2022 Summer Notes

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數值波分析 與 大氣模式計算效率
convergent rate + stiff problem

Analytical

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} = 0$$ 1D linear advection equation

$$q(x,t) = \sum_{k\omega} \hat{q}_{k\omega} e^{ikx - i\omega t}$$ Fourier series analysis

$$\omega = ck$$ Dispersion relationship

FD2 discretization

FD2

$$\frac{q(t+\triangle t, x) - q(t-\triangle t, x)}{2\triangle t} + c\frac{q(t, x+\triangle x) - q(t, x-\triangle x)}{2\triangle x} = 0$$ FD2中差分

$$\frac{q(t, x+\triangle x) - q(t, x-\triangle x)}{2\triangle x} = ik\frac{\sin(k\triangle x)}{k\triangle x}e^{ikx}$$

$$\frac{\sin(\omega t)}{\triangle t} = c\frac{\sin(k\triangle x)}{\triangle x}$$ Fourier analysis

$$\mu = \frac{c\triangle t}{\triangle x}$$ Courant number

$$\omega = \frac{1}{\triangle t}\sin^{-1}(\mu \sin(k\triangle x))$$ FD2 Dispersion relationship

Phase speed $$\frac{\omega}{kc} = \frac{c_k}{c} = \frac{1}{\mu k \triangle x}\sin^{-1}(\mu \sin(k\triangle x))$$

Group velocity $$c_g = \frac{d\omega}{dk} = \frac{1}{\triangle t}\frac{\mu \triangle x \cos(k\triangle x)}{\sqrt{1-\mu^2\sin^2(k\triangle x)}} = c\frac{\cos(k\triangle x)}{\sqrt{1-\mu^2\sin^2(k\triangle x)}}$$

$$\lambda = 4\triangle x, \quad k\triangle x = \frac{2\pi}{4\triangle x}\triangle x = \frac{\pi}{2}, \quad c_g = 0$$

Negative dispersion

$$\lambda = 2\triangle x, \quad k\triangle x = \frac{2\pi}{2\triangle x}\triangle x = \pi, \quad c_g = -c$$ 誤差往上游傳送

FD4

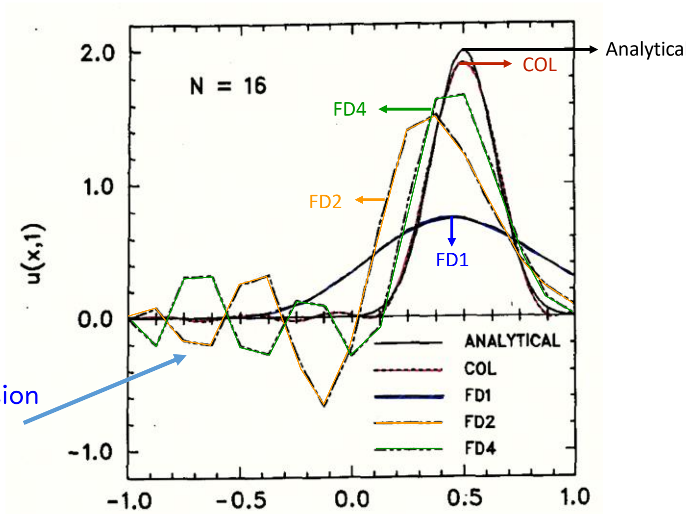
$$\frac{q(t+\triangle t, x) - q(t-\triangle t, x)}{2\triangle t}$$

$$+c\left(\frac{4}{3}\frac{q(t, x+\triangle x) - q(t, x-\triangle x)}{2\triangle x} - \frac{1}{3}\frac{q(t, x+2\triangle x) - q(t, x-2\triangle x)}{4\triangle x}\right) = 0$$

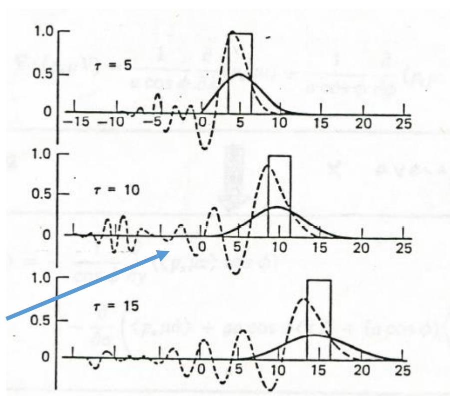
Advection – Finite Difference *vs.* Chebyshev Collocation

Fulton & Schubert (1987 a)

Physical space



Computational dispersion
Upstream transport



Computational dispersion
Upstream transport

Table 1

	$\frac{c\Delta t}{\Delta x}$:	$2\Delta x$	$4\Delta x$	$6\Delta x$	$8\Delta x$	$10\Delta x$	$12\Delta x$
Second order	0.2	0	0.64	0.83	0.91	0.94	0.96
	0.4	0	0.66	0.84	0.92	0.95	0.96
	0.6	0	0.68	0.87	0.93	0.96	0.97
	0.8	0	0.74	0.92	0.96	0.97	0.98
Fourth order	0.2	0	0.86	0.97	0.99	1.00	1.00
	0.4	0	0.89	0.99	1.00	1.01	1.01
	0.6	0	0.98	1.03	1.03	1.02	1.01
	0.8	0	Unstable	1.11	1.07	1.04	1.03

Performance of Chebyshev Collocation Method

Fulton & Schubert (1987 a)

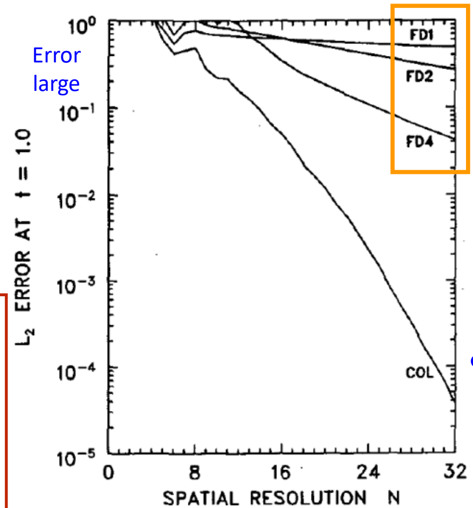
Linear Advection Equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=0$$

