Numerical Simulation of the Cosmic Microwave Background Formation

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Here we will write an abstract.

I. INTRODUCTION

II. MILESTONE I

A. Introduction

To begin with we will consider the evolution of the uniform background of the Universe. Our goal will be to calculate the evolution of the Hubble parameter, the energy content and the conformal time. We will model the Universe using the ΛCDM model, assuming the Universe to be flat.

B. Theory

The Friedmann equation for the ΛCDM Model can be written in terms of cartesian coordinates as

$$H = H_0\sqrt{(\Omega_{b0}+\Omega_{\mathrm{CDM}0})a^{-3}+(\Omega_{\gamma 0}+\Omega_{\nu 0})a^{-4}+\Omega_{k0}a^{-2}+\Omega_{\Lambda 0}}, \quad (1)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and Ω_{b0}, Ω_{CDM0}, $\Omega_{\gamma 0}$, $\Omega_{\nu 0}$, Ω_{k0} and $\Omega_{\Lambda 0}$ are the present day relative densities of baryonic matter, dark matter, radiation, neutrinos, curvature and dark energy, respectively. Note that the relative curvature density $\Omega_{k0} = -\frac{kc^2}{H_0^2}$ acts as if it were a normal matter fluid with equation of state $\omega \equiv P/\rho = -1/3$.

We also define a function $\mathcal{H} \equiv aH$. From (1) it then follows that

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{H_0^2}{2\mathcal{H}}\big(2\Omega_{\Lambda 0}e^{2x} &- (\Omega_{b0}+\Omega_{CDM0})e^{-x} \\ &-2(\Omega_{\gamma 0}+\Omega_{\nu 0})e^{-2x}\big),\end{aligned} \quad (2)$$

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} = \frac{H_0^2}{2\mathcal{H}}\big(4\Omega_{\Lambda 0}e^{2x} &+ (\Omega_{b0}+\Omega_{CDM0})e^{-x} \\ &+4(\Omega_{\gamma 0}+\Omega_{\nu 0})e^{-2x}\big) \\ -\frac{H_0^4}{4\mathcal{H}^3}\big(2\Omega_{\Lambda 0}e^{2x} &- (\Omega_{b0}+\Omega_{CDM0})e^{-x} \\ &-2(\Omega_{\gamma 0}+\Omega_{\nu 0})e^{-2x}\big)^2.\end{aligned} \quad (3)$$

The time evolution of the densities is given by

$$\dot{\rho} + 3H(\rho + P) = 0. \quad (4)$$

The solution can be written in terms of the equation of state as

$$\rho \propto a^{-3(1+\omega)}. \quad (5)$$

The equation of state is $\omega = 0$ for cold dark matter and baryons, $\omega = 1/3$ for relativistic matter and $\omega = -1$ for the cosmological constant. This gives

$$\begin{aligned}\rho_{\mathrm{CDM}} &= \rho_{\mathrm{CDM},0}a^{-3} \\ \rho_b &= \rho_{b,0}a^{-3} \\ \rho_\gamma &= \rho_{\gamma,0}a^{-4} \\ \rho_\nu &= \rho_{\nu,0}a^{-4} \\ \rho_\Lambda &= \rho_{\Lambda,0},\end{aligned}$$

where subscript 0 denotes present day values. The relative densities $\Omega_X(a) = \rho_X/\rho_c$ can be written

$$\begin{aligned}\Omega_k(a) &= \frac{\Omega_{k0}}{a^2H(a)^2/H_0^2}, \\ \Omega_{\mathrm{CDM}}(a) &= \frac{\Omega_{\mathrm{CDM}0}}{a^3H(a)^2/H_0^2}, \\ \Omega_b(a) &= \frac{\Omega_{b0}}{a^3H(a)^2/H_0^2}, \\ \Omega_\gamma(a) &= \frac{\Omega_{\gamma 0}}{a^4H(a)^2/H_0^2}, \\ \Omega_\nu(a) &= \frac{\Omega_{\nu 0}}{a^4H(a)^2/H_0^2}, \\ \Omega_\Lambda(a) &= \frac{\Omega_{\Lambda 0}}{H(a)^2/H_0^2}.\end{aligned}$$

The relative densities of the relativistic matter can be calculated from the CMB temperature through

$$\begin{aligned}\Omega_{\gamma 0} &= 2\cdot\frac{\pi^2}{30}\frac{(k_bT_{\mathrm{CMB}0})^4}{\hbar^3c^5}\cdot\frac{8\pi G}{3H_0^2}, \\ \Omega_{\nu 0} &= N_{\mathrm{eff}}\cdot\frac{7}{8}\cdot\left(\frac{4}{11}\right)^{4/3}\Omega_{\gamma 0},\end{aligned}$$

where N_{eff} is the effective number of massless neutrinos.

