Journal
of
The Franklin Institute

Devoted to Science and the Mechanic Arts

Vol. 228 DECEMBER, 1939 No. 6

A MECHANICAL HARMONIC SYNTHESIZER-ANALYZER.

BY

S. LEROY BROWN, Ph.D.,

University of Texas.

ABSTRACT.

A harmonic synthesizer is described that has thirty harmonic elements (fifteen sine components and fifteen cosine components) that operate simultaneously, and the sum of the thirty sinusoidal movements is recorded by a tracing point (pencil) on a drawing board that is driven uniformly past the pencil point.

Analysis is accomplished by setting the amplitudes of the thirty elements to values that are determined from the values of selected ordinates from the curve to be analyzed, and the harmonic components of this curve are determined from selected ordinates of the auxiliary curve that is traced by the machine. Thirty-two equispaced ordinates from the curve to be analyzed furnish sufficient data whereby analysis is accomplished, by a single trace of the machine, that includes fifteen harmonic components (fifteen sine components and fifteen cosine components); or, sixty-four equispaced ordinates may be taken from the curve and the analysis will include thirty-one harmonic components that are obtained from two traces made by the machine (odd harmonics are determined from one auxiliary curve and even harmonics from the other curve); or, one hundred-and-twenty ordinates may be selected from the curve and the analysis is extended to fifty-nine harmonic components by tracing four auxiliary curves with the machine.

Wave synthesis and wave analysis have been of theoretical and practical interest to mathematicians, physicists and engineers since the outstanding contributions of Fourier [1] in 1822. The solution of many problems in sound, electric waves,

[1] J. B. L. Fourier, "La Theorie Analytique de la Chaleur," Paris.

(Note—The Franklin Institute is not responsible for the statements and opinions advanced by contributors in the JOURNAL.)

electrical vibrations and mechanical vibrations depend upon the harmonic analyses of complex wave forms or the syntheses of complex waves from sinusoidal components.

The schemes that have been proposed and the electrical and mechanical methods that have been devised to aid in the processes of analysis and synthesis are so numerous that no attempt will be made to enumerate them. However, two machines have been built that warrant special mention on account of their mechanical perfection and the range of their multiple frequency components. One of these machines is the Henrici analyzer with thirty spherical integrators which was built for Professor D. C. Miller [2] at Case School of Applied Science, and the other machine is the synthesizer with forty harmonic components that was designed and built by Mr. B. E. Eisenhour [3] at Riverbank Laboratories.

The machine which is to be described herein is fundamentally a synthesizer with fifteen sine components and fifteen cosine components. The multiple ratios of the sinusoidal motions are accomplished with a train of gears of the proper ratios and the summation is accomplished by a chain and pulleys in much the same manner as is done in the tide predictor.[4] The amplitude of each sine and of each cosine component may be set independently and all thirty of the sinusoidal motions are communicated simultaneously to the chain. The sum of the harmonic motions is the resultant motion of the endless chain that causes a pencil point to move in a vertical line, the pencil being fastened to a weight that is suspended from the chain. By means of a gear, rack and pinion a drawing board (in a vertical plane) is moved uniformly in a horizontal direction past the pencil point.

All parts of this machine were built on a large scale with the idea that greater precision would be had and, also, that it would be more economical to construct a large machine that is rugged and dependable even though the mechanism is rather complicated. The complete machine weighs nearly a ton; it is about fifteen feet long and seven feet high.

---

[2] D. C. Miller, JOUR. FRANK. INST., **182**, 285–322.

[3] Frederick W. Kranz, JOUR. FRANK. INST., **204**, 245–262.

[4] Special Publication No. 32, U. S. Coast and Geodetic Survey, Washington, D. C.

The machine is mechanically excellent and, as a synthesizer, it is capable of accurately constructing the resultant wave from fifteen specified harmonic components. However, the principal interest is in its application as an analyzer. Frederick W. Kranz [5] showed that a forty-element synthesizer could be used to analyze a given wave for the first thirty-nine harmonic components. It was also recognized that the Michelson and Stratton [6] machine with its eighty elements could be used as an analyzer.

The derivations that follow show the possibility of using a synthesizer, with comparatively few harmonic elements, to make analyses that include many harmonic components. Namely, a fifteen-element synthesizer is described which may be used to determine, easily and quickly from the tracing of a single auxiliary curve, the first fifteen harmonic components of a given wave form; or, it may be used to determine thirty-one harmonic components from the tracing of two auxiliary curves; or, it may be used to determine fifty-nine harmonic components from four auxiliary curves.

The major portion of this paper is devoted to the mathematical proofs and derivations of working equations, as well as demonstrations of test analyses of specified wave forms (by equations of waves), that are necessary to establish the possibilities of a fifteen-element harmonic synthesizer as an analyzer. The several test analyses are intended to outline the procedure that is to be followed in each of the three cases. Note that the first case involves approximate integration of $f(x) \cos nx \, dx$ (and $f(x) \sin nx \, dx$) by increments of $11.25°$ (or 30 instead of 32 equispaced ordinates may be chosen and then the increments would be $12°$). The second case involves approximate integration by increments of $5.625°$ (or 60 instead of 64 equispaced ordinates may be chosen and then the increments would be $6°$). The third case involves approximate integration by increments of only $3°$.

