinite set of λ-variables, then A is dimension-complemented.

The following technical lemma is a corollary of the combinatory completeness lemma for LAA_Is; it complements Corollary 4.20.

Lemma 6.5. *Let A be a LAA_I and $J \subseteq I$ such that $|I\setminus J| \geqslant \omega$. Assume A is locally finite relative to $I\setminus J$. Let M be the minimal combinatory subreduct of A. Let $x_1, \ldots, x_n \in I\setminus J$. Finally, let y be any λ-variable in I.*

(i) *If $y\in I\setminus J$, then there exists a $p\in M$ such that, for all $a\in \mathrm{Nr}_J A$,*

$$\lambda y^A . a x_1^A \cdots x_n^A = p a x_1^A \cdots x_n^A .$$

(ii) *If $y\in J$, then there exists a $q\in M$ such that, for all $a\in \mathrm{Nr}_J A$,*

$$\lambda y^A . a x_1^A \cdots x_n^A = q(\lambda y^A . a) x_1^A \cdots x_n^A .$$

Proof. We omit the superscript ^{A} on x_i^A and λy^A.

(i) We consider two cases. Assume first of all that $y \neq x_1, \ldots, x_n$. Set $t(\xi, \mu_1, \ldots, \mu_n) = \lambda y . \xi \mu_1 \cdots \mu_n$. Since $a, x_1 \cdots x_n$ are independent of y, the only bound λ-variable of t, the hypothesis of the combinatory completeness lemma for LAA's holds and we can conclude that there is a $p\in M$ such that

$$\lambda y . a x_1 \cdots x_n = t^A(a, x_1, \ldots, x_n) = p a x_1 \cdots x_n .$$

Suppose now that $y = x_i$ for some i. Choose any $z \in I\setminus(J\cup\{x_1, \ldots, x_n\})$ and let $t(\xi, \mu_1, \ldots, \mu_n) = \lambda z . \xi \mu_1 \cdots \mu_{i-1} z \mu_{i+1} \cdots \mu_n$. Note that again $a, x_1, \ldots, x_n$ are independent of the only bound λ-variable of t. So there is a $p\in M$ such that

$$\begin{aligned} \lambda y . a x_1 \cdots x_n &= \lambda z . S^y_z(a x_1 \cdots x_n) \quad \text{by}(\alpha) \\ &= \lambda z . a x_1 \cdots x_{i-1} z x_{i+1} \cdots x_n \\ &= t^A(a, x_1, \ldots, x_n) \\ &= p a x_1 \cdots x_n . \end{aligned}$$

(ii) Let $z \in I\setminus(J\cup\{x_1, \ldots, x_n\})$.

$$\begin{aligned} \lambda y . a x_1 \cdots x_n &= \lambda z . S^y_z(a x_1 \cdots x_n) \quad \text{by } (\alpha) \\ &= \lambda z . S^y_z(a) S^y_z(x_1) \cdots S^y_z(x_n) \\ &= \lambda z . (\lambda y . a) z x_1 \cdots x_n ; \end{aligned}$$

the last equality holds since z is distinct from $x_1, \ldots, x_n$. Take $t(\xi, \mu_1, \ldots, \mu_n) = \lambda z . \xi z \mu_1 \cdots \mu_n$ and note that now $\lambda y . a, x_1, \ldots, x_n$ are all independent of the sole bound variable of t. Now proceed as before. $\square$

The following lemma is the key to a large part of the theory of neat reducts. It uses the combinatory completeness lemma (more specifically, Corollary 4.20 and Lemma 6.5) to give a simple and very useful characterization of the subalgebra generated by a neat subreduct.

Lemma 6.6. *Let $A \in \mathrm{LAA}_I$ and $J \subseteq I$ with $|I \setminus J| \geqslant \omega$. Let $\boldsymbol{B}$ be a subalgebra of $\mathbf{Nr}_J A$ and let*

$$B^+ = \{bx_1^A \cdots x_n^A : b \in B,\ x_1 \cdots x_n \in (I \setminus L)^*\}$$

Then B^+ includes B and is closed under the operations of A; hence $\boldsymbol{B}^+$ is the subalgebra of A generated by B.

