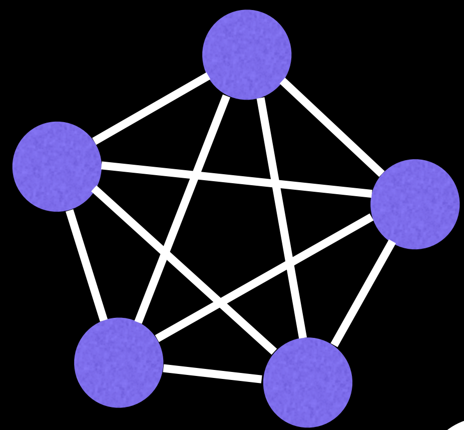
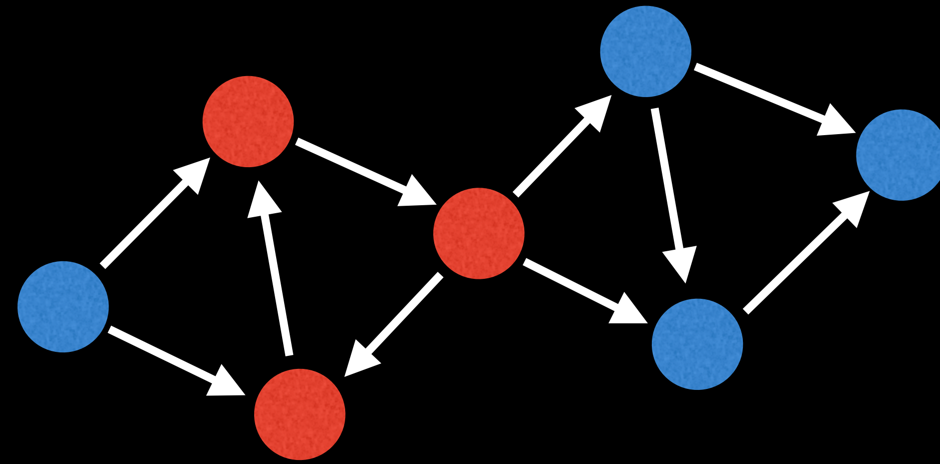
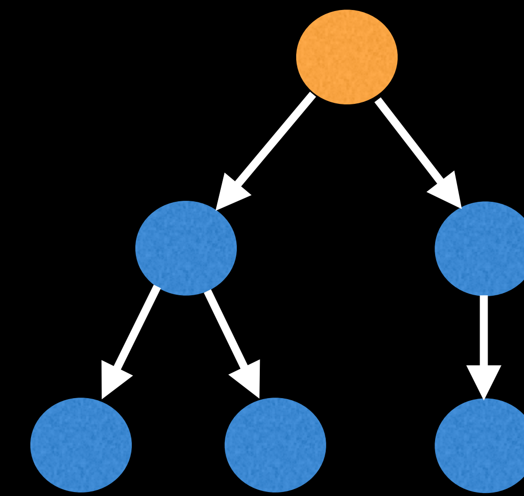
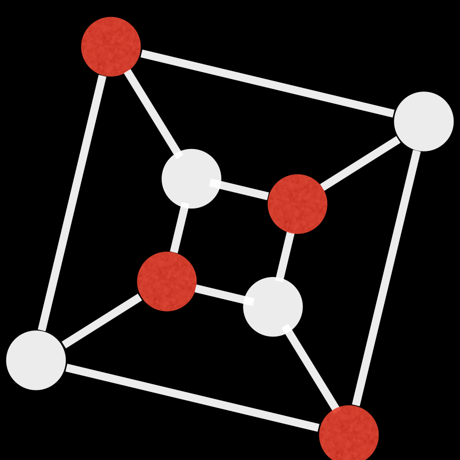
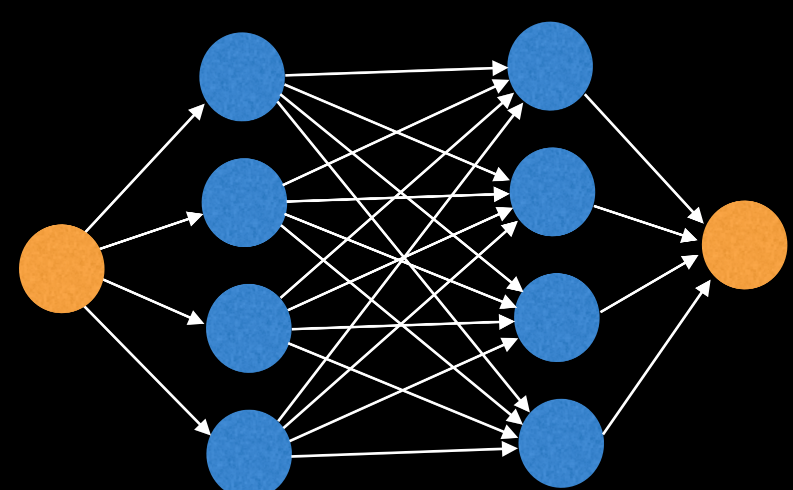


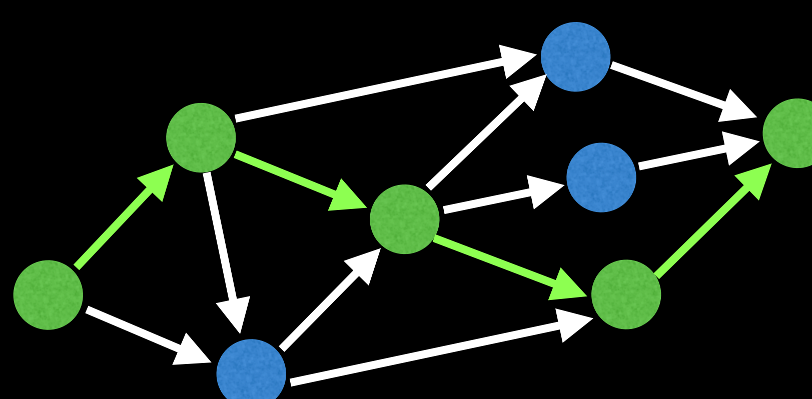




Graph Theory Video Series







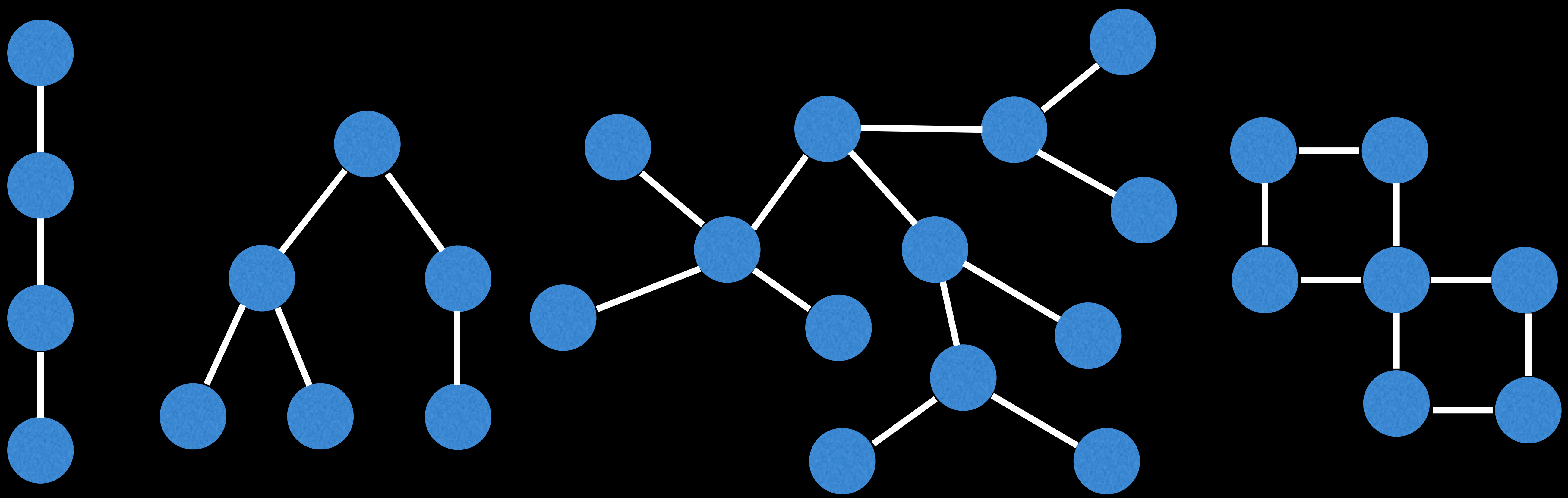
Storage and representation of trees

Definitions and storage representation

William Fiset

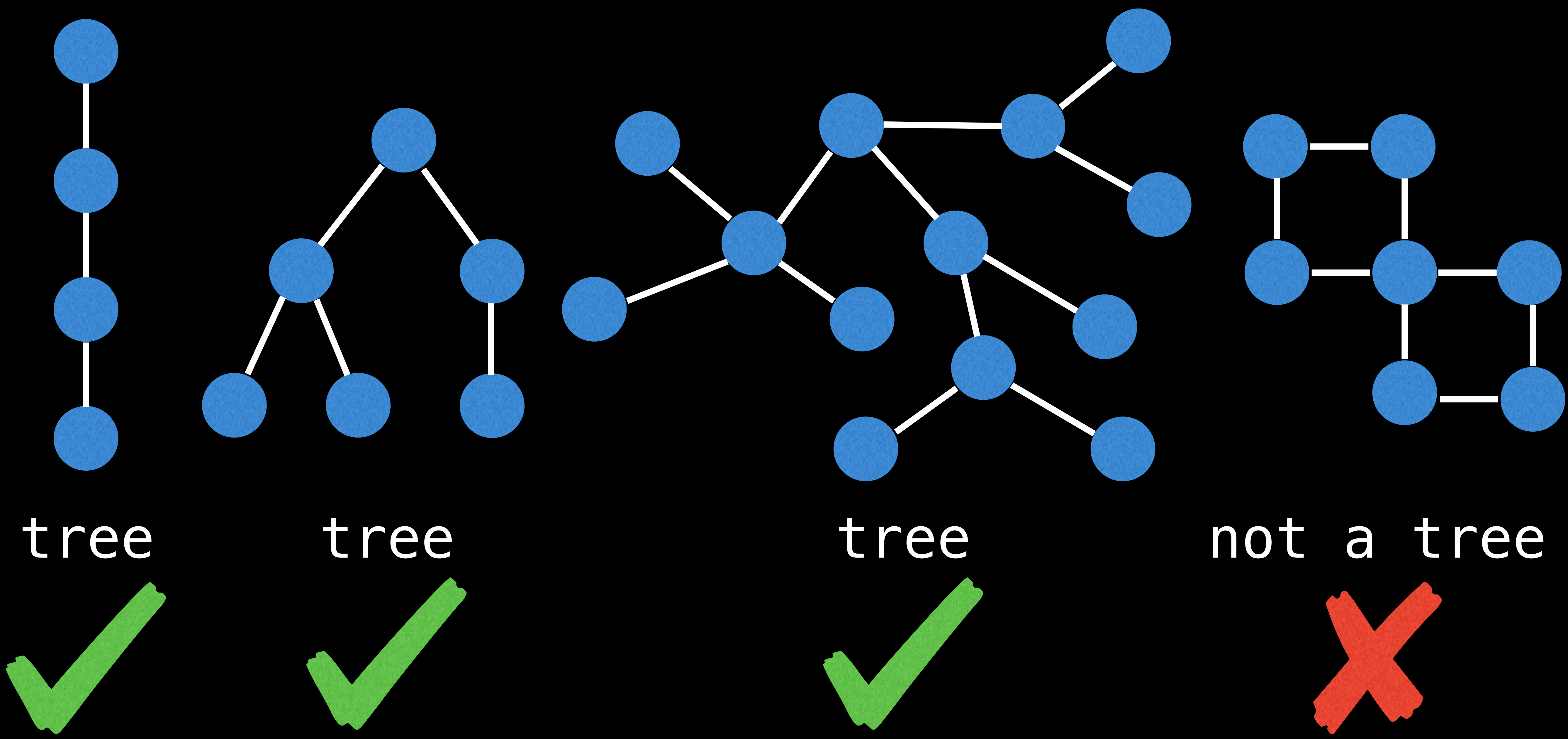
Trees!

