NANYANG TECHNOLOGICAL UNIVERSITY

# SC2001/ CX2101: Algorithm Design and Analysis

## Part 2

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# Topics

- Analysis Techniques (3 hours)
- Dynamic Programming (5 hours)
- String Matching (3 hours)
- Introduction to NP Completeness (2 hours)

# Lecture Delivery Method

1. Recorded lectures in **Course Media(Media Gallery) /Home**. Videos of one chapter in one PlayList.
2. Weekly review lectures/Q & As in **Zoom** on **Mondays 1.30pm – 2.30pm**, Week 8 to Week 13. No lecture on Fridays unless notified otherwise.

# Schedule

| Week | Lecture materials to be studied by end of the week | Tutorials | Example classes |
|---|---|---|---|
| 7 | Analysis techniques (up to slide 38) | Graphs | Project 1 |
| 8 | Analysis techniques (up to end), DP (up to DP slide 18) | Graphs | Project 2 |
| 9 | DP (up to DP slide 40) | Analysis techniques | Project 2 |
| 10 | DP (up to end) | DP | Quiz |
| 11 | String matching (up to end) | DP | Quiz |
| 12 | NP completeness (up to end) | String matching | Project 3 |
| 13 | | NP completeness | Project 3 |

Review lectures are from Week 8 to Week 13

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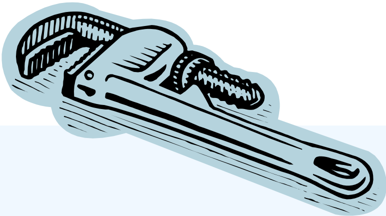


# Analysis Techniques

Huang Shell Ying

**Reference:** **Computer Algorithms: Introduction to Design and Analysis, 3rd Ed, by Sara Basse and Allen Van Gelder.**

# Outline

- Review of the big oh, big omega, big theta

- Solving recurrences (1)
  1. The substitution method
  2. The iteration method
  3. The master method.

- Solving recurrences (2)
  1. Solving linear homogeneous recurrences with constant coefficients

# Review of the big oh, big omega, big theta

## The Big-oh notation:

Definition: Let f and g be 2 functions such that

$f(n) : N \rightarrow R^+$ and $g(n) : N \rightarrow R^+$, if there exists positive constants c and $n_0$ such that

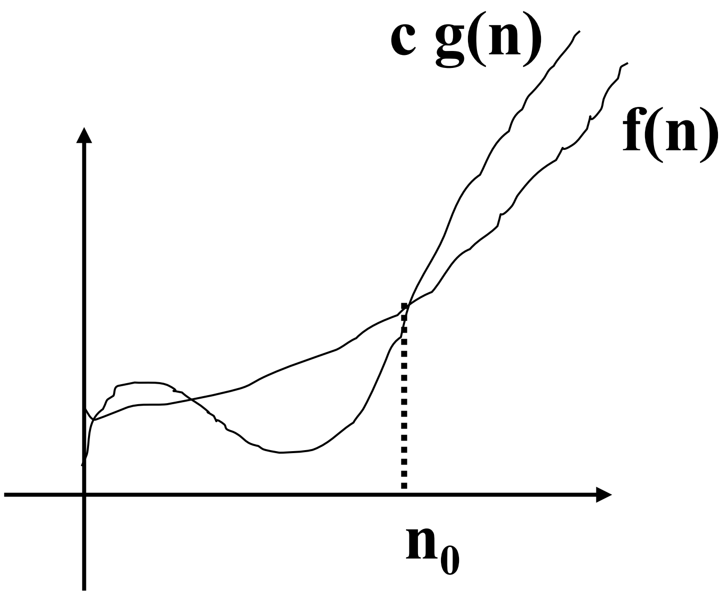
$f(n) \le c * g(n)$ for all $n > n_0$

then $f(n) = O(g(n))$.

Alternative definition: if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c < \infty$$

then $f(n) = O(g(n))$.



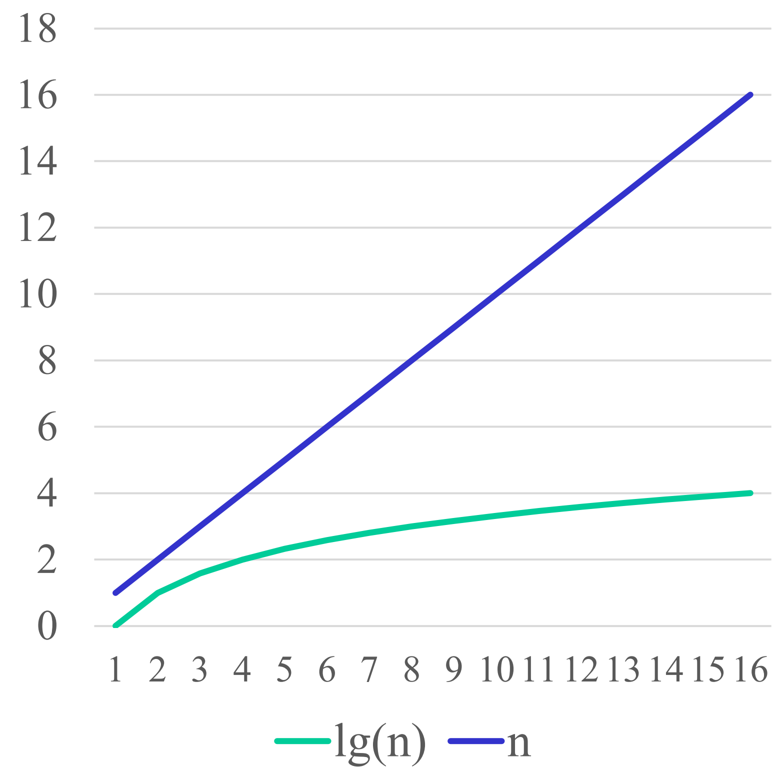
# Review of the big oh, big omega, big theta

Example: f(n)=lg(n), g(n)= n,

Let c=1, $n_0$ =1, then for all n>1

lg(n) <= n, i.e., f(n) <= g(n)

so f(n) = O(g(n)).




Another way: Since

$$\lim_{n\to\infty}\frac{f(x)}{g(x)} = \lim_{n\to\infty}\frac{\lg(n)}{n} = 0 < \infty$$

so f(n) = O(g(n)).

g(n) gives the asymptotic upper bound for f(n).

# Review of the big oh, big omega, big theta

## The big Omega notation

Definition: Let f and g be 2 functions such that

f(n) : N -> $R^+$ and g(n) : N -> $R^+$, if there exists positive

constants c and $n_0$ such that

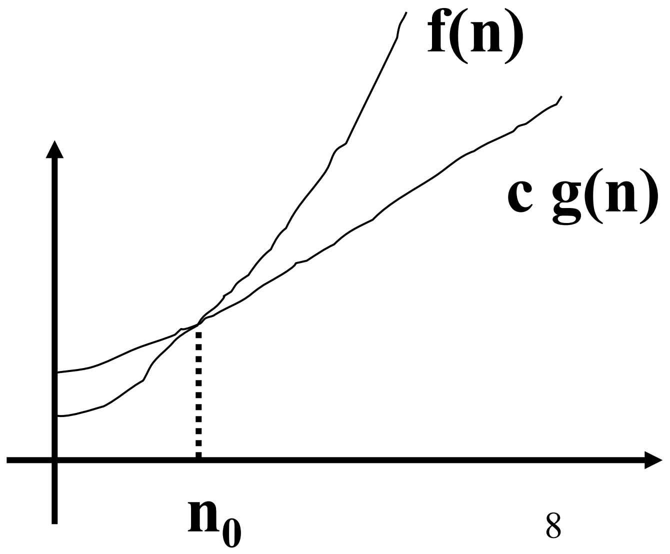
$f(n) >= c * g(n)$ for all $n > n_0$

then $f(n) = \Omega(g(n))$.

