# Numerical Solution of the gravitational N-body problem

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(Dated: October 29, 2020)

We have implemented the velocity Verlet algorithm for solving the gravitational N-body problem of the solar system. This method was compared with the Euler method for the system consisting of the Sun and Earth only, showing no remarkable difference in numerical precision. We have also tested alternative forms of Newton's law of gravitation by replacing the inverse square nature with a $1/r^{\beta}$ radial dependence. Our results show that energy and angular momentum is increasingly non-conserved as $\beta$ approaches 3, with orbits not being closed. Finally, adding a relativistic term to the gravitational force gave a $(47 \pm 19)$ arcsec/century precession for Mercury's orbit, in good accordance with physical observations.

## I. INTRODUCTION

Through Newton's law of gravitation, the seemingly different phenomena of an apple falling off a tree and the observed planetary motions in the night sky can elegantly be explained by one single equation.

Despite its simplicity, there is no closed form analytical solution of the planetary equations of motion for $N > 2$, $N$ being the number of planets [1]. This is more generally referred to as the gravitational N-body problem. It is therefore of great interest to find efficient ways of solving the gravitational N-body problem using numerical methods.

In this paper we will use numerical methods to study the gravitational N-body problem of our own solar system. Using the Euler method and the velocity Verlet method we will solve the three-dimensional N-body problem of the Sun and all its neighbouring planets, including Pluto. We also look at changes to the inverse square nature of gravity, and the effect of Jupiter's mass on the Earth's orbit. Finally, we will add a relativistic correction to the Newtonian gravitational force and examine the two-body problem of the Sun and Mercury.

## II. THEORY

### A. The many-body problem

We consider first the hypothetical solar system consisting of the Earth and Sun only. Newton's gravitational force $\mathbf{F}_G$ on the Earth from the Sun can be expressed as:

$$\mathbf{F}_G = -\frac{GM_{\odot}m}{r^3}\mathbf{r}, \tag{1}$$

where $M_{\odot}$ is the solar mass, $m$ is the mass of the Earth, $r$ is the distance between them and $\mathbf{r}$ is the position vector of the Earth relative to the Sun. We assume the mass of the Sun to be much larger than the Earth's, so we can chose to neglect the motion of the Sun and assume it to be the center of mass positioned at the origin. Newton's second law applied to the gravitational force (1) then gives the following second-order differential equations for the Earth's coordinates in the $xy$-plane:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{GM_{\odot}}{r^3}x, \tag{2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\frac{GM_{\odot}}{r^3}y. \tag{3}$$

Equations (2) and (3) can be rewritten to four coupled first order differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x, \tag{4}$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{GM_{\odot}}{r^3}x = a_x, \tag{5}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v_y, \tag{6}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{GM_{\odot}}{r^3}y = a_y. \tag{7}$$

Assuming a circular orbit, the magnitude of the gravitational force $F_G = |\mathbf{F}_G|$ can be expressed as a centripetal force:

$$F_G = \frac{mv^2}{r}, \tag{8}$$

where $v = |\mathbf{v}|$ is the magnitude of the a velocity corresponding to circular orbit. At a distance of 1 AU, this is 1 AU/yr, and inserting this into the gravitational force given by Eq. (1) gives

$$v^2r = GM_{\odot} = 4\pi^2 \text{ AU}^3/\text{yr}^2. \tag{9}$$

Furthermore, we extend our model of the solar system to include several objects and the motion of the Sun. This generalization is enabled by choosing the center of mass to be fixed at the origin, instead of the position of the Sun. We treat the Sun like the rest of the planets; a massive object with its motion affected by the gravitational force of all the other objects in it's system. The net

* Code repository: https://github.com/williameivikolsen/FYS4150

force acting on one object $j$ from the rest of the objects $k$ is

$$\mathbf{F}_j = -Gm_j \sum_{k\neq j} m_k \frac{\boldsymbol{r}_j - \boldsymbol{r}_k}{|r_j - r_k|^3}, \tag{10}$$

where $m_j$ is the mass of object $j$, $m_k$ is the masses of the other objects and the coordinate $\boldsymbol{r}_j - \mathbf{r}_k$ is the vector from object $k$ to $j$. We know from (9) that the gravitational parameter $Gm_j$ can be written

$$Gm_j = GM_\odot \left(\frac{m_j}{M_\odot}\right) = 4\pi^2 \left(\frac{m_j}{M_\odot}\right).$$

