Algebra Univers. 64 (2010) 15–20
DOI 10.1007/s00012-010-0082-3
Published online October 10, 2010

Algebra Universalis

A strong Mal'cev condition for locally finite varieties omitting the unary type

Mark H. Siggers

Abstract. We show that a finite algebra has a Taylor operation if and only if it has an operation satisfying a particular set of 6-ary Taylor identities. As a consequence we get the first strong Mal'cev condition for the family of locally finite varieties omitting the unary type. This is of interest to combinatorialists, as it is conjectured that a Constraint Satisfaction Problem defined by a core relational structure is polynomial time solvable exactly when a certain associated variety omits the unary type. Our result implies that the problem of deciding if a core relational structure meets this characterisation is itself in NP.

1. Introduction

In [6], Feder and Vardi redefined Constraint Satisfaction Problems (CSPs) in a graph theoretical setting by showing that they could be described as homomorphism problems for finite relational structures. This provided a natural class of restricted CSP: for any relational structure $\mathcal{H}$, CSP($\mathcal{H}$) is the problem of deciding whether or not a given relational structure $\mathcal{G}$, of the same type as $\mathcal{H}$, admits a homomorphism to $\mathcal{H}$. They conjectured that for any *relational structure* $\mathcal{H}$, CSP($\mathcal{H}$) is either polynomial time solvable or NP-complete.

In showing that the complexity of CSP($\mathcal{H}$) is determined by the algebra whose universe is the vertex set $V(\mathcal{H})$ and whose term operations are the *polymorphisms* of $\mathcal{H}$, Jeavons [9] brought a connection between CSPs and Universal Algebra onto firm footing. This connection was taken further in [5]. For a core relational structure $\mathcal{H}$, let $\mathbf{A}(\mathcal{H})$ be the algebra generated over the base set $V(\mathcal{H})$ by the idempotent polymorphisms of $\mathcal{H}$. In [5], Bulatov et. al. showed that for a core relational structure $\mathcal{H}$, CSP($\mathcal{H}$) is NP-complete if $\mathbf{A}(\mathcal{H})$ has any subalgebra with a non-trivial projective homomorphic image. They conjectured that CSP($\mathcal{H}$) is otherwise polynomial time solvable. We refer to the class of structures that they showed to be NP-complete as BJK. In [4] it was shown that a core $\mathcal{H}$ is in BJK if and only if, in the language of Tame Congruence Theory, the variety generated by $\mathbf{A}(\mathcal{H})$ admits Type 1, the unary type.

Presented by M. Valeriote.
Received December 2, 2008; accepted in final form January 6, 2009.
2000 *Mathematics Subject Classification*: 08B05.
Key words and phrases: Mal'cev condition, omitting Type 1, Taylor operation.
This work was supported by Kyungpook National University Research Fund.

This allows one to characterise the class BJK in terms of Mal'cev conditions for locally finite varieties. A *Taylor operation* is a d-ary operation ω ($d \geq 2$) that satisfies the idempotence identity $\omega(x, x, \ldots, x) = x$, and for each $i = 1, \ldots, d$, satisfies an identity of the form

$$\omega(-,-,\ldots,-,\underbrace{x}_{i},-,\ldots,-,-) = \omega(-,-,\ldots,-,\underbrace{y}_{i},-,\ldots,-,-), \tag{1}$$

where all empty spaces are filled with x or y.

A *weak near unanimity term* or *WNU term* is a d-ary term ω that satisfies the idempotence identity, and the identities

$$\omega(x, x, \ldots, x, x, y) = \omega(x, x, \ldots, x, y, x) = \cdots = \omega(y, x, \ldots, x, x, x). \tag{2}$$

In [8, Chapter 9], using [15], it was shown that a locally finite variety omits the unary type if and only if it has a Taylor operation. Recently, in [12], it was shown that a locally finite variety omits the unary type if and only if it has a WNU operation. Thus a core relational structure $\mathcal{H}$ is in BJK if and only if it has no *Taylor polymorphisms*, or equivalently, no *WNU polymorphisms.* These results have been used to great effect in problems of CSP complexity. See, for example, [10] and [1].

As useful as the algebraic approach has been to CSPs, we felt there was room for a graph theoretic characterisation of the class BJK. In [14], we showed that $\mathcal{H}$ is in BJK if and only if it has a K_3-partition, that is, if $\mathrm{CSP}(K_3)$ can be reduced to $\mathrm{CSP}(\mathcal{H})$ by a specific construction called the *fibre construction.* The main motivation of this paper was to give a single combinatorial proof that $\mathrm{CSP}(\mathcal{H})$ is NP-complete for any structure $\mathcal{H}$ whose core is in BJK.