Alternative definition: if

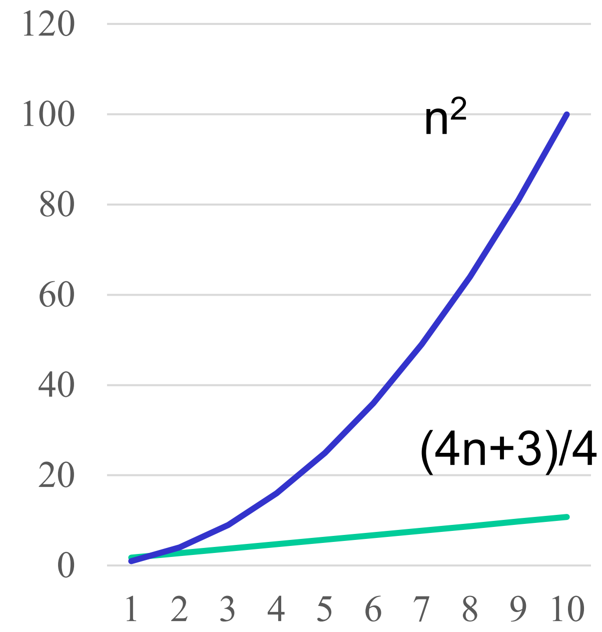
$$\lim_{n \to \infty} \frac{f(n)}{g(n)} > 0$$

then $f(n) = \Omega(g(n))$.



# Review of the big oh, big omega, big theta

Example: $f(n)= n^2$, $g(n)=4n+3$




Let $c=1/4$, $n_0 =1$, then for all $n>1$

$n^2 >=(4n+3)/4$ i.e., $f(n) >=(1/4)g(n)$

so $f(n) = \Omega(g(n))$.

Another way: Since

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^2}{4n+3} = \lim_{n\to\infty}\frac{n}{4+\frac{3}{n}} = \infty > 0$$

so $f(n) = \Omega(g(n))$.

g(n) gives the asymptotic lower bound for f(n).

# Review of the big oh, big omega, big theta

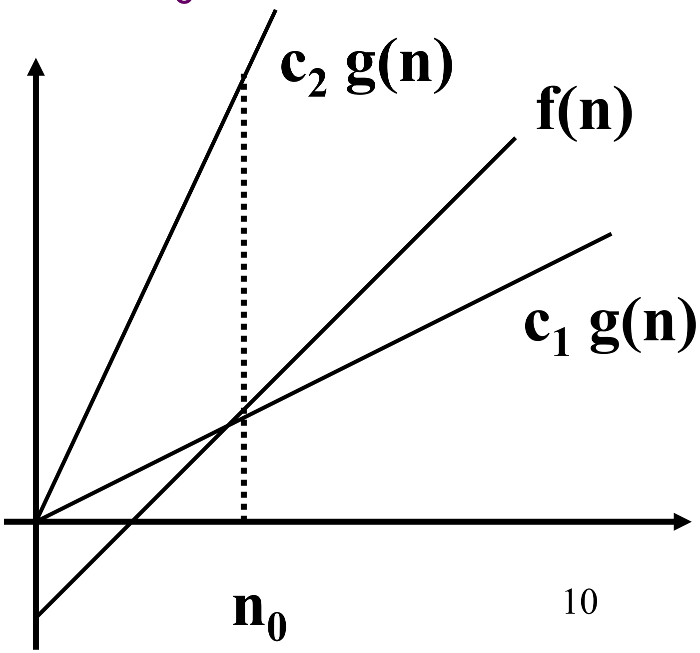
## The big Theta notation

Definition: Let f and g be 2 functions such that

$f(n) : N \rightarrow R^+$ and $g(n) : N \rightarrow R^+$, if there exists positive constants $c_1$, $c_2$ and $n_0$ such that

$c_1 * g(n) <= f(n) <= c_2 * g(n)$ for all $n > n_0$

then $f(n) = \theta(g(n))$.




Alternative definition: if

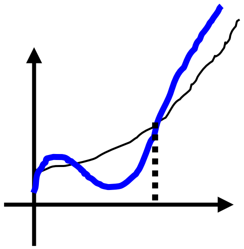
$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c \quad (0 < c < \infty)$$

then $f(n) = \theta(g(n))$.

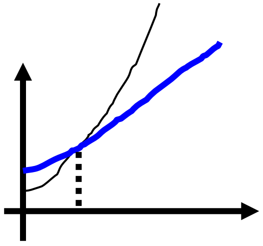
# Review of the big oh, big omega, big theta

- The idea of the O, $\Omega$ and $\theta$ definitions is to establish a relative order among functions.
- We compare the **relative rates of growth**.

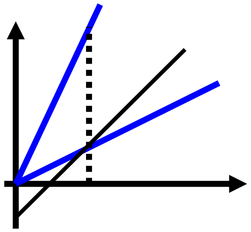
  - If f(n) = O(g(n)), g(n) gives the asymptotic upper bound



  - If f(n) = $\Omega$(g(n)), g(n) gives the asymptotic lower bound



  - If f(n) = $\theta$(g(n)), g(n) gives the asymptotic tight bound



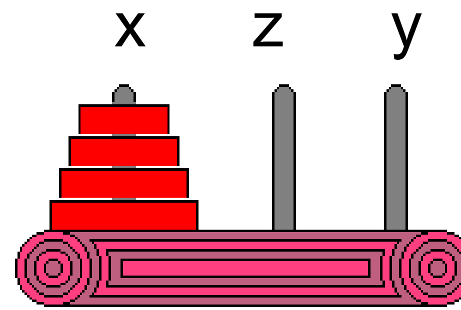
# Recursive algorithms and Recurrence relations

- Many problems have a recursive solution
- A common way of analysis for such solution algorithms will involve a recurrence relation that needs to be solved
- A recurrence is an equation or inequality that describe a function in terms of its value on smaller inputs, e.g.

$$M(n) = 2M(n-1) + 1$$

Example 1: Towers of Hanoi

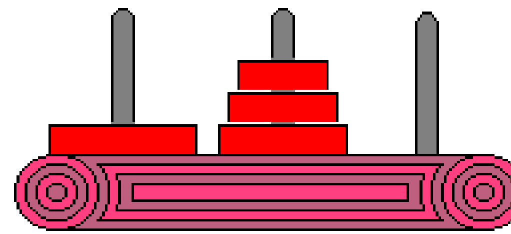
Move all disks from the first pole to the third pole subject to the condition that only one disk can be moved at a time and that no disk is ever placed on top of a smaller one.

