Modeling the Option Market by Solving the Black Scholes Equation Numerically

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The Black-Scholes equation, describing the price evolution of European options, has been solved numerically by transforming it to the one-dimensional diffusion equation, and solving it using both the explicit forward Euler scheme and the implicit Crank-Nicolson method. The analytical solution of the equation, the Black-Scholes formula, assumes zero yield of the underlying asset. In all other cases, numerical solutions are necessary, and these were compared with the Black-Scholes formula. The Crank-Nicolson scheme was chosen for further calculations as it is more numerically stable with smaller approximation errors. Moreover, we studied standardised options on the Oslo Stock Exchange, in particular the market prices of DNB call options. The option prices were used to estimate the implied volatility of the DNB stock, which was found to be $\sigma = 0.2 \text{ yr}^{-\frac{1}{2}}$. Finally, the options were studied more carefully through analysis of the partial derivatives of option prices known as the Greeks.

I. INTRODUCTION

A wide range of physical systems and processes can be described with partial differential equations (PDEs). Well known examples include diffusion conduction, particle diffusion and wave propagation. In addition, Maxwell's equations and the Schrödinger equation are fundamental PDEs in the fields of electrodynamics and quantum mechanics, respectively. PDEs also show up in fields outside of physics, and in this paper we will study the Black-Scholes equation, a famous partial differential equation from finance.

Options are popular financial instruments among investors, as they are used both to mitigate risk and to amplify bets on the market. The Black-Scholes equation describes how the price of these options change over time. Finding a fair price of options is difficult, and in 1997 the Nobel prize in economics was given for the development of the Black-Scholes(-Merton) model.

There is an analytical solution of the Black-Scholes equation, the Black-Scholes formula, however only under the assumption that the underlying asset has zero yield. Numerical solutions are necessary for all other cases. The main strategy in solving the Black-Scholes equation is by transforming it into a familiar physical equation, the one-dimensional diffusion equation. This will be solved numerically, first with a simpler explicit scheme (utilizing the forward Euler method), and then with the more sophisticated Crank-Nicolson method. Doing a reverse transformation of the 1D diffusion equation solution leads to a solution of the Black-Scholes equation. The Black-Scholes formula will then serve as a benchmark for our results.

The Black-Scholes equation includes parameters that reflect the current market situation. These are found from studying publicly traded options on the Oslo Stock Exchange (OSE) and interest rates from the Norwegian central bank (Norges Bank). We choose to study options from a large company (DNB), and see if we can use real market data in order to tune our model. We have also calculated the "Greeks", which are the rate of change between the option's price and the different parameters from the Black-Scholes equations. One key aspect of the equation is that you can hedge the option by buying and selling the underlying assess in the right way to minimize risk, which can be done by eliminating movement in one or several of the "Greeks".

This paper will be written from a physicists point of view, so all the necessary financial definitions is included in a brief glossary.

II. THEORY

A. Definitions of financial terms

- **Financial derivative.** A financial derivative (not to be mistaken for a mathematical derivative) is a contract that derives its value from the performance of an underlying entity.
- **Option.** An option is an example of a financial derivative. It is a contract which conveys its owner the right, but not the obligation, to buy an underlying asset at a specified price on a specified date, called the expiration date or maturity date. The agreed price for the underlying asset is called the exercise price. A right to buy an asset is called a *call*, and a right to sell a *put*.
- **Hedging.** The practice of reducing the risk of losing money on owned assets, by making other strategic investments. Often the goal is to let losses on one position (one amount of assets) being mitigated by earnings on another position.
- **Return.** The return is a way of measuring the profitability of an investment over time, often annually. It is the amount that was gained or lost from the investment over time.

* Code repository: https://github.com/williameivikolsen/FYS4150

- **Dividend yield.** The yield is another way of measuring the profitability of an investment over time. There are various type of yield. The dividend yield, as we will be using, measures the total annual dividend payments per share value of a company.

- **Volatility.** Volatility (σ) is the degree of variation of a trading price series over time. It is usually measured as the standard deviation of logarithmic returns. One differs between *historic volatility* (based on historical movements in price) and *implied volatility* (representing expected fluctuations in price).

Note that the option we have defined as giving the owner the right to buy an underlying asset on a specified date is called a European option. Another commonly used option is the American option, where the owner has the right to buy an underlying asset whenever the owner wants until an expiration date. We will assume European call options for our calculations.

