HW 8

DUE Friday, March 23.
Section 4.3: (29), 30, 32;
Section 4.4: 4, 8, (9), 12, 16, (18), 22, (23);
Section 4.5: 6, 8, 12, 14, 20, (25), (26), (32).

Section 4.3

Exercises: (29), 30, 32

4.3.29. (recommended)
Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k < n$. Use a theorem from Section 1.4 to explain why S cannot be a basis for $\mathbb{R}^n$.

4.3.30. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k > n$. Use a theorem from Chapter 1 to explain why S cannot be a basis for $\mathbb{R}^n$.

Remark: Exercises 4.3.29 and 4.3.30 show that every basis for $\mathbb{R}^n$ must contain exactly n vectors.

4.3.32. Suppose T is a one-to-one transformation. (Recall this means that, for all $\mathbf{u}$ and $\mathbf{v}$, we have $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.) Show that if the set $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$, of images is linearly dependent, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ linearly dependent. (This fact shows that a one-to-one linear transformation maps a linearly independent set onto a linearly independent set, because in this case the set of images cannot be linearly dependent).

Section 4.4

Exercises: 4, 8, (9), 12, 16, (18), 22, (23)

4.4.4. Find the vector $\mathbf{x}$ determined by the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$ and the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}.$$

4.4.8. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$ relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

4.4.9. (recommended)

Find the change-of-coordinates matrix from $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$ to the standard basis in $\mathbb{R}^n$.

4.4.12. Use an inverse matrix to find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$.

4.4.16. Mark each statement True or False. Justify each answer. Unless stated otherwise, $\mathcal{B}$ is a basis for a vector space V.

a. If $\mathcal{B}$ is the standard basis for $\mathbb{R}^n$, then the $\mathcal{B}$-coordinate vector of an $\mathbf{x}$ in $\mathbb{R}^n$ is $\mathbf{x}$ itself.

b. The correspondence $[\mathbf{x}]_{\mathcal{B}} \mapsto \mathbf{x}$ is called the coordinate mapping.

c. In some cases, a plane in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

4.4.18. (recommended)
Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Explain why the $\mathcal{B}$-coordinate vectors of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the $n \times n$ identity matrix.

4.4.22. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$. Describe an $n \times n$ matrix A that implements the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$.

4.4.23. (recommended)
Show that the coordinate mapping is one-to-one. [*Hint:* Suppose $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$ for some $\mathbf{u}$ and $\mathbf{w}$ in V and show that $\mathbf{u} = \mathbf{w}$.]

Section 4.5

Exercises: 6, 8, 12, 14, 20, (25), (26), (32)

4.5.6. Let $V = \left\{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$. Find the following:

a. a basis for V;

b. the dimension of V.

4.5.8. Let $V = \{(a, b, c, d) : a - 3b - +c = 0\}$. Find the following:

a. a basis for V;

b. the dimension of V.

4.5.12. Find the dimension of the subspace spanned by the vectors, $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$.

4.5.14. Determine the dimensions of $\operatorname{Nul} A$ and $\operatorname{Col} A$ for the matrix

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.5.20. Let V be a vector space. Mark each statement True or False. Justify each answer.

a. $\mathbb{R}^2$ is a two-dimensional subspace of $\mathbb{R}^3$.

b. The number of variables in the equation $A\mathbf{x} = \mathbf{0}$ equals the dimension of $\operatorname{Nul} A$.

c. A vector space is infinite-dimensional if it is spanned by an infinite set.

d. If $\dim V = n$ and if S spans V, then S is a basis of V.

e. The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

4.5.25. (recommended)
Let S be a subset of an n-dimensional vector space V, and suppose S contains fewer than n vectors. Explain why S cannot span V.

4.5.26. (recommended)
Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

4.5.32. (recommended)
Let H be a nonzero subspace of V, and suppose T is a one-to-one linear transformation of V into W. Prove that $\dim T(H) = \dim H$. (If T happens to be a one-to-one mapping of V *onto* W, then $\dim V = \dim W$. Isomorphic finite-dimensional vector spaces have the same dimension.)