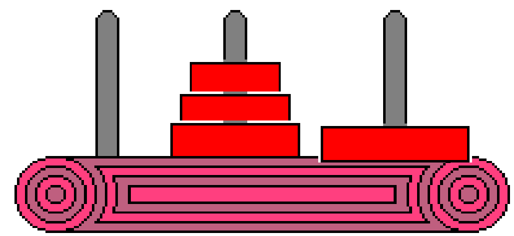


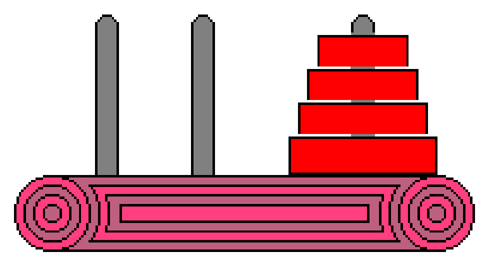

```
void TowersOfHanoi(int n, int x, int y, int z)
{ // Let M(n) be the total no. of disk moves
  if (n == 1)
      cout << "Move  disk from  " << x << " to  " << y << endl;
      // this has one disk move
  else {
     TowersOfHanoi(n-1, x, z, y);
     // this involves M(n-1) disk moves

     cout << "Move  disk from  " << x << " to  " << y << endl;
    // one disk move


     TowersOfHanoi(n-1, z, y, x);
     // another M(n-1) disk moves
  }
}
```







**The number of disk moves:** $M(1) = 1;$

$$M(n) = 2M(n-1) + 1$$

Example 2: Merge sort

```
void mergesort(int l, int m)
{
   int mid = (l+m)/2;
   if (m-l > 1) {
      mergesort(l, mid);
      mergesort(mid+1, m);
   }
   merge(l, m);
}
```

Let M(n) be the total no. of comparisons between array elements. n is a power of 2.

$M(2) = 1;$

$$M(n) = 2M(n/2) + n - 1$$

# Solving recurrences (1)

- We want to solve recurrences of the form

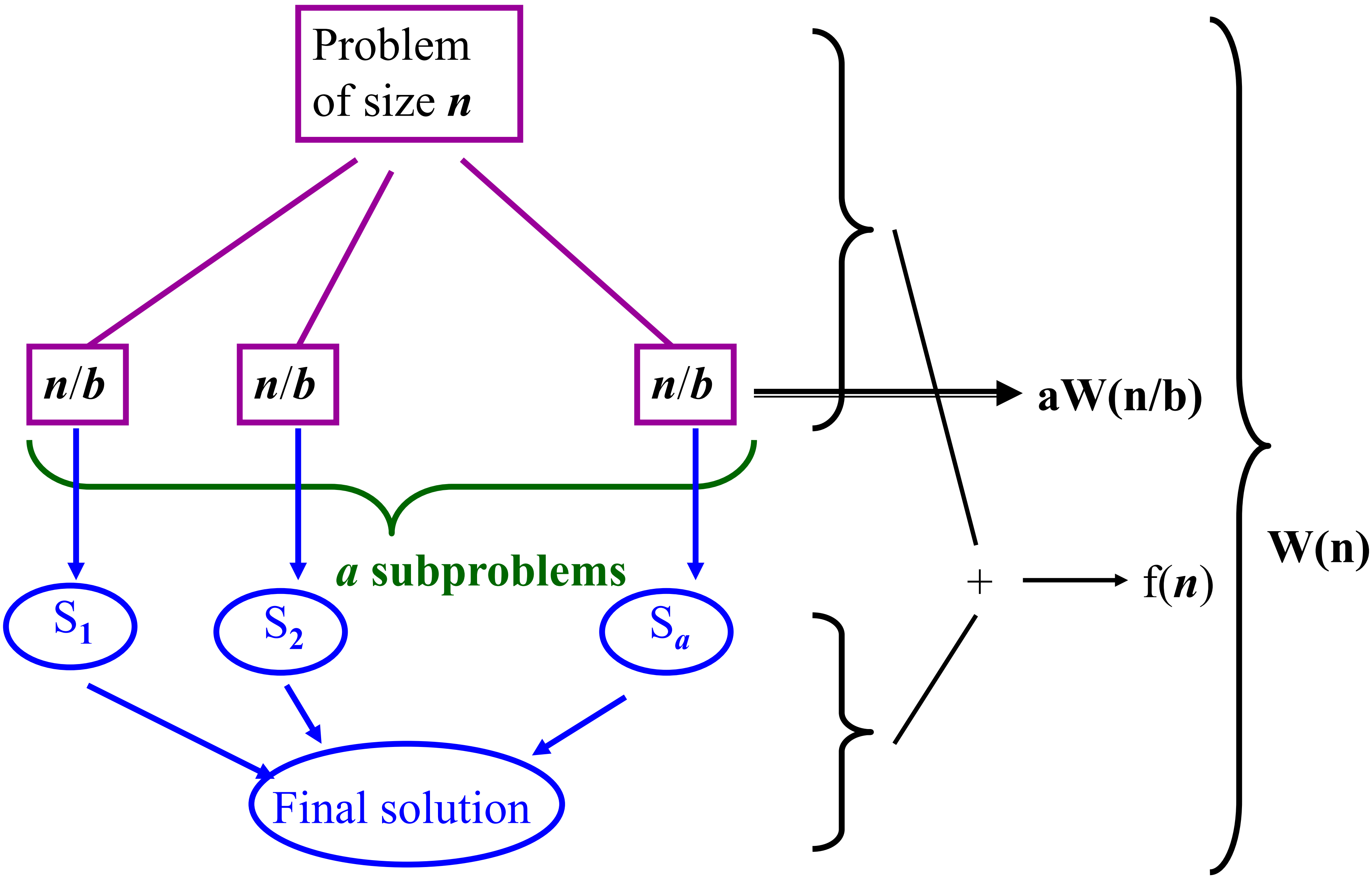
  $$\mathbf{W(\mathit{n}) = \mathit{a}W(\mathit{n/b}) + f(\mathit{n})}$$

  where $a \geq 1$ and $b > 1$ are constants, f($n$) is a function of $n$.

- The recurrence describes the computational cost of an algorithm that uses the "divide-and-conquer" approach.

- f($n$) is the cost of dividing the problem and combining the results of the subproblems.

- Usually the problem of size $n$ is divided into subproblems of sizes either $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$. However it does not change the asymptotic behaviour of the recurrence.


Problem
of size n
n/b
n/b
n/b
a subproblems
S1
S2
Sa
Final solution
aW(n/b)
+
f(n)
W(n)

# Solving recurrences (1)

- Examples

| | |
|---|---|
| **$W(n) = 2W(n/2) + 2$** | Finding the max and min from a sequence |
| **$W(n) = W(n/2) + 2$** | Binary search |
| **$W(n) = 3W(n/2) + cn$** | Multiplying two 2n-bits integers |
| **$W(n) = 2W(n/2) + n - 1$** | Merge sort |
| **$W(n) = 7W(n/2) + 15n^2/4$** | Multiplying two nxn matrices |

# Solving recurrences (1)

We describe three methods:

1) The substitution method
2) The iteration method
3) The master method.

## 1. The substitution method

- It is a "guess and check" strategy. First guess the form of the solution and then use mathematical induction to prove it.
- A powerful method because often it is easier to prove that a certain bound (in the form of the *O* notation) is valid than to compute the bound.

- but the method is only useful when it is easy to guess the form of the solution.
- **Mathematical Induction**: If $p(a)$ is true and, for some integer $k \geq a$, $p(k+1)$ is true whenever $p(k)$ is true, then $p(n)$ is true for all $n \geq a$.
- Example: The worst case for merge sort ($n = 2^k$)

  $W(2) = 1$

  $W(n) = 2\,W(n/2) + n - 1$

  Guess $W(n) = O(f(n))$ then prove it.

  Show (i) $W(2) <= f(2)$ (ii) for some integer $k \geq 2$, assume $W(n) = O(f(n))$ for $n \leq 2^k$, prove $W(2n) <= f(2n)$ then $W(n) = O(f(n))$ for all $n \geq 2$.