- Finite Difference Method:
Algebraic convergence（代數收斂）
誤差隨網格點增加呈算數減少

- Chebyshev Collocation Method:
Exponential convergence（指數收斂）
誤差隨網格點增加呈指數減少

Error large



Algebraic convergence

Exponential convergence

Convergence Diagram

思考重點

$\mu=\frac{c\triangle t}{\triangle x}$ Courant number < 1

FD2 with error of $O\left(\triangle x^2\right)$ $\mathrm{O}(\triangle t^2)$

FD2 differentiation response function $\frac{\sin(k\triangle x)}{k\triangle x}$

$2\triangle x$ wave stationary Computational dispersion

Negative advection of $2\triangle x$ pattern

Convergent rate for FD4, FD2 and Chebyshev or Fourier spectral method

Stiff Differential Equations

$$\frac{dy_1}{dt} = \frac{\lambda_1 + \lambda_2}{2} y_1 + \frac{\lambda_1 - \lambda_2}{2} y_2$$

$$\frac{dy_2}{dt} = \frac{\lambda_1 - \lambda_2}{2} y_1 + \frac{\lambda_1 + \lambda_2}{2} y_2$$

$$\frac{d\underset{\sim}{y}}{dt} = M\underset{\sim}{y}$$

Eigenvalues $\lambda_1, \lambda_2 < 0$

- The general solution is

$$\left.\begin{aligned} y_1(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ y_2(t) &= C_1 e^{\lambda_1 t} - C_2 e^{\lambda_2 t} \end{aligned}\right\} \quad t \geq 0$$

- Numerical solution by Euler's method

$$\eta_{1i} = C_1(1+\Delta t\lambda_1)^i + C_2(1+\Delta t\lambda_2)^i$$

$$\eta_{2i} = C_1(1+\Delta t\lambda_1)^i + C_2(1+\Delta t\lambda_2)^i$$

$$|1+\Delta t\lambda_i| < 1$$

$$\Delta t < \frac{2}{|\lambda_i|}$$

$\lambda_1 = -1$, $\lambda_2 = -1000$

e^{-t} , e^{-1000t}

$$\Delta t < \frac{2}{1000}$$

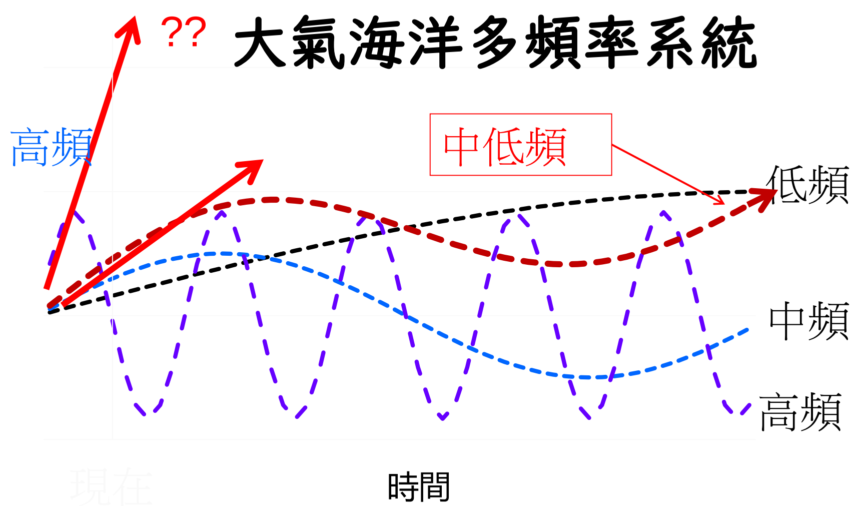
e^{-1000t}

對於解無重要性 但時不受限於它

Δt limited by stability rather than accuracy

時步被高頻率局限住　科學興趣在於低頻時間積分效率亟待提升

不同頻率時間尺度
穩定的Δ *t*大不同
大氣海洋多頻率系統



Use 2nd-order centred difference to discretize $\frac{\partial}{\partial x}$ $\frac{\partial}{\partial t}$

$$c^* \simeq c\left(1 - \frac{1}{6}k^2(\Delta x^2 - c^2 \Delta t^2)\right)$$

$$Error(\Delta x) = \frac{\sqrt{2}}{12} k^3 c A \Delta x^2 t$$

$$Error(\Delta t) = \frac{\sqrt{2}}{12} k^3 c^3 A \Delta t^2 t$$

$$\boxed{\frac{Error(\Delta t)}{Error(\Delta x)} = \frac{c^2 \Delta t^2}{\Delta x^2} = \frac{C^2}{C_0^2} \qquad C_0 = \frac{\Delta x}{\Delta t}}$$

Waves of meteorological interest

$$c = 15 \ m/s$$

However,

Stability requires $C_0 \geq C_1$
(CFL)

C_1 : fastest phase speed encountered

PE model (stiff system)

$C_1 \simeq 300 \ m/s$ (external gravity wave) $+$ $100 \ m/s$ (local wind)

$C_0 > 400 \ m/s$

$Error(\Delta t) = \frac{1}{700} Error(\Delta x)$

very inefficient !

Semi-implicit method (Robert 1972)

- Removes the stability constraint imposed by the fast gravity waves
- Most efficient in spectral model (Hoskins & Simmons 1975)

$C_0 \geq C_1 = 100 \ m/s$

$Error(\Delta t) = 1/40 \ Error(\Delta x)$

Wave + Advection
300 m/s + 100 m/s

Semi-implicit
以implicit處理waves
需解Poison Equation

Semi-Lagrangian
處理平流Δt問題

Semi-implicit method is much better,
Still room for improvement !

Problem : Time step limited by stability rather than by accuracy.

To make discretization efficient :

(1) Increase the accuracy of spatial discretization so that Δt is limited by accuracy

$\Rightarrow$ Chebyshev spectral method with 4th order Runge-Kutta time integration in non-stiff system.

(Fulton & Schubert 1987), (Kuo & Schubert 1988)

$\Rightarrow$ Efficient when high accuracy is needed.

(2) Semi-Lagrangian method (Robert 1981)

- Δt limited by accuracy only
- handle short waves and advection much better (Ritchie 1985)

eg. $\Delta x = 200$ km $\quad C = 15$ m/s $= C_0$

$\Delta t \simeq 3.7$ hr $\quad$ (Error(Δx) ~ Error(Δt))

Very efficient if not much overhead involved.

Semi-Lagrangian Solutions to the Inviscid Burgers Equation

MWR 1990

HUNG-CHI KUO

Naval Environmental Prediction Research Facility, Monterey, California

R. T. WILLIAMS

Naval Postgraduate School, Monterey, California

(Manuscript received 30 June 1989, in final form 4 December 1989)

ABSTRACT

We explore the use of semi-Lagrangian methods in a situation where the spatial scale of the flow collapses to zero during the time integration. The inviscid Burgers equation is used as the test model because it is the simplest equation that allows scale collapse (shock formation), and because it has analytic solutions. It is shown that despite the variable manner in which the gradient of the wind field approaches infinity in the neighborhood of the shock, the semi-Lagrangian method allows the error to be localized near the steep slope region. Comparisons with second-order finite difference and tau methods are provided. Moreover, the semi-Lagrangian method gives accurate results even with larger time steps (Courant number greater than 2 or 4) than are possible with the Eulerian methods. The semi-Lagrangian method, along with other recently developed numerical methods, is useful in simulating the development of steep gradients or near discontinuities in a numerical model. Some applications of the semi-Lagrangian method are discussed.