What ***is*** a tree?



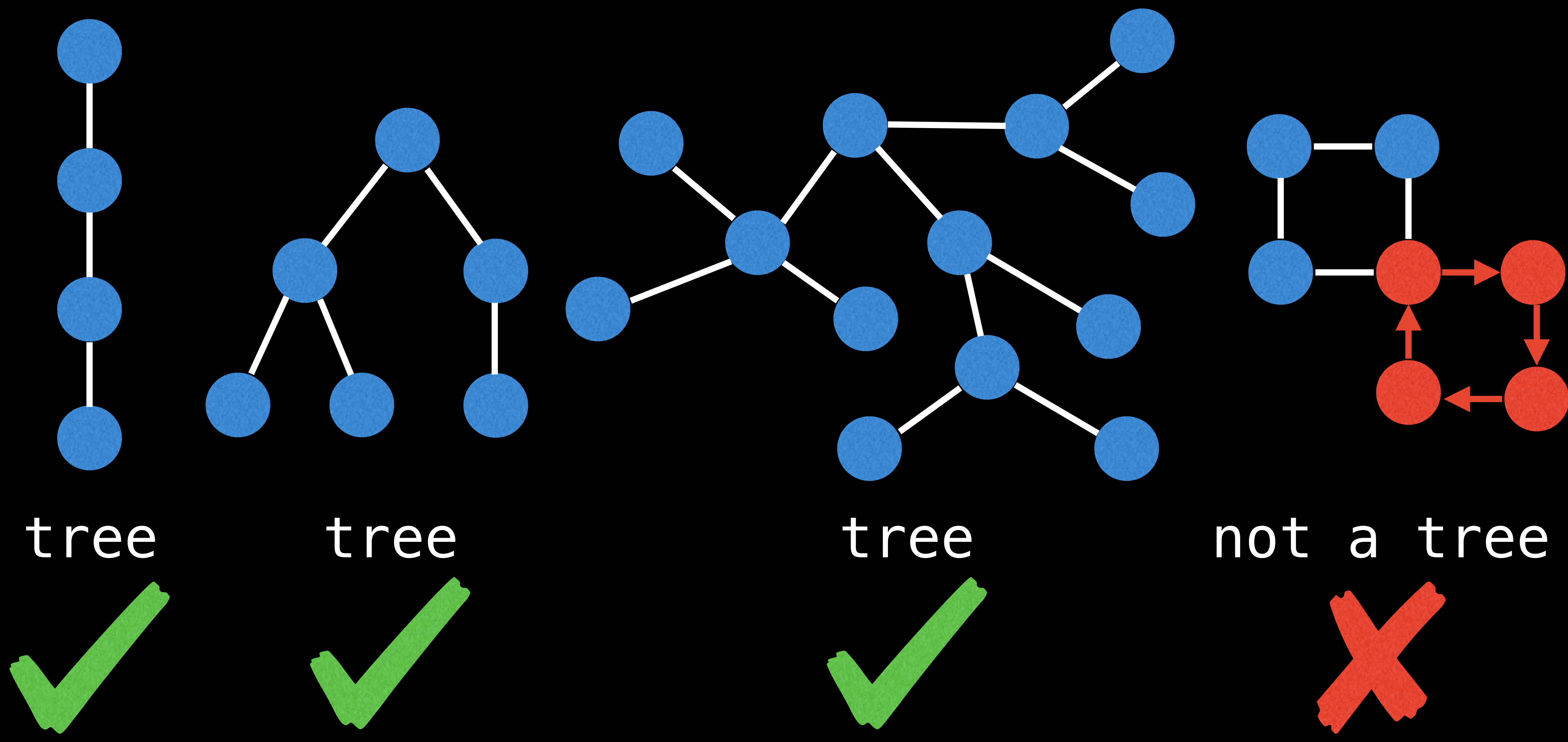
Trees!

What ***is*** a tree?



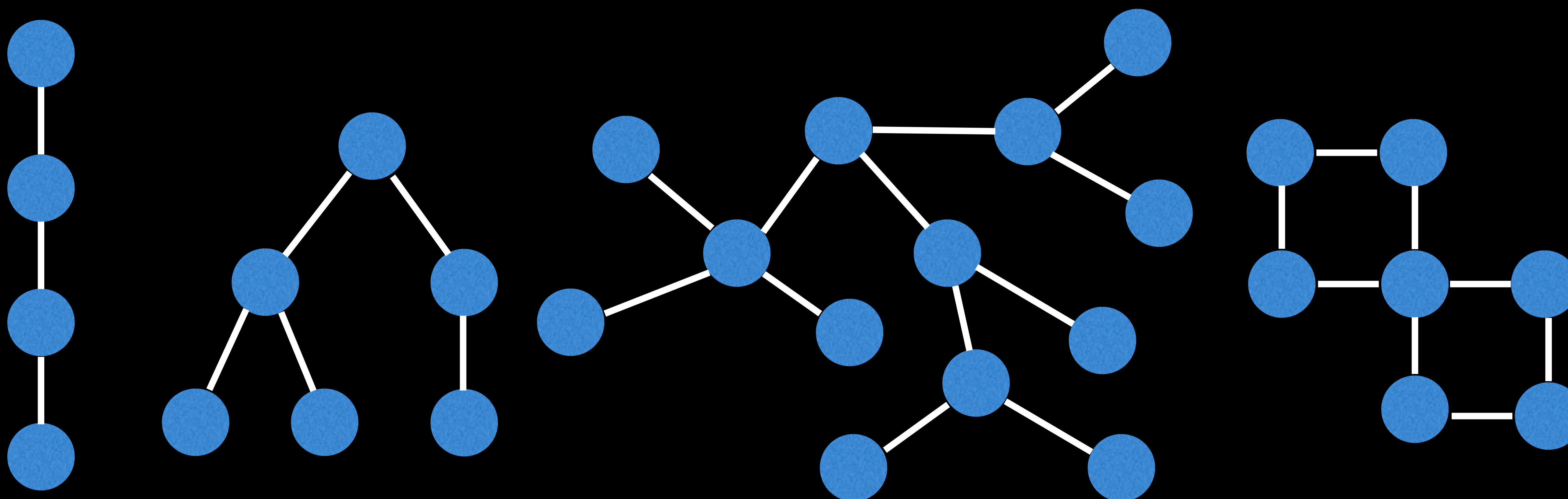
Trees!

A **tree** is a connected, undirected graph with **no cycles**.



Trees!

Equivalently, a **tree** it is a connected graph with N nodes and N–1 edges.



