SC2001/CE2101/CZ2101: Algorithm Design and Analysis

Quicksort

Instructor: Assoc. Prof. ZHANG Hanwang

Courtesy of Dr. Ke Yiping, Kelly’s slides

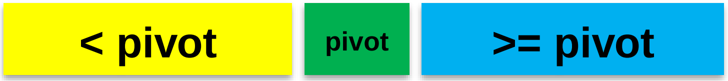
Learning Objectives

At the end of this lecture, students should be able to:

- Explain how “**Divide and Conquer**” approach is used in Quicksort
- Explain the pseudo code of Quicksort
- Manually execute Quicksort on an example input array
- Analyse time complexities of Quicksort in the best, average and worst cases

Quicksort

- **Fastest** general purpose in-memory sorting algorithm in the average case
- Implemented in Unix as qsort() which can be called in a program (see 'man qsort' for details)
- Main steps
  - Select one element in array as pivot
  - Partition list into two sublists with respect to pivot such that all elements in left sublist are less than pivot; all elements in right sublist are greater than or equal to pivot
  - Recursively partition until input list has one or zero element
- No merging is required because the pivot found during partitioning is already at its final position


< pivot | pivot | >= pivot

Quicksort (Pseudo Code)

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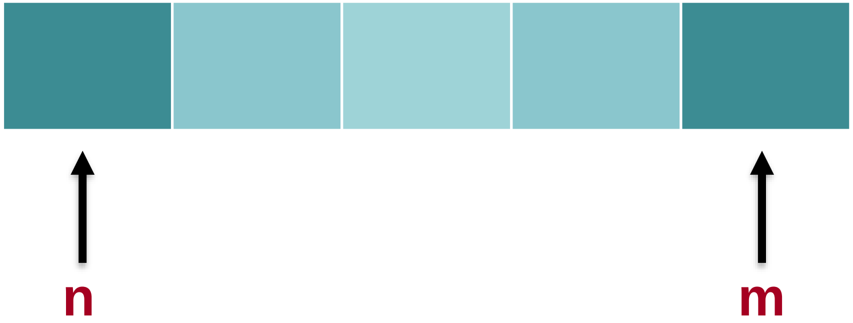
```
void quicksort(int n, int m)
{
    int pivot_pos;
    if (n >= m)
        return;
    pivot_pos = partition(n, m);
    quicksort(n, pivot_pos - 1);
    quicksort(pivot_pos + 1, m);
}
```



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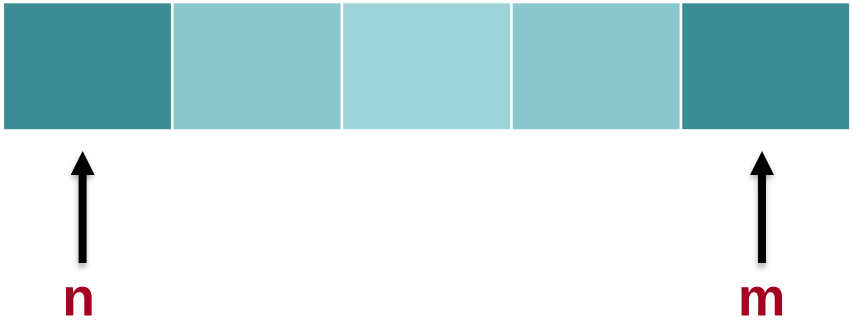

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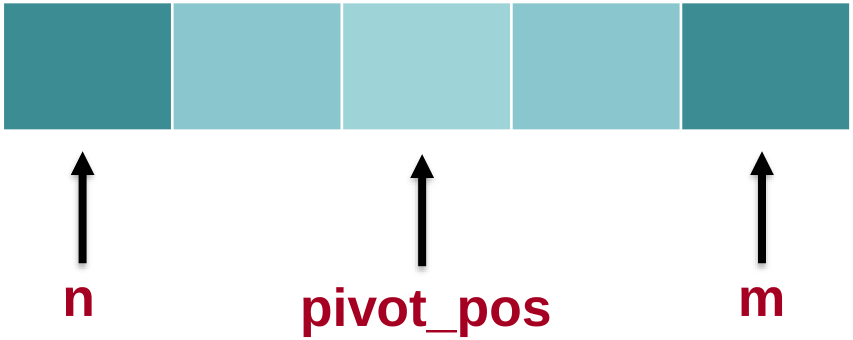

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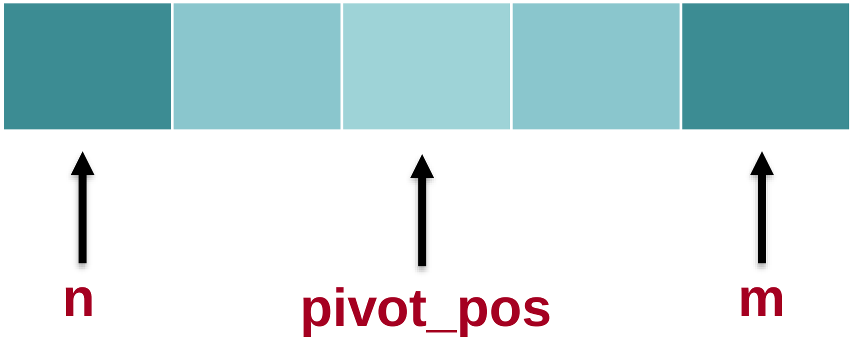

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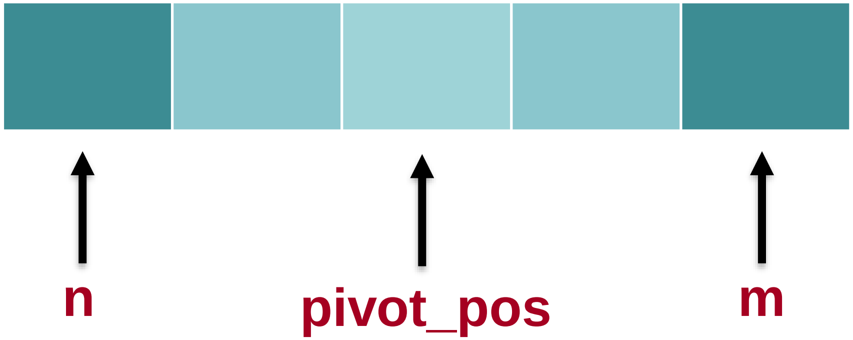

Partition Routine in Quicksort

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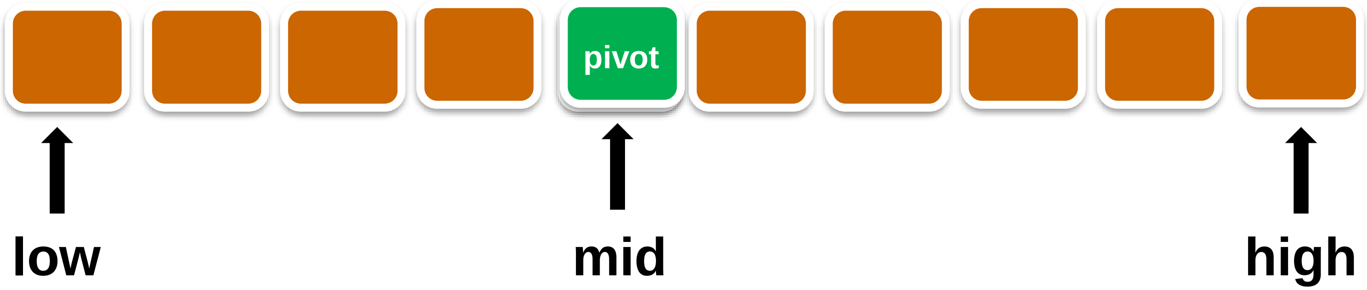
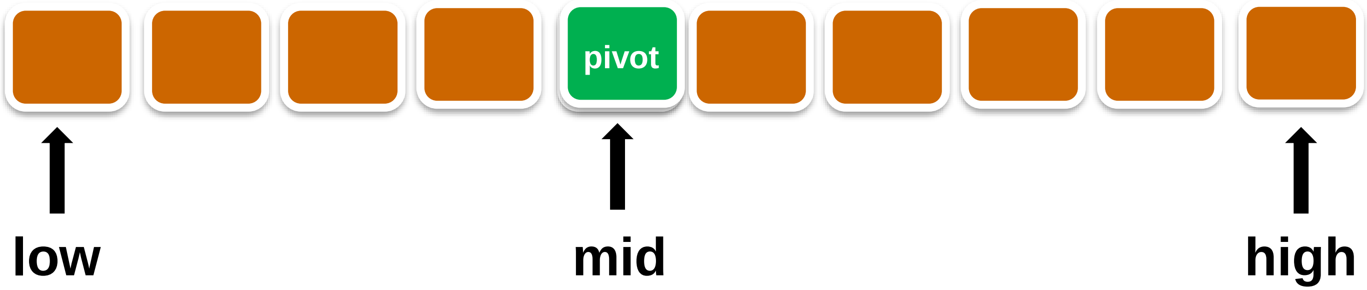


```
int partition(int low, int high)
{
    int i, last_small, pivot;
    int mid = (low+high)/2;
    swap(low, mid);
    pivot = slot[low];
    last_small = low;
```