For simplicity we have scaled all the masses of our solar system with the solar mass, so that $\tilde{m}_j = m_j/M_\odot$. The acceleration of object $j$ can then be written

$$\boldsymbol{a} = -4\pi^2 \sum_{k\neq j} \tilde{m}_k \frac{\boldsymbol{r}_j - \boldsymbol{r}_k}{|r_j - r_k|^3}. \tag{11}$$

The expressions for position coordinates of each object, (4) and (6), remain unchanged as they are solely velocity dependent. The differential equations to solve, together with the expression for the acceleration (11), for the planetary orbits are thus

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v_y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = v_z, \tag{12}$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = a_x, \quad \frac{\mathrm{d}v_y}{\mathrm{d}t} = a_y, \quad \frac{\mathrm{d}v_z}{\mathrm{d}t} = a_z. \tag{13}$$

These have been discretized and solved numerically for our solar system using the velocity Verlet method, as shown in the Method section.

### The perihelion precession of Mercury

One important confirmation of general relativity was to compare its prediction for the perihelion precession of Mercury to the observed value, which is 43 arc seconds per century. The Newtonian force proportional to $1/r^2$ leads to closed elliptical curves, so corrections to this would lead to non-closed orbits: after completing one round around the Sun, the planet wouldn't end up in the initial position. When the correction is small the orbit can be thought of as an ellipse whose orientation in space slowly rotates, which means the perihelion of the ellipse precesses around the Sun. We wish to study the system Sun-Mercury only, and add a general relativistic term to the force acting on Mercury from the Sun

$$F_G = \frac{GM_\odot M_M}{r^2}\left(1 + \frac{3l^2}{r^2c^2}\right). \tag{14}$$

where $M_M$ is the mass of Mercury, $M_\odot$ the mass of the Sun, $r$ is the distance between Mercury and the Sun, $l = |\boldsymbol{r} \times \boldsymbol{v}|$ is the magnitude of Mercury's orbital angular momentum per unit mass, and finally $c$ the vacuum speed of light. From the classical equation $\frac{dl}{dt} = \tau$ we know that the orbital angular momentum of Mercury is conserved as the torque $\tau$ from the Sun on Mercury is zero at all times. Hence we only need to calculate this quantity once using the initial conditions. The equations of motion for Mercury is the same as the general expressions for the many-body problem (Eqs. (12) and (13)), where an extra term is added to the acceleration (Eq. (11)) so that

$$\boldsymbol{a} = -4\pi^2 \frac{\boldsymbol{r}_M - \boldsymbol{r}_\odot}{|r_M - r_\odot|^3}\left(1 + \frac{3l^2}{|r_M - r_\odot|^2 c^2}.\right), \tag{15}$$

where $r_\odot$ is the position of the Sun and $r_M$ the position of Mercury. The equations of motion for Mercury can now be solved numerically in the exact same way as earlier in order to study the perihelion precession.

### Conservation of angular momentum

Kepler's second law states that a line connecting a planet and the Sun swipes out equal areas over equal times. Dividing the orbit into infinitesimal triangles of area

$$dA = \frac{1}{2}(r)(rd\theta) = \frac{1}{2}r^2 d\theta,$$

where we consider the radial distance $r$ to be a constant over the infinitesimal interval, we have that the rate at which the orbit swipes an area is

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega$$

where $\omega$ is the angular frequency. The angular momentum of the orbit is $\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = mr\omega$. Inserting in the expression for $dA/dt$, we have

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{m}.$$

Since equal areas must be swept over equal times must be the same, the rate $\frac{dA}{dt}$ must be constant. We therefore have that

$$L = \text{constant}.$$

This means that Kepler's second law provides a way to check if angular momentum of the orbit is conserved. If Kepler's second law holds, the angular momentum must be conserved.

