GEO441: Problem Set 0

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Problem 1:

1. $a^i b^i = c$
2. $\epsilon^k_{ij} a^i b^j = d^k$
3. $(\epsilon^k_{ij} a^i b^j) c^k$

Problem 2:

1. $\text{grad}(\phi) = \nabla_i \phi$
2. $\text{div}(\mathbf{v}) = \nabla_i v^i$
3. $\text{curl}(\mathbf{v}) = \epsilon^{ki}_j \nabla_i v^j$
4. $\nabla_i \nabla_i \phi = \nabla_i (\nabla_i \phi)$

Problem 3:

Writing $\nabla \times \nabla \phi = \mathbf{0}$ in index notation we get:

$$\epsilon^{ijk} \nabla_i \nabla_j \phi$$

Since $\nabla_i \nabla_j \phi = \nabla_j \nabla_i \phi$ by simply relabelling $j \to i$ and $i \to j$

$$\epsilon^{ijk} \nabla_i \nabla_j \phi = \epsilon^{jik} \nabla_i \nabla_j \phi$$

but we also know that

$$\epsilon^{ijk} \nabla_i \nabla_j \phi = -\epsilon^{jik} \nabla_i \nabla_j \phi$$

and hence the only way we can satisfy

$$\epsilon^{ijk} \nabla_i \nabla_j \phi = \epsilon^{jik} \nabla_i \nabla_j \phi = -\epsilon^{jik} \nabla_i \nabla_j \phi$$

is if they are all equal to 0.

To prove that $\nabla \cdot \nabla \times \mathbf{a} = 0$ it is easier if we introduce corresponding orthogonal bases for the vectors:

$$\nabla = \nabla_i \mathbf{e}^i$$

$$\mathbf{a} = a^j \mathbf{e}_j$$

Hence we can write the expression above as:

$$\nabla_i \mathbf{e}^i \cdot (\epsilon_j^{ik} \nabla_i a^j \mathbf{e}_k)$$

where $\mathbf{e}^i \cdot \mathbf{e}_k = \delta_k^i$ so we get

$$\epsilon_j^{ik} \nabla_i \nabla_i a^j \delta_k^i$$

where the delta leads to ϵ_j^{ii} which is always 0 because of the repeated index and hence the whole term is 0 (scalar).

Problem 4: If we write out $\epsilon_{ijk}\epsilon_{imn}$ explicitly as a summation over i then we get:

$$\epsilon_{ijk}\epsilon_{imn} = \epsilon_{1jk}\epsilon_{1mn} + \epsilon_{2jk}\epsilon_{2mn} + \epsilon_{3jk}\epsilon_{3mn}$$

where we know that in each case two of the terms will go to 0 because of repeated indices. So therefore taking just the first term on the RHS, $\epsilon_{1jk}\epsilon_{1mn}$, there are two options: either $j = m$ and $k = n$ or $j = n$ and $k = m$, with any other combinations $= 0$. In Option 1, the two epsilons the contribution is $+1$, i.e. $\epsilon_{123}\epsilon_{123} = 1$. In Option 2, the product can be written as $-\epsilon_{123}\epsilon_{123} = -1$. Therefore the overall summation is $1\times$ every case of Option 1 and $-1\times$ every case of Option 2. Hence we may write the summation as

$$\epsilon_{ijk}\epsilon_{imn} = +1 \times \delta_{jm}\delta_{kn} + (-1\times)\delta_{jn}\delta_{kn}$$

$$\text{ie. } \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} + \delta_{jn}\delta_{kn}$$

Problem 5:

1. ϵ_{lmn}
2. Using $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn}$ for $\epsilon_{ijk}\epsilon_{ikj}$ we get

   $$\delta_{jk}\delta_{kj} - \delta_{jj}\delta_{kk}$$

   which is equal to

   $$\delta_{kk} - \delta_{jj}\delta_{kk}$$

   where $\delta_{ii} = 3$ and so the answer is -6.