Partition Routine in Quicksort



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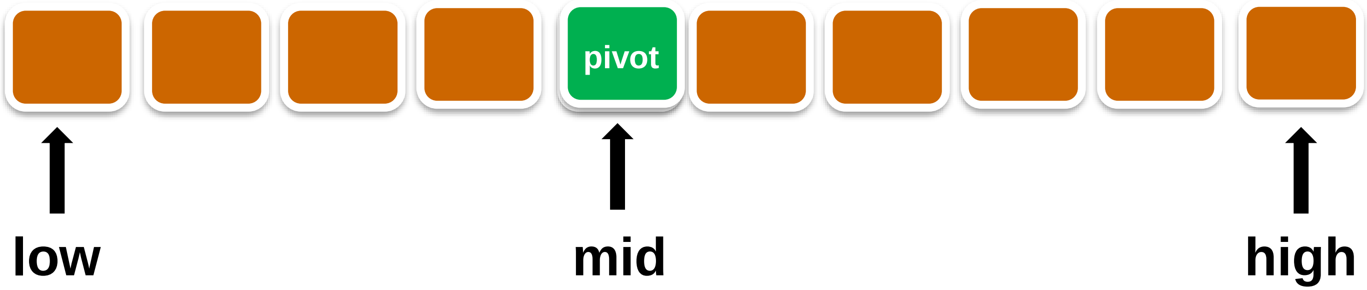

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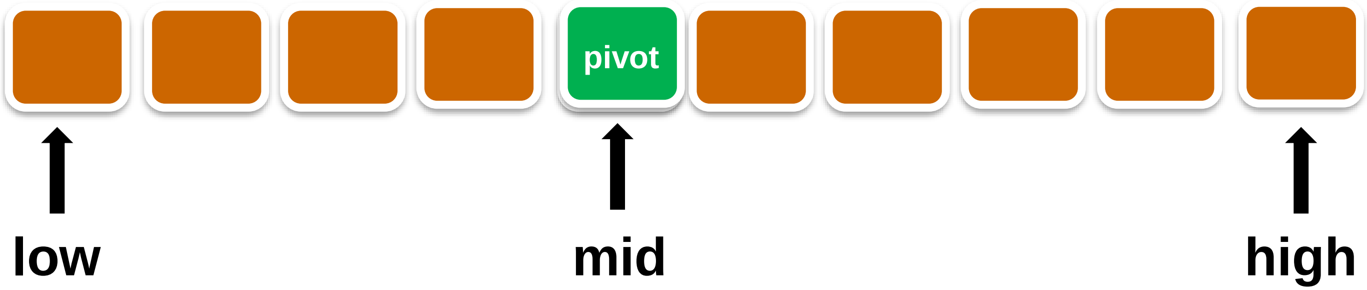

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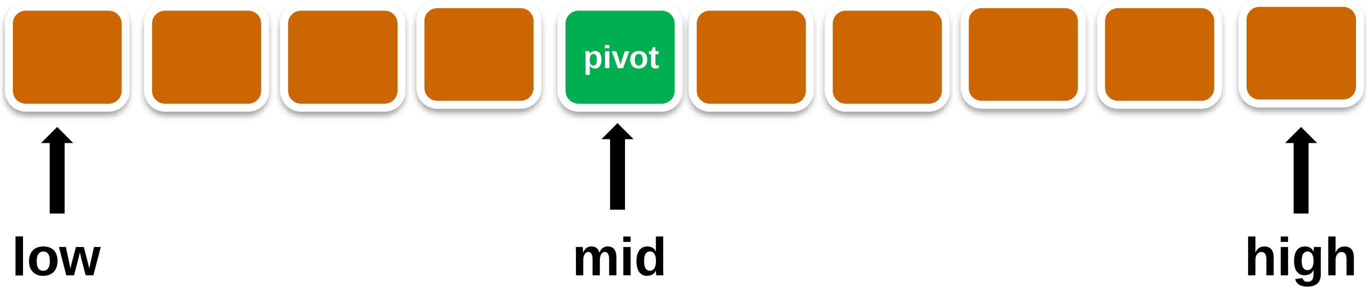

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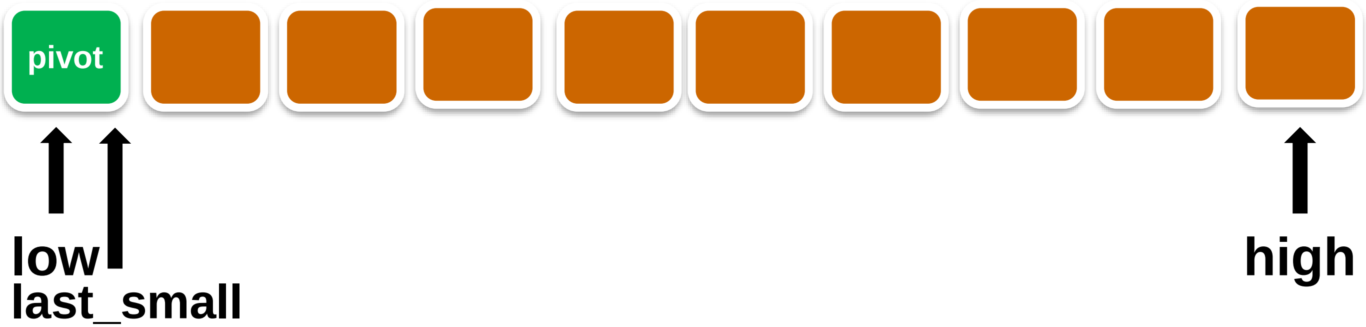
pivot

low

last_small

high

```
int partition(int low, int high)
{
    int i, last_small, pivot;
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    swap(low, mid);
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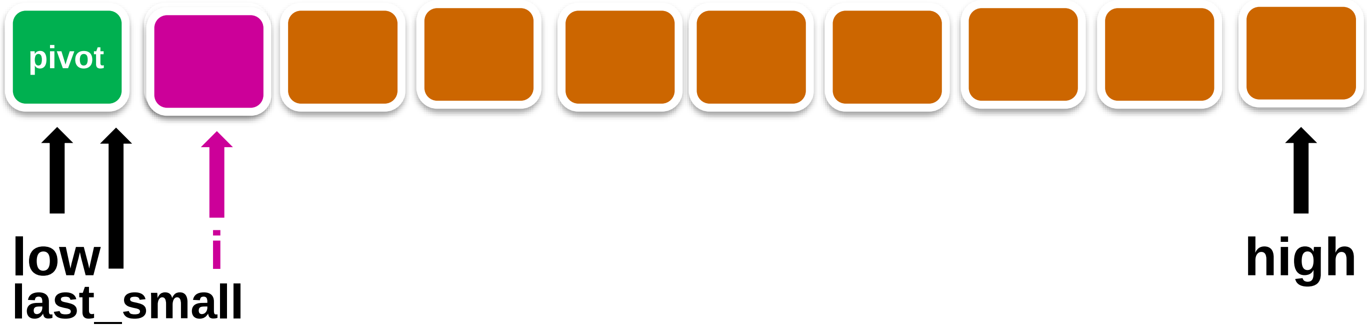
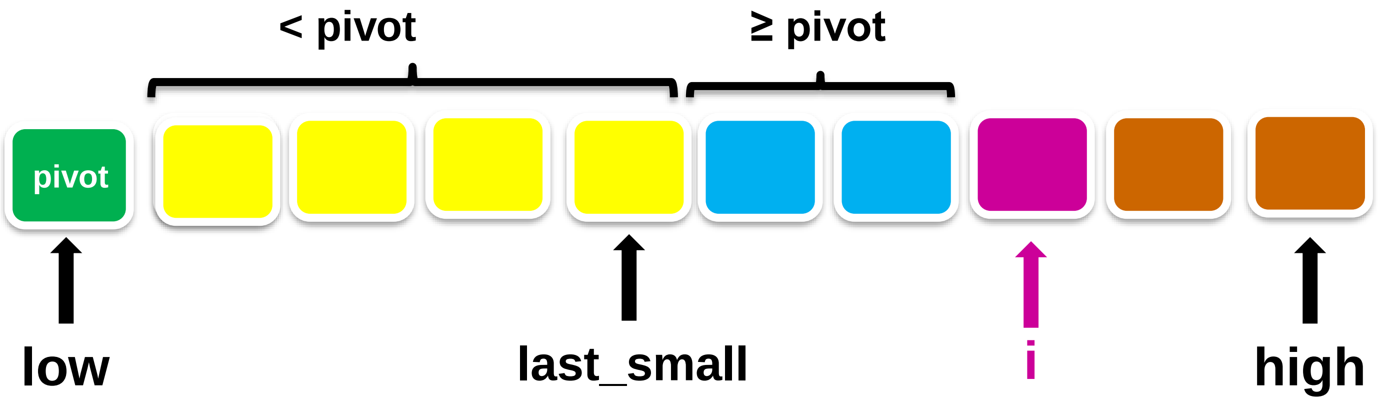
Partition Routine in Quicksort



