

# Basic Stats

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# Preface: Motivation

All the notes I have done here are about basic stats. While I have tried my best, probably there are still some typos and errors. Please feel free to let me know in case you find one. Thank you!



# Chapter 1

## 443

### 1.1 Some basic concepts

#### 1.1.1 Random variable

Any numerical realization of a random experiment results in a random variable.

In layman terms, it is a number that has a certain probability attached to it.

#### 1.1.2 Permutation

An ordered arrangement of a set of objects is known as a permutation.

e.g., The number of permutations of  $n$  distinguishable objects is  $n!$ . e.g., The number of permutations of  $n$  distinct objects taken  $r$  at a time is

$${}_nP_r = \frac{n!}{(n-r)!}$$

#### 1.1.3 Combinations

If the order of the objects is not important, then one may simply be interested in the number of combinations.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

#### 1.1.4 Partitioning

The number of ways of partitioning a set of  $n$  objects into  $k$  cells with  $r_1$  objects into the first cell,  $r_2$  in the second cell, and so forth is

$$\frac{n!}{r_1!r_2!\dots r_k!}$$

## 1.2 Discrete Random Variables

### 1.2.1 Binomial

$$X \sim \text{BIN}(n, p)$$

$$\binom{n}{x} P^x (1 - P)^{n-x}$$

mean:  $np$

variance:  $npq$

Support (or, possible values):  $0, 1, 2, \dots, n$

(Note that, Bernoulli is written as  $\text{BIN}(1, p)$ )

### 1.2.2 Poisson

$$X \sim \text{POI}(\mu)$$

$$\frac{e^{-\mu} \mu^x}{x!}$$

mean:  $\mu$

variance:  $\mu$

Support (or, possible values):  $0, 1, 2, \dots$  (i.e., 0+ positive integer; or Natural number)

## 1.3 Continuous Random Variables

### 1.3.1 Uniform

$$X \sim \text{UNIF}(a, b)$$

$$\frac{1}{b - a}$$

Mean:  $\frac{a+b}{2}$

Variance:  $\frac{(b-a)^2}{12}$

Support (or, possible values):  $a < x < b$



### 1.3.2 Exponential

$$X \sim EXP(\theta)$$

$$\frac{1}{\theta} e^{-x/\theta}$$

Mean:  $\theta$

Variance:  $\theta^2$

Support (or, possible values):  $0 < x$

### 1.3.3 Normal

$$X \sim N(\mu, \sigma^2)$$

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Mean:  $\mu$

Variance:  $\sigma^2$

Support (or, possible values):  $-\infty < x < +\infty$

## 1.4 Large Sample Theory

### 1.4.1 Convergence in distribution

[https://en.wikipedia.org/wiki/Law\\_of\\_large\\_numbers](https://en.wikipedia.org/wiki/Law_of_large_numbers)

$$\bar{X} \rightarrow \mu \quad (n \rightarrow \infty)$$

$$Var(\bar{X}) = Var\left(\frac{1}{n}(X_1 + \dots + X_n)\right) = \frac{1}{n^2} Var(X_1 + \dots + X_n) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

### 1.4.2 Weak law

There are two different versions of the Law of Large Numbers: Strong law of large numbers and Weak law of large numbers.

The weak law of large numbers: The sample average converges **in probability** towards the expected value.

$$\bar{X}_n \xrightarrow{p} \mu \quad (n \rightarrow \infty)$$

This, for any positive number  $\epsilon$

$$\lim_{n \rightarrow \infty} Pr(|\bar{X}_n - \mu| > \epsilon) = 0$$

### 1.4.3 Strong law

<http://maxim.ece.illinois.edu/teaching/fall12/handouts/LLN.pdf>

The sequence of sample mean  $\bar{X}_n$  converges to  $\mu$  **almost surely**.

$$\bar{X}_n \xrightarrow{a.s.} \mu \quad (n \rightarrow \infty)$$

This is,

$$Pr(\lim_{n \rightarrow \infty} \bar{X}_n = \mu) = 1$$

### 1.4.4 Central limit theorem

If  $X_1, \dots, X_n$  is a random sample from a distribution with mean  $\mu$  and variance  $\sigma^2 < \infty$ , then the limiting distribution of

$$Z_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}$$

is the standard normal,  $Z_n \xrightarrow{d} Z \sim N(0, 1)$  as  $n \rightarrow \infty$ .

#### 1.4.4.1 Bernoulli law of large number

$\hat{p}_n = \frac{Y_n}{n}$  converges stochastically to  $p$  as  $n$  approaches infinity. For example, if a coin is tossed repeatedly, and  $A = \{H\}$ , then the successive relative frequencies of  $A$  correspond to a sequence of random variables that will converge stochastically to  $p = 1/2$ .

#### 1.4.4.2 Normal approximation to Binomial

$$Z_n = \frac{Y_n - np}{\sqrt{npq}} \xrightarrow{d} Z \sim N(0, 1)$$

Example: The probability that a basketball player hits a shot is  $p = 0.5$ . If he takes 20 shots, what is the probability that he hits at least 9?

$$\begin{aligned} P[Y_{20} \geq 9] &= 1 - P[Y_{20} < 9] \\ &= 1 - \sum_{y=0}^8 \binom{20}{y} 0.5^y 0.5^{20-y} \\ &= 0.7483 \end{aligned}$$

A normal approximation is

$$\begin{aligned} P[Y_{20} \geq 9] &= 1 - P[Y_{20} < 8] \\ &= 1 - \Phi\left(\frac{8 - 10}{\sqrt{5}}\right) \\ &= 0.8133 \end{aligned}$$

#### 1.4.4.3 Normal approximation to Poisson

$$\begin{aligned} P[10 \leq Y_{20} \leq 30] &= P[Y_{20} \leq 30] - P[Y_{20} \leq 10] \\ &= \Phi\left[\frac{30.5 - 20}{\sqrt{20}}\right] - \Phi\left[\frac{9.5 - 20}{\sqrt{20}}\right] \\ &= 0.981 \end{aligned}$$

### 1.4.5 Poisson approximation to binomial

We know that the mean for binomial is

$$\mu = np \rightarrow p = \frac{\mu}{n}$$

The moment generating function for Binomial is

$$\begin{aligned} M_n(t) &= (1 - p + pe^t)^n = \left(1 + \frac{\mu(e^t - 1)}{n}\right)^n \\ \lim_{n \rightarrow \infty} M_n(t) &= e^{\mu(e^t - 1)} \end{aligned}$$

Note that the MGF for Poisson is as follows.

$$POI(\lambda) : e^{\lambda(e^t - 1)}$$

Thus,

$$Y_n \rightarrow Y \sim POI(\mu)$$



# Chapter 2

## 556

### 2.1 Statistics and Sampling Distributions

#### 2.1.1 Statistics

##### 2.1.1.1 Definition of Statistic

*P.264*

A function of observable random variables,  $T = t(X_1, \dots, X_n)$ , which does not depend on any unknown parameters is called statistic.

For example, let  $X_1, \dots, X_n$  represent a random sample from a population with pdf  $f(x)$ . The sample mean provides an example of a statistic with the function

$$t(x_1, \dots, x_n) = (x_1 + \dots + x_n)/n$$

This statistic usually is denoted by

$$\bar{X} = \sum_{i=1}^n \frac{X_i}{n}$$

When a random sample is observed, the value of  $\bar{X}$ , computed from the data, usually is denoted by lower case  $\bar{x}$ .

$$\bar{x} = \sum_{i=1}^n \frac{x_i}{n}$$

### 2.1.1.2 Sample and parameters

P.265

If  $X_1, \dots, X_n$  denotes a random sample from  $f(x)$  with  $E(X) = \mu$  and  $var(X) = \sigma^2$ , then

$$E(\bar{X}) = \mu$$

$$Var(\bar{X}) = \frac{\sigma^2}{n}$$

#### Example:

A random sample of size  $n$  from a Bernoulli distribution  $X_i \sim BIN(1, p)$ . We know Bernoulli has  $\mu = p$  and  $\sigma^2 = pq$ . In this case, the sample mean is

$$\bar{X} = Y/n = \hat{p}$$

Thus,

$$E(\hat{p}) = p$$

$$Var(\hat{p}) = \frac{pq}{n}$$

Thus, sample mean is the unbiased estimate for the population mean. However, the variance of the mean is not equal to population variance. That lead to definition of sample variance.

P.266

Sample variance:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

$$E(S^2) = \sigma^2$$

(Question: does it mean that sample variance is an unbiased estimator of population variance?)

**2.1.2**  $\chi^2, t, F, \text{beta}$ **2.1.2.1**  $\chi^2$ *P.271*

If  $X_1, \dots, X_n$  denotes a random sample from  $N(\mu, \sigma^2)$ , then

$$\sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2(1)$$

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2(n-1)$$

(Thus, we can see this is a bit weird, as the numerator is  $\bar{X}$  is from the sample, whereas  $\sigma^2$  is from the population. Thus, assume we know  $\sigma^2$ ?)

Thus, we can

$$\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$$

(You can compare  $\bar{X}$  with  $\mu$ , we can see the only difference is that the  $\chi^2$  has one less degree of freedom because we use this degree of freedom to calculate the mean.)

For the mean and variance of  $\chi^2$ :

Assume that

$$X \sim \chi^2(v)$$

$$\text{mean} : v$$

$$\text{variance} : 2v$$

**2.1.2.2**  $t$ 

Definition

$$t(k) = \frac{N(0, 1)}{\sqrt{\frac{\chi^2(k)}{k}}}$$

**Property 1**

$t$  distribution is symmetrical.

Given that  $t$  distribution is symmetrical, we can get

$$H(-c) = 1 - H(c)$$

### Property 2

$t$  distribution has heavier tails than the normal.

My note:  $t$  distribution only has a parameter of  $k$ , which is determined by the  $\chi^2$ 's degree of freedom. Of course,  $\chi^2$  also only has one parameter, namely the degree of freedom.

#### 2.1.2.3 $F$

If  $V_1 \sim \chi^2(v_1)$  and  $V_2 \sim \chi^2(v_2)$  are independent, then the random variable

$$\frac{V_1/v_1}{V_2/v_2} \sim F(v_1, v_2)$$

#### 2.1.2.4 Beta

If  $X \sim F(v_1, v_2)$

$$Y = \frac{(v_1/v_2)X}{1 + (v_1/v_2)X} \sim \text{Beta}(\alpha, \beta)$$

### 2.1.3 Large-sample approximations

*P.280*

If  $Y_v \sim x^2(x)$ , then

$$Z_v = \frac{Y_v - v}{\sqrt{2v}} \xrightarrow{d} Z \sim N(0, 1)$$

(The proof is based on CLT. In addition, as discussed above, we can get chi-square from normal distributions.)

## 2.2 Point Estimation

### 2.2.1 Method of moments estimators

#### 2.2.1.1 Definition about moments (chapter 2)

*P.73*



The **kth moment about the origin** of a random variable  $X$  is

$$\mu_k' = E(X^k)$$

and the **kth moment about the mean** is

$$\mu_k = E[X - E(X)]^k = E(X - \mu)^k$$

Thus,  $\mu_k' = E(X^k)$  may be considered as the  $k$ th moment of  $X$  or the first moment of  $X^k$ .

The first moment about the mean is zero,

$$\mu_1 = E[X - E(X)] = E(X) - E(X) = 0$$

The second moment about the mean is the variance,

$$\mu_2 = E[(X - \mu)^2] = \sigma^2$$

Note that the definition of variance:

*P.73*

$$Var(X) = E[(X - \mu)^2]$$

( **Note: Based on the information above, it seems important to understand the difference between the moment of the origin and the moment of the mean.** )

### 2.2.1.2 Definition

Based on the last chapter (i.e., Chapter 8), sample mean  $\bar{X}$  is an estimator of the population mean  $\mu$ . A more general approach, which produced estimators known as the **method of moments estimators (MMEs)**, can be developed.

If  $X_1, \dots, X_n$  is a random sample from  $f(x; \theta_1, \dots, \theta_k)$ , the first  $k$  sample moments are given by

$$M_j' = \frac{\sum_{i=1}^n X_i^j}{n}$$

where,

$$j = 1, 2, \dots, k$$

**Example 1:***P.291*

Consider a random sample from a distribution with two unknown parameters, the mean  $\mu$  and the variance  $\sigma^2$ . We know from earlier considerations that  $\mu = \mu'_1$  and  $\sigma^2 = E(X^2) - \mu^2 = \mu'_2 - (\mu'_1)^2$ .

Thus,

$$\hat{\sigma}^2 = \mu'_2 - (\mu'_1)^2 = \frac{\sum_{i=1}^n X_i^2}{n} - \bar{X}^2 = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n}$$

(Thus, we can see that the MME estimation of  $\sigma^2$  is not the same as the definition of sample variance  $S^2$ .  $\hat{\sigma}^2 = \frac{n-1}{n} S^2$ . So, the MME estimator is not an unbiased one?)

**Example 2:***P.292*

If a sample is from a Gamma distribution  $X_i \sim GAM(\theta, k)$ , and we want to estimate the  $\theta$  and  $k$ .

We know that for Gamma distribution, the mean is  $k\theta$ , and the variance is  $k\theta^2$ .

We also know that

$$\mu'_1 = \mu = k\theta$$

and

$$\mu'_2 = \sigma^2 + \mu^2 = k\theta^2 + k^2\theta^2 = k\theta^2(1 + k)$$

.

Thus, we can get

$$\bar{X} = k\theta$$

$$\sum \frac{X_i^2}{n} = k\theta^2(1 + k)$$

Thus, we can get

$$\hat{\theta} = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n\bar{X}} = \frac{(n-1)/n S^2}{\bar{X}}$$

$$\hat{k} = \frac{\bar{X}}{\hat{\theta}}$$

( Note: To sum up, (1) we know how to calculate moments based on sample. And, (2) we know that connection between moment and parameter of mean and variance. Thus, combining (1) and (2), we can get the estimators for the parameters. )

### 2.2.1.3 Property

The joint MGF of  $(X_1, \dots, X_n)$  is defined as  $M(t_1, \dots, t_n) = E(e^{\sum_{i=1}^n t_i X_i})$

When  $X_1, \dots, X_n$  are independent if and only if

$$M(t_1, \dots, t_n) = \prod_{i=1}^n M_{X_i}(t_i)$$

where  $M_{X_i}(t_i)$  is the MGF of  $X_i$

### 2.2.1.4 Well-known MGF

- (1) Bernoulli with success probability  $p$ :  $1 - p + pe^t$
- (2) Binomial  $Bin(n, p)$ :  $(1 - p + pe^t)^n$
- (3) Poisson  $POI(\lambda)$ :  $e^{\lambda(e^t - 1)}$
- (4) Normal  $N(\mu, \sigma^2)$ :  $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$
- (5) Gamma  $GAM(\theta, k)$ :  $(1 - \theta t)^{-k}$

**Two special cases:**

- (6) Chi-square  $\chi^2(v) = GAM(2, \frac{v}{2})$ :  $(1 - 2t)^{-\frac{v}{2}}$
- (7) Exponential  $EXP(\theta) = GAM(\theta, 1)$ :  $(1 - \theta t)^{-1}$

## 2.2.2 least squares estimators

## 2.2.3 likelihood function and maximum likelihood estimators

### 2.2.3.1 Likelihood function

*P.293*

The joint density function of  $n$  random variables  $X_1, \dots, X_n$  evaluated at  $x_1, \dots, x_n$ , say  $f(x_1, \dots, x_n; \theta)$ , is referred to as a *likelihood function*.

### 2.2.3.2 Maximum likelihood estimators

P.294

Let  $L(\theta) = f(x_1, \dots, x_n; \theta)$ ,  $\theta \in \Omega$ , be the joint pdf of  $X_1, \dots, X_n$ . For a given set of observations,  $(x_1, \dots, x_n)$ , a value  $\hat{\theta}$  in  $\Omega$  at which  $L(\theta)$  is a maximum is called a *maximum likelihood estimate (MLE)* of  $\theta$ . That is  $\hat{\theta}$  is a value of  $\theta$  that satisfies

$$f(x_1, \dots, x_n; \theta) = \text{MAX}_{\theta \in \Omega} f(x_1, \dots, x_n; \theta)$$

### 2.2.4 Invariance property of MLEs

P.296

If  $\hat{\theta}$  is the MLE of  $\theta$  and if  $u(\theta)$  is a function of  $\theta$ , then  $u(\hat{\theta})$  is an MLE of  $u(\theta)$ .

#### Example 1:

We know that the *pdf* of exponential distribution ( $X \sim \text{EXP}(\theta)$ ) is as follows:

$$\frac{1}{\theta} e^{-\frac{x}{\theta}}$$

Thus, its likelihood function is as follows

$$L(\theta) = \frac{1}{\theta^n} e^{-\sum \frac{x_i}{\theta}}$$

Thus, log-likelihood is as follows.

$$\ln L(\theta) = -n \ln(\theta) - \frac{\sum X_i}{\theta}$$

Thus,

$$\frac{d}{d\theta} \ln L(\theta) = -n \frac{1}{\theta} + \frac{\sum X_i}{\theta^2}$$

Thus, we can get the *MLE* for  $\theta$  is  $\hat{\theta} = \bar{x}$ .