B. The Black-Scholes equation and formula

The Black-Scholes equation is a partial differential equation describing the price of an option over time. It is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0, \tag{1}$$

where $V(S,T)$ is the value of the options at the expiration time T, S is the price of the underlying asset, σ is the volatility of the underlying asset, r is the "risk-free" interest rate, and D is the yield (dividend paying rate) of the underlying stock. The volatility stems from an assumption that the stock moves like a geometric Brownian motion:

$$\frac{dS}{S} = \mu dt + \sigma dW.$$

where μ is a constant and W is a stochastic variable. For European options there is an analytic solution, called the Black-Scholes formula [1], which has been used as a benchmark for our numerical calculations. It goes as

$$V(S,t) = N(d_1)S - N(d_2)PV(E), \tag{2}$$

where N is the cumulative normal distribution function, E is the exercise price of the option, and the present exercise price $PV(E)$ is given by

$$PV(E) = Ee^{-r(T-t)},$$

The parameters d_1 and d_2 are

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{E}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)\right], \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

C. Transformation to diffusion equation

When solving the Black-Scholes equation we instead of an initial value problem stand upon a terminal value problem at time T, the expiration date,

$$V(S,T) = \max\{0, S-E\}. \tag{3}$$

We assume the boundary conditions to be:

$$V(0,t) = 0, \tag{4}$$

$$V(\infty,t) \sim Se^{-D(T-t)} - E^{-r(T-t)}, \tag{5}$$

where we have included discount factors of S and E. The substitution $\tau = T-t$ can be made to convert the equation to a initial value problem where τ is defined as the time remaining until expiration. Further we introduce the transformed spatial variable $x = \log(S/E)$. The partial derivatives of the equation becomes

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\partial V}{\partial \tau}, \\ \frac{\partial V}{\partial S} &= \frac{\partial V}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial V}{\partial x}, \end{aligned}$$

and the transformed second derivative:

$$\begin{aligned} \frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial V}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial V}{\partial x} + \frac{1}{S}\frac{\partial^2 V}{\partial S \partial x} \\ &= -\frac{1}{S^2}\frac{\partial V}{\partial x} + \frac{1}{S^2}\frac{\partial^2 V}{\partial x^2}. \end{aligned}$$

V now is a function of the transformed variables $V(x,\tau)$. Inserted in the Black-Scholes equation Eq. (1),

$$-\frac{\partial V}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \left(r - D - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x} - rV = 0, \tag{6}$$

we obtain a parabolic equation containing constant coefficients. The transformed initial and boundary conditions are

$$\begin{aligned} &V(x,0) = E\max\{0, (e^x - 1)\}, \\ &V(-\infty,\tau) = 0, \\ &V(\infty,\tau) \sim E(e^x e^{-D\tau} - e^{-r\tau}). \end{aligned}$$

One final substitution can also be made:

$$u(x,\tau) = e^{\alpha x + \beta\tau}V(x,\tau), \tag{7}$$

so that we can write $V(x,\tau) = u(x,\tau)e^{-(\alpha x + \beta\tau)}$. Again we find the transformed derivatives:

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= -\beta\phi u + \phi\frac{\partial u}{\partial \tau}, \\ \frac{\partial V}{\partial x} &= -\alpha\phi u + \phi\frac{\partial u}{\partial x}, \\ \frac{\partial^2 V}{\partial x^2} &= \alpha^2\phi u - 2\alpha\phi\frac{\partial u}{\partial x} + \phi\frac{\partial^2 u}{\partial x^2}, \end{aligned}$$

where the exponential function is defined as $\phi \equiv e^{-(\alpha x+\beta\tau)}$. Inserted in Eq. (6), and divided by ϕ, we get

$$\frac{\partial V}{\partial x} = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \left(r - D - \frac{\sigma^2}{2} - \sigma^2\alpha\right)\frac{\partial u}{\partial x} + \left(\frac{\sigma^2}{2}\alpha^2 - \alpha(r - D - \frac{\sigma^2}{2}) - r + \beta\right)u.$$

The constants α and β are defined to be

$$\alpha = \frac{r-D}{\sigma^2} - \frac{1}{2}, \tag{8}$$

$$\beta = \frac{r+D}{2} + \frac{(r-D)^2}{2\sigma^2} + \frac{\sigma^2}{8}, \tag{9}$$

resulting in the terms containing u and $\partial u/\partial x$ cancels and we are left with the diffusion equation with a diffusion constant $C = \sigma^2/2$:

$$\frac{\partial u}{\partial \tau} = C\frac{\partial^2 u}{\partial x^2}. \tag{10}$$

The initial condition and boundary conditions for the diffusion equation becomes

$$u(x,0) = Ee^{\alpha x}\max\{0, e^x - 1\}, \tag{11}$$

$$u(-\infty,\tau) = 0, \tag{12}$$

$$u(\infty,\tau) \sim Ee^{\alpha x+\beta\tau}(e^x e^{-D\tau} - e^{-r\tau}). \tag{13}$$