```
int partition(int low, int high)
{.......
        for (i = low+1; i <= high; i++)
                if (slot[i] < pivot)
                        swap(++last_small, i);
        swap(low, last_small);
        return last_small;
 }
```

Partition Routine in Quicksort



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                    if (slot[i] < pivot)
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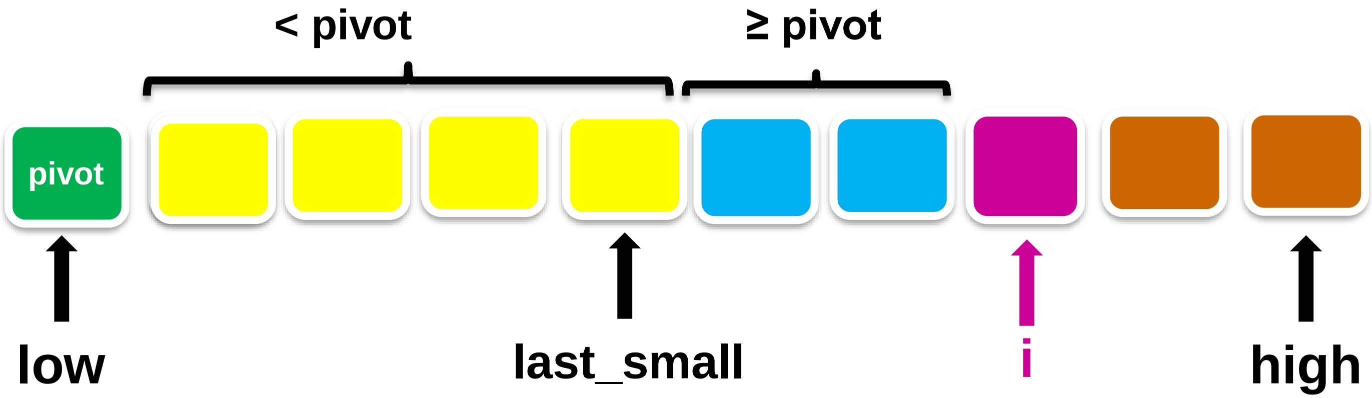
Partition Routine in Quicksort



```
int partition(int low, int high)
{.......
        for (i = low+1; i <= high; i++)
                if (slot[i] < pivot)  ✓
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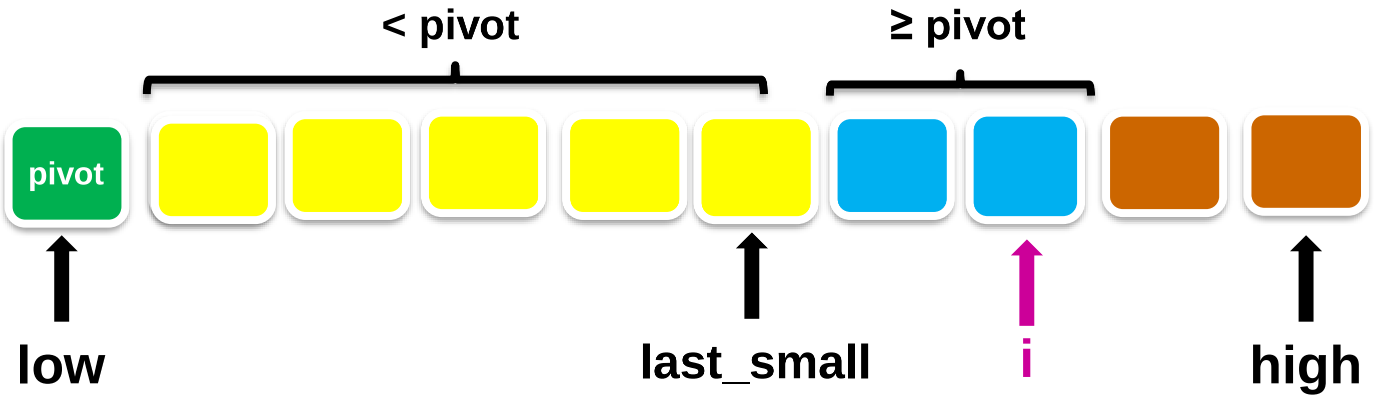
Partition Routine in Quicksort



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        for (i = low+1; i <= high; i++)
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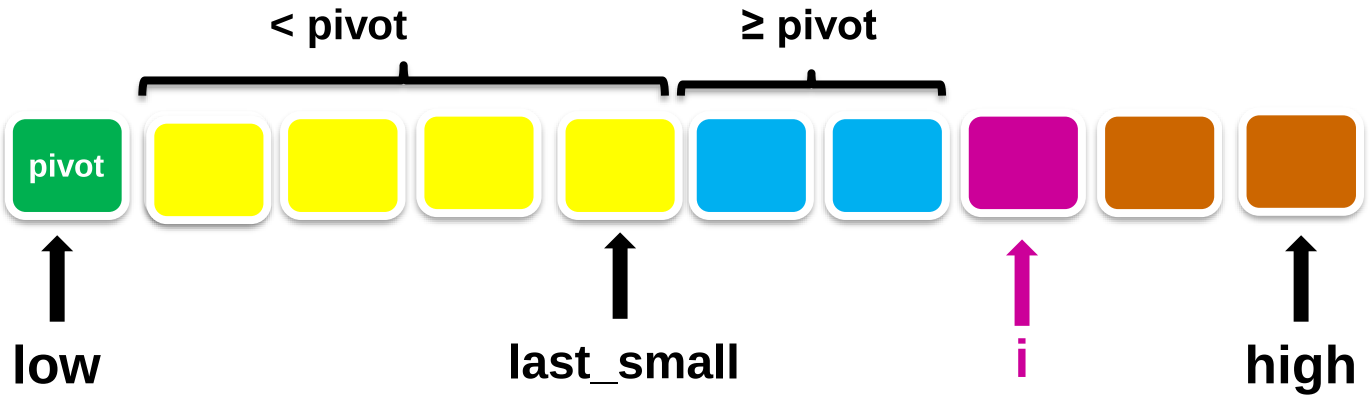

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        for (i = low+1; i <= high; i++)
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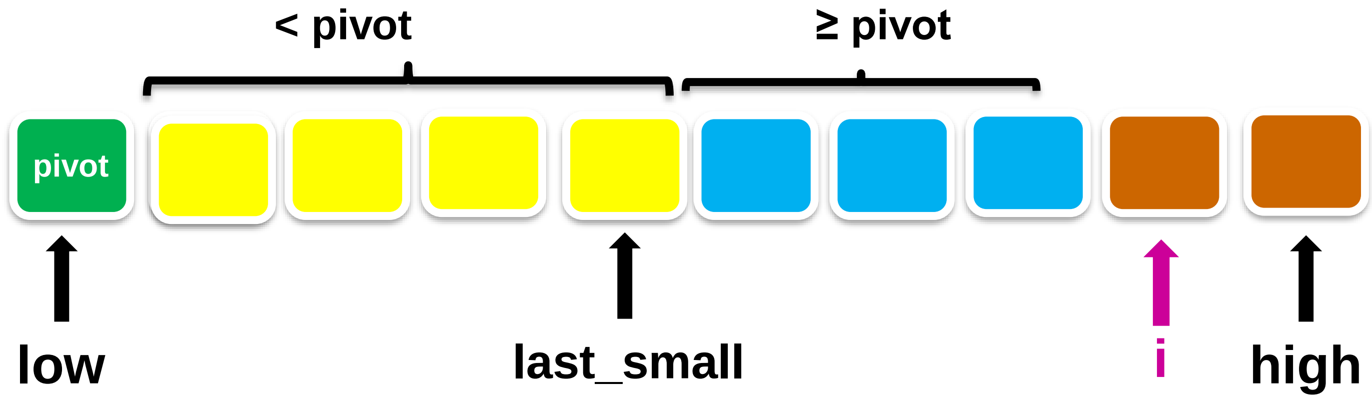