If we want to estimate  $\tau(\theta) = P(X \geq 1)$ :

$$\tau(\theta) = P(X \geq 1) = \int_1^\infty \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = - \int_1^\infty e^{-\frac{x}{\theta}} d\left(-\frac{x}{\theta}\right) = -[e^{-\frac{x}{\theta}}]_1^\infty = -[0 - e^{-\frac{1}{\theta}}] = e^{-\frac{1}{\theta}}$$

Thus, based on the invariance property, we know that the *MLE* for  $\tau(\theta)$  is as follows.

$$e^{-\frac{1}{\bar{x}}}$$

**Example 2: MLE vs. MME***P.296*

Assume a random sample from a two-parameter exponential distribution,  $X_i \sim EXP(1, \eta)$ . Thus, the pdf is  $e^{-(x-\eta)}$ . Thus, the likelihood function is

$$L(\eta) = e^{-\sum (x_i - \eta)}$$

Thus, the log likelihood,

$$\ln L(\eta) = -\sum (x_i - \eta) = n\eta - n\bar{X}$$

Thus, we know that as  $\eta$  increases, the log likelihood increases accordingly. Thus, we want to find the maximum  $\eta$ . Note that a two-parameter exponential distribution has the support of  $x_i \geq \eta$ . Thus, all  $\eta$  are smaller than any  $X_i$ . Thus, we can get the ML estimator is the first order statistic

$$\hat{\eta} = X_{1:n}$$

**Note that** the estimators above is based on ML. What would be the answer if using MME?

We know that for a two-parameter exponential distribution, its mean is  $\mu = 1 + \eta$ . And, we know that based on MME,  $\mu = \bar{X}$ . Thus, we can get the following,

$$\hat{\eta} = \bar{X} - 1$$

**Conclusion**

We can see that ML and MME have different estimators for the same  $\eta$  for a two-parameter exponential distribution.

**2.2.5 Unbiased estimators**

An estimator  $T$  is said to be an unbiased estimator of  $\tau(\theta)$  if

$$E(T) = \tau(\theta)$$

for all  $\theta \in \Omega$ . Otherwise, we said that  $T$  is biased stimator of  $\tau(\theta)$ .

For instance, if we want to estimate a percentile, say the 95th percentile of  $N(\mu, 9)$ . Note that the percentiles that we know are about standardized noraml

(i.e.,  $N(0, 1)$ ). Thus, we need to have some calculation to get the non-standard one.

$$\frac{X_{95 \text{ percentile}} - \mu}{\sigma} = 1.645$$

Thus, we can get

$$X_{95 \text{ percentile}} = 1.645 \times \sigma + \mu$$

We know that  $\bar{X}$  is the unbiased estimate for  $\mu$ . Thus, we can get

$$X_{95 \text{ percentile}} = 1.645 \times \sigma + \mu = 4.94 + \mu$$

We know that

$$E(T) = E(\bar{X} + 4.94) = \mu + 4.94$$

Thus,  $T = \bar{X} + 4.94$  is the unbiased estimator of  $\tau(\mu) = \mu + 4.94$ .

### 2.2.6 Unbiased estimators vs. Invariance property of MLEs

**Do not apply “Invariance property of MLEs” to the Unbiased estimators.**

( **Note that:** You can apply the Invariance property to “Unbiased estimators” when it is a linear combination. In that case,  $E(a\theta + b) = aE(\theta) + b$ . Thus, if you find a  $T$  that is a unbiased estimator for  $\theta$ , it should be unbiased estimator for  $a\theta + b$  as well. Thus, note that  $\frac{1}{\theta}$  is not a linear combination of  $\theta$ , thus  $\frac{1}{\theta}$  has a very different estimator, compared to  $\theta$ .)

*P.303*

For example, consider a random sample of size  $n$  from an exponential distribution,  $X_i \sim EXP(\theta)$ , with parameter  $\theta$ . We know that,  $\bar{X}$  is unbiased for  $\theta$  (which is the mean of an exponential distribution).

If we want to estimate  $\tau(\theta) = \frac{1}{\theta}$ , then by the invariance property of MLE is  $T_1 = \frac{1}{\bar{X}}$ .

However,  $T_1$  is a biased estimators of  $\frac{1}{\theta}$ . Specifically,

We know that if  $X \sim Gam(\theta, k)$ , then  $Y = \frac{2X}{\theta} \sim x^2(2k)$ . We know that exponential distributions are a special case of Gamma distribution,  $EXP(\theta) = Gam(\theta, 1)$ , thus we get the following,

$$Y = \frac{2n\bar{X}}{\theta} = \frac{2n}{\theta} \frac{\sum_{i=1}^n X_i}{n} = \frac{\sum_{i=1}^n 2X_i}{\theta} \sim \sum_{i=1}^n x^2(2 \cdot 1) = \sum_{i=1}^n x^2(2) = x^2(2n)$$

We further know that if  $Y \sim x^2(v)$ ,  $E(Y^r) = 2^r \frac{\Gamma(v/2+r)}{\Gamma(v/2)}$ . Thus, we know that,

$$E(Y^{-1}) = 2^{-1} \frac{\Gamma(2n/2 - 1)}{\Gamma(2n/2)} = \frac{1}{2} \cdot \frac{1}{n-1}$$

Thus,

$$E(Y^{-1}) = E\left(\frac{\theta}{2n\bar{X}}\right) = \frac{\theta}{2n} E\left(\frac{1}{\bar{X}}\right) = \frac{1}{2} \cdot \frac{1}{n-1}$$

Thus,

$$E\left(\frac{1}{\bar{X}}\right) = \frac{1}{n-1} \frac{n}{\theta} = \frac{n}{n-1} \frac{1}{\theta}$$

Thus,

$$E\left(\frac{n-1}{n} \frac{1}{\bar{X}}\right) = \frac{1}{\theta}$$

### Conclusion:

$\frac{1}{\bar{X}}$  is not the unbiased estimator for  $\frac{1}{\theta}$ . However, we can adjust it to  $\frac{n-1}{n} \frac{1}{\bar{X}}$ , which is the unbiased estimator for  $\frac{1}{\theta}$ . When the sample size is big enough, we know that  $\frac{n-1}{n}$  will be close to 1.

## 2.2.7 UMVUE and Cramer-Rao lower bound

### 2.2.7.1 UMVUE

Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from  $f(x; \theta)$ . An estimator  $T^*$  of  $\tau(\theta)$  is called a *uniformly minimum variance unbiased estimator* (UMVUE) of  $\tau(\theta)$  if

1.  $T^*$  is unbiased for  $\tau(\theta)$ .
2. For any other unbiased estimator  $T$  of  $\tau(\theta)$ ,  $Var(T^*) \leq Var(T)$  for all  $\theta \in \Omega$ .

### 2.2.7.2 Cramer-Rao lower bound

If  $T$  is an unbiased estimator of  $\tau(\theta)$ , then the Cramer-Rao lower bound (CRLB), based on a random sample, is

$$Var(T) = \frac{[\tau'(\theta)]^2}{nE[\frac{\partial}{\partial\theta} \ln f(X; \theta)]^2}$$

**Example:**

Consider a random sample from an exponential distribution,  $X_i \sim Exp(\theta)$ . Because

$$f(x; \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$$

Thus,

$$\ln(f(x; \theta)) = -\frac{x}{\theta} - \ln\theta$$

$$\frac{\partial}{\partial\theta} \ln(f(X; \theta)) = \frac{X}{\theta^2} - \frac{1}{\theta} = \frac{X - \theta}{\theta^2}$$

Thus,

$$E[\frac{\partial}{\partial\theta} \ln(f(X; \theta))]^2 = E[\frac{(X - \theta)^2}{\theta^4}] = \frac{1}{\theta^4} E(X - \theta)^2 = \frac{1}{\theta^4} Var(X) = \frac{\theta^2}{\theta^4} = \frac{1}{\theta^2}$$

Thus, the CRLB for  $\tau(\theta) = \theta$  is as follows.

$$Var(T) \geq \frac{[\tau'(\theta)]^2}{nE[\frac{\partial}{\partial\theta} \ln f(X; \theta)]^2} = \frac{[\frac{\partial}{\partial\theta} \theta]^2}{n \frac{1}{\theta^2}} = \frac{1^2}{\frac{n}{\theta^2}} = \frac{\theta^2}{n}$$

We know that the variance for the sample mean of the exponential distribution:

$$Var(\bar{X}) = \frac{Var(X)}{n} = \frac{\theta^2}{n}$$

In addition, we also know that sample mean  $\bar{X}$  is the unbiased estimator for population mean.

Thus,

$\bar{X}$  is the UMVUE of  $\theta$ .



### 2.2.8 Best linear unbiased estimation (BLUE or MVBLUE)

<https://statisticsbyjim.com/regression/gauss-markov-theorem-ols-blue/>

The Gauss-Markov theorem states that if you linear regression satisfies the first 6 classical assumption, the Ordinary least squares (OLS) produces unbiased estimates that have the smallest variance of all possible linear estimators.

The Gauss-Markov theorem famously states that OLS is (Best Linear Unbiased Estimator) BLUE.

In this context, **best** refers to the minimum variance.

**MVBLUE:** Minimum Variance Linear Unbiased Estimate

[https://link.springer.com/chapter/10.2991/978-94-91216-83-1\\_13](https://link.springer.com/chapter/10.2991/978-94-91216-83-1_13)

### 2.2.9 Consistency, asymptotic unbiasedness

**Simple consistency:**

*P.311*

Let  $\{T_n\}$  be a sequence of estimators of  $\tau(\theta)$ . These estimators are said to be **consistent** estimators of  $\tau(\theta)$  if for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} P[|T_n - \tau(\theta)| < \varepsilon] = 1$$

for every  $\theta \in \Omega$ .

In the terminology of Chapter 7,  $T_n$  converges **stochastically** to  $\tau(\theta)$ ,  $T_n \xrightarrow{P} \tau(\theta)$  as  $n \rightarrow \infty$ . Sometimes this also is referred to as **simple** consistency.

One interpretation of consistency is that for large sample size the estimator tends to be more concentrated about  $\tau(\theta)$ , and by making  $n$  sufficiently large  $T_n$  can be made as concentrated as desired.

**MSE consistency:**

If  $\{T_n\}$  is a sequence of estimator of  $\tau(\theta)$ , then they are called **mean squared error consistent** if

$$\lim_{n \rightarrow \infty} E[T_n - \tau(\theta)]^2 = 0$$

for every  $\theta \in \Omega$ .

**Asymptotic Unbiased**

A sequence  $\{T_n\}$  is said to be **asymptotically unbiased** for  $\tau(\theta)$  if

$$\lim_{n \rightarrow \infty} E(T_n) = \tau(\theta)$$

for every  $\theta \in \Omega$ .

**Example:**

*P.313*

For a sample from  $X_i \sim EXP(\theta)$ , we know that  $T_n = 1/\bar{X}$  is an MLE estimator for  $\tau(\theta) = 1/\theta$ . However,  $T_n$  is not a unbiased estimator for  $\tau(\theta)$ , as

$$E(T_n) = \frac{n}{n-1} \cdot \frac{1}{\theta}$$

We also know that

$$Var(T_n) = \frac{\left(\frac{n}{n-1}\right)^2}{(n-2)\theta^2}$$

Thus, while  $T_n$  is not unbiased, it is asymptotically unbiased and MSE consistent for  $\tau(\theta) = \frac{1}{\theta}$ .

**Note that**, MSE consistency is a stronger property than simple consistency. Thus, if a sequence  $\{T_n\}$  is mean squared error consistent, it is also simply consistent.

### 2.2.10 Efficiency

*P.308*

The relative efficiency of an **unbiased** estimator  $T$  of  $\tau(\theta)$  to another **unbiased** estimator  $T^*$  of  $\tau(\theta)$  is given by

$$re(T, T^*) = \frac{Var(T^*)}{Var(T)}$$

An unbiased estimator  $T^*$  of  $\tau(\theta)$  is said to be efficient if  $re(T, T^*) \leq 1$  for all unbiased estimators  $T$  of  $\tau(\theta)$ , and all  $\theta \in \Omega$ . The efficiency of an unbiased estimator  $T$  of  $\tau(\theta)$  is given by

$$e(T) = re(T, T^*)$$

if  $T^*$  is an efficient estimator of  $\tau(\theta)$ .

Thus, **an efficient estimator is just a UMVUE**.

**Example:**

*P.233 Example 7.2.2*

*P.303 Example 9.3.2*

P.309 Example 9.3.7

Let  $X_1, X_2, \dots, X_n$  sample from  $X_i \sim EXP(\theta)$ . We know that

$$X_{1:n} = EXP(\theta/n)$$

Thus,

$$E(nX_{1:n}) = nE(X_{1:n}) = n \frac{\theta}{n} = \theta$$

Thus,

$$nX_{1:n}$$

is the unbiased estimator for  $\theta$ .

Thus,

$$re(T, T^*) = \frac{Var(T^*)}{Var(T)} = \frac{Var(\bar{X})}{Var(nX_{1:n})} = \frac{\theta^2/n}{n^2 Var(X_{1:n})} = \frac{\theta^2/n}{n^2(\theta/n)^2} = \frac{\theta^2/n}{\theta^2} = \frac{1}{n}$$

Thus,  $T^* = \bar{X}$  is a more efficient estimator for  $\theta$  than  $T = nX_{1:n}$ .

### 2.2.11 Asymptotic efficiency

Let  $\{T_n\}$  and  $\{T_n^*\}$  be the two asymptotically unbiased sequences of estimators for  $\tau(\theta)$ . The **asymptotic relative efficiency** of  $T_n$  relative to  $T_n^*$  is given by

$$are(T_n, T_n^*) = \lim_{n \rightarrow \infty} \frac{Var(T_n^*)}{Var(T_n)}$$

The sequence  $\{T_n^*\}$  is said to asymptotically efficient if  $are\{T_n, T_n^*\} \leq 1$  for all other asymptotically unbiased sequences  $\{T_n\}$ , and all  $\theta \in \Omega$ .

### 2.2.12 Asymptotic properties of MLEs

P.316

If certain regularity conditions are satisfied, then the solutions,  $\hat{\theta}$ , of the MLE have the following properties:

- (1)  $\hat{\theta}_n$  exists and is unique.
- (2)  $\hat{\theta}_n$  is a consistent estimator of  $\theta$ .
- (3)  $\hat{\theta}_n$  is asymptotically normal with asymptotic mean  $\theta$  and variance

$$\frac{1}{2}E\left[\frac{\partial}{\partial\theta}\ln f(X;\theta)\right]^2$$

(4)  $\hat{\theta}$  is asymptotically efficient.

Note that the asymptotic efficiency of  $\hat{\theta}$  follows from the fact that the asymptotic variance is the same as the **CRLB** for unbiased estimators of  $\theta$ . Thus, for large  $n$ , approximately

$$\hat{\theta}_n \sim N(\theta, CRLB)$$

### Example

*P.317*

From Example 9.2.7, we know the MLE of the mean  $\theta$  of an exponential distribution is the sample mean,  $\hat{\theta}_n = \bar{X}$ .

We can infer the same asymptotic properties either from the discussion above or from the Central Limit Theorem. That is,  $\hat{\theta}_n$  is asymptotically normal with asymptotic mean  $\theta$  and variance  $\theta^2/n$ . From example 9.3.4, we know that  $CRLB = \theta^2/n$ .

Thus,

$$\sqrt{n}\frac{\hat{\theta}_n - \theta}{\theta} \sim N(0, 1)$$

## 2.3 Sufficient and completeness

### 2.3.1 Sufficiency and minimal sufficiency

*P.335*

#### Sufficient statistic:

A statistic  $S$  will be considered a “sufficient” statistic for a parameter  $\theta$  if the conditional distribution of any other statistic  $T$  given the value of  $S$  does not involve  $\theta$ .

#### Jointly sufficient statistics:

Let  $X = (X_1, \dots, X_n)$  have joint pdf  $f(x, \theta)$ , and let  $S = (S_1, \dots, S_k)$  be a  $k$ -dimensional statistic.

Then,  $S_1, \dots, S_k$  is a set of **jointly sufficient statistics** for  $\theta$  if for any other vector of statistics,  $T$ , the conditional pdf of  $T$  given  $S = s$ , denoted by  $f_{T|s}(t)$ , does not depend on  $\theta$ .

In the one-dimension case, we simply say that  $S$  is a sufficient statistic for  $\theta$ .

If  $k$  unknown parameters are present in the model, then quite often there will exist a set of  $k$  sufficient statistics. In some cases, the number of sufficient statistics will exceed the number of parameters.