4 nodes
3 edges

6 nodes
5 edges

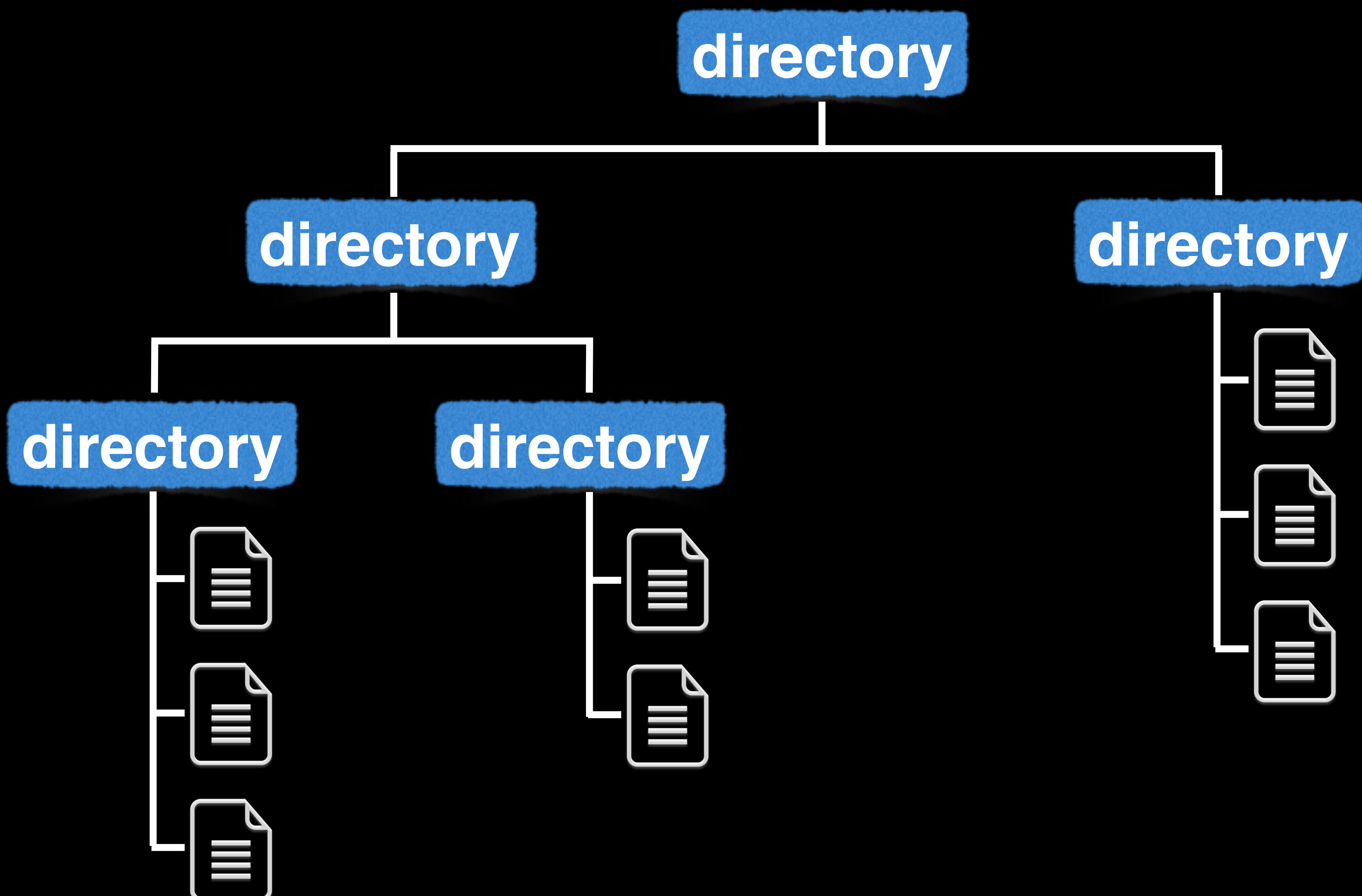
13 nodes
12 edges

7 nodes
8 edges

Trees out in the wild

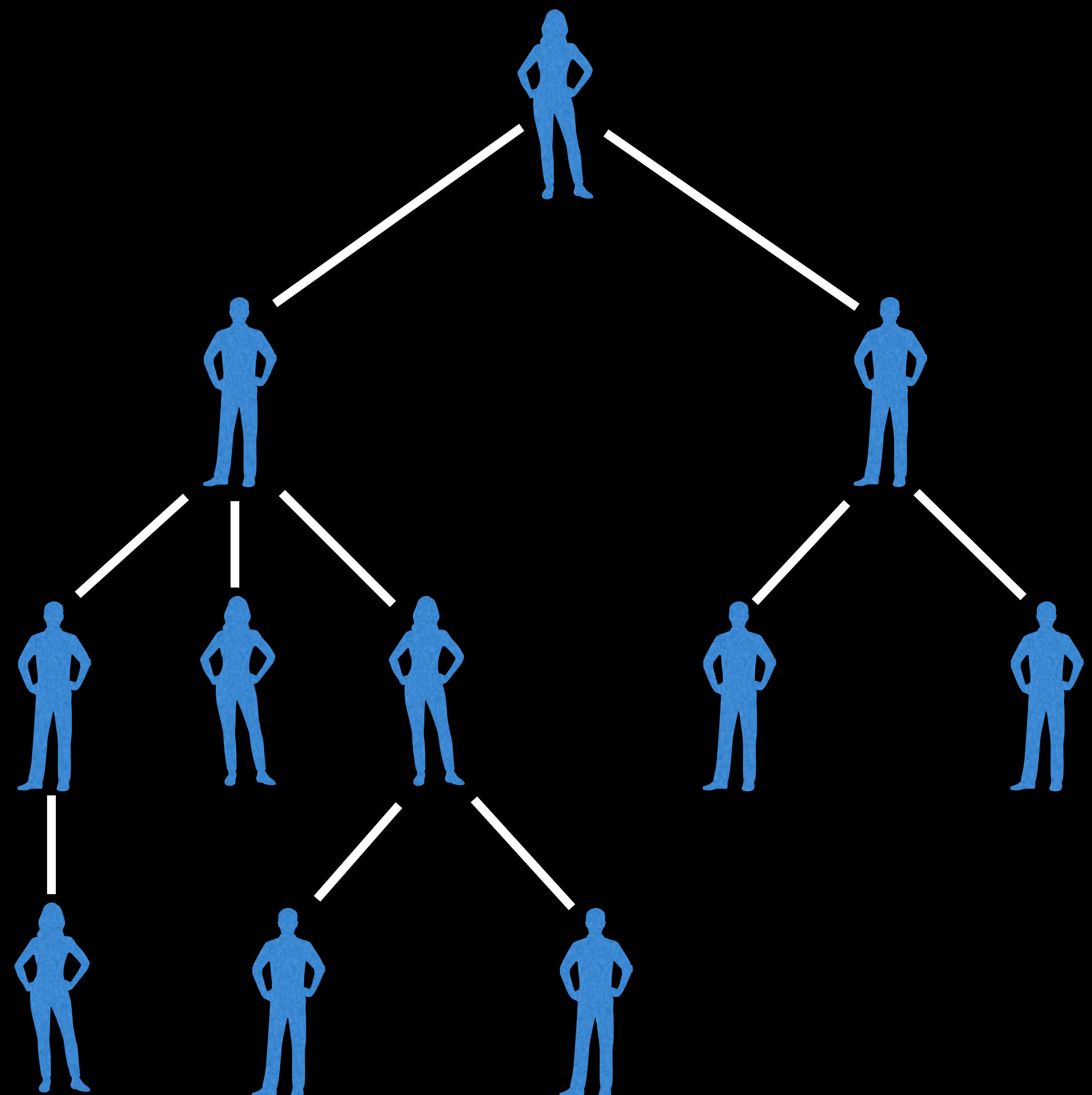
Trees out in the wild

Filesystem structures are inherently trees



Trees out in the wild

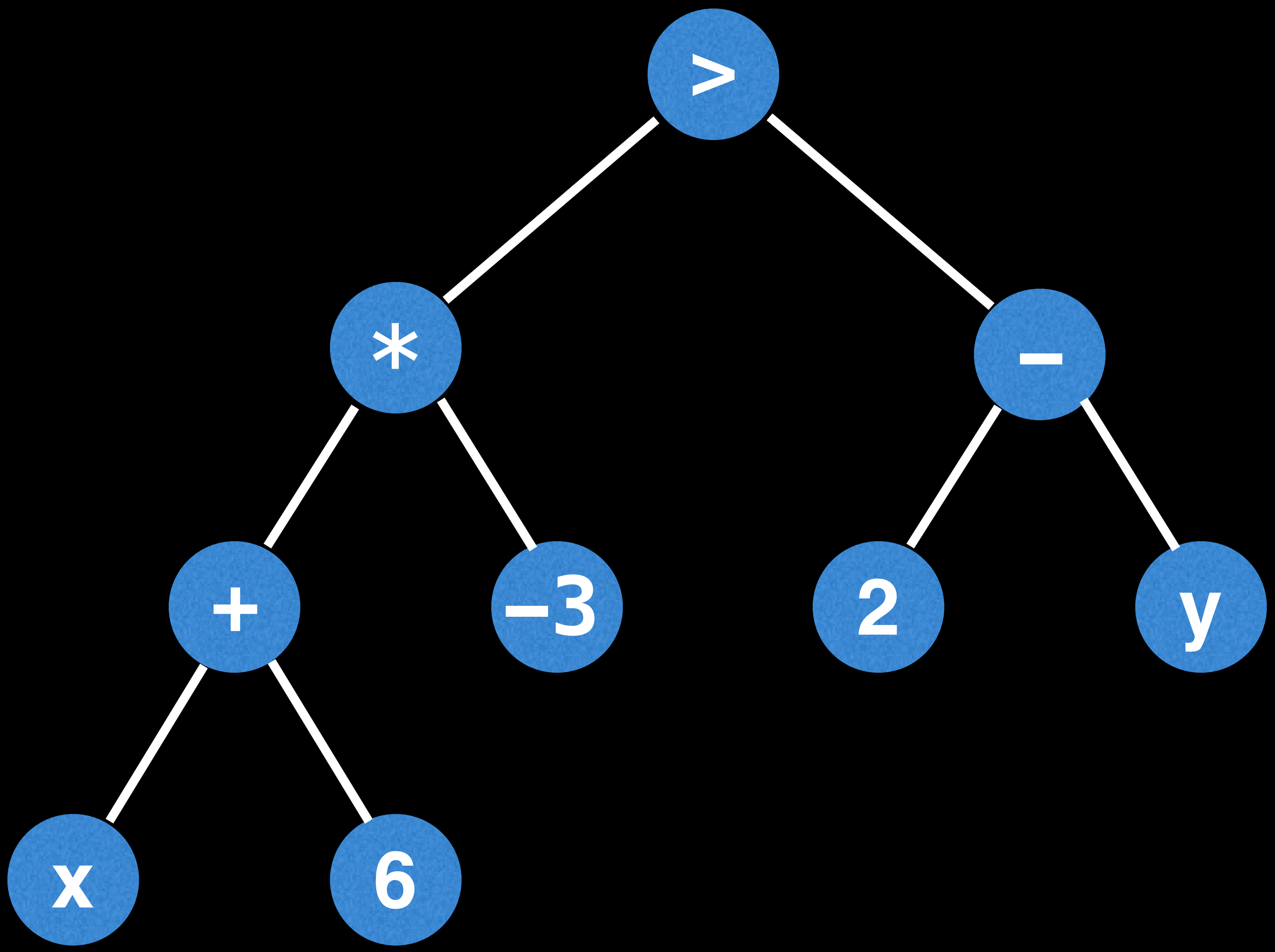
Social hierarchies



Trees out in the wild

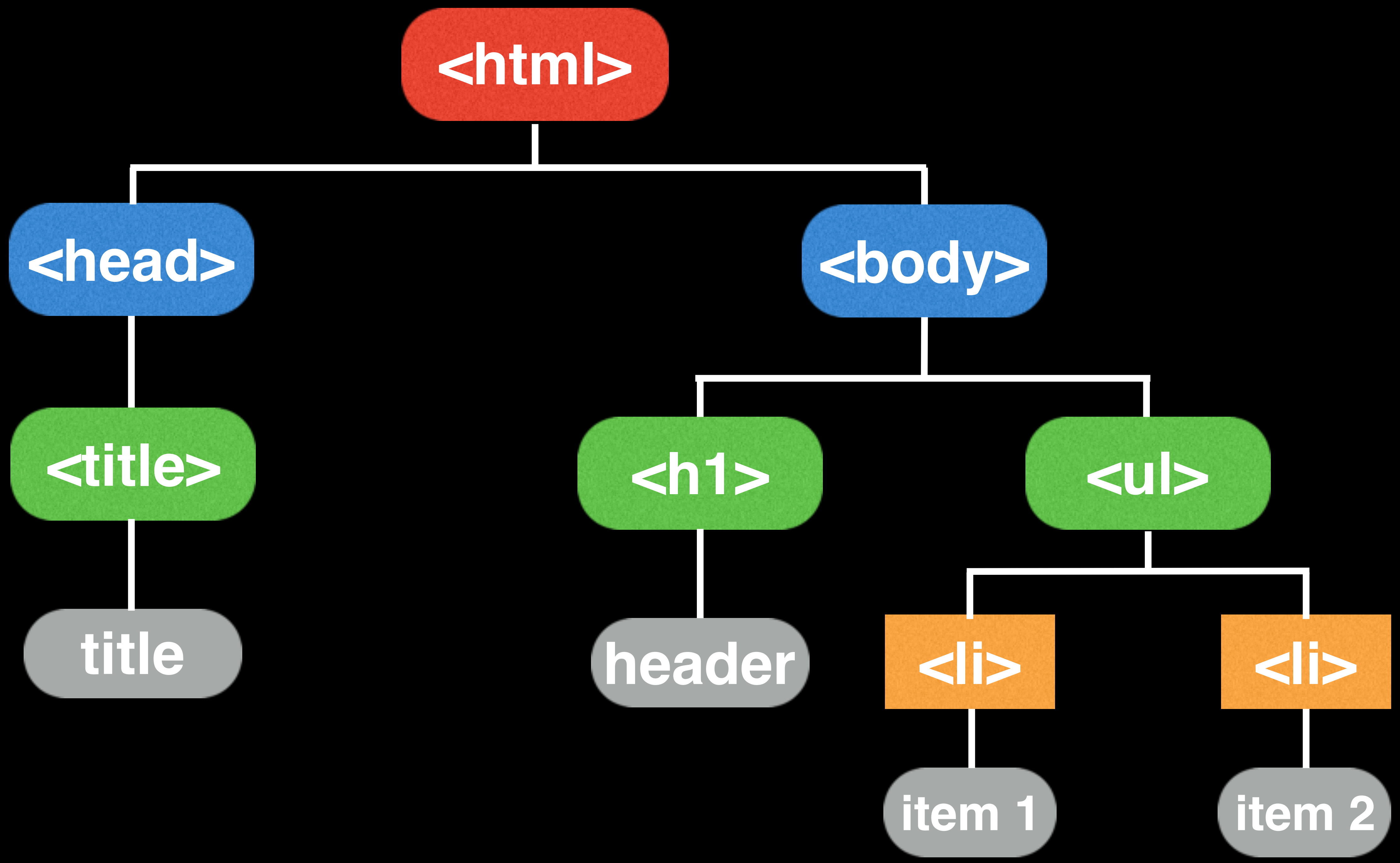