## First guess: $W(n) = O(n^2)$

Proof by mathematical induction that $W(n) \leq cn^2$:

(1) Base case: $W(2) = 1 \leq 2^2$;

(2) Inductive step: assume that $W(n) = O(n^2)$ for $n \leq 2^k$.
Now consider $n = 2^{k+1}$

$$
\begin{aligned}
W(2^{k+1}) &= 2W(2^k) + 2^{k+1} - 1 \\
&\leq 2 * (2^k)^2 + 2^{k+1} - 1 \\
&= 2 * (2^k)^2 + 2 * 2^k - 1 \\
&\leq 4 * (2^k)^2 \\
&= (2^{k+1})^2
\end{aligned}
$$

i.e. $W(2^{k+1}) \leq (2^{k+1})^2$

A lot is added from step 3 to step 4

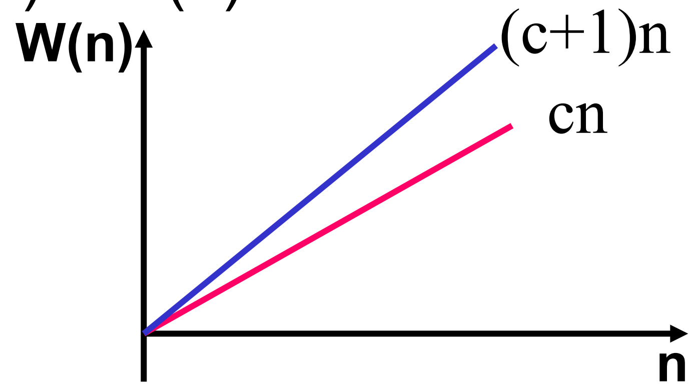
Thus $W(n) = O(n^2)$. But is this the best guess?

**Second guess: $W(n) = O(n)$, i.e. $W(n) \le c * n$**

Proof by mathematical induction:

(1) Base case: $W(2) = 1 \le 2c$;

(2) Inductive step: assume that $W(n) = O(n)$ for $n \le 2^k$.
Now consider $n = 2^{k+1}$




$$
\begin{aligned}
W(2^{k+1}) &= 2W(2^k) + 2^{k+1} - 1 \\
&\le 2 * c * 2^k + 2^{k+1} - 1 \\
&= c * 2^{k+1} + 2^{k+1} - 1
\end{aligned}
$$

Thus $W(2^{k+1}) \le (c+1) * 2^{k+1} - 1$ but we cannot say
$W(2^{k+1}) \le c * 2^{k+1}$ (note: $2^{k+1} - 1 > 0$ for all $k \ge 0$)

Thus $W(n) \ne O(n)$.

## Third guess: W(n) = O(nlgn)

Proof by mathematical induction:

(1) Base case: $W(2) = 1 \leq 2\lg 2$;

(2) Inductive step: assume that $W(n) \leq n\lg n$ for $n \leq 2^k$.
Now consider $n = 2^{k+1}$

$$
\begin{aligned}
W(2^{k+1}) &= 2W(2^k) + 2^{k+1} - 1 \\
&\leq 2 * k * 2^k + 2^{k+1} - 1 \\
&= k * 2^{k+1} + 2^{k+1} - 1 \\
&\leq (k + 1) * 2^{k+1}
\end{aligned}
$$

Thus W(n) = O(nlgn) is a very close upper bound.

# What if the base condition does not hold?

Consider the recurrence ($n = 2^k$) :

$W(1) = 1$

$W(n) = 2\,W(n/2) + n - 1$

Prove that $W(n) = O(n\lg n)$:

(1) Base case: $W(1) = 1 > c\lg 1$;

(2) Recall the big-O notation: for $f(n) = O(g(n))$, we need $f(n) <= c * g(n)$ for all $n > n_0$.

(3) Thus to prove $W(n) = O(n\lg n)$ , we may use another base case.

- We have $W(2) = 3 < c*2*\lg 2$ for any $c > 1$.
- We can assume that $W(n) \leq cn\lg n$ for $n \leq 2^k$ then prove $W(2^{k+1}) \leq c*(k + 1) * 2^{k+1}$

Then $W(n) = O(n\lg n)$.

## What can we say about the general case of n?

The worst case for merge sort :

$$W(2) = 1$$

$$W(n) = W(\lceil n/2 \rceil) + W(\lfloor n/2 \rfloor) + n - 1$$

Proof +:

(1) W(n) is a monotonically increasing function. So when n is not a power of 2, that is, $2^k < n < 2^{k+1}$, then $W(2^k) \leq W(n) \leq W(2^{k+1})$.

(2) We have proved that W(n) = O(nlgn) for powers of 2, so, $W(2^{k+1}) \leq c * (k+1) * 2^{k+1}$.

(3) For any $n < 2^{k+1}$ for some k, $W(n) \leq W(2^{k+1})$. Therefore

$$W(n) \leq c * (k+1) * 2^{k+1} < c * \lg(2n) * (2*n) < 4cn\lg n$$

Therefore W(n) = O(nlgn).

$\boxed{2^k < n, \text{ so } 2^{k+1} < 2n \text{ and } k+1 < \lg(2n)}$

+ See ***The design and analysis of Algorithms*** by Anany Levitin (pp481-483) about **Smoothness Rule.**

## 2. The iteration method

- The idea is to expand (iterate) the recurrence and express it as a summation of terms depending only on n and the initial condition.
- Techniques for evaluating summations can then be used to provide bounds on the solution.
- Example:

W(1) = 1, W(2) = 1, W(3) = 1,

$W(n) = 3W(\lfloor \frac{n}{4} \rfloor) + n$

we expand (iterate) it:

$$W(n) = 3W(\lfloor \frac{n}{4} \rfloor) + n$$

$$= 3(3W(\lfloor \frac{n}{4^2} \rfloor) + \lfloor \frac{n}{4} \rfloor) + n$$

$= 3^2\, W(\lfloor \frac{n}{4^2} \rfloor) + 3\lfloor \frac{n}{4} \rfloor + n$

$= 3^2(3W(\lfloor \frac{n}{4^3} \rfloor) + \lfloor \frac{n}{4^2} \rfloor) + 3\lfloor \frac{n}{4} \rfloor + n$

$= 3^3\, W(\lfloor \frac{n}{4^3} \rfloor) + 3^2\lfloor \frac{n}{4^2} \rfloor + 3\lfloor \frac{n}{4} \rfloor + n$

we need to iterate until we reach one of the boundary conditions, i.e $\lfloor \frac{n}{4^i} \rfloor = 1, 2$ or 3.

