16-642
PS4
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Q1.

a. Referring to the figure on the problem sheet, we have:

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad d_1^0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

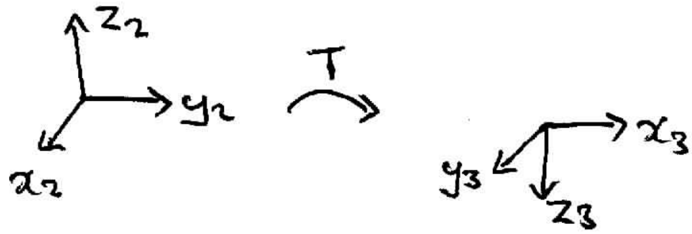
$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad d_2^1 = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = H_1^0 H_2^1$$

$$\therefore H_2^0 = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For R_3^2, we note that



So we have
$$x_3 \rightarrow y_2$$
$$y_3 \rightarrow x_2$$
$$z_3 \rightarrow -z_2$$

This transformation has basis vectors corresponding to

$$R_3^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ (view each column as a basis vector)

Also $d_3^2 = \begin{bmatrix} 0 \\ 0 \\ 1.9 \end{bmatrix}$ (in $O_2 x_2 y_2 z_2$ frame)

$$\therefore H_3^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = H_2^0 H_3^2$$

$$\therefore H_3^0 = \begin{bmatrix} 0 & 1 & 0 & -0.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If interpreting the "from" frame as $O_0x_0y_0z_0$, then we have

$$H_0^1 = (H_1^0)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^2 = (H_2^0)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & -1.5 \\ 0 & 0 & 1 & -1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^3 = (H_3^0)^{-1}$$

$$= \begin{bmatrix} 0 & 1 & 0 & -1.5 \\ 1 & 0 & 0 & 0.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b. For a pure rotation about the z-axis of $\frac{\pi}{2}$,

$$R_2^1 = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$d_2^1 = \begin{bmatrix} -1 \\ 0.5 \\ 0.1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So $H_2^0 = H_1^0 H_2^1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore H_2^0 = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If interpreting the "from" frame as $O_0 x_0 y_0 z_0$, then we have

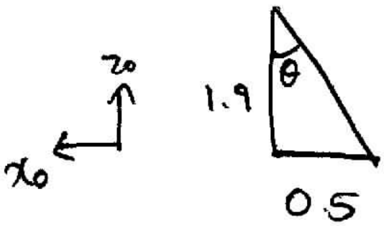
$$H_0^2 = (H_2^0)^{-1}$$

$$= \begin{bmatrix} 0 & 1 & 0 & -1.5 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the $O_0 x_0 y_0 z_0$ frame, we note that the block is at $\begin{bmatrix} -1 \\ 1.5 \\ 1.1 \end{bmatrix}$ and the camera is at $\begin{bmatrix} -0.5 \\ 1.5 \\ 3 \end{bmatrix}$

So they are on the same x_0-z_0 plane.

Their angle difference on the plane is related as follows:



So $\theta = \arctan\left(\frac{0.5}{1.9}\right) = 0.257$ rad

In the original $O_3x_3y_3z_3$ axis, this corresponds to a rotation about the x_3-axis.

So $$R_3^{old\,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 0.257 & -\sin 0.257 \\ 0 & \sin 0.257 & \cos 0.257 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.967 & -0.254 \\ 0 & 0.254 & 0.967 \end{bmatrix}$$

So $R_3^0 = R_{old\,3}^0 R_3^{old\,3}$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.967 & -0.254 \\ 0 & 0.254 & 0.967 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.967 & -0.254 \\ 1 & 0 & 0 \\ 0 & -0.254 & -0.967 \end{bmatrix}$$

and $d_3^0 = \begin{bmatrix} -0.5 \\ 1.5 \\ 3 \end{bmatrix}$

So $$H_3^0 = \begin{bmatrix} 0 & 0.967 & -0.254 & -0.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & -0.254 & -0.967 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$H_3^2 = H_0^2 H_3^0$