Since the relativistic matter density evolves with the scale factor raised to the most negative power, it is expected to dominate the early Universe as $a \to 0$. Likewise

* Code repository: https://github.com/williameivikolsen/AST5220

we expect dark energy to dominate the Universe in the distant future as $a \to \infty$ since its density is independent of the scale factor. Assuming the Universe to be dominated by a fluid with equation of state ω, the Friedmann equation takes the form

$$H = H_0 a^{-\frac{3}{2}(1+\omega)}.$$

This implies that $\mathcal{H}$ will satisfy

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = -\frac{1+3\omega}{2},$$
$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} = \frac{(1+3\omega)^2}{4}.$$

We therefore expect that as relativistic matter dominates in the early Universe, $\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = -1$ and $\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} = 1$. Likewise we expect that when dark energy dominates in the distant future, $\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = 1$ and $\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} = 1$.

The amount of matter (baryons and dark matter) is equal to the amount of radiation (photons and neutrinos) when $a = a_\gamma$, where

$$a_\gamma = \frac{\Omega_{\gamma 0} + \Omega_{\nu 0}}{\Omega_{b0} + \Omega_{CDM0}}. \tag{6}$$

Likewise, the amount of matter is equal to the amount of dark energy when $a = a_\Lambda$, where

$$a_\Lambda = \sqrt[3]{\frac{\Omega_{b0} + \Omega_{CDM0}}{\Omega_{\Lambda 0}}}. \tag{7}$$

The Universe will start to accelerate at the matter-dark energy equality.

The comoving horizon is the distance light may have travelled since the Big Bang. Since it is monotonically increasing it can be used as a time variable. We refer to this time variable as the conformal time η. The conformal time can be calculated through the differential equation

$$\frac{d\eta}{dx} = \frac{c}{\mathcal{H}}, \tag{8}$$

where $x \equiv \log a$ and $\mathcal{H} \equiv aH$. The initial condition of the equation is $\eta(-\infty) = 0$.

The age of the universe t can similarly be computed by solving the differential equation

$$\frac{dt}{dx} = \frac{1}{H}, \tag{9}$$

and evaluating it at $x = 0$. The initial condition is $t(x_{\mathrm{start}}) = \frac{1}{2H(x_{\mathrm{start}})}$.

In the radiation dominated era in the early Universe we have that $H^2 = H_0^2 a^{-4}$. Equation (8) then takes the form

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{c}{\mathcal{H}} = \frac{c}{aH} = \frac{c}{aH_0 a^{-2}} = \frac{c}{H_0}a = \frac{c}{H_0}e^x,$$

giving the solution

$$\eta = \frac{c}{H_0}e^x = \frac{c}{\mathcal{H}}.$$

We therefore expect that

$$\frac{\eta\mathcal{H}}{c} = 1 \tag{10}$$

in the early Universe.

The luminosity distance d_L is related to the conformal time η by

$$d_L = \frac{\eta_0 - \eta}{a} \tag{11}$$

where η_0 is the present day value of η (evaluated at $a = 1$).

The redshift z is defined as

$$1 + z \equiv \frac{1}{a}. \tag{12}$$

C. Implementation details

The background evolution was calculated numerically using a numerical solver that takes in h, Ω_{b0}, Ω_{CDM0}, Ω_{k0}, N_{eff} and T_{CMB0} as paramters. We assume space to be flat so that $\Omega_{k0} = 0$. We only consider the simplified model where there are no neutrinos, $N_{eff} = 0$. For the other parameters we use the best-fit values from the Planck 2018 data [1]:

$$h = 0.67$$
$$\Omega_{b0} = 0.05$$
$$\Omega_{CDM0} = 0.267$$
$$T_{CMB0} = 2.7255\ \mathrm{K}$$

The differential equations (8) and (9) were solved using the Runge-Kutta 4 method. We made cubic splines of the solutions.

The luminosity distance was calculated through (11) and compared to type Ia supernova observations [2].

