Problem Set 0: Refresher on index notation

Submission deadline: February 2, 2022

Note: Use following symbols, the permutation symbol, ϵ_{ijk}, and the Kronecker's delta, δ_{ij} wherever necessary.

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } i, j, \ldots \text{ is an even permutation of 1,2, } \ldots, & \text{i.e., (1,2,3), (3,1,2),} \ldots \quad , \\ 0 & \text{if some indices are identical,} & \text{i.e., } i = j,\ i = k, \ldots \quad , \\ -1 & \text{if } i, j, \ldots \text{is an odd permutation of 1,2,} \ldots, & \text{i.e., (1,3,2), (2,1,3),} \ldots \quad . \end{cases}$$

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

1. Rewrite the following expressions for the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ using index notation:
   (a) $\mathbf{a} \cdot \mathbf{b} = c$
   (b) $\mathbf{a} \times \mathbf{b} = \mathbf{d}$
   (c) $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \lambda$
2. Rewrite the following expressions using index notation:
   (a) $\mathrm{grad}\, \phi = \nabla \phi$
   (b) $\mathrm{div}\, \mathbf{v} = \nabla \cdot \mathbf{v}$
   (c) $\mathrm{curl}\, \mathbf{v} = \nabla \times \mathbf{v}$
   (d) $\nabla^2 \phi = \nabla \cdot \nabla \phi$
3. Use index notation to prove that $\nabla \times \nabla \phi = \mathbf{0}$ and $\nabla \cdot \nabla \times \mathbf{a} = 0$.
4. Prove the $\epsilon - \delta$ identity
   $\epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$
5. Simplify
   (a) $\epsilon_{ijk} \delta_{i\ell} \delta_{jm} \delta_{kn}$
   (b) $\epsilon_{ijk} \epsilon_{jik}$
6. Use index notation to prove the following vector identities:
   (a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
   (b) $\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0$
7. Given the Cauchy stress tensor, σ_{ij}, the deviatoric stress tensor is computed using the relation $s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij}$. Calculate s_{kk}.
8. A scalar quantity, r, is defined as $\sqrt{x_i\, x_i}$. Calculate $\frac{\partial r}{\partial x_i}$ and $\frac{\partial^2 r}{\partial x_i\, \partial x_j}$.

9. A scalar quantity, J, is given by the determinant of a second-order tensor s_{ij} as $J = |s_{ij}|$. Show that $\frac{\partial J}{\partial s_{ij}} = J\, s_{ji}^{-1}$. (*Hint*: $|s_{ij}| = \frac{1}{6}\,\epsilon_{ijk}\,\epsilon_{lmn}\,s_{il}\,s_{jm}\,s_{kn}$ and $s_{ji}^{-1} = \frac{1}{2\,J}\,\epsilon_{ikl}\,\epsilon_{jmn}\,s_{km}\,s_{ln}$.)

10. Use index notation to show that

$$\omega \cdot \mathrm{d}\mathbf{x} = -\frac{1}{2}\,(\nabla \times \mathbf{s}) \times \mathrm{d}\mathbf{x} \quad ,$$

where, the vorticity tensor, ω is given by

$$\omega = -\frac{1}{2}\left[\nabla\mathbf{s} - (\nabla\mathbf{s})^{\mathrm{T}}\right] \quad ,$$

$\mathbf{s}$ is the displacement vector, and $\mathrm{d}\mathbf{x}$ is the small change in position vector.

11. For the Cauchy stress tensor

$$\mathbf{T}_{ij} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & -2 \\ 3 & -2 & 5 \end{bmatrix}$$

find the traction on

(a) the x-y plane,

(b) the y-z plane,

(c) the plane with normal (3, 2, -1).