SC2001/CX2101
Algorithm Design and Analysis

Tutorial 3
Analysis Techniques

(Week 9)

This tutorial helps you develop skills in the learning outcome of the course: "Able to conduct complexity analysis of recursive algorithms: solve recurrences using the substitution method, the iteration method, the master theorem, the characteristic equation."

Question 1 (1)

Solve the following recurrence by the iteration method

$T(1) = 1$, and for $n \geq 2$, $T(n) = 3T(n-1) + 2$

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$T(1) = 1$, and for $n \geq 2$, $T(n) = 3T(n-1) + 2$

$$
\begin{aligned}
T(n) &= 3T(n-1) + 2 \\
&= 3(3T(n-2) + 2) + 2 \\
&= 3^2T(n-2) + 3*2 + 2 \\
&= 3^2\,(3T(n-3) + 2) + 3*2 + 2 \\
&= 3^3T(n-3) + 3^2*2 + 3*2 + 2 \\
&= 3^{n-1}T(n-(n-1)) + 3^{(n-1)-1}*2 + \ldots + 3^2*2 + 3*2 + 2 \\
&= 3^{n-1}T(1) + 3^{n-2}*2 + \ldots + 3^2*2 + 3*2 + 2
\end{aligned}
$$

Question 1 (1)

$= 3^{n-1} + 2(3^{n-2} + \ldots + 3^2 + 3 + 1)$

$= 3^{n-1} + 2(3^{n-2} + \ldots + 3^2 + 3 + 1)$

$= 3^{n-1} + 2(3^{n-1} - 1)/(3-1)$

$= 3^{n-1} + 3^{n-1} - 1$

$\leq 2 * 3^{n-1}$ // drop -1

$\leq 3^n$ // change 2 to 3

So $T(n) = O(3^n)$

Question 1 (2)

Solve the following recurrence by the iteration method

$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

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$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

$$\mathrm{T}(n) = 2\mathrm{T}(\tfrac{n}{2}) + 6n$$
$$= 2(2\mathrm{T}(\tfrac{n}{2^2}) + 6(\tfrac{n}{2})) + 6n$$
$$= 2^2\mathrm{T}(\tfrac{n}{2^2}) + 6n + 6n$$
$$= 2^2\,(2\mathrm{T}(\tfrac{n}{2^3}) + 6(\tfrac{n}{2^2})) + 6n + 6n$$
$$= 2^3\,\mathrm{T}(\tfrac{n}{2^3}) + 6n + 6n + 6n \qquad \text{// three } 6n\text{s}$$
$$= 2^{\mathrm{k}}\mathrm{T}(1) + 6\mathrm{k}n \qquad \text{assume } n = 2^{\mathrm{k}} \quad \text{so } k = \lg n$$
$$= n + 6n\lg n$$
$$= \mathrm{O}(n\lg n)$$

Question 2 (1)

Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, $T(n) = 3T(n-1) + 2$

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Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, $T(n) = 3T(n-1) + 2$

Guess that $T(n) = O(3^n)$

Proof: We try to prove $T(n) \leq 3^n$ for $n \geq 1$

(a) Base case: $T(1) = 1 \leq 3^1$.

(b) Inductive step: assume that $T(k) \leq 3^k$, prove that $T(k+1) \leq 3^{k+1}$.

$$\begin{aligned} T(k+1) &= 3T(k) + 2 \\ &\leq 3*3^k + 2 \\ &\leq 3^{k+1} + 2 \\ &\cancel{\leq} 3^{k+1} \end{aligned}$$

Note that in the proof, the expressions (those in red) we try to prove have to be the same

we only managed to prove $T(K+1) \leq 3^{k+1} + 2$, not $T(K+1) \leq 3^{k+1}$. We need to get rid of the "+2"

Question 2 (1)

Proof: We will prove $T(n) \leq 3^n - 2$ for $n \geq 1$

(a) Base case: $T(1) = 1 \leq 3^1 - 2$.

(b) Inductive step: assume that $T(k) \leq 3^k - 2$, prove that $T(k+1) \leq 3^{k+1} - 2$.

$$
\begin{aligned}
T(k+1) &= 3T(k) + 2 \\
&\leq 3(3^k - 2) + 2 \\
&\leq 3^{k+1} - 6 + 2 \\
&\leq 3^{k+1} - 2
\end{aligned}
$$

Thus $T(n) = O(3^n - 2) = O(3^n)$

Note that in the proof, the expressions (those in red) we try to prove have to be the same

Question 2 (2)

Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

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Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

We use c=1 for simplicity

Guess $T(n) = O(n \lg n)$

We choose n=2 because when n=1, lgn = 0

Proof: we try to prove $T(n) \leq n \lg n$ for any $n \geq 2$

a) Base case:

$T(2) = 2*1+6*2 = 14$, $n \lg n = 2$,

So c=1 will not have $T(2) \leq 2\lg 2$
so we try c=8 then $T(2) \leq 8*2\lg 2$

so we try to prove $T(n) \leq 8\, n \lg n$ for any $n \geq 2$

Question 2 (2)

Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

Guess $T(n) = O(n\lg n)$

Proof: we show that $T(n) \leq 8\, n\lg n$ for any $n \geq 2$

a) Base case:

$T(2) = 2*1+6*2 = 14$, $8n\lg n = 16$,

so $T(n) \leq 8\, n\lg n$ for $n=2$

Question 2 (2)

b) Inductive step: Assume that $T(2^k) \le 8\,k\,2^k$

Prove $T(2^{k+1}) \le 8\,(k+1)\,2^{k+1}$

$$T(2^{k+1}) = 2\,T(2^k) + 6 * 2^{k+1}$$
$$\le 2 * 8\,k\,2^k + 6 * 2^{k+1}$$
$$= (8\,k + 6) * 2^{k+1}$$
$$\le 8\,(k + 1) * 2^{k+1}$$

$T(n) = O(n \lg n)$

Question 3(1)

Solve the following recurrence by the master method.

$$W(n) = W(n/3) + 5$$

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Solve the following recurrence by the master method.

$$W(n) = W(n/3) + 5$$

$n^{\log_b a} = n^{\log_3 1} = n^0$, $f(n) = 5 = \theta(1) = \theta(n^0)$

So,

$W(n) = \theta(n^0 \lg n)$

Question 3(2)

Solve the following recurrence by the master method.

$$T(n) = 2T(n/2) + n/4$$

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Solve the following recurrence by the master method.

$T(n) = 2T(n/2) + n/4$

$n^{\log_b a} = n^{\log_2 2} = n^1$, $f(n) = n/4 = \theta(n) = \theta(n^1)$

So,

$W(n) = \theta(n\lg n)$

Question 3(3)

Solve the following recurrence by the master method.

$$W(n) = 2W(n/4) + \sqrt{n^3}$$

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Solve the following recurrence by the master method.

$$W(n) = 2W(n/4) + \sqrt{n^3}$$

$n^{\log_b a} = n^{\log_4 2} = n^{0.5}$, $f(n) = \sqrt{n^3} = n^{3/2} = \Omega(n^{0.5 + 0.1})$

And $a*f(n/b) = 2f(n/4) = 2*(n/4)^{3/2} = 2*n^{3/2}(1/4)^{3/2}$

$= (1/4)*n^{3/2} \leq (1/4)*n^{3/2} = c*f(n)$ $\quad$ $c = 1/4$

So,

$W(n) = \theta(n^{1.5})$

Question 4

Determine which of the following are linear homogeneous recurrence relations with constant coefficients. Also find the degree of those that are.

1) $a_n = 4a_{n-2} + 5a_{n-3}$ — Degree 3
2) $a_n = 2na_{n-1} + a_{n-2}$ — Not constant coefficient
3) $a_n = a_{n-1} + a_{n-4}$ — Degree 4
4) $a_n = a^2_{n-1} + a_{n-2}$ — Not linear
5) $a_n = a_{n-2} + n$ — Not homogeneous

Question 5 (no need to cover all parts if running out of time)

(1) Solve the following recurrence relation together with the initial conditions given.

$a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

Question 5(1)

Solve the following recurrence relation together with the initial conditions given.

$a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

The characteristic equation:

$$t^2 - 7t + 10 = 0 \Rightarrow (t - 2)(t - 5) = 0$$

Solution: two distinct roots: $t_1 = 2$, $t_2 = 5$

Thus $a_n = 2^n C + 5^n D$

Question 5(1)

Substitute the initial conditions into $a_n = 2^n C + 5^n D$ to find C and D:

$a_0 = 1 = C + D$,   => 2 = 2C + 2D

$a_1 = 0 = 2C + 5D$

Thus 2 = -3D, i.e. D = -2/3 then C = 5/3

So we have $a_n = (5/3)*2^n – (2/3)*5^n$

Question 5(2)

Solve the following recurrence relation together with the initial conditions given.

$a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

Question 5(2)

Solve the following recurrence relation together with the initial conditions given.

$a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

The characteristic equation:

$t^2 - 4 = 0 \quad \Rightarrow (t)^2 = 4$

Solution: two distinct roots: $t_1 = 2$, $t_2 = -2$

Thus $a_n = 2^n C + (-2)^n D$

Question 5(2)

Substitute the initial conditions into $a_n = 2^n C + (-2)^n D$ to find C and D:

$a_0 = 6 = C + D$, => $12 = 2C + 2D$

$a_1 = 8 = 2C - 2D$

Thus 20 = 4C, i.e. C = 5 then D = 1

So we have $a_n = 5 * 2^n + (-2)^n$

Question 5(3)

Solve the following recurrence relations together with the initial conditions given.

$a_n = 2a_{n-1} - a_{n-2}$ for all $n \geq 2$, $a_0 = 1$, $a_1 = 3$

Question 5(3)

Solve the following recurrence relations together with the initial conditions given.

$a_n = 2a_{n-1} - a_{n-2}$ for all $n \geq 2$, $a_0 = 1$, $a_1 = 3$

The characteristic equation:

$$t^2 - 2t + 1 = 0 \quad \Rightarrow (t-1)^2 = 0$$

Solution: single root: $t = 1$,

Thus $a_n = C + nD$, we have

$1 = C$, $D = 2$

$a_n = 1 + 2n$