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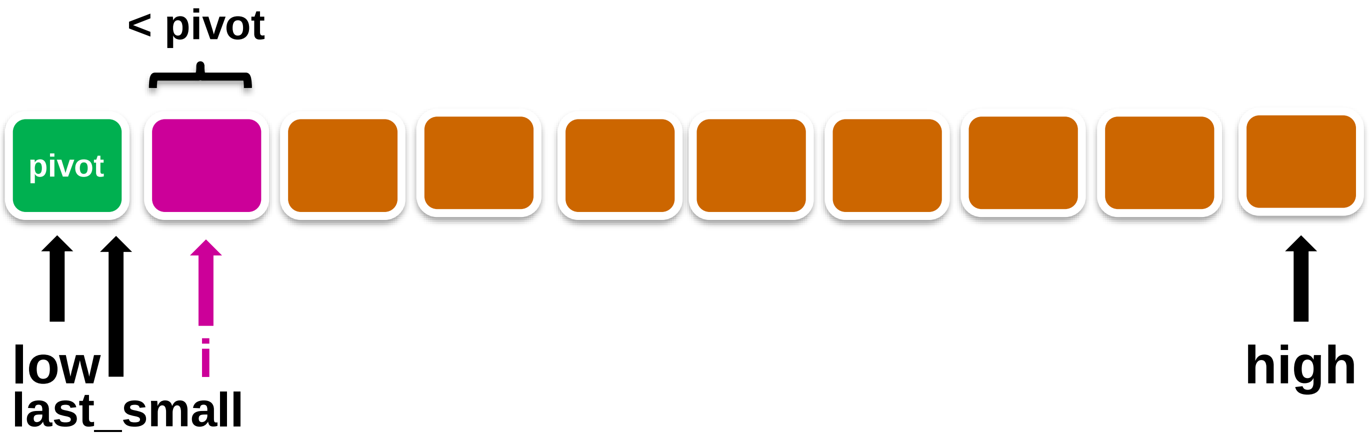

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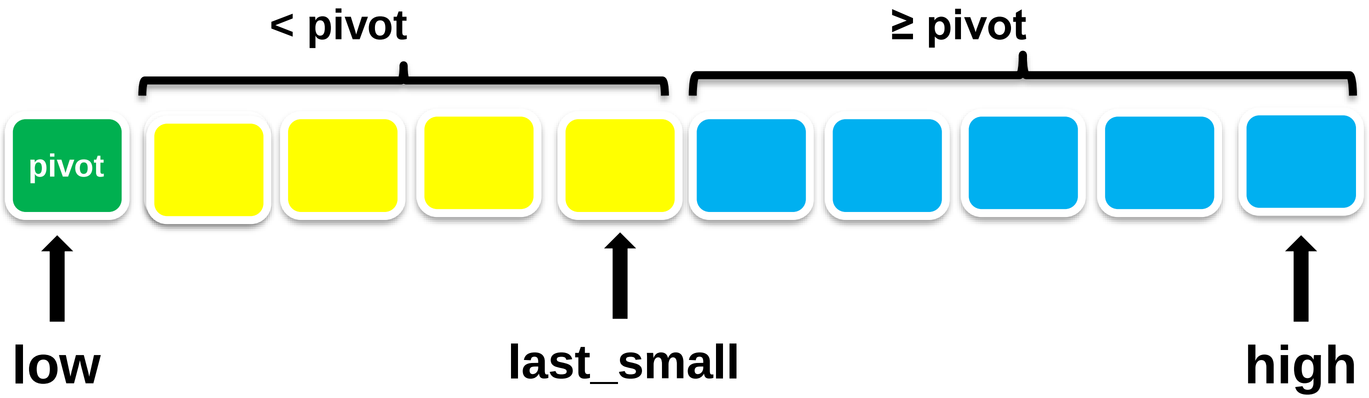
{…….
          for (i = low+1; i <= high; i++)
                    if (slot[i] < pivot)     ✓
                              swap(++last_small, i);
          swap(low, last_small);
          return last_small;
 }
```

Partition Routine in Quicksort



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{…….
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                if (slot[i] < pivot)
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        swap(low, last_small);
        return last_small;
 }
```

Note:

- Loop terminates when ***i*** reaches high;
- swap **pivot** from position low to position last_small, to obtain the final position of pivot element.

Quicksort (Example)

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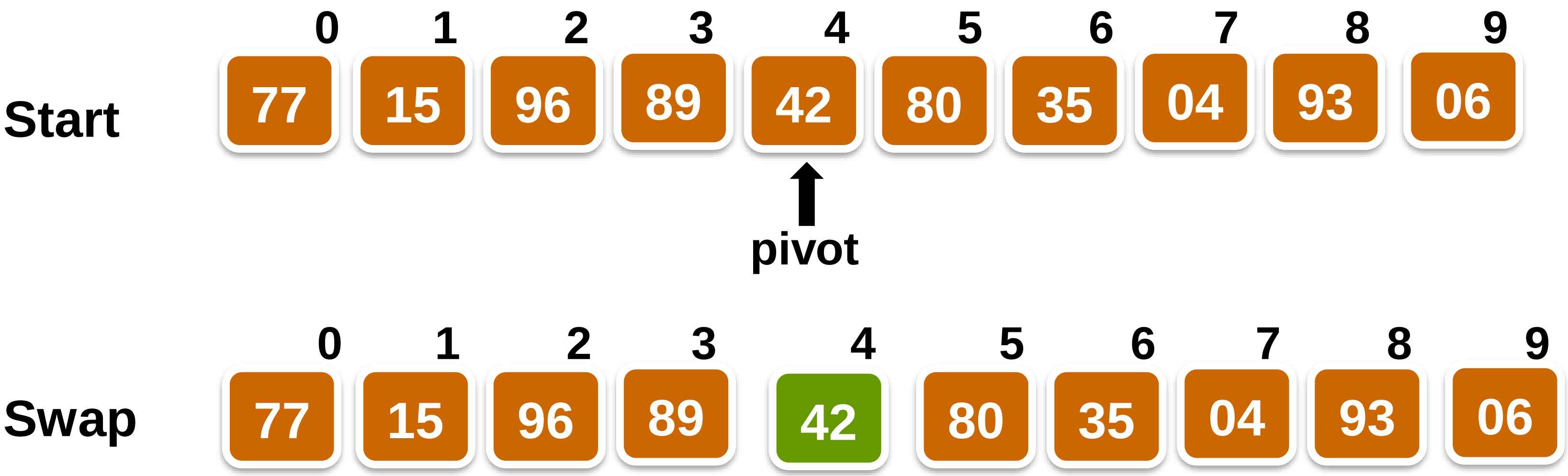


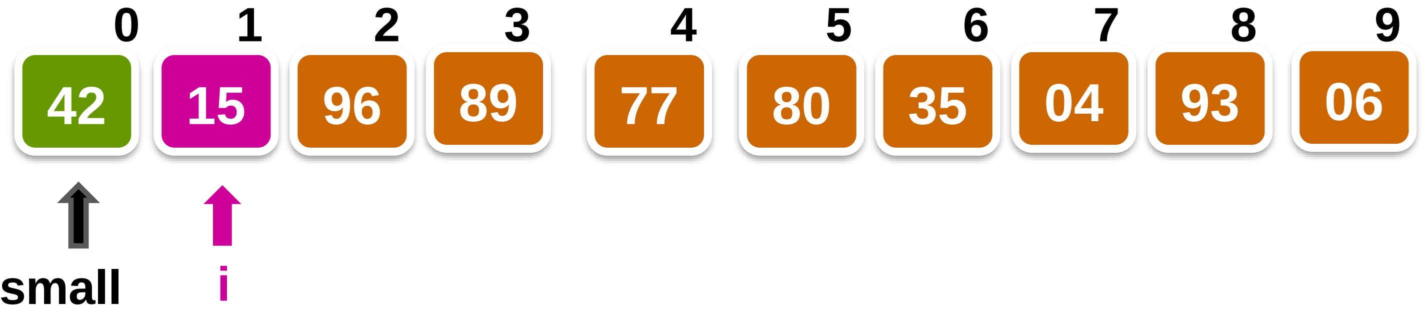
	0	1	2	3	4	5	6	7	8	9
Start	77	15	96	89	42	80	35	04	93	06

	0	1	2	3	4	5	6	7	8	9
Swap	77	15	96	89	42	80	35	04	93	06

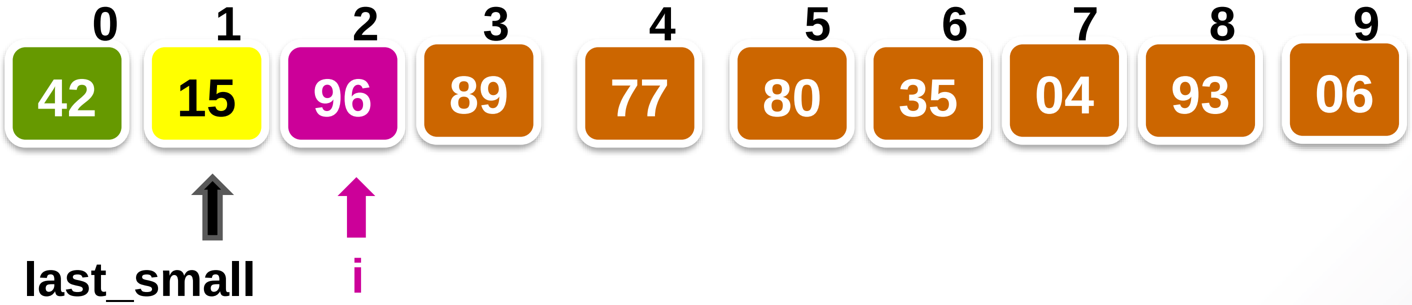
Partition the elements …

Quicksort (Example)

