Project Ideas ?

- Uses the idea that the following are equivalent:

$\{\text{CDF of } X\} \Leftrightarrow \{\text{MGF of } X\} \Leftrightarrow \{\text{Moments of } X\}$

- The Moment Generating Function of X is defined as

$$M_{\vec{X}}(\vec{s}) = \mathbb{E}(e^{\vec{s}^T \vec{X}})$$

With properties:

1. If $Y = aX + \vec{b}$, then $M_{\vec{Y}}(\vec{s}) = e^{\vec{s}^T b} M_{\vec{X}}(a\vec{s})$

2. If X and Y indep., then $M_{X+Y}(\vec{s}) = M_X(\vec{s}) M_Y(\vec{s})$

- We characterize the state using MGFs instead of the CDF (or PDF) of X. Their equivalency is guaranteed by the uniqueness theorem.

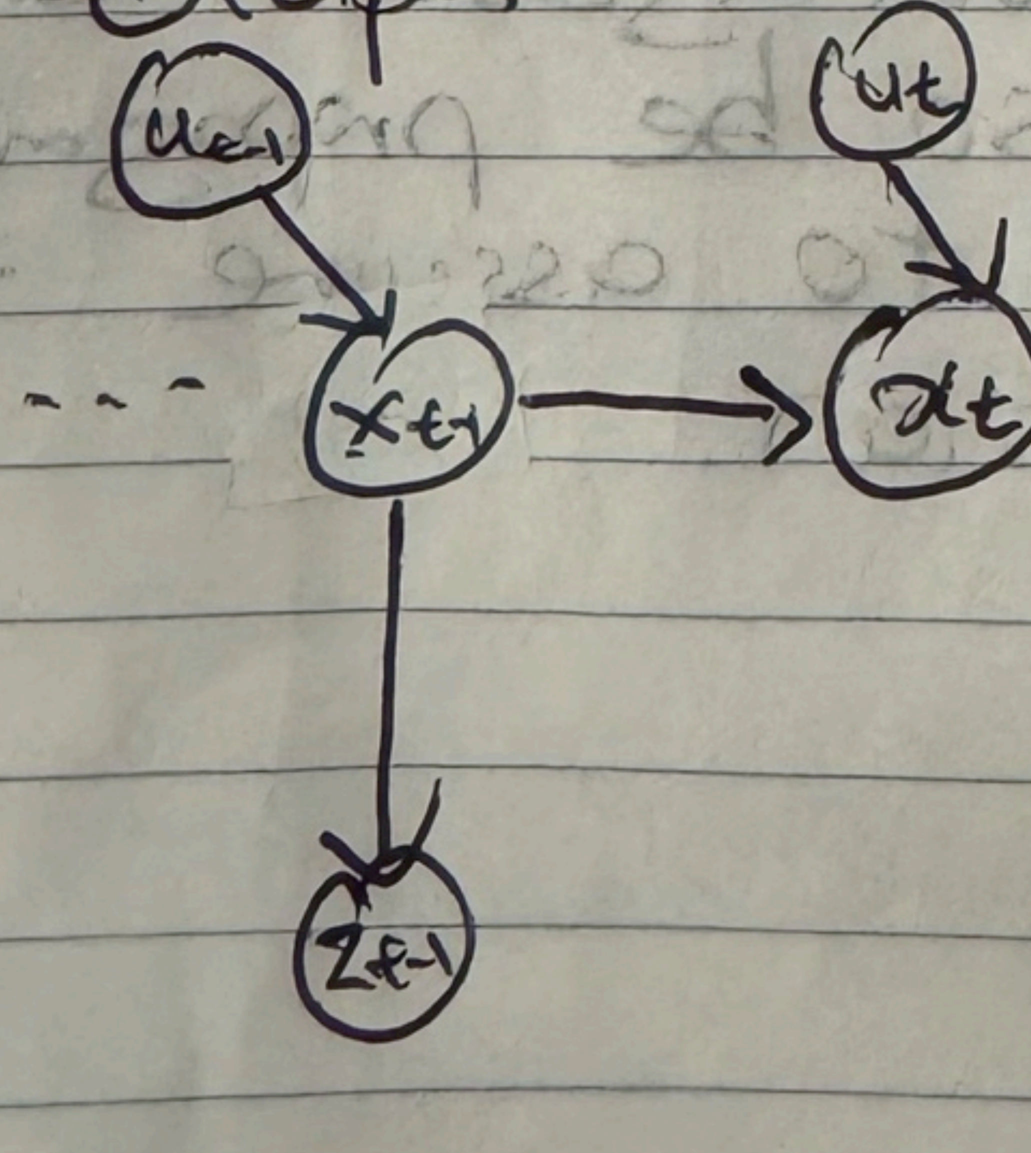
- Using MGFs is advantageous since higher moments can also be propagated. We no longer need to assume that the state is Gaussian.

