Chapter 1

First Examples

The purpose of this short chapter is to develop some simple examples of differential equations. This development motivates the linear algebra treated subsequently and moreover gives in an elementary context some of the basic ideas of ordinary differential equations. Later these ideas will be put into a more systematic exposition. In particular, the examples themselves are special cases of the class of differential equations considered in Chapter 3. We regard this chapter as important since some of the most basic ideas of differential equations are seen in simple form.

§1. The Simplest Examples

The differential equation

$$\frac{dx}{dt} = ax \tag{1}$$

is the simplest differential equation. It is also one of the most important. First, what does it mean? Here $x = x(t)$ is an unknown real-valued function of a real variable t and dx/dt is its derivative (we will also use x' or $x'(t)$ for this derivative). The equation tells us that for every value of t the equality

$$x'(t) = ax(t)$$

is true. Here a denotes a constant.

The solutions to (1) are obtained from calculus: if K is any constant (real number), the function $f(t) = Ke^{at}$ is a solution since

$$f'(t) = aKe^{at} = af(t).$$

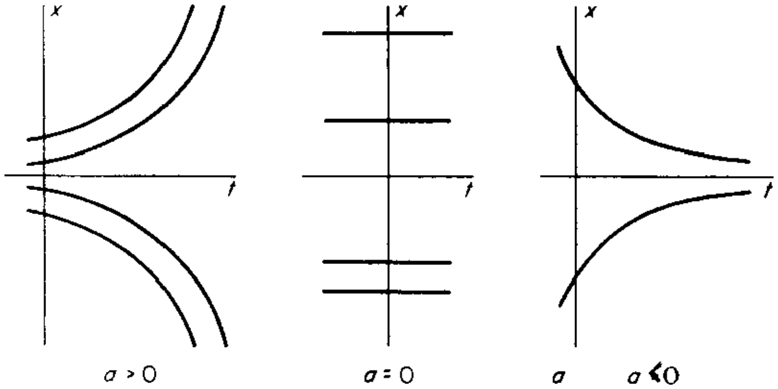

FIG. A

Moreover, *there are no other solutions*. To see this, let $u(t)$ be any solution and compute the derivative of $u(t)e^{-at}$:

$$\frac{d}{dt}\left(u(t)e^{-at}\right) = u'(t)e^{-at} + u(t)(-ae^{-at})$$

$$= au(t)e^{-at} - au(t)e^{-at} = 0.$$

Therefore $u(t)e^{-at}$ is a constant K, so $u(t) = Ke^{at}$. This proves our assertion.

The constant K appearing in the solution is completely determined if the value u_0 of the solution at a single point t_0 is specified. Suppose that a function $x(t)$ satisfying (1) is required such that $x(t_0) = u_0$, then K must satisfy $Ke^{at_0} = u_0$. Thus equation (1) has a unique solution satisfying a specified *initial condition* $x(t_0) = u_0$. For simplicity, we often take $t_0 = 0$; then $K = u_0$. There is no loss of generality in taking $t_0 = 0$, for if $u(t)$ is a solution with $u(0) = u_0$, then the function $v(t) = u(t - t_0)$ is a solution with $v(t_0) = u_0$.

It is common to restate (1) in the form of an *initial value problem*:

$$x' = ax, \qquad x(0) = K. \tag{2}$$

A solution $x(t)$ to (2) must not only satisfy the first condition (1), but must also take on the prescribed initial value K at $t = 0$. We have proved that the initial value problem (2) has a unique solution.

The constant a in the equation $x' = ax$ can be considered as a parameter. If a changes, the equation changes and so do the solutions. Can we describe qualitatively the way the solutions change?

The *sign* of a is crucial here:

if $a > 0$, $\lim_{t\to\infty} Ke^{at}$ equals ∞ when $K > 0$, and equals $-\infty$ when $K < 0$;
if $a = 0$, $Ke^{at} =$ constant;
if $a < 0$, $\lim_{t\to\infty} Ke^{at} = 0$.

The qualitative behavior of solutions is vividly illustrated by sketching the graphs of solutions (Fig. A). These graphs follow a typical practice in this book. The figures are meant to illustrate qualitative features and may be imprecise in quantitative detail.

The equation $x' = ax$ is *stable* in a certain sense if $a \neq 0$. More precisely, if a is replaced by another constant b sufficiently close to a, the qualitative behavior of the solutions does not change. If, for example, $|b - a| < |a|$, then b has the same sign as a. But if $a = 0$, the slightest change in a leads to a radical change in the behavior of solutions. We may also say that $a = 0$ is a *bifurcation point* in the one-parameter family of equations $x' = ax$, a in R.

Consider next a *system* of two differential equations in two unknown functions:

$$\begin{aligned} x_1' &= a_1x_1, \\ x_2' &= a_2x_2. \end{aligned} \tag{3}$$

This is a very simple system; however, many more-complicated systems of two equations can be reduced to this form as we shall see a little later.