**Minimal sufficient:**

A set of statistics is called a **minimal sufficient** set if the members of the set are jointly sufficient for the parameters and if they are a function of every other set of jointly sufficient statistics.

**In other words, once the value of a sufficient statistic is known, the observed value of any other statistic does not contain any further information about the parameter.**

**Example:**

*P.339*

Assume a random sample from an exponential distribution,  $X_i \sim EXP(\theta)$ . It follows that

$$f(x_1, \dots, x_n; \theta) = \frac{1}{\theta^n} e^{-\sum \frac{x_i}{\theta}}$$

which suggests checking the statistic  $S = \sum X_i$ . We know that  $S \sim GAM(\theta, n)$ , thus

$$f_S(s; \theta) = \frac{1}{\theta^n \Gamma(n)} s^{n-1} e^{-\frac{s}{\theta}}$$

If  $s = \sum x_i$ , then,

$$\frac{f(x_1, \dots, x_n; \theta)}{f_S(s; \theta)} = \frac{\Gamma(n)}{s^{n-1}}$$

which is free of  $\theta$ , and thus by definition  $S$  is sufficient for  $\theta$ .

### 2.3.2 Neyman factorization theorem, minimal sufficiency of MLEs

**Factorization criterion:**

If  $X_1, \dots, X_n$  have joint pdf  $f(x_1, \dots, x_n; \theta)$ , and if  $S = (S_1, \dots, S_n)$ , then  $S_1, \dots, S_k$  are jointly sufficient for  $\theta$  if and only if

$$f(x_1, \dots, x_n; \theta) = g(s; \theta)h(x_1, \dots, x_n)$$

where  $g(s; \theta)$  does not depend on  $x_1, \dots, x_n$ , except through  $s$ , and  $h(x_1, \dots, x_n)$  does not involve  $\theta$ .

**Example 1:***P.340*

$X_i \sim \text{Bin}(1, \theta)$ . We can use factorization criterion to check its sufficient statistic.

$$f(x_1, \dots, x_n; \theta) = \theta^{\sum x_i} (1 - \theta)^{n - \sum x_i} = \theta^s (1 - \theta)^{n-s} = g(s; \theta) h(x_1, \dots, x_n)$$

where  $s = \sum X_i$  and  $h(x_1, \dots, x_n) = 1$ . Thus,  $s$  is sufficient for  $\theta$ .

**Example 2:**

**It is important to specify the regions of zero probability. The following shows that care must be exercised in this matter.**

*P.340*

$X_i \sim \text{UNIF}(0, \theta)$ , where  $\theta$  is unknown. We get the joint *pdf* of  $X_1, \dots, X_n$  is

$$f(x_1, \dots, x_n; \theta) = \frac{1}{\theta^n}$$

where,

$$0 < x_i < \theta$$

It is easier to specify this *pdf* in terms of the minimum,  $x_{1:n}$ , and maximum,  $x_{n:n}$ , of  $x_1, \dots, x_n$ . In particular,

$$0 < x_{1:n} \leq x_{n:n} < \theta$$

Thus,  $x_{n:n}$  is sufficient for  $\theta$ .

**Example 3:***P.341*

Consider a random sample from a normal distribution,  $X_i \sim N(\mu, \sigma^2)$ .

We know the *pdf* for normal is

$$f = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

Thus, the joint *pdf* is as follows.

$$\begin{aligned}
f(x_1, \dots, x_n; \mu, \sigma^2) &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2} \\
&= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2} \sum (x_i^2 + \mu^2 - 2x_i\mu)} \\
&= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2} (\sum x_i^2 + n\mu^2 - 2\mu \sum x_i)}
\end{aligned}$$

Thus,  $S_1 = \sum X_i$  and  $S_2 = \sum X_i^2$  are jointly sufficient for  $\theta = \mu, \sigma^2$ .

**Note that,**

*P.341*

based on *P.298*, Example 9.2.10, ML estimation results in

$$\begin{aligned}
\hat{\mu} &= \bar{x} \\
\hat{\sigma}^2 &= \frac{\sum (x_i - \bar{x})^2}{n}
\end{aligned}$$

Further, note that

$$\begin{aligned}
\hat{\mu} &= \bar{x} = \frac{S_1}{n} \\
\hat{\sigma}^2 &= \frac{\sum (x_i - \bar{x})^2}{n} = \frac{\sum (x_i^2 + \bar{x}^2 - 2x_i\bar{x})}{n} \\
&= \frac{\sum x_i^2 + n\bar{x}^2 - 2\bar{x} \sum x_i}{n} \\
&= \frac{\sum x_i^2 + n\bar{x}^2 - 2n\bar{x}^2}{n} \\
&= \frac{\sum x_i^2 - n\bar{x}^2}{n} \\
&= \frac{\sum x_i^2}{n} - \bar{x}^2 \\
&= \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2 \\
&= \frac{S_2}{n} - \left(\frac{S_1}{n}\right)^2
\end{aligned}$$

Thus, we can see that  $\hat{\mu}$  and  $\hat{\sigma}^2$  correspond to a one-to-one transformation of  $S_1$  and  $S_2$ , thus  $\hat{\mu}$  and  $\hat{\sigma}^2$  also are jointly sufficient for  $\mu$  and  $\sigma^2$ .

### 2.3.3 Rao-Blackwell theorem

Let  $X_1, \dots, X_n$  have joint pdf  $f(x_1, \dots, x_n; \theta)$ , and let  $S = (S_1, \dots, S_k)$  be a vector of jointly sufficient statistics for  $\theta$ . If  $T$  is any unbiased estimator of  $\tau(\theta)$ , and if  $T^* = E(T|S)$ , then

- (1)  $T^*$  is an unbiased estimator of  $\tau(\theta)$ ,
- (2)  $T^*$  is a function of  $S$ , and
- (3)  $Var(T^*) \leq Var(T)$  for every  $\theta$ , and  $Var(T^*) < Var(T)$  for some  $\theta$  unless  $T^* = T$  with probability 1.

#### Discussion:

*P.345*

"It is clear from the Rao-Blackwell theorem that if we are searching for an unbiased estimator with small variance, we may as well restrict attention to function of sufficient statistics.

If any unbiased estimator exists, then there will be one that is a function of sufficient statistics, namely  $E(T|S)$ , which also is unbiased and has variance at least as small or smaller. In particular, we still are interested in knowing how to find a UMVUE for a parameter, and the above theorem narrows our problem down somewhat.

For instance, consider a one-parameter model  $f(x; \theta)$ , and assume that a single sufficient statistic,  $S$ , exists. We know that we must consider only unbiased functions of  $S$  in searching for a UMVUE. In some cases it may be possible to show that only one function of  $S$  is unbiased, and in that case we would know that it is a UMVUE.

The concept of "completeness" is helpful in determining unique unbiased estimators, and this concept is defined in the next section."

### 2.3.4 completeness

#### Completeness:

*P.345*

A family of density functions  $\{f_T(t, \theta); \theta \in \Omega\}$ , is called **complete** if  $E[u(T)] = 0$  for all  $\theta \in \Omega$  implies  $u(T) = 0$  with probability 1 for all  $\theta \in \Omega$ .

*P.345*

"This sometimes is expressed by saying that there are no nontrivial unbiased estimators of zero. In particular, it means that two different functions of  $T$  cannot have the same expected value."

For example, if  $E[u_1(T)] = \tau(\theta)$  and  $E[u_2(T)] = \tau(\theta)$ , which implies  $u_1(T) - u_2(T) = 0$ , or  $u_1(T) = u_2(T)$  with probability 1, if the family of density functions



is complete. That is, any unbiased estimator is unique in this case.

We mainly are interested in knowing that the family of density functions of a sufficient statistic is complete, because in that case an unbiased function of the sufficient statistic will be unique, and it must be a UMVUE by the Rao-Blackwell theorem.

### 2.3.5 Lehmann-Scheffe completeness theorem

Let  $X_1, \dots, X_n$  have joint pdf  $f(x_1, \dots, x_n; \theta)$  and let  $S$  be a vector of jointly complete sufficient statistics for  $\theta$ . If  $T^* = t^*(S)$  is a statistic that is unbiased for  $\tau(\theta)$  and a function of  $S$ , then  $T^*$  is a UMVUE of  $\tau(\theta)$ .

**Example:**

P.346

### 2.3.6 Exponential class, complete sufficient statistics

A density function is said to be a member of the **regular exponential class** if it can be expressed in the form

$$f(x; \theta) = c(\theta)h(x)e^{\sum_{j=1}^k q_j(\theta)t_j(x)}$$

where,

$$x \in A$$

And zero otherwise, where  $\theta = (\theta_1, \dots, \theta_k)$  is a vector of  $k$  unknown parameters, if the parameter space has the form

$$\Omega = \{\theta | a_i \leq \theta_i \leq b_i, i = 1, \dots, k\}$$

(note that  $a_i = -\infty$  and  $b_i = \infty$  are permissible values), and if it satisfies regularity conditions 1, 2, and 3a or 3b given by:

- (1) The set  $A = \{x : f(x; \theta) > 0\}$  does not depend on  $\theta$ .
- (2) The functions  $q_j(\theta)$  are nontrivial, functionally independent, continuous function of the  $\theta_i$ .

3a. For a continuous random variable, the derivatives  $t'_k(x)$  are linearly independent continuous functions of  $x$  over  $A$ .

3b. For a discrete random variable, the  $t_k(x)$  are nontrivial functions of  $x$  on  $A$ , and none is a linear function of the others.

**Example**

P.348

Consider Bernoulli distribution  $X \sim \text{Bin}(1, p)$ . It follows that

$$\begin{aligned} f(x; p) &= p^x(1-p)^{1-x} \\ &= P^x(1-p)(1-p)^{-x} \\ &= (1-p)(1-p)^{-x} P^x \\ &= (1-p)e^{\ln(1-p)^{-x} P^x} \\ &= (1-p)e^{x \ln(\frac{p}{1-p})} \end{aligned}$$

Compared to the definition of **Exponential Class** defined above, we can get the following.

$$\begin{aligned} c(\theta) &= (1-p) \\ h(x) &= 1 \\ q_1(\theta) &= \ln\left(\frac{p}{1-p}\right) \\ t_1(x) &= x \end{aligned}$$

#### Theorem 10.4.2

*P.348*

If  $X_1, \dots, X_n$  is a random sample from a member of the regular exponential class  $\text{REC}(q_1, \dots, q_k)$ , then the statistics

$$S_1 = \sum_{i=1}^n t_1(X_i), \dots, S_k = \sum_{i=1}^n t_k(X_i)$$

are a minimal set of complete sufficient statistics for  $\theta_1, \dots, \theta_k$ .

#### Example 1:

*P.348*

Again, consider Bernoulli distribution  $X \sim \text{Bin}(1, p)$ . For a random sample of size  $n$ ,  $t(x_i) = x_i$  and thus based on **Theorem 10.4.2** (see above)  $S = \sum_{i=1}^n X_i$  is a complete sufficient statistic for  $p$ .

**How about we want to find UMVUE for  $\text{Var}(X) = p(1-p)$ ?**

We might try  $\bar{X}(1 - \bar{X})$ .

$$\begin{aligned} E[\bar{X}(1 - \bar{X})] &= E(\bar{X}) - E(\bar{X}^2) \\ &= p - (p^2 + \text{var}(\bar{X})) \\ &= p - p^2 - \frac{p(1-p)}{n} \\ &= p(1-p)\left(1 - \frac{1}{n}\right) \end{aligned}$$

Thus,

$$E\left[\frac{1}{1 - \frac{1}{n}} \bar{X}(1 - \bar{X})\right] = p(1 - p)$$

Thus,  $\frac{1}{1 - \frac{1}{n}} \bar{X}(1 - \bar{X})$  is the UMVUE of the  $p(1 - p)$ .

but **why?**

**Example 2:**

*P.349*

If  $X \sim N(\mu, \sigma^2)$ , then

$$\begin{aligned} f(x; \mu, \sigma) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^2 + \mu^2 - 2x\mu)} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} + \frac{x\mu}{\sigma^2}} \end{aligned}$$

Thus,  $S_1 = \sum X_i^2$  and  $S_2 = \sum X_i$  are jointly complete and sufficient statistics of  $\mu$  and  $\sigma^2$ . **Refer to the normal example in the section of “Neyman factorization theorem, minimal sufficiency of MLEs”**

**Theorem 10.4.3**

*P.349*

If a CRLB estimator  $T$  exists for  $\tau(\theta)$ , then a single sufficient statistic exists, and  $T$  is a function of the sufficient statistic. Conversely, if a single sufficient statistic exists and the CRLB exists, then a CRLB estimator exists for some  $\tau(\theta)$ .



# Chapter 3

530\_\_533

<https://www.ssc.wisc.edu/sscc/pubs/RegressionDiagnostics.html>

<http://www.sthda.com/english/articles/39-regression-model-diagnostics/161-linear-regression-assumptions-and-diagnostics-in-r-essentials/>

## 3.1 Definition of the general linear model

$$Y = X\beta + \varepsilon$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \dots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{12} & x_{13} & \dots & x_{1m} \\ 1 & x_{21} & x_{22} & x_{23} & \dots & x_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_{n1} & x_{n2} & x_{n3} & \dots & x_{nm} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \dots \\ \beta_m \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \dots \\ \varepsilon_n \end{bmatrix}$$

Where,

$Y$ : Response vector

$X$ : Design matrix

$\beta$ : parameter vector

$\varepsilon$ : error vector

If  $\varepsilon$  follows a mutivariate normal distribution then we will be under the General Linear Model (GLM) framework.

$$\varepsilon \sim N(0, \sigma^2 I_{x \times n})$$

If  $X$  is continuous we have regression.

If  $X$  is categorical we have ANOVA.

If  $X$  is a mix of both, we have ANCOVA.

## 3.2 Simple Linear Regression

Simple linear regression is a linear regression model with a single explanatory variable. In addition, we typically assume that this is under the GLM framework and thus we also assume that the residuals follow normal distribution.

### 3.2.1 Least squares and propoerties of the regression parameters

#### 3.2.1.1 Basic idea

**Of note** The following method can always calculate the vector of  $\beta$ , not related to any specific methods.

$$Y_{n \times 1} = X_{n \times m} \beta_{m \times 1}$$

→

$$[X^T]_{m \times n} Y_{n \times 1} = [X^T]_{m \times n} X_{n \times m} \beta_{m \times 1}$$

→

$$[X^T X]_{m \times m}^{-1} [X^T Y]_{m \times 1} = \beta_{m \times 1}$$

→

$$\beta_{m \times 1} = [(X^T X)^{-1} \cdot (X^T Y)]_{m \times 1}$$

Thus,

$$Var(\hat{\beta}) = (X^T X)^{-1} X^T \cdot \sigma^2 \cdot [(X^T X)^{-1} X^T]^T$$

**Residual is given by  $e_i$** , which is an estimate of  $\varepsilon_i$ .

Properties of the residuals  $e_i$  :

1. The mean of resituals should be 0.
2. The variance of  $e_i$ :

$$\begin{aligned}
Var(e) &= Var(Y - X\hat{\beta}) \\
&= Var(Y - X(X^T X)^{-1} X^T Y) \\
&= Var(I - X(X^T X)^{-1} X^T)Y \\
&= (I - X(X^T X)^{-1} X^T)\sigma^2
\end{aligned}$$

Thus,

$$Var(e) \neq \sigma^2$$

### 3.2.1.2 Least Squares

Assume the following model:

$$Y = X\beta + \varepsilon$$

When  $\beta$  only has a dimension of  $2 \times 1$ , we can write it as follows.

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon$$

Thus,

$$Q = \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 X_i)^2$$

We can calculate the partial derivatives as follows.

$$\frac{\partial}{\partial \beta_0} Q \rightarrow \sum_{i=1}^n [Y_i - \beta_0 - \beta_1 X_i] = 0$$

$$\frac{\partial}{\partial \beta_1} Q \rightarrow \sum_{i=1}^n [Y_i - \beta_0 - \beta_1 X_i] X_i = 0$$

Combining the two pieces of information, we can get

$$b_1 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$$

$$b_0 = \frac{1}{n} (\sum Y_i - b_1 \sum X_i) = \bar{Y} - b_1 \bar{X}$$

**Properties of Least Squares Estimators (Gauss- Markov theorem):**

P.18

Under the conditions of regression model shown above, the least squares estimator  $b_0$  and  $b_1$  are unbiased and have minimum variance among all unbiased linear estimators.

That is,

$$E(b_0) = \beta_0$$

$$E(b_1) = \beta_1$$

### Point Estimation of Mean Response:

Given sample estimators  $b_0$  and  $b_1$  in the regression function

$$E\{Y\} = \beta_0 + \beta_1 X$$

We estimate the regression function as follows:

$$\hat{Y} = b_0 + b_1 X$$

We call a value of the response variable a response and  $E\{Y\}$  the *mean response*.