D. The Greeks

The "Greeks" measure sensitivity of the value of the derivative of a portfolio to changes in parameter value while holding other parameters fixed. The derivatives are called the Greeks as they are represented by Greek letters[1]. Some of the most widely used Greeks are defined as follows:

$$\text{Delta}: \quad \Delta = \frac{\partial V}{\partial S} \tag{14}$$

$$\text{Gamma}: \quad \Gamma = \frac{\partial^2 V}{\partial S^2} \tag{15}$$

$$\text{Vega}: \quad \nu = \frac{\partial V}{\partial \sigma} \tag{16}$$

$$\text{Theta}: \quad \Theta = -\frac{\partial V}{\partial \tau} \tag{17}$$

$$\text{Rho}: \quad \rho = \frac{\partial V}{\partial r}. \tag{18}$$

III. METHOD

A. Solvers

The Black Scholes equation has been solved numerically using both an explicit scheme, the Forward Euler method, and one implicit scheme, Crank-Nicholson, both of which are described below. When deriving the diffusion equation we found x to be unbounded, $x \in [-\infty, \infty]$, thus we have to approximate with a bounded interval $x \in [-L, L]$ where L is sufficiently large. The boundary conditions, Eq. (12) and Eq. (13), are rewritten to

$$u(-L,\tau) = 0 \equiv a(\tau),$$
$$u(L,\tau) = Ee^{\alpha L+\beta\tau}(e^L e^{-D\tau} - e^{-r\tau}) \equiv b(\tau).$$

We discretize Eq. (10) by applying equally spaced mesh points

$$x_i = x_0 + i\Delta x, \qquad i = 0, ..., Nx,$$
$$\tau_j = j\Delta\tau, \qquad j = 0, ..., N_\tau,$$

where

$$\Delta x = \frac{2L}{N_x} \quad \text{and} \quad \Delta\tau = \frac{T}{N_\tau},$$

and N_x and N_τ are the number of the respective intervals. We rename the expression for the initial condition Eq. (11) as $u(x,0) \equiv I(x)$. The discrete approximated solution to the diffusion equation is written as $u(x_i,\tau_j) \equiv u_{ij}$.

1. The forward Euler

For the explicit scheme we use forward approximation for the derivatives:

$$u_\tau \approx \frac{u_{i,j+1} - u_{i,j}}{\Delta\tau},$$

with local approximation error $\mathcal{O}(\Delta\tau)$, and

$$u_{xx} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

with error $\mathcal{O}(\Delta x^2)$. We define the the diffusion constant $C = \sigma^2/2$ and introduce the mesh Fourier number

$$F \equiv C\frac{\Delta\tau}{\Delta x^2}. \tag{19}$$

The diffusion equation, Eq. (10), can now be written:

$$u_{i,j+1} = Fu_{i-1,j} + (1-2F)u_{i,j} + Fu_{i+1,j}. \tag{20}$$

From the initial condition $u_{i,0} = I(x_i)$ for all x_i we can easily by using Eq. (20) calculate the solution for the next time step. By continuing using the previous time step and the boundary conditions $a(\tau)$ and $b(\tau)$ to calculate the next we obtain all the solutions. The method is summarized in the Algorithm 1. Even though the explicit scheme is easy to implement, it is constrained by a stability condition (proved by Langtangen and Linge [7]):

$$F = C\frac{\Delta\tau}{\Delta x^2} \le \frac{1}{2}. \tag{21}$$

[1] Neglecting the fact that "Vega" is not a Greek letter.

Algorithm 1 The forward Euler method

```
for i = 0, ..., N_x do
    u[i, 0] = I(x_i)                          ▷ Calculating initial conditions
for j = 0, ..., N_t do
    for i = 1, ..., N_x do
        u_n[i] = u[i] + F * (u[i − 1]) − 2u[i] + u[i + 1])
    u_n[0] = a(τ_j)                           ▷ Setting boundary conditions
    u_n[N_x] = b(τ_j)
    u = u_n                                   ▷ Preparing old values for next step
```

2. *The Crank-Nicolson method*

Using the θ-rule, as explained by Langtangen and Linge [7], applied on the diffusion equation Eq. (10) we can set up the generalized approximated equation:

$$C\left[\frac{\theta}{\Delta x^2}\left(u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right) + \frac{1-\theta}{\Delta x^2}\left(u_{i+1,j-1}-2u_{i,j-1}+u_{i-1,j-1}\right)\right] = \frac{1}{\Delta t}\left(u_{i,j}-u_{i,j-1}\right),$$

where $\theta = 0$ would give the forward Euler scheme, $\theta = 1$ the implicit backward Euler and $\theta = 1/2$ the Crank-Nicolson scheme, named after its inventors. By again inserting the mesh Fourier number F to the Crank-Nicolson scheme we obtain the equation

$$-Fu_{i-1,j} + (2+2F)\,u_{i,j} - Fu_{i+1,j} = Fu_{i-1,j-1} + (2-2F)\,u_{i,j-1} + Fu_{i+1,j-1},$$

where the errors of the approximations goes as $\mathcal{O}(\Delta\tau^2)$ and $\mathcal{O}(\Delta x^2)$. This could also be written in matrix-form:

$$\left(2\hat{I}+F\hat{B}\right)V_j = \left(2\hat{I}-F\hat{B}\right)V_{j-1},$$

where the vector V_j is defined below and the matrix B is a tridiagonal matrix:

$$V_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N_x} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & -1 \\ 0 & 0 & \dots & -1 & 2 \end{bmatrix}.$$