In the current note, we use the fibre construction to show the surprising fact that there is a *strong* Mal'cev condition for the family of locally finite varieties omitting the unary type. We see this as further justification for our combinatorial approach.

Our main result is the following.

Theorem 1.1. *A finite algebra* $\mathbf{A}$ *admits Taylor operations if and only if it admits an idempotent* 6*-ary operation satisfying the identities*

$$\begin{aligned} \omega(x, x, x, x, y, y) &= \omega(x, y, x, y, x, x) \text{ and} \\ \omega(y, y, x, x, x, x) &= \omega(x, x, y, x, y, x). \end{aligned} \tag{3}$$

As one can verify that a given 6-ary function on $V(\mathcal{H})$ is an operation of $\mathbf{A}(\mathcal{H})$ in time polynomial in $|V(\mathcal{H})|$, we immediately get the following corollary.

Corollary 1.2. *The decision problem "Is a given core relational structure in* BJK*?" is in co-NP.*

The two Mal'cev conditions given above for the family of locally finite varieties omitting the unary type, both consist of an infinite family of identities. A Mal'cev condition is *strong* if it consists of a finite set of equations. Using our theorem, standard techniques will give us the first strong Mal'cev condition for locally finite varieties omitting the unary type.

Corollary 1.3. *A locally finite variety omits the unary type if and only if it has an idempotent operation ω satisfying* (3)*.*

I would like to express my gratitude to László Zádori for informing me that this result is of great interest to universal algebraists.

2. Proofs

Before we get to the proofs, we need to recall a definition and a result.

Definition 2.1. Given a finite base set V, a d-ary relation on V is a subset of V^d. A d-ary relation R on V is an *invariant* relation of an algebra $\mathbf{A}$ with universe V if for each operation f of $\mathbf{A}$ and each set of m d-tuples $\overline{v_1}, \dots, \overline{v_m}$ from R, where $\overline{v_i} = (v_{i,1}, \dots, v_{i,d})$, the d-tuple

$$(f(v_{1,1}, \dots, v_{m,1}), \dots, f(v_{1,d}, \dots, v_{m,d}))$$

is in R.

The following rephrasing of the fact that CSP(H) is NP-complete for any graph H containing a triangle follows from Bulatov's reproof [3] of Hell and Nešetřil's H-colouring Dichotomy [7].

Lemma 2.2. *If some irreflexive symmetric binary relation (i.e. graph) containing a triangle is an invariant relation of a finite algebra* $\mathbf{A}$*, then* $\mathbf{A}$ *omits Taylor operations.*

The idea of the proof of Theorem 1.1 is that if a relational structure $\mathcal{H}$ omits idempotent polymorphisms satisfying (3), then we can use the fibre construction of [14] to reduce CSP(T) for some graph T containing a triangle, to CSP($\mathcal{H}$). To avoid the definitions associated with the fibre construction, we work directly with algebras.

Proof of Theorem 1.1. Let $\mathbf{A}$ be a finite algebra admitting an idempotent operation satisfying (3). This operation is a Taylor operation, so $\mathbf{A}$ admits Taylor operations.

Now let $\mathbf{A}$ be a finite idempotent algebra omitting operations that satisfy (3), and let $A = \{v_1, \dots, v_n\}$ be the base set of $\mathbf{A}$. Let W^1 and W^2 be the ordered subsets of $w = 4 \cdot \binom{n}{2}$ elements of A^6 shown in Figure 1.

Consider the algebra $\mathbf{A}^w$, viewing each base element as a mapping from an ordered w-element set to A. Define a symmetric binary relation T on $\mathbf{A}^w$ by letting $(f, g) \in T$ if there exist idempotent term operations $s, t \colon \mathbf{A}^6 \to \mathbf{A}$ such that $s|_{W^1} = f$, $s|_{W^2} = g$, $t|_{W^1} = g$, and $t|_{W^2} = f$. It is clear that this relation is invariant under the term operations of $\mathbf{A}^w$. Indeed let ϕ be a d-ary