E.g. $n$=64, $4^3 \le 64 < 4^4$ and $\lfloor \frac{64}{4^3} \rfloor = 1$;

$n$=255, $4^3 \le 255 < 4^4$ and $\lfloor \frac{255}{4^3} \rfloor = 3$;

This means if $4^i \le n < 4^{i+1}$ then $i = \lfloor \log_4 n \rfloor$. So

$W(n) = 3^i\, W(a) + 3^{i-1}\lfloor \frac{n}{4^{i-1}} \rfloor + \ldots + 3^2\lfloor \frac{n}{4^2} \rfloor + 3\lfloor \frac{n}{4} \rfloor + n$

a = 1,2 or 3

$$W(n) = 3^i W(a) + 3^{i-1} \left\lfloor \frac{n}{4^{i-1}} \right\rfloor + \ldots + 3^2 \left\lfloor \frac{n}{4^2} \right\rfloor + 3 \left\lfloor \frac{n}{4} \right\rfloor + n$$

$$\leq 3^{\log_4 n} W(a) + 3^{i-1} \frac{n}{4^{i-1}} + \ldots + 3^2 \frac{n}{4^2} + 3 \frac{n}{4} + n$$

Let $x = 3^{\log_4 n}$ then

$\log_4 x = \log_4 n \log_4 3$ then

$4^{\log_4 x} = 4^{\log_4 n \log_4 3}$ then

$x = n^{\log_4 3}$, i.e. $3^{\log_4 n} = n^{\log_4 3}$

$$\sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i = 4$$

$$W(n) \leq n^{\log_4 3} + n \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i \quad + = O(n)$$

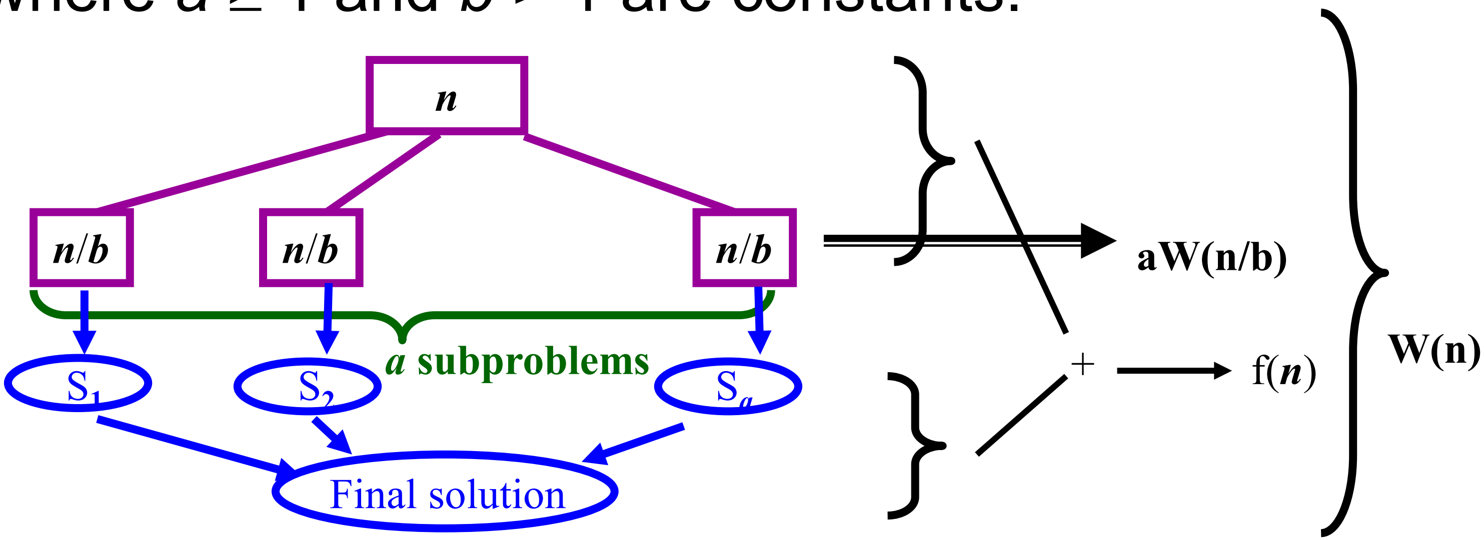
- The iteration method usually leads to lots of algebra.
- We should focus on how many times the recurrence needs to be iterated to reach the boundary condition.

# 3. The master method

- The master method provides a “manual” for solving recurrences of the form

$$\mathbf{W}(n) = a\mathbf{W}(n/b) + \mathbf{f}(n)$$

where $a \geq 1$ and $b > 1$ are constants.




- We are able to determine the asymptotic tight bound in the following three cases

# The master theorem

For $\mathbf{W(n) = aW(n/b) + f(n)}$ $a \geq 1$ and $b > 1$

The manual:

1. If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $W(n) = \theta(n^{\log_b a})$.

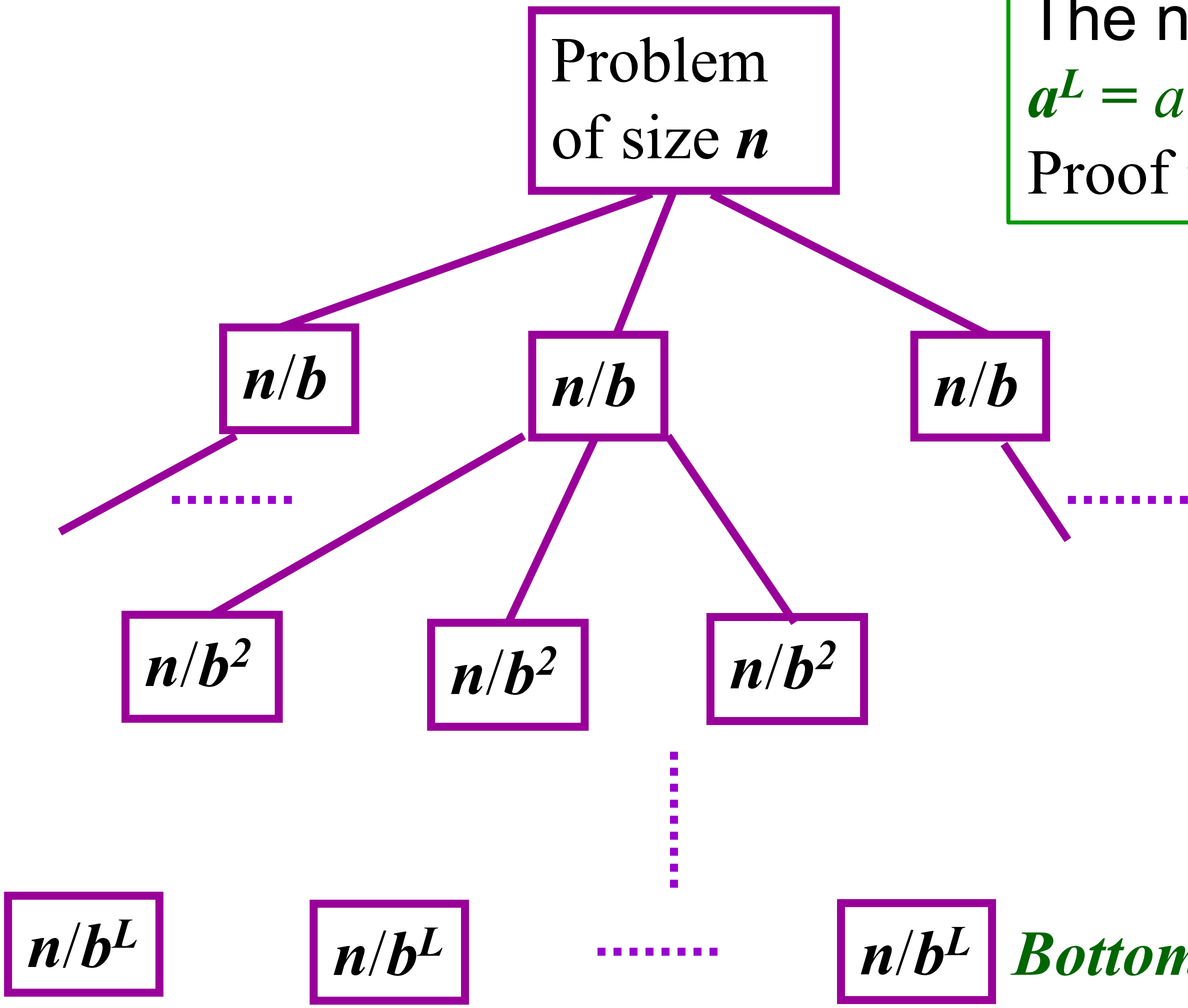
2. If $f(n) = \theta(n^{\log_b a})$, then $W(n) = \theta(n^{\log_b a} \log n)$.

