Remarks on Subset Notation

$A \subseteq B$ means that A is a subset of B. That is,

$$(\forall x)(x \in A \longrightarrow x \in B)$$

Notice that it is possible for A to be equal to B in this case.

Some authors use the notation $A \subset B$ to mean that A is a subset of B and $A \neq B$. That is, both

$$(\forall x)(x \in A \longrightarrow x \in B) \;\; \textit{and} \;\; (\exists y)(y \in B \wedge y \notin A).$$

In other words, $A \subset B$ means that A is *strictly* contained in B, so there is at least one element in B that is not in A; whereas $A \subseteq B$ means that A is contained in *or equal to* B.

This notation is completely analogous to notation you are already familiar with involving inequality. $A \leq B$ means that A is less than or equal to B, whereas $A < B$ means that A is *strictly* less than B.

Unfortunately, there is at least one well known book that uses $\subset$ to mean "subset or equal." So, personally, I never write $A \subset B$ because of the confusion that this inconsistent usage has caused. Instead, on the rare occasions when I want to emphasize that a set is not only a subset, but also a *proper* subset, then I might use $A \subsetneqq B$. This means *exactly the same thing* as what some authors denote by $A \subset B$.

To summarize $A \subsetneqq B$ is equivalent to $(A \subseteq B) \wedge (A \neq B)$. (In fact, we might just write this conjunction if we want to be exceedingly careful and clear.)

Another equivalent way to say this is "A is a *proper* subset of B." Alternatively, "A is *strictly* contained in B," which means the same thing.

Finally, whenever you see a line through an *entire* symbol, it means the negation of the statement involving that symbol without the line. So, for example,

- $a \notin A$ means that a is not an element of A;
- $A \not\subset B$ means that A is not a proper subset B;
- $A \nsubseteq B$ means that A is not a subset nor equal to B.