$$W^1 = \Big(\begin{array}{llllll}
(v_1, & v_1, & v_1, & v_1, & v_2, & v_2), \\
(v_2, & v_2, & v_1, & v_1, & v_1, & v_1), \\
(v_2, & v_2, & v_2, & v_2, & v_1, & v_1), \\
(v_1, & v_1, & v_2, & v_2, & v_2, & v_2), \\ \cline{2-5}
(v_1, & v_1, & v_1, & v_1, & v_3, & v_3), \\
(v_3, & v_3, & v_1, & v_1, & v_1, & v_1), \\
(v_3, & v_3, & v_3, & v_3, & v_1, & v_1), \\
(v_1, & v_1, & v_3, & v_3, & v_3, & v_3), \\ \cline{2-5}
 & \multicolumn{4}{c}{\cdots} & \\ \cline{2-5}
(v_{n-1}, & v_{n-1}, & v_{n-1}, & v_{n-1}, & v_n, & v_n), \\
(v_n, & v_n, & v_{n-1}, & v_{n-1}, & v_{n-1}, & v_{n-1}), \\
(v_n, & v_n, & v_n, & v_n, & v_{n-1}, & v_{n-1}), \\
(v_{n-1}, & v_{n-1}, & v_n, & v_n, & v_n, & v_n)
\end{array} \Big)$$

$$W^2 = \Big(\begin{array}{llllll}
(v_1, & v_2, & v_1, & v_2, & v_1, & v_1), \\
(v_1, & v_1, & v_2, & v_1, & v_2, & v_1), \\
(v_2, & v_1, & v_2, & v_1, & v_2, & v_2), \\
(v_2, & v_2, & v_1, & v_2, & v_1, & v_2), \\ \cline{2-5}
(v_1, & v_3, & v_1, & v_3, & v_1, & v_1), \\
(v_1, & v_1, & v_3, & v_1, & v_3, & v_1), \\
(v_3, & v_1, & v_3, & v_1, & v_3, & v_3), \\
(v_3, & v_3, & v_1, & v_3, & v_1, & v_3), \\ \cline{2-5}
 & \multicolumn{4}{c}{\cdots} & \\ \cline{2-5}
(v_{n-1}, & v_n, & v_{n-1}, & v_n, & v_{n-1}, & v_{n-1}), \\
(v_{n-1}, & v_{n-1}, & v_n, & v_{n-1}, & v_n, & v_{n-1}), \\
(v_n, & v_{n-1}, & v_n, & v_{n-1}, & v_n, & v_n), \\
(v_n, & v_n, & v_{n-1}, & v_n, & v_{n-1}, & v_n)
\end{array} \Big)$$

FIGURE 1. Subsets W^1 and W^2 of A^6: in each of W^1 and W^2, the same four 6-tuples are repeated for each pair of elements in A.

operation of $\mathbf{A}^w$. Let (f_i, g_i) be in T for $i = 1, \ldots, d$ and let s_i and t_i be the 6-ary operations of $\mathbf{A}$ witnessing this. Then

$$\bar{s} = \phi \circ (s_1 \times \cdots \times s_d) \quad \text{and} \quad \bar{t} = \phi \circ (t_1 \times \cdots \times t_d)$$

are 6-ary operations witnessing that $(\phi(f_1, \ldots, f_d), \phi(g_1, \ldots, g_d))$ is in T. That is, for example,

$$\bar{s}|_{W^1} = \phi(s_1|_{W^1}, \ldots, s_d|_{W^1}) = \phi(f_1, \ldots, f_d).$$

Because $\mathbf{A}$ omits 6-ary operations satisfying (3), any idempotent term operation $s\colon \mathbf{A}^6 \to \mathbf{A}$ restricts to different functions on W^1 and W^2, so T is loopless. Furthermore, the six projections of $\mathbf{A}^6$ to $\mathbf{A}$ witness a triangle in T

on the vertices

$$t_1 = (v_1, v_1, v_2, v_2,\ v_2, v_2, v_1, v_1,\ v_1, v_1, v_3, v_3,\ \ldots,\ v_n, v_n, v_{n-1}, v_{n-1}),$$
$$t_2 = (v_2, v_1, v_1, v_2,\ v_1, v_2, v_2, v_1,\ v_3, v_1, v_1, v_3,\ \ldots,\ v_{n-1}, v_n, v_n, v_{n-1}),\ \text{and}$$
$$t_3 = (v_1, v_2, v_2, v_1,\ v_2, v_1, v_1, v_2,\ v_1, v_3, v_3, v_1,\ \ldots,\ v_n, v_{n-1}, v_{n-1}, v_n).$$

Indeed we have, for example, that $\pi_1|_{W^1} = \pi_3|_{W^2} = t_3$ and $\pi_1|_{W^2} = \pi_3|_{W^1} = t_1$, which witnesses that (t_1, t_3) is in T. As T is invariant under the term operations of $\mathbf{A}^w$, Lemma 2.2 implies that $\mathbf{A}^w$ omits Taylor operations. As $\mathbf{A}^w$ is in the variety generated by $\mathbf{A}$, $\mathbf{A}$ also omits Taylor operations. □

Proof of Corollary 1.3. Let $\mathcal{V}$ be a locally finite variety. If $\mathcal{V}$ admits the unary type, then it omits Taylor operations, so omits operations satisfying (3).

