Q1.1.3

Let $\vec{x}_1^i = \begin{bmatrix} x_1^i \\ y_1^i \\ 1 \end{bmatrix}$, $\vec{x}_2^i = \begin{bmatrix} x_2^i \\ y_2^i \\ 1 \end{bmatrix}$, and $\vec{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$

with $\vec{x}_1^i \equiv \vec{H}\vec{x}_2^i$

In non-homogenous coordinates, we have

$$x_1^i = \frac{h_{11}x_2^i + h_{12}y_2^i + h_{13}}{h_{31}x_2^i + h_{32}y_2^i + h_{33}} \qquad (1.1)$$

$$y_1^i = \frac{h_{21}x_2^i + h_{22}y_2^i + h_{23}}{h_{31}x_2^i + h_{32}y_2^i + h_{33}} \qquad (1.2)$$

Multiplying by the denominator for (1.1) and (1.2) and making one side equal to zero, we have

$$h_{11}x_2^i + h_{12}y_2^i + h_{13} - (h_{31}x_2^i + h_{32}y_2^i + h_{33})x_1^i = 0$$
$$h_{21}x_2^i + h_{22}y_2^i + h_{23} - (h_{31}x_2^i + h_{32}y_2^i + h_{33})y_1^i = 0$$

$$\Rightarrow x_2^i h_{11} + y_2^i h_{12} + h_{13} - x_1^i x_2^i h_{31} - x_1^i y_2^i h_{32} - x_1^i h_{33} = 0$$
$$x_2^i h_{21} + y_2^i h_{22} + h_{23} - y_1^i x_2^i h_{31} - y_1^i y_2^i h_{32} - y_1^i h_{33} = 0$$

In matrix form, we set

$$\underbrace{\begin{bmatrix} x_2^i & y_2^i & 1 & 0 & 0 & 0 & -x_1^i x_2^i & -x_1^i y_2^i & -x_1^i \\ 0 & 0 & 0 & x_2^i & y_2^i & 1 & -y_1^i x_2^i & -y_1^i y_2^i & -y_1^i \end{bmatrix}}_{\vec{A_i}} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = 0$$

Q1.2.1

Let $\vec{x}_1 = \vec{K}_1 [\vec{I}\ \vec{0}] \vec{X}$ and $\vec{x}_2 = \vec{K}_2 [\vec{R}\ \vec{0}] \vec{X}$

Where $\vec{K}_1$ and K_2 are 3×3 intrinsic matrices

$\vec{I}$ is the 3×3 identity matrix

$\vec{0}$ is a 3×1 zero vector

$\vec{X} \in \mathbb{R}^4$ is the homogenous coordinates of a point in space

$\vec{R}$ is a 3×3 rotation matrix

Let $\tilde{X} \in \mathbb{R}^3$ be the non-homogenous coordinates of $\vec{X}$

Then
$$\vec{x}_1 = \vec{K}_1 [\vec{I}\ \vec{0}] \vec{X} = \vec{K}_1 \tilde{X} \qquad (1.3)$$

and
$$\vec{x}_2 = \vec{K}_2 [\vec{R}\ \vec{0}] \vec{X} = \vec{K}_2 \vec{R} \tilde{X} \qquad (1.4)$$

Noting that $\vec{K}_2$ is upper-triangular with $\det(\vec{K}_2) \neq 0$ and $\vec{R}$ is orthonormal with $\det(\vec{R}) = 1$, so both $\vec{K}_2^{-1}$ and $\vec{R}^{-1}$ exists

From (1.4):

$$\vec{K}_2^{-1} \vec{x}_2 = \overbrace{\vec{K}_2^{-1} \vec{K}_2}^{I} \vec{R} \tilde{X}$$

$$\Rightarrow \vec{R}^{-1} \vec{K}_2^{-1} \vec{x}_2 = \overbrace{\vec{R}^{-1} \vec{R}}^{I} \tilde{X}$$

$$\Rightarrow \tilde{X} = \vec{R}^{-1} \vec{K}_2^{-1} \vec{x}_2$$

Into (1.3):

$$\vec{x}_1 = \vec{K}_1 \vec{R}^{-1} \vec{K}_2^{-1} \vec{x}_2$$

So $\vec{H} = \vec{K}_1 \vec{R}^{-1} \vec{K}_2^{-1}$ satisfies $\vec{x}_1 \equiv \vec{H} \vec{x}_2$

Q1.2.2

From Q1.2.1, the homography H for a rotation is given by

$$\vec{H} = \vec{K}_1 \vec{R}^{-1} \vec{K}_2^{-1}$$

Since the intrinsic parameters are constant, we have $\vec{K} = \vec{K}_1 = \vec{K}_2$

So $\vec{H} = \vec{K}\vec{R}^{-1}\vec{K}^{-1}$

$$\begin{aligned}
\text{So } \vec{H}^2 &= (\vec{K}\vec{R}^{-1}\underbrace{\vec{K}^{-1})(\vec{K}}_{I}\vec{R}^{-1}\vec{K}^{-1}) \\
&= \vec{K}\vec{R}^{-1}\vec{K}^{-1}\vec{K}\vec{R}^{-1}\vec{K}^{-1} \quad \text{(associativity of matrix multiplication)} \\
&= \vec{K}\vec{R}^{-1}\vec{R}^{-1}\vec{K}^{-1} \\
&= \vec{K}(\vec{R}^{-1})^2\vec{K}^{-1} \\
&= \vec{K}(\vec{R}^2)^{-1}\vec{K}^{-1} \quad \text{(commutivity of matrix inverse and multiplication)}
\end{aligned}$$

Since $\vec{R}$ is a rotation of θ, $\vec{R}^2$ is a rotation of 2θ

So $\vec{H}^2 = \vec{K}(\vec{R}^2)^{-1}\vec{K}^{-1}$ is the homography corresponding to a rotation of 2θ.

$\square$