Abstract syntax trees to decompose source code and mathematical expressions for easy evaluation.

$$((x + 6) * -3) > (2 - y)$$



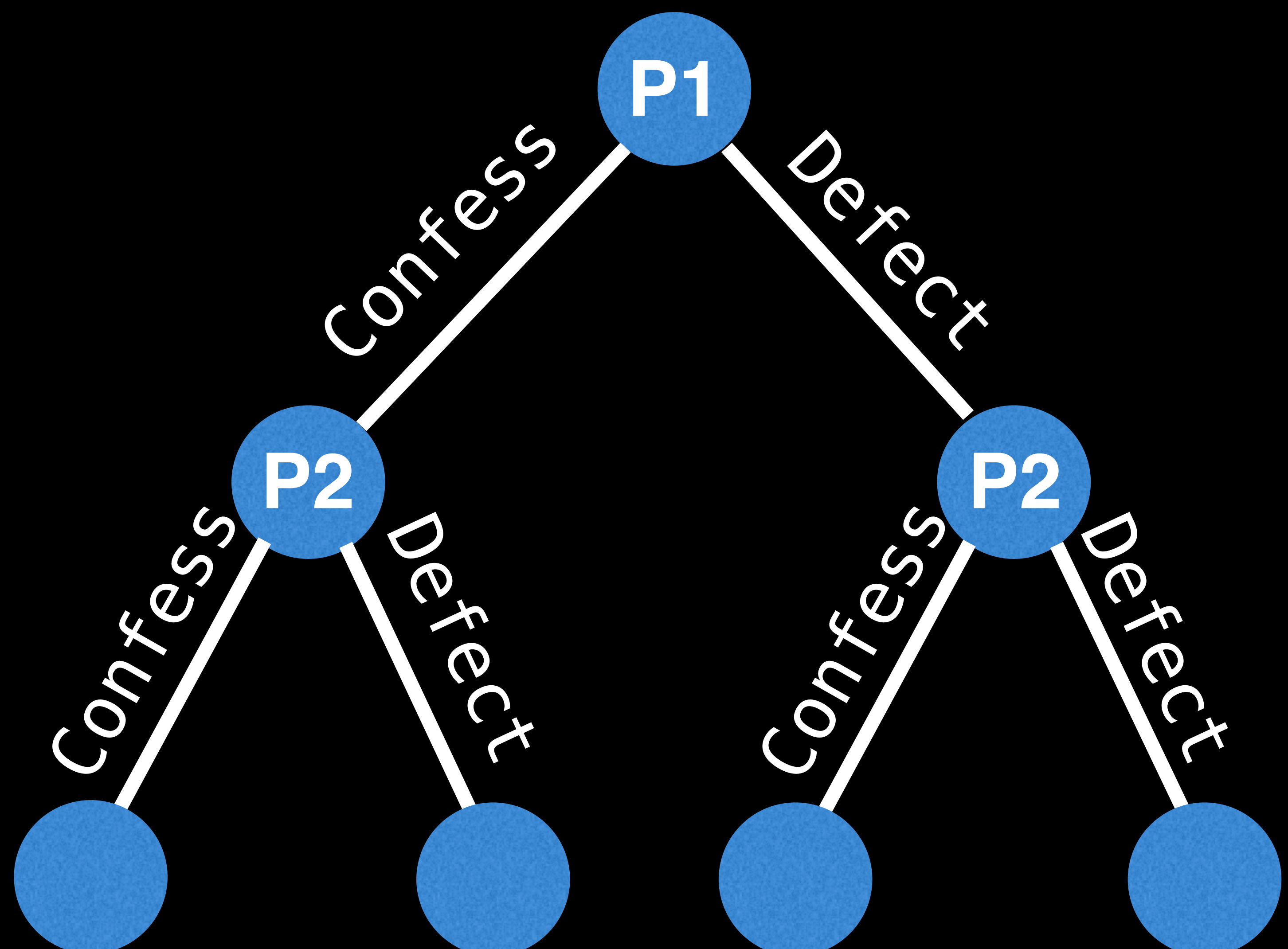
Trees out in the wild

Every webpage is a tree as an HTML DOM structure



Trees out in the wild

The decision outcomes in game theory are often modeled as trees for ease of representation.