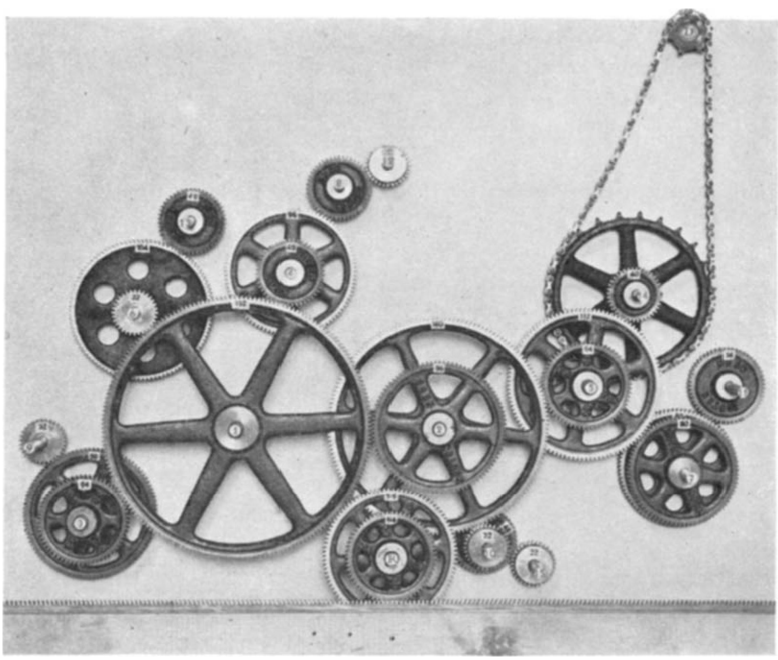
Details of the Machine. The driving mechanism is a train of twenty-two spur gears (shown in Fig. 1.) that are high grade but selected from commercial stock. These gears drive fifteen shafts with rotational ratios ranging from one to

[5] Frederick W. Kranz, op. cit.

[6] Michelson and Stratton, *Am Jour. of Sci.*, 5, 1–13.

fifteen. A Scotch crosshead is operated from each end of each shaft with the pin of each crosshead set in quadrature with the pin of the crosshead at the other end of the respective shaft. As the pin of each crosshead executes uniform circular motion it transmits sinusoidal motion to a vertical rod (in guides), and near the top of each vertical rod there is a small pulley over (or under) which passes a fine chain. The sum of the thirty sinusoidal motions is communicated to a tracer point

FIG. 1.

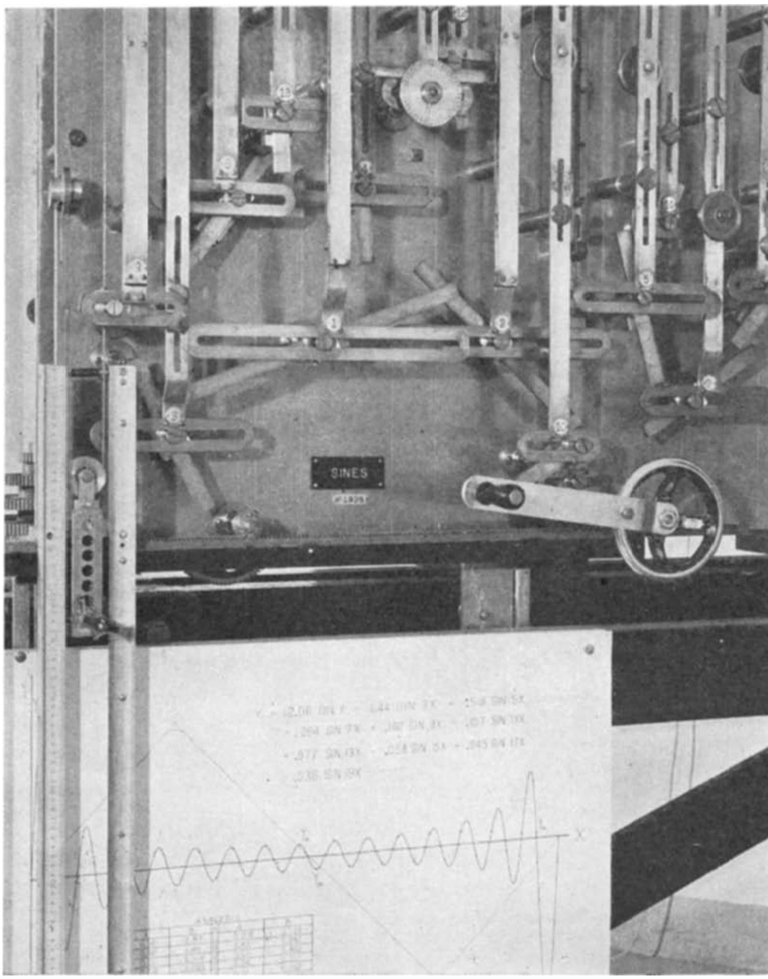


Train of gears.

(pencil) by means of the pulleys and the continuous fusee chronometer chain. The motion of the pencil point is in a vertical line and the pencil is carried by a metal block (in guides) that is suspended by the chain. By means of a rack and pinion, one shaft of the gear train drives the drawing board (in a vertical plane) with uniform horizontal motion past the pencil point. The rack is adjustable to different sized pinions and, thereby, the wave-length of the curve traced on the drawing board may be adjusted.

Figure 2 is a front view of the machine and it shows the eccentrics that furnish the fifteen sine components, the tracer block that is suspended by a loop of the endless chain around

FIG. 2.