Proof. $\mathbf{Nr}_J A$ is included in the largest subalgebra of A that is locally finite relative to $I \setminus J$. Hence we can replace A by this subalgebra without loss of generality. Consequently we assume A itself is locally finite relative to $I \setminus J$. In particular, A is dimension-complemented.

Suppose $ax_1 \cdots x_n,\ by_1 \cdots y_m \in B^+$. By the corollary of the combinatory completeness lemma for LAAs there is an l in the minimal combinatory subreduct of A such that

$$(ax_1 \cdots x_n)(by_1 \cdots y_m) = labx_1 \cdots x_n y_1 \cdots y_m.$$

Since B is closed under application and contains $\mathbf{k}^A$ and $\mathbf{s}^A$, it includes the minimal combinatory subreduct of A. So $l \in B$ and hence $lab \in B$. So B^+ is closed under application.

Let $y \in I$. If $y \notin J$, the hypothesis of Lemma 6.5(i) is satisfied and

$$\lambda y.ax_1 \cdots x_n = pax_1 \cdots x_n \quad \text{for some } p \in B.$$

If $y \in J$, then Lemma 6.5(ii) applies and

$$\lambda y.ax_1 \cdots x_n = q(\lambda y.a)x_1 \cdots x_n \quad \text{with } q \in B.$$

Since B is closed under λy, we have $q(\lambda y.a) \in B$. So B^+ is closed under λ-abstraction with respect to y for all $y \in I$. Finally note that, for each $x \in I \setminus J$, $x = \mathbf{i}^A x$ and $\mathbf{i} \in B$. □

Definition 6.7. Let $A \in \mathrm{LAA}_I$ and $J \subseteq I$ with $|I \setminus J| \geqslant \omega$. For each subalgebra $\boldsymbol{B}$ of $\mathbf{Nr}_J A$, the subalgebra $\boldsymbol{B}^+$ of A considered in the preceding lemma is called the *I-dilation* of $\boldsymbol{B}$ in A, in symbols $\mathbf{Di}_I^A \boldsymbol{B}$. Its universe B^+ is denoted by $\mathrm{Di}_I^A \boldsymbol{B}$. If A is clear from context we write simply $\mathbf{Di}_I \boldsymbol{B}$ and $\mathrm{Di}_I \boldsymbol{B}$.

Theorem 6.8. *Assume $A \in \mathrm{LAA}_I$ and $J \subseteq I$* with *$|I \setminus J| \geqslant \omega$. Then for every $\boldsymbol{B} \subseteq \mathbf{Nr}_J A$ we have $\boldsymbol{B} = \mathbf{Nr}_J \mathbf{Di}_J \boldsymbol{B}$.*

Proof. Trivially, $B \subseteq \mathrm{Nr}_J \mathbf{Di}_I \boldsymbol{B}$. For the reverse inclusion assume $b = ax_1 \cdots x_n \in \mathrm{Nr}_J \mathbf{Di}_I \boldsymbol{B}$, with $a \in B$ and $x_1, \ldots, x_n \in I \setminus J$. Since $x_1, \ldots, x_n \notin \Delta b$,

$$b = S^{x_1 \cdots x_n}_{\mathbf{k} \cdots \mathbf{k}}(b) = a\mathbf{k} \cdots \mathbf{k} \in B. \qquad \square$$

Corollary 6.9. *If $J \subseteq I$ and $|I \setminus J| \geqslant \omega$, then $\mathrm{Nr}_J \mathrm{LAA}_I$ is closed under the formation of subalgebras, i.e., $\mathbb{S}\mathrm{Nr}_J \mathrm{LAA}_I = \mathrm{Nr}_J \mathrm{LAA}_I$.*

Theorem 6.10. *Let A be a LAA_I and let $J \subseteq I$ with $|I \setminus J| \geqslant \omega$.*
(i) *$\mathbf{Di}_I \mathbf{Nr}_J A$ is the largest subalgebra of A locally finite relative to $I \setminus J$.*
(ii) *The mapping $\boldsymbol{B} \mapsto \mathbf{Di}_I \boldsymbol{B}$ is a one–one correspondence between subalgebras of $\mathbf{Nr}_J A$ and subalgebras of A locally finite relative to $I \setminus J$. Its inverse is $\mathbf{Nr}_J$.*