Partitioning…



0	1	2	3	4	5	6	7	8	9
42	15	96	89	77	80	35	04	93	06
last_small	i								

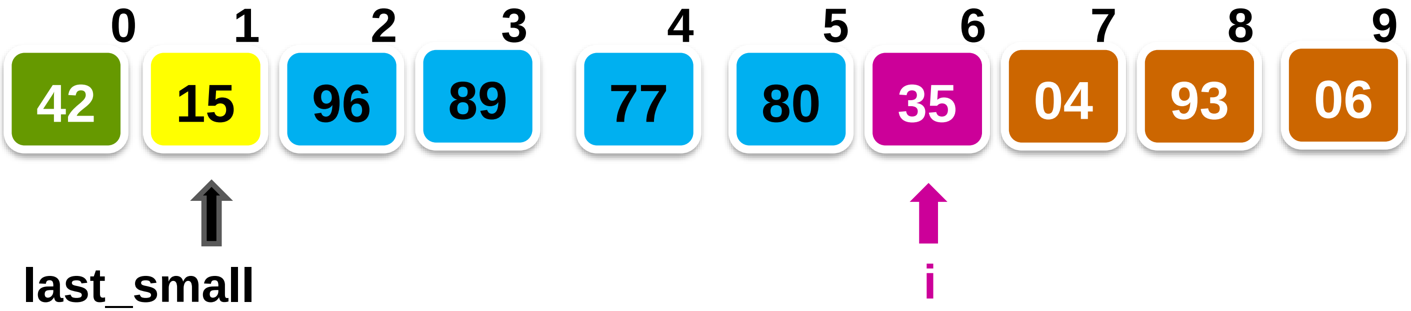


0	1	2	3	4	5	6	7	8	9
42	15	96	89	77	80	35	04	93	06
	last_small	i							

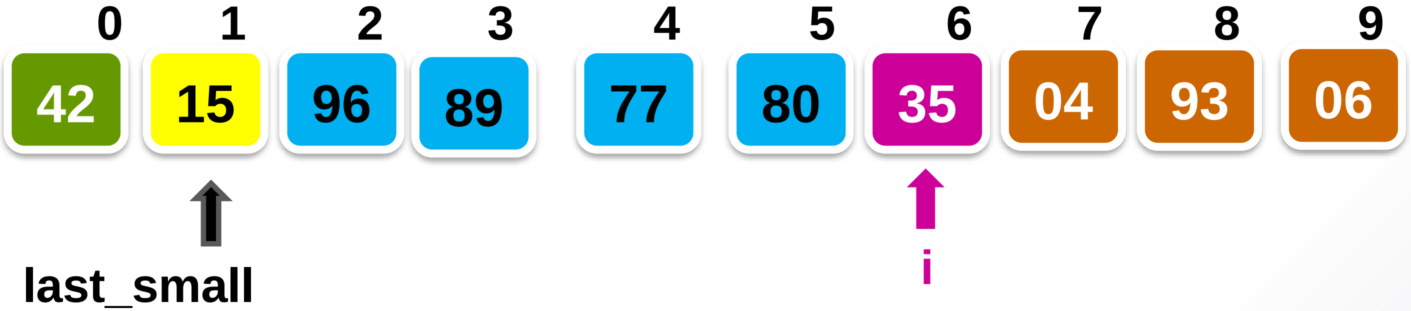
Quicksort (Example)

Partitioning…

Carry on checking if (item ≥ pivot) …



0	1	2	3	4	5	6	7	8	9
42	15	96	89	77	80	35	04	93	06

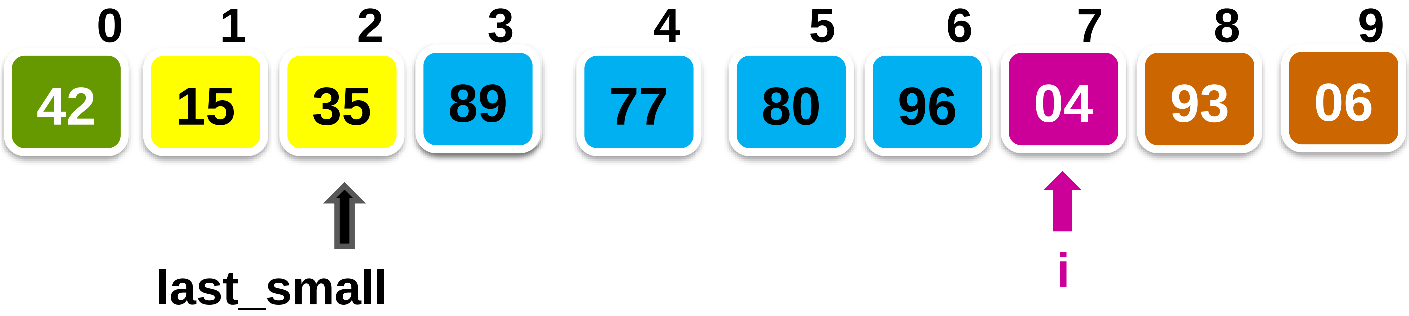


0	1	2	3	4	5	6	7	8	9
42	15	96	89	77	80	35	04	93	06

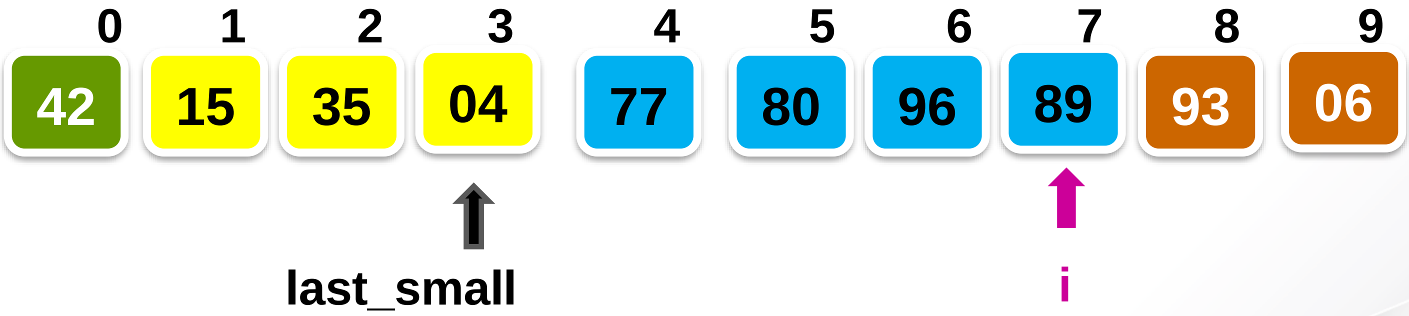
Quicksort (Example)

Partitioning…

Carry on checking if (item ≥ pivot) …



0	1	2	3	4	5	6	7	8	9
42	15	35	89	77	80	96	04	93	06

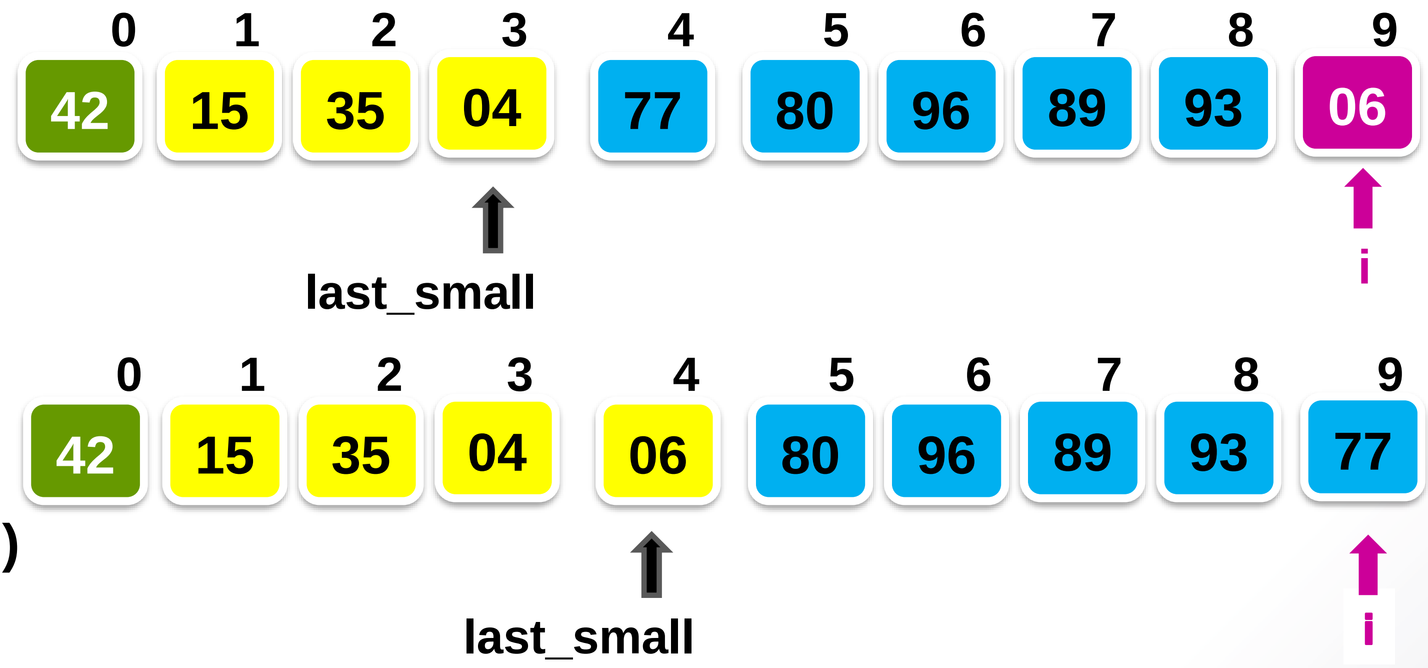


0	1	2	3	4	5	6	7	8	9
42	15	35	04	77	80	96	89	93	06

Quicksort (Example)