Problem 6:

1. Writing the LHS in index notation we get:

   $$\epsilon_{lk}^{m} a^l \epsilon_{ij}^{k} b^i c^j$$

   where we can combined the epsilons using the delta identity to make the LHS:

   $$(\delta_{mi}\delta_{lj} - \delta_{mj}\delta_{li}) a^l b^i c^j$$

   which can be simplified to

   $$a^j b^m c^j - a^i b^i c^m$$

   $$(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

2. This is the same proof as in Problem 3 except replacing $\nabla \to \mathbf{a}$ and $\mathbf{a} \to \mathbf{b}$ by virtue of the fact that the cross product must be done first in both cases and the contraction of two vectors is commutative

Problem 7 If $s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}$ then

$$s_{kk} = \sigma_{kk} - \frac{\sigma_{kk}}{3}\delta_{kk}$$
$$s_{kk} = 0$$

Problem 8:

$$r = (x_i x_i)^{0.5}$$
$$\frac{\partial r}{\partial x_i} = x_i(x_i x_i)^{-0.5}$$
$$\frac{\partial^2 r}{\partial x_i x_j} = \delta_{ij}((x_i x_i)^{-\frac{1}{2}} - x_i x_i (x_i x_i)^{-\frac{3}{2}})$$
$$\frac{\partial^2 r}{\partial x_i x_j} = 0$$

Problem 9: Starting with writing $J = |s_{ij}| = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmn}s_{il}s_{jm}s_{kn}$ then we may differentiate with respect to some s tensor s_{pq}:

$$\frac{\partial J}{\partial s_{pq}} = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmn}(\delta_{ip}\delta_{lq}s_{jm}s_{kn} + \delta_{jp}\delta_{mq}s_{il}s_{kn} + \delta_{kp}\delta_{nq}s_{il}s_{jm})$$
$$\frac{\partial J}{\partial s_{pq}} = \frac{1}{6}(\epsilon_{pjk}\epsilon_{qmn}s_{jm}s_{kn} + \epsilon_{ipk}\epsilon_{lqn}s_{il}s_{kn} + \epsilon_{ijp}\epsilon_{lmq}s_{il}s_{jm})$$

It is necessary to now rearrange each of these terms on the RHS individually:

Term 1: Taking

$$\epsilon_{pjk}\epsilon_{qmn}s_{jm}s_{kn}$$

and setting $j \to l$ and switching dummy variables m and n we get

$$\epsilon_{plk}\epsilon_{qnm}s_{ln}s_{km}$$
$$= \epsilon_{pkl}\epsilon_{qmn}s_{ln}s_{km}$$

by reversing the positions of k, l and n, m.

Term 2: Similarly for

$$\epsilon_{ipk}\epsilon_{lqn}s_{il}s_{kn}$$

by changing dummy variables $i \to l$, $l \to m$ and then switching m and n around we get

$$\epsilon_{lpk}\epsilon_{nqm}s_{ln}s_{km}$$

and then cycling the indices of the levi-civitas we get

$$\epsilon_{pkl}\epsilon_{qmn}s_{ln}s_{km}$$

Term 3: Finally, for

$$\epsilon_{ijp}\epsilon_{lmq}s_{il}s_{jm}$$

by changing the dummy indices $i \to k$, $j \to l$ and $l \to n$ it becomes

$$\epsilon_{klp}\epsilon_{nmq}s_{kn}s_{lm}.$$

Then switching the m and n around and cycling both levi-civitas we get

$$\epsilon_{pkl}\epsilon_{qmn}s_{km}s_{ln}$$

Hence overall we find that

$$\frac{\partial J}{\partial s_{pq}} = \frac{3}{6}(\epsilon_{pkl}\epsilon_{qmn}s_{km}s_{ln})$$

Replacing $p \to i$ and $q \to j$ it becomes

$$\frac{\partial J}{\partial s_{ij}} = \frac{1}{2}(\epsilon_{ikl}\epsilon_{jmn}s_{km}s_{ln}) = Js_{ji}^{-1}$$

as per the hint given.

Problem 10

$$\begin{aligned}
-\frac{1}{2}(\nabla \times \mathbf{s}) \times d\mathbf{x} &= -\frac{1}{2}\epsilon_{kml}(\epsilon_{ijk}\nabla_i s_j)\mathrm{d}x_m \\
&= -\frac{1}{2}\epsilon_{kml}\epsilon_{kij}\nabla_i s_j \mathrm{d}x_m \\
&= -\frac{1}{2}(\delta_{mi}\delta_{lj} - \delta_{mj}\delta_{li})\nabla_i s_j \mathrm{d}x_m \\
&= -\frac{1}{2}(\nabla_m s_l - \nabla_l s_m)\mathrm{d}x_m \\
&= -\frac{1}{2}(\nabla\mathbf{s} - (\nabla\mathbf{s})^T)\cdot \mathrm{d}\mathbf{x} \\
&= \omega \cdot \mathrm{d}\mathbf{x}
\end{aligned}$$

Problem 11:

1. $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & -2 \\ 3 & -2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$
2. $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & -2 \\ 3 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$
3. $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & -2 \\ 3 & -2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$