Math 2001 HW 9

Due 1pm Fri **26 Apr** 2019.

1. Write the principle of complete induction using the notation of symbolic logic. Also write the least element principle this way, and use logical manipulations to show that the two are equivalent.

2. Show that for every n, $0^2 + 1^2 + 2^2 + \ldots n^2 = \frac{1}{6}n(n+1)(2n+1)$.

3. Recall the definition of the Fibonacci numbers, $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

   Prove Cassini's identity: for every $n \geq 0$, $F_{n+1}^2 - F_{n+2}F_n = (-1)^n$.

   Hint: in the induction step, write F_{n+2}^2 as $F_{n+2}(F_{n+1} + F_n)$.

4. Prove that every natural number can be written as a sum of *distinct* powers of 2. For this problem, $1 = 2^0$ is counted as power of 2.

5. Let V be a non-empty set of integers such that the following two properties hold:

   - If $x, y \in V$, then $x - y \in V$.
   - If $x \in V$, then every multiple of x is an element of V.

   Prove that there is some $d \in V$, such that V is equal to the set of multiples of d. Hint: use the least element principle.

6. Show that multiplication distributes over addition. In other words, prove that for natural numbers m, n, and k, $m(n + k) = mn + mk$. Use the definitions of addition and multiplication and facts proved in 17.4 (but nothing more).

7. Prove the multiplication is associative, in the same way. Use the facts proved in 17.4 and the previous exercise.

8. Prove that multiplication is commutative.

9. Prove that if n and m are natural numbers and $nm = 1$, then $n = m = 1$, using only properties listed in 17.5.

   This is tricky. First show that n and m are greater than 0, and hence greater than or equal to 1. Then show that if either one of them is greater than 1, then $nm > 1$.

Extra Credit Exercise

Suppose you have an infinite chessboard with a natural number written in each square. The value in each square is the average of the values of the four neighboring squares. Prove that all the values on the chessboard are equal.