Proof. (i) $\mathbf{Di}_I \mathbf{Nr}_J A$ is clearly locally finite relative to $I \setminus J$. Suppose $a \in A$ and $(I \setminus J) \cap \Delta a = \{x_1, \ldots, x_n\}$. Then $\mathrm{a} = (\lambda x_1 \cdots x_n . a) x_1 \cdots x_n \in \mathrm{Di}_I \mathbf{Nr}_J A$.

(ii) In view of Theorem 6.8 it suffices to show that for each $\boldsymbol{B} \supseteq \mathbf{Nr}_J A$ that is locally finite relative to $I \setminus J$ we have $\boldsymbol{B} = \mathbf{Di}_I \mathbf{Nr}_J \boldsymbol{B}$. But the inclusion $\mathbf{Di}_I \mathbf{Nr}_J \boldsymbol{B} \subseteq \boldsymbol{B}$ is obvious and thus the equality follows by part (i). $\square$

Corollary 6.11. *If A is a locally finite LAA_I, then $A = \mathbf{Di}_I \mathbf{Zd}\, A$.*

6.1. Homomorphisms of neat reducts

In this section we show that $\mathrm{Nr}_J \mathrm{LAA}_I$ is closed under homomorphic images and thus is a variety. The key lemma is a characterization of the tolerance on a dilation that is generated by a congruence on the neat reduct (Proposition 6.13). We begin with a simple, well-known result of a general algebraic character.

A binary relation Θ on an algebra A has the *substitution property* if $\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \in \Theta$ implies $\langle O(a_1 \cdots a_n), O(b_1 \cdots b_n) \rangle \in \Theta$, where O is any fundamental operation of A and n is its rank. A reflexive and symmetric binary relation Θ on A with the substitution property is called a *tolerance.*

Lemma 6.12. *Let Θ be a tolerance on an algebra of arbitrary similarity type. Then Θ^* (the transitive closure of Θ) is a congruence relation on A.*

Proof. It is easy to see Θ^* is an equivalence relation. To see that Θ^* has the substitution property, let O be a fundamental operation of A and assume $\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \in \Theta^*$, where n the rank of O. There exist sequences $c_{i1} \cdots c_{im} \in A^m$ for each $i = 1, \ldots, n$ such that

$$a_i = c_{i1} \equiv_\Theta c_{i2} \equiv_\Theta \cdots \equiv_\Theta c_{im} = b_i.$$

(We can take these sequences to all be of the same length since Θ is reflexive.) Set $c_j = c_{1j}c_{2j}\cdots c_{nj}$ for each $j=1,\dots,m$. Then

$$O(a_1\cdots a_n) = O(c_1) \equiv_\Theta O(c_2) \equiv_\Theta \cdots \equiv_\Theta O(c_m) = O(b_1\cdots b_n).$$

This shows Θ^* is a congruence relation. □

Proposition 6.13. *Let A be a* LAA_I *and $J\subseteq I$ such that $|I\backslash J|\geqslant\omega$. Let $\boldsymbol{B}\subseteq \mathbf{Nr}_J A$ and Θ be a congruence of $\boldsymbol{B}$. Define*

$$\Theta^+ = \{\langle a x_1^A\cdots x_n^A, b x_1^A \cdots x_n^A\rangle : \langle a,b\rangle\in\Theta, x_1,\dots,x_n\in I\backslash J\}.$$

Then Θ^+ is a tolerance on $\mathbf{Di}_I \boldsymbol{B}$.