## III. METHOD

The differential equations presented for the many-body system, Eqs. (11), (12) and (13), have been solved using two different numerical methods for solving ordinary coupled differential equations. First, we calculated the orbits of the system consisting of the Sun and Earth using both Euler and velocity Verlet solvers, followed by a comparison of the two methods. When solving the equations of motions for systems containing more than two objects, we have exclusively used the velocity Verlet method.

The masses, initial positions and initial velocities for all the objects in our solar system in which we have studied have been gathered from [2], and are based on real observations dated midnight 2020-Oct-09 Barycentric Dynamical Time. These initial conditions are used in all simulations, unless explicitly stated. The masses have for simplicity been scaled using the solar mass. Note that we have done the calculations with astronomical units (AU) and years (yr) as units for length and time, so that $GM_\odot$ can be replaced with $4\pi^2$ (Equation (9)).

In addition to the ordinary Newtonian gravitational force, we have studied what happens with the Earth-Sun system when the inverse square law is replaced by a force on the form:

$$F_G = \frac{GM_\odot M_{earth}}{r^\beta}, \tag{16}$$

where $\beta$ is tested for different values in the interval $[2, 3]$.

### Euler method

The equations of motion for the Sun-Earth system, Eqs. (4) to (7), have been discretized and solved numerically using the Euler method, shown in Algorithm 1.

**Algorithm 1** Euler method

**for** $i = 1, 2, ..., N$ **do**
  $x_{i+1} = x_i + hv_i$
  $v_{i+1} = v_i + ha_i$

Here, $n$ is the number of time steps and $h = (t_{n+1} - t_n)/n$ is the step size. The global error of Euler's method goes as $\mathcal{O}(h)$.

### The velocity Verlet method

Using the velocity Verlet algorithmm, our coupled ordinary differential equations for the planetary motions, Eqs. (11), (12) and (13), can be discretized and approximated as shown in Algorithm 2 below.

**Algorithm 2** The velocity Verlet method

**for** $i = 1, 2..., N$ **do**
  **for** $j = 1, 2, ..., n$ **do**
    calculate $a_i$
    $x_{i+1} = x_i + hv_i + (h^2/2)a_i$
  **for** $j = 1, 2, ..., n$ **do**
    calculate $a_{i+1}$
    $v_{i+1} = v_i + (h/2)(a_{i+1} + a_i)$

Here $N$ and $h$ are defined in the same manner as in the Euler method. $n$ is the total number of objects in the system considered. Note that we have a split loop in every time step calculating positions and velocities for the planets separately. This has to be done as we require the acceleration in time step $i + 1$ to calculate the velocities in time step $i$, and the acceleration of an object $j$ depends on the position of all the other objects (Eq. (11)). In other words; we first need to obtain the positions of all the planets in time step $i + 1$ before calculating the acceleration and velocities. Positions and velocities for the two remaining dimensions $y$ and $z$ are calculated the exact same way. The global error of the Verlet method goes as $\mathcal{O}(h^3)$ both for the position and velocity.

## IV. RESULTS

The calculated planetary orbits using the velocity Verlet method over a time interval $T = 500$ yr with $N = 5 \cdot 10^6$ time steps has been plotted in Figure 1. The same plot zoomed in on inner planets is shown in Figure 2.

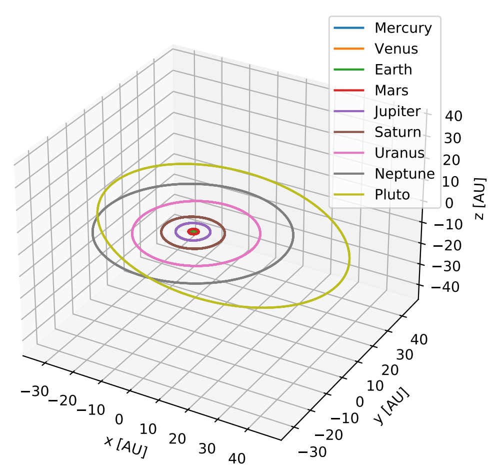


Figure 1. Planetary orbits simulated for $T = 500$ yr with $N = 5 \cdot 10^6$ time steps using velocity Verlet.