$\vec{b} = b\vec{u}_{t-1}$

Assumptions:

1. Motion Model: $\tilde{X}_t = a\vec{X}_{t-1} + \vec{b}_{t-1}$
   - For now, a is scalar (Matrix A is also possible I think)
   - $\vec{b} = b\vec{u}_{t-1}$ is the control, it is a random variable with a distribution
2. Sensor Model: $\vec{Z}_{t-1} \sim$ some measurement model
   - $\vec{Z}_{t-1}$ is transformed to the frame of $\vec{X}_t$
     E.g. if laser rangefinder, $\{r, \beta\} \to \{x, y, \theta\}$
   - $\vec{Z}_{t-1}$ is stochastic with a distribution
3. Fusion: $\vec{X}_t = \alpha\vec{Z}_{t-1} + (1-\alpha)\tilde{X}_t$
   - Linear combination, α could be based on information criterion, etc...

- Graph:



Is this reasonable?

Assuming $\vec{z}_{t-1}$ and $\tilde{x}_t$ independent given $\vec{x}_{t-1}$, in MGF space, we have:

$$M_{\vec{x}_t}(\vec{s}) = M_{\vec{z}_{t-1}}(\alpha\vec{s})\, M_{\tilde{x}_t}((1-\alpha)\vec{s})$$
$$= M_{\vec{z}_{t-1}}(\alpha\vec{s})\, M_{\vec{x}_{t-1}}((1-\alpha)a\vec{s})\, M_{\vec{b}_{t-1}}((1-\alpha)\vec{s})$$
$$= \underbrace{M_{\vec{b}_{t-1}}((1-\alpha)\vec{s})}_{f}\, \underbrace{M_{\vec{z}_{t-1}}(\alpha\vec{s})}_{g}\, \underbrace{M_{\vec{x}_{t-1}}((1-\alpha)a\vec{s})}_{h}$$

· A property of MGFs is that

$$\mathbb{E}[X^k] = M_X^{(k)}\big|_{s=0}$$ (also version for vector $\vec{x}$)

So differentiation gives moments of X

For scalar X:

$$\mathbb{E}[X_t^n] = \frac{\partial^n}{\partial s^n}[f(s)g(s)h(s)]\Big|_{s=0}$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{n!}{i!\,j!\,(n-i-j)!} f^{(i)}(s)\, g^{(j)}(s)\, h^{(n-i-j)}(s)\Big|_{s=0}$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{n!}{i!\,j!\,(n-i-j)!} (1-\alpha)^i M_{b_{t-1}}^{(i)}((1-\alpha)s)\, \alpha^j M_{z_{t-1}}^{(j)}(\alpha s)\, [a(1-\alpha)]^{n-i-j} M_{x_{t-1}}^{(n-i-j)}((1-\alpha)as)\Big|_{s=0}$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{n!}{i!\,j!\,(n-i-j)!} (1-\alpha)^i\, \mathbb{E}(b_{t-1}^i)\, \alpha^j\, \mathbb{E}(z_{t-1}^j)\, [a(1-\alpha)]^{n-i-j}\, \mathbb{E}(x_{t-1}^{n-i-j})$$

Letting $n=1$, we recover

$$\mathbb{E}(X_t) = \alpha \mathbb{E}(Z_{t-1}) + (1-\alpha)[a\mathbb{E}(X_{t-1}) + \mathbb{E}(b_{t-1})]$$

Letting $n=2$, we recover (with b fixed)

$$\mathbb{E}(X_t^2) = \underbrace{\alpha^2[\mathbb{E}(Z_{t-1}^2) - \mathbb{E}(Z_{t-1})^2] + (1-\alpha)^2 a^2 [\mathbb{E}(X_{t-1}^2) - \mathbb{E}(X_{t-1})^2]}_{var(X_t)}$$
$$+ \underbrace{[\alpha\mathbb{E}(Z_{t-1}) + (1-\alpha)[a\mathbb{E}(X_{t-1}) + b]]^2}_{\mathbb{E}(X_t)^2}$$

Which is the equations for the 1st and 2nd moments of a linear combination

Idea for algorithm:

Prior: Represent prior at $t-1$ with
$\{\mathbb{E}(X), \mathbb{E}(X^2), \ldots \mathbb{E}(X^n), \mathbb{E}(b), \mathbb{E}(b^2) \ldots,$
$\mathbb{E}(Z), \mathbb{E}(Z^2) \ldots \mathbb{E}(Z^n)\}$

Note: Taylor exp. of MGF gives

$$M_X(t) = \sum_{n=0}^{\infty} \frac{\mathbb{E}[X^n]}{n!} t^n$$

So if we want to represent distribution of X to be more precise, we can include more terms of $\mathbb{E}(X^n)$. In the limit $n \to \infty$, the dist. of X is exact

Some terms can be 0 if the higher order moments are negligible.