To solve the equation we first calculate the matrix multiplication:

$$\tilde{V}_{j-1} = \left(2\hat{I}-F\hat{B}\right)V_{j-1}. \tag{22}$$

We define $\left(2\hat{I}+F\hat{B}\right) \equiv A$ and can rewrite the Crank-Nicolson scheme as

$$AV_j = \tilde{V}_{j-1},$$

where we have to solve this tridiagonal matrix equation to obtain the solutions V_j. This scheme is convenient as it is stable for all $\Delta\tau$ and Δx. The method is summarized in Algorithm 2. A comparison of the Crank-Nicolson and forward Euler schemes is presented in Table I.

Algorithm 2 The Crank-Nicolson method

```
for i = 0, ..., N_x do
    u[i, 0] = I(x_i)                          ▷ Set initial conditions
Initialize the matrix A
for j = 0, ..., N_τ do
    for i = 1, ..., N_x do                    ▷ Calculate the RHS of equation
        Ṽ[i] = Fu[i − 1] + (2 − 2F)u[i] + Fu[i + 1]
    Ṽ[0] = a(τ_j)                             ▷ Set boundary conditions
    Ṽ[N_x] = b(τ_j)
    Solve Au_n = Ṽ for u_n
    u_n = u                                   ▷ Prepare for next iteration
```

Table I. The table shows a comparison of the two schemes used for solving the diffusion equation.

Scheme	Truncation error	Stability requirements
Crank-Nicolson	$\mathcal{O}(\Delta x^2)$ and $\mathcal{O}(\Delta\tau^2)$	Stable for all $\Delta\tau$ and Δx
Forward Euler	$\mathcal{O}(\Delta x^2)$ and $\mathcal{O}(\Delta\tau)$	$C\Delta\tau/\Delta x^2 \leq 1/2$

B. Externally sourced data

We use the interest rate on government securities as an estimate of the "risk-free" interest rate r. More specifically, we base r on the interest rate of treasury bills ("statskasseveksler") with maturity of 12 months. These are updated daily, and $r = 0.10$ as of Dec. 15$^{\text{th}}$ 2020 [5]. Note that the closing prices in the next paragraphs refers to the following day, Dec. 16$^{\text{th}}$ 2020.

In addition, the Black-Scholes equation utilizes parameters that are asset-specific. This includes the yield D and volatility σ. The estimates of these variables are based on the stock and option prices of DNB, the largest financial services provider in Norway [3]. The closing price for DNB is 160.00 NOK, and the 2019 dividend per share was 9.00 NOK [3]. This leads to a yield $D = 0.056$.

The closing price of option with exercise price $E = 155$ NOK and maturity Jan. 2021[2] is 8.82 NOK. This data point will be used in order to tune the volatility σ in the Black-Scholes model. Note that the standardized stock options traded at OSE are American options.

All market prices were sourced from OSE's website [4]. Since only the closing prices from the last trading day are publicly available for free, we provide a snapshot of the option prices on the Github repository (URL found on page 1).

[2] All stock option maturity dates at OSE are set to be the third Friday of the month, so the January maturity date is coincidentally a full month away.

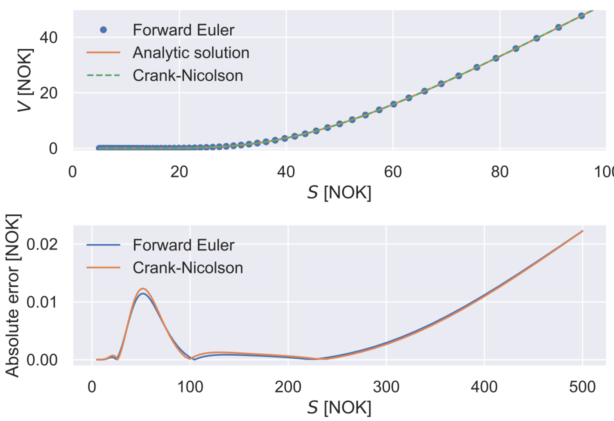

Figure 1. The Black Scholes equation solved using forward Euler and Crank-Nicolson compared with the analytical expression Eq. (2) for $\tau = 1.0$ yr. The parameters used are $D = 0$, $\sigma = 0.4\ \mathrm{yr}^{-\frac{1}{2}}$, $r = 0.04\ \mathrm{yr}^{-1}$ and $E = 50$. The lower figure shows the absolute error between the two methods and the analytical solution.