Proof. Without loss of generality we assume $A=\mathbf{Di}_I\boldsymbol{B}$. Clearly Θ^+ is symmetric and reflexive. Suppose $a^+, b^+\in \mathrm{Di}_I\boldsymbol{B}$ and $a^+\equiv_{\Theta^+} b^+$. Then $a^+ = ax_1\cdots x_n$, $b^+ = bx_1\cdots x_n$ with $a\equiv_\Theta b$. Let $y\in I$. If $y\notin J$, then by Lemma 6.5(i) there is a p in $\boldsymbol{M}$, the minimal combinatory subreduct of A, such that

$$\lambda y^A.a^+ = pax_1\cdots x_n \quad\text{and}\quad \lambda y^A.b^+ = pbx_1\cdots x_n.$$

Clearly $pa\equiv_\Theta pb$, so $\lambda y^A.a^+\equiv_{\Theta^*}\lambda y^A.b^+$. If $y\in J$, then by Lemma 6.5(ii) there is a $q\in M$ such that

$$\lambda y^A.a^+ = q(\lambda y^B.a)x_1\cdots x_n \quad\text{and}\quad \lambda y^A.b^+ = q(\lambda y^B.b)x_1\cdots x_n.$$

Again we get $\lambda y^A.a^+\equiv_{\Theta^*}\lambda y^A b_1^+$ and $a_2^+\equiv_{\Theta^*} b_2^+$ imply $q(\lambda y^B.a)\equiv_\Theta q(\lambda y^B.b)$. In a similar way we get that $a_1^+\equiv_{\Theta^+} b_1^+$ and $a_2^+\equiv_{\Theta^+} b_2^+$ imply $a_1^+a_2^+\equiv_{\Theta^*} b_1^+ b_2^+$; for this we use Corollary 4.20(i). □

We call Θ^{+*} the *I-dilation* of Θ and denote it by $\mathrm{Di}_I\Theta$. It is a congruence on $\mathbf{Di}_I\boldsymbol{B}$ by Lemma 6.12. We show that when restricted to $\boldsymbol{B}$ it coincides with Θ.

Lemma 6.14. *Let A and $\boldsymbol{B}$ be as in the last proposition. Let $a,b\in B$ and $x_1,\dots,x_n$, $y_1,\dots,y_m\in I\backslash J$. If $ax_1^A\cdots x_n^A\equiv_{\mathrm{Di}_I\Theta} by_1^A\cdots y_m^A$, then $a(\mathbf{k}^A)^n\equiv_\Theta b(\mathbf{k}^A)^m$.*

Proof. For some $c_0^+,\dots,c_k^+\in\mathrm{Di}_I\boldsymbol{B}$ we have

$$ax_1\cdots x_n = c_0^+\equiv_{\Theta^*}\cdots c_{k-1}^+\equiv_{\Theta^*} c_k^+ = by_1\cdots y_m.$$

We prove $a\mathbf{k}^n\equiv_{\Theta^*} b\mathbf{k}^m$ by induction on k. If $k=0$, then $ax_1\cdots x_n = by_1\cdots y_m$. Let $z=z_1\cdots z_l\in I^*$ be an enumeration of $\{x_1,\dots,x_n, y_1,\dots,y_m\}$. Then

$$a\mathbf{k}^n = S_{\mathbf{k}^l}^z(ax_1\cdots x_n) = S_{\mathbf{k}^l}^z(by_1\cdots y_m) = b\mathbf{k}^m.$$

Assume $k>0$. Since $c_{k-1}^+\equiv_{\Theta^*} c_k^+$, there are $c_{k-1}, c_k\in B$ and $z_1,\dots,z_l\in I\backslash J$ such that

$$c_{k-1}^+ = c_{k-1}z_1\cdots z_l,\quad c_k^+ = c_k z_1\cdots z_l \quad\text{and}\quad c_{k-1}\equiv_\Theta c_k.$$

Thus

$$a\mathbf{k}^n \equiv_\Theta c_{k-1}\mathbf{k}^l \quad \text{by induction hypothesis}$$
$$\equiv_\Theta c_k\mathbf{k}^l \quad \text{since } c_{k-1} \equiv_\Theta c_k$$
$$= b\mathbf{k}^m \quad \text{by base step of induction.}$$

Since Θ is transitive, we get the desired result. □

Proposition 6.15. *Let* $A \in \mathrm{LAA}_I$ and $J \subseteq I$ *such that* $|I \setminus J| \geqslant \omega$. *Let* $\boldsymbol{B} \subseteq \mathbf{Nr}_J A$ *and* Θ *be a congruence of* $\boldsymbol{B}$. *Then* $\mathrm{Di}_I \Theta$ *is a congruence of* $\mathbf{Di}_I \boldsymbol{B}$ *and* $(\mathrm{Di}_I \Theta) \cap B^2 = \Theta$.