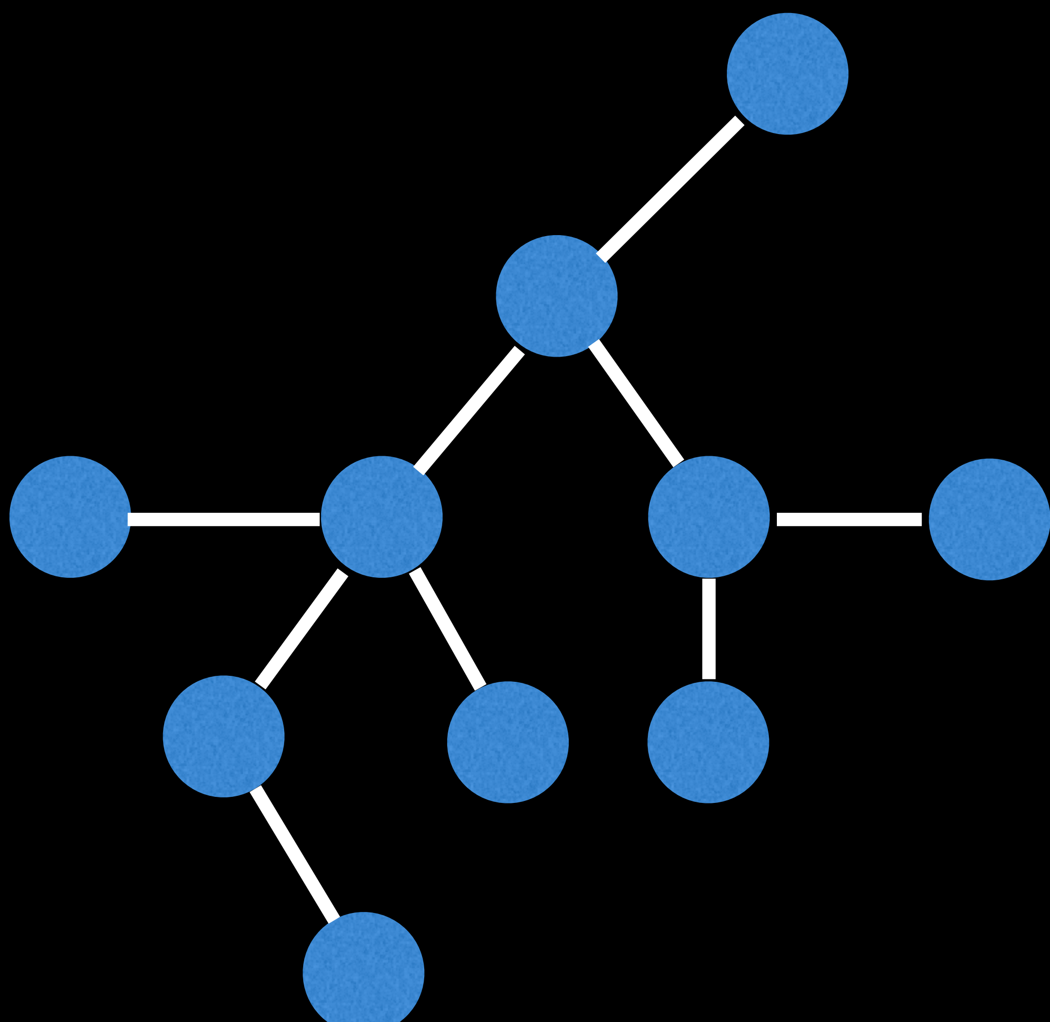
Tree of the prisoner’s dilemma

Trees out in the wild

There are many many more applications…

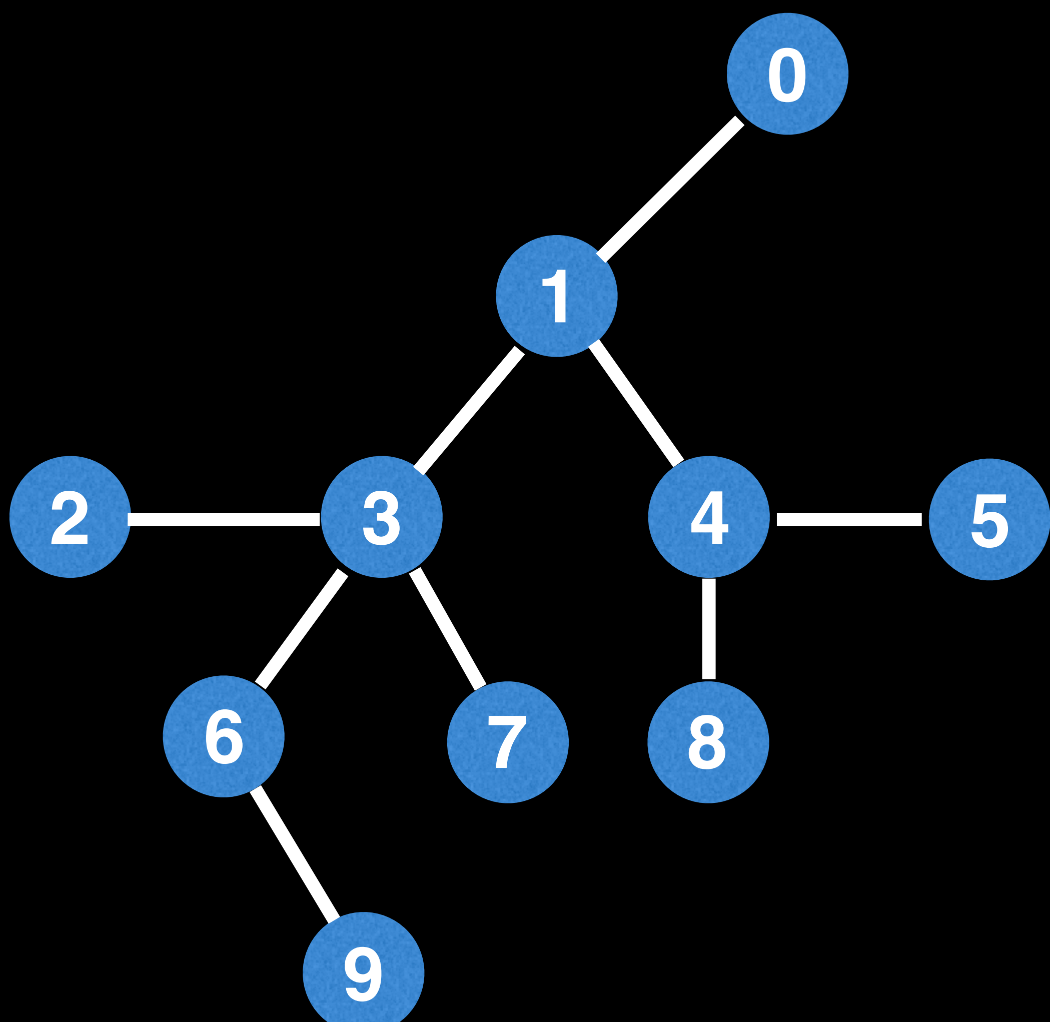
- Family trees
- File parsing/HTML/JSON/Syntax trees
- Many data structures use/are trees:
  - AVL trees, B-tree, red-black trees, segment trees, fenwick trees, treaps, suffix trees, tree maps/sets, etc…
- Game theory decision trees
- Organizational structures
- Probabilty trees
- Taxonomies
- etc…

Storing undirected trees

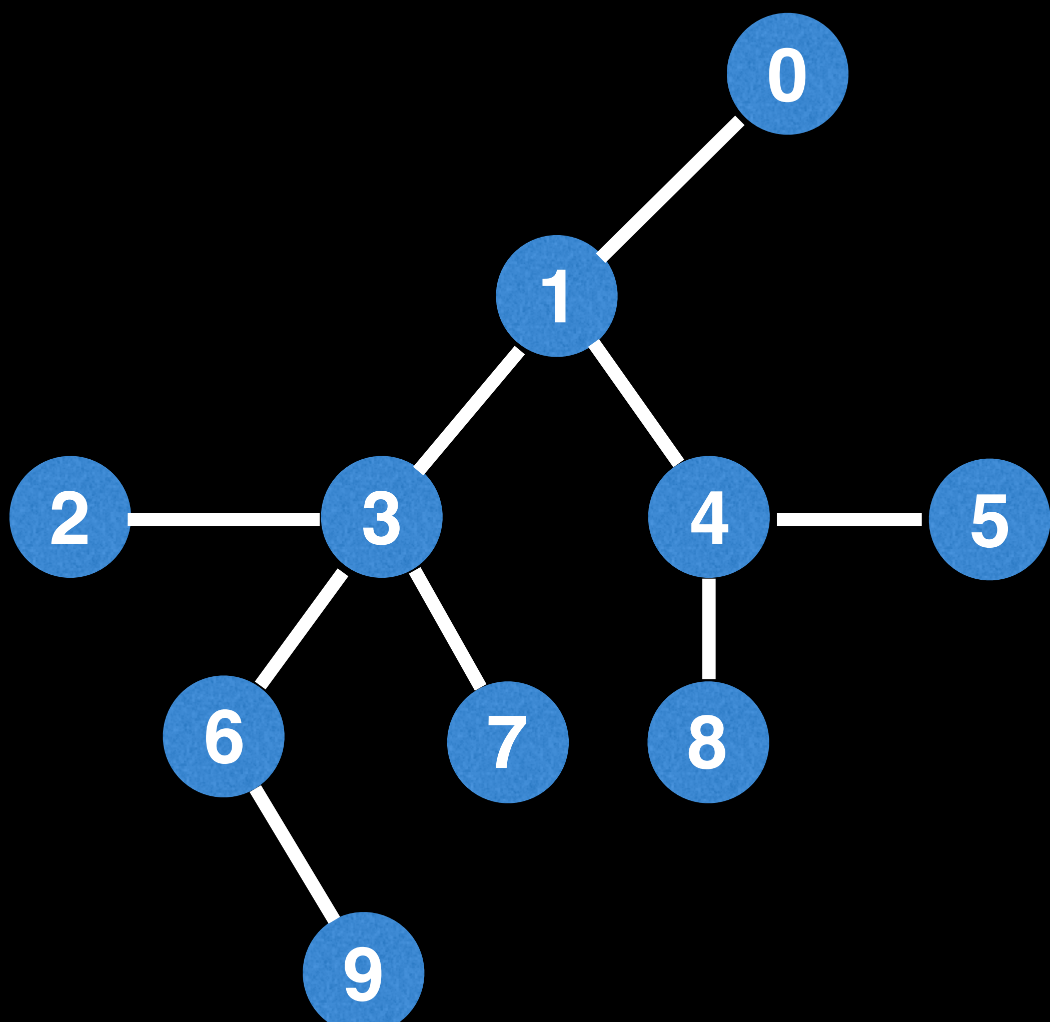


Storing undirected trees

Start by labelling the tree
nodes from [0, n)



Storing undirected trees

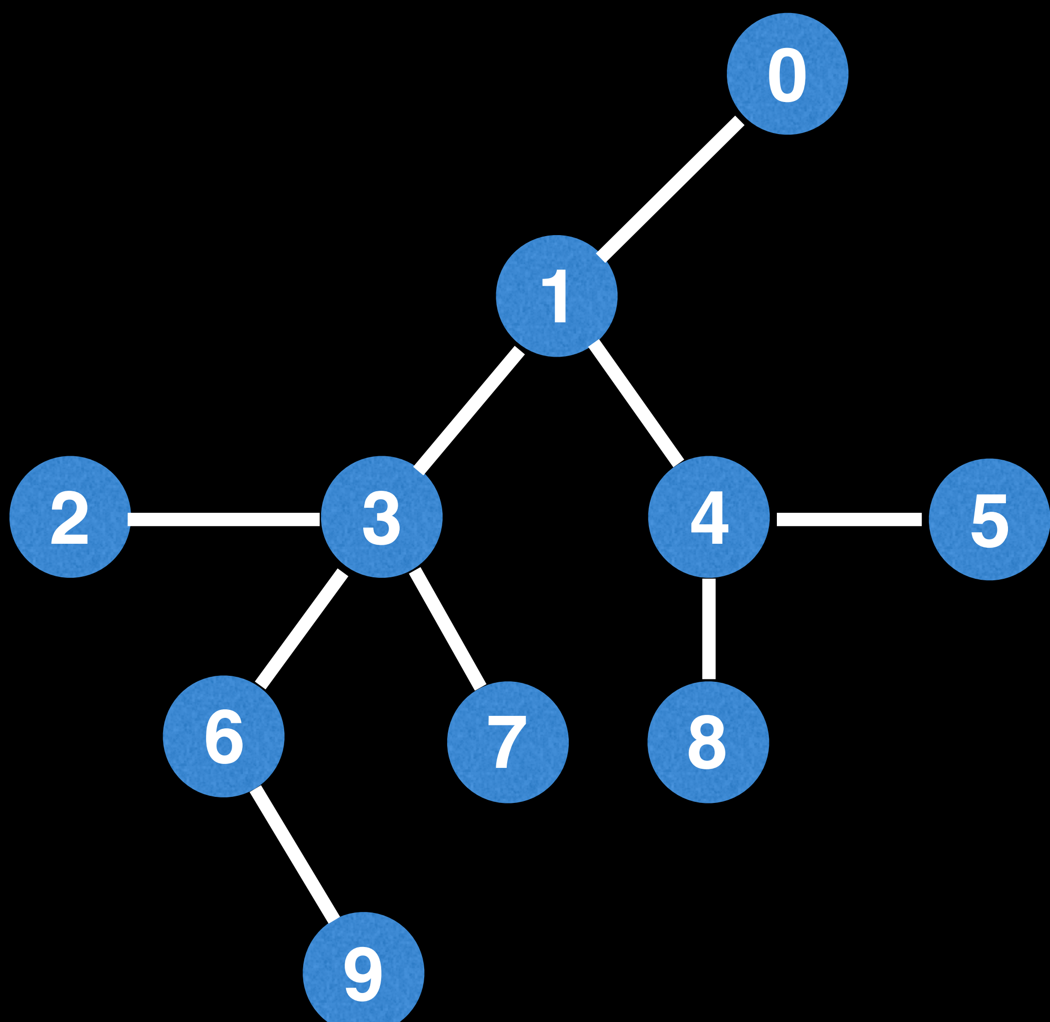


edge list storage representation:

```
[(0, 1),
 (1, 4),
 (4, 5),
 (4, 8),
 (1, 3),
 (3, 7),
 (3, 6),
 (2, 3),
 (6, 9)]
```

pro: super fast and easy to iterate over.