The calculated orbit of the Earth for the two-body problem of the Earth and the Sun for $T = 1$ yr and $N = 10^6$ time steps using the Euler method and the

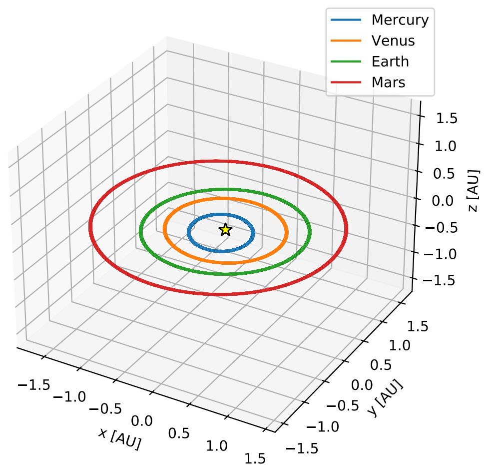


Figure 2. Planetary orbits simulated for $T = 500$ yr with $N = 5 \cdot 10^6$ time steps using velocity Verlet, zoomed in to show inner planets.

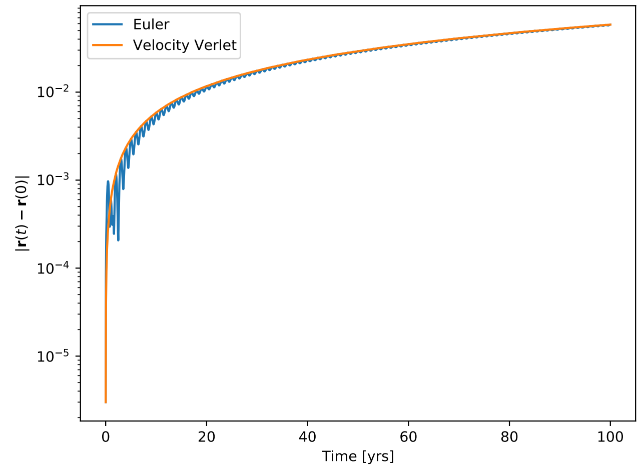


Figure 4. Distance from expected circular orbits for initial conditions of $\mathbf{r}_0 = (1, 0, 0)$ AU and $\mathbf{v}_0 = (0, 2\pi, 0)$ for the Earth in the Earth-Sun system. The calculation is done over $T = 100$ yr with $N = 10^6$ time steps.

velocity Verlet method has been plotted in the $xy$-plane in Figure 3.

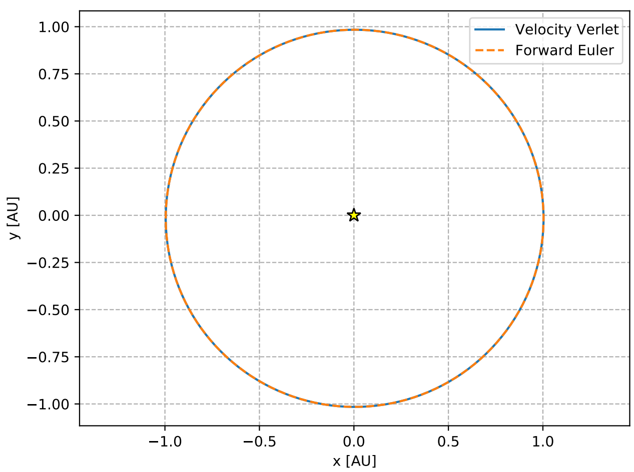


Figure 3. Two-body problem of Earth and Sun solution using Euler method and velocity Verlet method for $T = 1$ yr and $N = 1 \cdot 10^6$ time steps.

Figure 4 shows the the distance from the expected circular orbit of the Earth with initial values set to $\mathbf{r}_0 = (1, 0, 0)$ AU and $\mathbf{v}_0 = (0, 2\pi, 0)$ AU/yr in the Earth-Sun system. The calculation was done for a period of $T = 100$ yr with $N = 10^6$ time steps with the Euler method and the velocity Verlet method. Table I compares the maximal difference found in the kinetic energy, potential energy, total energy and angular momentum for the methods. The deviation from a perfectly circular orbit is shown in Figure 4.