Since there is no relation specified between the two unknown functions $x_1(t)$, $x_2(t)$, they are "uncoupled"; we can immediately write down all solutions (as for (1)):

$$x_1(t) = K_1 \exp(a_1t), \qquad K_1 = \text{constant},$$
$$x_2(t) = K_2 \exp(a_2t), \qquad K_2 = \text{constant}.$$

Here K_1 and K_2 are determined if initial conditions $x_1(t_0) = u_1$, $x_2(t_0) = u_2$ are specified. (We sometimes write exp a for e^a.)

Let us consider equation (2) from a more geometric point of view. We consider two functions $x_1(t)$, $x_2(t)$ as specifying an unknown *curve* $x(t) = (x_1(t), x_2(t))$ in the (x_1, x_2) plane $\mathbf{R}^2$. That is to say, x is a map from the real numbers R into $\mathbf{R}^2$, x: $\mathbf{R} \to \mathbf{R}^2$. The right-hand side of (3) expresses the *tangent vector* $x'(t) = (x_1'(t), x_2'(t))$ to the curve. Using vector notation,

$$x' = Ax, \tag{3'}$$

where Ax denotes the vector (a_1x_1, a_2x_2), which one should think of as being based at x.

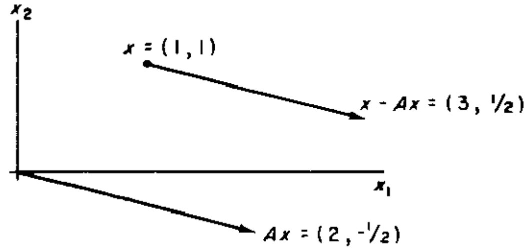

FIG. B

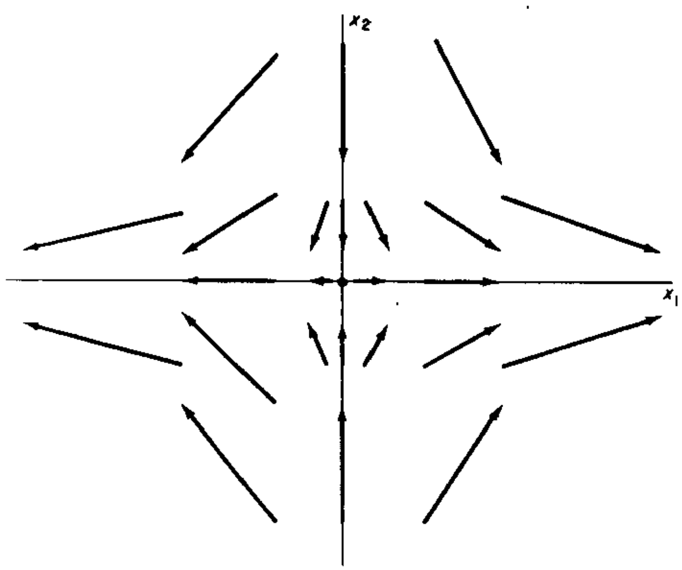

FIG. C. $Ax = (2x_1, -\frac{1}{2}x_2)$.

Initial conditions are of the form $x(t_0) = u$ where $u = (u_1, u_2)$ is a given point of $\mathbf{R}^2$. Geometrically, this means that when $t = t_0$ the curve is required to pass through the given point u.

The map (that is, function) $A: \mathbf{R}^2 \to \mathbf{R}^2$ (or $x \to Ax$) can be considered a *vector field* on $\mathbf{R}^2$. This means that to each point x in the plane we assign the vector Ax. For purposes of visualization, we picture Ax as a vector "based at x"; that is, we assign to x the directed line segment from x to $x + Ax$. For example, if $a_1 = 2$, $a_2 = -\frac{1}{2}$, and $x = (1, 1)$, then at $(1, 1)$ we picture an arrow pointing from $(1, 1)$ to $(1, 1) + (2, -\frac{1}{2}) = (3, \frac{1}{2})$ (Fig. B). Thus if $Ax = (2x_1, -\frac{1}{2}x_2)$, we attach to each point x in the plane an arrow with tail at x and head at $x + Ax$ and obtain the picture in Fig. C.

Solving the differential equation (3) or (3′) with initial conditions (u_1, u_2) at $t = 0$ means finding in the plane a curve $x(t)$ that satisfies (3′) and passes through the point $u = (u_1, u_2)$ when $t = 0$. A few solution curves are sketched in Fig. D. The trivial solution $(x_1(t), x_2(t)) = (0, 0)$ is also considered a "curve."

The family of all solution curves as subsets of $\mathbf{R}^2$ is called the "phase portrait" of equation (3) (or (3′)).