Storing undirected trees

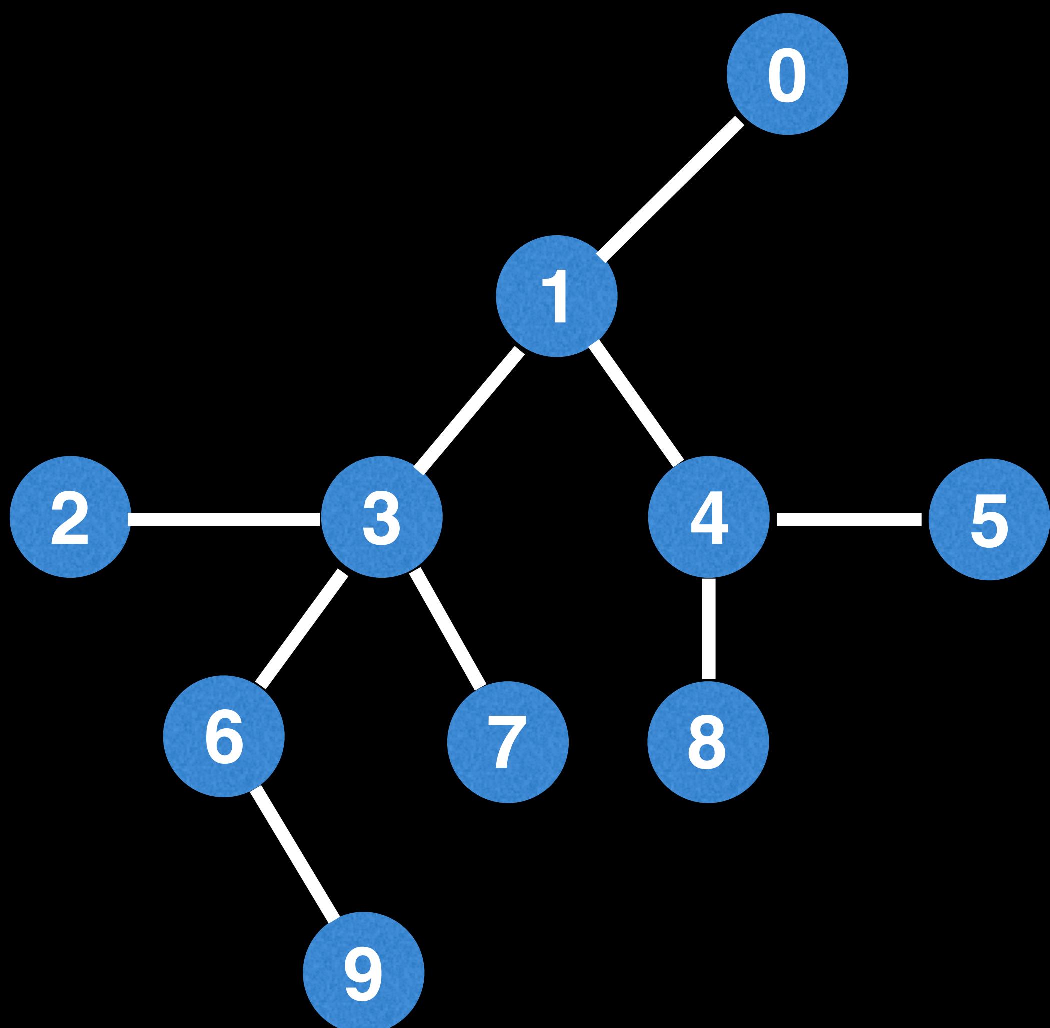


edge list storage representation:

[(0, 1),
 (1, 4),
 (4, 5),
 (4, 8),
 (1, 3),
 (3, 7),
 (3, 6),
 (2, 3),
 (6, 9)]

con: storing a tree as a list lacks the structure to efficiently query all the neighbors of a node.

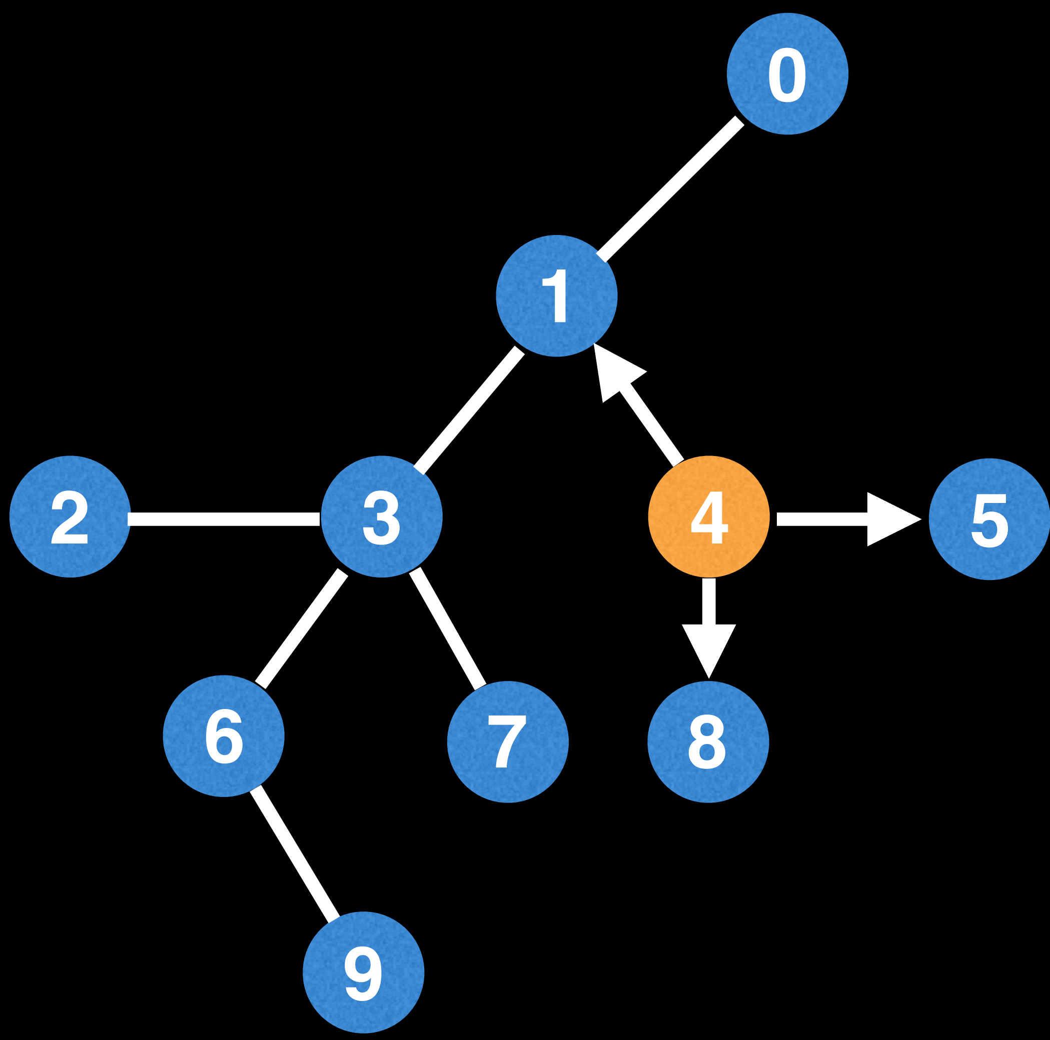
Storing undirected trees



adjacency list representation

```
0 -> [1]
1 -> [0,3,4]
2 -> [3]
3 -> [1,2,6,7]
4 -> [1,5,8]
5 -> [4]
6 -> [3,9]
7 -> [3]
8 -> [4]
9 -> [6]
```

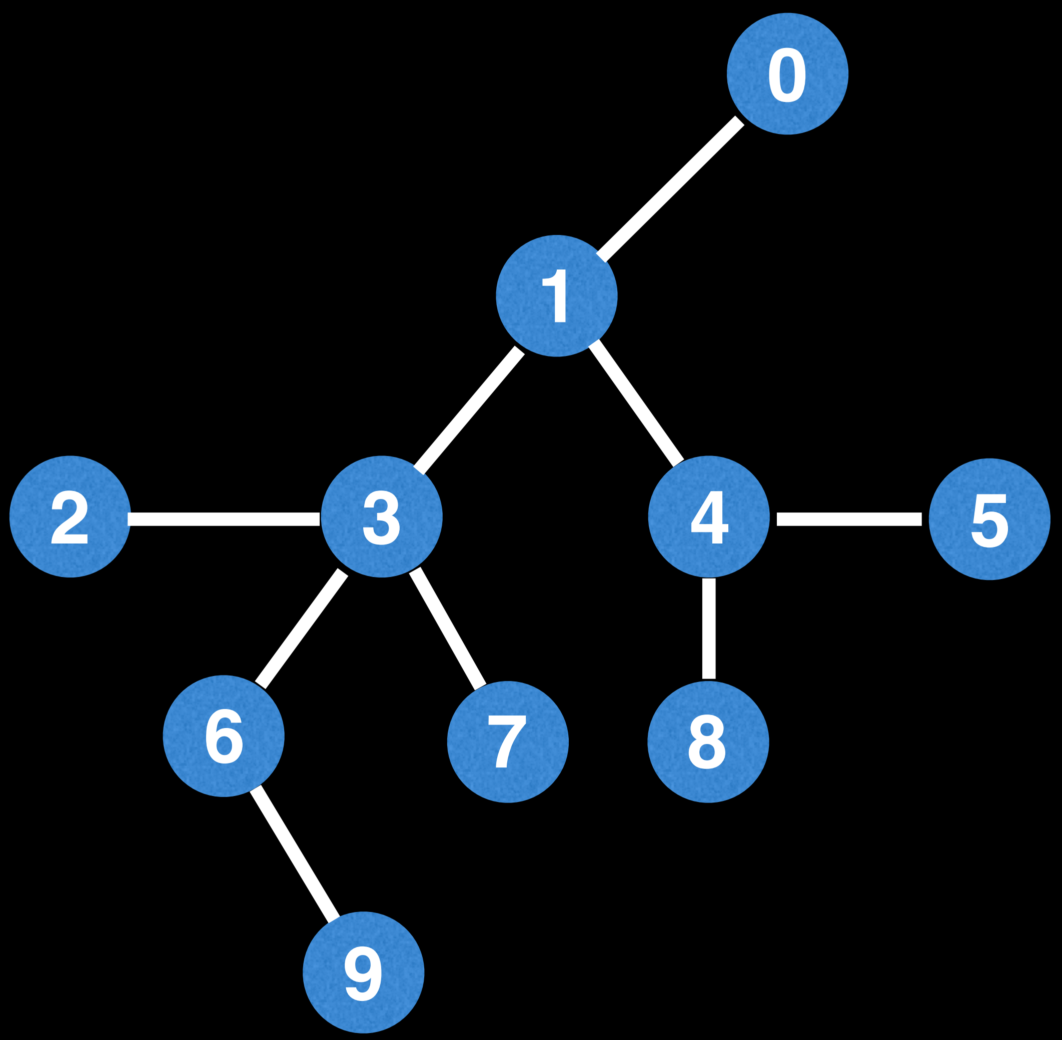
Storing undirected trees



adjacency list representation

0 -> [1]
1 -> [0,3,4]
2 -> [3]
3 -> [1,2,6,7]
4 -> [1,5,8]
5 -> [4]
6 -> [3,9]
7 -> [3]
8 -> [4]
9 -> [6]