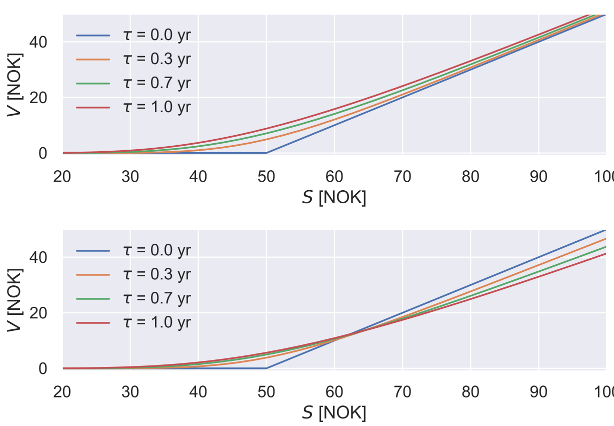

Figure 2. Both figures shows the Black-Scholes equation solved using the Crank-Nicolson scheme for parameters $\sigma = 0.4\ \mathrm{yr}^{-\frac{1}{2}}$, $r = 0.04\ \mathrm{yr}^{-1}$, $E = 50$ NOK for different values of τ. The upper figure shows option value when the yield D is set to zero, and the one below for yield $D = 0.12$.

IV. RESULTS

The Black-Scholes equation has been solved using the Crank-Nicolson scheme for a test-case with arbitrarily chosen parameter values. The calculated option price V is presented against the price of the underlying stock S for different times until expiration date in Fig. 2. We have also compared the Crank-Nicolson scheme and the forward Euler scheme to the analytical solution Eq. (2) for time until expiration $\tau = 1.0$ yr. These results are shown in Fig. 1.

In order to find parameters reflecting real-world cases, we used the yield D and the closing price of DNB options

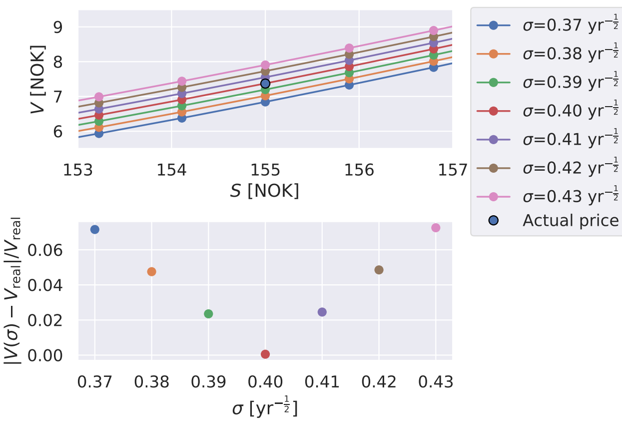

Figure 3. Above: Graphically showing how σ effects the theoretical option price with $E = 155$ NOK and maturity 1 month ($\tau = 1/12$ yr). Below: Relative difference between the calculated option prices $V(\sigma)$ and the market price V_{real}.

to determine the implied volatility σ. Since the closing price of options with exercise price E and maturity one month is known (see Section III B), we solved the Black Scholes equation with varying values of σ, and found the value of σ that gave the best accordance. We tested for $\sigma \in [0.37, 0.43]\ \mathrm{yr}^{-\frac{1}{2}}$, in intervals $\Delta\sigma = 0.01\ \mathrm{yr}^{-\frac{1}{2}}$. Setting $\sigma = 0.40\ \mathrm{yr}^{-\frac{1}{2}}$ gave the best results, with an error $< 1\%$. These calculations are shown in Fig. 3.

Next, having chosen $\sigma = 0.40\ \mathrm{yr}^{-\frac{1}{2}}$, we have compared more market prices of DNB options with calculated solutions of the Black-Scholes equation. We have studied varying strike prices $E \in [140, 175]$ NOK with maturities January, February and March 2021 ($\tau \in \left[\frac{1}{12}, \frac{2}{12}, \frac{3}{12}\right]$ yr). The results are shown in Fig. 4.

To make our Black-Scholes predictions reflect reality better, we then tried to adjust the volatility to $\sigma = 0.20\ \mathrm{yr}^{-\frac{1}{2}}$. This gave much better results, shown in Fig. 5. This new value of σ is kept going on.

Finally, the Greeks were calculated (also using the parameters specified in section III B) for maturities $\tau \in [0.01, 0.25, 0.5, 0.75, 1]$ yrs. These are plotted in Figures 6 to 10.

V. DISCUSSION

The computed solutions using the forward Euler and Crank-Nicolson methods versus the analytical solution to the Black-Scholes equation are shown in Fig. 1. The analytical solution assumes both European call options and that the underlying stock does not pay any dividend [1], which is why we had to set the dividend yield to zero to be able to compare with analytical values.

