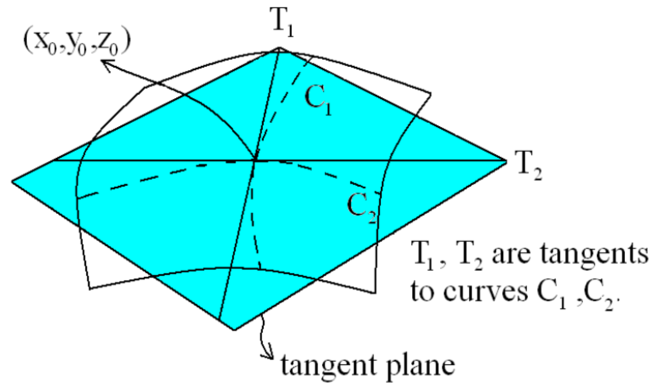
Q1F.

Given $z = f(x,y)$, the directional tangents at a point is given by:

$$T_x = \left(1, 0, \frac{\partial f(x,y)}{\partial x}\right) = \left(1, 0, \frac{\partial z}{\partial x}\right)$$

$$T_y = \left(0, 1, \frac{\partial f(x,y)}{\partial y}\right) = \left(0, 1, \frac{\partial z}{\partial y}\right)$$




(Plot from my undergrad calculus class)

In essence, it calculates how much we move in z-direction for a small change in x- or y-direction.

The span of the directional tangents give the tangent plane, which is perpendicular to the surface normal.

The cross product gives the perpendicular vector:

$$\vec{n} = T_x \times T_y = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix}$$

$$= \hat{x}\left(-\frac{\partial z}{\partial x}\right) - \hat{y}\left(\frac{\partial z}{\partial y}\right) + \hat{z}$$

$$= \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)$$

Normalizing, we get a unit normal

$$\tilde{n} = \frac{\left(-\frac{\partial z}{\partial x}, \frac{-\partial z}{\partial y}, 1\right)}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}} \overset{\text{defn.}}{\equiv} (n_1, n_2, n_3)$$

Denoting $f_x = \frac{\partial f(x,y)}{\partial x} = \frac{\partial z}{\partial x}$ and observing the above vector $\tilde{n}$, we have

$$\frac{-n_1}{n_3} = \frac{\dfrac{-\left(-\frac{\partial z}{\partial x}\right)}{\cancel{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}}}{\dfrac{1}{\cancel{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}}}$$

$$= \frac{\partial z}{\partial x}$$

$$= f_x$$

$$\frac{-n_2}{n_3} = \frac{\dfrac{-\left(-\frac{\partial z}{\partial y}\right)}{\cancel{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}}}{\dfrac{1}{\cancel{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}}}$$

$$= \frac{\partial z}{\partial y}$$

$$= f_y$$

So $f_x = \frac{-n_1}{n_3}$, $f_y = \frac{-n_2}{n_3}$ is the relationship between $\tilde{n}$ and the partial derivatives of $f$ at $(x,y)$.

Q16.

Let $g(x,y) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$

with $(0,0)$ as top left, $x$ in horizontal direction, and $y$ in vertical direction

Defining $g_x(x_i, y_j) = g(x_{i+1}, y_j) - g(x_i, y_j)$ and similarly for $g_y$, we have:

$g_x(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $g_y(x,y) = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$

1. Given $g(0,0) = 1$, we reconstruct the first row of $g$ using $g(x_{i+1}, 0) = g_x(x_i, 0) + g(x_i, 0)$

$\Rightarrow g(1,0) = 1+1 = 2$

$g(2,0) = 1+2 = 3$

$g(3,0) = 1+3 = 4$

$\therefore$ The first row is $g(x,0) = [1 \quad 2 \quad 3 \quad 4]$

Using $g_y$, we reconstruct $g$ using the relation $g(x_i, y_{j+1}) = g_y(x_i, y_j) + g(x_i, y_j)$

With $g_y(x,y) = 4 \cdot I^{3 \times 4}$, we basically add 4 to each element $y_j$ to set $y_{j+1}$.

We obtain

$$g(x,y) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

2. Reconstructing the first column first:

$$g(0,1) = g_y(0,0) + g(0,0) = 4+1 = 5$$

$$g(0,2) = g_y(0,1) + g(0,1) = 4+5 = 9$$

$$g(0,3) = g_y(0,2) + g(0,2) = 4+9 = 13$$

Reconstructing $g$, we use $g(x_{i+1}, y_j) = g_x(x_i, y_j) + g(x_i, y_j)$

Since $g_x(x,y) = I^{4\times3}$, we basically add 1 to each element $x_i$ to set $x_{i+1}$.

We obtain

$$g(x,y) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Both reconstructions are the same.

Imposing integrability means that the gradients are intrinsically linked by the equation

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

(One can show that by imposing the above and using Fubini's theorem, the reconstruction of $g$ is the same)

We see that this is the case for our function since

$$g_x \circ g_y = g_y \circ g_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can change the gradients by simply making $g_x \circ g_y \neq g_y \circ g_x$ to make $g_y$ and $g_x$ non-integrable. Concretely, we can simply alter one element in either $g_x$ or $g_y$ to make it non-integrable.

Some reasons as to why the gradient estimates are non-integrable could be:

- Recall we are using the Moore-Penrose pseudoinverse to calculate $\tilde{n}$. It is a least squares approx. solution. So it may deviate from a "true surface" as noise in the data will cause errors.
- Slopes and edges may have undefined gradients, causing the reconstruction to deviate
- Discretization of a continuous surface to each pixel will cause "averaging" and numerical errors, causing $g_x \circ g_y \neq g_y \circ g_x$ and non-integrability.