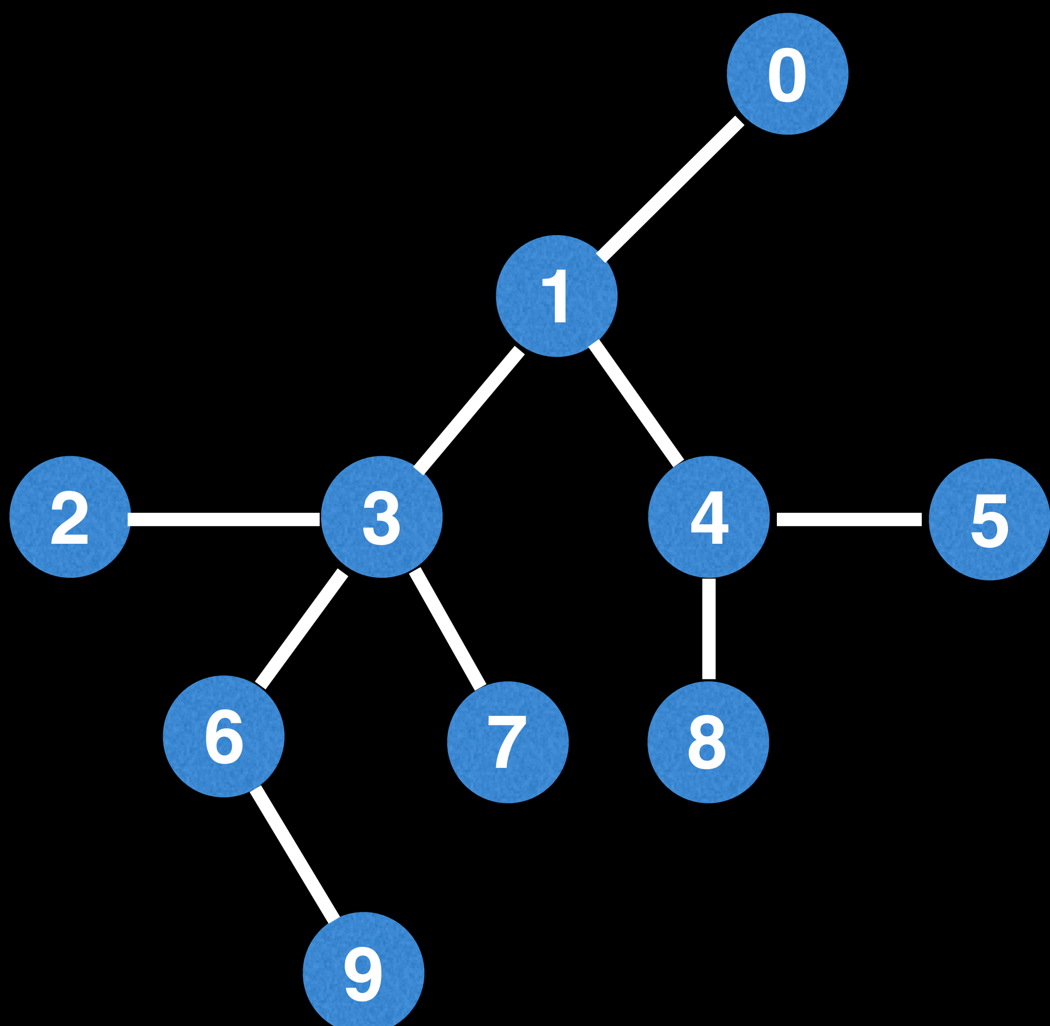
Storing undirected trees



adjacency matrix representation

	0	1	2	3	4	5	6	7	8	9
0	0	1	0	0	0	0	0	0	0	0
1	1	0	0	1	1	0	0	0	0	0
2	0	0	0	1	0	0	0	0	0	0
3	0	1	1	0	0	0	1	1	0	0
4	0	1	0	0	0	1	0	0	1	0
5	0	0	0	0	1	0	0	0	0	0
6	0	0	0	1	0	0	0	0	0	1
7	0	0	0	1	0	0	0	0	0	0
8	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	1	0	0	0

Storing undirected trees



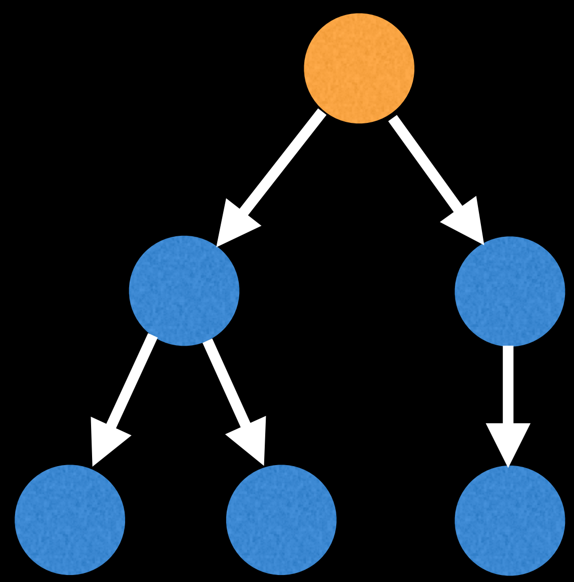
adjacency matrix representation

	0	1	2	3	4	5	6	7	8	9
0	0	1	0	0	0	0	0	0	0	0
1	1	0	0	1	1	0	0	0	0	0
2	0	0	0	1	0	0	0	0	0	0
3	0	1	1	0	0	0	1	1	0	0
4	0	1	0	0	0	1	0	0	1	0
5	0	0	0	0	1	0	0	0	0	0
6	0	0	0	1	0	0	0	0	0	1
7	0	0	0	1	0	0	0	0	0	0
8	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	1	0	0	0

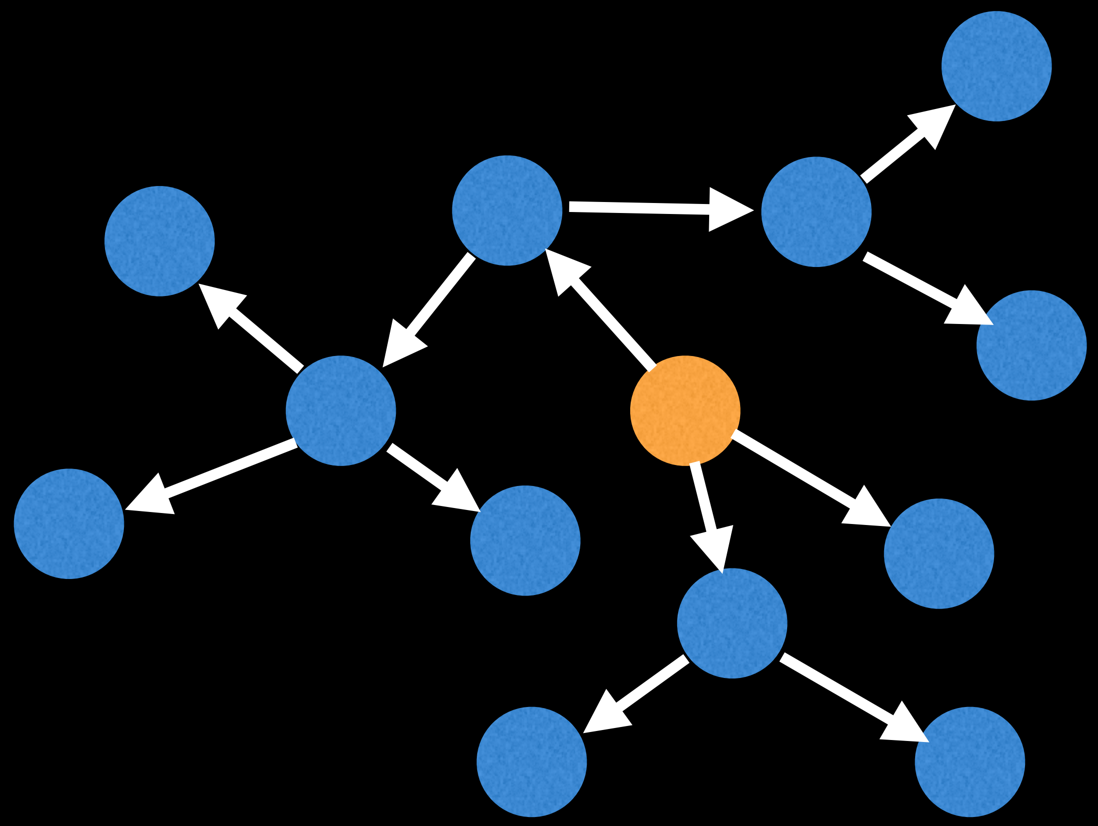
In practice, avoid storing a tree as an adjacency matrix! It’s a huge **waste of space** to use n^2 memory and only use $2(n-1)$ of the matrix cells.

Rooted Trees!

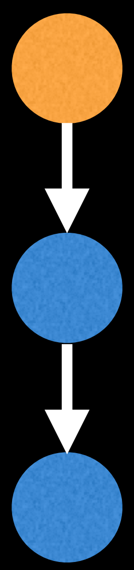
One of the more interesting types of trees is a **rooted tree** which is a tree with a designated **root node**.