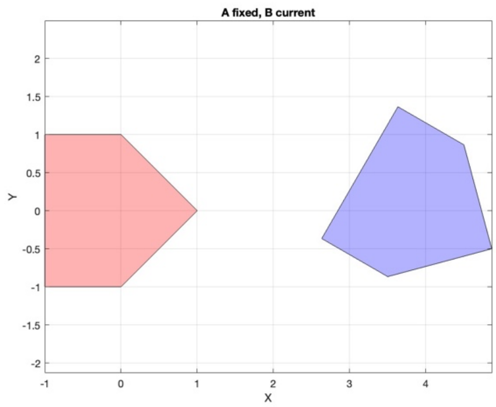
$$= \begin{bmatrix} 0 & 1 & 0 & -1.5 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0.967 & -0.254 & -0.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & -0.254 & -0.967 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore H_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.967 & 0.254 & -0.5 \\ 0 & -0.254 & -0.967 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

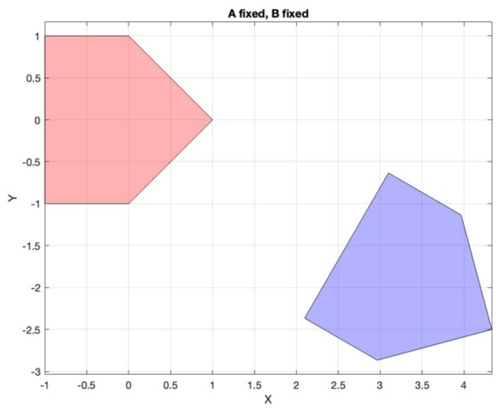
Q2.

The red shape denotes the original rigid body, and the blue shape denotes the transformed rigid body. The rigid body is transformed as follows:

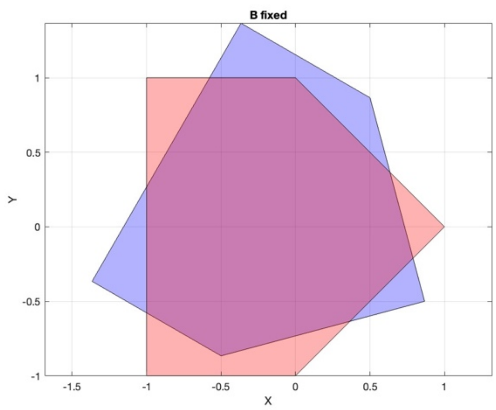
A.



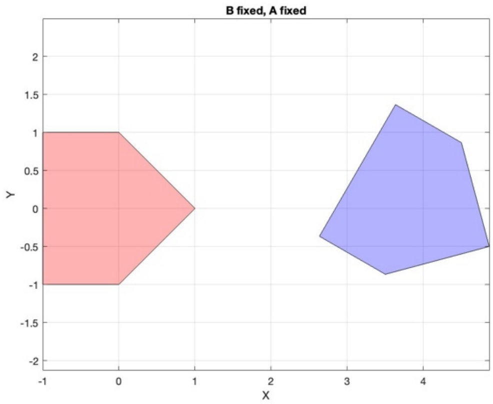
B.



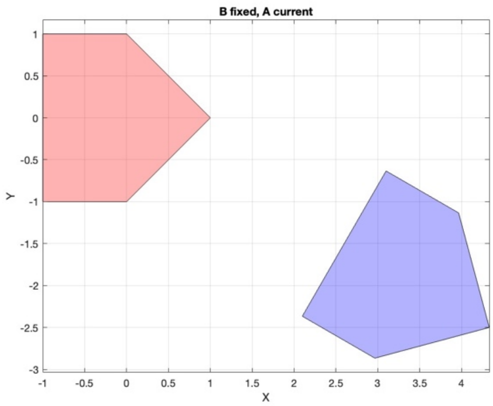
C.



D.