### Properties of fitted regression line

P.22

The residual:

$$e_i = Y_i - \hat{Y}_i$$

The sum of the residuals is zero

$$\sum_{i=1}^n e_i = 0$$

#### 3.2.1.3 Point estimator of $\sigma^2$

Note that, there are two  $S^2$  below, and they have different formulas, even though the only difference is the degree of freedom and the logic of calculating df is the same across these two cases (i.e., **Single Population vs. Regression Model**).

#### 1. Single Population

**Sum of squares:**



P.25

$$\sum_{i=1}^n e_i^2 = \sum_{i=1}^n (Y_i - \bar{Y})^2$$

The sum of squares is then divided by the degrees of freedom associated with it.

$$S^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n - 1}$$

$S^2$  is an unbiased estimator of the variance  $\sigma^2$ . The sample variance  $S^2$  is often called a mean square, because a sum of squares has been divided by the appropriate number of degrees of freedom.

$$E(S^2) = \sigma^2$$

## 2. Regression model

**Sum of Squares Error (SSE) (or, Sum of Squares Residual):**

$$SSE = \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^n e_i^2$$

**Mean Square Error (MSE) (or, Mean Square Residual):**

$$S^2 = MSE = \frac{SSE}{n - 2} = \frac{\sum (Y_i - \hat{Y}_i)^2}{n - 2} = \frac{\sum e_i^2}{n - 2}$$

$MSE$  is an unbiased estimator of  $\sigma^2$  for regression model.

$$E(MSE = S^2) = \sigma^2$$

### 3.2.2 MLE

P.30

Note that the **Normal Error Regression Model** is as follows.

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon$$

(This is a special case of the definition provided earlier, i.e.,  $Y = X\beta + \varepsilon$ )

For this model, each  $Y_i$  observation is normally distributed with mean  $\beta_0 + \beta_1 x_i$  and a standard deviation  $\sigma$ .

Given  $Y_i$  follows normal distribution, we can use *pdf* of normal distributions and MLE to estimate parameters.

$$\begin{aligned} L(\beta_0, \beta_1, \sigma^2) &= \prod_{i=1}^n \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[-\frac{1}{2\sigma^2}(Y_i - \beta_0 - \beta_1 X_i)^2\right] \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 X_i)^2\right] \end{aligned}$$

Since the variance  $\sigma^2$  is usually unknown, the likelihood function is a function of three parameters,  $\beta_0$ ,  $\beta_1$ , and  $\sigma^2$ .

Thus, we can calculate  $\beta_0$ ,  $\beta_1$ , and  $\sigma^2$  analytically, and the results of  $\beta_0$  and  $\beta_1$  are the same as least squares estimators (see above).

The variance  $\sigma^2$  is as follows.

$$\hat{\sigma}^2 = \frac{\sum (Y_i - \hat{Y}_i)^2}{n}$$

As noted in the least square estimation, we know that  $\hat{\sigma}^2$  is biased. Thus, we know that:

$$S^2 = MSE = \frac{n}{n-2} \hat{\sigma}^2$$

### 3.3 Full rank, less than full rank

<http://www.biostat.jhsph.edu/~iruczins/teaching/140.751/notes/ch7.pdf>

$$Y_{n \times 1} = X_{n \times m} \beta_{m \times 1}$$

If the rank  $r$  of  $X_{n \times m}$  is smaller than  $m$ , i.e.,  $r < m$ , there is not a unique solution  $\hat{\beta}$ . We have three ways to find a solution  $\hat{\beta}$  and the orthogonal projection  $\hat{Y}$ :

#### 1. Reducing the model to the full rank.

Let  $X_1$  consist of  $r$  linear independent columns from  $X$  and let  $X_2$  consist of the remaining columns. Then,  $X_2 = X_1 F$  because the columns of  $X_2$  are linearly dependent on the columns of  $X_1$ .

$$X = (X_1, X_2) = (X_1, X_1 F) = X_1 [I_{r \times r}, F]$$

This is a special case of the factorization  $X = KL$ , where  $\text{rank}(K_{n \times r}) = r$  and  $\text{rank}(L_{r \times p}) = r$ .

$$E[Y] = x\beta = KL\beta = k\alpha$$

Since  $K$  has full rank, the Least Squares Estimate of  $\alpha$  is  $\hat{\alpha} = (K^T K)^{-1} \cdot K^T Y$ .

The orthogonal project,

$$\hat{Y} = K \cdot \alpha = K \cdot (K^T K)^{-1} \cdot K^T Y = X_1 \cdot (X_1^T X_1)^{-1} \cdot X_1^T Y$$

$$\begin{bmatrix} Y_{11} \\ \dots \\ Y_{1n_1} \\ Y_{21} \\ \dots \\ Y_{2n_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ \dots & \dots & \dots \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \dots & \dots & \dots \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} \\ \dots \\ \varepsilon_{1n_1} \\ \varepsilon_{21} \\ \dots \\ \varepsilon_{2n_2} \end{bmatrix}$$

Let  $X_1$  consist of the first 2 columns of  $X$ , then

$$X = X_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Thus,

$$X_1 = K = \begin{bmatrix} 1 & 1 \\ \dots & \dots \\ 1 & 1 \\ 1 & 0 \\ \dots & \dots \\ 1 & 0 \end{bmatrix}$$

$$\hat{\alpha} = (K^T K)^{-1} \cdot K^T Y = \begin{bmatrix} n & n_1 \\ n_1 & n_1 \end{bmatrix}^{-1} \begin{bmatrix} \sum Y_{1j} + \sum y_{2j} \\ \sum y_{1j} \end{bmatrix} = \begin{bmatrix} \bar{Y}_2 \\ \bar{Y}_1 - \bar{Y}_2 \end{bmatrix}$$

$$\hat{Y} = X_1 \hat{\alpha} = \begin{bmatrix} 1 & 1 \\ \dots & \dots \\ 1 & 1 \\ 1 & 0 \\ \dots & \dots \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{Y}_2 \\ \bar{Y}_1 - \bar{Y}_2 \end{bmatrix} = \begin{bmatrix} \bar{Y}_1 \\ \dots \\ \bar{Y}_1 \\ \bar{Y}_2 \\ \dots \\ \bar{Y}_2 \end{bmatrix}$$

## 2. Find the generalized inverse $(X^T X)^{-1}$ .

As noted above, when  $X^T X$  has a full rank, we can directly calculate the inverse of  $X^T X$ . That is,

$$\beta_{m \times 1} = [(X^T X)^{-1} \cdot (X^T Y)]_{m \times 1}$$

We can just find **some columns** within  $X$  that are independent, and then calculate the inverse of it.

This is because if a matrix  $W$  with a rank  $r$  and can be partitioned as follows.

$$W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Assume  $A$  has rank  $r$ , then

$$W^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, let  $X = (X_1, X_2)$ , where  $X_1$  consists of  $r$  linearly independent columns from  $X$ . Then a generalized inverse of  $X^T X$  is

$$[X^T X]^{-1} = \begin{bmatrix} [X_1^T X_1]^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, all other steps are similar to the full rank case.

### 3. Impose identifiability constraints

(Not very sure this one.)

## 3.4 Assumptions, checking assumptions

The following are some of the possible violations of assumptions:

### 3.4.1 Regression function is nonlinear

Plot explanatory and response variables to see whether their relationship is linear, it is not, suggesting that a linear regression function is not appropriate.

**Stat Methods:**

F-test for lack of fit.

### 3.4.2 Non-constant variance

Plots of the residuals against the predictor variable (i.e.,  $X$ ) or against the fitted values (i.e.,  $\hat{Y}$ ), to study whether the linear model is appropriate and whether the **variance of the error terms** is constant.

**Stat Methods:**

**(1) Modified Levene's test (Brown-Forsythe test):**

To check and see if the variability for  $Y$  for the smaller  $X$ 's are different than the variability of  $Y$  for the larger  $X$ 's.

Thus, we can break up the data into two groups based  $X$ 's values. Then, we test to see if the variability of these two groups significantly differ.

**Note that** Modified Levene's test does not depend on normality of the error terms. That is, this test is robust against serious departures from normality.

**(2) Breusch-Pagan test:**

Want to see if there is any relationship between  $\sigma_i$  and the  $X_i$ 's. To do this, we can fit a regression of  $\log(\sigma_i)$  on  $X_i$ 's:

$$\log(\sigma_i) = b_0 + b_1 X_i$$

**3.4.3 Error terms not independent**

We want a random pattern. Any type of pattern indicates time ordered problems indicates that the model is not appropriate.

**Stat Methods:**

**Durbin Watson Test** Test for presence of autocorrelation amongst observations.

**3.4.4 Possibility of outliers**

Scatter plot the dependent variable and independent variables (each  $X_i$  is plotted separately.)

**3.4.5 Non normal distribution of error**

Use histogram to check whether the residual follows normal distribution.

**Stat Methods:**

*P.115*

Correlation between ordered residuals and their expected values under normality.

*P.70 512 note*

You can use Shapiro-Wilk, Kolmogorov-Smirnov, Cramer-von Mises, Anderson-Darling in SAS to assess the normal errors.

### 3.4.6 Omission of some of the important predictors

Plots should be made between variables omitted from the model and the dependent variable. Such omitted variables may have important effects on the response.

## 3.5 Bootstrapping used in linear models

*P.458*

### Intro

For standard fitted regression models, methods described earlier chapters are available for evaluating the precision of estimated regression coefficients, fitted values, and predictions of new observations.

However, in many nonstandard situations, such as when nonconstant error variance are estimated by iteratively reweighted least square or when robust regression estimation is used, standard methods for evaluating the precision may not be available or may only approximately applicable when the sample size is large.

Bootstrapping was developed by Efron to provide estimates of the precision of sample estimates for these complex cases.

### Conceptual

Suppose that we have fitted a regression model (simple or multiple) by some procedure and get the coefficient  $b_1$ . We now wish to evaluate the precision of this estimate by bootstrap method.

- (1) In essence, the bootstrap method call for the selection from the observed sample data of a random sample of size  $n$  with replacement.
- (2) Next, the bootstrap method calculates the estimated regression coefficient from the bootstrap sample, using the same fitting procedure as employed for the original fitting. This leads to the first bootstrap estimate  $b_1^*$ .

This procedure repeated a large number of times, each time a bootstrap sample of size  $n$  is selected with replacement from the original sample and the estimated coefficient is obtained.

- (3) The estimated standard deviation of all the bootstrap estimates  $b_1^*$  denoted by  $S^*\{b_1^*\}$ , is an estimate of the variability of the sampling distribution of  $b_1$  and therefore a measure of the precision of  $b_1$ .

### Some Math

Bootstrapping sampling for regression can be done in two basic ways.

- (1) When the regression function being fitted is a good model for the data, the error terms have constant variance, and the predictor variables can be regarded as fixed, fixed X sampling is appropriate.

Here, the residuals  $e_i$  from the original fitting are regarded as the sample data to be sampled with replacement. After bootstrap sample of the residuals of size  $n$  have been obtained, denoted by  $e_1^*, \dots, e_n^*$ . The bootstrap sample residuals are added to the fitted values from the original fitting to obtain new bootstrap  $Y$  values, denoted by  $Y_1^*, \dots, Y_n^*$ :

$$Y_i^* = \hat{Y}_i + e_i^*$$

These bootstrap  $Y^*$  values are then regressed on the original  $X$  variables by the same procedure used initially to obtain the bootstrap estimate  $b_1^*$ .

- (2) When there is some doubt about the adequacy of the regression model, or the error variances are not constant, or the predictor variables can not be regarded as fixed, random  $X$  sampling is appropriate.

For simple regression, the pair  $X$  and  $Y$  data in the original sample are considered to be the data to be sampled with replacement. This second procedure samples cases with replacement  $n$  times, yielding a bootstrap sample of  $n$  pairs of  $(X^*, Y^*)$  values. This bootstrap sample is then used for obtaining the bootstrap estimate  $b_1$ , as with fixed  $X$  sampling.

#### Bootstrap confidence intervals

P.460

## 3.6 Generalized linear models

### 3.6.1 Definition, similarities and differences from general linear models

Generalized Linear Model (GLM) loosens this assumption that  $\varepsilon$  follows a multivariate normal distribution, and allows for a variety of other distributions from the exponential family for the residuals.

Of note, the GLM is a special case of the GLiM in which the distribution of the residuals follow a conditionally normal distribution.

### 3.6.2 Advantage and disadvantages

### 3.6.3 logistic and poisson regression

#### Logistic regression

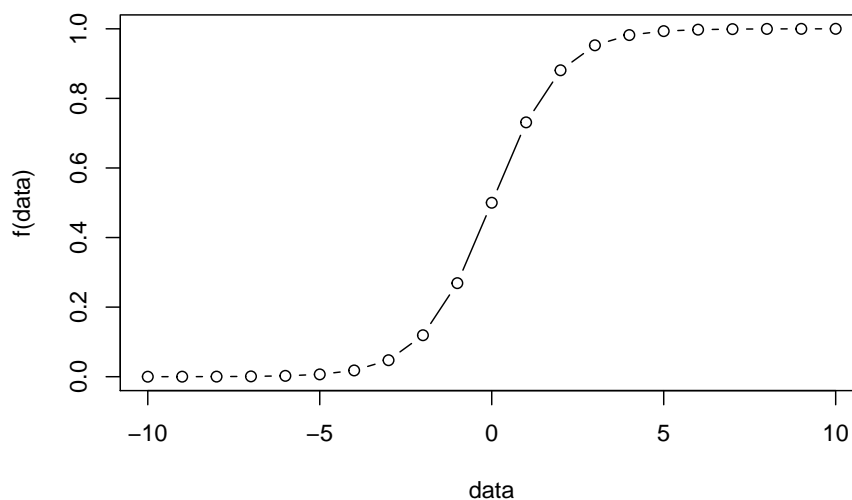
Logistic regression is a special case of GLiM with binomial distribution on the  $Y$ 's and the link function

The basic idea of logistic regression:

$$p(y = 1) = \frac{1}{1 + e^{-(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n)}} = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}}$$

Thus,  $\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n$  can be from  $-\infty$  to  $+\infty$ , and  $p(y = 1)$  will be always within the range of  $(0, 1)$ .

```
f<-function(x){exp(x)/(1+exp(x))}
data<-seq(-10,10,1)
plot(data,f(data),type = "b")
```



We can also write the function into another format as follows:

$$\log \frac{p(y = 1)}{1 - p(y = 1)} = \beta_0 + \beta_1 x_1 + \dots + \beta_n x_n$$

Thus, we know that the regression coefficients of  $\beta_i$  actually change the “log-odds” of the event. Of course, note that the magnitude of  $\beta_i$  is dependent upon the units of  $x_i$ .



# Chapter 4

## 512

### 4.1 Basics

#### 4.1.1 Random vs. Fixed

A factor is **random** if its levels represent a random sample from a population consisting of a large number of possible levels.

A factor is **fixed** if its levels are selected by a non-random process or if its levels consist of the entire population of possible levels.

#### 4.1.2 Crossed vs. Nested

A factor, A, is said to be **crossed** with respect to a second factor, say B, if each level of factor A is exactly the same for each level of factor B, and each level of factor B is exactly the same for each level of factor A.

Otherwise, the factor is said to be **nested**. A factor, say B, is **nested** within another factor, say A, if the levels of factor B are **not** the same across all levels of factor A.

### 4.2 Completely randomized designs

Completely random design (CRD):

Randomly select 9 farms. Randomly assign 3 farms to Variety 1, 3 farms to Variety 2, and 3 farms to Variety 3. This arrangement might appear as in the following table.

All experimental units (i.e., 9 farms) are **randomly assigned** to all treatment (i.e., 3 varieties) and therefore the design is a CRD design. That is, across

Variety 1	farm 7	farm 5	farm 4
Variety 2	farm 3	farm 9	farm 6
Variety 3	farm 8	farm 1	farm 2

Figure 4.1: CRD

Farm 1	variety 2	variety 3	variety 1
Farm 2	variety 3	variety 1	variety 2
Farm 3	variety 3	variety 2	variety 1

Figure 4.2: RCBD

**all experimental units**, (i.e., 9 farms) they are assumed to be homogeneous before getting the **assigned treatments**.

### 4.3 Randomized complete block designs

**Randomized complete block design (RCBD):**

Randomly select 3 farms. Each farm is divided into 3 subunits with all three varieties, and all 3 varieties are randomly assigned into the 3 subunits.

The  $t$  treatments are randomly assigned to the  $t$  subunits within each block. Thus, there are **t-treatments**, with each to be observed **b times** (i.e.,  $t \times b$  experimental units).

In the table above, **t-treatments** are the 3 varieties, whereas **b times** are the 3 blocks. Thus, **3 farms** are the **3 blocks**.

This is a **Randomized Complete Block Design**. That is, we have a one-way treatment structure (Variety) in a randomized complete block design structure.

We call the **farms** blocks or **blocking factor**. Note that block and treatment (Variety) are crossed.

