GEO441: Problem Set 2

Will Eaton

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1 Tsunami equations

$$\partial_t^2 P = \partial_x v^2 \partial_x P + \partial_y v^2 \partial_y P + v^2(\partial_x^2 P + \partial_y^2 P)$$

1.1 2nd order time, 2nd order space

$$\partial_t^2 P \approx \frac{P_{i,j}^{t+1} - 2P_{i,j}^t + P_{i,j}^{t-1}}{(\Delta t)^2}$$

Using a centred approximation for the first order terms:

$$\partial_x P \approx \frac{P_{i+1,j}^t - P_{i-1,j}^t}{2\Delta x}$$

Overall we then get:

$$\frac{P_{i,j}^{t+1} - 2P_{i,j}^t + P_{i,j}^{t-1}}{(\Delta t)^2} = \frac{(P_{i+1,j}^t - P_{i-1,j}^t)({v_{i+1,j}^t}^2 - {v_{i-1,j}^t}^2)}{4(\Delta x)^2} + \frac{(P_{i,j+1}^t - P_{i,j-1}^t)({v_{i,j+1}^t}^2 - {v_{i,j-1}^t}^2)}{4(\Delta y)^2} + {v_{i,j}^t}^2\left(\frac{P_{i+1,j}^t - 2P_{i,j}^t + P_{i-1,j}^t}{(\Delta x)^2} + \frac{P_{i,j+1}^t - 2P_{i,j}^t + P_{i,j-1}^t}{(\Delta y)^2}\right)$$

Hence rearranging for the next timestep in P we get

$$P_{i,j}^{t+1} = (\Delta t)^2\Bigg[\frac{(P_{i+1,j}^t - P_{i-1,j}^t)({v_{i+1,j}^t}^2 - {v_{i-1,j}^t}^2)}{4(\Delta x)^2} + \frac{(P_{i,j+1}^t - P_{i,j-1}^t)({v_{i,j+1}^t}^2 - {v_{i,j-1}^t}^2)}{4(\Delta y)^2} + {v_{i,j}^t}^2\left(\frac{P_{i+1,j}^t - 2P_{i,j}^t + P_{i-1,j}^t}{(\Delta x)^2} + \frac{P_{i,j+1}^t - 2P_{i,j}^t + P_{i,j-1}^t}{(\Delta y)^2}\right)\Bigg] + 2P_{i,j}^t - P_{i,j}^{t-1}$$

Where the homogenous case is

$$P_{i,j}^{t+1} = (\Delta t)^2 {v_{i,j}^t}^2\left(\frac{P_{i+1,j}^t - 2P_{i,j}^t + P_{i-1,j}^t}{(\Delta x)^2} + \frac{P_{i,j+1}^t - 2P_{i,j}^t + P_{i,j-1}^t}{(\Delta y)^2}\right) + 2P_{i,j}^t - P_{i,j}^{t-1}$$

1.2 2^{nd} order time, 4^{th} order space

$$\partial_t^2 P \approx \frac{P_{i,j}^{t+1} - 2P_{i,j}^t + P_{i,j}^{t-1}}{(\Delta t)^2}$$

$$\partial_x v^2 \partial_x P \approx \frac{-P_{i+2,j}^t + 8P_{i+1,j}^t - 8P_{i-1,j}^t + P_{i-2,j}^t}{12\Delta x} \frac{-{v_{i+2,j}^t}^2 + 8{v_{i+1,j}^t}^2 - 8{v_{i-1,j}^t}^2 + {v_{i-2,j}^t}^2}{12\Delta x}$$

$$v^2(\partial_x^2 P + \partial_y^2 P) \approx {v_{i,j}^t}^2 \Big[\frac{-P_{i+2,j}^t + 16P_{i+1,j}^t - 30P_{i,j}^t + 16P_{i-1,j}^t - P_{i-2,j}^t}{12(\Delta x)^2} + \frac{-P_{i,j+2}^t + 16P_{i,j+1}^t - 30P_{i,j}^t + 16P_{i,j-1}^t - P_{i,j-2}^t}{12(\Delta y)^2}\Big]$$

So putting it all together we get

$$\begin{aligned}\frac{P_{i,j}^{t+1} - 2P_{i,j}^t + P_{i,j}^{t-1}}{(\Delta t)^2} &= \frac{(-P_{i+2,j}^t + 8P_{i+1,j}^t - 8P_{i-1,j}^t + P_{i-2,j}^t)(-{v_{i+2,j}^t}^2 + 8{v_{i+1,j}^t}^2 - 8{v_{i-1,j}^t}^2 + {v_{i-2,j}^t}^2)}{144(\Delta x)^2} \\ &+ \frac{(-P_{i,j+2}^t + 8P_{i,j+1}^t - 8P_{i,j-1}^t + P_{i,j-2}^t)(-{v_{i,j+2}^t}^2 + 8{v_{i,j+1}^t}^2 - 8{v_{i,j-1}^t}^2 + {v_{i,j-2}^t}^2)}{144(\Delta y)^2} \\ &+ {v_{i,j}^t}^2 \Big[\frac{-P_{i+2,j}^t + 16P_{i+1,j}^t - 30P_{i,j}^t + 16P_{i-1,j}^t - P_{i-2,j}^t}{12(\Delta x)^2} \\ &+ \frac{-P_{i,j+2}^t + 16P_{i,j+1}^t - 30P_{i,j}^t + 16P_{i,j-1}^t - P_{i,j-2}^t}{12(\Delta y)^2}\Big]\end{aligned}$$