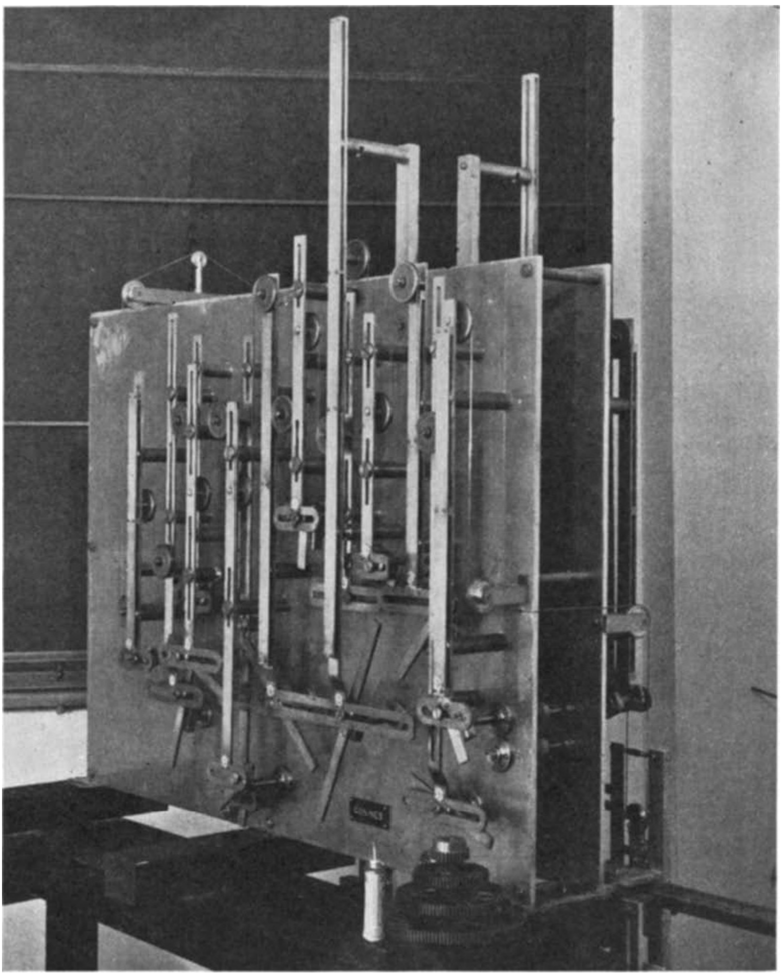


Front view—sine components.

a pulley on the block, the drawing board, and the crank by which the train of gears is driven.

Figure 3 is a back view of the machine and it shows the eccentrics that furnish the fifteen cosine components. The chain threads around the pulleys on the vertical rods, it is transferred from front to back and from back to front over

FIG. 3.

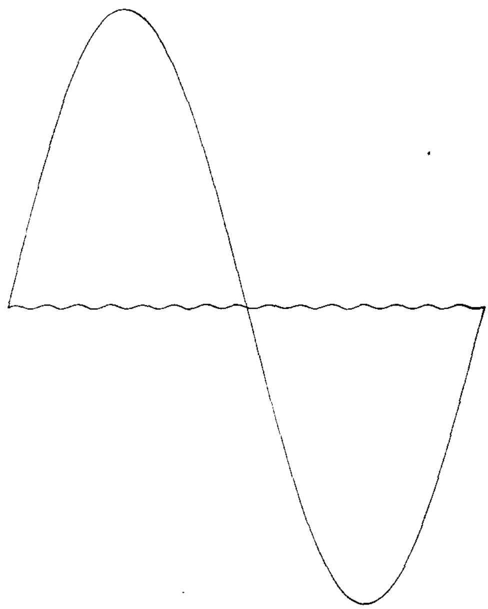


Back view—cosine components.

stationary pulleys, and tension is maintained in the continuous chain by the weight of the block that carries the tracing point. A more uniform tension is maintained throughout the length of the chain by subjecting the entire machine to slight vibrations. The vibration of the machine is produced by a small electric motor with eccentric shaft and this method of reducing friction might be termed "mechanical lubrication."

The pin of each crosshead may be slid in either direction from a center-position on a crossbar and, thereby, positive or negative amplitudes are possible. The greatest amplitude that may be set on the fundamental is 16 cm. and the crossbar

FIG. 4.



Sensitivity test.

of a fifteenth element is long enough to permit amplitude settings of as much as 4 cm.

Syntheses. An example of the use of the machine as a synthesizer is shown by the zig-zag curve in Fig. 7 where the trace was made by adding the first fifteen harmonics of the Fourier series that represents a zig-zag periodic curve of straight-line sections. The rounded tops are evidence that the higher harmonics are missing.

Figure 4 shows a fundamental along with a fifteenth harmonic, both of them traced by the machine. The amplitude of the fundamental is 10 cm. and the amplitude of the harmonic is only .5 mm., or the ratio of amplitudes is 200. The curves of Fig. 4 are given to show the mechanical perfection that has been attained and to indicate the sensitivity of the machine. An amplitude as small as .1 mm. is discernible by motion of the tracing point. The tracing point used to record the curves in this figure was a beam of light sharply focused on photographic paper.

Analyses. The possibilities of the synthesizer as an analyzer depend on two basic considerations. One of these basic facts is the repeating properties of sine and cosine functions such as

$$\sin x = \sin(\pi - x) = -\sin(\pi + x) = -\sin(2\pi - x), \quad (1)$$

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2} + x\right) = -\sin\left(\frac{3\pi}{2} - x\right) = -\sin\left(\frac{3\pi}{2} + x\right), \quad (2)$$

and

$$\cos x = \cos(2\pi - x) = -\cos(\pi - x) = -\cos(\pi + x), \quad (3)$$

$$\cos\left(\frac{\pi}{2} - x\right) = -\cos\left(\frac{\pi}{2} + x\right) = -\cos\left(\frac{3\pi}{2} - x\right) = \cos\left(\frac{3\pi}{2} + x\right). \quad (4)$$

The other basic consideration which is utilized is the negative symmetry of the sum of multiple sine curves about the half-wave-length position and the positive symmetry of the sum of multiple cosine curves about the half-wave-length position.

In the derivations to follow the ordinates of the curve to be analyzed are always designated by the small letters (y_1, y_2, etc.), and the ordinates of the auxiliary curves that are traced by the machine are always designed by capitals (Y_1, Y_2, Y_1', etc.). The amplitude setting of the nth cosine component on the machine is designated by a_n, a_n' or by α_n, α_n', etc., while the amplitude setting of the nth sine component is designated by b_n, b_n' or by β_n, β_n', etc. The sum of the contributions from a series of multiple cosine components is abbreviated by C_1, C_2, or C_1', C_2', etc., and the sum of the contributions from a series of multiple sine components is abbreviated by S_1, S_2, or S_1', S_2', etc.