Model Problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$u(x, 0) = f(x) = \bar{u} - \tan^{-1}(x - x_0).$$

$$u = \bar{u} - \tan^{-1}(x - ut - x_0).$$

$$\frac{\partial u}{\partial x} = -\frac{1 - t\dfrac{\partial u}{\partial x}}{1 + (x - x_0 - ut)^2}.$$

At $x = x_0 + \bar{u}t$, $u = \bar{u}$ so that we obtain

$$\left(\frac{\partial u}{\partial x}\right)_{x=x_0+\bar{u}t} = -\left(1 - t\left(\frac{\partial u}{\partial x}\right)_{x=x_0+\bar{u}t}\right),$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=x_0+\bar{u}t} = \frac{1}{t-1} \rightarrow -\infty \quad \text{as} \quad t \rightarrow 1.$$

Semi-Lagrangian solution

$$\frac{d}{dt}u(x(t), t) = 0. \quad \frac{dx}{dt} = u$$

$$u(x_j, t + \Delta t) = u(x_j - 2\alpha_j, t - \Delta t).$$

$$\alpha_j = \Delta t u(x_j - \alpha_j, t),$$

$$\alpha_j^{(n+1)} = \Delta t u(x_j - \alpha_j^{(n)}, t).$$

$$\|\alpha_j^{(n+1)} - \alpha_j\| = \Delta t \left\|\frac{\partial u}{\partial x}\right\| \|\alpha_j^{(n)} - \alpha_j\|.$$

$$\boxed{\Delta t \left\|\frac{\partial u}{\partial x}\right\| < 1.}$$

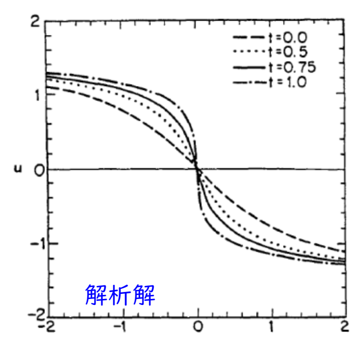
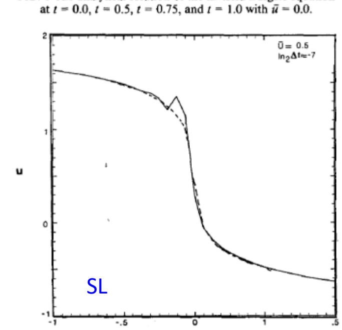


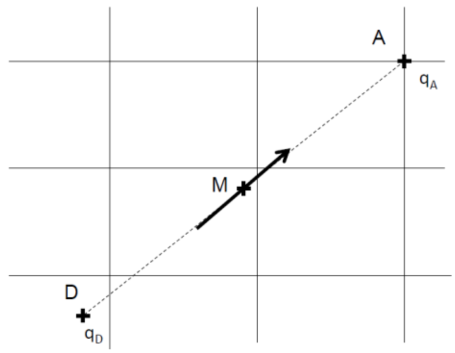
FIG. 1 The analytical solution of the inviscid Burgers equation at $t = 0.0$, $t = 0.5$, $t = 0.75$, and $t = 1.0$ with $\bar{u} = 0.0$.



Semi-Lagrangian Method
迭代軌跡追蹤 內差求值 無Δt限制

Backward scheme (tradition scheme)

Starting from arrival point at model grid point



$$\frac{\partial q}{\partial t} + V \bullet \nabla q = 0$$

$$\frac{Dq}{Dt} = 0$$

$$q_A^{n+1} = q_D^n$$

Requires guessing and iteration to find mid- or departure point

Two iterations for three interpolations at mid-point for wind

One interpolation for variables at departure point

Eulerian advection

$$\frac{\partial q}{\partial t} + V \bullet \nabla q = 0$$

$$|V_{\max}| \frac{\Delta t}{\Delta x} < 1$$

CFL condition
Numerical stability

Lagrangian advection

$$\frac{dq}{dt} = 0$$

$$q_A^{n+1} = q_D^n$$

No restriction of delta t

The solution is on
(1) the determination of advection trajectory and
(2) the interpolation between model grid and arrival or departure grids.

Global Spectral Model
(Spherical Harmonics basis functions)

- Eliminate Pole problem.
- High efficiency and accuracy (exponential convergence)
- Semi-implicit + semi-Lagrangian method
  (larger Δ t allowed)
- Discrete energy/enstrophy conservations and NO aliasing
- Accurate Geostrophic Adjustment in 2 Δx scale
- No Fast Legendre Transform (slow)N^3 vs. N^2(fast)
- Moisture advection (negative values).

Linear 綫性

淺 錢 盞 箋 棧 踐 賤 濺 綫

Linear Waves
(大氣)綫性波動

Wave Equation
波動方程

Euler, D’Alembert, and Bernoulli
Linear wave dynamics
Partial differential equations
Fourier series and analysis
Dispersion and polarization

Daniel Bernoulli

Daniel Bernoulli 1700-1782

分離變數法，流力

Euler 1707-1783

D' Alembert 1717-1783

$f(x + ct) + g(x-ct)$

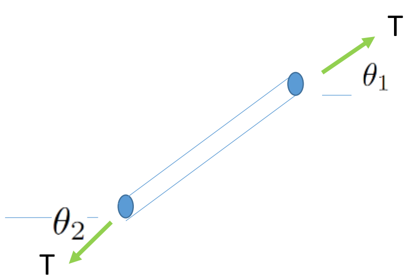
Lagrange 1736-1813

Laplace 1740-1827

Fourier 1768-1830

Dirichlet 1805-1839

1-D Wave Equation



m a = F

$$\rho \triangle x \frac{\partial^2 y}{\partial t^2} = T(\sin\theta_1 - \sin\theta_2)$$

$$\simeq T(\tan\theta_1 - \tan\theta_2)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{1}{\triangle x}\left((\frac{\triangle y}{\triangle x})_1 - (\frac{\triangle y}{\triangle x})_2 \right)$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} \qquad \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

D' Alembert wave solution

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < t < \infty$$

$$-\infty < x < \infty,$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t} = g(x)$$

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$0 < x < \pi \quad \& \quad t > 0$

邊界條件 u((0,t) = u(π ,t) = 0

u(x,t) = f(ct−x) + h(ct +x)

u(0,t) = 0 = f(ct) + h(ct)
u(x,t) = f(ct−x) – f(ct+x)
u(π ,t) = 0 = f(ct – π) – f(ct + π)

f(ct – π) = f(ct + π)

邊界條件產生週期解

Bernoulli wave solution

解由正模組合，正模一定空間結構對應一定時間結構。

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}\,|_{t=0} = g(x)$$

$$\lambda_n = \frac{cn\pi}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

$$B_n^* = \frac{2}{cn\pi}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \ldots\ldots$$

$$B_n^* \lambda_n = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

D’ Alembert

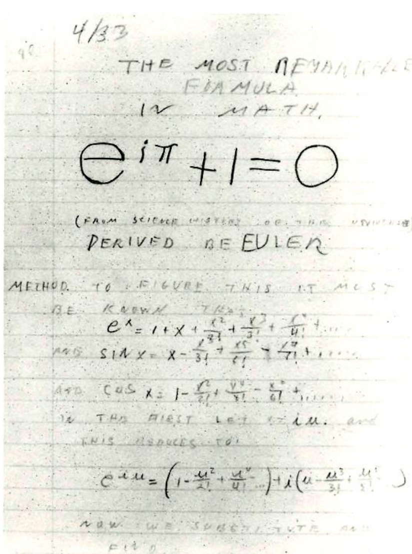
Euler

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

弦樂器端點固定 不僅是提供張力，
基本結構與對應基本頻率，所有頻率組成完整空間?