### 4.4 Row-column design (Latin square designs)

( **My own short summary:** Latin Square Designs basically are a double-block design.)

Randomly select **3 farms** and divide each farm into **3 subunits** with all three Varieties to be assigned to each farm.

	Fertility		
	High	Medium	Low
Farm 1	variety 1	variety 2	variety 3
Farm 2	variety 3	variety 1	variety 2
Farm 3	variety 2	variety 3	variety 1

Figure 4.3: LSD

In addition, suppose that these farms have fertility gradients (high, medium and low). The treatments are assigned so that each farm (row) receives all three Varieties, but so does each fertility (column) level.

This is a **Latin Square Design**. That is, we have blocked both on **rows (farms)** and **columns (fertility)** such that each treatment occurs once, and only once, within a row or column (e.g., a block).

## 4.5 Comparing RCBD versus CRD

When is it to use RCBD or CRD more efficient?

If there is “substantial” amount of variation in the units of replications (i.e., blocks for RCBD; replicats for CRD), then the block design is more efficient and has higher power than CRD.

In contrast, if the variation among units of replication is small, CRD is more efficient and has more power than RCBD.

## 4.6 Incomplete block designs

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These designs are similar to the RCBD except that each block (e.g., each farm) contains fewer than  $t$  units and thus, only a portion of the  $t$  treatments are applied to the units within each block. (e.g., not all the farms will get all the varieties of wheat).

## 4.7 Split-plot designs

( **My own short summary:** The split-plot design is a bit similar to Latin Square Designs, but not exactly the same.)

Farm, Variety 1	Farm, Variety 2	Farm, Variety 3
1, $V_1$	5, $V_2$	9, $V_3$
2, $V_1$	6, $V_2$	10, $V_3$
3, $V_1$	7, $V_2$	11, $V_3$
4, $V_1$	8, $V_2$	12, $V_3$

Figure 4.4: RCD2

Source of Variation	Degrees of Freedom	Sum of Squares
Variety	$3 - 1 = 2$	$SS_{\text{Variety}}$
Farm(Variety) = Whole Plot Error	$3(4 - 1) = 9$	$SS_{\text{Farm(Variety)}}$

Figure 4.5: RCD2\_ANOVA

Consider a completely randomized design with a one-way treatment structure. In this case, suppose the treatment structure consists of 3 varieties of wheat ( $V_1, V_2, V_3$ ), each planted on 4 randomly selected farms.

The ANOVA table for this experiment would appear as follows. As shown, **farms** are nested within **Variety**. As we can see, we do not need to model specifically the **variety** within each farm.

### Whole Plot and Treatment Structure

However, now suppose that the researcher is also interested in the effect of two different fertilizers ( $F_1, F_2$ ) on yield. The completely randomized design can be modified by splitting each farm in half. In this experiment, there are two different sizes of experimental units: the large units, which are the farms and the small units, which are the Farm Halves (i.e., fertility levels).

### Sub-plot Design and Treatment Structure

Split plot vs split block design

See 512 P.363

Farm, Variety 1		Farm, Variety 2		Farm, Variety 3	
1, V <sub>1</sub> , F <sub>1</sub>	1, V <sub>1</sub> , F <sub>2</sub>	5, V <sub>2</sub> , F <sub>1</sub>	5, V <sub>2</sub> , F <sub>2</sub>	9, V <sub>3</sub> , F <sub>2</sub>	9, V <sub>3</sub> , F <sub>1</sub>
2, V <sub>1</sub> , F <sub>2</sub>	2, V <sub>1</sub> , F <sub>1</sub>	6, V <sub>2</sub> , F <sub>1</sub>	6, V <sub>2</sub> , F <sub>2</sub>	10, V <sub>3</sub> , F <sub>1</sub>	10, V <sub>3</sub> , F <sub>2</sub>
3, V <sub>1</sub> , F <sub>1</sub>	3, V <sub>1</sub> , F <sub>2</sub>	7, V <sub>2</sub> , F <sub>2</sub>	7, V <sub>2</sub> , F <sub>1</sub>	11, V <sub>3</sub> , F <sub>1</sub>	11, V <sub>3</sub> , F <sub>2</sub>
4, V <sub>1</sub> , F <sub>1</sub>	4, V <sub>1</sub> , F <sub>2</sub>	8, V <sub>2</sub> , F <sub>2</sub>	8, V <sub>2</sub> , F <sub>1</sub>	12, V <sub>3</sub> , F <sub>1</sub>	12, V <sub>3</sub> , F <sub>2</sub>

Figure 4.6: splitting

Fertilizer	$2 - 1 = 1$	$SS_{\text{Fertilizer}}$
Fertilizer*Variety	$(3 - 1)(2 - 1) = 2$	$SS_{\text{Fertilizer*Variety}}$
Subplot Error = Fertilizer*Farm(Variety)	$3(4 - 1)(2 - 1) = 9$	$SS_{\text{Sub-plot Error}}$

Figure 4.7: splitting\_ANOVA



## Chapter 5

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### 5.1 12.1

#### 5.1.1 Theory

If  $X \sim f(x; \theta)$ , a **statistical hypothesis** is a statement about the distribution of  $X$ . If the hypothesis completely specifies  $f(x; \theta)$ , then it is referred to as a **simple hypothesis**; otherwise it is called **composite**.

Quite often the distribution in question has a **known parametric form** with a single unknown parameter  $\theta$ , and the hypothesis consists of a statement about  $\theta$ .

In this framework, a statistical hypothesis corresponds to a subset of the parameter space, and the objective of a test would be to decide whether the true value of the parameter is in the subset. Thus, a null hypothesis would correspond to a subset  $\Omega_0$  of  $\Omega$ , and the alternative hypothesis would correspond to its complement,  $\Omega - \Omega_0$ .

In the case of **simple hypothesis**, these sets consist of only one element each,  $\Omega_0 = \{\theta_0\}$  and  $\Omega - \Omega_0 = \{\theta_1\}$ , where  $\theta_0 \neq \theta_1$ .

We now must consider sample data, and decide on the basis of the data whether we have sufficient statistical evidence to reject  $H_0$  in favor of the alternative  $H_a$ , or whether we do not have sufficient evidence.

That is, our philosophy will be to divide the sample space into two regions, the “critical region” or “rejection region”  $C$ , and the nonrejection region  $S - C$ . If the observed sample data fall in  $C$ , then we will reject  $H_0$ .

$\bar{X}$  is a sufficient statistic for  $\mu$ , so we may conveniently express the critical region directly in terms of the univariate variable  $\bar{X}$ , and we will refer  $\bar{X}$  as the test statistic. Because  $\mu_1 > \mu_0$ , a natural form for the critical region in this problem

is to let  $C = \{(x_1, \dots, x_n) | \bar{x} \geq c\}$ , for some appropriate constant  $c$ . That is, we will reject  $H_0$  if  $\bar{x} \geq c$ , and we not reject  $H_0$  if  $\bar{x} < c$ .

There are two possible error:

1. Type I error: reject a true  $H_0$ .  $P[\text{Type I Error}] = \alpha$ .
2. Type II error: fail to reject a false  $H_0$ .  $P[\text{Type II Error}] = \beta$ .

Note that,  $P[\text{Type I Error}] = \alpha$  is also referred to as the **significance level** of the test.

### 5.1.2 Example

For instance, if  $n = 25$ , then  $\alpha = 0.05$  gives

$$c = \mu_0 + Z_{1-\alpha}\sigma/\sqrt{n} = 10 + 1.645 \cdot 4/5 = 11.316$$

We can verify as follows.

$$\begin{aligned} P[\bar{X} \geq c | \mu = \mu_0 = 10] &= P\left[\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq \frac{c - \mu_0}{\sigma/\sqrt{n}}\right] \\ &= P\left[Z \geq \frac{11.316 - 10}{4/5}\right] \\ &= P[Z \geq 1.645] \\ &= 0.05 \end{aligned}$$

Thus, a size 0.05 test of  $H_0 : \mu = 10$  against the alternative  $H_a : \mu = 11$  is to reject  $H_0$  if the observed value  $\bar{X} \geq 11.316$ .

Note that, the calculation of  $C$  has nothing to do with the alternative. However,  $c$  impacts type II error:

$$\begin{aligned} \beta = P[TII] &= P[\bar{X} < 11.316 | \mu = \mu_1 = 11] \\ &= P\left[\frac{\bar{X} - 11}{4/5} < \frac{11.316 - 11}{4/5}\right] \\ &= P[Z < 0.395] \\ &= 0.645 \end{aligned}$$

#### My personal note:

- (1)  $C$  is only determined by  $H_0$ , but the choose of  $C$  can impact type II error. The basic idea of the calculation a region where the probability there will be 0.05. And if the observed mean falls into that region, we can reject  $H_0$ .



- (2) Type II error is to calculate the **probability** that **NOT** in the **critical region** defined by **null hypothesis**.

Thus, we can set another critical area to illustrate this.

$$P[10 < \bar{X} < 10.1006] = P[0 < \frac{\bar{X} - 10}{4/5} < 0.1257] = 0.05$$

However,  $P[TypeIIerror]$  for this critical region is:

$$\begin{aligned} P[TII] &= 1 - P[10 < \bar{X} < 10.1006 | \mu = 11] \\ &= 1 - P[\frac{10 - 11}{4/5} < Z < \frac{10.1006 - 11}{4/5}] \\ &= 1 - P[-1.25 < Z < -1.12425] \\ &= 0.9752 \end{aligned}$$

Thus, this critical region is much worse than using the right-hand tail of the distribution of  $\bar{X}$  under  $H_0$ .

If we increase the sample size from 25 to 100, we can reduce the Type II error. Specifically,

$$C_2 = \mu_0 + Z_{1-\alpha}\sigma/\sqrt{n} = 10 + 1.645 \cdot 4/10 = 10.658$$

Thus, the  $P[TII]$  is:

**Type II is the opposite of Type I regarding directions: < vs. >**

$$\begin{aligned} P[\bar{X} < 10.658 | \mu = 11] &= P[\frac{\bar{X} - 11}{4/10} < \frac{10.658 - 11}{4/10}] \\ &= P[Z < -0.855] \\ &= 0.196 \end{aligned}$$

**Note that, we need to choose the direction of the tail based on  $H_1$  in order to avoid a big type II error.** To illustrate, I calculate the following.

**Original, right-hand side**

$$c = \mu_0 + Z_{1-\alpha}\sigma/\sqrt{n} = 10 + 1.645 \cdot 4/5 = 11.316$$

.

**changed, left hand side**

$$c = \mu_0 - Z_{1-\alpha}\sigma/\sqrt{n} = 10 - 1.645 \cdot 4/5 = 8.648$$

Then, we can declare the critical region as  $[-\infty, 8.648]$ . We can verify whether the probability is 0.05

$$\begin{aligned} P[\bar{X} < c | \mu = \mu_0 = 10] &= P\left[\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < \frac{c - \mu_0}{\sigma/\sqrt{n}}\right] \\ &= P\left[Z < \frac{8.648 - 10}{4/5}\right] \\ &= P[Z < -1.645] \\ &= 0.05 \end{aligned}$$

However, in the case, the type II error is as follows.

$$\begin{aligned} \beta = P[TII] &= P[\bar{X} > 8.648 | \mu = \mu_1 = 11] \\ &= P\left[\frac{\bar{X} - 11}{4/5} > \frac{8.648 - 11}{4/5}\right] \\ &= P[Z > -2.94] \\ &= 1 - P[Z < -2.94] \\ &= 1 - \Phi(-2.94) \\ &= 1 - 0.00164 \\ &= 0.9984 \end{aligned}$$

```
pnorm(-2.94)
```

```
## [1] 0.001641061
```

```
1-pnorm(-2.94)
```

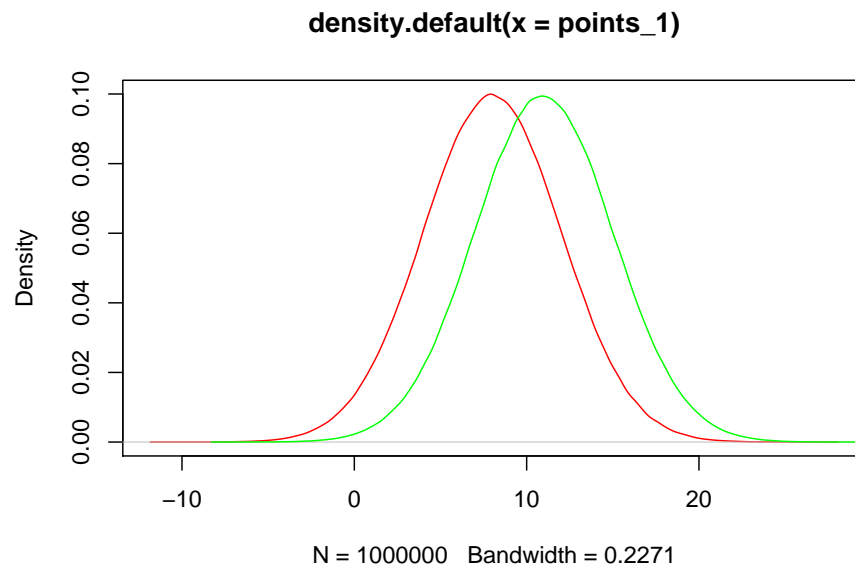
```
## [1] 0.9983589
```

Thus, we can see that it is worse than choose the middle, which has a type II error or 0.9752.

**To summarize:**

- (1) While  $C$  is determined by  $H_0$ , since you can calculate a  $c$  based on a given  $\alpha$  value (which determine the percentile), it is important to incorporate the  $H_1$  into the consideration of determining  $C$ .
- (2) If  $H_1$  is greater than  $H_0$ , it is better to choose the right-hand side, namely “+”. In contrast, if  $H_1$  is smaller than  $H_0$ , it is better to choose the left-hand side, namely “-”. It seems it is always better than to choose a middle. (I am not sure about this, actually).
- (3) The calculation of Type II error is always use the opposite  $<$  or  $>$  in the writing of  $P(\dots)$ .

```
points_1<-rnorm(1000000,mean=8,sd=4)
points_2<-rnorm(1000000,mean=11,sd=4)
plot(density(points_1),col="red")+lines(density(points_2),col="green")
```



```
## integer(0)
```

### 5.1.3 Power function

As above, the computation of  $H_0$  is based on standard normal  $Z$ . If critical region is on the right-tail (e.g.,  $H_0 : \mu = \mu_0, H_1 : \mu = \mu_1, \mu_1 > \mu_0$ ), and we reject  $H_0$  if  $z_0 \geq z_{1-\alpha}$ . That is,

$$P[z_0 \geq z_{1-\alpha}] = \alpha$$

$$\begin{aligned}
P[TII] &= P\left[\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < z_{1-\alpha} \mid \mu = \mu_1\right] \\
&= P\left[\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} < z_{1-\alpha} + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} \mid \mu = \mu_1\right] \\
&= P\left[\frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} < z_{1-\alpha} + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} \mid \mu = \mu_1\right] \\
&= \Phi\left(z_{1-\alpha} + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}}\right) \\
&= \beta
\end{aligned}$$

Thus, we can get the following:

$$z_{1-\alpha} + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} = z_\beta = -z_{1-\beta}$$

Thus, we can get the following:

$$n = \frac{(z_{1-\alpha} + z_{1-\beta})^2 \sigma^2}{(\mu_0 - \mu_1)^2}$$

Thus, for instance,  $\alpha = 0.05, \beta = 0.1, \mu_0 = 10, \mu_1 = 11, \sigma = 4$ , we can get:

$$n = \frac{(1.645 + 1.282)^2 \cdot 16}{1} = 137$$

### Power function:

The power function,  $\pi(\theta)$ , of a test of  $H_0$  is the probability of rejection  $H_0$  when then true value of the parameter is  $\theta$ .

For instance, for simple hypothesis,  $H_0 : \theta = \theta_0$  versus  $H_a : \theta = \theta_1$ , we have:

$$\pi(\theta_0) = P[TI] = \alpha$$

and

$$\pi(\theta_1) = 1 - P[TII] = 1 - \beta$$

**Type II error:** Fix C, then calculate the opposite of  $H_0$ , based on  $H_a$ . Thus, power function based on  $H_a$  is  $\pi(\theta_1) = 1 - P(TII) = 1 - \beta$ .

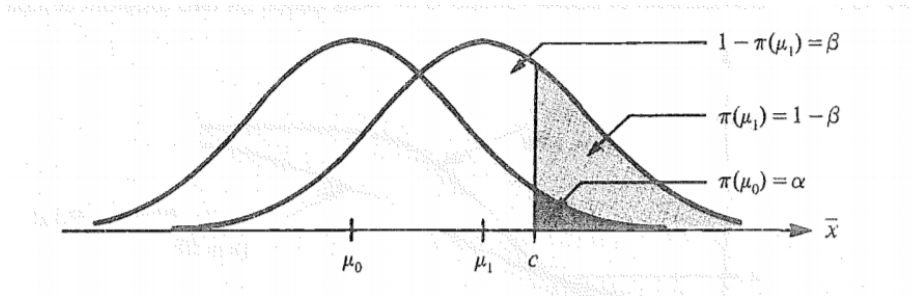


Figure 5.1: From The Textbook

## 5.2 12.2

Note that,  $\pi(\mu)$  (see the following) is that, given the  $\mu$ , as well as given the  $\alpha$ , the probability of rejecting  $H_0$  is  $\pi(\mu)$ .