   If $f(n) = \theta(n^{\log_b a} \log^k n)$, $k \geq 0$,
   then $W(n) = \theta(n^{\log_b a} \log^{k+1} n)$

3. If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $a\,f(n/b) \leq c\,f(n)$ for some constant $c < 1$ and all sufficiently large $n$, then $W(n) = \theta(f(n))$.

# What is $n^{\log_b a}$?

The depth of the tree
$L = \log_b n$
The number of leaves is
$a^L = a^{\log_b n} = n^{\log_b a}$
Proof in slide 54

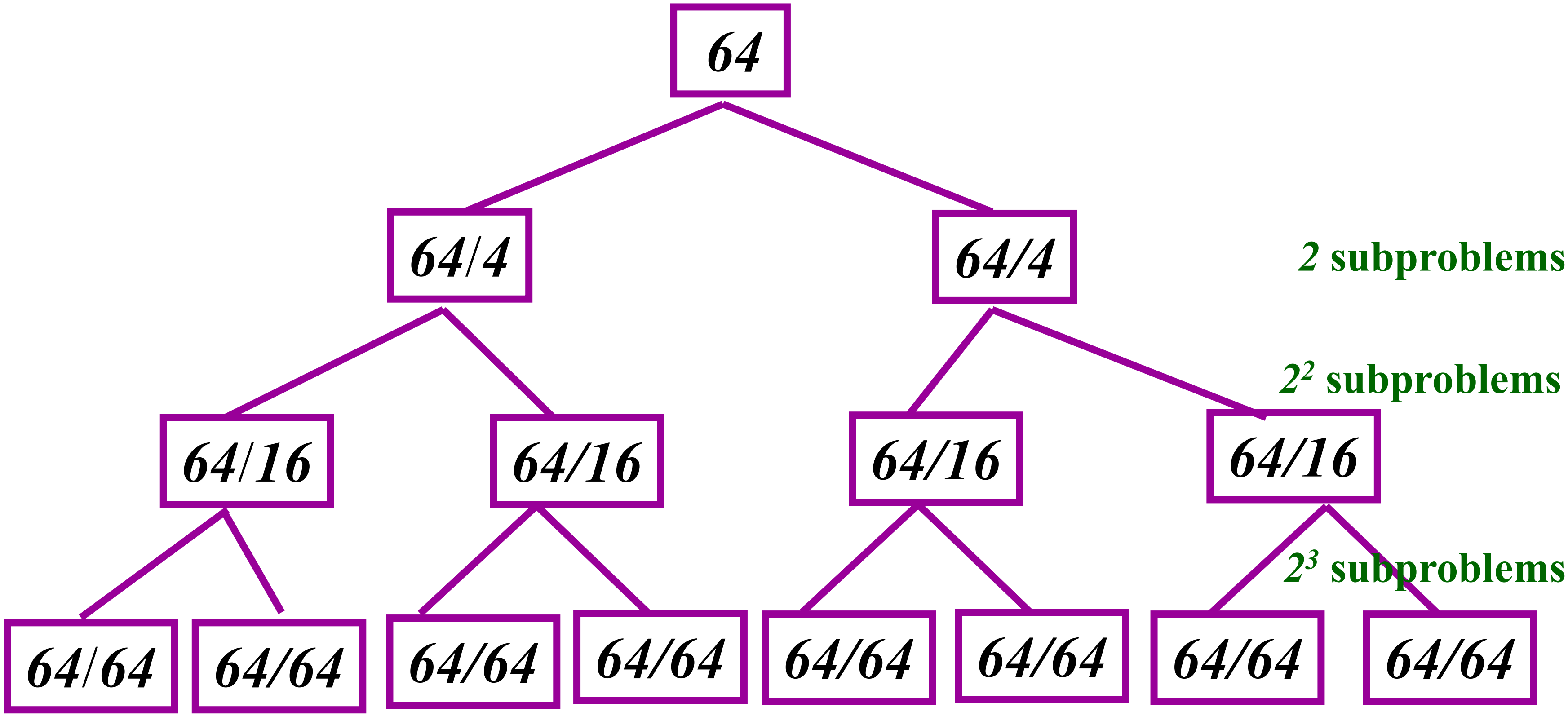



***a* subproblems**

**$a^2$ subproblems**

***Bottom level: $a^L$* subproblems**

E.g. $\boldsymbol{n} = 64$, $\boldsymbol{a} = 2$, $\boldsymbol{b} = 4$




Depth of tree L = $\log_4 64$, Number of leaves = $8 = 2^{\log_4 64} = 64^{\log_4 2}$

$(a^{\log_b n} = n^{\log_b a})$

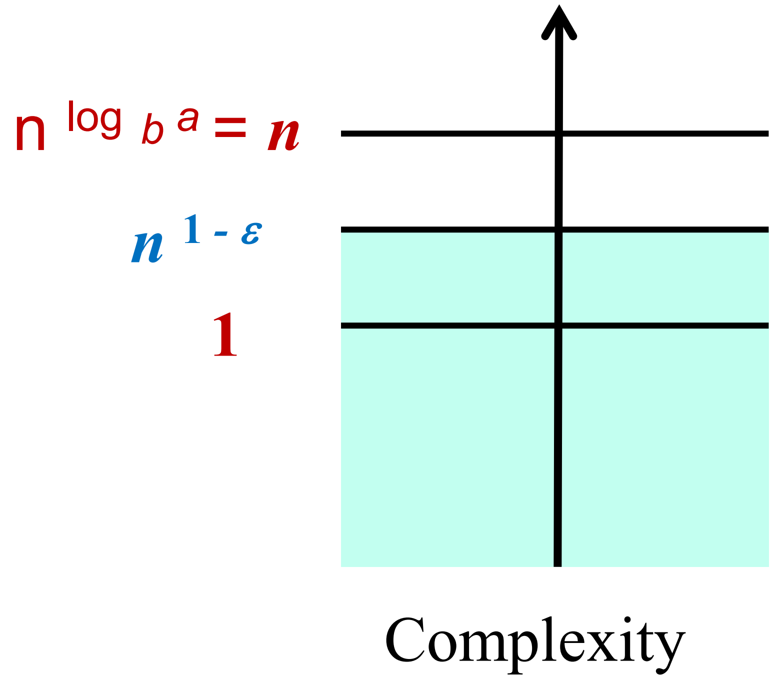
## Examples

1) W(n) = 3W(n/3) + 2,

so a = 3, b = 3,

$n^{\log_b a} = n^1$

$f(n) = 2 = \theta(1) = O(n^1)$




We may let $\varepsilon = 0.5$ then we confirm $2 = O(n^{1-0.5})$,

i.e. $f(n) = O(n^{1-\varepsilon})$

$\Rightarrow f(n) = O(n^{\log_b a - \varepsilon})$ (case 1)

thus $W(n) = \theta(n^{\log_b a})$

$W(n) = \theta(n)$.