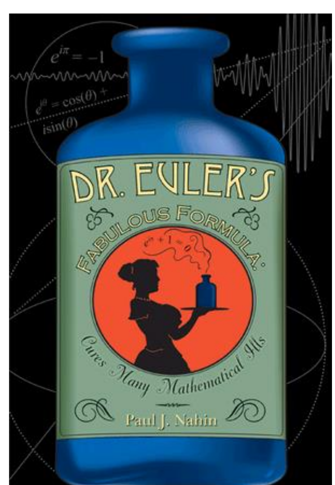
D’ Alembert 1717-1783
f(x + ct) + g(x-ct)
信號傳遞 固定邊界產生週期解
Daniel Bernoulli 1700-1782
流體力學，分離變數法，BC
固定結構固定的頻率
許多正模組成解 特徵值
Euler 1707-1783
通解? 不斷的質疑、討論與支持
Euler formula=Fourier Transform

Note of Feynman at age of 15
費曼15歲的筆記



Euler治療數學疑難雜症的神奇藥方

$$e^{i\theta} = \cos\theta + i\sin\theta$$



Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$

$$a + ib = re^{i\theta}$$

振幅與相位 $e^{\alpha+i\theta}$

$$-\alpha + i\omega$$

調振幅與調頻率 $e^{(-\alpha+i\omega)t}$

Wave formula $e^{i(kx-\omega t)} = e^{ik(x-ct)}$

Fourier 傅立葉

Fourier, Jean Baptiste Joseph

1768-1830

地球若從太陽取的能量，也必須散熱不然溫度會一直上升。

溫室效應
他的計算顯示地表溫度太低
（溫室效應低估）

The profound study of nature is the most fertile source of mathematical discoveries.
自然研究是數學發展最肥沃的土壤

$$f(x) = \sum \hat{f}_k e^{ikx}$$ 傅立葉級數

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$ 傅立葉轉換

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}_k e^{ikx}dx$$

$$\hat{f}_k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

不連續與不是週期函數皆可以使用Fourier分析

1807年，傅立葉 39歲; 因為以cosine 和 sin級數表達三角形狀波動，計算熱傳導，而被偉大的數學家如 Lagrange(71歲)，Laplace(77歲)，Cauchy所責備與攻擊，罵他是「騙子」。

f(x) does not have to be analytical;
f(x) does not have to be periodic.

$x \in [0, 2\pi]$

展開 $$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \; coskx + b_k \; sinkx)$$ Fourier Series

投影
內積

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \; coskx \; dx, \quad k = 0, 1, 2......$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \; sinkx \; dx, \quad k = 1, 2......$$

Fourier Transform

正交
垂直

$$\frac{1}{\pi} \int_0^{2\pi} coskx \;\; sinlx \; dx = 0, \quad k, l, integers$$

Orthogonal relationship

$$\frac{1}{\pi} \int_0^{2\pi} coskx \;\; coslx \; dx = \begin{cases} \delta_{kl}, & k \neq 0 \\ 2, & k = l = 0, \end{cases}$$

$$\frac{1}{\pi} \int_0^{2\pi} sinkx \;\; sinlx \; dx = \begin{cases} \delta_{kl}, & k, l \neq 0 \\ 0, & k = 0, \end{cases}$$

$$\delta_{kl} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

Complex form Fourier series and transform

$$e^{ikx} = cos \; kx + i \; sin \; kx,$$

$$cos \; kx = \frac{e^{ikx} + e^{-ikx}}{2},$$

$$sin \; kx = \frac{e^{ikx} - e^{-ikx}}{2i},$$

Fourier級數與轉換
常用的複數型

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \left(\frac{e^{ikx} + e^{-ikx}}{2}\right) + b_k \left(\frac{e^{ikx} - e^{-ikx}}{2i}\right) \right)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[\left(\frac{a_k - ib_k}{2} \right) e^{ikx} + \left(\frac{a_k + ib_k}{2} \right) e^{-ikx} \right].$$

Fourier series

$$f(x) = \sum_{-\infty}^{\infty} \hat{c}_k e^{ikx}.$$

$$\hat{c}_0 = \frac{a_0}{2}$$

$$\hat{c}_k = \begin{cases} \frac{a_k - ib_k}{2}, & k > 0 \\ \frac{a_k + ib_k}{2}, & k < 0 \end{cases}$$

Fourier Transform (inner product, projection)

$$\hat{c}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \; dx$$

Orthogonal relationship

$$\delta_{kl} = \frac{1}{2\pi} \int_0^{2\pi} e^{ikx} \; e^{-ilx} \; dx$$

內積是函數和共軛複數積分

$$\vec{V} = \sum_{i=1}^{3} a_i \vec{e_i} = a_1\vec{e_1} + a_2\vec{e_2} + a_3\vec{e_3}$$

$$\vec{e_i} \cdot \vec{e_j} = \delta_{ij}$$

$$a_i = \vec{V} \cdot \vec{e_i}$$

Expansion with basis (complete)