Partitioning…

Carry on checking if (item ≥ pivot) …



i ++

(93 >= 42)

0	1	2	3	4	5	6	7	8	9
42	15	35	04	77	80	96	89	93	06

Swapped

(06 and 77)

0	1	2	3	4	5	6	7	8	9
42	15	35	04	06	80	96	89	93	77

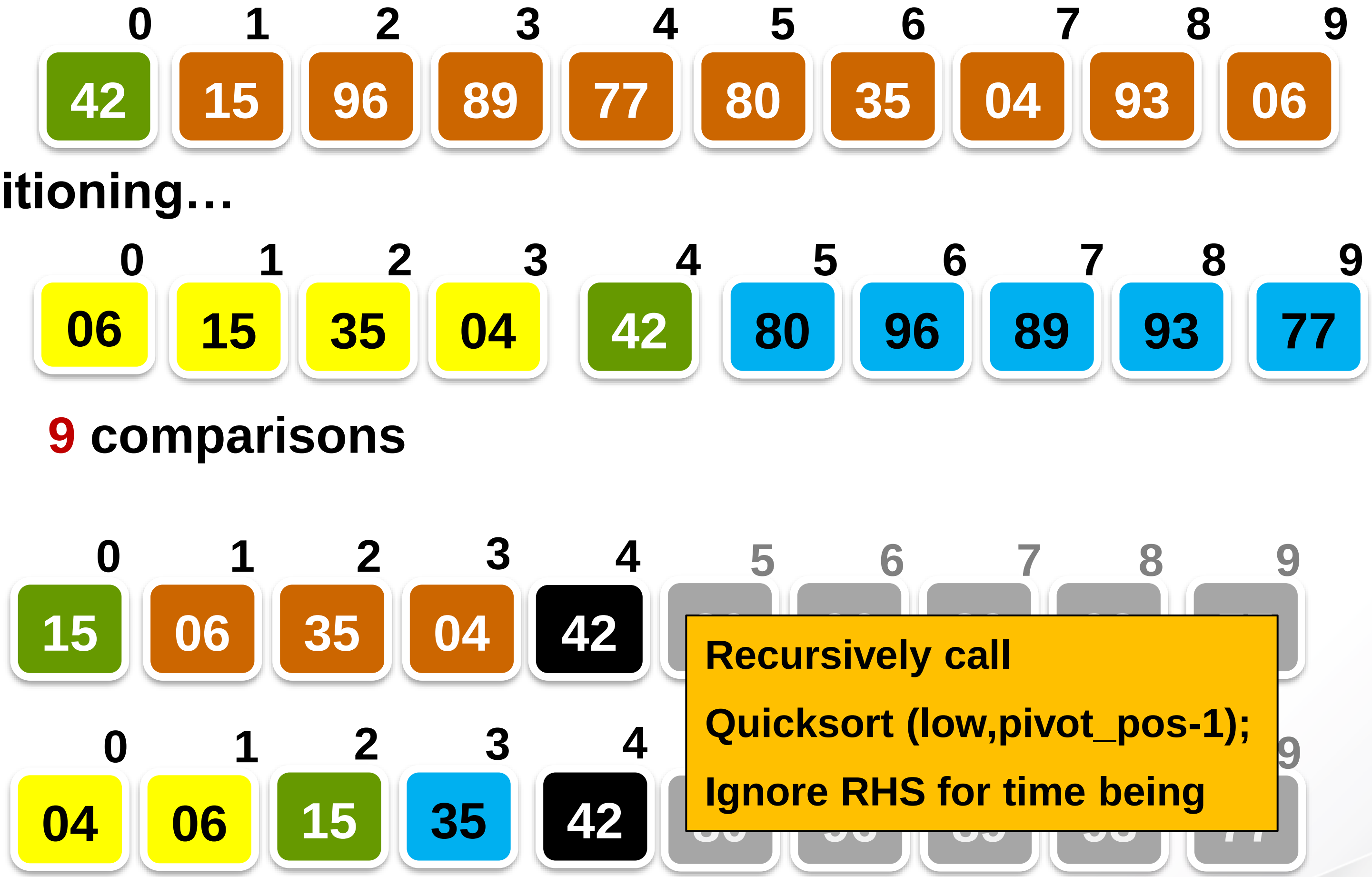
Quicksort (Example)

Finally, swap “last_small” (i.e. the final position where the pivot should be) with pivot.

0	1	2	3	4	5	6	7	8	9
42	15	35	04	06	80	96	89	93	77

We have done 9 comparisons in the partition.

Quicksort (Example)



Step 1:

0	1	2	3	4	5	6	7	8	9
42	15	96	89	77	80	35	04	93	06

After partitioning…

0	1	2	3	4	5	6	7	8	9
06	15	35	04	42	80	96	89	93	77

9 comparisons

Step 2:

Swap

0	1	2	3	4	5	6	7	8	9
15	06	35	04	42					

Insert

0	1	2	3	4					9
04	06	15	35	42					

Recursively call

Quicksort (low,pivot_pos-1);

Ignore RHS for time being

3 comparisons

Quicksort (Example)

Step 3:

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	80	96	89	93	77

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	80	96	89	93	77

1 comparison

Step 4:

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	80	96	89	93	77

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	80	96	89	93	77

0 comparison

Quicksort (Example)

Step 5:

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	80	96	89	93	77

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	80	96	89	93	77

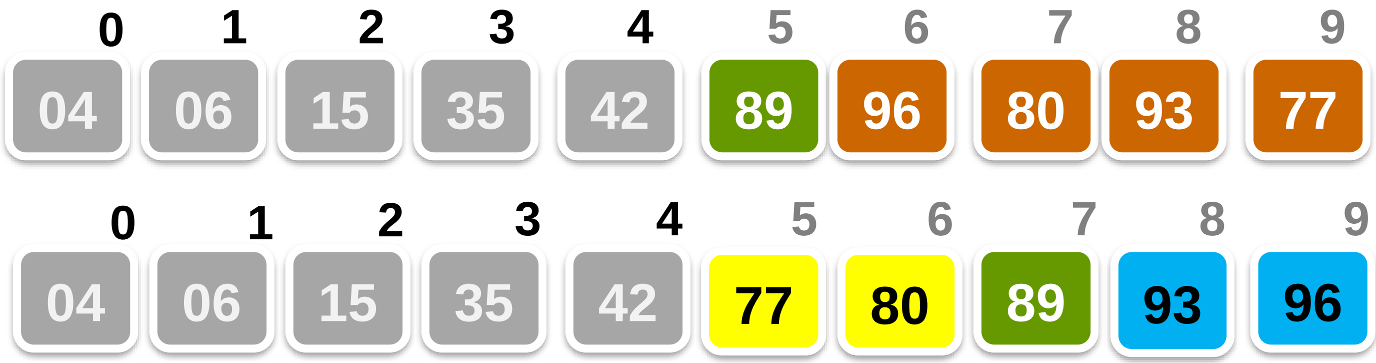
0 comparison

Sorting of LHS completed

Quicksort (Example)

Dealing with right half of the array:

Step 6:

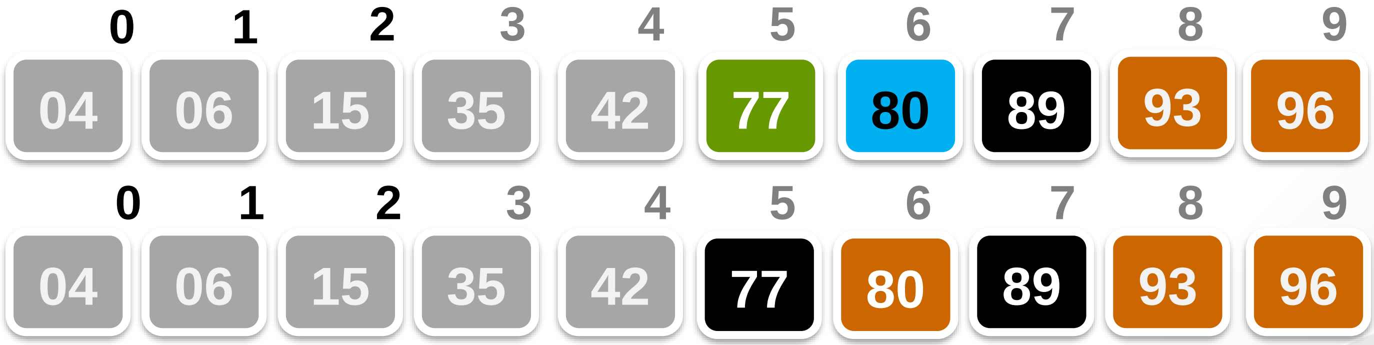


0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	89	96	80	93	77

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

4 comparisons

Step 7:



0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

1 comparison

Quicksort (Example)

Dealing with right half of the array:

Step 8:

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

0 comparison

Step 9:

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

1 comparison

Quicksort (Example)

Step 10:

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

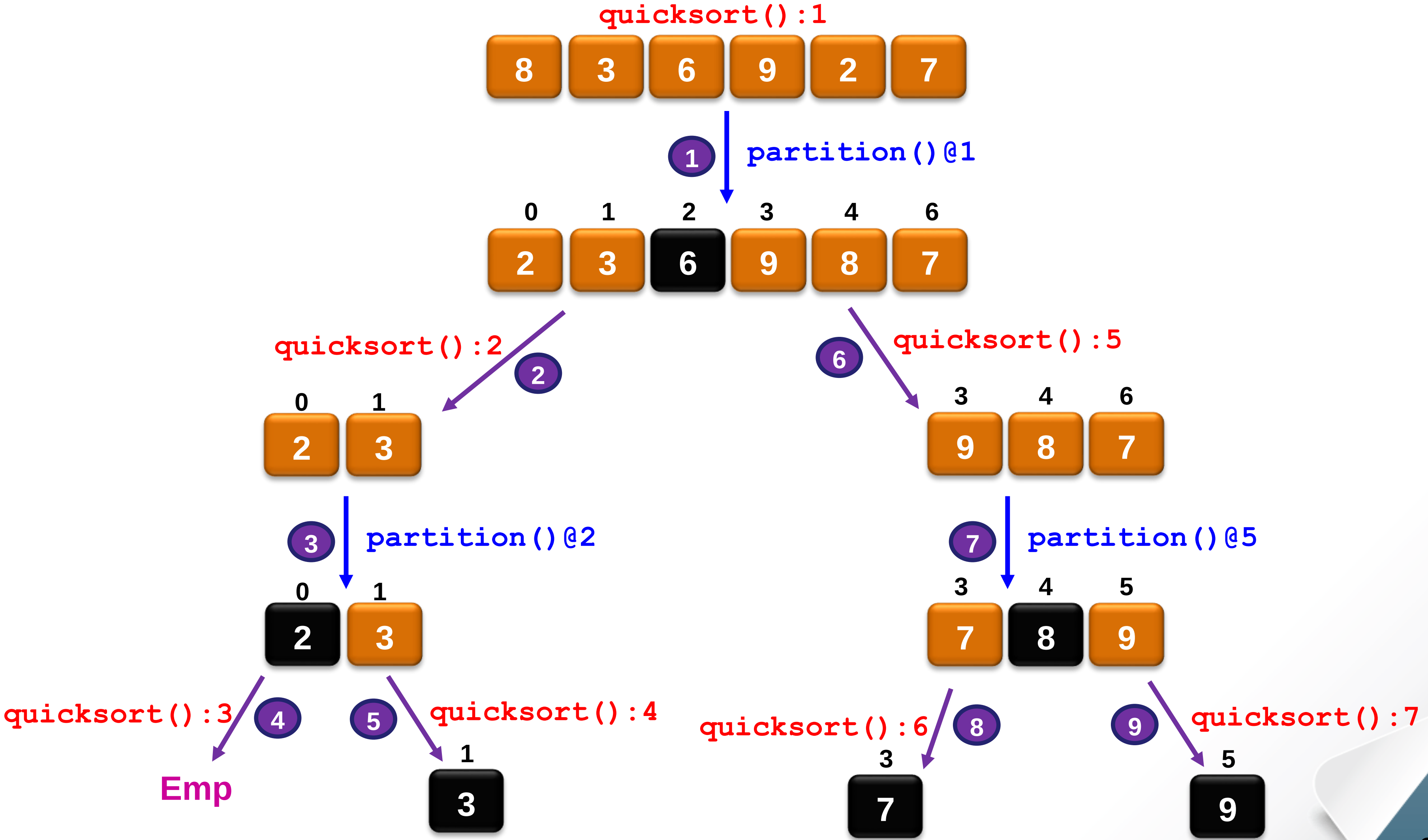
0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

0 comparison

Final outcome:

0	1	2	3	4	5	6	7	8	9
04	06	15	35	42	77	80	89	93	96

Execution Order of Quicksort



Comments on Quicksort

- **Which element of array should be pivot?** In this implementation, we take the middle element as pivot (other choices possible).

- Use quicksort(0, size − 1) to invoke quick sort; ‘size’ is the number of elements in array slot[ ].

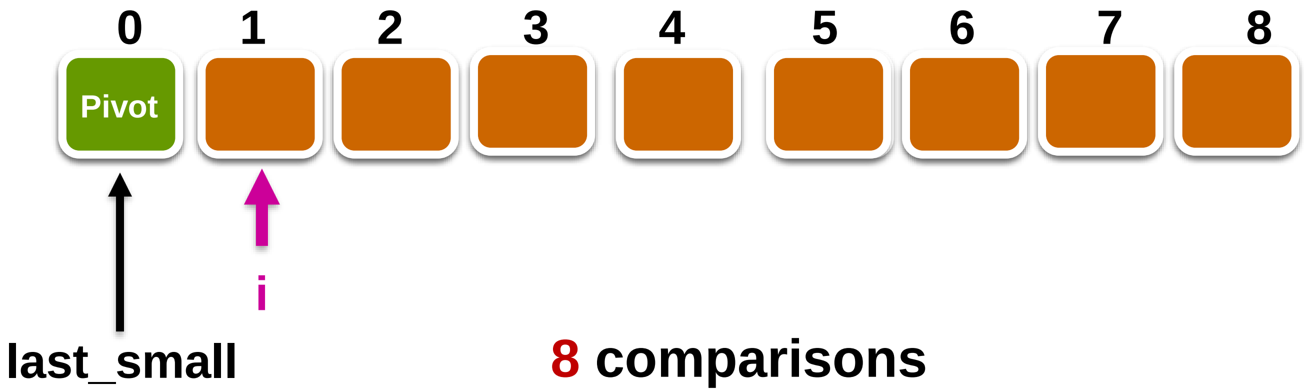
- During partitioning, the middle element (pivot) is moved to the 1st position (i.e. slot[0]).

- A ‘for’ loop goes through the rest of array to split it into two portions.