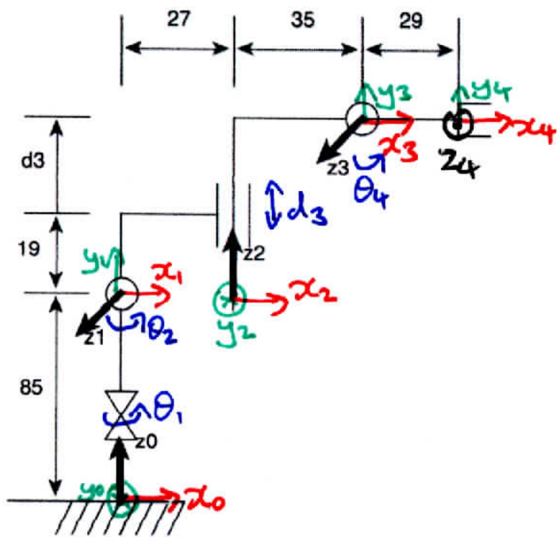


E.



Q3.

The locations of the DH frames are as follows:
Note that ⊗ indicates into the page and ⊙ indicates out of the page



A table of DH parameters are as follows:

i	a_i	α_i	d_i	θ_i
1	0	90°	85	θ_1
2	27	-90°	0	θ_2
3	35	90°	$19+d_3$	0
4	29	0	0	θ_4

The freedom for the frames are as follows:

Frame 0: The frame is not uniquely defined. The x_0- and y_0-axis can be any orientation that is orthogonal in the normal plane of z_0. The direction of x_0 is chosen so that x_1 and x_2 are in the same direction. y_0 is from the right-hand rule.

Frame 1: The frame is uniquely defined up to bidirectional ambiguity (pointing one way or the other).
This is because x_1 needs to be perpendicular to z_0, so it needs to be in a shifted plane spanned by x_0, y_0. Since z_1 is already given, by orthogonality we fix x_1. We arbitarily choose x_1 to be in the rightwards direction, y_1 is resolved using the right-hand rule.

Frame 2: The frame is uniquely defined up to bidirectional ambiguity. This is because x_2 needs to be perpendicular to z_1, so it needs to be in a shifted plane spaned by x_1, y_1. Since z_2 is already given, by orthogonality we fix x_2. We arbitarily choose x_2 to be in the rightwards direction, y_2 is resolved using the right-hand rule.

Frame 3: The frame is uniquely defined up to bidirectional ambiguity. The argument is the same as Frame 2.