D. Results

The age of the Universe t, the matter-radiation equality a_γ and the matter-dark energy equality a_Λ were found to be

$$t = 13.8508 \times 10^9 \text{ years},$$
$$a_\gamma = 1.74 \times 10^{-4},$$
$$a_\Lambda = 7.74 \times 10^{-1}.$$

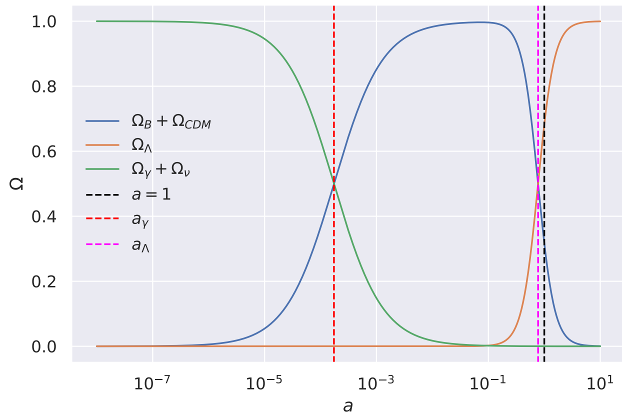

Figure 1. Evolution of the relative densities. Radiation-matter equality is marked by a_γ. Matter-dark energy equality is marked by a_Λ.

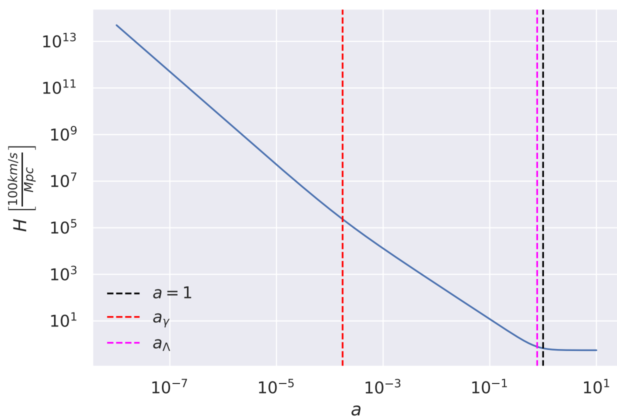

Figure 2. Evolution of the Hubble parameter H. Radiation-matter equality is marked by a_γ. Matter-dark energy equality is marked by a_Λ.

E. Discussion

In figure 1 we see that our expectation that the Universe was be dominated by relativistic particles in the beginning is indeed correct. The baryons and dark matter then takes over as the dominating matter at $a_\gamma = 1.74 \times 10^{-4}$. Finally, just before today, the cosmological constant takes over as the largest quantity at $a_\Lambda = 7.74 \times 10^{-1}$. The cosmological constant then goes over to dominate the energy density of the Universe, as we also expected.

In figure 2 we see how the Hubble parameter changes as the Universe expands. We see that it decreases monotonically until the matter-dark energy equality. Here it seems to stabilize to a constant value. In figure 3 we see that at the same point $\mathcal{H}$, which is equivalent to $\dot{a}$, is at a minimum after which it increases: The Universe begins accelerating, as we expected when the cosmological constant took over as the dominating form of energy.

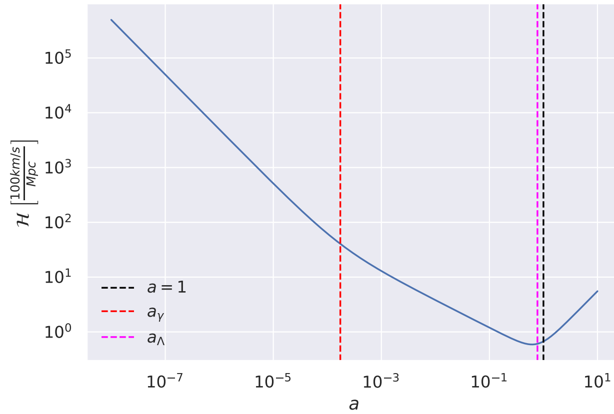

Figure 3. Evolution of $\mathcal{H}$. Radiation-matter equality is marked by a_γ. Matter-dark energy equality is marked by a_Λ.

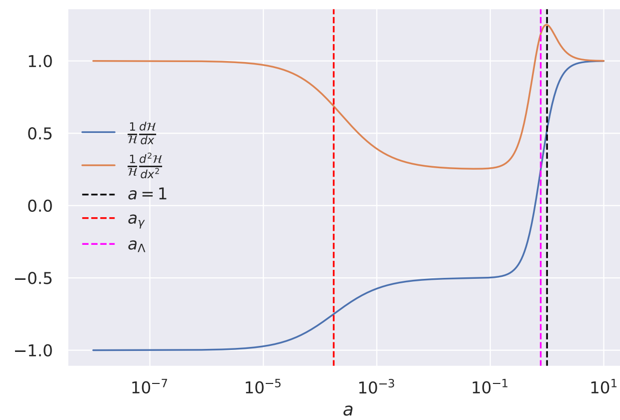

Figure 4. Evolution of $\frac{1}{\mathcal{H}}\frac{\mathrm{d}H}{\mathrm{d}x}$ and $\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2H}{\mathrm{d}x^2}$. Radiation-matter equality is marked by a_γ. Matter-dark energy equality is marked by a_Λ.