Orthogonal

Projection; Inner product

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(k) \exp^{ikx}$$

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \exp^{-ikx} dx$$

$$\frac{1}{2\pi} \int_0^{2\pi} \exp^{ikx} \exp^{-ilx} dx = \delta_{kl}$$

$$\vec{f} = \sum_i \hat{f}_i \vec{\psi}_i$$

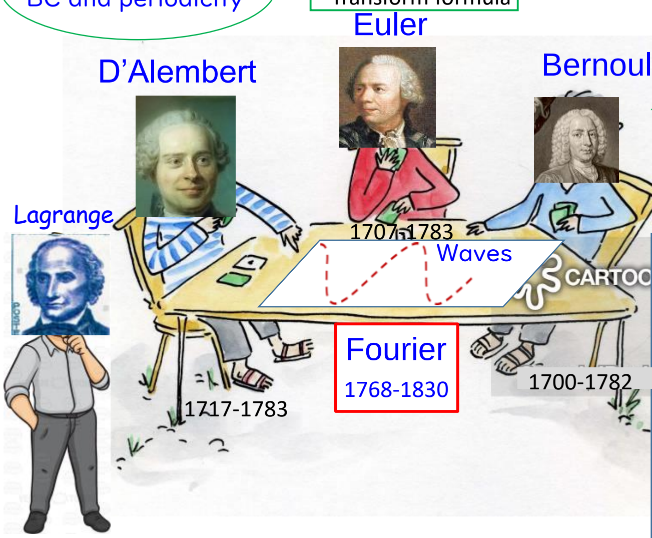
$$\left(\vec{\psi}_i, \vec{\psi}_j\right) = \delta_{ij}$$

$$\hat{f}_i = \left(\vec{f}, \vec{\psi}_i\right)$$

f(x + ct) + g(x−ct)
BC and periodicity

Complete?
Sin and Cosin
Transform formula

BC; Fixed structure with fixed frequency
Solution composes of normal modes
Separation of variables
Eigenvalues and eigenfunctions



1736-1813
Rosie Brooks via CartoonStock

Fourier series

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(k) \exp^{ikx}$$

Fourier transform

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \exp^{-ikx} dx$$

$$\frac{1}{2\pi} \int_0^{2\pi} \exp^{ikx} \exp^{-ilx} dx = \delta_{kl}$$

週期函數
sin cos

Periodic phenomena are actually everywhere in the biological world

東京台北的愛情故事
20xx友達以上戀人未滿數學版

第1場東京告白 A :[😍 喜歡] B: [

]

第2場台北告白 A :[

] B: [😍 喜歡]

第3場台大冷戰 A :[🖕 討厭] B: [

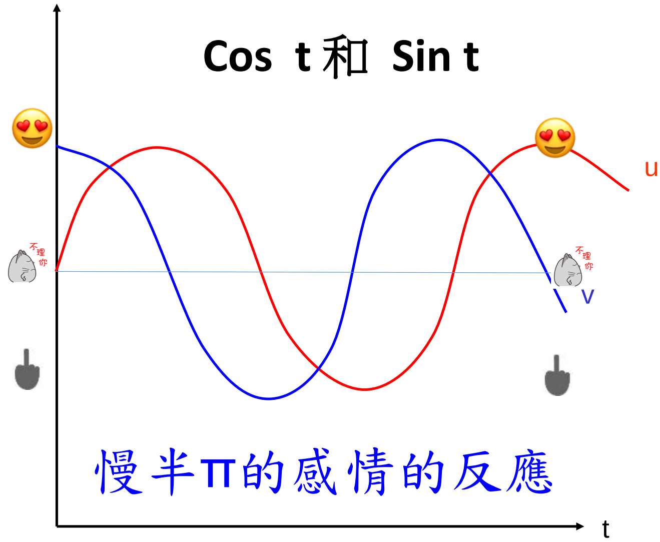
]

第4場台大持續冷戰 A :[

] B: [🖕 討厭]

第5場東京再度告白 A :[😍 喜歡] B: [

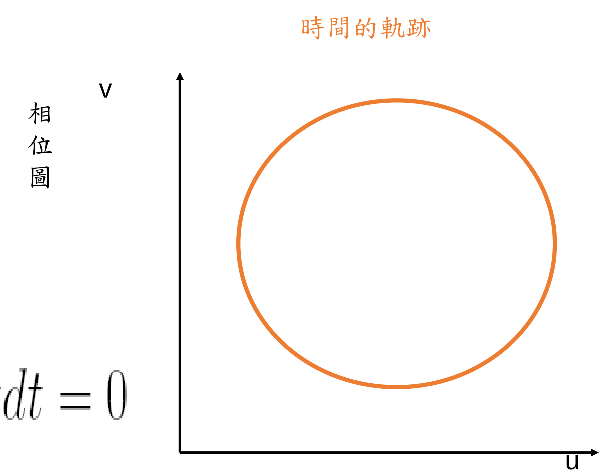
]



不相關的兩個時間序列

Cos t 和 Sin t 零相關、不來電！

$$\int_0^{2\pi} \underset{u}{\cos t}\,\underset{v}{\sin t}\,dt = 0$$



Romantic Romeo and Fickle Juliet

(Strogatz 1988)

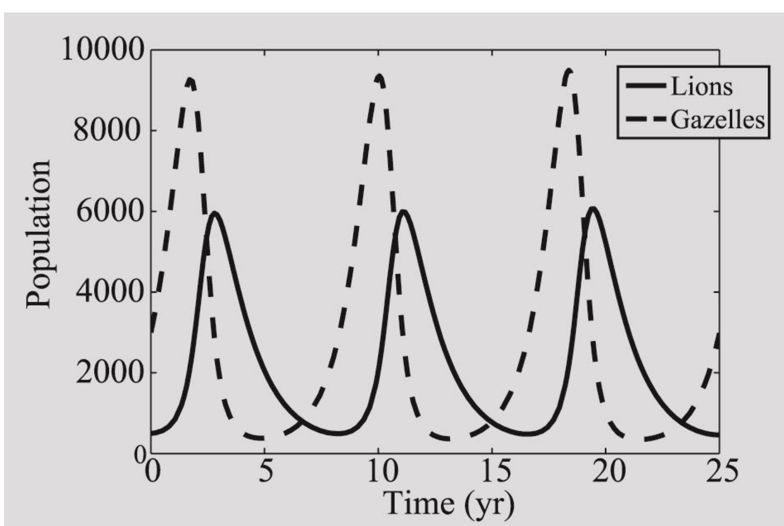
$$\frac{dR}{dt} = J \qquad \frac{dJ}{dt} = -R$$

J = Cos t

R = Sin t

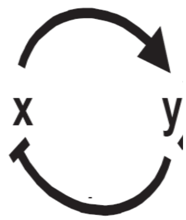
慢半π的感情的反應

獅子與羚羊



Negative Feedback Oscillators

X Cost Y Sin t



$$\frac{dy}{dt} = x$$

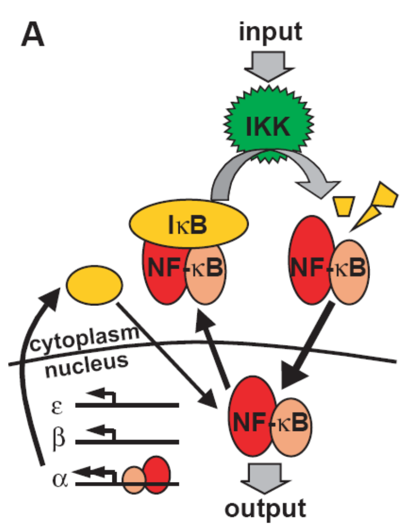
$$\frac{dx}{dt} = -y$$

物廉價美　顧客增加消費　高需求價格上揚

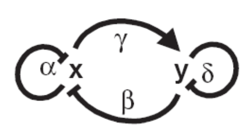
價格上揚　顧客減少消費　低需求價格下滑

負回饋　反者 道之動也

NF-κB and IκB Model



Science **298:** 1241-1245.