The three-body-problem of the Sun, Earth and Jupiter is calculated using the velocity Verlet method for $N = 10^6$ time steps over a period of $T = 100$ years. The orbits in the $xy$-plane are presented in Figure 5.

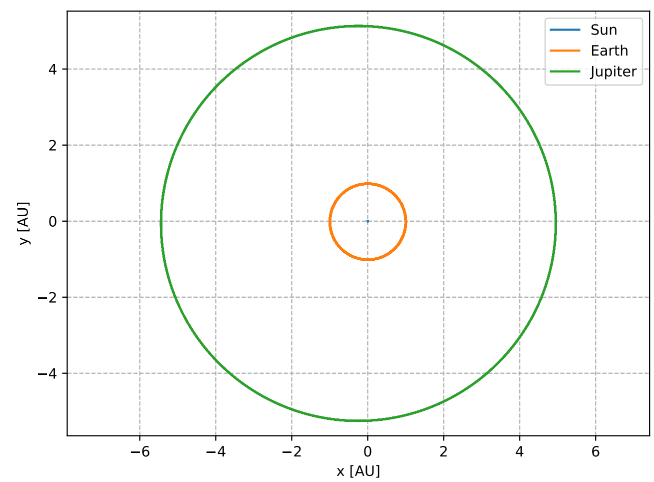


Figure 5. The three-body problem solved using velocity Verlet for the Sun, Earth and Jupiter over $T = 100$ yr with $N = 10^6$ time steps.

Figure 6 shows the system of Sun, Earth and Jupiter for same period and number of time steps, but simulated with the mass of Jupiter multiplied with 10. In Figure 7 we study the same system, but with the mass of Jupiter increased by a factor of 1000 and over a time period of 10 years.

The two-body problem of the Sun-Earth system was calculated for $T = 100$ yr with $N = 10^6$ time steps using the Euler method and the Verlet method. The maximum relative difference from the initial value of the total energy and angular momentum is presented in Table II. The same system was also calculated with the Earth initially at a position $\mathbf{r}_0 = (1, 0, 0)$ with velocity $\mathbf{v}_0 = (0, 5, 0)$,

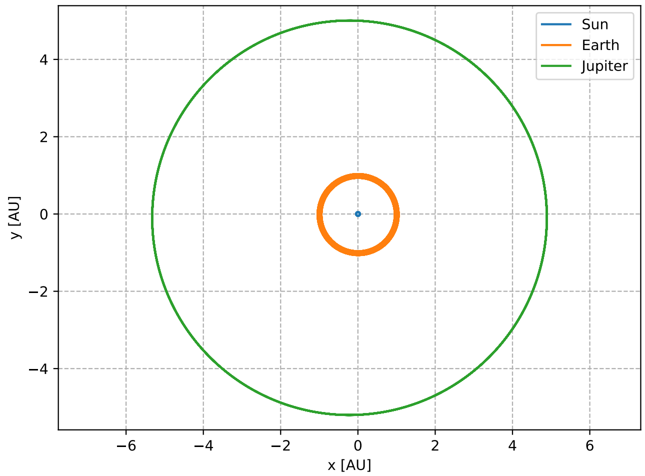


Figure 6. The three-body problem solved using velocity Verlet for the Sun, Earth and Jupiter over $T = 100$ yr with $N = 10^6$ time steps. The mass of Jupiter is multiplied by 10.

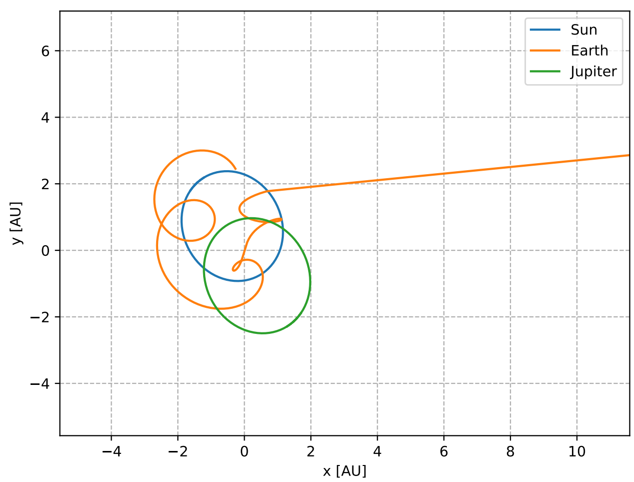


Figure 7. The three-body problem solved using velocity Verlet for the Sun, Earth and Jupiter over $T = 10$ yr with $N = 10^6$ time steps. The relative mass of Jupiter is multiplied by 1000.

which should correspond to an elliptical orbit. For this case, the maximum relative difference from the initial value of the total energy and angular momentum is presented in Table III.