However, the catch here is that,  $c$  is not determined by  $\mu$ , but actually by  $\mu_0$ .

Also, note that, from the figure above, the **direction** of power calculation  $(1-\beta)$  is the same **direction** of type one error  $\alpha$ . Remember, the only opposite is  $\beta$  and  $\alpha$ , but power of  $1-\beta$  and  $\alpha$  actually are the same region!

### 5.2.1 H\_a is composite

For

$$H_0 : \mu = \mu_0$$

$$H_a : \mu > \mu_0$$

$$\begin{aligned} \pi[\mu] &= P\left[\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq z_{1-\alpha} | \mu\right] \\ &= P\left[\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \geq z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} | \mu\right] \\ &= P\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} | \mu\right] \\ &= 1 - P\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} | \mu\right] \\ &= 1 - \Phi\left(z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right) \end{aligned}$$

Thus, based on the formula above, we know that:

For  $\mu = \mu_0$ , we have  $\pi(\mu_0) = \alpha$ , and for  $\mu > \mu_0$ , we have  $\pi(\mu) = 1 - P[TII]$ .

**My interpretation:**

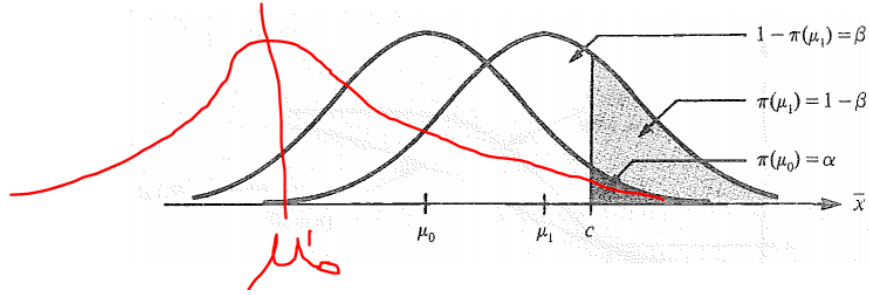


Figure 5.2: From The Textbook

The statement above is about power of the test. Note that,  $\mu$  is always the true  $\mu$ . However, we do not know what the actual value is, thus we have two different statements namely  $H_0$  and  $H_a$ . Thus, do **NOT** think that  $\mu$  stated in  $H_a$  is the true  $\mu$ !

Thus, when we calculate the type II error, we assume that  $\mu = \mu_1$ , which is stated in  $H_a$ . But, in reality, we never know the true type II error. Thus, the calculation of type 2 error is based on an assumption!

Anyhow, link to the figure above, when  $\mu = \mu_0$ , that means that the true  $\mu$  is equal to  $\mu_0$  (note that,  $\mu$  is always true!). That is, the true  $\mu$  is the same as the one stated in the  $H_0$ . In this case,  $\pi(\mu) = \pi(\mu_0) = \alpha$

$$\begin{aligned}
 1 - \Phi\left(z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right) &= 1 - \Phi\left(z_{1-\alpha} + \frac{\mu_0 - \mu_0}{\sigma/\sqrt{n}}\right) \\
 &= 1 - \Phi(z_{1-\alpha}) \\
 &= 1 - (1 - \alpha) \\
 &= \alpha
 \end{aligned}$$

### 5.2.2 Both $H_0$ and $H_a$ are composite

$$H_0 : \mu \leq \mu_0$$

$$H_a : \mu > \mu_0$$

The probability of rejecting  $H_0$  for any  $\mu \leq \mu_0$  is  $\pi(\mu)$ , and  $\pi(\mu) \leq \pi(\mu_0) = \alpha$  for  $\mu \leq \mu_0$ . Thus,  $\text{MAX}_{\mu \leq \mu_0} \pi(\mu) = \alpha$ .

**That is, if the critical region is chosen to have a size of  $\alpha$  at  $\mu_0$ , then the Type I error will be less than  $\alpha$ .**

To understand that, you can visualize that there is  $\mu'_0$ , which is smaller than  $\mu_0$ . If that is the case, the  $c$  set by  $\mu_0$  will result in a smaller  $\pi(\mu) < \alpha$ .

### 5.2.3 P value

To test

$$H_0 : \mu \geq \mu_0$$

$$H_a : \mu < \mu_0$$

Note that, the direction of rejection region is determined by alternative hypothesis  $H_a$ . That is, it is consistent with the direction of  $H_a$  related to  $H_0$ .

If  $H_a$  is the left-hand side of  $H_0$ , the rejection region is on the left-hand side.

If  $H_a$  is the right-hand side of  $H_0$ , the rejection region is on the right-hand side.

Thus, since  $H_a$  here is on the left-hand side of  $H_0$ , we should choose the rejection region to be on the left-hand side as well.

These tests are known one-sided tests of hypotheses.

Another common type of test involves a two-sided alternative. For instance,

$$H_0 : \mu = \mu_0$$

$$H_a : \mu \neq \mu_0$$

In this case, we choose a two-sided test, and the rejection region is as follows:

$$Z_0 \leq -Z_{1-\alpha/2} \text{ or } Z_0 \geq Z_{1-\alpha/2}$$

The type II error is

$$P[-Z_{1-\alpha/2} < Z_0 < Z_{1-\alpha/2} | \mu]$$

Thus, the power function for the two-sided test is as follows.

$$\begin{aligned}
\pi(\mu) &= 1 - P[-Z_{1-\alpha/2} < Z_0 < Z_{1-\alpha/2} | \mu] \\
&= 1 - P[-Z_{1-\alpha/2} < Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < Z_{1-\alpha/2} | \mu] \\
&= 1 - P[-Z_{1-\alpha/2} < Z_0 = \frac{\bar{X} - \mu + \mu - \mu_0}{\sigma/\sqrt{n}} < Z_{1-\alpha/2} | \mu] \\
&= 1 - P[-Z_{1-\alpha/2} < \frac{\bar{X} - \mu + \mu - \mu_0}{\sigma/\sqrt{n}} < Z_{1-\alpha/2} | \mu] \\
&= 1 - P[-Z_{1-\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}} < Z_{1-\alpha/2} | \mu] \\
&= 1 - P[-Z_{1-\alpha/2} - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < Z_{1-\alpha/2} - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} | \mu] \\
&= 1 - P[-Z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < Z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} | \mu] \\
&= 1 - \Phi(Z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}) + \Phi(-Z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}})
\end{aligned}$$

### 5.3 12.3 Tests for the normal distribution

#### 5.3.1 Test for mean ( $\sigma^2$ known)

(1)

$$H_0 : \mu \leq \mu_0$$

$$H_a : \mu > \mu_0$$

The power function is as follows.

$$\begin{aligned}
\pi(\mu) &= P\left[\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha} | \mu\right] = P[\bar{X} > z_{1-\alpha} \cdot (\sigma/\sqrt{n}) + \mu_0 | \mu] \\
&= P\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > \frac{z_{1-\alpha} \cdot (\sigma/\sqrt{n}) + \mu_0 - \mu}{\sigma/\sqrt{n}} | \mu\right] \\
&= P\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} | \mu\right] \\
&= 1 - P\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} | \mu\right] \\
&= 1 - \Phi\left(z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right)
\end{aligned}$$

Thus, the the type II error is as follows:



$$1 - P\left[\frac{\bar{X} - \mu_0}{\sigma/n} > z_{1-\alpha} | \mu\right] = \Phi\left(z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/n}\right)$$

(2)

$$H_0 : \mu \geq \mu_0$$

$$H_a : \mu < \mu_0$$

Note that  $H_a$  is on the left-hand side of  $H_0$ , thus the rejection region is on the left-hand side (i.e., left-tail).

$$\begin{aligned} P[\bar{X} < -Z_{1-\alpha} \cdot (\sigma/n) + \mu_0] &= P\left[\frac{\bar{X} - \mu}{\sigma/n} < -Z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/n}\right] \\ &= \Phi\left(-Z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/n}\right) \end{aligned}$$

(3)

$$H_0 : \mu = \mu_0$$

$$H_a : \mu \neq \mu_0$$

Reject  $H_0$  if

$$Z_0 \leq -Z_{1-\alpha/2} \text{ or } Z_0 \geq Z_{1-\alpha/2}$$

The power function is as follows.

$$\pi(\mu) = 1 - \Phi\left(Z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right) + \Phi\left(-Z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right)$$

(4)

Sample size, see book *P.399*

### 5.3.2 Test for mean ( $\sigma^2$ unknown)

Let  $x_1, x_2, \dots, x_n$  be an observed random sample from  $N(\mu, \sigma^2)$ , where  $\sigma^2$  is unknown, and let

$$(1) \quad t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

$$H_0 : \mu \leq \mu_0$$

$$H_a : \mu > \mu_0$$

Reject  $H_0$  if

$$(2) \quad t_0 \geq t_{1-\alpha}(n-1)$$

$$H_0 : \mu \geq \mu_0$$

$$H_a : \mu < \mu_0$$

Reject  $H_0$  if

$$(3) \quad t_0 \leq -t_{1-\alpha}(n-1)$$

$$H_0 : \mu = \mu_0$$

$$H_a : \mu \neq \mu_0$$

Reject  $H_0$  if

$$(4) \quad t_0 \leq -t_{1-\alpha/2}(n-1) \text{ or } t_0 \geq t_{1-\alpha/2}(n-1)$$

For power function, please see *P.400* .

### 5.3.3 Test for variance

P.401

### 5.3.4 Paired-sample t test

P.403

## 5.4 12.6 Most powerful test

Let  $X_1, \dots, X_n$  have joint pdf  $f(x_1, \dots, x_n; \theta)$ , and consider a critical region  $C$ . The notation for the power function corresponding to  $C$  is

$$\pi_C(\theta) = P[(X_1, \dots, X_n) \in C | \theta]$$

A test  $H_0 : \theta = \theta_0$  versus  $H_a : \theta = \theta_1$  based on a critical region  $C^*$  is said to be a **most powerful test** of size  $\alpha$  if

- (1)  $\pi_{C^*}(\theta_0) = \alpha$  and
- (2)  $\pi_{C^*}(\theta_1) > \pi_C(\theta_1)$  for any other critical region  $C$  of size  $\alpha$  [that is,  $\pi_C(\theta_0) = \alpha$ ].

The following theorem shows how to derive a most powerful critical region for testing simple hypotheses.

### Neyman-Pearson Lemma

Suppose that  $X_1, \dots, X_n$  have joint pdf  $f(x_1, \dots, x_n; \theta)$ . Let

$$\lambda(x_1, \dots, x_n; \theta_0, \theta_1) = \frac{f(x_1, \dots, x_n; \theta_0)}{f(x_1, \dots, x_n; \theta_1)}$$

and let  $C^*$  be the set

$$C^* = \{(x_1, \dots, x_n) | \lambda(x_1, \dots, x_n; \theta_0, \theta_1) \leq k\}$$

where  $k$  is a constant such that

$$P[(X_1, \dots, X_n) \in C^* | \theta_0] = \alpha$$

Then  $C^*$  is a most powerful critical region of size  $\alpha$  for testing  $H_0 : \theta = \theta_0$  versus  $H_a : \theta = \theta_1$ .

The general philosophy of the Neyman-Pearson approach to hypothesis testing is to put sample points into the critical region until it reaches size  $\alpha$ . To maximize power, points should be put into the critical region that are more likely under

$H_a$  than under  $H_0$ . In particular, the Neyman-Pearson lemma says that the criterion for choosing sample points to be included should be based on the magnitude of the ratio of the likelihood functions under  $H_0$  and  $H_a$ .

## 5.5 12.7 Uniformly most powerful tests

If a test is most powerful against every possible value in a composite alternative, then it will be called a uniformly most powerful test.

Let  $X_1, \dots, X_n$  have joint pdf  $f(x_1, \dots, x_n; \theta)$  for  $\theta \in \Omega$ , and consider hypotheses of the form  $H_0 : \theta \in \Omega_0$  versus  $H_a : \theta \in \Omega - \Omega_0$ , where  $\Omega_0$  is a subset of  $\Omega$ . A critical region  $C^*$ , and the associated test, are said to be **uniformly most powerful (UMP)** of size  $\alpha$  if

$$\text{MAX}_{\theta \in \Omega_0} \pi_{C^*}(\theta) = \alpha$$

and

$$\pi_{C^*}(\theta) \geq \pi_C(\theta)$$

for all  $\theta \in \Omega - \Omega_0$  and all critical regions  $C$  of size  $\alpha$ .

## 5.6 12.8 Generalized likelihood ratio tests

The Neyman-Pearson lemma provides a method for deriving a most powerful test of simple hypotheses, and quite often this test also will be UMP for a one-sided composite alternative.

The generalized likelihood ratio test is a generalization for the Neyman-Pearson test, and it provides a desirable test in many applications.

Let  $X = (X_1, \dots, X_n)$  where  $X_1, \dots, X_n$  have joint pdf  $f(x; \theta)$  for  $\theta \in \Omega$ , and consider the hypothesis  $H_0 : \theta \in \Omega_0$  versus  $H_a : \theta \in \Omega - \Omega_0$ . The **Generalized likelihood Ratio (GLR)** is defined by

$$\lambda(x) = \frac{\text{MAX}_{\theta \in \Omega_0} f(x; \theta)}{\text{MAX}_{\theta \in \Omega} f(x; \theta)} = \frac{f(x; \hat{\theta}_0)}{f(x; \hat{\theta})}$$

where  $\hat{\theta}$  denotes the usual MLE of  $\theta$ , and  $\hat{\theta}_0$  denotes the MLE under the restriction that  $H_0$  is true.

## Chapter 6

# Logit and Probit

### 6.1 Logit

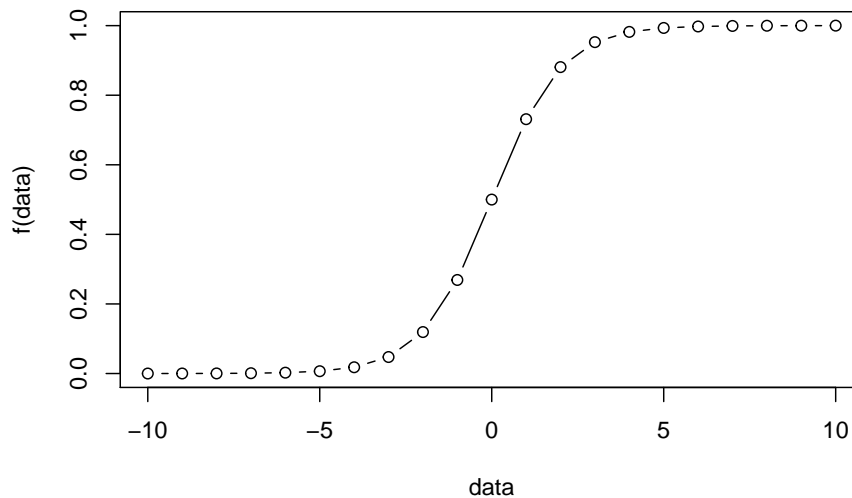
$$f(x) = \log\left(\frac{p(y=1)}{1-p(y=1)}\right)$$

The basic idea of logistic regression:

$$p(y=1) = \frac{1}{1 + e^{-(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n)}} = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}}$$

Thus,  $\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n$  can be from  $-\infty$  to  $+\infty$ , and  $p(y=1)$  will be always within the range of  $(0, 1)$ .

```
f<-function(x){exp(x)/(1+exp(x))}  
data<-seq(-10,10,1)  
plot(data,f(data),type = "b")
```



We can also write the function into another format as follows:

$$\log \frac{p(y=1)}{1-p(y=1)} = \beta_0 + \beta_1 x_1 + \dots + \beta_n x_n$$

Thus, we know that the regression coefficients of  $\beta_i$  actually change the “log-odds” of the event. Of course, note that the magnitude of  $\beta_i$  is dependent upon the units of  $x_i$ .