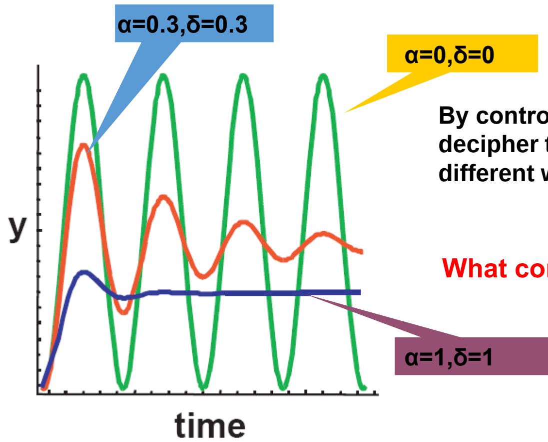
X: nucleus NF-κB

Y: IκB

$$\frac{d}{dt}x = S - \alpha x \boxed{- \beta y}$$

$$\frac{d}{dt}y = \boxed{\gamma x} - \delta y$$

Summary of
NF-κB and IκB Model



By controlling α and δ, a cell can decipher the same stimulus in different ways.

What controls α and δ?

正交 完整 特徵函數(基底)

$$\mathbf{A}^{\mathbf{T}*} = \mathbf{A}^{\mathbf{H}}$$

$$\mathbf{A}^{\mathbf{H}} = \mathbf{A}$$ Hermitian

$$\mathbf{A}\mathbf{E} = \mathbf{E}\mathbf{D}$$

$$\mathbf{A}^{\mathbf{H}} = -\mathbf{A}$$ Skew-Hermitian

$$(Lf, g) = (f, Lg)$$ Hermitian

$$L\psi_j = \lambda_j \psi_j$$

$$(Lf, g) = -(f, Lg)$$ Skew-Hermitian

$$(f, g) = \frac{1}{L}\int_0^L f_j g_j^* w dx$$

成分分析

Principle Component Analysis

結構成分 頻率成分

正交成分 分析完整


Daniel Bernoulli

Daniel Bernoulli 1700-1782
分離變數法，BC固定結構固定的頻率
許多正模組成解 特徵值

Euler 1707-1783
通解? 不斷的質疑、討論與支持
Euler formula=Fourier Transform
D' Alembert 1717-1783
f(x + ct) + g(x-ct)
信號傳遞 BC與週期解

弦樂器端點固定 不僅是提供張力而已
許多基本結構與對應頻率

$$\vec{V} = \sum_{i=1}^{3} a_i \vec{e_i} = a_1 \vec{e_1} + a_2 \vec{e_2} + a_3 \vec{e_3}$$ Expansion with basis

$$\vec{e_i} \cdot \vec{e_j} = \delta_{ij}$$ Orthogonal

$$a_i = \vec{V} \cdot \vec{e_i}$$ Projection; Inner product

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(k) \exp^{ikx}$$

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \exp^{-ikx} dx$$

$$\frac{1}{2\pi} \int_0^{2\pi} \exp^{ikx} \exp^{-ilx} dx = \delta_{kl}$$

$$\vec{f} = \sum_i \hat{f}_i \vec{\psi_i}$$

$$\left(\vec{\psi_i}, \vec{\psi_j}\right) = \delta_{ij}$$

$$\hat{f}_i = \left(\vec{f}, \vec{\psi_i}\right)$$

$$\mathbf{A}^{\mathbf{T}*} = \mathbf{A}^{\mathbf{H}}$$

$$\mathbf{A}^{\mathbf{H}} = \mathbf{A}$$ Hermitian

$$\mathbf{AE} = \mathbf{ED}$$

$$\mathbf{A}^{\mathbf{H}} = -\mathbf{A}$$ Skew-Hermitian

$$(Lf, g) = (f, Lg)$$ Hermitian

$$L\psi_j = \lambda_j \psi_j$$

$$(Lf, g) = -(f, Lg)$$ Skew-Hermitian

$$(f, g) = \frac{1}{L} \int_0^L f_j g_j^* w dx$$

Complete Orthogonal

Sturm Liouville Equation

Eigenvalue and eigenfunction

Chebyshev, Hermit, …….

Convergent Rate

Cooley–Tukey FFT algorithm 1965
Tukey (Princeton) + Cooley (IBM Watson)

Fast Fourier Transform 二十世紀最偉大的發明

Tukey reportedly came up with the idea during a meeting of President Kennedy's Science Advisory Committee discussing ways to detect nuclear-weapon tests in the Soviet Union by employing seismometers located outside the country. These sensors would generate seismological time series however analysis of this data would require fast algorithms
for computing DFT due to number of sensors and length of time.

1D	N variables	N^2 vs N *log* N
2D	N^2 variables	N^3 vs N^2 *log* N

波動基礎

Wave Fundamentals

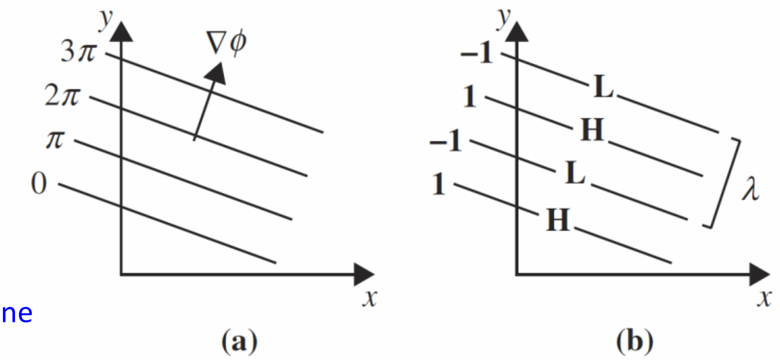


FIGURE 5.5 Two-dimensional plane wave at a fixed time: (a) phase angle, ϕ, and (b) phase, $e^{i\phi}$. Wavelength is denoted λ. Note that if $\nu > 0$, the wave travels in the direction of the wave vector, $\nabla\phi$.