We also had some expectations as to how the derivative and second derivative of $\mathcal{H}$ would evolve during the expansion. In figure 4 we see that these expectations were indeed satisfied: In the radiation dominated era at the beginning, $\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = -1$ and $\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} = 1$. In the distant future when the cosmological constant dominates, $\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = 1$ and $\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} = 1$.

Our main target of these calculations, the conformal time η, increases monotonically over the entire evolution of the Universe, however much slower after the matter-dark energy equality. We see that our expectation that $\eta\mathcal{H}/c = 1$ in the early Universe is fulfilled in 6.

Finally, we see that our predicted luminosity distance fits well with the observed type Ia supernova in figure 7.

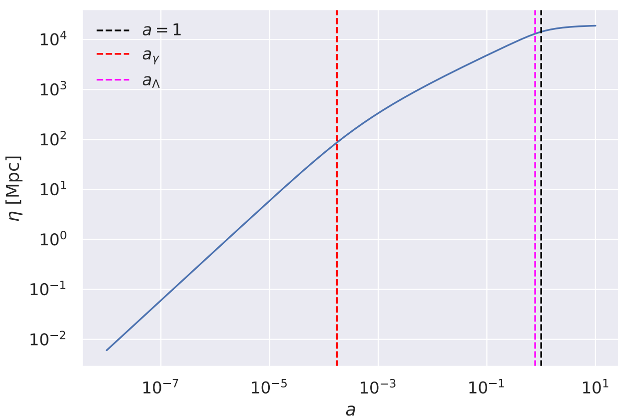

Figure 5. Evolution of η. Radiation-matter equality is marked by a_γ. Matter-dark energy equality is marked by a_Λ.

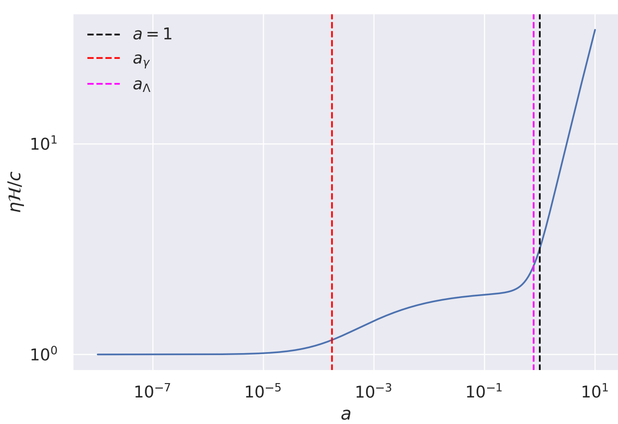

Figure 6. Evolution of $\eta\mathcal{H}/c$. Radiation-matter equality is marked by a_γ. Matter-dark energy equality is marked by a_Λ.

[1] Aghanim, N., Akrami, Y., Ashdown, M., Aumont, J., Baccigalupi, C., Ballardini, M., Banday, A. J., Barreiro, R. B., Bartolo, N., et al. Planck 2018 results. *Astronomy & Astrophysics 641* (Sep 2020), A6.

[2] Betoule, M., Kessler, R., Guy, J., Mosher, J., Hardin, D., Biswas, R., Astier, P., El-Hage, P., Konig, M., Kuhlmann, S., et al. Improved cosmological constraints from a joint analysis of the sdss-ii and snls supernova samples. *Astronomy & Astrophysics 568* (Aug 2014), A22.

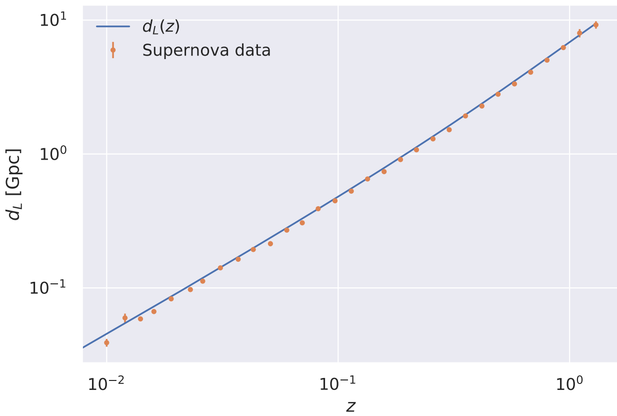

Figure 7. Computed luminosity distance d_L and data.