MATH 2001: SPRING 2019 **EXAM 2**

RULES

- **No books**, **no notes**, and **no calculators**.
- **No bathroom breaks** until after you have completed and submitted the exam.
- **All phones must be completely silent** for the duration of the exam, so please *turn off your phone now!*
- Out of consideration for your classmates, do not make disturbing noises during the exam.

Cheating will not be tolerated. If there is any indication that a student may have given or received unauthorized aid on this test, the case will be referred to the Office of the Chair of the Mathematics Department. When you finish the exam, you must sign the following pledge:

"On my honor as a student I, ______________________________, have neither given nor received unauthorized aid on this exam." (print name clearly)

Signature: ____ Date: April 5, 2019

Page:	2	3	4	5	6	7	Total
Points:	20	18	12	15	25	10	100
Score:							

1. **Definitions.** Let X and Y be sets. Give *precise and complete definitions* of the italicized terms below by completing the given sentence. If you use words like *reflexive*, *symmetric*, etc., *you must provide the formulas that define these terms!*

(5pts) (a) A *binary relation* on X is...

(5pts) (b) A *partial order relation* on a set A is...

(10pts) (c) Let A be a set and let $\leq$ be a partial order on A. An element x of A is called a *minimum element* of A if $x \leq y$ for every y in A. Show with an ordinary mathematical proof (using complete English sentences) that a minimum element is unique. (In other words, any two minimum elements have to be equal.)

Score for this page: ____________ out of 20

2. Let $f\colon S \to T$ and $g\colon T \to U$ be functions.

(5pts) (a) Say precisely what it means for the function $g\colon T \to U$ to be *one-to-one* (i.e., injective).

(5pts) (b) Say precisely what it means for the function $g \circ f\colon S \to U$ to be *onto* (i.e., surjective).

(8pts) (c) Write an ordinary mathematical proof (using complete English sentences) to show that if g is one-to-one and $g \circ f$ is onto, then, then f is onto.

Score for this page: ____________ out of 18

(12pts) 3. Give a natural deduction proof of $\forall x\, B(x)$ from hypotheses $\forall x\, (A(x) \vee B(x))$ and $\forall y\, \neg A(y)$.

Score for this page: __________ out of 12

(15pts) 4. Give a natural deduction proof of $\exists x A(x)$ from $\exists x(A(x) \vee B(x))$ and $\forall x(B(x) \to A(x))$. That is, *assume* that both $\exists x(A(x) \vee B(x))$ and $\forall x(B(x) \to A(x))$ hold, and use that to *derive* $\exists x A(x)$.

Score for this page: ______________ out of 15

5. In a first-order language with a binary relation $R(x, y)$ on a set U, consider the following sentences:

   (a) $\exists x\, \forall y\, R(x, y)$ (b) $\forall x\, \exists y\, R(x, y)$ (c) $\exists y\, \forall x\, R(x, y)$.

   For the following models, determine whether each sentence above is true or false, and fill in the blanks below with T if the sentence is true and F if the sentence is false. (No justification is required.)

   Example. If $(U, R) = (\mathbb{N}, \leq)$, then (a) is ___T___, (b) is ___T___, (c) is___F___.

(6pts) (a) If $(U, R) = (\mathbb{Z}, \leq)$, then (a) is _______, (b) is _______, (c) is _______.

(3pts) (b) If $(U, R) = (\mathbb{N} \setminus \{0\}, |)$, then (a) is _______, (b) is _______, (c) is _______.
(Here the interpretation is the set $U = \{1, 2, \dots\}$ of natural numbers *without zero* and the relation $R =|$, the "divides" relation.)

(6pts) (c) If $(U, R) = (P(\mathbb{N}), \subseteq)$, then (a) is _______, (b) is _______, (c) is _______.
(Here the interpretation is the set U of subsets of natural numbers and the relation $R =\subseteq$, the subset inclusion relation.)

6. Using the symbols $\vDash$ and $\vdash$,

(5pts) (a) say what it means for a logical system to be *sound*;

(5pts) (b) say what it means for a logical system to be *complete*.

Score for this page: ____________ out of 25

(10pts) 7. Fill in the blanks to complete the following Lean proofs.
(Each blank is worth one point.)

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variable  U : Type
variables A B C : U → Prop

example : (¬ ∃ x, A x) → ∀ x, ¬ A x :=

assume hnAx : ¬ ∃ x, A x, show ∀ x, ¬ A x, from

assume y, show ¬ A y, from

assume h : __________, show false, from

    have hAx : ∃ x, A x, from exists.intro ______  ______,

    hnAx ______
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example : (∀ x, ¬ A x) → ¬ ∃ x, A x :=

assume hnA : ∀ x, ¬ A x, show ¬ ∃ x, A x, from

  assume hA : ∃ x, A x, show ____________, from

  exists.elim ______

    (assume y (h : A y),

    have h' : ¬ A y, from ______  ______,

      ______  ______ )
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– scratch –

– scratch –