The one-dimensional equation $x' = ax$ can also be interpreted geometrically: the phase portrait is as in Fig. E, which should be compared with Fig. A. It is clearer to picture the graphs of (1) and the solution curves for (3) since two-dimensional pictures are better than either one- or three-dimensional pictures. The *graphs* of

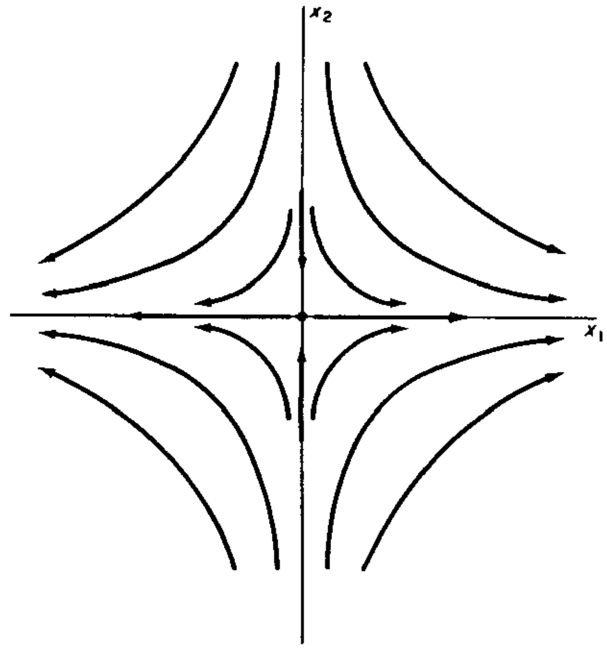

FIG. D. Some solution curves to $x' = Ax$, $A = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

solutions to (3) require a three-dimensional picture which the reader is invited to sketch!

Let us consider equation (3) as a *dynamical system*. This means that the independent variable t is interpreted as *time* and the solution curve $x(t)$ could be thought of, for example, as the path of a particle moving in the plane $\mathbf{R}^2$. We can imagine a particle placed at any point $u = (u_1, u_2)$ in $\mathbf{R}^2$ at time $t = 0$. As time proceeds the particle moves along the solution curve $x(t)$ that satisfies the initial condition $x(0) = u$. At any later time $t > 0$ the particle will be in another position $x(t)$. And at an earlier time $t < 0$, the particle was at a position $x(t)$. To indicate the dependence of the position on t and u we denote it by $\phi_t(u)$. Thus

$$\phi_t(u) = (u_1 \exp(a_1 t), u_2 \exp(a_2 t)).$$

We can imagine particles placed at each point of the plane and all moving simultaneously (for example, dust particles under a steady wind). The solution curves are spoken of as trajectories or orbits in this context. For each fixed t in $\mathbf{R}$, we have a transformation assigning to each point u in the plane another point $\phi_t(u)$. This transformation denoted by $\phi_t: \mathbf{R}^2 \to \mathbf{R}^2$ is clearly a *linear* transformation, that is,

FIG. E

$\phi_t(u + v) = \phi_t(u) + \phi_t(v)$ and $\phi_t(\lambda u) = \lambda\phi_t(u)$, for all vectors u, v, and all real numbers λ.

As time proceeds, every point of the plane moves simultaneously along the trajectory passing through it. In this way the collection of maps $\phi_t\colon \mathbf{R}^2 \to \mathbf{R}^2$, $t \in \mathbf{R}$, is a one-parameter family of transformations. This family is called the *flow* or *dynamical system* or $\mathbf{R}^2$ determined by the vector field $x \to Ax$, which in turn is equivalent to the system (3).

The dynamical system on the real line $\mathbf{R}$ corresponding to equation (1) is particularly easy to describe: if $a < 0$, all points move toward 0 as time goes to ∞; if $a > 0$, all points except 0 move away from 0 toward $\pm\infty$; if $a = 0$, all points stand still.

We have started from a differential equation and have obtained the dynamical system ϕ_t. This process is established through the fundamental theorem of ordinary differential equations as we shall see in Chapter 8.

Later we shall also reverse this process: starting from a dynamical system ϕ_t, a differential equation will be obtained (simply by differentiating $\phi_t(u)$ with respect to t).

It is seldom that differential equations are given in the simple uncoupled form (3). Consider, for example, the system:

$$x_1' = 5x_1 + 3x_2, \qquad (4)$$

$$x_2' = -6x_1 - 4x_2$$

or in vector notation

$$x' = (5x_1 + 3x_2, -6x_1 - 4x_2) \equiv Bx. \qquad (4')$$

Our approach is to find a linear *change of coordinates* that will transform equation (4) into uncoupled or diagonal form. It turns out that new coordinates (y_1, y_2) do the job where

$$y_1 = 2x_1 + x_2,$$

$$y_2 = x_1 + x_2.$$

(In Chapter 3 we explain how the new coordinates were found.)