Fig. 2 shows the option value for different times until expiration, and illustrates the difference of the yield being set to zero and having a positive value. We observe that for no dividend yield the option values for $\tau > 0$ always

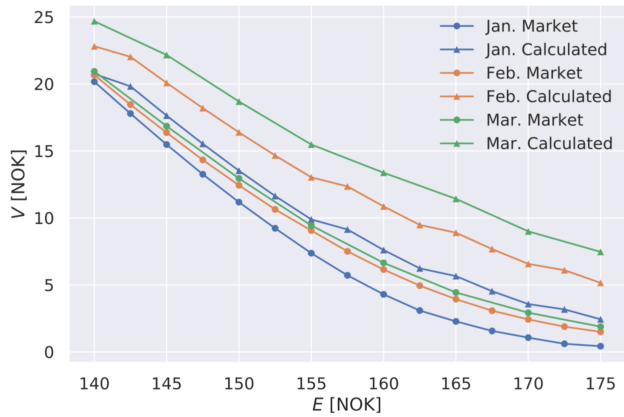

Figure 4. Comparing closing prices of DNB call options with $E \in [140, 175]$ NOK and maturities with $\tau \in \left[\frac{1}{12}, \frac{2}{12}, \frac{3}{12}\right]$ yr, using parameters specified in section III B and $\sigma = 0.40\ \mathrm{yr}^{-\frac{1}{2}}$.

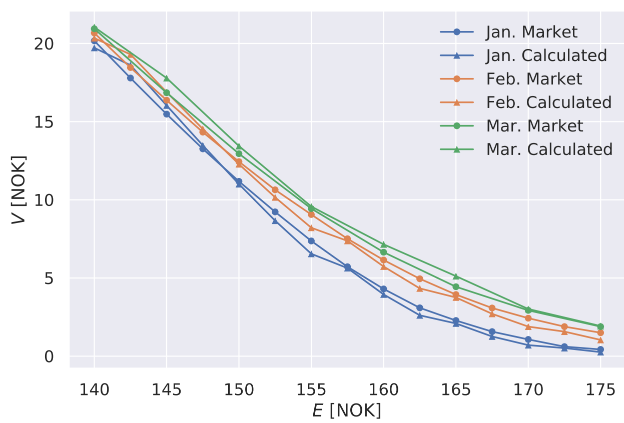

Figure 5. Comparing closing prices of DNB call options with $E \in [140, 175]$ NOK and maturities with $\tau \in \left[\frac{1}{12}, \frac{2}{12}, \frac{3}{12}\right]$ yr, using parameters specified in section III B and $\sigma = 0.20\ \mathrm{yr}^{-\frac{1}{2}}$.

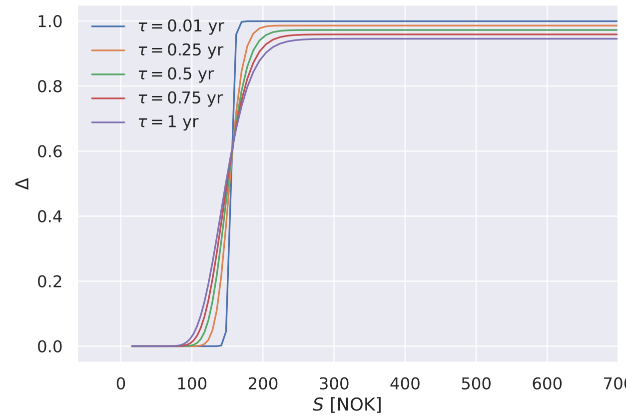

Figure 6. Delta calculated using parameters specified in section III B and $\sigma = 0.20\ \mathrm{yr}^{-\frac{1}{2}}$.

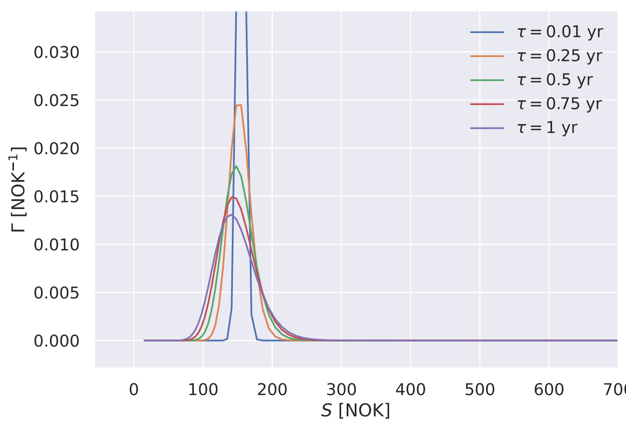

Figure 7. Gamma calculated using parameters specified in section III B and $\sigma = 0.20\ \mathrm{yr}^{-\frac{1}{2}}$.

lies above the $\tau = 0$ solution, but for $D = 0.12\ \mathrm{yr}^{-1}$ they cross the $\tau = 0$ solution and decrease with S. A stock that pays out dividends is more attractive to own in the present, and waiting for the call's expiration day effectively leads to negative returns. The option value is thus diminished.

We initially chose arbitrary parameter values for the comparison to be able to check the reliability of our solver. We observe that both methods are quite accurate, with the largest absolute error being 0.01 NOK for both. For further calculations we chose to use the Crank-Nicolson scheme, as it is more stable in choices of $\Delta\tau$ and Δx, and the approximation errors are in general smaller compared to the forward Euler scheme (Table I).