Proof. We have already seen that $\mathrm{Di}_I \Theta$ is a congruence on $\mathbf{Di}_I \boldsymbol{B}$, and obviously $\Theta \subseteq \mathrm{Di}_I \Theta$. Suppose $a^+, b^+ \in B$ and $a^+ \equiv_{\mathrm{Di}_I \Theta} b^+$. Then apply Lemma 6.14 with $n = m = 0$. □

A class K of LAA_Is is a *variety* if it is defined by a set of identities, i.e. there is a set E of sentences of the form $\forall \xi_1 \cdots \forall \xi_n (t(\xi_1, \ldots, \xi_n) = s(\xi_1, \ldots, \xi_n))$, with $t(\xi_1, \ldots, \xi_n)$, $s(\xi_1, \ldots, \xi_n) \in \Lambda_I$, such that K is the class of models of E.

Theorem 6.16. *Assume* $J \subseteq I$ *and* $|I \setminus J| \geqslant \omega$.
(i) $\mathbb{H}\mathrm{Nr}_J \mathrm{LAA}_J = \mathrm{Nr}_J \mathrm{LAA}_I$, *i.e.*, $\mathrm{Nr}_J \mathrm{LAA}_I$ *is closed under formation of homomorphic images.*
(ii) $\mathrm{Nr}_J \mathrm{LAA}_I$ *is a variety.*

Proof. (i) Let $\boldsymbol{B} = \mathrm{Nr}_J \boldsymbol{A}$ with $\boldsymbol{A} \in \mathrm{LAA}_I$. By Theorem 6.8 we assume without loss of generality that $\boldsymbol{A} = \mathbf{Di}_I \boldsymbol{B}$. Let $\boldsymbol{C}$ be a homomorphic image of $\boldsymbol{B}$. Then $\boldsymbol{C} \cong \boldsymbol{B}/\Theta$ for some congruence Θ on $\boldsymbol{B}$. Let $\Psi = \mathrm{Di}_I \Theta$. Since $\Psi \cap B^2 = \Theta$, the correspondence $b/\Theta \mapsto b/\Psi$ establishes a one–one mapping from B/Θ into A/Ψ that preserves application, λ-abstraction for $y \in J$, and the constants $y \in J$, i.e., it is an isomorphism from $\boldsymbol{B}/\Theta$ into a subalgebra of the J-reduct of $\boldsymbol{A}/\Psi$. For each $b \in B$ we have, by Proposition 1.8(i) and (ii),

$$\Delta^{\boldsymbol{B}/\Theta}(b/\Theta) = \Delta^{\boldsymbol{A}/\Psi}(b/\Psi) \subseteq \Delta^{\boldsymbol{A}} b \subseteq J.$$

Thus $\boldsymbol{B}/\Theta$, and hence $\boldsymbol{C}$, is isomorphic to a subalgebra of $\mathbf{Nr}_J(\boldsymbol{A}/\Psi)$. We now conclude by Corollary 6.9 that $\boldsymbol{C} \in \mathrm{Nr}_J \mathrm{LAA}_I$.

(ii) By Theorem 6.3, Corollary 6.9, and part (i) we have $\mathbb{H}\mathbb{S}\mathbb{P}\,\mathrm{Nr}_J \mathrm{LAA}_I = \mathrm{Nr}_J \mathrm{LAA}_I$. So $\mathrm{Nr}_J \mathrm{LAA}_I$ is a variety by Birkhoff's Theorem (see [14, p. 171]). □

Corollary 6.17. *The class of all zero-dimensional subreducts of infinite dimensional* LAA_I's *forms a variety.*