Solving for x in terms of y, we have

$$x_1 = y_1 - y_2,$$

$$x_2 = -y_1 + 2y_2.$$

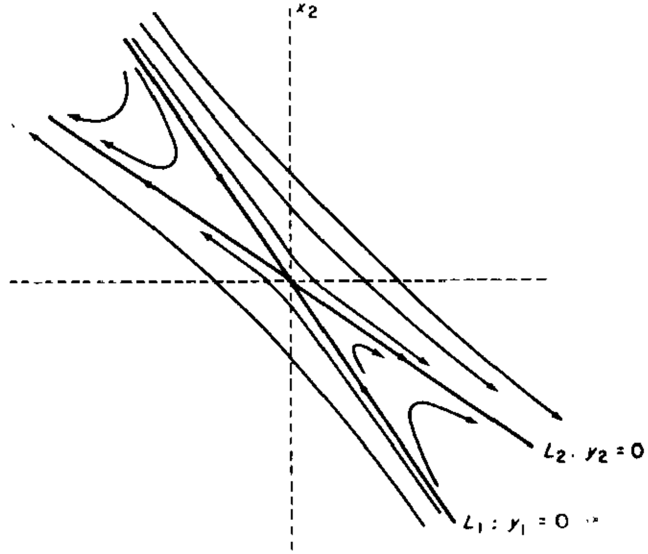

FIG. F

To find y_1', y_2' differentiate the equations defining y_1, y_2 to obtain

$$y_1' = 2x_1' + x_2',$$

$$y_2' = x_1' + x_2'.$$

By substitution

$$y_1' = 2(5x_1 + 3x_2) + (-6x_1 - 4x_2) = 4x_1 + 2x_2,$$

$$y_2' = (5x_1 + 3x_2) + (-6x_1 - 4x_2) = -x_1 - x_2.$$

Another substitution yields

$$y_1' = 4(y_1 - y_2) + 2(-y_1 + 2y_2),$$

$$y_2' = -(y_1 - y_2) - (-y_1 + 2y_2),$$

or

$$y_1' = 2y_1, \qquad (5)$$

$$y_2' = -y_2.$$

The last equations are in *diagonal form* and we have already solved this class of systems. The solution $(y_1(t), y_2(t))$ such that $(y_1(0), y_2(0)) = (v_1, v_2)$ is

$$y_1(t) = e^{2t}v_1,$$

$$y_2(t) = e^{-t}v_2.$$

The phase portrait of this system (5) is given evidently in Fig. D. We can find the phase portrait of the original system (4) by simply plotting the new coordinate axes $y_1 = 0$, $y_2 = 0$ in the (x_1, x_2) plane and sketching the trajectories $y(t)$ in these coordinates. Thus $y_1 = 0$ is the line L_1: $x_2 = -2x_1$ and $y_2 = 0$ is the line L_2: $x_2 = -x_1$.

Thus we have the phase portrait of (4) as in Fig. F, which should be compared with Fig. D.

Formulas for the solution to (4) can be obtained by substitution as follows. Let (u_1, u_2) be the initial values $(x_1(0), x_2(0))$ of a solution $(x_1(t), x_2(t))$ to (4). Corresponding to (u_1, u_2) is the initial value (v_1, v_2) of a solution $(y_1(t), y_2(t))$ to (5) where

$$v_1 = 2u_1 + u_2,$$
$$v_2 = u_1 + u_2.$$

Thus

$$y_1(t) = e^{2t}(2u_1 + u_2),$$
$$y_2(t) = e^{-t}(u_1 + u_2)$$

and

$$x_1(t) = e^{2t}(2u_1 + u_2) - e^{-t}(u_1 + u_2),$$
$$x_2(t) = -e^{2t}(2u_1 + u_2) + 2e^{-t}(u_1 + u_2).$$

If we compare these formulas to Fig. F, we see that the diagram instantly gives us the qualitative picture of the solutions, while the formulas convey little geometric information. In fact, for many purposes, it is better to forget the original equation (4) and the corresponding solutions and work entirely with the "diagonalized" equations (5), their solution and phase portrait.

PROBLEMS

1. Each of the "matrices"

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = [a_{ij}]$$

given below defines a vector field on $\mathbf{R}^2$, assigning to $x = (x_1, x_2) \in \mathbf{R}^2$ the vector $Ax = (a_{11}x_1 + a_{12}x_2,\ a_{21}x_1 + a_{22}x_2)$ based at x. For each matrix, draw enough of the vectors until you get a feeling for what the vector field looks

like. Then sketch the phase portrait of the corresponding differential equation $x' = Ax$, guessing where necessary.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{2} & -2 \\ 2 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ (h) $\begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$ (i) $\begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}$

2. Consider the one-parameter family of differential equations

$$x_1' = 2x_1,$$
$$x_2' = ax_2; \qquad -\infty < a < \infty.$$

(a) Find all solutions $(x_1(t), x_2(t))$.

(b) Sketch the phase portrait for a equal to $-1, 0, 1, 2, 3$. Make some guesses about the stability of the phase portraits.

§2. Linear Systems with Constant Coefficients

This section is devoted to generalizing and abstracting the previous examples. The general problem is stated, but solutions are postponed to Chapter 3.

Consider the following set or "system" of n differential equations:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,$$
$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \tag{1}$$
$$\vdots$$
$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n.$$

Here the a_{ij} $(i = 1, \ldots, n;\ j = 1, \ldots, n)$ are n^2 constants (real numbers), while each x_i denotes an unknown real-valued function of a real variable t. Thus (4) of Section 1 is an example of the system (1) with $n = 2$, $a_{11} = 5$, $a_{12} = 3$, $a_{21} = -6$, $a_{22} = -4$.

At this point we are not trying to solve (1); rather, we want to place it in a geometrical and algebraic setting in order to understand better what a solution means.

At the most primitive level, a solution of (1) is a set of n differentiable real-valued functions $x_i(t)$ that make (1) true.