THIRTY-TWO INCREMENTS.

$$A_n = \frac{2}{\lambda}\int_0^\lambda y \cos\frac{2\pi n x}{\lambda}\,dx, \tag{5}$$

$$\begin{aligned}
16\,A_1 &\doteq y_1 \cos\frac{2\pi}{32} \cdots + y_{15}\cos\frac{30\pi}{32} - y_{16} \\
&\qquad + y_{17}\cos\frac{34\pi}{32} \cdots + y_{31}\cos\frac{62\pi}{32} + y_{32} \\
&\doteq (y_1 + y_{31})\cos\frac{2\pi}{32} + (y_2 + y_{30})\cos\frac{4\pi}{32} \cdots \\
&\qquad + (y_{15} + y_{17})\cos\frac{30\pi}{32} + y_{32} - y_{16} \\
&\doteq a_1\cos\frac{2\pi}{32} + a_2\cos\frac{4\pi}{32} \cdots \\
&\qquad + a_{15}\cos\frac{30\pi}{32} + y_{32} - y_{16} \\
&\doteq C_1 + y_{32} - y_{16},
\end{aligned} \tag{6}$$

$$16\,A_2 \doteq a_1\cos\frac{4\pi}{32} + a_2\cos\frac{8\pi}{32} \cdots + a_{15}\cos\frac{60\pi}{32} + y_{32} + y_{16},$$

$$16\,A_2 \doteq C_2 + y_{32} + y_{16}, \tag{7}$$

$$16\,A_3 \doteq C_3 + y_{32} - y_{16}, \tag{8}$$

. . . .

$$16\,A_{15} \doteq C_{15} + y_{32} - y_{16}, \tag{20}$$

$$a_1 = y_1 + y_{31}, \tag{21}$$

$$a_2 = y_2 + y_{30}, \tag{22}$$

$$a_3 = y_3 + y_{29}, \tag{23}$$

$$a_4 = y_4 + y_{28}, \tag{24}$$

$$a_5 = y_5 + y_{27}, \tag{25}$$

$$a_6 = y_6 + y_{26}, \tag{26}$$

. . . .

$$a_{15} = y_{15} + y_{17}, \tag{35}$$

$$B_n = \frac{2}{\lambda}\int_0^\lambda y \sin\frac{2\pi n x}{\lambda}\,dx, \tag{36}$$

$$16\,B_1 \doteq y_1 \sin\frac{2\pi}{32} \cdots + y_{15}\sin\frac{30\pi}{32} + 0$$

$$+ y_{17}\sin\frac{34\pi}{32} \cdots + y_{31}\sin\frac{62\pi}{32} + 0$$

$$\doteq (y_1 - y_{31} \sin\frac{2\pi}{32} + (y_2 - y_{30})\sin\frac{4\pi}{32} \cdots$$

$$+ (y_{15} - y_{17})\sin\frac{30\pi}{32}$$

$$\doteq b_1 \sin\frac{2\pi}{32} + b_2\sin\frac{4\pi}{32} \cdots + b_{15}\sin\frac{30\pi}{32}$$

$$\doteq S_1, \tag{37}$$

$$16\,B_2 \doteq b_1\sin\frac{4\pi}{32} + b_2\sin\frac{8\pi}{32} \cdots + b_{15}\sin\frac{60\pi}{32},$$

$$16\,B_2 \doteq S_2, \tag{38}$$
$$16\,B_3 \doteq S_3, \tag{39}$$
$$\cdots\cdots$$
$$16\,B_{15} \doteq S_{15}, \tag{51}$$

$$b_1 = y_1 - y_{31}, \tag{52}$$
$$b_2 = y_2 - y_{30}, \tag{53}$$
$$b_3 = y_3 - y_{29}, \tag{54}$$
$$b_4 = y_4 - y_{28}, \tag{55}$$
$$b_5 = y_5 - y_{27}, \tag{56}$$
$$b_6 = y_6 - y_{26}, \tag{57}$$
$$\cdots\cdots$$
$$b_{15} = y_{15} - y_{17}, \tag{66}$$

$$16\,(A_1 + B_1) \doteq C_1 + S_1 + y_{32} - y_{16} \doteq Y_1 + y_{32} - y_{16}, \tag{67}$$

$$16\,(A_1 - B_1) \doteq C_1 - S_1 + y_{32} - y_{16} \doteq Y_{31} + y_{32} - y_{16}, \tag{68}$$

$$A_1 \doteq \frac{1}{32}[Y_1 + Y_{31} + 2(y_{32} - y_{16}],$$

$$B_1 \doteq \frac{1}{32}[Y_1 - Y_{31}], \tag{69}$$

$$A_2 \doteq \frac{1}{32}[Y_2 + Y_{30} + 2(y_{32} + y_{16})],$$

$$B_2 \doteq \frac{1}{32}[Y_2 - Y_{30}], \tag{70}$$

$$A_3 \doteq \frac{1}{32}\left[Y_3 + Y_{29} + 2(y_{32} - y_{16})\right],$$

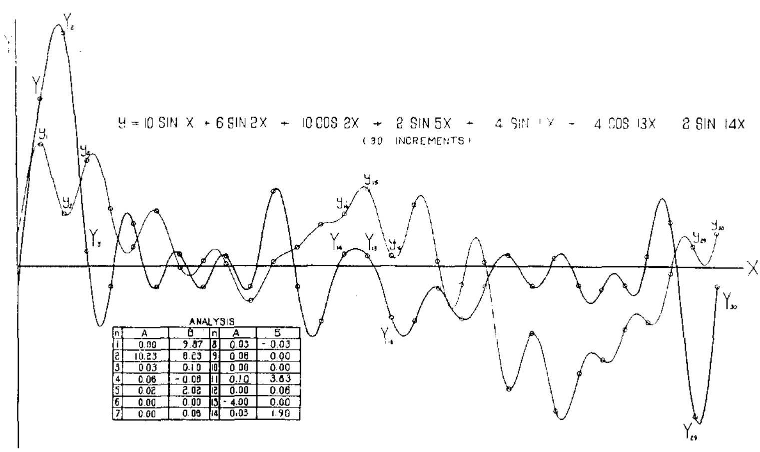
$$B_3 \doteq \frac{1}{32}\left[Y_3 - Y_{29}\right], \quad (71)$$

. . . .

$$A_{15} \doteq \frac{1}{32}\left[Y_{15} + Y_{17} + 2(y_{32} - y_{16})\right],$$

$$B_{15} \doteq \frac{1}{32}\left[Y_{15} - Y_{17}\right]. \quad (83)$$

FIG. 5.



n	A	B	n	A	B
1	0.00	9.87	8	0.03	- 0.03
2	10.23	8.23	9	0.08	0.00
3	0.03	0.10	10	0.00	0.00
4	0.06	- 0.08	11	0.10	3.83
5	0.02	2.02	12	0.00	0.06
6	0.00	0.00	13	- 4.00	0.00
7	0.00	0.08	14	0.03	1.90

Synthesis and analysis (increments of 12°).

