Math 654 Homework #2 (due on 9/16/2016)

Your name:

Problem 8. Prove that a set, X, is finite iff there is a relation such that it and its inverse order both well-order X.

Definition 2. A set of strings over alphabet A is a subset of $\bigcup_{n\in\omega} A^n$. In other words, it is a set of functions whose domains are natural numbers and whose ranges are contained in A. We denote by λ the unique function with domain 0. The alphabet will not be mentioned if it is clear from context or does not need to be specified. If S is a set of strings and $x, y \in S$, we write $x \preceq y$ if $y{\restriction}\mathrm{dom}(x) = x$. If $\neg(x \preceq y \vee y \preceq x)$, then we write $x \perp y$ and say that x and y are incomparable; otherwise, we write $x \parallel y$ and say that x and y are comparable. For a set of strings, S, $T[S] = \{x : (\exists y \in S)\big(x \preceq y\big)\}$ is the prefix closure of S.

Definition 3. Let S and P be two sets of strings.

- $P * S = \{xy : x \in P \wedge y \in S\}$.
- $P^{-1}S = \{y : (\exists x \in P)\big(xy \in S\big)\}$.

For notational simplicity, we define $x^{-1}S = \{x\}^{-1}S$, $P^{-1}x = P^{-1}\{x\}$, $x * S = \{x\} * S$ and $P * x = P * \{x\}$ for a string x.

Definition 4. Given a set of strings, S, we call $P \subseteq T[S]$ a *maximal antichain of* S if $(\forall x, y \in P)\big(x \perp y \vee x = y\big)$ and $(\forall x \in S)(\exists y \in P)(y \parallel x)$. P is a *valid antichain of* S if P is a maximal antichain of S and $(\forall x, y \in P)\big(x^{-1}T[S] = y^{-1}T[S]\big)$. We define, $\mathsf{Vac}(S) = \{P : P \text{ is a valid antichain of } S\}$.

Example 1. Consider the following set of strings over the alphabet $\{a, b\}$:

$$S = \{a^5, a^4b, a^2ba, a^2b^2, ba^4, ba^3b, baba, bab^2, b^2a^3, b^2a^2b, b^3a, b^4\}.$$

Graphically, we can represent S as a tree where branching left indicates an a and branching right indicates a b. In the picture below to the right, we highlight the four valid antichains of S: $P_0 = \{\lambda\}$, $P_1 = \{a^2, ba, b^2\}$, $P_2 = \{a^4, a^2b, ba^3, bab, b^2a^2, b^3\}$ and $P_3 = S$. Note that S is only a valid antichain of itself because it contains no comparable strings. The members of the four valid antichains are connected via dotted lines in the right picture (P_0 has only one member and therefore includes no dotted lines). For reference a maximal antichain that is not valid is included in the picture on the left and its members are joined with a dotted line.

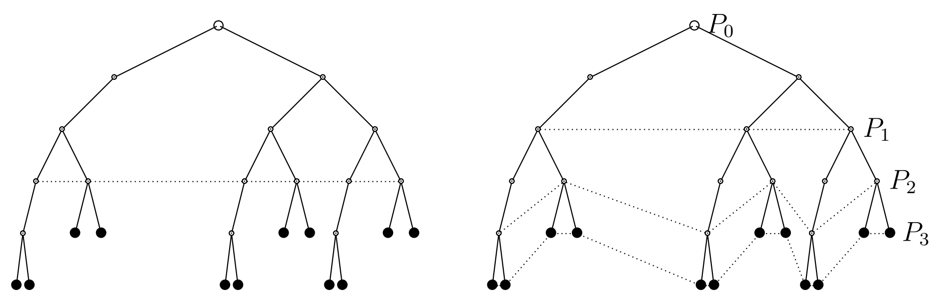

Figure 1. On the left, a maximal antichain that is not valid; on the right, all the valid antichains.

In the next figure, we focus on the valid antichain P_1.

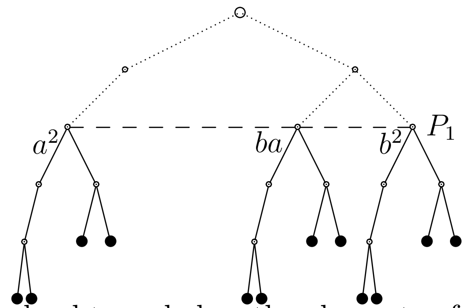

FIGURE 2. The identical subtrees below the elements of the valid antichain P_1.

Observe that the portions of the tree below each of a^2, ba and b^2 are identical; the terminal nodes of all three sub-trees are $\{a^3, a^2b, ab, b^2\}$. It is this equivalence of suffixes that makes P_1 a valid antichain.

Problem 9. Suppose that P is a valid antichain of a set of strings S and Q is a valid antichain of P. Prove that Q is a valid antichain of S.