In order to reach a more conceptual understanding of (1) we introduce *real n-dimensional Cartesian space* $\mathbf{R}^n$. This is simply the set of all n-tuples of real numbers. An element of $\mathbf{R}^n$ is a "point" $x = (x_1, \ldots, x_n)$; the number x_i is the ith *coordinate* of the point x. Points x, y in $\mathbf{R}^n$ are added coordinatewise:

$$x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

Also, if λ is a real number we define the *product* of λ and x to be

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n).$$

The *distance* between points x, y in $\mathbf{R}^n$ is defined to be

$$|x - y| = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{1/2}.$$

The *length* of x is

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

A vector *based at* $x \in \mathbf{R}^n$ is an ordered pair of points x, y in $\mathbf{R}^n$, denoted by $\overrightarrow{xy}$. We think of this as an arrow or line segment directed from x to y. We say $\overrightarrow{xy}$ is *based* at x.

A vector $\overrightarrow{0x}$ based at the *origin*

$$\mathbf{0} = (0, \ldots, 0) \in \mathbf{R}^n$$

is identified with the point $x \in \mathbf{R}^n$.

To a vector $\overrightarrow{xy}$ based at x is associated the vector $y - x$ based at the origin $\mathbf{0}$. We call the vectors $\overrightarrow{xy}$ and $y - x$ *translates* of each other.

From now on a vector based at $\mathbf{0}$ is called simply a vector. Thus an element of $\mathbf{R}^n$ can be considered either as an n-tuple of real numbers or as an arrow issuing from the origin.

It is only for purposes of visualization that we consider vectors based at points other than 0. For computations, all vectors are based at $\mathbf{0}$ since such vectors can be added and multiplied by real numbers.

We return to the system of differential equations (1). A candidate for a solution is a *curve* in $\mathbf{R}^n$:

$$(*) \qquad x(t) = (x_1(t), \ldots, x_n(t)).$$

By this we mean a map

$$x: \mathbf{R} \to \mathbf{R}^n.$$

Such a map is described in terms of coordinates by $(*)$. If each function $x_i(t)$ is differentiable, then the map x is called differentiable; its derivative is defined to be

$$\frac{dx}{dt} = x'(t) = (x_1'(t), \ldots, x_n'(t)).$$

Thus the derivative, as a function of t, is again a map from $\mathbf{R}$ to $\mathbf{R}^n$.

The derivative can also be expressed in the form

$$x'(t) = \lim_{h \to 0} \frac{1}{h}(x(t + h) - x(t)).$$

It has a natural geometric interpretation as the vector $v(t)$ based at $x(t)$, which is a translate of $x'(t)$. This vector is called the *tangent vector* to the curve at t (or at $x(t)$).

If we imagine t as denoting time, then the length $|x'(t)|$ of the tangent vector is interpreted physically as the speed of a particle describing the curve $x(t)$.

To write (1) in an abbreviated form we call the doubly indexed set of numbers a_{ij} an $n \times n$ *matrix* A, denoted thus:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Next, for each $x \in \mathbf{R}^n$ we define a vector $Ax \in \mathbf{R}^n$ whose ith coordinate is

$$a_{i1}x_1 + \cdots + a_{in}x_n;$$

note that this is the ith row in the right-hand side of (1). In this way the matrix A is interpreted as a map

$$A: \mathbf{R}^n \to \mathbf{R}^n$$

which to x assigns Ax.

With this notation (1) is rewritten

$$(2) \qquad x' = Ax.$$

Thus the system (1) can be considered as a single "vector differential equation" (2). (The word *equation* is classically reserved for the case of just one variable; we shall call (2) both a system and an equation.)

We think of the map $A: \mathbf{R}^n \to \mathbf{R}^n$ as a *vector field* on $\mathbf{R}^n$: to each point $x \in \mathbf{R}^n$ it assigns the vector based at x which is a translate of Ax. Then a solution of (2) is a curve $x: \mathbf{R} \to \mathbf{R}^n$ whose tangent vector at any given t is the vector $Ax(t)$ (translated to $x(t)$). See Fig. D of Section 1.

In Chapters 3 and 4 we shall give methods of explicitly solving (2), or equivalently (1). In subsequent chapters it will be shown that in fact (2) has a unique solution $x(t)$ satisfying any given initial condition $x(0) = u_0 \in \mathbf{R}^n$. This is the fundamental theorem of linear differential equations with constant coefficients; in Section 1 this was proved for the special case $n = 1$.

PROBLEMS

1. For each of the following matrices A sketch the vector field $x \to Ax$ in $\mathbf{R}^3$. (Missing matrix entries are 0.)

(a) $\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & & \\ & -2 & \\ & & 0\end{bmatrix}$ (c) $\begin{bmatrix}1 & & \\ & -2 & \\ & & 2\end{bmatrix}$

(d) $\begin{bmatrix}0 & & \\ & -1 & \\ & & 0\end{bmatrix}$ (e) $\begin{bmatrix}0 & -1 & \\ 1 & 0 & \\ & & -\frac{1}{2}\end{bmatrix}$ (f) $\begin{bmatrix}-1 & & \\ 1 & 1 & \\ & 1 & 1\end{bmatrix}$

2. For A as in (a), (b), (c) of Problem 1, solve the initial value problem

$$x' = Ax, \qquad x(0) = (k_1, k_2, k_3).$$

3. Let A be as in (e), Problem 1. Find constants a, b, c such that the curve $t \to (a \cos t, b \sin t, ce^{-t/2})$ is a solution to $x' = Ax$ with $x(0) = (1, 0, 3)$.