So then if we rearrange for the next timestep in P we have

$$\begin{aligned}P_{i,j}^{t+1} = (\Delta t)^2 \Bigg[&\frac{(-P_{i+2,j}^t + 8P_{i+1,j}^t - 8P_{i-1,j}^t + P_{i-2,j}^t)(-{v_{i+2,j}^t}^2 + 8{v_{i+1,j}^t}^2 - 8{v_{i-1,j}^t}^2 + {v_{i-2,j}^t}^2)}{144(\Delta x)^2} \\ &+ \frac{(-P_{i,j+2}^t + 8P_{i,j+1}^t - 8P_{i,j-1}^t + P_{i,j-2}^t)(-{v_{i,j+2}^t}^2 + 8{v_{i,j+1}^t}^2 - 8{v_{i,j-1}^t}^2 + {v_{i,j-2}^t}^2)}{144(\Delta y)^2} \\ &+ {v_{i,j}^t}^2 \Big[\frac{-P_{i+2,j}^t + 16P_{i+1,j}^t - 30P_{i,j}^t + 16P_{i-1,j}^t - P_{i-2,j}^t}{12(\Delta x)^2} \\ &+ \frac{-P_{i,j+2}^t + 16P_{i,j+1}^t - 30P_{i,j}^t + 16P_{i,j-1}^t - P_{i,j-2}^t}{12(\Delta y)^2}\Big]\Bigg] + 2P_{i,j}^t - P_{i,j}^{t-1}\end{aligned}$$

Note that in the homogenous case there are no changes in velocity spatially such that the first two terms on the RHS go to 0 and we get

$$\begin{aligned}P_{i,j}^{t+1} = (\Delta t)^2 {v_{i,j}^t}^2 \Big[&\frac{-P_{i+2,j}^t + 16P_{i+1,j}^t - 30P_{i,j}^t + 16P_{i-1,j}^t - P_{i-2,j}^t}{12(\Delta x)^2} \\ &+ \frac{-P_{i,j+2}^t + 16P_{i,j+1}^t - 30P_{i,j}^t + 16P_{i,j-1}^t - P_{i,j-2}^t}{12(\Delta y)^2}\Big] + 2P_{i,j}^t - P_{i,j}^{t-1}\end{aligned}$$

1.3 Chain Rule variations

Alternatively we can replace $\partial_x v^2 \rightarrow 2v\partial_x v$ and incorporate this into the equations for the heterogenous cases. In doing so we get for second and fourth order, respectively:

$$
\begin{aligned}
P_{i,j}^{t+1} = (\Delta t)^2 \Bigg[\Big(& \frac{(P_{i+1,j}^t - P_{i-1,j}^t)(v_{i+1,j}^t - v_{i-1,j}^t)}{2(\Delta x)^2} + \frac{(P_{i,j+1}^t - P_{i,j-1}^t)(v_{i,j+1}^t - v_{i,j-1}^t)}{2(\Delta y)^2} \Big) v_{i,j}^t \\
& + {v_{i,j}^t}^2 \Big(\frac{P_{i+1,j}^t - 2P_{i,j}^t + P_{i-1,j}^t}{(\Delta x)^2} + \frac{P_{i,j+1}^t - 2P_{i,j}^t + P_{i,j-1}^t}{(\Delta y)^2} \Big) \Bigg] + 2P_{i,j}^t - P_{i,j}^{t-1}
\end{aligned}
$$

$$
\begin{aligned}
P_{i,j}^{t+1} = (\Delta t)^2 \Bigg[& \frac{(-P_{i+2,j}^t + 8P_{i+1,j}^t - 8P_{i-1,j}^t + P_{i-2,j}^t)(-v_{i+2,j}^t + 8v_{i+1,j}^t - 8v_{i-1,j}^t + v_{i-2,j}^t)}{72(\Delta x)^2} v_{i,j}^t \\
& + \frac{(-P_{i,j+2}^t + 8P_{i,j+1}^t - 8P_{i,j-1}^t + P_{i,j-2}^t)(-v_{i,j+2}^t + 8v_{i,j+1}^t - 8v_{i,j-1}^t + v_{i,j-2}^t)}{72(\Delta y)^2} v_{i,j}^t \\
& + {v_{i,j}^t}^2 \Big[\frac{-P_{i+2,j}^t + 16P_{i+1,j}^t - 30P_{i,j}^t + 16P_{i-1,j}^t - P_{i-2,j}^t}{12(\Delta x)^2} \\
& + \frac{-P_{i,j+2}^t + 16P_{i,j+1}^t - 30P_{i,j}^t + 16P_{i,j-1}^t - P_{i,j-2}^t}{12(\Delta y)^2} \Big] \Bigg] + 2P_{i,j}^t - P_{i,j}^{t-1}
\end{aligned}
$$

These are the formula that I actually use in my code.

2 Running the code

Currently the code seems to be stable for the homogenous case with both a 2^{nd} and 4^{th} order spatial discretisation. However the heterogenous version appears to be unstable for both cases. To try and rectify this I attempted to implement a gaussian smoothing kernel to the velocity (bathymetry) model. This can be switched on and off in the same way as heterogenity can be by setting:

```
int hetero   = 1      /* 0 for homogenous; 1 for heterogenous*/
int smooth   = 1      /* 1 to apply smoothing kernel; no smoothing otherwise */
```

I unfortunately can not find the issue that is causing these instabilities but will continue to try and resolve it.