January 19th 2023

Working on understanding Majorana polarization. So we want to find two Hermitian Majoranas γ^+ and γ^- (left/right Majorana) and form a non-local fermion

$$f = \frac{1}{2}(\gamma^+ + i\gamma^-), \tag{1}$$

such that $f^\dagger |o\rangle = |e\rangle$. Then we get $|e\rangle = \gamma^+ |o\rangle$ and $|e\rangle = -i\gamma^- |o\rangle$. We can expand these Majoranas in the Hermitian site-Majoranas $\gamma_j^+ = d_j^\dagger + d_j$ and $\gamma_j^- = i(d_j^\dagger - d_j)$

$$\begin{aligned} \gamma^+ &= \sum_{j,s} a_j^s \gamma_j^s \\ \gamma^- &= \sum_{j,s} b_j^s \gamma_j^s. \end{aligned} \tag{2}$$

We then calculate e.g.,

$$\begin{aligned} \langle o|\gamma_j^+|e\rangle = \langle o|\gamma_j^+\gamma^+|o\rangle = \langle o|\gamma_j^+ \sum_{k,s} a_k^s \gamma_k^s|o\rangle = \langle o|a_j^+ + \sum_{(k,s)\neq(j,+)} a_k^s \gamma_j^+ \gamma_k^s|o\rangle \\ = a_j^+ + \sum_{(k,s)\neq(j,+)} a_k^s \langle o|\gamma_j^+\gamma_k^s|o\rangle \end{aligned} \tag{3}$$

but $\langle o|\gamma_j^+\gamma_k^s|o\rangle^* = \langle o|\gamma_k^s\gamma_j^+|o\rangle = -\langle o|\gamma_j^+\gamma_k^s|o\rangle$ (by using anti-commutation). Hence the second term is purely imaginary and we can write (analogously for the other coefficients)

$$\begin{aligned} a_j^+ &= \mathrm{Re}\{\langle o|\gamma_j^+|e\rangle\} \\ a_j^- &= \mathrm{Re}\{\langle o|\gamma_j^-|e\rangle\} \\ b_j^+ &= -\mathrm{Im}\{\langle o|\gamma_j^+|e\rangle\} \\ b_j^- &= -\mathrm{Im}\{\langle o|\gamma_j^-|e\rangle\} \end{aligned} \tag{4}$$

since the a, b coefficients are real (γ^+ and γ^- are Hermitian). If the Hamiltonian is real (?), $a_j^- = b_j^+ = 0$ and we can skip taking the real part.

1. What have we assumed when doing this? Have we assumed that we have good left/right majoranas and then derived their expansion?

January 20th 2023

The Majorana polarization is defined in e.g., Aksenov 2020 as roughly (though not taking into account taking the real part as above)

$$\mathrm{MP} = \frac{\sum_j' {a_j^+}^2 - {b_j^-}^2}{\sum_j' {a_j^+}^2 + {b_j^-}^2} \tag{5}$$

where the prime means to sum over half of the chain. This seems to depend on how much of γ^- has leaked into the left side.

1. What is the meaning of the MP? Relation to original paper?
2. Is it reasonably defined? What if we have an imaginary part? Spin?
3. What if one Majoranas leaks to the opposite half but the other does not?
4. Could you do a normal scalar product to calculate overlap?
5. If no overlap, the denominator is not necessary since the sum of the coefficients squared is unity (see below).

Proved that $\sum_k {a_k^+}^2 = 1$ by induction, anti-commutation relations and that the majoranas square to one, assuming that we only get the + coefficients. Also tried out the MP on the Poor mans geometry.

January 23rd, 2023

1. Confused about Fourier transform of second quantization operators. Turns out it can be seen as a basis change between position and momentum space bases. See p. 16 to 18 in Flensberg Many body.
2. Started trying to derive bulk energies of Kitaev chain with periodic boundary condition (Alicea).
3. Discussed MP with Viktor, seems like the normalization is not good. One special case with overlap only on right side, still unity MP. To include spin one can subtract the spin part, such that it measures how lonely one of the spin species is.
4. Then I started looking through the derivation in Akhmerov to try and understand how the Kitaev chain emerges from local coupling.

January 24th, 2023

Working on the derivation in Akhmerov, struggling with Maple. Might have a problem with one of my rules, giving the wrong sign in the new operators.

Januray 25th, 2023

Found problem with sign, just took other branch of Δ. Struggling with getting the correct phases of the a and b operators in Maple, but I think I got it. Expressed the old c operators in the new a and b. Inserted it in Hamiltonian. The dot hamiltonian of course reduces to only $a^\dagger a$ and $b^\dagger b$ so only spin-orbit/rotation left. Found out what $e^{i\boldsymbol{\lambda}\boldsymbol{\sigma}}$ actually means, it is the matrix representation of the rotation operator $D(n, 2\lambda)$ which rotates the spin 2λ around the n axis. Also, very convenient to use Pauli matrix exponent formula. Very messy algebra however, try to get to Akhmerov. Don't understand how to/the reason why you can project onto the a operators.

Janurary 26th

Looked at job openings before lunch. After lunch I managed to get to the expression for the tunneling in Akhmerov. Still confused about how you can remove all b operators and what the conditions of doing it are. Thinking about the conditions on the parameters to get to the topological phase.

January 27th

Rewrote the derivation of Akhmerov more neatly. Discussed with Viktor:

- We can neglect the b operators at $V \gg w$ since the b-states are far off in energy and do not affect the ground state (which is what we are interested in). Perhaps one can think about it in terms of line width?
- The limit $V \gg w$ also implies $\Delta_{ind} \propto \Gamma_{SC}, \mu_n \gg w$. So in essence we need large Zeeman splitting, SC-dot coupling and chemical potential to obtain Kitaev chain with local SC-coupling.
- Also interesting that the particles in the emerging Kitaev chain are these a operators (Bogoliubov-like), not electrons.
- Next up is to
  1. Write up the Akhmerov Hamiltonian with constant SC-phase, spin-orbit rotation and spin-orbit direction, and maybe do a gauge transformation to only get one phase.
  2. Implement the model as well as the Kitaev chain in code.
  3. Implement a MP function for each model.
  4. Compare MP and energy gap in Kitaev and local coupling model.
- I don't understand how to do the scanning of MP and gap. I need to scan λ probably such that it takes linear steps in Δ?
- For the MP I now need to sum over three indices.