4. Find two different matrices A, B such that the curve

$$x(t) = (e^t, 2e^{2t}, 4e^{2t})$$

satisfies both the differential equations

$$x' = Ax \qquad \text{and} \qquad x' = Bx.$$

5. Let $A = [a_{ij}]$ be an $n \times n$ diagonal matrix, that is, $a_{ij} = 0$ if $i \neq j$. Show that the differential equation

$$x' = Ax$$

has a unique solution for every initial condition.

6. Let A be an $n \times n$ diagonal matrix. Find conditions on A guaranteeing that

$$\lim_{t\to\infty} x(t) = 0$$

for all solutions to $x' = Ax$.

7. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Denote by $-A$ the matrix $[-a_{ij}]$.
   (a) What is the relation between the vector fields $x \to Ax$ and $x \to (-A)x$?
   (b) What is the geometric relation between solution curves of $x' = Ax$ and of $x' = -Ax$?

8. (a) Let $u(t)$, $v(t)$ be solutions to $x' = Ax$. Show that the curve $w(t) = \alpha u(t) + \beta v(t)$ is a solution for all real numbers α, β.

(b) Let $A = \begin{bmatrix}1 & \\ & -2\end{bmatrix}$. Find solutions $u(t)$, $v(t)$ to $x' = Ax$ such that every solution can be expressed in the form $\alpha u(t) + \beta v(t)$ for suitable constants α, β.

Notes

The background needed for a reader of Chapter 1 is a good first year of college calculus. One good source is S. Lang's *Second Course in Calculus* [12, Chapters I, II, and IX]. In this reference the material on derivatives, curves, and vectors in $\mathbf{R}^n$ and matrices is discussed much more thoroughly than in our Section 2.

Chapter 2

Newton's Equation and Kepler's Law

We develop in this chapter the earliest important examples of differential equations, which in fact are connected with the origins of calculus. These equations were used by Newton to derive and unify the three laws of Kepler. These laws were found from the earlier astronomical observations of Tycho Brahe. Here we give a brief derivation of two of Kepler's laws, while at the same time setting forth some general ideas about differential equations.

The equations of Newton, our starting point, have retained importance throughout the history of modern physics and lie at the root of that part of physics called classical mechanics.

The first chapter of this book dealt with linear equations, but Newton's equations are nonlinear in general. In later chapters we shall pursue the subject of nonlinear differential equations somewhat systematically. The examples here provide us with concrete examples of historical and scientific importance. Furthermore, the case we consider most thoroughly here, that of a particle moving in a central force gravitational field, is simple enough so that the differential equations can be solved explicitly using exact, classical methods (just calculus!). This is due to the existence of certain invariant functions called *integrals* (sometimes called "first integrals"; we do not mean the integrals of elementary calculus). Physically, an integral is a conservation law; in the case of Newtonian mechanics the two integrals we find correspond to conservation of energy and angular momentum. Mathematically an integral reduces the number of dimensions.

We shall be working with a particle moving in a *field of force* F. Mathematically F is a *vector field* on the (configuration) space of the particle, which in our case we suppose to be Cartesian three space $\mathbf{R}^3$. Thus F is a map $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ that assigns to a point x in $\mathbf{R}^3$ another point $F(x)$ in $\mathbf{R}^3$. From the mathematical point of view, $F(x)$ is thought of as a vector based at x. From the physical point of view, $F(x)$ is the force exerted on a particle located at x.

The example of a force field we shall be most concerned with is the gravitational field of the sun: $F(x)$ is the force on a particle located at x attracting it to the sun.

We shall go into details of this field in Section 6. Other important examples of force fields are derived from electrical forces, magnetic forces, and so on.

The connection between the physical concept of force field and the mathematical concept of differential equation is *Newton's second law*: $F = ma$. This law asserts that a particle in a force field moves in such a way that the force vector at the location of the particle, at any instant, equals the acceleration vector of the particle times the mass m. If $x(t)$ denotes the position vector of the particle at time t, where $x\colon \mathbf{R} \to \mathbf{R}^3$ is a sufficiently differentiable curve, then the acceleration vector is the second derivative of $x(t)$ with respect to time

$$a(t) = \ddot{x}(t).$$

(We follow tradition and use dots for time derivatives in this chapter.) Newton's second law states

$$F(x(t)) = m\ddot{x}(t).$$

Thus we obtain a second order differential equation:

$$\ddot{x} = \frac{1}{m} F(x).$$

In Newtonian physics it is assumed that m is a positive constant. Newton's law of gravitation is used to derive the exact form of the function $F(x)$. While these equations are the main goal of this chapter, we first discuss simple harmonic motion and then basic background material.