Proof. $\{\mathbf{Zd}\boldsymbol{A} \colon \boldsymbol{A} \in \mathrm{LAA}_I\} = \mathrm{Nr}_\emptyset \mathrm{LAA}_I$. Now apply the theorem. □

In [28] it is shown that the class of zero-dimensional subreducts of LAA_Is coincides with the class of λ-algebras [3, 5.2.2]. Lambda algebras are considered by many to be the most natural models of the lambda calculus, partly because they form a variety. This has been established by verifying that they coincide with the models of a certain finite set of complicated identities due to Curry; see [3, p. 94]. Corollary 6.17 provides an alternative proof that λ-algebras form a variety, via Birkhoff's theorem, that does not depend on Curry's axiomatization.

7. Functional representation of neat reducts

The precise connections between FLAs, RFAs, and neat reducts are made in this section. The situation is similar to that for cylindric algebras. The novelty here is that it is the RFAs that seem to be the proper notion of representable LAAs rather than the FLAs. First of all they are, up to isomorphism, exactly the LAAs that can be neatly embedded in LAAs with infinitely more dimensions, and thus they form a variety. Secondly, they are more general than FLAs in the sense that every FLA is isomorphic to a RFA. We do not know if the converse holds. However, as we have already seen, every RFA is a homomorphic image of a FLA, and thus FLA_I and RFA_I generate the same variety.

Theorem 7.1 (Functional representation of $\mathrm{Nr}_J\mathrm{LAAs}$). $\mathrm{Nr}_J\mathrm{LAA}_I \subseteq \mathbb{I}\mathrm{RFA}_J$ *for every* $I \supseteq J$ *with* $|I \setminus J| \geqslant \omega$, *i.e., every* $\mathrm{Nr}_J\mathrm{LAA}_I$ *is isomorphic to a* RFA_J.

Proof. Let $\boldsymbol{A} = \mathbf{Nr}_J \boldsymbol{B}$ where $\boldsymbol{B} \in \mathrm{LAA}_I$. We assume without loss of generality that $\boldsymbol{B}$ is locally finite relative to $I \setminus J$, in particular, dimension-complemented. By the functional representation theorem for DCA_Is (Theorem 3.14), there is an environment model $\boldsymbol{V}$ and an $r \in V^I$ such that $\boldsymbol{B}$ is isomorphic to a total subalgebra of $\boldsymbol{V}_{I,r}$. Without loss of generality we assume $\boldsymbol{B}$ itself is a total subalgebra of $\boldsymbol{V}_{I,r}$. Thus $\boldsymbol{B}$ is a LAA_I of total functions from V^I_r to V where $V^I_r = \{p \in V^I : p \equiv_\omega r\}$. Note that $V^J_{r \restriction J} = \{p \in V^J ; p \equiv_\omega (r \restriction J)\}$. By the equivalence of algebraic and functional dependency (the analogue of Proposition 3.6 for RFA_Is) we have that $p \restriction J = q \restriction J$ implies $b(p) = b(q)$ for every $b \in \mathrm{Nr}_J \boldsymbol{B}$ and all $p, q \in V^I_r$. Thus the mapping $h : \mathrm{Nr}_J \boldsymbol{B} \to V_{J, r \restriction J}$ defined by $h(b)(p) = b(p^+)$ for every $p \in V^J_{r \restriction J}$, where p^+ is any environment in V^I_r such that $p^+ \restriction J = p$, is well defined. The verification that h is an isomorphism from $\mathbf{Nr}_J \boldsymbol{B}$ onto a total subalgebra of $\boldsymbol{V}_{J, r \restriction J}$ is routine. $\square$

There is a converse of this result (see Theorem 7.3); we first establish the converse for FLA_I's. It uses the notion of *functional dilation*, a functional analogue of the abstract algebraic notion of dilation studied in the previous section.

Theorem 7.2. $\mathrm{FLA}_J \subseteq \mathrm{Nr}_J\mathrm{LAA}_I$ *for every* $I \supseteq J$.