Therefore, if the fifteen multiple cosine components of the synthesizer are set at the respective amplitudes as given by Eqs. 21–35 and the sine components are set at the respective amplitudes as given by Eqs. 52–66, the auxiliary curve with equispaced ordinates Y_1, Y_2, $\cdots$ Y_{32} is traced by the machine and the amplitudes of the fifteen harmonics of the curve which has equispaced ordinates y_1, y_2, $\cdots$ y_{32} are given by Eqs. 69–83. The auxiliary curve need not be actually traced since the values of Y_1, Y_2, etc. may be read from a stationary vertical scale. A dial on the shaft of the 8th multiple element indicates the angle through which the machine has been turned; that is, Y_1 is read from the vertical scale when this dial shows one-quarter of a revolution, Y_2 when dial shows half a revolution, etc. Figure 5 shows the synthesis and the analysis of a curve represented by a specific equation for fourteen harmonics (increments of 12°).

SIXTY-FOUR INCREMENTS.

The analyses may be extended to thirty-one harmonics if the curve to be analyzed is divided into sixty-four equispaced ordinates (y_1, y_2, $\cdots y_{64}$), and a derivation similar to that used for thirty-two equispaced ordinates gives

$$64\,A_1 \doteq Y_1 + Y_{63} + 2(y_{64} - y_{32}), \tag{84}$$

$$64\,A_3 \doteq Y_3 + Y_{61} + 2(y_{64} - y_{32}), \tag{85}$$

$$64\,A_5 \doteq Y_5 + Y_{59} + 2(y_{64} - y_{32}), \tag{86}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$64\,A_{31} \doteq Y_{31} + Y_{33} + 2(y_{64} - y_{32}), \tag{99}$$

and

$$64\,B_1 \doteq Y_1 - Y_{63} + 2(y_{16} - y_{48}), \tag{100}$$

$$64\,B_3 \doteq Y_3 - Y_{61} + 2(y_{48} - y_{16}), \tag{101}$$

$$64\,B_5 \doteq Y_5 - Y_{59} + 2(y_{16} - y_{48}), \tag{102}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$64\,B_{31} \doteq Y_{31} - Y_{33} + 2(y_{48} - y_{16}), \tag{115}$$

where the auxiliary curve of equispaced ordinates Y_1, Y_2, $\cdots Y_{64}$ is traced by the machine with the amplitudes of the fifteen cosine components set at

$$a_1 = y_1 - y_{31} - y_{33} + y_{63}, \tag{116}$$

$$a_2 = y_2 - y_{30} - y_{34} + y_{62}, \tag{117}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$a_{15} = y_{15} - y_{17} - y_{47} + y_{49}, \tag{130}$$

and the amplitudes of the fifteen sine components set at

$$b_1 = y_1 + y_{31} - y_{33} - y_{63}, \tag{131}$$

$$b_2 = y_2 + y_{30} - y_{34} - y_{62}, \tag{132}$$

. . . .

. . . .

$$b_{15} = y_{15} + y_{17} - y_{47} - y_{49}. \tag{145}$$

And the amplitudes of the even numbered components are given by

$$64\,A_2 \doteq Y_2' + Y_{62}' + 2(y_{64} - y_{48} + y_{32} - y_{16}), \tag{146}$$

$$64\,A_4 \doteq Y_4' + Y_{60}' + 2(y_{64} + y_{48} + y_{32} + y_{16}), \tag{147}$$

$$64\,A_6 \doteq Y_6' + Y_{58}' + 2(y_{64} - y_{48} + y_{32} - y_{16}), \tag{148}$$

. . . .

. . . .

$$64\,A_{30} \doteq Y_{30}' + Y_{34}' + 2(y_{64} - y_{48} + y_{32} - y_{16}), \tag{160}$$

and

$$64\,B_2 \doteq Y_2' - Y_{62}', \tag{161}$$

$$64\,B_4 \doteq Y_4' - Y_{60}', \tag{162}$$

$$64\,B_6 \doteq Y_6' - Y_{58}', \tag{163}$$

. . . .

. . . .

$$64\,B_{30} \doteq Y_{30}' - Y_{34}', \tag{175}$$

where the second auxiliary curve of equispaced ordinates Y_1', Y_2', $\cdots$ Y_{64}' is traced by the machine with the amplitudes of the fifteen cosine components set at

$$a_1' = y_1 + y_{31} + y_{33} + y_{63}, \tag{176}$$

$$a_2' = y_2 + y_{30} + y_{34} + y_{62}, \tag{177}$$

. . . .