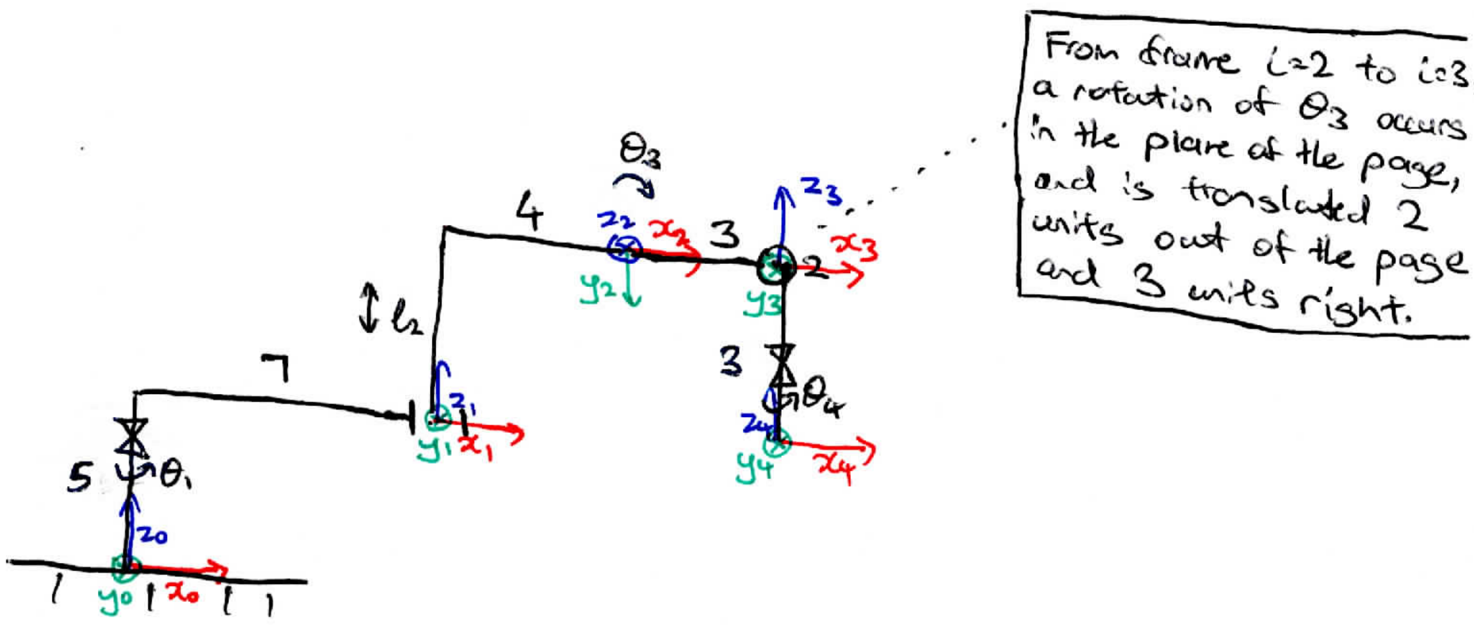
Frame 4: The frame is not uniquely defined. However, following gripper convention and putting O_4 at the gripper, the x_4-direction can only point left or right so that it is perpendicular to z_3 and intersects the span of z_3. The y_4- and z_4 can be in any orientation that is orthogonal in the normal plane of x_4. We choose it to be the same orientation as y_3, z_3 for simplicity.

Q4. Given the following DH parameters, and letting ⊗ denote into the page and ⊙ denote out of the page,

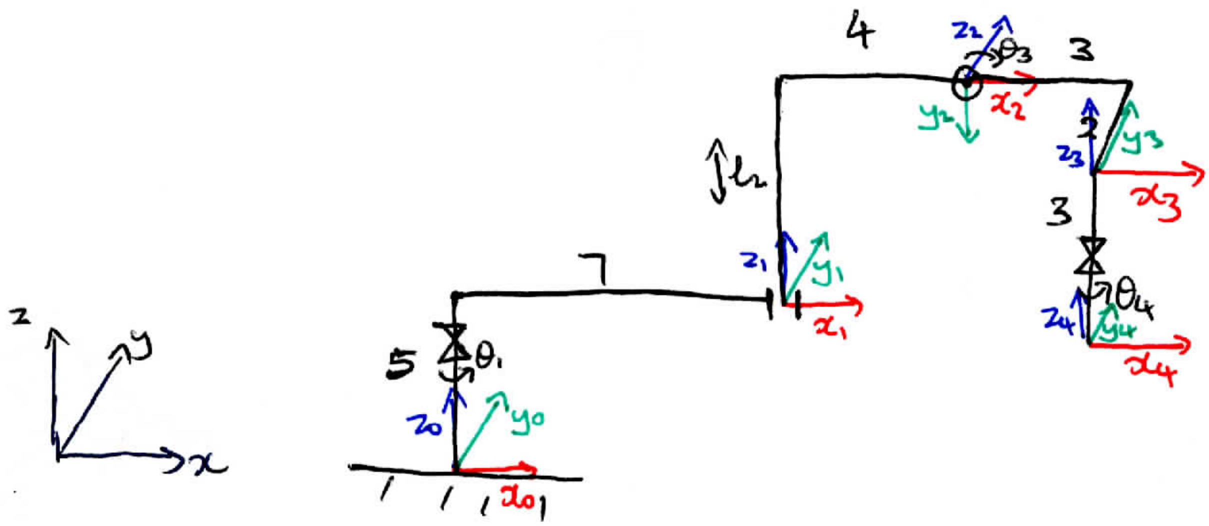
i	θ_i	d_i	a_i	α_i
1	θ_1	5	7	0
2	0	l_2	4	$-90°$
3	θ_3	-2	3	$90°$
4	θ_4	-3	0	0