Proof. Let $A \in \mathrm{FLA}_J$. Then A is a total subalgebra of $\boldsymbol{V}_J$ for some environment model $\boldsymbol{V}$. Define $h: A \to \boldsymbol{V}_I$ by $h(a)(p) = a(p \restriction J)$, for all $p \in V^I$. Clearly h is one–one and $h(x^A)$ $(= h(x^{V_J})) = x^{V_I}$ for $x \in J$. It is also clear that the range of h contains only total functions. We show h preserves application and λ-abstraction over variables in J.

$$\begin{aligned} h(a \cdot^A b)(p) &= (a \cdot^{V_J} b)(p \restriction J) \\ &= a(p \restriction J) \cdot^V b(p \restriction J) \\ &= h(a)(p) \cdot^V h(b)(p) \\ &= (h(a) \cdot^{V_I} h(b))(p). \end{aligned}$$

For $x \in J$,

$$\begin{aligned} h(\lambda x^A . a) &= (\lambda x^{V_J} . a)(p \restriction J) \\ &= \lambda^V \langle a((p \restriction J)(v/x)) : v \in V \rangle \\ &= \lambda^V \langle a(p(v/x) \restriction J) : v \in V \rangle \\ &= \lambda^V \langle h(a)(p(v/x)) : v \in V \rangle \\ &= (\lambda x^{V_I} . h(a))(p). \end{aligned}$$

For $x \in I \backslash J$ and $v, u \in V$,

$$\begin{aligned} h(a)(p(v/x)) &= a(p(v/x) \restriction J) \\ &= a(p(u/x) \restriction J) \\ &= h(a)(p(u/x)). \end{aligned}$$

Thus $h(a)$ is functionally independent of x for each $x \in I \backslash J$, and hence the image $h(A)$ of A is a subalgebra of $\mathbf{Nr}_J \boldsymbol{V}_I$. It still remains, however, to show that it is a subalgebra of $\mathbf{Nr}_J \boldsymbol{B}$ for some total subalgebra of $\boldsymbol{V}_I$. We will in fact show this for $\boldsymbol{B} = \boldsymbol{V}_I^{\mathrm{T}}$, the largest total subalgebra of $\boldsymbol{V}_I$. Then we have that $A \cong h(A) \subseteq \mathbf{Nr}_J \boldsymbol{V}_I^{\mathrm{T}}$ and hence $A \in \mathrm{Nr}_J \mathrm{LAA}_I$.

We show that for each $a \in A$, $h(a)$ satisfies the condition of Theorem 5.8, i.e., for each $p \in V^I$ and $\boldsymbol{x} = x_1 \cdots x_n \in I^\star$, there exists a $u_{a,p,\boldsymbol{x}} \in V$ such that

$$h(a)(p(v_1/x_1, \ldots, v_n/x_n)) = u_{a,p,\boldsymbol{x}} v_1 \cdots v_n, \quad \text{for all } v_1 \cdots v_n \in V. \tag{7.1}$$

Fix $a \in A$, $p \in V^I$, and $\boldsymbol{x} = x_1 \cdots x_n \in I^\star$. Let $\{x_{i_1}, \ldots, x_{i_m}\} = \{x_1, \cdots, x_n\} \cap J$. Let $\boldsymbol{x}' = x_{i_1} \ldots x_{i_m}$. Then by Theorem 5.8, there exists a $u_{a,p \restriction J, \boldsymbol{x}'} \in V$ such that, for all $v_{i_1}, \ldots, v_{i_m} \in V$, $a(p \restriction J)(v_{i_1}/x_{i_1}, \ldots, v_{i_m}/x_{i_m}) = u_{a,p \restriction J, \boldsymbol{x}'} v_{i_1} \cdots v_{i_m}$. By Corollary 5.4 and the combinatory completeness lemma (for combinatory algebras) there exists a $w \in V$

such that, for all $v_1, \dots, v_n \in V$,

$$h(a)(p(v_1/x_1, \dots, v_n/x_n)) = a(p \restriction J)(v_{i_1}/x_{i_1}, \dots, v_{i_m}/x_{i_m})$$
$$= u_{a,p \restriction J, x'} v_{i_1} \cdots v_{i_m}$$
$$= w v_1 \cdots v_n.$$

Thus (7.1) holds with $u_{a,p,x} = w$. □

Corollary 7.3. $\mathrm{RFA}_J \subseteq \mathrm{Nr}_J \mathrm{LAA}_I$ *for every* $I \supseteq J$.