. . . .

$$a_{15}' = y_{15} + y_{17} + y_{47} + y_{49}, \tag{190}$$

and the amplitudes of the fifteen sine components are set at

$$b_1' = y_1 - y_{31} + y_{33} - y_{63}, \tag{191}$$

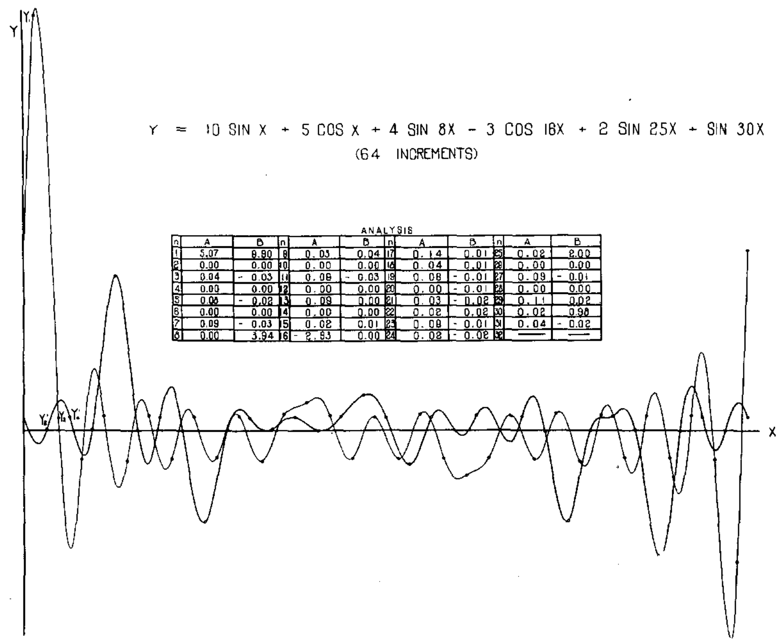
$$b_2' = y_2 - y_{30} + y_{34} - y_{62}, \tag{192}$$

. . . .

. . . .

$$b_{15}' = y_{15} - y_{17} + y_{47} - y_{49}. \tag{205}$$

FIG. 6.



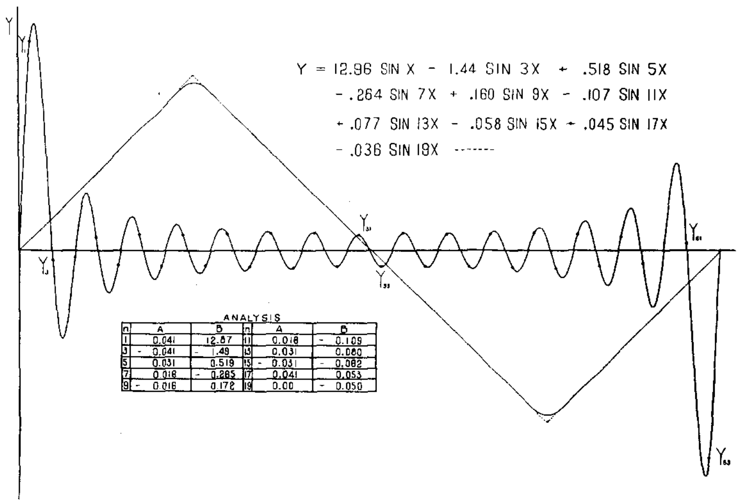
n	A	B	n	A	B	n	A	B	n	A	B
1	5.07	9.90	9	0.03	0.04	17	0.14	0.01	25	0.02	2.00
2	0.00	0.00	10	0.00	0.00	18	0.04	0.01	26	0.00	0.00
3	0.04	- 0.03	11	0.08	- 0.03	19	0.08	- 0.01	27	0.09	- 0.01
4	0.00	0.00	12	0.00	0.00	20	0.00	- 0.01	28	0.00	0.00
5	0.06	- 0.02	13	0.09	0.00	21	0.03	- 0.02	29	0.11	0.02
6	0.00	0.00	14	0.00	0.00	22	0.02	0.02	30	0.02	0.98
7	0.09	- 0.03	15	0.02	0.01	23	0.08	- 0.01	31	0.04	- 0.02
8	0.00	3.94	16	- 2.93	0.00	24	0.02	- 0.02	32	——	——

Analysis (increments of 5.625°).

If the curve is to be analyzed from sixty (increments of 6°), instead of sixty-four equispaced ordinates, only the first fourteen harmonic components of the machine are utilized ($a_{15} = a_{15}' = b_{15} = b_{15}' = 0$), and only thirty ordinates from each of the two auxiliary curves enter the calculations of the harmonic components of the curve that is being analyzed.

Figure 6 shows the analysis of a curve represented by a specific equation for thirty-one harmonics (increments of 5.625°). Figure 7 illustrates the determination of the first

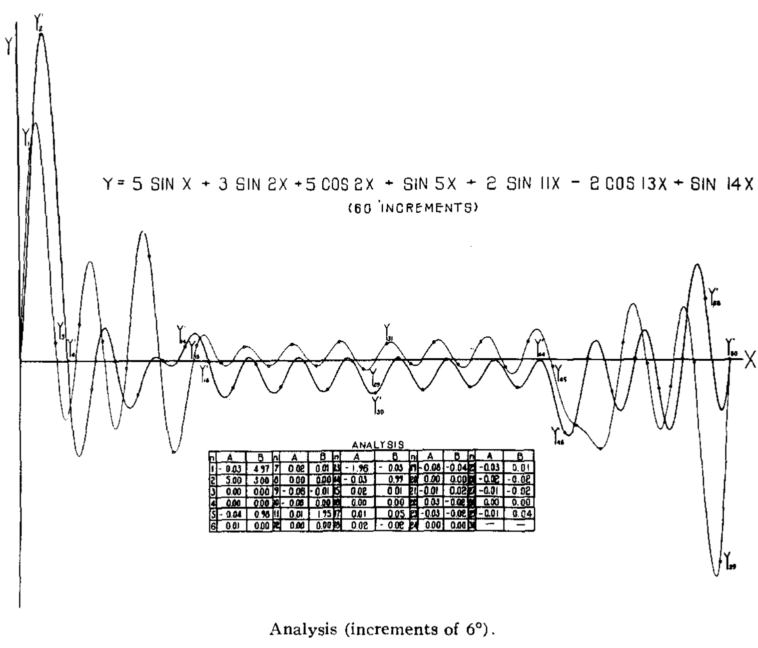
FIG. 7.



n	A	B	n	A	B
1	0.041	12.87	11	0.018	− 0.109
3	− 0.041	− 1.49	13	0.031	0.080
5	0.031	0.519	15	− 0.031	− 0.082
7	0.018	− 0.285	17	0.041	0.053
9	− 0.016	0.172	19	0.00	− 0.050

Synthesis and analysis (infinite series).