The two-body problem of the Earth-Sun system was calculated for 11 evenly spaced values of $\beta$ on the interval $[2, 3]$ using the velocity Verlet method over a period of $T = 100$ yr with $N = 10^6$ time steps. The resulting distance from the initial position after 1 year of orbit is shown in figure 8. The same values of $\beta$ were used to solve the Earth-Sun system for the elliptical orbit case with $\mathbf{r}_0 = (1, 0, 0)$ and $\mathbf{v}_0 = (0, 5, 0)$ for the Earth over a period of $T = 100$ yr with $N = 10^6$ time steps. The resulting relative difference from the initial value of total energy and angular momentum is presented in Figure 9.

Table I. Maximum relative difference from initial value of kinetic energy, potential energy, total energy and angular momentum for circular orbit with initial position $\mathbf{r}_0 = (1, 0, 0)$ AU and velocity $\mathbf{v}_0 = (0, 2\pi, 0)$ AU/yr.

| | Euler method | Velocity Verlet method |
|---|---|---|
| Kinetic energy | $1.93 \cdot 10^{-3}$ | $1.31 \cdot 10^{-3}$ |
| Potential energy | $3.2 \cdot 10^{-4}$ | $5.85 \cdot 10^{-6}$ |
| Total energy | $1.32 \cdot 10^{-3}$ | $1.31 \cdot 10^{-3}$ |
| Angular momentum | $2.28 \cdot 10^{-8}$ | $1.23 \cdot 10^{-8}$ |

Table II. Maximum relative difference from initial value of total energy and angular momentum for Earth orbit for an elliptical orbit

| | Euler method | velocity Verlet method |
|---|---|---|
| Total energy | $1.42 \cdot 10^{-3}$ | $1.43 \cdot 10^{-3}$ |
| Angular momentum | $1.26 \cdot 10^{-7}$ | $9.82 \cdot 10^{-8}$ |

We have also added a relativistic correction term (Eq. (14)) to the gravitational force while studying the Sun-Mercury system. We let Mercury start at perihelion on the $x$-axis, at $\mathbf{r}_0 = (x_\mathrm{p}, y_\mathrm{p}) = (0.3075, 0)$ AU, with an initial velocity $\mathbf{v}_0 = (0, 12.44)$ AU/yr. Simulating for 100 yr with $N = 2 \cdot 10^7$ integration points, we registered the position where the distance to the sun was shortest, and found the corresponding angle $\theta_\mathrm{p} = \arctan \frac{y_\mathrm{p}}{x_\mathrm{p}}$. How $\theta_\mathrm{p}$ changes with time is shown in Figure 10. From the slope of the resulting linear regression, we found the precession rate to be $(1.3 \pm 0.5) \cdot 10^{-4}$ deg/yr or $(47 \pm 19)$ arcsec/century.

## V. DISCUSSION

The calculated orbits of the solar system presented in Figures 1 and 2 seem to be elliptical and closed at first sight, as they should be. From the closer view of the Earth's orbit in the $xy$-plane shown in Figure 3 we see that the Earth has an orbit of low eccentricity at a distance of around 1 AU from the Sun. This is the expected result and is indicated by both algorithms.

Figure 4 gives an indication of the algorithms stability over time. We see that the algorithms conserves the circular orbit approximately well. This preservation could be due to the algorithms performing with equal precision, but could also be due to a numerical imprecision dominating the instability we observe. This could be examined by trying to vary the number of time steps in the calculations. Table I shows how both of the methods conserve kinetic, potential and total energy and angular momentum. Since the calculations are done for a circular orbit, the potential energy should not vary, and all the energy quantities should be conserved. We see that both methods perform well in this test: All of the quantities are conserved down to a maximum relative deviation $\sim 10^{-3}$.