The reason we chose to study DNB in particular is because it is one of only 21 companies (as of December 2020) which has standardized option trading available at OSE. In addition, we wanted the company to have a high turnover of stocks ($5.6 \cdot 10^9$ NOK in Oct. 2020 [6]) and frequent call option trades (in average 55 contracts per trading day, Nov. 2020 [6]).

The estimate of $\sigma = 0.40\ \mathrm{yr}^{-\frac{1}{2}}$ found in Fig. 3 turns out to be too high; this is most easily seen by studying Fig. 4. Unsurprisingly, the error between calculated and real option prices increases with longer expiration time. However, we also see that the error increases with higher strike prices E. Looking at the January maturity call options with $E = 175.00$ NOK, we see that with a current closing price of 160.00 NOK, the market thinks it's highly unlikely that the stock will rise to 175 NOK. This leads to a very low option price V (< 1 NOK).

We have translated the market's belief in lower upward mobility in stock price to a reduction of σ. Setting $\sigma = 0.20\ \mathrm{yr}^{-\frac{1}{2}}$ leads to much better accuracy for all expiration dates, as shown in Fig. 5.

It's worth noting that even though we have made a

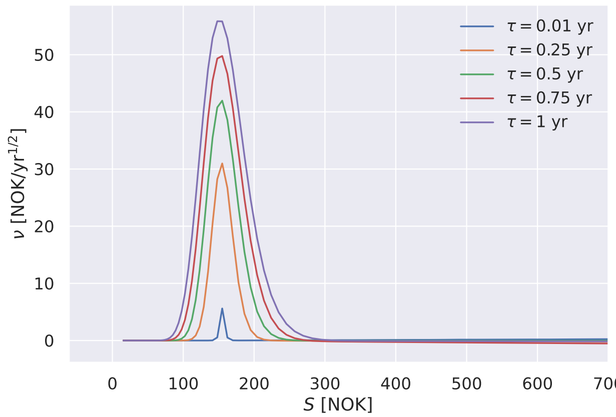

Figure 8. Vega calculated using parameters specified in section III B and $\sigma = 0.20\ \mathrm{yr}^{-\frac{1}{2}}$.

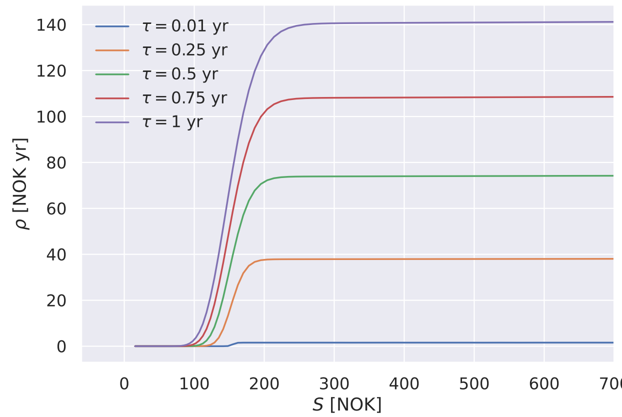

Figure 10. Rho calculated using parameters specified in section III B and $\sigma = 0.20\ \mathrm{yr}^{-\frac{1}{2}}$.

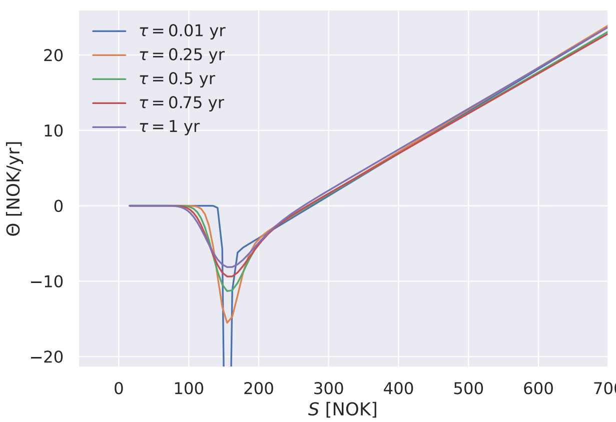

Figure 9. Theta calculated using parameters specified in section III B and $\sigma = 0.20\ \mathrm{yr}^{-\frac{1}{2}}$.

model that closely models the (call) options market, we have to be careful about claiming to have *solved* it. The assumptions behind the Black-Scholes model are not necessarily met. Some examples includes the assumption about the market being frictionless [1]. We have neglected transaction costs in this report, but over the last 12 months, the premium of buying DNB options has been 5.8 NOK in average [6].

In addition, the Black-Scholes model assumes European options, while the single stock options at OSE are American. One can show that if the dividend yield is zero, it's advantageous to wait until the expiration date before using the right to exercises the option (it's better to sell the option rather than exercise a call early) [2]. This means that European and American options behave similarly if $D = 0$. In the case where $D \neq 0$ and the underlying pays out dividends, it may be preferable to use the possibility to buy it earlier.