Slope of phase line

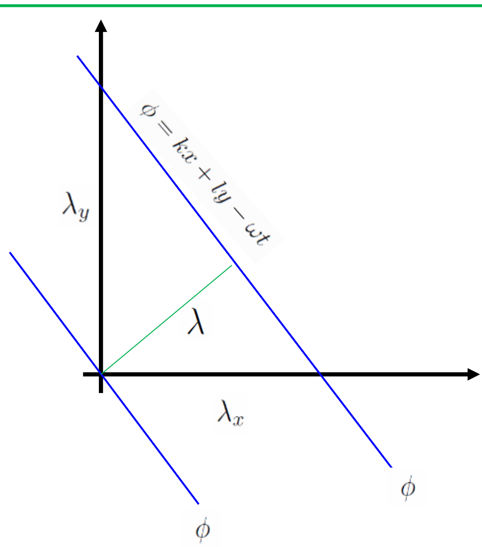
$$\frac{\delta y}{\delta x}\Big|_{\phi} = -k/l$$

$\mathbf{K} = \nabla\phi$ *wave vector*

$\mathcal{K} = |\mathbf{K}|$ is the *total wave number*

wave *period* $\frac{2\pi}{|\nu|}$ $\nu = -\frac{\partial\phi}{\partial t}$

λ λ_x λ_y	T	$\omega = \frac{2\pi}{T}$	$k = \frac{2\pi}{\lambda_x}$ $l = \frac{2\pi}{\lambda_y}$	$(k^2+l^2)^{\frac{1}{2}} = \frac{2\pi}{\lambda}$
wavelength	period	angular frequency	angular wavenumber	total wavenumber



$$c_x = \frac{\omega}{k}$$

$$c_y = \frac{\omega}{l}$$

$$c = \frac{\omega}{(k^2+l^2)^{\frac{1}{2}}}$$

phase speed
相速(率)

$$\boxed{c^2 \neq c_x^2 + c_y^2}$$

$$\vec{c_g} = c_{gx}\vec{i} + c_{gy}\vec{j}$$

$$c_{gx} = \frac{\partial\omega}{\partial k}$$

$$c_{gy} = \frac{\partial\omega}{\partial l}$$

group velocity
群速(度)

Group velocity
群速

$$\frac{\partial k}{\partial t}+\boxed{\frac{\partial \omega}{\partial k}}\frac{\partial k}{\partial x}+\boxed{\frac{\partial \omega}{\partial l}}\frac{\partial k}{\partial y}+\boxed{\frac{\partial \omega}{\partial m}}\frac{\partial k}{\partial z}=0.$$

$$\frac{\partial \omega}{\partial t}+\frac{\partial \omega}{\partial k}\frac{\partial \omega}{\partial x}+\frac{\partial \omega}{\partial l}\frac{\partial \omega}{\partial y}+\frac{\partial \omega}{\partial m}\frac{\partial \omega}{\partial z}=0.$$

跟隨群速運動
波數與頻率守恆

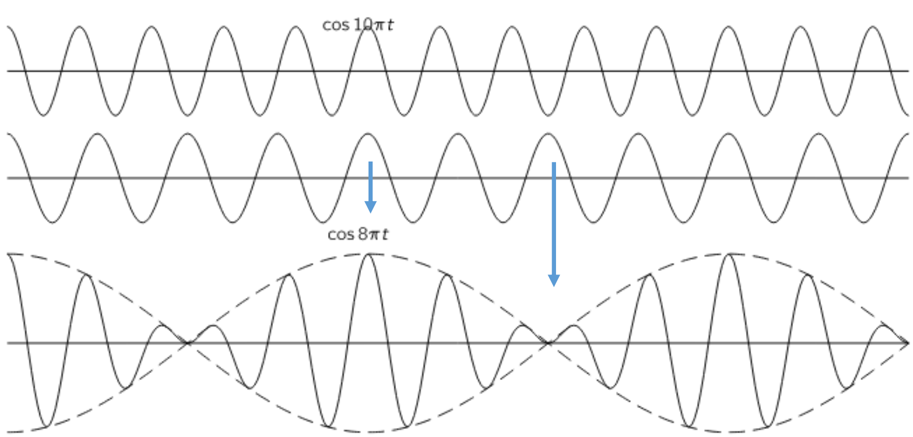
$$\Psi(x,t)=\exp\{i[(k+\delta k)x-(\nu+\delta\nu)t]\}+\exp\{i[(k-\delta k)x-(\nu-\delta\nu)t]\}$$

$$\Psi=\left[e^{i(\delta kx-\delta\nu t)}+e^{-i(\delta kx-\delta\nu t)}\right]e^{i(kx-\nu t)}$$
$$=2\cos(\delta kx-\delta\nu t)\,e^{i(kx-\nu t)}$$

$$\frac{\partial \mathbf{K}}{\partial t}+(\mathbf{C}_g\cdot\nabla)\mathbf{K}=0$$

$$\mathbf{K}=(k,\ l,\ m)$$

The wave vector is conserved; if wavelength and frequency are fixed if we follow a group of waves

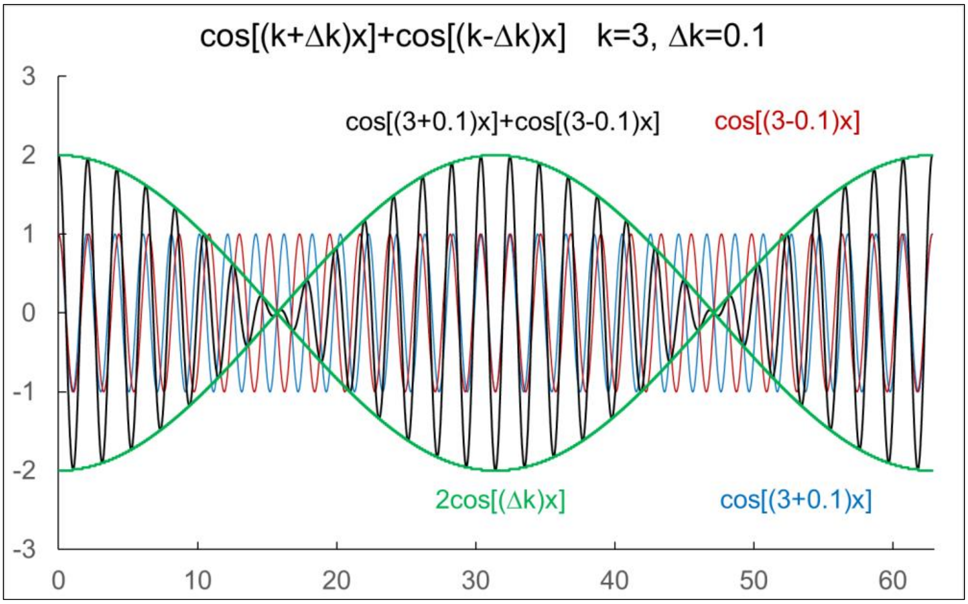


cos 10 π t + cos 8 π t = 2 cos π t cos 9 π t

cos 12 π t + cos 6 π t = 2 cos 3 π t cos 9 π t

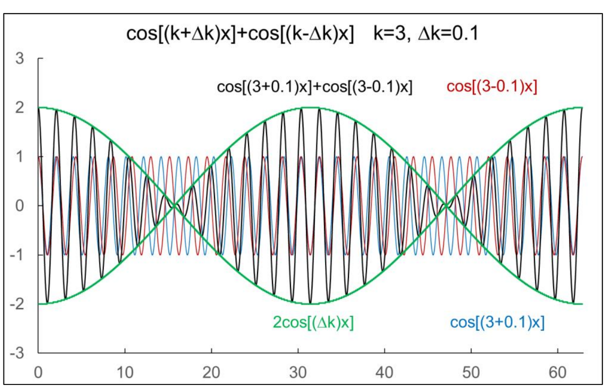
兩個波長近似(較不近似)的波疊加 波的抵銷與增強
產生波長較長(較短)的波胞 [物理空間與波譜空間的寬窄範圍相反]

