Math 2001 Homework 8

Due 12 April 2019 (start of class)

Exercises

1. (13.4.1) Suppose $<$ is a strict partial order on a domain A, and define $a \leq b$ to mean that $a < b$ or $a = b$.

   - Show that $\leq$ is a partial order.
   - Show that if $<$ is moreover a strict total order, then $\leq$ is a total order.

   (In Chapter 13 of the text, the analogous theorem going in the other direction is proved.)

2. (13.4.2) Suppose $<$ is a strict partial order on a domain A. (In other words, it is transitive and asymmetric.) Suppose that $\leq$ is defined so that $a \leq b$ if and only if $a < b$ or $a = b$. We saw in class that $\leq$ is a partial order on a domain A, i.e.~it is reflexive, transitive, and antisymmetric. Prove that for every a and b in A, we have $a < b$ iff $a \leq b$ and $a \neq b$, using the facts above.

3. (13.4.3) An *ordered graph* is a collection of vertices (points), along with a collection of arrows between vertices. For each pair of vertices, there is at most one arrow between them: in other words, every pair of vertices is either unconnected, or one vertex is "directed" toward the other. Note that it is possible to have an arrow from a vertex to itself.

   Define a relation $\leq$ on the set of vertices, such that for two vertices a and b, $a \leq b$ means that there is an arrow from a pointing to b.

   On an arbitrary graph, is $\leq$ a partial order, a strict partial order, a total order, a strict total order, or none of the above? If possible, give examples of graphs where $\leq$ fails to have these properties.

4. (13.4.4) Let $\equiv$ be an equivalence relation on a set A. For every element a in A, let $[a]$ be the equivalence class of a: that is, the set of elements $\{c \mid c \equiv a\}$. Show that for every a and b, $[a] = [b]$ if and only if $a \equiv b$.

   (Hints and notes:

   - Remember that since you are proving an $$if and only if" statement, there are two directions to prove.
   - Since that $[a]$ and $[b]$ are sets, $[a] = [b]$ means that for every element c, c is in $[a]$ if and only if c is in $[b]$.
   - By definition, an element c is in $[a]$ if and only if $c \equiv a$. In particular, a is in $[a]$.)

5. (13.4.5) Let the relation $\sim$ on the natural numbers $\mathbb{B}$ be defined as follows: if n is even, then $n \sim n+1$, and if n is odd, then $n \sim n-1$. Furthermore, for every n, $n \sim n$. Show that $\sim$ is an equivalence relation. What is the equivalence class of the number 5? Describe the set of equivalence classes $\{[n] \mid n \in \mathbb{N}\}$.

6. (13.4.7) A binary relation $\leq$ on a domain A is said to be a *preorder* it is is reflexive and transitive. This is weaker than saying it is a partial order; we have removed the requirement that the relation is asymmetric. An example is the ordering on people currently alive on the planet defined by setting $x \leq y$ if and only if x's birth date is earlier than y's. Asymmetry fails, because different people can be born on the same day. But, prove that the following theorem holds:

   Theorem. Let $\leq$ be a preorder on a domain A. Define the relation $\equiv$, where $x \equiv y$ holds if and only if $x \leq y$ and $y \leq x$. Then $\equiv$ is an equivalence relation on A.