In Figure 6 we see that $\Delta > 0$ for all S, meaning that the value of the option always increases for a positive change in the value of the underlying asset as it should. We see that for every remaining times τ, the value of Δ seems to converge for some $S > E$. In particular, for $\tau = 0.01$ yrs, the value of Δ seems to converge to ~ 1 exactly as $S > E$. This result is according to our expectations: Since the option is very close to expiring, an investor will expect the price of the underlying asset to be the same when it expires. Therefore, for $S > E$, the value of the option behaves like the value of the asset, giving a slope $\Delta = 1$. As τ gets larger, the probability for the underlying asset to have its value decrease below E increases. Therefore, V should still be proportional to E for some $S > E$, but with a lower slope. This means that Δ should converge to some value < 1, as we see is reflected in our results. Note also that the value of S necessary for convergence increases with τ: For higher expiration time, the value of the underlying asset must be higher for there to be a simple linear relationship between V and S.

The sensitivity of Delta itself to changes in S can be extracted from the behavior of Gamma, which is displayed in Figure 7. We see that Γ has a peak around $S = E$ where Δ increases, before approaching 0 when Δ converges, as discussed above. The peaks are narrower and taller for lower values of τ, shortening down the interval of S at which Δ changes. We see that when τ approaches 0, the value of Δ can only change for values of S around E. The height of the peak approaches ∞, meaning that Δ converges to 1 on an infinitesimal interval around $S = E$.

In Figure 10, we see that Vega, the sensitivity to changes in the volatility, also comes in peaks around $S = E$. The peaks shrink as τ decreases, meaning that the value of the option has low sensitivities to the volatility when the time till expiration is short. This behavior is what we would expect as the Brownian motion of the stock is limited by the short amount of time till expiration.

For Theta, shown in Figure 9, we see negative peaks around $S = E$. These peaks become deeper and narrower as τ approaches 0. The value of Θ for a given option will therefore become smaller at an accelerating rate as the expiration approaches. Note that for $S \gg E$, the value of Θ seems to increase linearly with S. We can interpret this as with τ decreasing with all other parameters fixed, including S, an option will be more likely to give positive returns. Therefore a decrease in τ will cause an increase in the option value.

As for Rho, we see in Figure 10 that the $\tau = 0.01$ solution is $\rho \approx 0$ for $S < E$, as expected since the option has 0 value for $S < E$, which makes changes in interest irrelevant. With increasing τ, the rent has a higher impact even for options with 0 value. For $S > E$ on the other hand, we see that ρ seems to reach a constant value, giving a linear relationship between the option value and the interest rate.

Knowledge about the Greeks are crucial for investors aiming to make the value of their portfolios resistant to market changes (hedging). Using the Greeks one tries to reduce directional risk, by letting the total value of a set of assets have a zero partial derivative for one single parameter.

For example, one common method to Delta-hedge is to own a combination of options and the underlying asset, such that the value of the total position is unchanged by price-variations of the underlying ($\Delta = 0$ overall). Put options have negative values of Δ, so by buying stocks of the underlying, an investor can offset any losses on the puts (from an increased market price S) by owning stocks that increase in value. Should $\Delta = -0.8$, a Delta-neutral position is obtained by owning stocks amounting to 0.8 times the value of the puts. For call options, $\Delta \geq 0$ (Fig. 6) for all S, so instead of buying stocks in order to Delta-hedge, it's necessary to *short* stocks (betting against the stock).

Delta-hedging is useful when an investor wants to profit from bets on the underlying connected to τ, σ or other parameters, but be safe-guarded against how the price S of the underlying fluctuates. The downside of this strategy is that the Greeks don't tend to be constant around the exercise price (shown in figures related to the Greeks, Figs. 6 to 10), so keeping i.e. a Delta-neutral position requires monitoring and adjusting the position as needed. This easily leads to large expenses incurring from transaction fees.

VI. CONCLUSION

The Black-Scholes equation has been solved by transforming it to the one-dimensional diffusion equation. We solved this numerically using the explicit forward Euler scheme, as well as the implicit Crank-Nicolson method, and by a reversed transformation the solution to Black-Scholes equation was obtained. The reliability of the numerical solvers were tested by comparing option values to the analytical solution, the Black-Scholes formula. After testing for arbitrary parameters, the Crank-Nicolson scheme was chosen for further analysis of the equation, as it is numerically stable for any $\Delta\tau$ and Δx, and has smaller local approximation errors than the explicit scheme.

After confirming the reliability of the solver, we were able to encounter real standardised options on Oslo Stock exchange from a large company, which were chosen to be DNB. The market given option prices were used to estimate the implied volatility of the underlying stock to be $\sigma = 0.20\ \mathrm{yr}^{-\frac{1}{2}}$.

Finally, the Greeks (Delta, Gamma, Vega, Theta and Rho) were used to study the option price evolution closer.

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