Δk Δ x=π 物理空間與波譜空間的窄寬相反

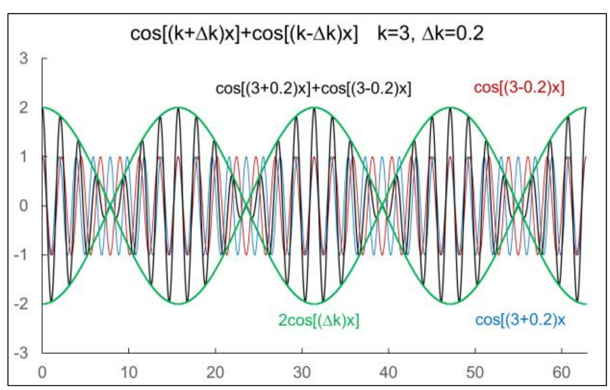


Δk Δ x=π 物理空間寬則波譜空間窄

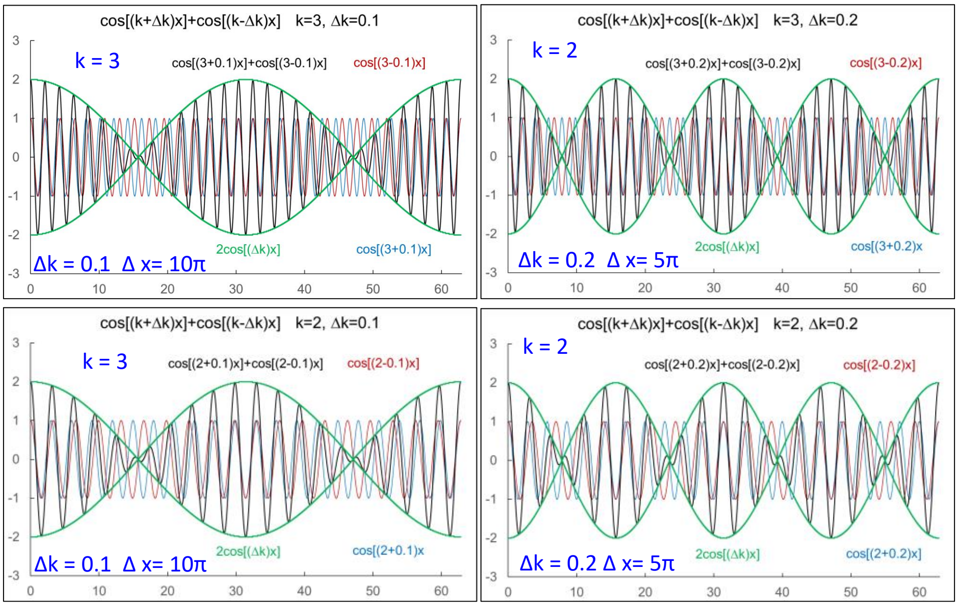
Δk = 0.1 Δ x= 10π



Δk = 0.2 Δ x= 5π



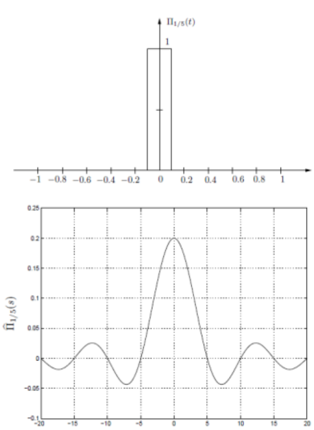
Δk Δ x=π   Wave envelop 和k沒關係



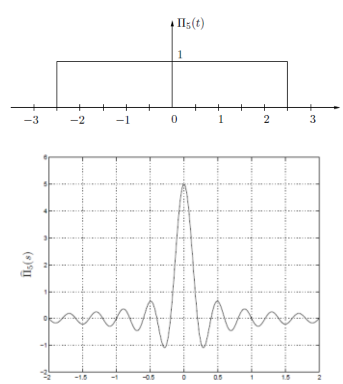
時間域的寬度和頻率域的寬度成相反形態；
一邊寬另一邊就窄

Uncertainty Principle

頻率域窄



時間域寬



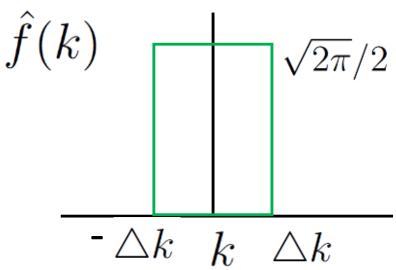
頻率域寬

時間域窄

Uncertainty Principle

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

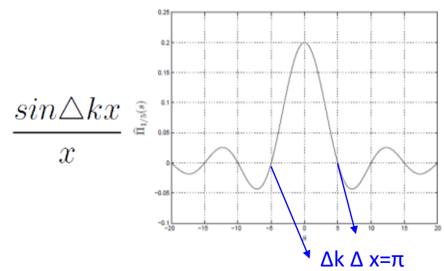
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$



$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk = \frac{1}{2} \int_{k-\triangle k}^{k+\triangle k} e^{ikx} dk$$

$$= \frac{1}{2} \left(\frac{e^{i(k+\triangle k)x} - e^{i(k-\triangle k)x}}{ix} \right)$$

$$= \frac{sin \triangle k x}{x} e^{ikx}$$



Uncertainty Principle

$$\frac{h}{T} = E$$

deBroglie frequency/energy

$$\frac{h}{\lambda} = p$$

deBroglie wavelength/momentum

$$\triangle k \triangle x = \pi$$

$$\hbar \triangle k = \triangle p$$

$$\hbar = \frac{h}{2\pi}$$

$$\boxed{\begin{aligned} \triangle p \triangle x &= \pi \hbar = \frac{h}{2} \\ \triangle E \triangle t &= \frac{h}{2} \end{aligned}}$$

Werner Heisenberg. The Physical Principles of the Quantum Theory. Courier Dover Publications. 1949. ISBN 978-0-486-60113-7.

Wave-particle duality

$$\frac{h}{\lambda} = p \quad hf = E$$

$$\hbar k = p \quad \hbar\omega = E$$

$$\psi \sim e^{ikx - i\omega t}$$

Schrodinger Equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\frac{(\hbar k)^2}{2m} \qquad V \qquad \hbar\omega$$

T + V = E

動能加位能等於粒子總能量