FIG. 8.



n	A	B	n	A	B	n	A	B	n	A	B	n	A	B
1	− 0.03	4.97	7	0.02	0.01	13	− 1.96	− 0.03	19	− 0.06	− 0.04	[illegible]	− 0.03	0.01
2	5.00	3.00	8	0.00	0.00	14	− 0.03	0.99	20	0.00	0.00	[illegible]	− 0.02	− 0.02
3	0.00	0.00	9	− 0.06	− 0.01	15	0.02	0.01	21	− 0.01	0.02	[illegible]	− 0.01	− 0.02
4	0.00	0.00	10	− 0.06	0.00	[illegible]	0.00	0.00	[illegible]	0.03	− 0.02	[illegible]	0.00	0.00
5	− 0.04	0.98	11	0.01	1.95	17	0.01	0.05	23	− 0.03	− 0.02	[illegible]	− 0.01	0.04
6	0.01	0.00	12	0.00	0.00	18	0.02	− 0.02	24	0.00	0.00	[illegible]	—	—

Analysis (increments of 6°).

nineteen harmonics (odd numbered) of a curve that is represented by an infinite series. Figure 8 illustrates the analysis of a curve represented by a specific equation for twenty-nine harmonics (increments of 6°).

ONE HUNDRED-AND-TWENTY INCREMENTS.

Finally, the analysis is extended to include fifty-nine harmonics by dividing a curve into one hundred-and-twenty equispaced ordinates ($y_1, y_2, \cdots y_{120}$), and then the derivation gives

$$120\,A_1 \doteq Y_1 + Y_{119}' + 2(y_{120} - y_{60}), \tag{206}$$

$$120\,B_1 \doteq Y_1 - Y_{119}' + 2(y_{30} - y_{90}), \tag{207}$$

$$120\,A_3 \doteq Y_3' + Y_{117} + 2(y_{120} - y_{60}), \tag{208}$$

$$120\,B_3 \doteq Y_3' - Y_{117} + 2(y_{90} - y_{30}), \tag{209}$$

. . . .

. . . .

$$120\,A_{57} \doteq Y_{57} + Y_{63}' + 2(y_{120} - y_{60}), \tag{262}$$

$$120\,B_{57} \doteq Y_{57} - Y_{63}' + 2(y_{30} - y_{90}), \tag{263}$$

$$120\,A_{59} \doteq Y_{59}' + Y_{61} + 2(y_{120} - y_{60}), \tag{264}$$

$$120\,B_{59} \doteq Y_{59}' - Y_{61} + 2(y_{90} - y_{30}), \tag{265}$$

and

$$120\,A_2 \doteq Y_2'' + Y_{118}'' + 2(y_{120} + y_{60} - y_{30} - y_{90}), \tag{266}$$

$$120\,B_2 \doteq Y_2'' - Y_{118}'', \tag{267}$$

$$120\,A_4 \doteq Y_4''' + Y_{116}''' + 2(y_{120} + y_{60} + y_{30} + y_{90}), \tag{268}$$

$$120\,B_4 \doteq Y_4''' - Y_{116}''', \tag{269}$$

. . . .

. . . .

$$120\,A_{56} \doteq Y_{56}''' + Y_{64}''' + 2(y_{120} + Y_{60} + y_{30} + y_{90}), \tag{320}$$

$$120\,B_{56} \doteq Y_{56}''' - Y_{64}''', \tag{321}$$

$$120\,A_{58} \doteq Y_{58}'' + Y_{62}'' + 2(y_{120} + y_{60} - y_{30} - y_{90}), \tag{322}$$

$$120\,B_{58} \doteq Y_{58}'' - Y_{62}'', \tag{323}$$

where the four auxiliary curves of equispaced ordinates Y_1, Y_2, $\cdots$ Y_{120}; Y_1', Y_2', $\cdots$ Y_{120}'; Y_1'', Y_2'', $\cdots$ Y_{120}'' and Y_1''', Y_2''', $\cdots$ Y_{120}''' are traced by the machine with the amplitudes of the fifteen cosine components set at the respective values (α), (α'), (α'') and (α''').

(α)

$$\alpha_1 = a_1 + k_1 = (y_1 - y_{59} - y_{61} + y_{119}) + (y_{29} + y_{31} - y_{89} - y_{91}), \quad (324)$$

$$\alpha_2 = a_2 + k_2 = (y_2 - y_{58} - y_{62} + y_{118}) + (y_{28} + y_{32} - y_{80} - y_{92}), \quad (325)$$

. . . .

$$\alpha_{14} = a_{14} + k_{14} = (y_{14} - y_{46} - y_{74} + y_{106}) + (y_{16} + y_{44} - y_{76} - y_{104}), \quad (337)$$

$$\alpha_{15} = a_{15} + 0 = (y_{15} - y_{45} - y_{75} + y_{105}). \quad (338)$$

(α'')

$$\alpha_1'' = a_1' + b_1' = (y_1 + y_{59} + y_{61} + y_{119}) + (-y_{29} - y_{31} - y_{89} - y_{91}), \quad (339)$$

$$\alpha_2'' = a_2' + b_2' = (y_2 + y_{58} + y_{62} + y_{118}) + (-y_{28} - y_{32} - y_{88} - y_{92}), \quad (340)$$

. . . .

$$\alpha_{14}'' = a_{14}' + b_{14}' = (y_{14} + y_{46} + y_{74} + y_{106}) + (-y_{16} - y_{44} - y_{76} - y_{104}), \quad (352)$$

$$\alpha_{15}'' = a_{15}' + 0 = (y_{15} + y_{45} + y_{75} + y_{105}). \quad (353)$$

(α')

$$\alpha_1' = a_1 - k_1, \quad (354)$$

$$\alpha_2' = a_2 - k_2, \quad (355)$$

. . . .