The following is an example testing whether that home teams are more likely to win in NFL games. The results show that the odd of winning is the same for both home and away teams.

```
mydata = read.csv(url('https://raw.githubusercontent.com/nfl-football-ops/Big-Data-Bow'))
mydata$result_new<-ifelse(mydata$HomeScore>mydata$VisitorScore,1,0)
summary(mydata$result_new)
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
## 0.0000 0.0000 0.0000 0.4945 1.0000 1.0000
```

```
mylogit1 = glm(result_new~1, family=binomial, data=mydata)
summary(mylogit1)
```

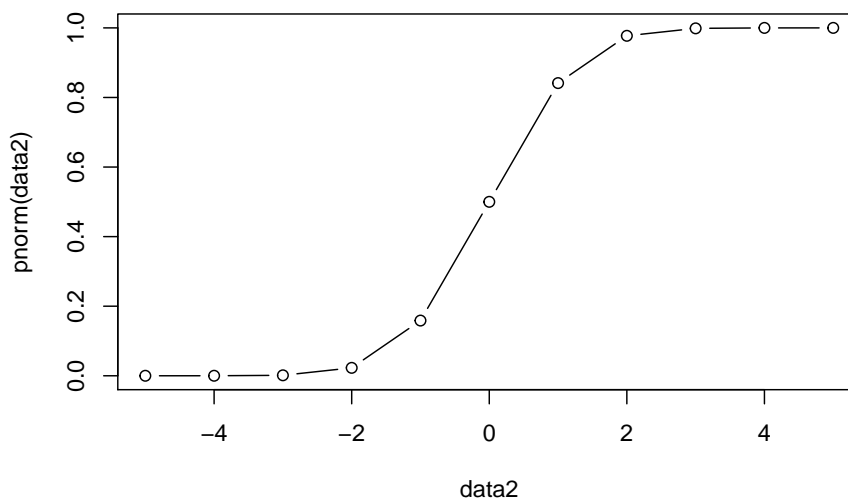
```
##
## Call:
## glm(formula = result_new ~ 1, family = binomial, data = mydata)
##
## Deviance Residuals:
```

```
##      Min      1Q  Median      3Q      Max
## -1.168 -1.168 -1.168   1.187   1.187
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -0.02198    0.20967  -0.105   0.917
##
## (Dispersion parameter for binomial family taken to be 1)
##
##      Null deviance: 126.14  on 90  degrees of freedom
## Residual deviance: 126.14  on 90  degrees of freedom
## AIC: 128.14
##
## Number of Fisher Scoring iterations: 3
```

## 6.2 Probit

As noted above, logit  $f(x) = \log\left(\frac{p(y=1)}{1-p(y=1)}\right)$  provides the resulting range of  $(0,1)$ . Another way to provide the same range is through the cdf of normal distribution. The following R code is used to illustrate this process.

```
data2<-seq(-5,5,1)
plot(data2,pnorm(data2),type = "b")
```



Thus, the cdf of normal distribution can be used to indicate the probability of  $p(y = 1)$ .

$$\Phi(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n) = p(y = 1)$$

Similar to logit model, we can also write the inverse function of the cdf to get the function that can be from  $-\infty$  to  $+\infty$ .

$$\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n = \Phi^{-1}(p(y = 1))$$

Thus, for example, if  $X\beta = -2$ , based on  $\Phi(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n) = p(y = 1)$  we can get that the  $p(y = 1) = 0.023$ .

In contrast, if  $X\beta = 3$ , the  $p(y = 1) = 0.999$ .

```
pnorm(-2)
```

```
## [1] 0.02275013
```

```
pnorm(3)
```

```
## [1] 0.9986501
```

Let's assume that there is a latent variable called  $Y^*$  such that

$$Y^* = X\beta + \epsilon, \epsilon \sim N(0, \sigma^2)$$

You could think of  $Y^*$  as a kind of “proxy” between  $X\beta + \epsilon$  and the observed  $Y(1 \text{ or } 0)$ . Thus, we can get the following. Note that, it does not have to be zero, and can be any constant.

$$Y^* = \begin{cases} 0 & \text{if } y_i^* \leq 0 \\ 1 & \text{if } y_i^* > 0 \end{cases}$$

Thus,

$$y_i^* > 0 \Rightarrow \beta' X_i + \epsilon_i > 0 \Rightarrow \epsilon_i > -\beta' X_i$$

Thus, we can write it as follows. Note that  $\frac{\epsilon_i}{\sigma} \sim N(0, 1)$

$$p(y = 1|x_i) = p(y_i^* > 0|x_i) = p(\epsilon_i > -\beta' X_i) = p\left(\frac{\epsilon_i}{\sigma} > \frac{-\beta' X_i}{\sigma}\right) = \Phi\left(\frac{\beta' X_i}{\sigma}\right)$$

We thus can get:

$$p(y = 0|x_i) = 1 - \Phi\left(\frac{\beta' X_i}{\sigma}\right)$$



For  $p(y = 1|x_i) = \Phi(\frac{\beta'X_i}{\sigma})$ , we can not really estimate both  $\beta$  and  $\sigma$  as they are in a ratio. We can assume  $\sigma = 1$ , then  $\epsilon \sim N(0, 1)$ . We know  $y_i$  and  $x_i$  since we observe them. Thus, we can write it as follows.

$$p(y = 1|x_i) = \Phi(\beta'X_i)$$



## Chapter 7

# Normal distribution

### 7.1 Basics

$\mu$  and  $\sigma$  determine the center and spread of the distribution.

The empirical rule holds for all normal distributions:

- (1) 68% of the area under the curve lies between  $(\mu - \sigma, \mu + \sigma)$ .
- (2) 95% of the area under the curve lies between  $(\mu - 2\sigma, \mu + 2\sigma)$ .
- (3) 99.7% of the area under the curve lies between  $(\mu - 3\sigma, \mu + 3\sigma)$ .

### 7.2 Confidence intervals for normal distributions

$$\bar{X} \pm Z \frac{\sigma}{\sqrt{n}}$$

where,

$\bar{X}$  is the mean

$Z$  is the Z value (see the table below)

$\sigma$  is the standard deviation

$n$  is the number of observations

(We can see the connection between this formula and information shown in the *Basics* section.)

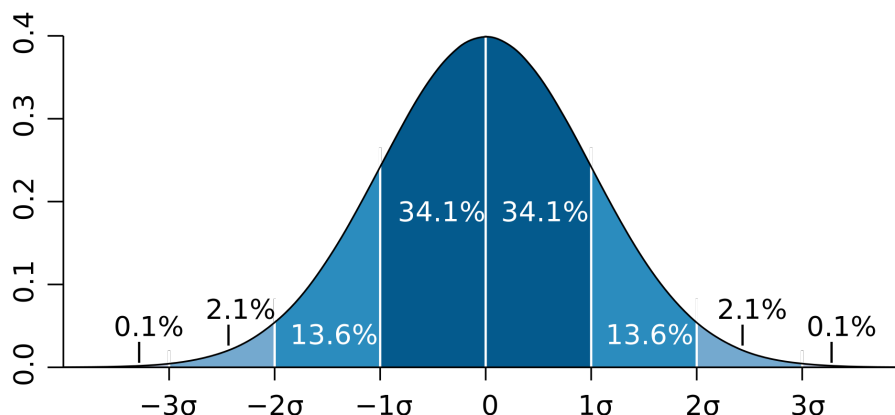


Figure 7.1: Normal

<i>Confidence Levels</i>	<i>Z</i>
80	1.282
85	1.440
90	1.645
95	1.960
99	2.576
99.5	2.807
99.9	3.291

### 7.3 Percentile

A percentile is a measure used in statistics indicating the value below which a given percentage of observations in a group of observations falls.

For example, the 20th percentile is the value (or score) below which 20% of the observations may be found.

For normal distribution,

$-3\sigma$  is the 0.13th percentile (i.e.,  $\frac{100-99.7}{2} = 0.15$ );

$-2\sigma$  is the 2.28th percentile ((i.e.,  $\frac{100-95}{2} = 2.50$ ));

$-1\sigma$  is the 15.87th percentile (i.e.,  $\frac{100-68}{2} = 16$ );

$0\sigma$  is 50th percentile.

$+2\sigma$  is the 97.72nd percentile (i.e.,  $100 - \frac{100-95}{2} = 100 - 2.5 = 97.50$ );

$+3\sigma$  is the 99.87th percentile (i.e.,  $100 - \frac{100-99.70}{2} = 100 - 0.15 = 99.85$ ).

This is related to the 68-95-99.7 rule or the three-sigma rule.

(Note that, it is *related*, not *direct* 68-95-99.7 rule, which is about symmetric situations. See the figure above)

<i>Percentile</i>	<i>Z</i>
90	1.282
—	1.440
95	1.645
—	1.960
—	2.576
—	2.807
99.9	3.000



# Chapter 8

## MLE

### 8.1 Basic idea of MLE

Suppose that we flip a coin,  $y_i = 0$  for tails and  $y_i = 1$  for heads. If we get  $p$  heads from  $n$  trials, we can get the proportion of heads is  $p/n$ , which is the sample mean. If we do not do any further calculation, this is our best guess.

Suppose that the true probability is  $\rho$ , then we can get:

$$\mathbf{L}(y_i) = \begin{cases} \rho & y_i = 1 \\ 1 - \rho & y_i = 0 \end{cases}$$

Thus, we can also write it as follows.

$$\mathbf{L}(y_i) = \rho^{y_i} (1 - \rho)^{1-y_i}$$

Thus, we can get:

$$\prod \mathbf{L}(y_i|\rho) = \rho^{\sum y_i} (1 - \rho)^{\sum (1-y_i)}$$

Further, we can get a log-transformed format.

$$\log(\prod \mathbf{L}(y_i|\rho)) = \sum y_i \log \rho + \sum (1 - y_i) \log(1 - \rho)$$

To maximize the log-function above, we can calculate the derivative with respect to  $\rho$ .

$$\frac{\partial \log(\prod \mathbf{L}(y_i|\rho))}{\partial \rho} = \sum y_i \frac{1}{\rho} - \sum (1 - y_i) \frac{1}{1 - \rho}$$

Set the derivative to zero and solve for  $\rho$ , we can get

$$\begin{aligned}
& \sum y_i \frac{1}{\rho} - \sum (1 - y_i) \frac{1}{1 - \rho} = 0 \\
& \Rightarrow (1 - \rho) \sum y_i - \rho \sum (1 - y_i) = 0 \\
& \Rightarrow \sum y_i - \rho \sum y_i - n\rho + \rho \sum y_i = 0 \\
& \Rightarrow \sum y_i - n\rho = 0 \\
& \Rightarrow \rho = \frac{\sum y_i}{n} = \frac{p}{n}
\end{aligned}$$

Thus, we can see that the  $\rho$  maximizing the likelihood function is equal to the sample mean.

## 8.2 Coin flip example, probit, and logit

In the example above, we are not really trying to estimate a lot of regression coefficients. What we are doing actually is to calculate the sample mean, or intercept in the regression sense. What does it mean? Let's use some data to explain it.

Suppose that we flip a coin 20 times and observe 8 heads. We can use the R's `glm` function to estimate the  $\rho$ . If the result is consistent with what we did above, we should observe that the *cdf* of the estimate of  $\beta_0$  (i.e., intercept) should be equal to  $8/20 = 0.4$ .

```
coins<-c(rep(1,times=8),rep(0,times=12))
table(coins)
```

```
## coins
##  0  1
## 12  8
```

```
coins<-as.data.frame(coins)
```

### 8.2.1 Probit

```
probitresults <- glm(coins ~ 1, family = binomial(link = "probit"), data = coins)
probitresults
```

```
##
## Call:  glm(formula = coins ~ 1, family = binomial(link = "probit"),
##       data = coins)
##
## Coefficients:
## (Intercept)
##      -0.2533
```



```
##
## Degrees of Freedom: 19 Total (i.e. Null); 19 Residual
## Null Deviance: 26.92
## Residual Deviance: 26.92 AIC: 28.92
pnorm(probitresults$coefficients)
```

```
## (Intercept)
## 0.4
```

As we can see the intercept is  $-0.2533$ , and thus  $\Phi(-0.2533471) = 0.4$

### 8.2.2 Logit

We can also use logit link to calculate the intercept as well. Recall that

$$p(y = 1) = \frac{1}{1 + e^{-(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n)}} = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}}$$

Thus,

$$p(y = 1) = \frac{e^{\beta_0}}{1 + e^{\beta_0}}$$

```
logitresults <- glm(coins ~ 1, family = binomial(link = "logit"), data = coins)
logitresults$coefficients
```

```
## (Intercept)
## -0.4054651
```

```
exp(logitresults$coefficients)/(1+exp(logitresults$coefficients))
```

```
## (Intercept)
## 0.4
```

Note that, the default link for the binomial in the glm function is logit.

## 8.3 Further on logit

The probability of  $y = 1$  is as follows:

$$p = p(y = 1) = \frac{1}{1 + e^{-(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n)}} = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}}$$

Thus, the likelihood function is as follows:

$$L = \prod p^{y_i} (1-p)^{1-y_i} = \prod \left( \frac{1}{1 + e^{-(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n)}} \right)^{y_i} \left( \frac{1}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}} \right)^{1-y_i}$$

$$= \prod (1 + e^{-(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n)})^{-y_i} (1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n})^{-(1-y_i)}$$

Thus, the log-likelihood is as follows:

$$\log L = \sum (-y_i \cdot \log(1 + e^{-(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n)}) - (1 - y_i) \cdot \log(1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_n x_n}))$$

Typically, optimisers minimize a function, so we use negative log-likelihood as minimising that is equivalent to maximising the log-likelihood or the likelihood itself.

*#Source of R code: <https://www.r-bloggers.com/logistic-regression/>*

```
mle.logreg = function(fmla, data)
{
  # Define the negative log likelihood function
  logl <- function(theta,x,y){
    y <- y
    x <- as.matrix(x)
    beta <- theta[1:ncol(x)]

    # Use the log-likelihood of the Bernoulli distribution, where p is
    # defined as the logistic transformation of a linear combination
    # of predictors, according to logit(p)=(x%%beta)
    loglik <- sum(-y*log(1 + exp(-(x%%beta))) - (1-y)*log(1 + exp(x%%beta)))
    return(-loglik)
  }

  # Prepare the data
  outcome = rownames(attr(terms(fmla),"factors"))[1]
  dfrTmp = model.frame(data)
  x = as.matrix(model.matrix(fmla, data=dfrTmp))
  y = as.numeric(as.matrix(data[,match(outcome,colnames(data))]))

  # Define initial values for the parameters
  theta.start = rep(0,(dim(x)[2]))
  names(theta.start) = colnames(x)

  # Calculate the maximum likelihood
  mle = optim(theta.start,logl,x=x,y=y, method = 'BFGS', hessian=T)
  out = list(beta=mle$par,vcov=solve(mle$hessian),ll=2*mle$value)
}

mydata = read.csv(url('https://stats.idre.ucla.edu/stat/data/binary.csv'))
mylogit1 = glm(admit~gre+gpa+as.factor(rank), family=binomial, data=mydata)
```

```
mydata$rank = factor(mydata$rank) #Treat rank as a categorical variable
fmla = as.formula("admit~gre+gpa+rank") #Create model formula
mylogit2 = mle.logreg(fmla, mydata) #Estimate coefficients

print(cbind(coef(mylogit1), mylogit2$beta))
```

```
##                [,1]      [,2]
## (Intercept)    -3.989979073 -3.772676422
## gre             0.002264426  0.001375522
## gpa             0.804037549  0.898201239
## as.factor(rank)2 -0.675442928 -0.675543009
## as.factor(rank)3 -1.340203916 -1.356554831
## as.factor(rank)4 -1.551463677 -1.563396035
```

## 8.4 References

[http://www.columbia.edu/~so33/SusDev/Lecture\\_9.pdf](http://www.columbia.edu/~so33/SusDev/Lecture_9.pdf)



## Chapter 9

# Score, Gradient and Jacobian

### 9.1 Score

The score is the gradient (the vector of partial derivatives) of  $\log L(\theta)$ , with respect to an  $m$ -dimensional parameter vector  $\theta$ .

$$S(\theta) = \frac{\partial \ell}{\partial \theta}$$

Typically, they use  $\nabla$  to denote the partial derivative.

$$\nabla \ell$$

Such differentiation will generate a  $m \times 1$  row vector, which indicates the sensitivity of the likelihood.

Quote from Steffen Lauritzen's slides: "Generally the solution to this equation must be calculated by iterative methods. One of the most common methods is the Newton–Raphson method and this is based on successive approximations to the solution, using Taylor's theorem to approximate the equation."

For instance, using logit link, we can get the first derivative of log likelihood logistic regression as follows. We can not really find  $\beta$  easily to make the equation to be 0.

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= \sum_{i=1}^n x_i^T \left[ y_i - \frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}} \right] \\ &= \sum_{i=1}^n x_i^T [y_i - \hat{y}_i]\end{aligned}$$

## 9.2 Fisher scoring

[I will come back to this later.]