# Examples

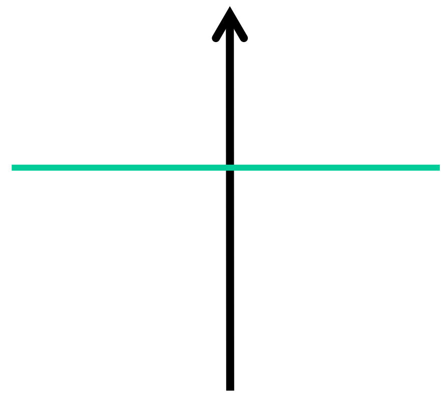
2) $W(n) = 4W(n/4) + n - 1$,

so $a = 4$, $b = 4$,

$n^{\log_b a} = n^1$

$f(n) = n - 1$

$n^{\log_b a} = n$



Complexity

We have

$f(n) = n-1$

$= \theta(n^1)$,

$= \theta(n^{\log_b a})$, (case 2)

thus

$W(n) = \theta(n^{\log_b a} \log n)$

$= \theta(n \log n)$

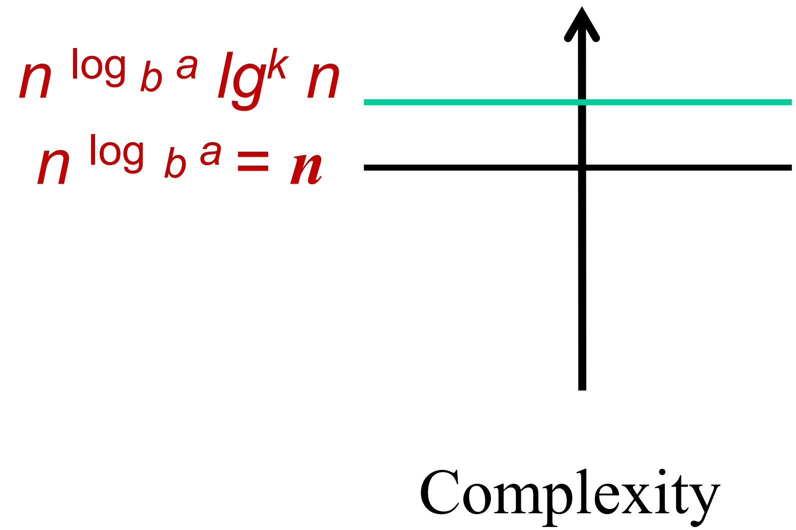
## Examples

3) W(n) = 2W(n/2) + n lg n,

so a = 2, b = 2,

$f(n) = n \lg n$

$n^{\log_b a} = n^1$




We have

$f(n) = \theta(n^1 \lg n)$,

$= \theta(n^{\log_b a} \lg^k n)$, (case 2: $k = 1$)

thus

$W(n) = \theta(n^{\log_b a} \lg^2 n)$

$= \theta(n (\lg n)^2)$

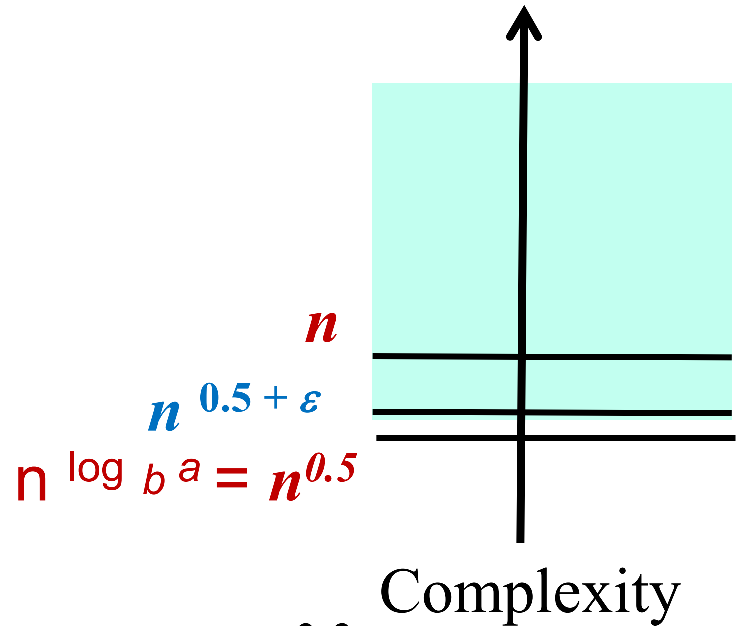
## Examples

4) $W(n) = 2W(n/4) + n$,

so $a = 2$, $b = 4$,

$n^{\log_b a} = n^{\log_4 2} = n^{0.5}$

$f(n) = n = \theta(n)$




We may let $\varepsilon = 0.1$ then we have $n = \Omega(n^{0.6})$

i.e. $f(n) = \Omega(n^{\log_b a + \varepsilon})$, and

for all sufficiently large $n$, we can find a value for $c$, say, $c = ¾$, to show that $a\, f(n/b) \le c\, f(n)$. (case 3)

$a*f(n/b) = 2*f(n/4) = n/2 \le c*n$

thus $W(n) = \theta(n)$.

# Sometimes the master method cannot apply

Example 1: W(n) = 3W(n/3) + n/lgn, $n^{\log_b a} = n^1$

f(n) = $n/\lg n = O(n^1)$ because $\lim_{n\to\infty} \frac{n/lgn}{n^1} = \lim_{n\to\infty} \frac{1}{lgn} = 0$

f($n$) = $O(n^{1-\varepsilon})$ ?

(L'Hôpital's rule, slide 55)

i.e. $n/\lg n = O(n^{1-\varepsilon})$ ?

No, because asymptotically, $n/\lg n > n^{1-\varepsilon}$ for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{n/lgn}{n^{1-\varepsilon}} = \lim_{n\to\infty} \frac{n^{\varepsilon}}{lgn} = \infty$$

This recurrence falls into the gap between case 2 and case 3.

So the Master Theorem cannot apply.

# Sometimes the master method cannot apply

Example 2: W(n) = W(n/3) + f(n)

where f($n$) = $\begin{cases} 3n + 2^{3n} & for\ n = 2^i \\ 3n & otherwise \end{cases}$

so a = 1, b = 3 then $n^{\log_b a} = n^0$

let $\varepsilon = 1$ then f($n$) = $\Omega(n^{0+1})$, case 3?

$a$ f($n$/$b$) ≤ $c$ f($n$) for all sufficiently large $n$?

*When* $n = 3 * 2^i$, $a$ f($n$/$b$) = f($2^i$) = $n + 2^n$, but cf($n$) = c(3$n$)

i.e. $a$ f($n$/$b$) > c f($n$) . E.g. for n = 6 or greater

So the Master Theorem cannot apply.

- Notice that when we want to find the order of a recurrence, the initial conditions are not important. This is because the running costs of the terminating conditions are small constants that do not affect the order.

# Solving recurrences (2)

- Definition: A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

  $$a_n = c_1 a_{n-l} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

  where $c_1, c_2, \dots, c_k$ are real constants and $c_k \neq 0$.

The two different notations: A($n$) and $a_n$

- When using A($n$), we mean the function value with parameter $n$
- When using $a_n$, we mean the $n$th term in a sequence $a_1, a_2, \dots, a_n$.
- If we list A(1), A(2), …, A($n$) in a sequence, we can write them as $a_1, a_2, \dots, a_n$. They are equivalent.

# Solving recurrences (2)

- <u>Definition</u>: A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

  $$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k},$$

  where $c_1, c_2, \ldots, c_k$ are real constants and $c_k \neq 0$.

