Lecture 1: Introduction to the Complex Plane

MAST30021 Complex Analysis: semester 1, 2023

Dr Mario Kieburg

m.kieburg@unimelb.edu.au

School of Mathematics and Statistics,
University of Melbourne

Please note the following lecture arrangements

- Handwritten material will be recorded on the document camera and available on the lecture capture.
- The half-empty slides are all available on the LMS in advance of the lectures.
- Fill in the spaces in the slides during the lecture or from the lecture recording.

Although all material is available to you for remote study, the subject consistently sees a strong correlation between lecture attendance and high marks.

The Imaginary Unit

Questions: What is the general solution of

(1) $p(x) = x^n + \sum_{j=0}^{n-1} c_j x^j = 0$ with $c_0, \ldots, c_{n-1} \in \mathbb{R}$

(2) $x^2 - 2ax + b = 0$ with $a, b \in \mathbb{R}$

(3) $x^2 + 1 = 0$

Generally, we need the imaginary unit "i" to solve these equations.

Answer: (Let us start backwards)

(3) $x = \pm i$

(2) $x = a \pm \begin{cases} \sqrt{a^2 - b}, & b < a^2, \\ i\sqrt{b - a^2}, & b > a^2, \end{cases}$

(1) Fundamental Theorem of Algebra: $p(x) = (x - x_1) \cdots (x - x_n)$

Defining property: $i^2 = -1$

Fundamental Theorem of Algebra

Theorem 1.1. A polynomial

$$p(x) = x^n + \sum_{j=0}^{n-1} c_j x^j, \quad \text{with } c_0, \ldots, c_{n-1} \in \mathbb{C},$$

of degree $n \in \mathbb{N}$ has exactly n complex roots counted with their multiplicities.

Recall:

- x_0 is a root of the polynomial $p(x)$ if $p(x_0) = 0$.
- m is the multiplicity of the root x_0 of $p(x)$ if

$$q(x) = \frac{p(x)}{(x - x_0)^m}$$

is a polynomial and x_0 is **not** a root of $q(x)$.

Hamilton's construction of the complex numbers

Definition 1.2. A **complex number** is an ordered pair $\langle x, y\rangle$ of real numbers $x, y \in \mathbb{R}$, with

- the vector addition and scalar multiplication of $\mathbb{R}^2$:

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle,$$
$$\lambda\langle x_1, y_1\rangle = \langle \lambda x_1, \lambda y_1\rangle,$$

  for any $\lambda, x_1, x_2, y_1, y_2 \in \mathbb{R}$

- and the product rule

$$\langle x_1, y_1\rangle \cdot \langle x_2, y_2\rangle = \langle x_1x_2 - y_1y_2, x_1y_2 + x_2y_1\rangle.$$

Essentially, $\mathbb{C}$ is equal to $\mathbb{R}^2$ with an additional multiplicative structure!

Properties of the complex numbers I

1. **Zero element:** $\langle 0,0\rangle \hat{=} 0$

$$\langle 0,0\rangle + \langle x,y\rangle = \langle 0+x, 0+y\rangle = \langle x,y\rangle,$$
$$\langle 0,0\rangle \cdot \langle x,y\rangle = \langle 0\cdot x - 0\cdot y, 0\cdot y + x\cdot 0\rangle = \langle 0,0\rangle$$

2. **Negative element:** $\langle -x,-y\rangle = -\langle x,y\rangle$

$$\langle -x,-y\rangle + \langle x,y\rangle = \langle -x+x, -y+y\rangle = \langle 0,0\rangle$$

Properties of the complex numbers II

3. **Unit element:** $\langle 1, 0\rangle \hat{=} 1$

$$\langle 1, 0\rangle \cdot \langle x, y\rangle = \langle 1 \cdot x - 0 \cdot y, 1 \cdot y + x \cdot 0\rangle = \langle x, y\rangle$$

4. **Reciprocal element:** $\dfrac{1}{\langle x, y\rangle} = \left\langle \dfrac{x}{x^2+y^2}, -\dfrac{y}{x^2+y^2} \right\rangle$

$$\frac{1}{\langle x, y\rangle} \cdot \langle x, y\rangle = \left\langle \frac{x \cdot x - (-y) \cdot y}{x^2+y^2}, \frac{x \cdot y + x \cdot (-y)}{x^2+y^2} \right\rangle = \langle 1, 0\rangle$$

only for $x^2 + y^2 \neq 0$

Properties of the complex numbers III

5. **Modulus of a complex number:**

Definition 1.3. The complex conjugation is defined by

$$\langle x, y\rangle^* = \langle x, -y\rangle.$$

(Sometimes also denoted by $\overline{\langle x, y\rangle}$.)

$$\begin{aligned}
|\langle x, y\rangle|^2 =& \langle x, y\rangle^* \cdot \langle x, y\rangle \\
=& \langle x \cdot x - (-y) \cdot y, x \cdot y + x \cdot (-y)\rangle \\
=& \langle x^2 + y^2, 0\rangle \hat{=} x^2 + y^2
\end{aligned}$$