The manipulator is as follows:

2D schematic:



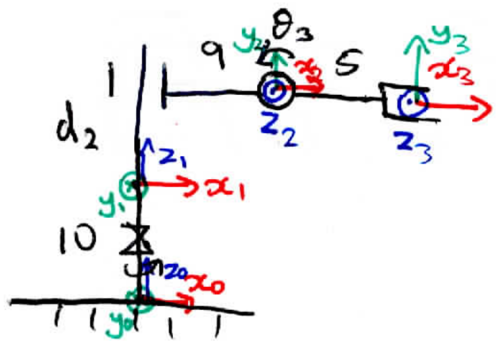
3D drawing:



z
y
x
5
θ_1
z_0
y_0
x_0
7
z_1
y_1
x_1
l_2
4
z_2
θ_3
y_2
x_2
3
2
z_3
y_3
x_3
3
θ_4
z_4
y_4
x_4

Q5.

a. The DH table of the configeration is given as follows:



i	θ_i	d_i	a_i	α_i
1	θ_1	10	0	0
2	0	d_2	9	90°
3	θ_3	0	5	0

By the DH convention,

$$A_i = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We set

$$A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & 5c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & 5s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So $T_3^0 = A_1 A_2 A_3$

$$= \begin{bmatrix} c\theta_1 c\theta_3 & -c\theta_1 s\theta_3 & s\theta_1 & 5c\theta_1 c\theta_3 + 9c\theta_1 \\ s\theta_1 c\theta_3 & -s\theta_1 s\theta_3 & -c\theta_1 & 5s\theta_1 c\theta_3 + 9s\theta_1 \\ s\theta_3 & c\theta_3 & 0 & d_2 + 5s\theta_3 + 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So $o_3^0 = \begin{bmatrix} 5c\theta_1 c\theta_3 + 9c\theta_1 \\ 5s\theta_1 c\theta_3 + 9s\theta_1 \\ d_2 + 5s\theta_3 + 10 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

From velocity dynamics,

$\dot{x} = \dot{o}_3^0 = J_a(q)\dot{q}$ with $q = \begin{bmatrix} \theta_1 \\ d_2 \\ \theta_3 \end{bmatrix}$

whee $J_a(q) = \dfrac{\partial o_3^0}{\partial \Theta}$

$$= \begin{bmatrix} \dfrac{\partial o_3^0}{\partial q_1} & \dfrac{\partial o_3^0}{\partial q_2} & \dfrac{\partial o_3^0}{\partial q_3} \end{bmatrix}$$

$$= \begin{bmatrix} -5s\theta_1 c\theta_3 - 9s\theta_1 & 0 & -5c\theta_1 s\theta_3 \\ 5c\theta_1 c\theta_3 + 9c\theta_1 & 0 & -5s\theta_1 s\theta_3 \\ 0 & 1 & 5c\theta_3 \end{bmatrix}$$

At $t=0$, $\theta_1=\theta_3=0$. S_0

$$J_a(q)\big|_{t=0} = \begin{bmatrix} -5s_0c_0-9s_0 & 0 & -5c_0s_0 \\ 5c_0c_0+9c_0 & 0 & -5s_0s_0 \\ 0 & 1 & 5c_0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 14 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

b. For the linear velocity term, we build up the geometric Jacobian according to

$$J_{v_i} = \begin{cases} z_{i-1}^0 & \text{, if joint } i \text{ is P} \\ z_{i-1}^0 \times (o_n^0 - o_{i-1}^0), & \text{if joint } i \text{ is R} \end{cases}$$

For joint $i=1$, we have R joint.

