Math 374 – Interpretations, Models, Completeness

First, let us briefly summarize some of the things we have learned so far. The language of propositional logic consists of a set of symbols – namely,

- *propositional symbols*, like A, B, P, Q, etc.,
- *logical connectives*, $'$, $\wedge$, $\vee$, $\rightarrow$
- parentheses

A *sentence* in our language is simply a string of symbols. A statement, or *well formed formula* (wff), is not just some arbitrary string of symbols. Rather, a wff is a sentence that is constructed according to the following rules:

(1) Each propositional symbol, A, B, P, Q, etc. is itself a wff.
(2) If P is a wff, then P' is a wff.
(3) If P and Q are both wffs, then so are $P \wedge Q$, $P \vee Q$ and $P \rightarrow Q$.

Any string of symbols that cannot be constructed by applying these three rules a finite number of times (along with some parentheses if necessary) is not a well formed formula.

Now, we can use propositional symbols and logical connectives to construct a statement (wff) in the language, but whether or not this statement is true depends on the *interpretation* that we assign to the propositional symbols. An interpretation is merely an assignment of T or F to each propositional symbol in the language. If we know the truth value of each propositional symbol appearing in a wff, then we can determine whether that wff is true of false.

More formally, if $\mathcal{P}$ is the set of all propositional symbols in our language, then an **interpretation** is a function $I : \mathcal{P} \rightarrow \{T, F\}$ that assigns to each propositional symbol P in $\mathcal{P}$, a value: either $I(P) = T$ (true), or $I(P) = F$ (false).

Suppose φ is a wff, and I an interpretation. If φ is true in I, then we write $I \models \varphi$ and we say that I is a **model** of φ.

If we have a whole set Σ of wffs and if I is an interpretation in which every statement in Σ is true, then we write $I \models \Sigma$ and we say that I is a *model* of Σ.

Now, given a set Σ of wffs, if there exists an interpretation I that is a model of Σ, that is, $I \models \Sigma$, the we say that Σ is **satisfiable**. (If there is no such model, we say that Σ is unsatisfiable.) Let φ be a wff. We say that Σ **logically implies** φ, and we write $\Sigma \models \varphi$, if the set $\Sigma \cup \{\varphi'\}$ is unsatisfiable. In other words, $\Sigma \models \varphi$ means that φ is true in every interpretation that models Σ.

Every logical system has a set S of **deduction rules** that allow us to start from some wff and derive other wffs in a *truth preserving* manner. That is, if φ is true and if we can derive ψ from φ using our rules, then ψ must also be true. We use the symbol $\vdash$ to mean "leads to" and we write $\varphi \vdash \psi$ to mean φ leads to ψ, that is, ψ can be **logically deduced** from φ. Sometimes we write $\varphi \vdash_S \psi$ to emphasize that this is a property that depends on S. If Σ is a set of wffs, and if we can deduce φ using Σ and the deduction rules in S, then we write $\Sigma \vdash_S \varphi$.

A set S of deduction rules is called **complete** if every wff that is logically implied by some set Σ can be logically deduced from Σ using the rules in S. That is, a system is complete provided for all Σ and φ the following is true: If $\Sigma \models \varphi$, then $\Sigma \vdash_S \varphi$. Conversely, a system is called **correct** provided $\Sigma \vdash_S \varphi$ implies $\Sigma \models \varphi$.