Math 2001 Homework 7

Section 10.6 all exercises (1–4)

Due: Wednesday, 3 April 2019 (start of class)

1. In a first-order language with a binary relation, $R(x, y)$, consider the following sentences:

   - $\exists x \, \forall y \; R(x, y)$
   - $\exists y \, \forall x \; R(x, y)$
   - $\forall x, y \; (R(x, y) \wedge x \neq y \to \exists z \; (R(x, z) \wedge R(z, y) \wedge x \neq z \wedge y \neq z))$

   For each of the following structures, determine whether of each of those sentences is true or false.

   - the structure $(\mathbb{N}, \leq)$, that is, the interpretation in the natural numbers where R is $\leq$
   - the structure $(\mathbb{Z}, \leq)$
   - the structure $(\mathbb{Q}, \leq)$
   - the structure $(\mathbb{N}, |)$, that is, the interpretation in
   - the natural numbers where R is the "divides" relation
   - the structure $(P(\mathbb{N}), \subseteq)$, that is, the
   - interpretation where variables range over sets of natural numbers,
   - where R is interpreted as the subset relation.

2. Create a 4 x 4 "dots" world that makes all of the following sentences true:

   - $\forall x \; (\mathit{green}(x) \vee \mathit{blue}(x))$
   - $\exists x, y \; (\mathit{adj}(x, y) \wedge \mathit{green}(x) \wedge \mathit{green}(y))$
   - $\exists x \; (\exists z \; \mathit{right\text{-}of}(z, x) \wedge \forall y \; (\mathit{left\text{-}of}(x, y) \to \mathit{blue}(y) \vee \mathit{small}(y)))$
   - $\forall x \; (\mathit{large}(x) \to \exists y \; (\mathit{small}(y) \wedge \mathit{adj}(x, y)))$
   - $\forall x \; (\mathit{green}(x) \to \exists y \; (\mathit{same\text{-}row}(x, y) \wedge \mathit{blue}(y)))$
   - $\forall x, y \; (\mathit{same\text{-}row}(x, y) \wedge \mathit{same\text{-}column}(x, y) \to x = y)$
   - $\exists x \, \forall y \; (\mathit{adj}(x, y) \to \neg \mathit{same\text{-}size}(x, y))$
   - $\forall x \, \exists y \; (\mathit{adj}(x, y) \wedge \mathit{same\text{-}color}(x, y))$
   - $\exists y \, \forall x \; (\mathit{adj}(x, y) \to \mathit{same\text{-}color}(x, y))$
   - $\exists x \; (\mathit{blue}(x) \wedge \exists y \; (\mathit{green}(y) \wedge \mathit{above}(x, y)))$

3. Fix a first-order language L, and let A and B be any two sentences in L. Remember that $\models A$ means that A is valid. Unpacking the definitions, show that if $\models A \wedge B$, then $\models A$ and $\models B$.

4. Give a concrete example to show that $\models A \vee B$ does not necessarily imply $\models A$ or $\models B$. In other words, pick a language L and choose particular sentences A and B such that $A \vee B$ is valid but neither A nor B is valid.