So $J_1 = z_0^0 \times (o_3^0 - o_0^0)$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} 5c\theta_1c\theta_3+9c\theta_1 \\ 5s\theta_1c\theta_3+9s\theta_1 \\ d_2+5s\theta_3+10 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ 5c\theta_1c\theta_3+9c\theta_1 & 5s\theta_1c\theta_3+9s\theta_1 & d_2+5s\theta_3+10 \end{vmatrix}$$

$$= \hat{x}(-5s\theta_1 c\theta_3 - 9s\theta_1) - \hat{y}(-5c\theta_1 c\theta_3 - 9c\theta_1) + \hat{z}(0)$$

$$\therefore J_1 = \begin{bmatrix} -5s\theta_1 c\theta_3 - 9s\theta_1 \\ 5c\theta_1 c\theta_3 + 9c\theta_1 \\ 0 \end{bmatrix}$$

For joint $i=2$, we have P joint

so $J_2 = z_1^0$

$$= R_1^0 z_1^1$$

$$= \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore J_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For joint $i=3$, we have R joint

So $J_3 = z_2^0 \times (O_3^0 - O_2^0)$

Since $T_0^2 = A_1 A_2$

$$= \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 9c\theta_1 \\ s\theta_1 & 0 & -c\theta_1 & 9s\theta_1 \\ 0 & 1 & 0 & d_2+10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have $O_2^0 = \begin{bmatrix} 9c\theta_1 \\ 9s\theta_1 \\ d_2+10 \end{bmatrix}$

$$z_2^0 = R_2^0 z_2^2$$

$$= \begin{bmatrix} c\theta_1 & 0 & s\theta_1 \\ s\theta_1 & 0 & -c\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} s\theta_1 \\ -c\theta_1 \\ 0 \end{bmatrix}$$

So $$J_3 = \begin{bmatrix} s\theta_1 \\ -c\theta_1 \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} 5c\theta_1 c\theta_3 + 9c\theta_1 \\ 5s\theta_1 c\theta_3 + 9s\theta_1 \\ d_2 + 5s\theta_3 + 10 \end{bmatrix} - \begin{bmatrix} 9c\theta_1 \\ 9s\theta_1 \\ d_2 + 10 \end{bmatrix} \right)$$

$$= \begin{bmatrix} s\theta_1 \\ -c\theta_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 5c\theta_1 c\theta_3 \\ 5s\theta_1 c\theta_3 \\ 5s\theta_3 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ s\theta_1 & -c\theta_1 & 0 \\ 5c\theta_1 c\theta_3 & 5s\theta_1 c\theta_3 & 5s\theta_3 \end{bmatrix}$$

$$= \hat{x}(-5c\theta_1 s\theta_3) - \hat{y}(5s\theta_1 s\theta_3) + \hat{z}(5s^2\theta_1 c\theta_3 + 5c^2\theta_1 c\theta_3)$$

$$\therefore J_3 = \begin{bmatrix} -5c\theta_1 s\theta_3 \\ -5s\theta_1 s\theta_3 \\ 5c\theta_3 \end{bmatrix}$$

Since $J(q) = [J_1 \; J_2 \; J_3]$, we have

$$J(q) = \begin{bmatrix} -5s\theta_1 c\theta_3 - 9s\theta_1 & 0 & -5c\theta_1 s\theta_3 \\ 5c\theta_1 c\theta_3 + 9c\theta_1 & 0 & -5s\theta_1 s\theta_3 \\ 0 & 1 & 5c\theta_3 \end{bmatrix}$$

(which equals $J_a(q)$)

At $t=0$, $\theta_1=\theta_3=0$. So

$$J(q)\big|_{t=0} = \begin{bmatrix} 0 & 0 & 0 \\ 14 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

c. Since $J_a(q) = J(q)$, it suffices to determine the singularities of $J(q)$.