Quicksort’s Performance

Quicksort’s Performance

Worst-case



0 1 2 3 4 5 6 7 8

Pivot

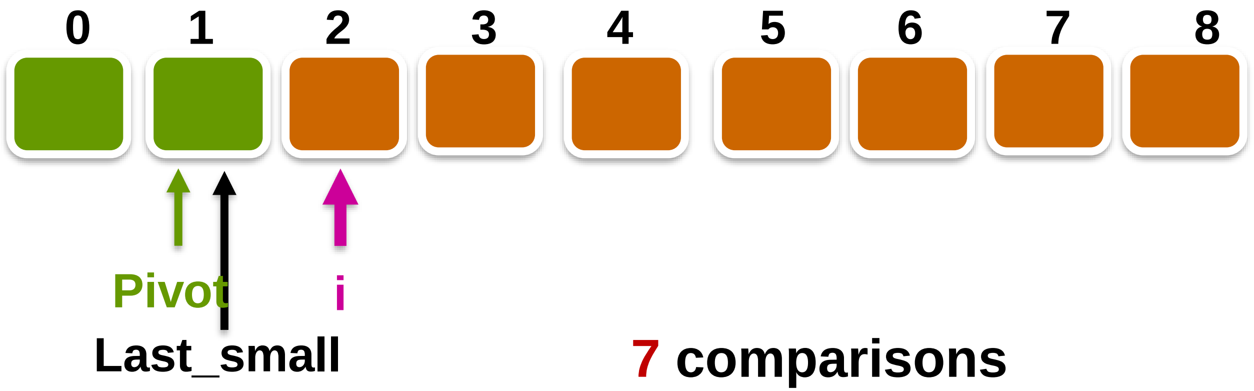
i

last_small

8 comparisons

Quicksort’s Performance

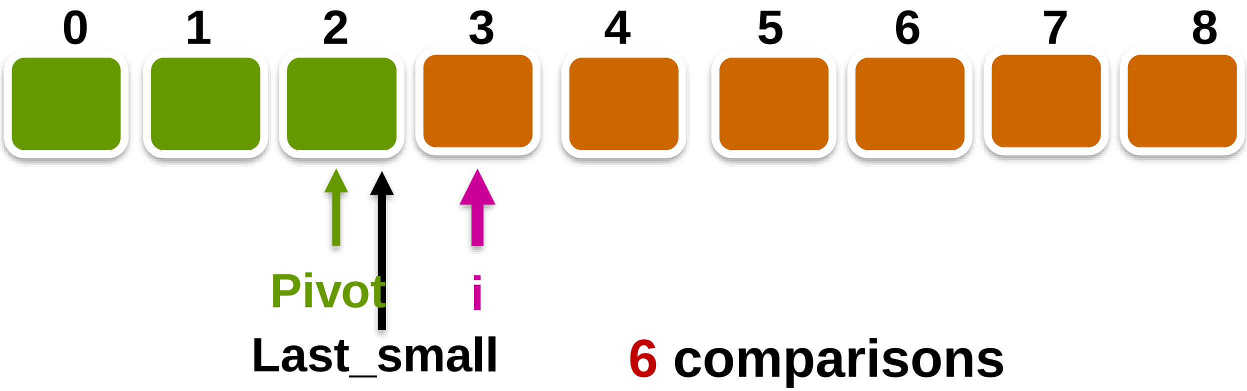
Worst-case



7 comparisons

Quicksort’s Performance

Worst-case



6 comparisons

Quicksort's Performance

Worst case happens when the pivot does a bad job at splitting the array evenly, if pivot is the smallest or the largest key each time, then the total no. of key comparisons is O(n^2).

$$\sum_{k=2}^{n}(k-1)=\sum_{k=1}^{n-1}k=\frac{n(n-1)}{2}$$

Quicksort’s Performance

Best case happens when the pivot happens to divide the array into two sub-arrays of equal length, in every partitioning.

For simplicity, let’s assume:

- $n = 2^k$, i.e. $k = \lg n$.
- Each step, the pivot divides the array of length n into two sub-arrays each of length approximately $n/2$.

Quicksort's Performance

The recurrence equation is:

$T(1) = 0,$

$T(n) = 2T(n/2) + cn$, where c is a constant

$T(n) = 2\,(2T(n/4) + cn/2) + cn$

$= 2^2T(n/4) + 2cn$

$= 2^3T(n/8) + 3cn$

$\cdots$

$= 2^k\,T(n/2^k) + kcn$

$= nT(1) + cn\lg n = cn\lg n$

Because $n = 2^k$, i.e. $k = \lg n$, and $T(1) = 0$

$$\therefore\ T(n) = \Theta(n\lg n)$$

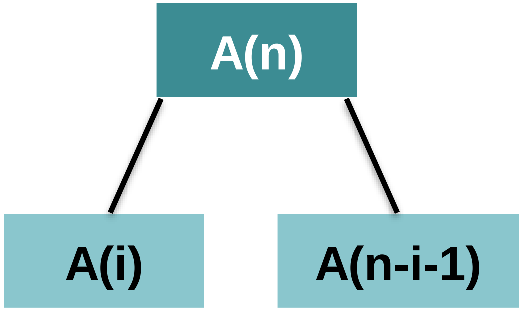
Quicksort’s Performance

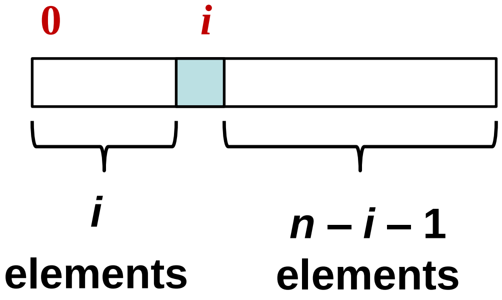
Average case: assume that the keys are distinct and that all permutations of the keys are equally likely.

k = no. of elements in the range of the array being sorted,

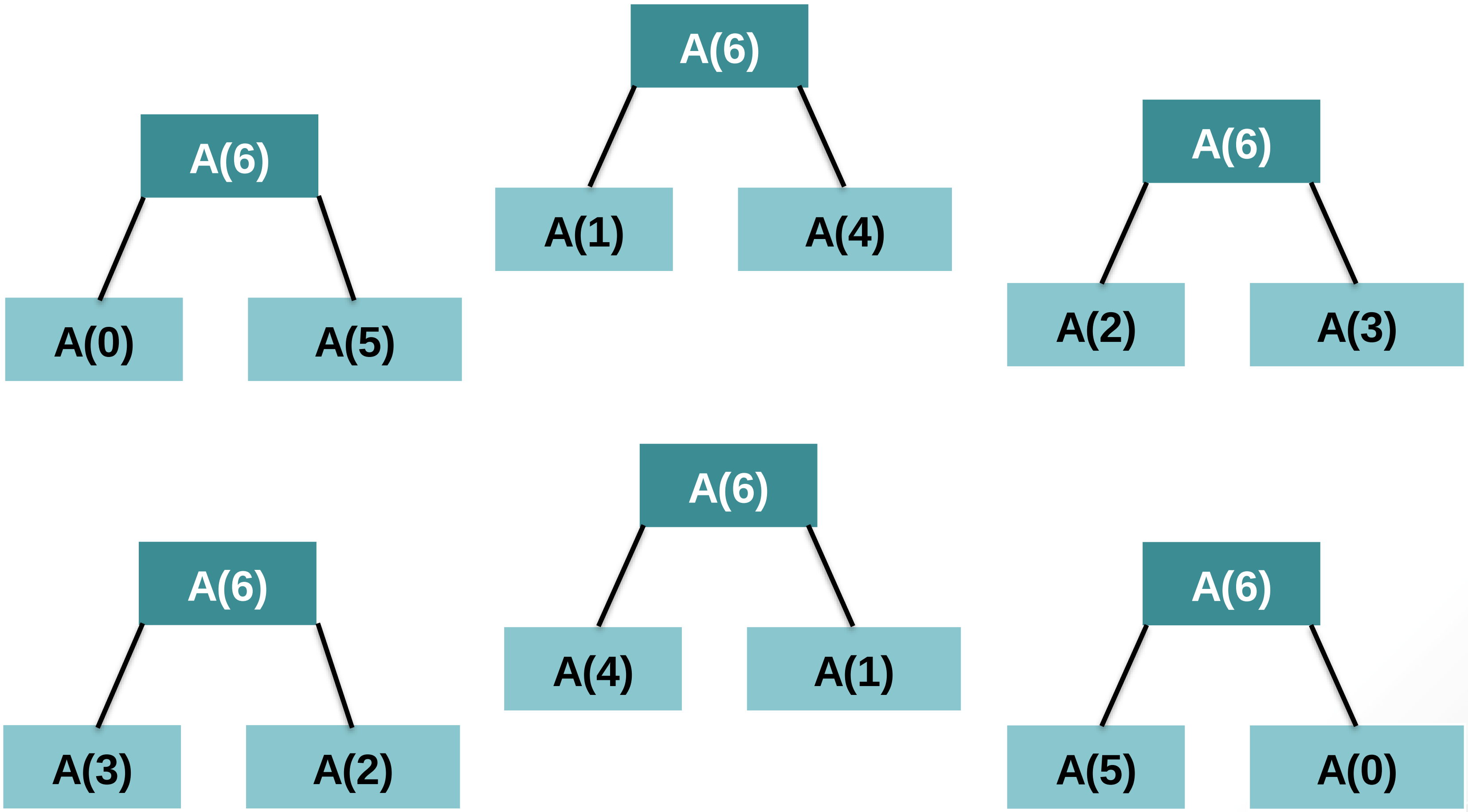
A(*k*) = no. of comparisons done for this range,

i = final position of the pivot, counting from 0,





Quicksort’s Performance



Quicksort’s Performance

Thus,

$A(6) = 5 + 1/6(\,\underline{A(0) + A(5)} + \underline{A(1) + A(4)} + \underline{A(2) + A(3)} + \ldots + \underline{A(5) + A(0)})$

$$A(0) = A(1) = 0$$

$$A(n) = n - 1 + \frac{1}{n}\sum_{i=0}^{n-1}\left[A(i) + A(n-i-1)\right] = \Theta\left(n \lg n\right)$$

Proof is not required

Proof in text-book (MIT book) by Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest.

Quicksort’s Performance

☺ Strengths:

- ☞ Fast on average
- ☞ No merging required
- ☞ Best case occurs when pivot always splits array into equal halves

☹ Weaknesses:

- ☞ Poor performance when pivot does not split the array evenly
- ☞ Quicksort also performs badly when the size of list to be sorted is small
- ☞ If more work is done to select pivot carefully, the bad effects can be reduced

Summary

- Quicksort uses the “**Divide and Conquer**” approach.
- Partition function splits an input list into two sub-lists by comparing all elements with the pivot:
  - Elements in the left sub-list are < pivot and
  - Elements in the right sub-list are ≥ pivot.
- Quicksort is called recursively on each sub-list.

- The worst-case time complexity of Quicksort is $\Theta(n^2)$.
- The best-case and average-case time complexities of Quicksort are both $\Theta(n\lg n)$.