$$\alpha_{14}' = a_{14} - k_{14}, \quad (367)$$

$$\alpha_{15}' = a_{15} - 0, \quad (368)$$

(α''')

$$\alpha_1''' = a_1' - b_1', \quad (369)$$

$$\alpha_2''' = a_2' - b_2', \quad (370)$$

. . . .

$$\alpha_{14}''' = a_{14}' - b_{14}', \quad (382)$$

$$\alpha_{15}''' = a_{15}' - 0, \quad (383)$$

and with the fifteen sine components of the machine set at the respective values (β), (β'), (β'') and (β''')

(β)

$$\beta_1 = b_1 + j_1 = (y_{29} - y_{31} - y_{89} + y_{91}) + (y_1 + y_{59} - y_{61} - y_{119}), \quad (354)$$

$$\beta_2 = b_2 + j_2 = (y_{28} - y_{32} - y_{88} + y_{92}) + (y_2 + y_{58} - y_{62} - y_{118}), \quad (355)$$

. . . .

$$\beta_{14} = b_{14} + j_{14} = (y_{16} - y_{44} - y_{76} + y_{104}) + (y_{14} + y_{46} - y_{74} - y_{106}), \quad (397)$$

$$\beta_{15} = 0 + j_{15} = (y_{15} + y_{45} - y_{75} - y_{105}). \quad (398)$$

(β'')

$$\beta_1'' = j_1' + k_1' = (y_1 - y_{59} + y_{61} - y_{119}) + (y_{29} - y_{31} + y_{89} - y_{91}), \quad (399)$$

$$\beta_2'' = j_2' + k_2' = (y_2 - y_{58} + y_{62} - y_{118}) + (y_{28} - y_{32} + y_{88} - y_{92}), \quad (400)$$

. . . .

$$\beta_{14}'' = j_{14}' + k_{14}' = (y_{16} - y_{46} + y_{74} - y_{106}) + (y_{16} - y_{44} + y_{76} - y_{104}), \quad (412)$$

$$\beta_{15}'' = j_{15}' + 0 = (y_{15} - y_{45} + y_{75} - y_{105}). \quad (413)$$

(β')

$$\beta_1' = -b_1 + j_1, \quad (414)$$

$$\beta_2' = -b_2 + j_2, \quad (415)$$

. . . .

$$\beta_{14}' = -b_{14} + j_{14}, \quad (427)$$

$$\beta_{15}' = -0 + j_{15}. \quad (428)$$

(β''')

$$\beta_1''' = j_1' - k_1', \quad (429)$$

$$\beta_2''' = j_2' - k_2', \quad (430)$$

. . . .

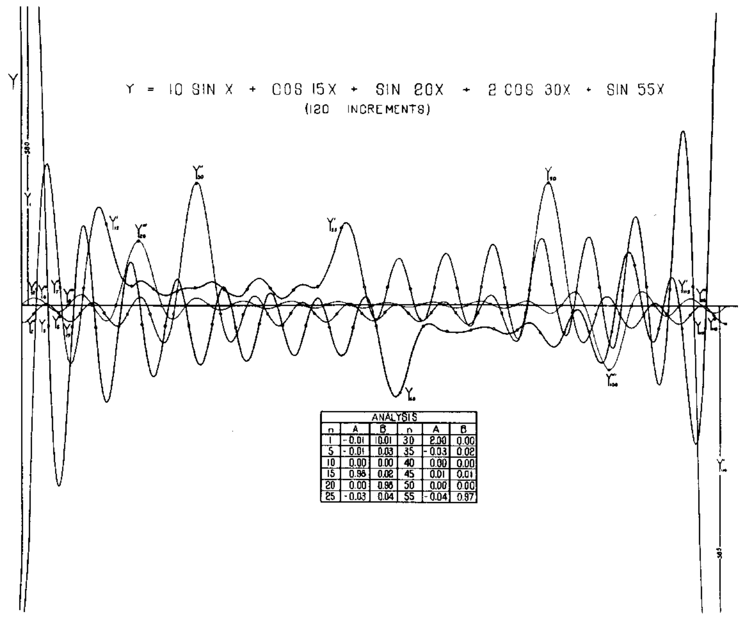
$$\beta_{14}''' = j_{14}' - k_{14}', \quad (442)$$

$$\beta_{15}''' = j_{15}' - 0. \quad (443)$$

Figure 9 shows the analysis of a curve represented by a specific equation for fifty-nine harmonics (increments of 3°). The complete traces of the four auxiliary curves are shown in this figure but, of course, the analysis only requires that the values of specified ordinates from each curve be recorded (read from fixed scale on the machine), and no curves really need to be traced.

The computations for the amplitude settings of the machine can be greatly facilitated by the proper schedule in tabulated values of the ordinates y_n of the curve being analyzed, and the computations of the harmonic components,

FIG. 9.



ANALYSIS					
n	A	B	n	A	B
1	-0.01	10.01	30	2.00	0.00
5	-0.01	0.03	35	-0.03	0.02
10	0.00	0.00	40	0.00	0.00
15	0.98	0.02	45	0.01	0.01
20	0.00	0.98	50	0.00	0.00
25	-0.03	0.04	55	-0.04	0.97

Analysis (increments of 3°).

A_n and B_n, are facilitated by the proper schedule in tabulated values of the ordinates Y_n, Y_n', etc. from the auxiliary curves.

In addition to the usual applications of synthesizers and analyzers, this machine has been used for the interpretation of harmonic solutions of differential equations and it has been used to synthesize crystal structure data to determine interatomic distances by the Patterson [7] method. Some specific applications of the machines as an analyzer are: hidden

[7] A. L. Patterson, *Phys. Rev.*, **46**, 372–376.

periodicities in strata of cores from oil wells or in weather conditions, etc., and analyses of cardiograms and of wave forms of tones from organ pipes as influenced by the construction of the pipes.

The author is indebted to the mechanicians, Mr. L. H. Gruber and Mr. Robert Brandt, for the mechanical perfection which they achieved in the construction of this machine, and to Dr. Lucien LaCoste for his valuable suggestions.