<https://www2.stat.duke.edu/courses/Fall00/sta216/handouts/diagnostics.pdf>

<https://stats.stackexchange.com/questions/176351/implement-fisher-scoring-for-linear-regression>

## 9.3 Gradient and Jacobian

**Remarks:** This part discusses gradient in a more general sense.

When  $f(x)$  is only in a single dimension space:

$$\mathbb{R}^n \rightarrow \mathbb{R}$$

$$\nabla f(x) = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right]$$

When  $f(x)$  is only in a m-dimension space (i.e., Jacobian):  $\mathbb{R}^n \rightarrow \mathbb{R}^m$

$$Jac(f) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \frac{\partial f_m}{\partial x_3} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

For instance,

$$\mathbb{R}^n \rightarrow \mathbb{R}:$$

$$f(x, y) = x^2 + 2y$$

$$\nabla f(x, y) = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] = [2x, 2]$$

$$\mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f(x, y) = (x^2 + 2y, x^3)$$

$$Jac(f) = \begin{bmatrix} 2x & 2 \\ 3x^2 & 0 \end{bmatrix}$$

## 9.4 Hessian and Fisher Information

Hessian matrix or Hessian is a square matrix of second-order partial derivatives of a scalar-valued function, or scalar field.

$$\mathbb{R}^n \rightarrow \mathbb{R}$$

$$Hessian = \nabla^2(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \cdots & & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

As a special case, in the context of logit:

Suppose that the log likelihood function is  $\ell(\theta)$ .  $\theta$  is a  $m$  dimension vector.

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \cdots \\ \theta_m \end{bmatrix}$$

$$Hessian = \nabla^2(\ell) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \theta_1^2} & \frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_3} & \cdots & \frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_m} \\ \frac{\partial^2 \ell}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ell}{\partial \theta_2^2} & \frac{\partial^2 \ell}{\partial \theta_2 \partial \theta_3} & \cdots & \frac{\partial^2 \ell}{\partial \theta_2 \partial \theta_m} \\ \frac{\partial^2 \ell}{\partial \theta_3 \partial \theta_1} & \frac{\partial^2 \ell}{\partial \theta_3 \partial \theta_2} & \frac{\partial^2 \ell}{\partial \theta_3^2} & \cdots & \frac{\partial^2 \ell}{\partial \theta_3 \partial \theta_m} \\ \cdots & & & & \\ \frac{\partial^2 \ell}{\partial \theta_m \partial \theta_1} & \frac{\partial^2 \ell}{\partial \theta_m \partial \theta_2} & \frac{\partial^2 \ell}{\partial \theta_m \partial \theta_3} & \cdots & \frac{\partial^2 \ell}{\partial \theta_m \partial \theta_m} \end{bmatrix}$$

“In statistics, the observed information, or observed Fisher information, is the negative of the second derivative (the Hessian matrix) of the “log-likelihood” (the logarithm of the likelihood function). It is a sample-based version of the Fisher information.” (Direct quote from Wikipedia.)

Thus, the observed information matrix:

$$-Hessian = -\nabla^2(\ell)$$

Expected (Fisher) information matrix:

$$E[-\nabla^2(\ell)]$$





## Chapter 10

# Canonical link function

Inspired by a Stack Exchange post, I created the following figure:

$$\frac{\text{Paramter}}{\theta} \longrightarrow \gamma'(\theta) = \mu \longrightarrow \frac{\text{Mean}}{\mu} \longrightarrow g(\mu) = \eta \longrightarrow \frac{\text{Linearpredictor}}{\eta}$$

For the case of  $n$  time Bernoulli (i.e., Binomial), its canonical link function is logit. Specifically,

$$\frac{\text{Paramter}}{\theta = \beta^T x_i} \longrightarrow \gamma'(\theta) = \frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}} \longrightarrow \frac{\text{Mean}}{\mu = \frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}}} \longrightarrow g(\mu) = \log \frac{\frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}}}{1 - \frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}}} \longrightarrow \frac{\text{Linearpredictor}}{\eta = \beta^T x_i}$$

Thus, we can see that,

$$\theta \equiv \eta$$

The link function  $g(\mu)$  relates the linear predictor  $\eta = \beta^T x_i$  to the mean  $\mu$ .

**Remarks:**

- (1) Parameter is  $\theta = \beta^T x_i$  (Not  $\mu$ !).
- (2)  $\mu = p(y = 1) = \frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}}$  (Not logit!).
- (3) Link function (i.e.,  $g(\mu)$ ) = logit = logarithm of odds =  $\log \frac{\text{Event-Happened}}{\text{Event-Not-Happened}}$ .
- (4)  $g(\mu) = \log \frac{\mu}{1-\mu} = \beta^T x_i$ . Thus, link function = linear predictor = log odds!

- (5) Quote from the Stack Exchange post “Newton Method and Fisher scoring for finding the ML estimator coincide, these links simplify the derivation of the MLE.”

(Recall, we know that  $\mu$  or  $p(y = 1)$  is the mean function. Recall that,  $n$  trials of coin flips, and get  $p$  heads. Thus  $\mu = \frac{p}{n}$ .)

## Chapter 11

# Ordinary Least Squares (OLS)

Suppose we have  $n$  observation, and  $m$  variables.

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2m} \\ \dots & & & & \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nm} \end{bmatrix}$$

Thus, we can write it as the following  $n$  equations.

$$y_1 = \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + \dots + \beta_m x_{1m}$$

$$y_2 = \beta_0 + \beta_1 x_{21} + \beta_2 x_{22} + \dots + \beta_m x_{2m}$$

$$y_3 = \beta_0 + \beta_1 x_{31} + \beta_2 x_{32} + \dots + \beta_m x_{3m}$$

...

$$y_n = \beta_0 + \beta_1 x_{n1} + \beta_2 x_{n2} + \dots + \beta_m x_{nm}$$

We can combine all the  $n$  equations as the following one:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_m x_{im} (i \in [1, n])$$

We can further rewrite it as a matrix format as follows.

$$y = X\beta$$

Where,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \dots \\ y_n \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & x_{11} & x_{12} & x_{13} & \dots & x_{1m} \\ 1 & x_{21} & x_{22} & x_{23} & \dots & x_{2m} \\ \dots & & & & & \\ 1 & x_{n1} & x_{n2} & x_{n3} & \dots & x_{nm} \end{bmatrix}$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \dots \\ \beta_m \end{bmatrix}$$

Since later we need the inverse of  $X$ , we need to make it into a square matrix.

$$X^T y = X^T X \hat{\beta} \Rightarrow \hat{\beta} = (X^T X)^{-1} X^T y$$

We can use R to implement this calculation. As we can see, there is no need to do any iterations at all, but rather just pure matrix calculation.

```
X<-matrix(rnorm(1000),ncol=2) # we define a 2 column matrix, with 500 rows
X<-cbind(1,X) # add a 1 constant
beta_true<-c(2,1,2) # True regression coefficients
beta_true<-as.matrix(beta_true)
y=X%%beta_true+rnorm(500)

transposed_X<-t(X)
beta_hat<-solve(transposed_X%%X)%%transposed_X%%y
beta_hat

##           [,1]
## [1,] 1.947625
## [2,] 1.003846
## [3,] 2.009860
```

**Side Notes** The function of `as.matrix` will automatically make `c(2,1,2)` become the dimension of  $3 \times 1$ , you do not need to transpose the  $\beta$ .

## 11.1 Taylor series

$$\begin{aligned} f(x)|_a &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f'(a)}{2!}(x-a)^2 + \frac{f''(a)}{3!}(x-a)^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \end{aligned}$$

For example:

$$\begin{aligned} e^x|_{a=0} &= e^a + \frac{e^a}{1!}(x-a) + \frac{e^a}{2!}(x-a)^2 + \dots + \frac{e^a}{n!}(x-a)^n \\ &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots + \frac{1}{n!}x^n \end{aligned}$$

if  $x = 2$

$$e^2 = 7.389056$$

$$e^2 \approx 1 + \frac{1}{1!}x = 1 + \frac{1}{1!}2 = 3$$

$$e^2 \approx 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 = 1 + \frac{1}{1!}2 + \frac{1}{2!}2 = 5 \dots$$

$$e^2 \approx 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^2 + \frac{1}{4!}x^2 + \frac{1}{5!}x^2 = 7.2666\dots$$

## 11.2 References

1. Steffen Lauritzen's slides:

<http://www.stats.ox.ac.uk/~steffen/teaching/bs2HT9/scoring.pdf>

2. The Stack Exchange post:

<https://stats.stackexchange.com/questions/40876/what-is-the-difference-between-a-link-function-and-a-canonical-link-function>

3. Wikipedia for OLS

[https://en.wikipedia.org/wiki/Ordinary\\_least\\_squares](https://en.wikipedia.org/wiki/Ordinary_least_squares)

4. Gradient and Jacobian

<https://math.stackexchange.com/questions/1519367/difference-between-gradient-and-jacobian>

[https://www.youtube.com/watch?v=3xVMVT-2\\_t4](https://www.youtube.com/watch?v=3xVMVT-2_t4)

<https://math.stackexchange.com/questions/661195/what-is-the-difference-between-the-gradient-and-the-directional-derivative>

5. Hessian

[https://en.wikipedia.org/wiki/Hessian\\_matrix](https://en.wikipedia.org/wiki/Hessian_matrix)

6. Observed information

[https://en.wikipedia.org/wiki/Observed\\_information](https://en.wikipedia.org/wiki/Observed_information)

## 7. Fisher information

[https://people.missouristate.edu/songfengzheng/Teaching/MTH541/Lecture%20notes/Fisher\\_\\_info.pdf](https://people.missouristate.edu/songfengzheng/Teaching/MTH541/Lecture%20notes/Fisher__info.pdf)

## 8. Link function

[https://en.wikipedia.org/wiki/Generalized\\_linear\\_model#Link\\_function](https://en.wikipedia.org/wiki/Generalized_linear_model#Link_function)

<https://stats.stackexchange.com/questions/40876/what-is-the-difference-between-a-link-function-and-a>

## Chapter 12

# Cholesky decomposition

### 12.1 Example 1

Use Cholesky decomposition to generate 1,000 trivariate normal deviates  $X_1, \dots, x_{1000}$  with mean  $\mu = (-2, 4, 3)$  and covariance matrix

$$X = \begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 4 & 1 \\ 0.5 & 1 & 5 \end{bmatrix}$$

```
Nsim = 10
means = c(-2,4,3)
N_columns = 3

# Generating random standard normal distribution numbers
Generated_numbers = matrix(rnorm(N_columns * Nsim), nrow = N_columns)

# The provided covariance matrix
cov_matrix = rbind(c(2, -1, 0.5), c(-1, 4, 1), c(0.5, 1, 5))

# Cholesky decomposition
Cholesky_decom_results = chol(cov_matrix)

# Data is transformed using the Cholesky decomposition
adjusted_data = t(Generated_numbers) %*% Cholesky_decom_results

Final_data = t(t(adjusted_data) + means)
```

```

# calculating column means
colMeans(Final_data)

## [1] -2.228181  4.315511  2.899584

# calculating column variances
apply(Final_data,2,var)

## [1] 1.080893 4.034614 1.429978

# calculating covariance matrix
cov(Final_data)

##           [,1]      [,2]      [,3]
## [1,] 1.0808929 -0.4839923 0.2055892
## [2,] -0.4839923  4.0346136 1.3871988
## [3,] 0.2055892  1.3871988 1.4299777

```

## 12.2 Example 2

AR(1) Covariance Matrix with Correlation Rho and Variance SigmaSq. Note that, there is only one individual or participant in this data simulation.

```

n = 10;
SigmaSq = 5;
Rho = 0.8;

V = matrix(rep(n*n,0),n,n);

for (i in 1:n)
{
  for (j in i:n)
  {
    V[i,j]=SigmaSq*Rho^(j-i)
    V[j,i]=V[i,j]
  }
}

set.seed(123)
random_normal<-rnorm(n,2,1)
#chol(V) %*% random_normal
#colSums (chol(V))
b2<-t(as.matrix(random_normal))%*%chol(V)

pi = exp(b2)/(1 + exp(b2));

y<-ifelse(pi>runif(1),1,0)

```



```

y

##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
## [1,]    1    1    1    1    1    1    1    1    1    1

# The code above basically completes the generating job!
# The code below is to check

b = b2[2:n]
c = b2[1:(n-1)]
cor(b,c)

## [1] 0.8967058

sd(as.vector(b2))

## [1] 3.535119

# note that, you can not use var, as the mean is not zero, but rather it is 2
var(as.vector(b2))

## [1] 12.49707

#Not sure why the means are not the same ?
mean(as.vector(b2))

## [1] 10.01925

mean(random_normal)

## [1] 2.074626

```

## 12.3 Example 3

The following code very similar to the code shown above. However, it had only one observation. To illustrate the situation where there are more than one individual (or, participant), I did the code below.

```

n =25;   #the number of time points
m= 15;   # the number of participants or individuals, whichever ways you would like to think
SigmaSq = 5;
Rho = 0.8;

filling_numbers<-rep(n*n,0)
V = matrix(filling_numbers,n,n);

for (i in 1:n)
{
  for (j in i:n)

```

```

{
  V[i,j]=SigmaSq*Rho^(j-i)
  V[j,i]=V[i,j]
}
}

set.seed(2345)
random_normal<-matrix(rnorm(m*n),nrow = m)
#chol(V) %%% random_normal
#colSums (chol(V))
b2<-random_normal%%chol(V)

pi = exp(b2)/(1 + exp(b2));

random_unfirom<-matrix(runif(m*n),nrow = m)

y<-ifelse(pi>random_unfirom,1,0)
y

```

```

##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12] [,13]
## [1,]  0    0    0    0    0    0    1    1    1    1    1    0    1
## [2,]  0    1    1    1    1    1    0    1    0    0    0    0    0
## [3,]  1    0    0    0    0    1    0    1    0    1    1    0    0
## [4,]  1    0    0    0    0    0    0    0    1    0    0    0    0
## [5,]  0    0    0    0    1    0    0    0    0    0    0    0    1
## [6,]  0    0    1    1    0    0    0    1    0    0    0    0    0
## [7,]  0    0    0    0    0    1    0    0    1    0    0    0    1
## [8,]  1    1    1    1    0    0    0    0    0    1    1    0    0
## [9,]  1    1    1    1    0    1    1    1    0    1    0    0    0
## [10,] 1    0    1    1    1    1    1    1    1    1    0    0    0
## [11,] 1    1    1    1    1    1    1    1    1    0    1    0    0
## [12,] 1    1    1    1    1    1    0    1    1    1    1    1    0
## [13,] 0    1    0    0    0    0    0    0    1    1    1    1    0
## [14,] 1    1    0    1    1    1    1    1    1    0    0    0    0
## [15,] 1    1    0    1    1    1    1    0    0    1    0    0    1
##      [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24]
## [1,]  1    1    1    1    0    1    0    0    1    0    0
## [2,]  0    1    1    1    1    1    0    0    0    1    0
## [3,]  0    1    0    0    1    1    1    1    0    0    1
## [4,]  1    0    0    0    0    0    1    0    1    0    0
## [5,]  0    0    1    1    1    0    1    1    1    0    1
## [6,]  1    1    0    0    1    0    0    1    0    1    1
## [7,]  0    0    0    0    0    1    0    0    1    0    0
## [8,]  0    0    0    0    0    0    0    0    0    1    1
## [9,]  0    1    1    1    0    0    1    0    0    0    0

```

```
## [10,] 1 1 1 1 1 1 1 1 1 1 1
## [11,] 0 0 0 1 0 0 0 0 1 1 1
## [12,] 0 0 0 0 1 1 1 0 0 0 1
## [13,] 0 1 0 0 0 1 0 1 1 1 1
## [14,] 0 0 0 0 1 0 1 0 0 0 0
## [15,] 0 0 0 0 0 0 1 0 1 0 0
##      [,25]
## [1,] 0
## [2,] 1
## [3,] 1
## [4,] 0
## [5,] 1
## [6,] 1
## [7,] 0
## [8,] 1
## [9,] 0
## [10,] 1
## [11,] 0
## [12,] 0
## [13,] 1
## [14,] 0
## [15,] 1

# The code above basically completes the generating job! The code below is to check

# The following calculates variance
# calculate variance of each column
mean(apply(b2, 2, var))

## [1] 4.330903

# calculate variance of each row
mean(apply(b2, 1, var))

## [1] 3.568107

# The whole table
var(as.vector(b2))

## [1] 4.299165

# The following code calculates the correlation
b = b2[,2:n]
c = b2[,1:(n-1)]

collected_cor<-rep(0,m-1) #creating an empty vector to collect correlation.
for (i in 1:(m-1))
{collected_cor[i]<-cor(b[i,],c[i,])}
collected_cor
```

```
## [1] 0.8473037 0.7065013 0.6376223 0.5481540 0.7851062 0.6576329 0.4844481  
## [8] 0.6950847 0.6731673 0.6409116 0.7966547 0.7184030 0.8001861 0.7913736
```

```
mean(collected_cor)
```

```
## [1] 0.6987535
```

```
mean(y)
```

```
## [1] 0.456
```

```
log(mean(y)/(1-mean(y)))
```

```
## [1] -0.1764564
```

```
# It will always get a value close to zero, since we set the mean to be zero when simu
```