Since velocity Verlet should be an energy conserving

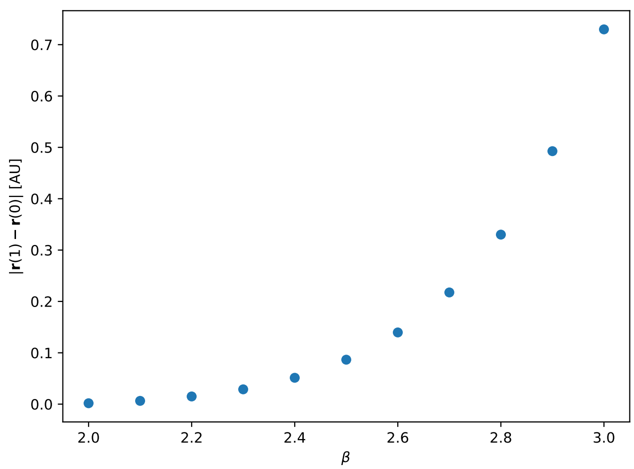


Figure 8. Earth's distance from initial position after 1 year of orbit with varying value of $\beta$. Simulation was done with the velocity Verlet method with $N = 10^5$ time steps.

Table III. Maximum relative difference from initial value of total energy and angular momentum for elliptical orbit with initial position $\mathbf{r}_0 = (1, 0, 0)$ AU and velocity $\mathbf{v}_0 = (0, 5, 0)$ AU/yr.

| | Euler method | velocity Verlet method |
|---|---|---|
| Total energy | $2.82 \cdot 10^{-2}$ | $2.82 \cdot 10^{-2}$ |
| Angular momentum | $5.51 \cdot 10^{-3}$ | $5.51 \cdot 10^{-3}$ |

method, we would expect to see it performing somewhat better than the Euler method at conserving the total energy. This can not be seen for any of the Tables I, II and III. This could be a result of the algorithms both performing well with regards to energy conservation for these problems, but it could also be due to a numerical imprecision dominating the instability, as discussed above.

The three-body problem of the Sun, Earth and Jupiter has been calculated over a period of $T = 100$ yr using the velocity Verlet solver, and the orbits in the $xy$-plane are shown in Figure 5. We know that Jupiter uses about 12 years to complete an orbit and from the figure both the motion of Jupiter and the Earth seems stable. With Jupiter being twice as massive as all the other planets combined, we observe in Figure 6 that increasing the relative mass of Jupiter with a factor of 10 will slowly alter the motion of Earth. The orbit of Earth seems to move outwards against Jupiter. When increasing the mass of Jupiter with a factor of 1000, we see in Figure 7 that the motion of the earth is totally unstable. The magnitude of Jupiter's mass is now on the size of the Sun, and the Earth gets thrown away from the gravitational field of the two massive objects orbiting around their common center of mass.

From observations we know that the planets follow closed orbits; this follows from Bertrand's theorem as well [3]. We would therefore expect the position of the Earth to return to its initial position after orbiting for 1 year. Figure 8 shows how well this is realized for the different values of $\beta$. We see that for the usual value of $\beta = 2$, the Earth successfully returns to its initial position. For increasing values of $\beta$, we see that the orbit gets further away from closure. Figure 9 shows how well the total energy and angular momentum is conserved when varying the value of $\beta$, and we see that the conservation of total energy breaks when increasing $\beta$. This loss of conservation is less clear for angular momentum, except

