HW 7

Due Monday, March 19, 9am.
Section 4.1: 2, 10, 12, (15), (17), 18, (20), 24;
Section 4.2: 4, 8, 10, 14, 26, (30), (35);
Section 4.3: 2, (3), 6, 8, (9), (13), 16, 22, (31), (32).

Section 4.1

Exercises: 2, 10, 12, (15), (17), 18, (20), 24

4.1.2. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid xy \geq 0 \right\}$.

a. If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

b. Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} + \mathbf{v}$ is not in W. (This is enough to show that W is *not* a vector space.)

4.1.10. Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that H is a subspace of $\mathbb{R}^3$. (*Hint:* Find a vector $\mathbf{v} \in \mathbb{R}^3$ such that $H = \mathrm{Span}\{\mathbf{v}\}$.)

4.1.12. Let W be the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$ for arbitrary real numbers s and t. Show that H is a subspace of $\mathbb{R}^4$. (*Hint:* Use the method of Exercise 4.1.11.)

4.1.15. (recommended)

Let W be the set of all vectors of the form $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$ for arbitrary real numbers a and b. Find a set of vectors that spans W, or say why (or give an example to show) that W is *not* a vector space.

4.1.17. (recommended)

Let W be the set of all vectors of the form $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$ for arbitrary real numbers a, b, and c. Find a set of vectors that spans W, or say why (or give an example to show) that W is *not* a vector space.

4.1.18. Let W be the set of all vectors of the form $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$ for arbitrary real numbers a, b, and c. Find a set of vectors that spans W, or say why (or give an example to show) that W is *not* a vector space.

4.1.20. (recommended)
The set of all continuous real-valued functions defined on a closed interval $[a,b]$ of $\mathbb{R}$ is denoted by $C[a,b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a,b]$.

a. What facts about continuous functions should be proved in order to demonstrate that $C[a,b]$ is indeed a subspace as claimed? (These facts are usually discussed in a calcu- lus class.)

b. Show that $\{\mathbf{f} \in C[a,b] \mid \mathbf{f}(a) = \mathbf{f}(b)\}$ is a subspace of $C[a,b]$.

4.1.24. Mark each statement True or False. Justify each answer.

a. A vector is any element of a vector space.

b. If $\mathbf{u}$ is a vector in a vector space V, then $(-1)\mathbf{u}$ is the same as the negative of $\mathbf{u}$.

c. A vector space is also a subspace.

d. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

e. A subset H of a vector space V is a subspace of V if the following conditions are satisfied: (i) the zero vector of V is in H; (ii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ are in H; (iii) c is a scalar and $c\mathbf{u}$ is in H. (If this is false, give the correct conditions.)

Section 4.2

Exercises: 4, 8, 10, 14, 26, (30), (35)

4.2.4. Let $A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$. Find an explicit description of $\operatorname{Nul} A$ by listing vectors that span the null space.

4.2.8. Let $W = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}$. Use an appropriate theorem to show that W is a vector space, or give a specific reason or example demonstrating that W is not a subspace.

4.2.10. Let

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + 3b = c \text{ and } a + b + c = d \right\}.$$

Use an appropriate theorem to show that W is a vector space, or give a specific reason or example demonstrating that W is not a subspace.

4.2.14. Let $W = \left\{ \begin{bmatrix} -a + 2b \\ a - 2b \\ 3a - 6b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Use an appropriate theorem to show that W is a vector space, or give a specific reason or example demonstrating that W is not a subspace.

4.2.26. Let A be an $m \times n$ matrix. Mark each statement True or False. Justify each answer.

a. A null space is a vector space.

b. The column space of an $m \times n$ matrix is in $\mathbb{R}^m$.

c. Col A is the set of all solutions to $A\mathbf{x} = \mathbf{b}$.

d. Nul A is the kernel of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

e. The range of a linear transformation is a vector space.

f. The set of all solutions of a homogeneous linear differential equation is the kernel of a linear transformation.

4.2.30. (recommended)
Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. [*Hint:* Typical elements of the range have the form $T(\mathbf{x})$ and $T(\mathbf{y})$ for $\mathbf{x}$, $\mathbf{y}$ in V.]

4.2.35. (recommended)
Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Given a subspace U of V, let $T(U)$ denote the image of U under T. That is, $T(U) = \{T(\mathbf{u}) : \mathbf{u} \in U\}$. Show that $T(U)$ is a subspace of W.

Section 4.3

Exercises: 2, (3), 6, 8, (9), (13), 16, 22, (31), (32)

Determine which sets in Exercises 1–8 are bases for $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

4.3.2 $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

4.3.3. (recommended)

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}.$$

4.3.6. $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix}.$

4.3.8. $\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}.$

4.3.9. (recommended)

Find a basis for the null space of the matrix $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}$.

4.3.13. (recommended)

Let $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and assume that A is row equivalent to B. Find bases for $\mathrm{Nul}\, A$ and $\mathrm{Col}\, A$.

4.3.16. Find a basis for the space spanned by the following set of vectors:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} \right\}$$

4.3.22. Mark each statement True or False. Justify each answer.

a. A linearly independent set in a subspace H is a basis for H.

b. If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

c. A basis is a linearly independent set that is as large as possible.

d. The standard method for producing a spanning set for $\mathrm{Nul}\, A$, described in Section 4.2, sometimes fails to produce a basis for $\mathrm{Nul}\, A$.

e. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for $\mathrm{Col}\, A$.

4.3.31. (recommended)
Let V and W be vector spaces, let $T: V \to W$ be a linear transformation, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a subset of V. Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ linearly dependent in V, then the set of images, $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$, is linearly dependent in W. (This fact shows that if a linear transformation maps a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ onto a linearly independent set, then S is linearly independent, too.)

4.3.32. (recommended)
Suppose T is a one-to-one transformation, so that an equation $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$. Show that if the set $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$, of images is linearly dependent, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ linearly dependent. (This fact shows that a one-to-one linear transformation maps a linearly independent set onto a linearly independent set, because in this case the set of images cannot be linearly dependent).