Since q is singular iff $\det J = 0$, we calculate it as follows:

$$\det J = \det \begin{bmatrix} -5s\theta_1 c\theta_3 - 9s\theta_1 & 0 & -5c\theta_1 s\theta_3 \\ 5c\theta_1 c\theta_3 + 9c\theta_1 & 0 & -5s\theta_1 s\theta_3 \\ 0 & 1 & 5c\theta_3 \end{bmatrix}$$

Cofactor expansion at element (3,2) and equating to zero:

$$\det J = -1 \begin{vmatrix} -5s\theta_1 c\theta_3 - 9s\theta_1 & -5c\theta_1 s\theta_3 \\ 5c\theta_1 c\theta_3 + 9c\theta_1 & -5s\theta_1 s\theta_3 \end{vmatrix} = 0$$

$$\Rightarrow 0 = (5s\theta_1 c\theta_3 + 9s\theta_1)(5s\theta_1 s\theta_3) + (5c\theta_1 c\theta_3 + 9c\theta_1)(5c\theta_1 s\theta_3)$$

$$0 = 25s^2\theta_1 c\theta_3 s\theta_3 + 45s^2\theta_1 s\theta_3 + 25c^2\theta_1 c\theta_3 s\theta_3 + 45c^2\theta_1 s\theta_3$$

$$0 = 25c\theta_3 s\theta_3(s^2\theta_1 + c^2\theta_1) + 45s\theta_3(c^2\theta_1 + s^2\theta_1)$$

$$0 = s\theta_3(25c\theta_3 + 45)$$

So $\det 3 = 0$ iff

$s\theta_3 = 0$ or $25c\theta_3 + 45 = 0$

$\Rightarrow \theta_3 = 0$ or π

which is impossible since $c\theta_3 \in \mathbb{R}_{[-1,1]}$

So the singular configurations are

$\theta_1 \in \mathbb{R}_{[0, 2\pi)}$

$d_2 \in \mathbb{R}$ (or its maximum extension limits)

$\theta_3 \in \{0, \pi\}$

It physically represents the end effector being fully extended or fully folded, at which the forward/aft movement is not possible, hence losing one degree of freedom (i.e. when viewed at end effector frame, it can only move up/down or sideways, but never front and back).

Appendices:

A.1. Code for Q2

```matlab
% MEC
% Q2
clear;

A = [1, 0, 4;
     0, 1, 0;
     0, 0, 1];

B = [0.866, 0.5, 0;
     -0.5, 0.866, 0;
     0, 0, 1];

rigid_body = [-1  0  1  0 -1;
               1  1  0 -1 -1;
               1  1  1  1  1];

% A
rigid_body_a = A * B * rigid_body;

% B
rigid_body_b = B * A * rigid_body;

% C
rigid_body_c = B * rigid_body;

% D
rigid_body_d = A * B * rigid_body;

% E
rigid_body_e = B * A * rigid_body;


% Plots
plot_rigid_body(rigid_body, rigid_body_a, 'A fixed, B current');

plot_rigid_body(rigid_body, rigid_body_b, 'A fixed, B fixed');

plot_rigid_body(rigid_body, rigid_body_c, 'B fixed');

plot_rigid_body(rigid_body, rigid_body_d, 'B fixed, A fixed');

plot_rigid_body(rigid_body, rigid_body_e, 'B fixed, A current');


% Function to plot the rigid body
function plot_rigid_body(original_vertices, vertices, title_text)
    figure;
    fill(vertices(1, :), vertices(2, :), 'b', 'FaceAlpha', 0.3, 'EdgeColor',
'k');
    hold on;
    fill(original_vertices(1, :), original_vertices(2, :), 'r', 'FaceAlpha',
0.3, 'EdgeColor', 'k');
    title('SE2 Transformation');
    xlabel('X');
    ylabel('Y');
    title(title_text);
    grid on;
```

```
    axis equal;
end
```