Properties of the complex numbers IV

6. **Decomposition into "Cartesian form":**

$$\langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle = x\langle 1, 0\rangle + y\langle 0, 1\rangle.$$

Definition 1.4. Define the **real unit** and the **imaginary unit**

$$1 = \langle 1, 0\rangle \qquad \text{and} \qquad i = \langle 0, 1\rangle.$$

In Summary

$$\text{So if } x, y \in \mathbb{R}, \quad \langle x, y \rangle = x + iy, \quad \text{where} \quad i^2 = -1;$$

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \quad \text{if} \quad x + iy \neq 0.$$

Standard algebra works in $\mathbb{C}$, provided we remember that

$$i^2 = -1 \quad \text{and} \quad \frac{1}{i} = -i.$$

Definition 1.5. If $z = x + iy$ $(x, y \in \mathbb{R})$ we write

$$\begin{aligned} x &= \mathrm{Re}\{z\} = \text{"the real part of } z\text{"}, \\ y &= \mathrm{Im}\{z\} = \text{"the imaginary part of } z\text{"}, \\ \text{so} \quad z &= \mathrm{Re}\{z\} + i\mathrm{Im}\{z\}. \end{aligned}$$

n.b. $\mathrm{Im}\{z\}$ is **real**.

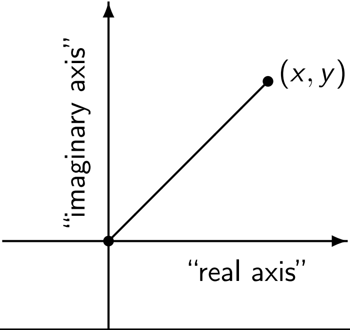
The complex plane (and Argand diagrams)

$\mathbb{C}$ has a bijective correspondence with the Cartesian plane $\mathbb{R}^2$:

$$z = x + iy \in \mathbb{C} \quad \leftrightarrow \quad (x, y) \in \mathbb{R}^2$$

The **origin**: $0 \in \mathbb{C} \quad \leftrightarrow \quad (0, 0) \in \mathbb{R}^2$

"the set $\mathbb{C}$" $\Leftrightarrow$ "the complex numbers" $\Leftrightarrow$ "the complex plane"

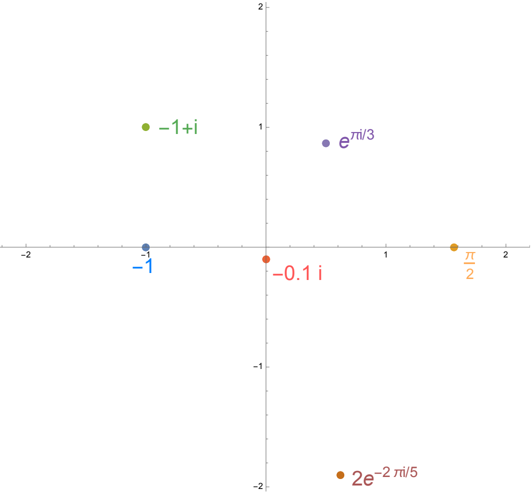


Definition 1.6. **Modulus** of $z = x + iy \in \mathbb{C}$ is

$$|z| = \sqrt{x^2 + y^2}.$$

(Distance of (x, y) from the origin.)

Objects in higher dimensions than 1 cannot be easily ordered!



1. We can say $a < b$ if $a, b \in \mathbb{R}$, e.g., $-1 < \pi/2$.
2. We <u>**cannot**</u> say $-1 + i < -0.1i$ or $e^{i\pi/3} < 2e^{-2i\pi/5}$, even when the modulus of one complex number is bigger than the other.

Complex numbers <u>cannot</u> be ordered!

Complex Conjugation Revisited

You are expected to *know and to be able to prove* all of the following results:

For any $z \in \mathbb{C}$, $w \in \mathbb{C}$:

1. $(z^*)^* = z$
2. $(z + w)^* = z^* + w^*$
3. $(zw)^* = z^* w^*$
4. $z^* = z$ if and only if $\mathrm{Im}\{z\} = 0$; $z + z^* = 2\mathrm{Re}\{z\}$
5. $z^* = -z$ if and only if $\mathrm{Re}\{z\} = 0$; $z - z^* = 2i\,\mathrm{Im}\{z\}$

Modulus Revisited

You are expected to *know and to be able to prove* all of the following results:

For any $z \in \mathbb{C}$, $w \in \mathbb{C}$:

1. $|z| = 0$ if and only if $z = 0$
2. $|z^*| = |z|$
3. $|zw| = |z||w|$
4. $|z| \geq \max\big\{|\mathrm{Re}\{z\}|, |\mathrm{Im}\{z\}|\big\}$
5. $|z + w| \leq |z| + |w|$ (the Triangle Inequality)
6. $|(|z| - |w|)| \leq |z \pm w|$ (reverse Triangle Inequality)