HW 11

DUE Friday, April 27, 9am

List of Exercises

Section 5.4: 2, 12, (21), (25), 26, (27)
Section 6.1: 6, 10, 14, 16, 20, (27), 28, (30)
Section 6.2: 10, 12, (13), 16, 24, (33)

Section 5.4

Exercises: 2, 12, (21), (25), 26, (27)

5.4.2. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \to W$ be a linear transformation satisfying

$$T(\mathbf{d}_1) = 2\mathbf{b}_1 - 3\mathbf{b}_2 \quad \text{and} \quad T(\mathbf{d}_2) = -4\mathbf{b}_1 + 5\mathbf{b}_2.$$

Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$.

5.4.12. Find the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$, when $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

5.4.21. (recommended)
Prove the following statement for square matrices, A, B, C. If B is similar to A and C is similar to A, then B is similar to C.

5.4.25. (recommended)
The *trace* of a square matrix A is the sum of the diagonal entries in A and is denoted by $\operatorname{tr} A$. It can be verified that $\operatorname{tr}(FG) = \operatorname{tr}(GF)$ for any two $n \times n$ matrices F and G. Show that if A and B are similar, then $\operatorname{tr} A = \operatorname{tr} B$.

5.4.26. It can be shown that the *trace* (see Exercise 5.4.25 for definition) of a matrix A equals the sum of the eigenvalues of A. Verify this statement for the case when A is diagonalizable.

5.4.27. (recommended)
Let V be $\mathbb{R}^n$ with a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, let W be $\mathbb{R}^n$ with a the standard basis, denoted $\mathcal{E}$. Consider the identity transformation $I : V \to W$, where $I(\mathbf{x}) = \mathbf{x}$. Find the matrix for I relative to $\mathcal{B}$ and $\mathcal{E}$. What was this matrix called in Section 4.4?

Section 6.1

Exercises: 6, 10, 14, 16, 20, (27), 28, (30)

6.1.6. Let $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. Compute $\left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right) \mathbf{x}$

6.1.10. Find a unit vector in the direction of the vector $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

6.1.14. Find the distance between $\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$

6.1.16. Determine whether $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ are orthogonal vectors.

6.1.20. Mark each statement True or False. Justify each answer. All vectors are assumed to be in $\mathbb{R}^n$.

a. $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$

b. For any scalar c, $\|c\mathbf{v}\| = c\|\mathbf{v}\|$.

c. If $\mathbf{x}$ is orthogonal to every vector in a subspace W, then $\mathbf{x}$ is in $W^\perp$.

d. If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u}+\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

e. For an $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A.

6.1.27. (recommended)
Suppose a vector $\mathbf{y}$ is orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to the vector $\mathbf{u} + \mathbf{v}$.

6.1.28. Suppose $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$. [*Hint:* An arbitrary $\mathbf{w}$ in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ has the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$; show that $\mathbf{y}$ is orthogonal to every such a vector.]

6.1.30. (recommended)
Let W be a subspace of $\mathbb{R}^n$, and let $W^\perp$ be the set of all vectors orthogonal to W. Show that $W^\perp$ is a subspace of $\mathbb{R}^n$ using the following steps.

a. Take $\mathbf{z}$ in $W^\perp$, and let $\mathbf{u}$ represent any element of W. Then $\mathbf{z} \cdot \mathbf{u} = 0$. Take any scalar c and show that $c\mathbf{z}$ is orthogonal to $\mathbf{u}$. (Since $\mathbf{u}$ was an arbitrary element of W, this will show that $c\mathbf{z}$ is in $W^\perp$.)

b. Take $\mathbf{z}_1$ and $\mathbf{z}_2$ in $W^\perp$, and let $\mathbf{u}$ be any element of W. Show that $\mathbf{z}_1 + \mathbf{z}_2$ is orthogonal to $\mathbf{u}$. What can you conclude about $\mathbf{z}_1 + \mathbf{z}_2$? Why?

c. Finish the proof that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Section 6.2

Exercises: 10, 12, (13), 16, 24, (33)

6.2.10. Let $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

6.2.12. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ onto the line through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

6.2.13. (recommended)
Let $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in $\mathrm{Span}\{\mathbf{u}\}$ and the other orthogonal to $\mathbf{u}$.

6.2.16. Let $\mathbf{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line passing through $\mathbf{u}$ and the origin.

6.2.24. Mark each statement True or False. Justify each answer. All vectors are assumed to belong to $\mathbb{R}^n$.

a. Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b. If a set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.

c. If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths.

d. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

e. An orthogonal matrix is invertible.

6.2.33. (recommended)
Suppose $\mathbf{u}$ is a nonzero vector in $\mathbb{R}^n$, and let $L = \mathrm{Span}\{\mathbf{u}\}$. Show that the mapping $\mathbf{x} \mapsto \mathrm{proj}_L\,\mathbf{x}$ is a linear transformation.