Math 2001: Fall 2018 **FINAL EXAM**

RULES

- **No books**, **no notes**, and **no calculators**.
- **No bathroom breaks** until after you have completed and submitted the exam.
- **All phones must be completely silent** for the duration of the exam, so please *turn off your phone now!*
- Out of consideration for your classmates, do not make disturbing noises during the exam.

Cheating will not be tolerated. If there is any indication that a student may have given or received unauthorized aid on this test, the case will be referred to the Office of the Chair of the Mathematics Department. When you finish the exam, you must sign the following pledge:

"On my honor as a student I, ________________________________, have neither given nor received unauthorized aid on this exam." (print name clearly)

Signature: ______________________________________ Date: May 6, 2019

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Points:	20	14	14	19	14	21	14	9	125
Score:									

1. **Definitions.** Let A, B, and X be sets. Give *precise and complete* definitions of the italicized terms below by completing the given sentence.

(4pts) (a) A *binary relation* on A is...

(4pts) (b) A *equivalence relation* on A is...

(4pts) (c) An *partial order relation* on A is...

(4pts) (d) Given two sets A, B, write down the precise definition of $A \subseteq B$. That is, give the formula that defines $\subseteq$. (*Hint:* use some of the symbols $\in$, $\forall$, $\exists$, $\vee$, $\wedge$, $\rightarrow$, $\neg$.)

(4pts) (e) The subset relation $\subseteq$ on the collection $\mathscr{P}(X)$ of all subsets of X is...

- ◯ a binary relation.
- ◯ an equivalence relation.
- ◯ a partial order relation.
- ◯ none of the above.

(check all that apply)

Score for this page: __________ out of 20

2. Complete the definitions in parts (a) and (b), then answer part (c).

(3pts) (a) A function $f\colon X \to Y$ is called *one-to-one* provided for all...

(3pts) (b) A function $f\colon X \to Y$ is called *onto* provided for all...

(8pts) (c) Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. Assume f is onto and $g \circ f$ is one-to-one. Prove that g is one-to-one.

(4pts) (d) Now suppose $g \circ f$ is one-to-one but f is not onto. Must g be one-to-one? If yes, prove it. If no, provide a specific counter-example.

(10pts) 3. Let X be a set. Suppose $\mathscr{S} = \{A, B, C, \dots\}$ is some collection of subsets of X. Let us call $M \in \mathscr{S}$ a *minimum subset* if for every set $S \in \mathscr{S}$ we have $M \subseteq S$.
Show that if a minimum subset of $\mathscr{S}$ exists, then it is unique.

Score for this page: ____________ out of 14

4. Let $W(x)$ be "x is a website," $A(x)$ be "x has audio," and $V(x)$ be "x has video." Let the domain of interpretation be the whole world.

(5pts) (a) Write a complete English sentence that is equivalent to the following formulas of first-order logic. (Use plain English prose—no variable or logical symbols.)

$$(\forall x)(W(x) \to (V(x) \to A(x)).$$

(6pts) (b) Write formulas in first-order logic (using logical symbols) that express the following statements.

i. "Every website has audio."

ii. "Some websites don't have video."

(8pts) 5. Consider a language with variables ranging over people, and predicates $\texttt{trusts}(x, y)$, $\texttt{politician}(x)$, $\texttt{crazy}(x)$, $\texttt{knows}(x, y)$, and $\texttt{related}(x, y)$, and $\texttt{rich}(x)$. Consider the following statements in this language:

1. Nobody trusts a politician.
2. Anyone who trusts a politician is crazy.
3. Everyone knows someone who is related to a politician.
4. Everyone who is rich is either a politician or knows a politician.

Match each statement above with the right formula in first order logic by filling in the blanks below with the appropriate numbers.

(a) __________ $\forall x \exists y(\texttt{knows}(x, y) \land \exists z(\texttt{politician}(z) \land \texttt{related}(y, z)))$

(b) __________ $\forall y(\texttt{politician}(y) \to \forall x(\neg\texttt{trusts}(x, y)))$

(c) __________ $\forall x \forall y(\texttt{politician}(y) \land \texttt{trusts}(x, y) \to \texttt{crazy}(x))$

(d) __________ $\forall x(\texttt{rich}(x) \to \texttt{politician}(x) \lor \exists z(\texttt{politician}(z) \land \texttt{knows}(x, z)))$

Score for this page: __________ out of 19

(14pts) 6. Give a natural deduction proof of $\exists x\, A(x) \vee \exists x\, B(x) \to \exists x\, (A(x) \vee B(x))$.

Score for this page: ______________ out of 14

(12pts) 7. Fill out the truth table, then answer the True/False questions below.

A	B	$\neg A$	$\neg B$	$A \to B$	$\neg B \to \neg A$	$\neg A \to (\neg B \to \neg A)$

(a) $\neg A \to (\neg B \to \neg A)$ is a tautology True False

(b) $A \to (\neg B \to \neg B)$ is a tautology True False

(c) $\neg A \wedge A \to B$ is a tautology True False

(d) $\neg B \to \neg A$ and $A \to B$ are equivalent True False

8. Assume P is a propositional formula.

(3pts) (a) What does $\vdash P$ mean?

(3pts) (b) What does $\models P$ mean?

(3pts) (c) A logical system is called *complete* provided the following holds: for every formula P, if P is true in every model, then P is provable. What does it mean to call a logical system *sound*?

(14pts) 9. Prove by induction that addition is commutative. In other words, prove that for all natural numbers m and n, we have $m + n = n + m$.

Hints.

- Be sure to clearly *state your induction hypothesis* and point out where it is used.
- Also point out where *associativity of addition* is used.
- You may use without proof the following:

  Fact 1. $m + 1 = 1 + m$.

10. Fill in the blanks to complete the following Lean proofs.
(Each blank is worth a half-point.)

(2pts) (a)

```
variable U : Type
variables A B C : set U

example : ∀ x, x ∈ A ∩ C → x ∈ A ∪ B :=
assume x (h: x ∈ A ∩ C), show x ∈ A ∪ B, from

  have ha : x ∈ A, from __________,

  or.inl __________
```

(4pts) (b)

```
variable  U : Type
variables A B C : U → Prop

example : (¬ ∃ x, A x) → ∀ x, ¬ A x :=
assume hnAx : ¬ ∃ x, A x, show ∀ x, ¬ A x, from
  assume y, show ¬ A y, from

    assume h : ____________, show false, from

      have hAx : ∃ x, A x, from exists.intro ________   ________,

      hnAx __________
```

(3pts) (c)

```
variable  U : Type
variables A B C : U → Prop

example : (∀ x, ¬ A x) → ¬ ∃ x, A x :=
assume hnA : ∀ x, ¬ A x, show ¬ ∃ x, A x, from

  assume hA : ∃ x, A x, show ____________, from

    exists.elim ___________

      (assume y (h : A y),

      have h' : ¬ A y, from ________   ________,

      ________    ________ )
```

Score for this page: ____________ out of 9

– scratch –

– scratch –