Posterior: Represent state using

$$\mathbb{E}[X_t^n] = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{n!}{i!\,j!\,(n-i-j)!} (1-\alpha)^i \mathbb{E}(b_{t-1}^i) \alpha^j \mathbb{E}(z_{t-1}^j) \cdot [a(1-\alpha)]^{n-i-j} \mathbb{E}(X_{t-1}^{n-i-j})$$

- So far the model is linear motion model, arbitrary state model. For non-linear motion model + arbitrary state, we can Taylor expand the motion model:

$$\tilde{X}_t = m(X_{t-1}) = a X_{t-1} + b X_{t-1}^2 + \cdots z X_{t-1}^m$$

This introduces $M_{X_{t-1}^2}(s), \ldots M_{X_{t-1}^m}(s)$ into the MGF propagation. So for state model of n-terms, need $n+m$ for non-linear motion + arbitrary state.

+: No matrix inverses, only lookup tables
Arbitrary state (no assumption of Gaussian)
Maybe extendable to non-linear motion

-: Arbitrary α value, not using Bayesian updating ratio
Lookup tables explode for higher dimensional $\vec{X}$, $\vec{b}$ and $\vec{Z}$

/: a is currently a scalar, could extend to Matrix A
I think

(Sketch of Proof)

I haven't worked out the details yet..

For $X \in \mathbb{R}^d$, the MGF is still valid:

$$E[X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_d^{\alpha_d}] = \frac{\partial^{(\alpha_1 + \cdots + \alpha_d)}}{\partial s_1^{\alpha_1} \cdots \partial s_d^{\alpha_d}} M_{\vec{X}}(\vec{s}) \Big|_{\vec{s}=0}$$

However, the moments are:

1st moment: $\mathbb{R}^d$
2nd moment: $\mathbb{R}^{d \times d}$
⋮
nth moment: $\mathbb{R}^{d \overset{n}{\otimes} d}$ - dimensionality may get out of hand

Deriving in vector form gives (using scalar A)

$$M_{\vec{X}_t}(\vec{s}) = \underbrace{M_{\vec{b}}((1-\alpha)\vec{s})}_{f} \underbrace{M_{\vec{Z}_{t-1}}(\alpha \vec{s})}_{g} \underbrace{M_{\vec{X}_{t-1}}((1-\alpha)A\vec{s})}_{h}$$

So, for the nth moments in a $\mathbb{R}^{d \overset{n}{\otimes} d}$ dimensional tensor:

$$D^n[fgh](\vec{s}) = \sum_{a+b+c=n} \frac{n!}{a!\,b!\,c!} D^a f(\vec{s}) \otimes D^b g(\vec{s}) \otimes D^c h(\vec{s})$$

Where $\otimes$ is outer product (tensor)
D^α is nth derivative tensor

The moments are when we eval. $\vec{s}=0$

So:

$$D^a f(\vec{\xi})|_{\vec{\xi}=0} = D^a M_{\vec{b}}((1-\alpha)\vec{\xi})$$
$$= \underbrace{(1-\alpha)^a}_{\text{scaler}} \times \underbrace{a\text{'th moments of } \vec{b}}_{\text{known from prior}}$$

$$D^b g(\vec{\xi})|_{\vec{\xi}=0} = D^b M_{\vec{z}_{t-1}}(\alpha\vec{\xi})$$
$$= \underbrace{\alpha^b}_{\text{scaler}} \times \underbrace{b\text{'th moments of } \vec{z}_{t-1}}_{\text{known from prior}}$$

$$D^c h(\vec{\xi})|_{\vec{\xi}=0} = D^c M_{\vec{x}_{t-1}}((1-\alpha)A\vec{\xi})$$
$$= \underbrace{A^c}_{\text{scaler}} \times \underbrace{c\text{'th moments of } \vec{x}_{t-1}}_{\text{known from prior}}$$

We have $(\mathbb{R}^{d \overset{a}{\otimes} d}) \otimes (\mathbb{R}^{d \overset{b}{\otimes} d}) \otimes (\mathbb{R}^{d \overset{c}{\otimes} d})$

$= \mathbb{R}^{d \overset{n}{\otimes} d}$ since $a+b+c=n$

For $n=1$, gives

$$\mathbb{E}[\vec{x}_t] = (1-\alpha)\mathbb{E}(\vec{b}_{t-1}) + \alpha\,\mathbb{E}(\vec{z}_{t-1}) + (1-\alpha)A\,\mathbb{E}(\vec{x}_{t-1}$$