§1. Harmonic Oscillators

We consider a particle of mass m moving in one dimension, its position at time t given by a function $t \to x(t)$, $x\colon \mathbf{R} \to \mathbf{R}$. Suppose the force on the particle at a point $x \in \mathbf{R}$ is given by $-mp^2x$, where p is some real constant. Then according to the laws of physics (compare Section 3) the motion of the particle satisfies

$$\ddot{x} + p^2x = 0. \tag{1}$$

This medel is called the harmonic oscillator and (1) is the equation of the harmonic oscillator (in one ~~dimension~~).

An example of the harmonic oscillator is the simple pendulum moving in a plane, when one makes an approximation of $\sin x$ by x (compare Chapter 9). Another example is the case where the force on the particle is caused by a spring.

It is easy to check that for any constants A, B, the function

$$x(t) = A\cos pt + B\sin pt \tag{2}$$

is a solution of (1), with initial conditions $x(0) = A$, $\dot{x}(0) = pB$. In fact, as is proved often in calculus courses, (2) is the only solution of (1) satisfying these initial conditions. Later we will show in a systematic way that these facts are true.

Using basic trigonometric identities, (2) may be rewritten in the form

$$x(t) = a\cos(pt + t_0), \tag{3}$$

where $a = (A^2 + B^2)^{1/2}$ is called the amplitude, and $\cos t_0 = A(A^2 + B^2)^{-1/2}$.

In Section 6 we will consider equation (1) where a constant term is added (representing a constant disturbing force):

$$\ddot{x} + p^2x = K. \tag{4}$$

Then, similarly to (1), every solution of (4) has the form

$$x(t) = a\cos(pt + t_0) + \frac{K}{p^2}. \tag{5}$$

The two-dimensional version of the harmonic oscillator concerns a map $x: \mathbf{R} \to \mathbf{R}^2$ and a force $F(x) = -mkx$ (where now, of course, $x = (x_1, x_2) \in \mathbf{R}^2$). Equation (1) now has the same form

$$\ddot{x} + k^2x = 0 \tag{1'}$$

with solutions given by

$$\begin{aligned} x_1(t) &= A\cos kt + B\sin kt, \\ x_2(t) &= C\cos kt + D\sin kt. \end{aligned} \tag{2'}$$

See Problem 1.

Planar motion will be considered more generally and in more detail in later sections. But first we go over some mathematical preliminaries.

§2. Some Calculus Background

A path of a moving particle in $\mathbf{R}^n$ (usually $n \le 3$) is given by a map $f: I \to \mathbf{R}^n$ where I might be the set R of all real numbers or an interval (a, b) of all real numbers strictly between a and b. The derivative of f (provided f is differentiable at each point of I) defines a map $f': I \to \mathbf{R}^n$. The map f is called C^1, or *continuously differentiable*, if f' is continuous (that is to say, the corresponding coordinate functions $f_i'(t)$ are continuous, $i = 1, \ldots, n$). If $f': I \to \mathbf{R}^n$ is itself C^1, then f is said to be C^2. Inductively, in this way, one defines a map $f: I \to \mathbf{R}^n$ to be C^r, where $r = 3, 4, 5$, and so on.

The *inner product*, or "dot product," of two vectors, x, y in $\mathbf{R}^n$ is denoted by $\langle x, y\rangle$ and defined by

$$\langle x, y\rangle = \sum_{i=1}^n x_iy_i.$$

Thus $\langle x, x\rangle = |x|^2$. If $x, y: I \to \mathbf{R}^n$ are C^1 functions, then a version of the Leibnis product rule for derivatives is

$$\langle x, y\rangle' = \langle x', y\rangle + \langle x, y'\rangle,$$

as can be easily checked using coordinate functions.

We will have occasion to consider functions $f: \mathbf{R}^n \to \mathbf{R}$ (which, for example, could be given by temperature or density). Such a map f is called C^1 if the map $\mathbf{R}^n \to \mathbf{R}$ given by each partial derivative $x \to \partial f/\partial x_i(x)$ is defined and continuous (in Chapter 5 we discuss continuity in more detail). In this case the gradient of f, called *grad* f, is the map $\mathbf{R}^n \to \mathbf{R}^n$ that sends x into $(\partial f/\partial x_1(x), \ldots, \partial f/\partial x_n(x))$. Grad f is an example of a vector field on $\mathbf{R}^n$. (In Chapter 1 we considered only linear vector fields, but grad f may be more general.)

Next, consider the composition of two C^1 maps as follows:

$$I \xrightarrow{f} \mathbf{R}^n \xrightarrow{g} \mathbf{R}.$$

The chain rule can be expressed in this context as

$$\frac{d}{dt}g(f(t)) = \langle \operatorname{grad} g(f(t)), f'(t)\rangle;$$

using the definitions of gradient and inner product, the reader can prove that this is equivalent to

$$\sum_{i=1}^n \frac{\partial g}{\partial x_i}(f(t))\frac{df_i}{dt}(t).$$

§3. Conservative Force Fields

A vector field $F: \mathbf{R}^3 \to \mathbf{R}^3$ is called a force field if the vector $F(x)$ assigned to the point x is interpreted as a force acting on a particle placed at x.

