Discretised Equations

In my code I use equation (1a) or (2) for the displacement form and (1b) for all velocity - stress forms.

$$\partial_t^2 u = c^2 \partial_x^2 u$$

$$\frac{u_x^{t+1} - 2u_x^t + u_x^{t-1}}{(\Delta t)^2} = c^2 \frac{u_{x+1}^t - 2u_x^t + u_{x-1}^t}{(\Delta x)^2}$$

$$\text{(1A)} \quad u_x^{t+1} = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(u_{x+1}^t + u_{x-1}^t - 2u_x^t\right) + 2u_x^t - u_x^{t-1}$$

If we instead consider the homogenous case for velocity - stress:

$$\rho \partial_t v = \partial_x T$$

$$\partial_t T = K \partial_x v \quad \text{assuming } K \sim K(x) \text{ only}$$

$$\frac{\rho_x}{2\Delta t}\left(v_x^{t+1} - v_x^{t-1}\right) = \frac{1}{2\Delta x}\left(T_{x+1}^t - T_{x-1}^t\right) \longrightarrow v_x^{t+1} = \frac{\Delta t}{\cancel{2}\rho_x \Delta x}\left(T_{x+1}^t - T_{x-1}^t\right) + v_x^{t-1}$$

$$\frac{1}{2\Delta t}\left(T_x^{t+1} - T_x^{t-1}\right) = \frac{K}{2\Delta x}\left(v_{x+1}^t - v_{x-1}^t\right) \longrightarrow T_x^{t+1} = \frac{K\Delta t}{\cancel{2}\Delta x}\left(v_{x+1}^t - v_{x-1}^t\right) + T_x^{t-1}$$

(1b)

↑ could alternatively use $\frac{1}{\Delta t}\left(T_x^{t+1} - T_x^t\right)$

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for heterogenous case we need to edit the displacement formulation:

$$\rho \partial_t^2 u = \partial_x (K \partial_x u) = \partial_x K \partial_x u + K \partial_x^2 u$$

$$\partial_x K \partial_x u \approx \frac{1}{(2\Delta x)^2}\left(K_{x+1}^t - K_{x-1}^t\right)\left(u_{x+1}^t - u_{x-1}^t\right)$$

so the inhomogenous eqn becomes:

$$\text{(2)} \quad u_x^{t+1} = \left(\frac{\Delta t^2}{\rho(\Delta x)^2}\right)\left[K\left(u_{x+1}^t - 2u_x^t + u_{x-1}^{t+1}\right) + \underbrace{\frac{1}{4}\left(K_{x+1}^t - K_{x-1}^t\right)\left(u_{x+1}^t - u_{x-1}^t\right)}_{\text{inhomogenous part}}\right] + 2u_x^t - u_x^{t-1}$$