  - **Linear**: $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$ appear in separate terms and to the first power
  - **Homogeneous**: the total degree of each term is the same, e.g. no constant term
  - **Constant coefficients**: $c_1, c_2, \ldots, c_k$ are fixed real constants that do not depend on $n$
  - ***Degree k***: the expression for $a_n$ contains the previous $k$ terms $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$, ($c_k \neq 0$)

- Examples
  - A *linear homogeneous recurrence relation of degree 2:* $a_n = a_{n-1} + a_{n-2}$
  - A *linear homogeneous recurrence relation of degree 1:* $a_n = 1.04a_{n-1}$
  - A *linear homogeneous recurrence relation of degree 3 :* $a_n = a_{n-3}$
- Non-examples
  - $a_n = a_{n-1} + a_{n-2} + 1$: non-homogeneous
  - $a_n = a_{n-1}a_{n-2}$ : not linear
  - $a_n = na_{n-1}$ : coefficient not constant

- A linear homogeneous recurrence relation of degree $k$ can be systematically solved, i.e. find the explicit expression for $a_n$
- The basic approach is to look for solutions of the form $a_n = t^n$ where $t$ is a constant
- If $a_n = t^n$ is a solution for

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

Then

$$t^n = c_1 t^{n-1} + c_2 t^{n-2} + \dots + c_k t^{n-k}$$

$\Rightarrow t^k = c_1 t^{k-1} + c_2 t^{k-2} + \dots + c_k$ (divide both side by $t^{n-k}$)

$\Rightarrow t^k - c_1 t^{k-1} - c_2 t^{k-2} - \dots - c_k = 0$

- This means if we can solve the equation

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - \dots - c_k = 0$$

to find t, then $a_n = t^n$ is a solution for

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

We call

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - \dots - c_k = 0$$

the **characteristic equation** of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

The solutions to the characteristic equation are called the **characteristic roots**

- We consider a linear homogeneous recurrence relation of degree 2

  $a_n = Aa_{n-l} + Ba_{n-2}$ *for all* $n \geq 2$

  where A and B are real constants

- The **characteristic equation**

  $t^2 - At - B = 0$

  may have

  1) two distinct roots

  2) a single root

- **Theorem 1 (Distinct Roots Theorem)**

Suppose a sequence $a_0, a_1, a_2, \ldots.$ satisfies a recurrence relation

$$a_n = Aa_{n-l} + Ba_{n-2} \text{ for all } n \geq 2$$

where A and B are real constants and B $\neq$ 0. If the characteristic equation

$$t^2 - At - B = 0$$

has two distinct roots $r$ and $s$, then $a_0, a_1, a_2, \ldots.$ is given by the explicit formula

$$\boldsymbol{a_n = Cr^n + Ds^n}$$

where C and D are determined by the values of $a_0$ and $a_1$.

- Example 1:

$$F_n = F_{n-l} + F_{n-2} \text{ for all } n \geq 2, \quad \text{and } F_0 = F_1 = 1$$

The characteristic equation is

$$t^2 - t - 1 = 0$$

The roots are

For $ax2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \begin{cases} \frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{cases}$$

$$F_n = \quad C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$$

- To find C and D, we have

$$F_0 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^0 + D\left(\frac{1-\sqrt{5}}{2}\right)^0 = C \cdot 1 + D \cdot 1 = C + D$$

$$F_1 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^1 + D\left(\frac{1-\sqrt{5}}{2}\right)^1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right)$$

To solve this system of 2 equations with 2 unknowns, from

$C + D = 1$

$\Rightarrow \left(\frac{1+\sqrt{5}}{2}\right)C + \left(\frac{1+\sqrt{5}}{2}\right)D = \left(\frac{1+\sqrt{5}}{2}\right)$

Then

$$D\left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right) = \left(\frac{1+\sqrt{5}}{2}\right) - 1$$

$$\Rightarrow \quad D\sqrt{5} = \left(\frac{1+\sqrt{5}}{2}\right) - 1$$

$$\Rightarrow D = \left(\frac{-1+\sqrt{5}}{2\sqrt{5}}\right)$$

Then $C = 1 - D = 1 - \left(\frac{-1+\sqrt{5}}{2\sqrt{5}}\right)$

$$\Rightarrow C = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

We can write

$$D = \left(\frac{-(1-\sqrt{5})}{2\sqrt{5}}\right)$$

- So

$$F_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{-(1-\sqrt{5})}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

After simplifying it, we get

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

for all $n \geq 0$.

- Example 2:

$$a_n = 5a_{n-l} - 6a_{n-2}\ ,\ a_0 = 9,\ a_1 = 20$$

The characteristic equation is

$$t^2 - 5t + 6 = 0$$

$\Rightarrow \quad (t-2)(t-3) = 0 \quad \Rightarrow$ two roots: $t = 2,\ t = 3$

$a_n = C2^n + D3^n$ for all n $\geq$ 0.

To find $C$ and $D$:

$9 = C + D, \Rightarrow 18 = 2C + 2D$

$20 = 2C + 3D$

Thus $D = 2$, $C = 7$ So $a_n = 7*2^n + 2*3^n$ for all n $\geq$ 0

- **Theorem 2 (Single-Root Theorem)**

  Suppose a sequence $a_0, a_1, a_2, \ldots.$ satisfies a recurrence relation

  $$a_n = Aa_{n-l} + Ba_{n-2} \text{ for all } n \geq 2$$

  where A and B are real constants and $B \neq 0$. If the characteristic equation

  $$t^2 - At - B = 0$$

  has a single (real) root, then $a_0, a_1, a_2, \ldots.$ is given by the explicit formula

  $$\boldsymbol{a_n = Cr^n + Dnr^n}$$

  where C and D are determined by the values of $a_0$ and any other known value of the sequence.

- Example

  $b_n = 4b_{n-l} - 4b_{n-2}$ *for all* $n \geq 2$

  with $b_0 = 1$, $b_1 = 3$.

  The characteristic equation is

  $$t^2 - 4t + 4 = 0$$

  $\Rightarrow (t-2)^2 = 0 \quad \Rightarrow$ single root $t = 2$

  The explicit formula is

  $$b_n = C2^n + Dn2^n$$

  where $C$ and $D$ are determined by the values of $b_0$ and $b_1$.

  We have $1 = C$ and $3 = 2C + 2D$, so $D = ½$ and $C = 1$.

- Therefore

$$b_n = 2^n + (½)n2^n = (1 + n/2)\ 2^n$$

Theorem 1 can be generalised to the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

with characteristic equation

$$t^k - c_1\, t^{k-1} - c_2\, t^{k-2} - \dots - c_k = 0$$

having *k* distinct roots.

Theorem 2 can be generalised to less than *k* distinct roots.

# Proof of $a^{\log_b n} = n^{\log_b a}$

- Let $\mathrm{L} = \log_b n$, i.e. $b^{\mathrm{L}} = n$

$$\Rightarrow (b^{\mathrm{L}})^{\log_{\mathrm{b}} a} = n^{\log_b a}$$

$$\Rightarrow (b^{\log_b a})^{\mathrm{L}} = n^{\log_b a}$$

$$\Rightarrow a^{\mathrm{L}} = n^{\log_b a}$$

$$\Rightarrow a^{\log_b n} = \mathrm{n}^{\log_b a}$$

# L'Hôpital's rule

L'Hôpital's rule states that for functions $f(x)$ and $g(x)$, if:

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} g(x) = \pm\infty$$

then:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

where the prime (') denotes the derivative.