Many force fields appearing in physics arise in the following way. There is a C^1 function

$$V: \mathbf{R}^3 \to \mathbf{R}$$

such that

$$\begin{aligned} F(x) &= -\left(\frac{\partial V}{\partial x_1}(x), \frac{\partial V}{\partial x_2}(x), \frac{\partial V}{\partial x_3}(x)\right) \\ &= -\operatorname{grad} V(x). \end{aligned}$$

(The negative sign is traditional.) Such a force field is called *conservative*. The function V is called the *potential energy* function. (More properly V should be called *a* potential energy since adding a constant to it does not change the force field $-\operatorname{grad} V(x)$.) Problem 4 relates potential energy to *work*.

The planar harmonic oscillation of Section 1 corresponds to the force field

$$F: \mathbf{R}^2 \to \mathbf{R}^2, \qquad F(x) = -mkx.$$

This field is conservative, with potential energy

$$V(x) = \tfrac{1}{2}mk\,|x|^2$$

as is easily verified.

For any moving particle $x(t)$ of mass m, the *kinetic energy* is defined to be

$$T = \tfrac{1}{2}m\,|\dot{x}(t)|^2.$$

Here $\dot{x}(t)$ is interpreted as the *velocity vector* at time t; its length $|\dot{x}(t)|$ is the *speed* at time t. If we consider the function $x: \mathbf{R} \to \mathbf{R}^3$ as describing a curve in $\mathbf{R}^3$, then $\dot{x}(t)$ is the *tangent vector* to the curve at $x(t)$.

For a particle moving in a conservative force field $F = -\operatorname{grad} V$, the potential energy at x is defined to be $V(x)$. Note that whereas the kinetic energy depends on the velocity, the potential energy is a function of position.

The *total energy* (or sometimes simply *energy*) is

$$E = T + V.$$

This has the following meaning. If $x(t)$ is the trajectory of a particle moving in the conservative force field, then E is a real-valued function of time:

$$E(t) = \tfrac{1}{2}\,|m\dot{x}(t)|^2 + V(x(t)).$$

Theorem (Conservation of Energy) *Let $x(t)$ be the trajectory of a particle moving in a conservative force field $F = -\operatorname{grad} V$. Then the total energy E is independent of time.*

Proof. It needs to be shown that $E(x(t))$ is constant in t or that

$$\frac{d}{dt}(T + V) = 0,$$

or equivalently,

$$\frac{d}{dt}\left(\frac{1}{2}m\,|\dot{x}(t)|^2 + V(x(t))\right) = 0.$$

It follows from calculus that

$$\frac{d}{dt}|\dot{x}|^2 = 2\langle\ddot{x}, \dot{x}\rangle$$

(a version of the Leibniz product formula); and also that

$$\frac{d}{dt}(V(\dot{x})) = \langle\operatorname{grad} V(x), \dot{x}\rangle$$

(the chain rule).

These facts reduce the proof to showing that

$$m\langle\ddot{x}, \dot{x}\rangle + \langle\operatorname{grad} V, \dot{x}\rangle = 0$$

or $\langle m\ddot{x} + \operatorname{grad} V, \dot{x}\rangle = 0$. But this is so since Newton's second law is $m\ddot{x} + \operatorname{grad} V(x) = 0$ in this instance.

§4. Central Force Fields

A force field F is called *central* if $F(x)$ points in the direction of the line through x, for every x. In other words, the vector $F(x)$ is always a scalar multiple of x, the coefficient depending on x:

$$F(x) = \lambda(x)x.$$

We often tacitly exclude from consideration a particle at the origin; many central force fields are not defined (or are "infinite") at the origin.

Lemma *Let F be a conservative force field. Then the following statements are equivalent:*

(a) *F is central,*
(b) *$F(x) = f(|x|)x$,*
(c) *$F(x) = -\operatorname{grad} V(x)$ and $V(x) = g(|x|)$.*

Proof. Suppose (c) is true. To prove (b) we find, from the chain rule:

$$\frac{\partial V}{\partial x_j} = g'(|x|)\frac{\partial}{\partial x_j}(x_1^2 + x_2^2 + x_3^2)^{1/2}$$

$$= \frac{g'(|x|)}{|x|}x_j;$$

this proves (b) with $f(|x|) = g'(|x|)/|x|$. It is clear that (b) implies (a). To show that (a) implies (c) we must prove that V is constant on each sphere.

$$S_\alpha = \{x \in \mathbf{R}^3 \mid |x| = \alpha\}, \qquad \alpha > 0.$$

Since any two points in S_α can be connected by a curve in S_α, it suffices to show that V is constant on any curve in S_α. Hence if $J \subset \mathbf{R}$ is an interval and $u: J \to S_\alpha$ is a C^1 map, we must show that the derivative of the composition $V \circ u$

$$J \xrightarrow{u} S_\alpha \subset \mathbf{R}^3 \xrightarrow{V} \mathbf{R}$$

is identically 0. This derivative is

$$\frac{d}{dt}V(u(t)) = \langle\operatorname{grad} V(u(t)), u'(t)\rangle$$