On the other hand, assume $\mathcal{V}$ omits operations satisfying (3). We may further assume that $\mathcal{V}$ is idempotent, (see [8, Chapter 9]). As (3) is an identity in 2 variables, the algebra $F = F_{\mathcal{V}}(\{x, y\})$ generated freely over two variables in $\mathcal{V}$ also omits term operations satisfying (3). As $\mathcal{V}$ is locally finite, F is finite, so omits Taylor operations by Theorem 1.1. Thus $\mathcal{V}$ admits the unary type. □

3. Concluding remarks

After sending an initial version of this result to Ralph McKenzie [13], he soon sent back an alternate proof. Petar Marković [11] then observed that his proof could be altered to replace the 6-ary Taylor identities (3) with any of several pairs of 4-ary Taylor identities. This replacement works with the present proof also, with only small changes. It would depend on [1] and [12] for an analogue of Lemma 2.2 instead of [3].

One such pair of 4-ary identities is the following:

$$\omega(y, y, x, x) = \omega(x, x, x, y) \quad \text{and} \quad \omega(x, x, x, y) = \omega(y, x, y, x). \tag{4}$$

An idempotent term that satisfies the identities

$$\omega(y, y, x, x) = \omega(x, x, x, y) = \omega(y, x, y, x) = x$$

was called a 3-*edge term* in [2]. In analogy to the relationship between weak near-unanimity functions and near-unanimity functions, it would seem appropriate to call an idempotent term satisfying (4) a *weak* 3-*edge term.* I thank an anonymous referee for bringing this to my attention.

References

[1] Barto, L., Kozik, M., Niven, T.: The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to the conjecture of Bang-Jensen and Hell). SIAM J. Comput. **38**, 1782–1802 (2008/09)
[2] Berman, J., Idziak, P., Marković, P., McKenzie, R., Valeriote, M., Willard, R.: Varieties with few subalgebras of powers. Trans. Amer. Math. Soc. **362**, 1445–1473 (2010)
[3] Bulatov, A.: *H*-coloring dichotomy revisited. Theoret. Comput. Sci. **349**, 31–39 (2005)

[4] Bulatov, A., Jeavons, P.: Algebraic structures in combinatorial problems. Tech. Rep. MATH-AL-4-2001, Technische universitt Dresden, Dresden, Germany
[5] Bulatov, A., Jeavons, P., Krokhin, A.: Classifying the complexity of constraints using finite algebras. SIAM J. Comput. **34**, 720–742 (2005)
[6] Feder, T., Vardi, M.: The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory. SIAM J. Comput. **28**, 57–104 (1999). doi: 10.1137/S0097539794266766
[7] Hell, P., Nešetřil, J.: On the complexity of H-colouring. J. Combin. Theory B **48**, 92–100 (1990)
[8] Hobby, D., McKenzie, R.: The Structure of Finite Algebras. Contemporary Mathematics, vol. 76. American Mathematical Society, Providence (1988)
[9] Jeavons, P.G.: On the algebraic structure of combinatorial problems. Theoret. Comput. Sci. **200**, 185–204 (1998). doi: 10.1016/S0304-3975(97)00230-2
[10] Larose, B.: Taylor operations on finite reflexive structures. Int. J. Math. Comput. Sci. **1**, 1–26 (2006)
[11] Markovič, P.: Personal communication
[12] Maróti, M., McKenzie, R.: Existence theorems for weakly symmetric operations. Algebra Universalis **59**, 463–489 (2008). doi: 10.1007/s00012-008-2122-9
[13] McKenzie, R.: Personal communication
[14] Nešetřil, J., Siggers, M., Zadori, L.: A combinatorial CSP dichotomy classification conjecture. European Journal of Combinatorics **31**, 280-296 (2010)
[15] Taylor, W.: Varieties obeying homotopy laws. Canad. J. Math. **29**, 498–527 (1977)

Mark H. Siggers
Department of Mathematics, College of Natural Sciences, Kyungpook National University, 1370 Sankyuk-dong, Buk-gu, Daegu 702-701, Republic of Korea
e-mail: mhsiggers@knu.ac.kr