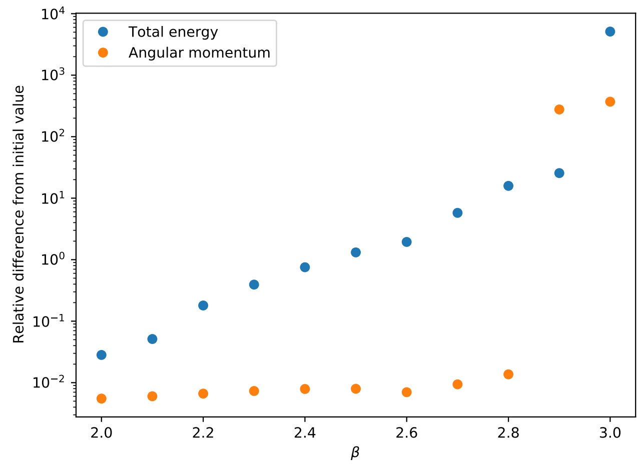


Figure 9. Relative differences from initial value of total energy and angular momentum for Earth put in elliptical orbit with initial position $\mathbf{r}_0 = (1, 0, 0)$ and $\mathbf{v}_0 = (0, 5, 0)$ with varying values of $\beta$, simulated using velocity Verlet method over a period of $T = 100$ yr with $N = 10^6$ time steps.

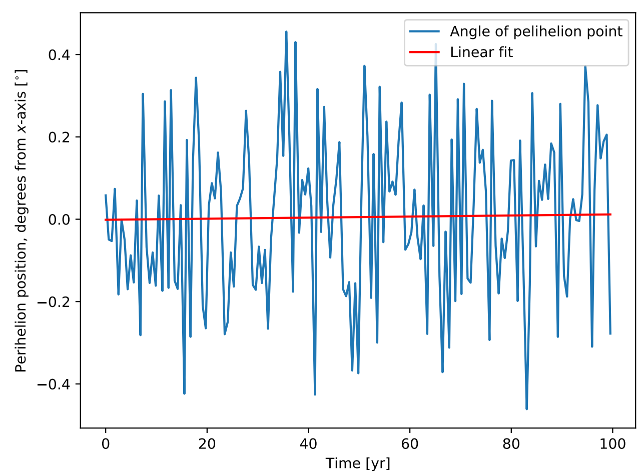


Figure 10. Changes in perihelion angle $\theta_\mathrm{p}$ for Mercury's orbit when accounting for relativistic effects. The simulation time was 100 yr and the number of integration points $N = 2 \cdot 10^7$, using velocity Verlet. Note that only every 100th precession angle is plotted. The precession rate was found using linear regression with least squares.

when $\beta$ approaches 3. From these results we can not determine if Nature deviates from a perfect square law. We can only conclude that the deviation is by a small amount, if any.

Regarding the linear regression shown in Figure 10, we see that even though our calculated precession rate of 47" per century is in full accordance with theory [4], the value has an uncertainty of around 40%. Our method of calculating the perihelion angle $\theta_\mathrm{p}$ was based on finding local minima of Mercury's distance to the Sun, but from Figure 10 we see that the perihelion position varies by up to $\pm 0.4°$ from the $x$-axis.

Although lowering the simulation time (while keeping the number of integration points constant) would ensure better precision in the measurements of $\theta_\mathrm{p}$, our experience was that this led to the opposite outcome. More data points are beneficial as it helps smooth out fluctuations in $\theta_\mathbf{p}$. Most of the computational time while simulating Mercury was spent processing large data files, with the one used to make Figure 10 reaching 1.6 GB. Having a more tailored code that only stored perihelion coordinates would have allowed us to reduce the time steps and the uncertainty in the precession rate.

## VI. CONCLUSION

We have solved the many-body problem of our solar system using the velocity Verlet algorithm for solving coupled ordinary differential equations. The algorithm was compared to Euler's method for solving the equations of motion for the system consisting of the Sun and Earth only, with the result of no observable difference in stability. The difference could however be due to numerical imprecision dominating the instability, which should be examined more closely by performing the same calculations with different numbers of time steps.

We have also for the Sun-Earth system tested the motion of Earth for a gravitational force that is not perfectly inverse squared, such that $F \propto 1/r^\beta$. $\beta$ has been varied between 2 and 3, with the result that conservation of energy breaks down when the value of $\beta$ becomes any larger than 2. From our calculations we can see that Nature deviates from a perfect square law by only a small amount, if any.

By adding a relativistic correction to the gravitational force on Mercury we have been able to show a perihelion precession rate of $(47 \pm 19)''$ per century. This is in accordance with the observed value of of $43''$ per century. Even though are result contains a large uncertainty, it is a result that could not have been obtained from the Newtonian law of gravitation. Our results therefore show that the observed perihelion precession is explainable by the general theory of relativity.


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