Proof. Let $A \in \mathrm{RFA}_J$. By Proposition 5.13, Theorem 6.16, and the last theorem we have $A \in \mathbb{H}\mathrm{FLA}_I \subseteq \mathbb{H}\mathrm{Nr}_J \mathrm{LAA}_I = \mathrm{Nr}_J \mathrm{LAA}_I$.

The corollary can also be proved directly. Let A be a total subalgebra of $V_{J,r}$ for some environment model V and thread $r \in V^J$. Choose any $r^+ \in V^I$ such that $r^+ \restriction J = r$ and define $h(a)(p) = a(p \restriction J)$, for any $p \in V^I_{r^+}$. The proof that h is an isomorphism for A into $\mathbf{Nr}_J\, V_{I,r^+}$ is similar to the proof of the last theorem. □

These results suggest that RFAs rather than FLAs constitute the proper analogue of the notion of a representable cylindric algebra. Theorems 7.1 and 7.2 give the following theorem.

Theorem 7.4. $\mathbb{I}\mathrm{RFA}_J = \mathrm{Nr}_J \mathrm{LAA}_I$ *for any* $I \supseteq J$ *such that* $|I \setminus J| \geqslant \omega$.

Corollary 7.5. $\mathbb{I}\mathrm{RFA}_I$ *is a variety.*

Proof. By Theorems 6.16(ii) and 7.4. □

Corollary 7.6. $\mathrm{FLA}_I \subseteq \mathbb{I}\mathrm{RFA}_I$.

Proof. By Theorems 7.2 and 7.4. □

Let A be a FLA_I over the value domain V. Then A is isomorphic to a RFA_I over a value domain more complex than V. By Proposition 5.12 A is also isomorphic to a subdirect product of RFA_Is over the same value domain V.

Theorem 7.7. $\mathbb{I}\mathrm{RFA}_I$ *is generated as a variety by* FLA_I.

Proof. By Proposition 5.13 and Theorem 7.6. □

8. Conclusion

It is an open problem if $\mathbb{I}\mathrm{FLA}_I$ is also a variety and hence coincides with $\mathbb{I}\mathrm{RFA}_I$. Since $\mathbb{I}\mathrm{RFA}_I$ is a variety, it is axiomatized by some set of identities by Birkhoff's theorem. It is conjectured that it is finitely axiomatizable and, moreover, that Curry's equational axioms for λ-algebras [3, 5.2.5], together with those of LAAs, are sufficient for this purpose. In contrast the representable cylindric algebras are not finitely axiomatizable. The reason for the conjecture is the categorical equivalence between LFA_Is and λ-algebras, which is established in [28], and the fact that $\mathbb{I}\mathrm{RFA}_I$ is generated as a variety by LFA_I.

We have not investigated the subvariety of extensional LAA_Is that are obtained by adjoining the algebraic version of the extensionality axiom:

(η) $\lambda x.\xi x = \xi$.

Diskin and Beylin [10, 11] have considered them, however, and have obtained some interesting results. We would expect the appropriate analogues of most of the results presented here would continue to hold.

We are also planning to begin investigating multiple-sorted LAAs and their applications to the typed lambda calculus. We believe a pure algebraic theory of the typed lambda calculus following the algebraic-logic model would prove useful.

Connections with other work. The importance of abstract substitution, and lambda abstraction, has been recognized for some time among computer scientists because it leads among other things to more natural term rewriting systems, which are useful in the analysis of processes of computations. See for example [1]. In the *transformation algebras and substitution algebras* of LeBlanc [21] and Pinter [30] substitution is primitive and abstract quantification is defined in terms of it. A pure theory of abstract substitution has been developed by Feldman [12, 13]. This work parallels ours in many respects and we acknowledge our indebtedness to it.

Finally, we mention some recent work of ours connecting a theory of substitution in combination with abstract variable-binding operators that has been recently done. See [29, 31].

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