MATH 321 **Homework 5 (due 11/2)** **NAME:**

(2pts) 1. (V 4.1.2) What are the truth sets of the following statements? List a few elements of each truth set.

(a) "x lives in y," where x ranges over the set P of all people and y ranges over the set C of all cities.

(b) "The population of x is y," where x ranges over the set C of all cities and y ranges over $\mathbb{N}$.

(2pts) 2. (V 4.1.10)

Prove that for any sets A, S, C, and D, if $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

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(3pts) 3. (V 4.2.9) Suppose R and S are relations from A to B. Must the following statements be true? Justify your answers with proofs or counterexamples.

(a) $R \subseteq \mathrm{Dom}(R) \times \mathrm{Ran}(R)$.

(b) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

(c) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

(4pts) 4. For each of the following binary relations ρ on $\mathbb{Z}$, decide which of the given ordered pairs belong to ρ.

(a) $x \, \rho \, y \leftrightarrow x \mid y$; $(2,6), (3,5), (8,4), (4,8)$

(b) $x \, \rho \, y \leftrightarrow x$ and y are relatively prime; $(5,8), (9,16), (6,8), (8,21)$

(c) $x \, \rho \, y \leftrightarrow \gcd(x,y) = 7$; $(28,14), (7,7), (10,5), (21,14)$

(d) $x \, \rho \, y \leftrightarrow x^2 + y^2 = z^2$ for some integer z; $(1,0), (3,9), (2,2), (3,4)$

(e) $x \, \rho \, y \leftrightarrow x$ is a Fibonacci number; $(4,3), (7,6), (7,12), (20,20)$

Example: (a) $\{(2,6),(4,8)\} \subseteq \rho$. (Present your answers in the same format.)

Answers:

(b)

(c)

(d)

(e)

(3pts) 5. Let $S = \{0,1,2,4,6\}$. Test the following binary relations on S for reflexivity, symmetry, antisymmetry, and transitivity.

(a) $\rho = \{(0,0),(1,1),(2,2),(4,4),(6,6),(0,1),(1,2),(2,4),(4,6)\}$

(b) $\rho = \{(0,1),(1,0),(2,4),(4,2),(4,6),(6,4)\}$

(c) $\rho = \{(0,1),(1,2),(0,2),(2,0),(2,1),(1,0),(0,0),(1,1),(2,2)\}$

(d) $\rho = \emptyset$

Example: (a) reflexive, antisymmetric. (Present your answers in the same format.)

Answers:

(b)

(c)

(d)

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(2pts) 6. Find the reflexive, symmetric, and transitive closure of each of the relations in the last two parts of Exercise 5.

Example Solution: (a) reflexive closure(ρ) $= \rho$;

symmetric closure(ρ) $= \rho \cup \{(1,0),(2,1),(4,2),(6,4)\}$;

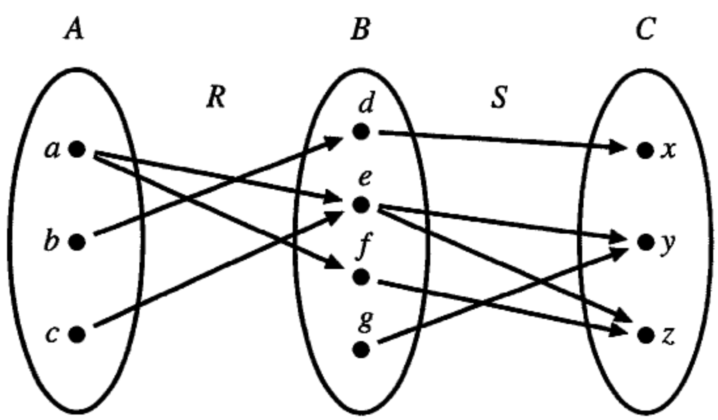
transitive closure(ρ) $= \rho \cup \{(0,2),(1,4),(2,6),(0,4),(0,6),(1,6)\}$.

Answers:

(c)

(d)

(2pts) 7. (V 4.3.5) The following diagram shows two relations R and S. Find $S \circ R$.



(3pts) 8. (V 4.4.2) In each case, say whether or not R is a partial order on A. If so, is it a total order?

(a) $A =$ the set of all words of English, $R = \{(x, y) \in A \times A \mid$ the word y occurs at least as late in alphabetical order as the word $x\}$.

(b) $A =$ the set of all words of English, $R = \{(x, y) \in A \times A \mid$ the first letter of the word y occurs at least as late in the alphabet as the first letter of the word $x\}$.

(c) $A =$ the set of all countries in the world, $R = \{(x, y) \in A \times A \mid$ the population of the country y is at least as large as the population of the country $x\}$.

(3pts) 9. Draw the Hasse diagram for each of the partially ordered sets.

(a) $S = \{a, b, c\}$,
$\rho = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$.

(b) $S = \{a, b, c, d\}$,
$\rho = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c)\}$.

(c) $S = \{\emptyset, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{b\}\}$,
$\rho = \subseteq$. That is, $A \, \rho \, B \leftrightarrow A \subseteq B$.

Recommended Exercises

The remaining exercises in this assignment are recommended but will not be graded. Students are encouraged to solve these problems and ask questions about them. We will solve some of them in class.

10. (V 4.6.2) Find the set $\mathrm{Eq}(A)$ of all equivalence relations on the set $A = \{1, 2, 3\}$ and draw the Hasse diagram of the subset inclusion relation on $\mathrm{Eq}(A)$.

11. (V 4.2.10) Suppose R is a relation from A to B and S is a relation from B to C. Prove that $S \circ R = \emptyset$ iff $\mathrm{Ran}(R)$ and $\mathrm{Dom}(S)$ are disjoint.

12. (V 4.3.18) Suppose R and S are transitive relations on A. Prove that if $S \circ R \subseteq R \circ S$, then $R \circ S$ is transitive.

13. (V 4.4.17) If a subset of a partially ordered set has exactly one minimal element, must that element be a smallest element? Give either a proof or a counter example to justify your answer.

14. (V 4.4.23) Prove the following

    Theorem 4.4.11 Suppose A is a set, $\mathcal{F} \subseteq \mathcal{P}(A)$, and $\mathcal{F} \neq \emptyset$. Then the least upper bound of $\mathcal{F}$ (in the subset partial order) is $\cup\mathcal{F}$ and the greatest lower bound of $\mathcal{F}$ is $\cap\mathcal{F}$.

15. (V 4.6.19) Suppose R and S are equivalence relations on a set A. Let $T = R \cap S$.
    (a) Prove that T is an equivalence relation on A.
    (b) Prove that for all $x \in A$, $[x]_T = [x]_R \cap [x]_S$.

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