Rooted tree



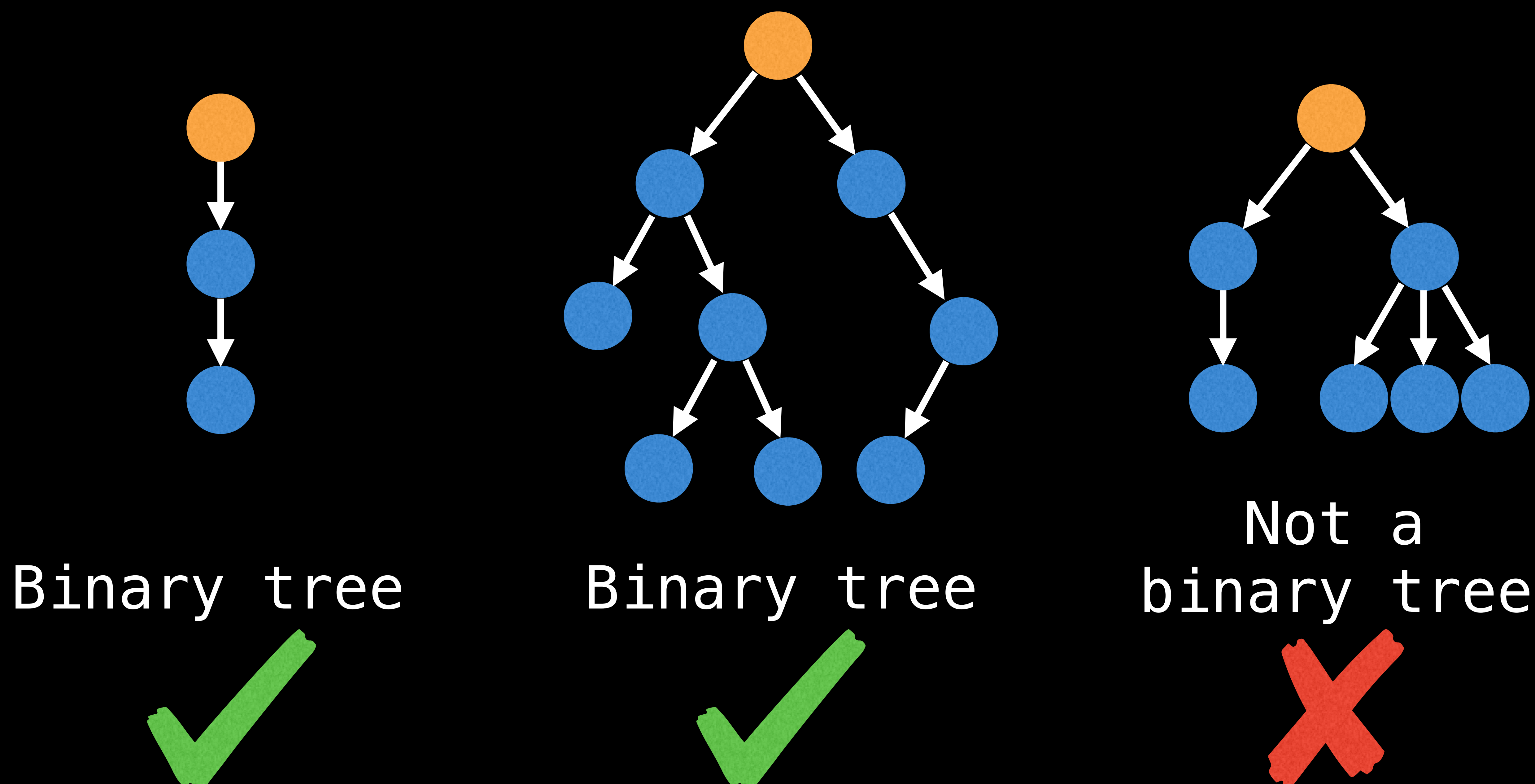
Rooted tree



Rooted tree

Binary Tree (BT)

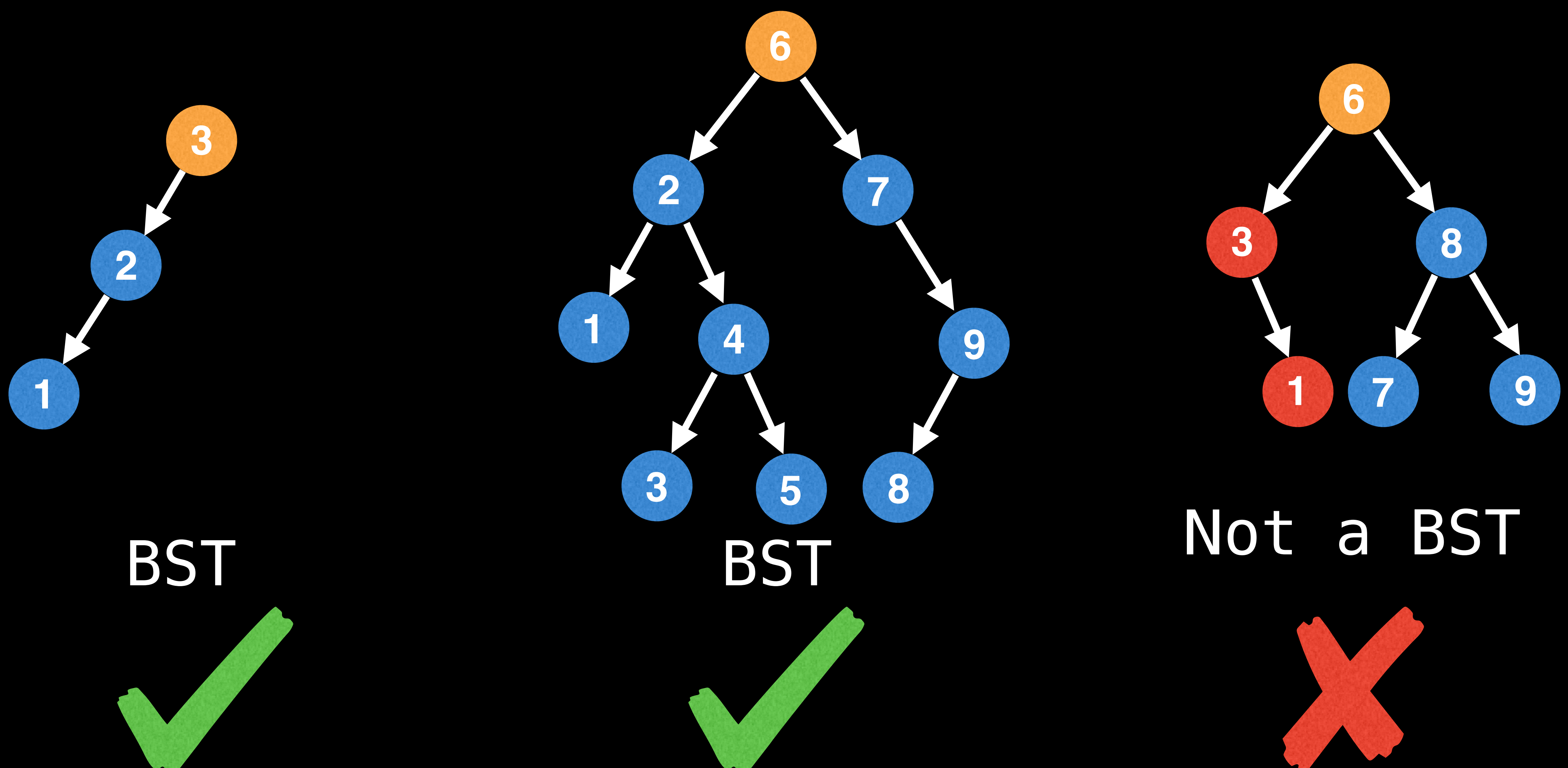
Related to rooted trees are **binary trees** which are trees for which every node has **at most two child nodes**.



Binary Search Trees (BST)

Related to binary trees are **binary search trees** which are trees which satisfy the BST invariant which states that for every node x:

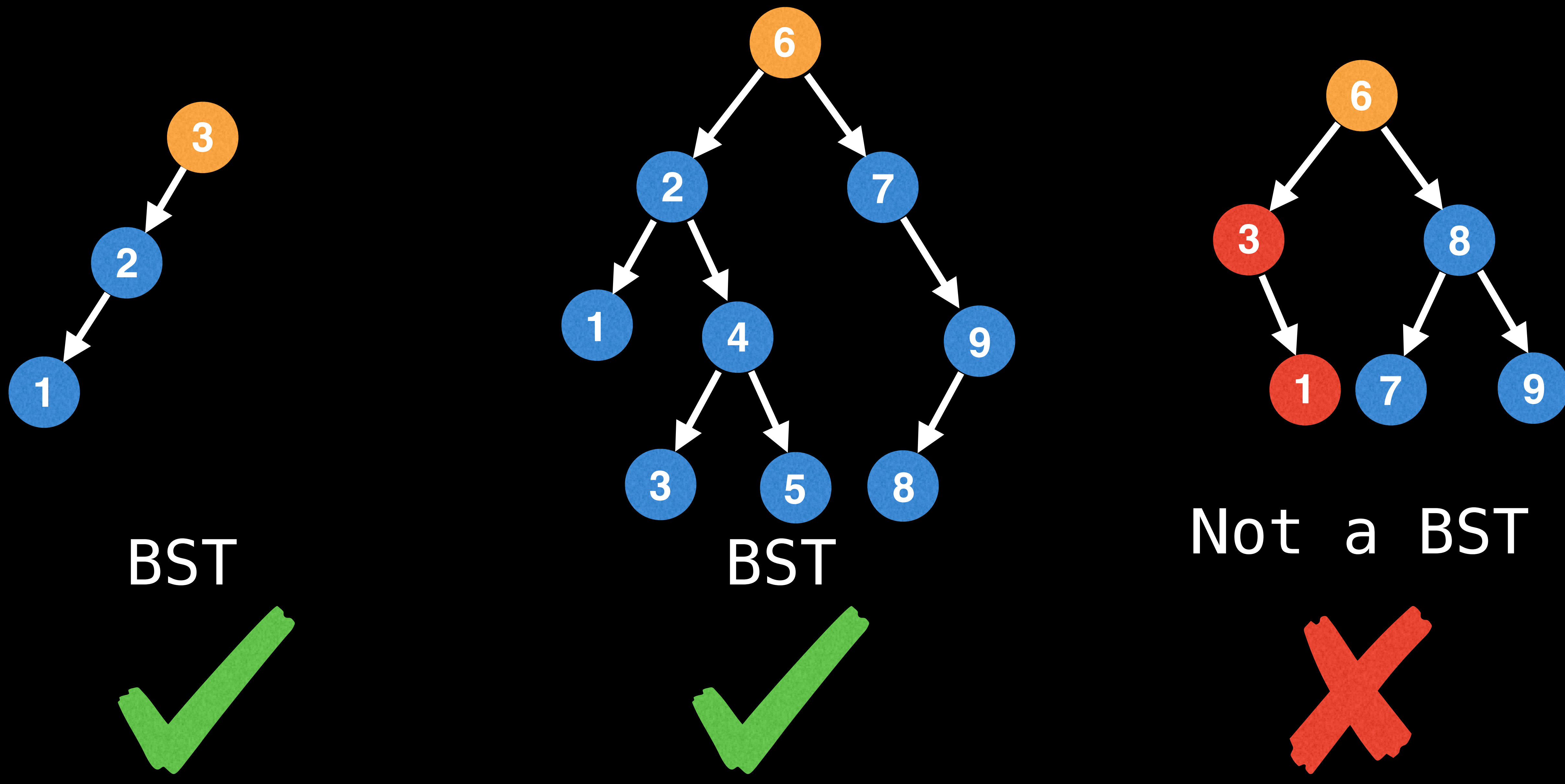
$$x.left.value \leq x.value \leq x.right.value$$



Binary Search Trees (BST)

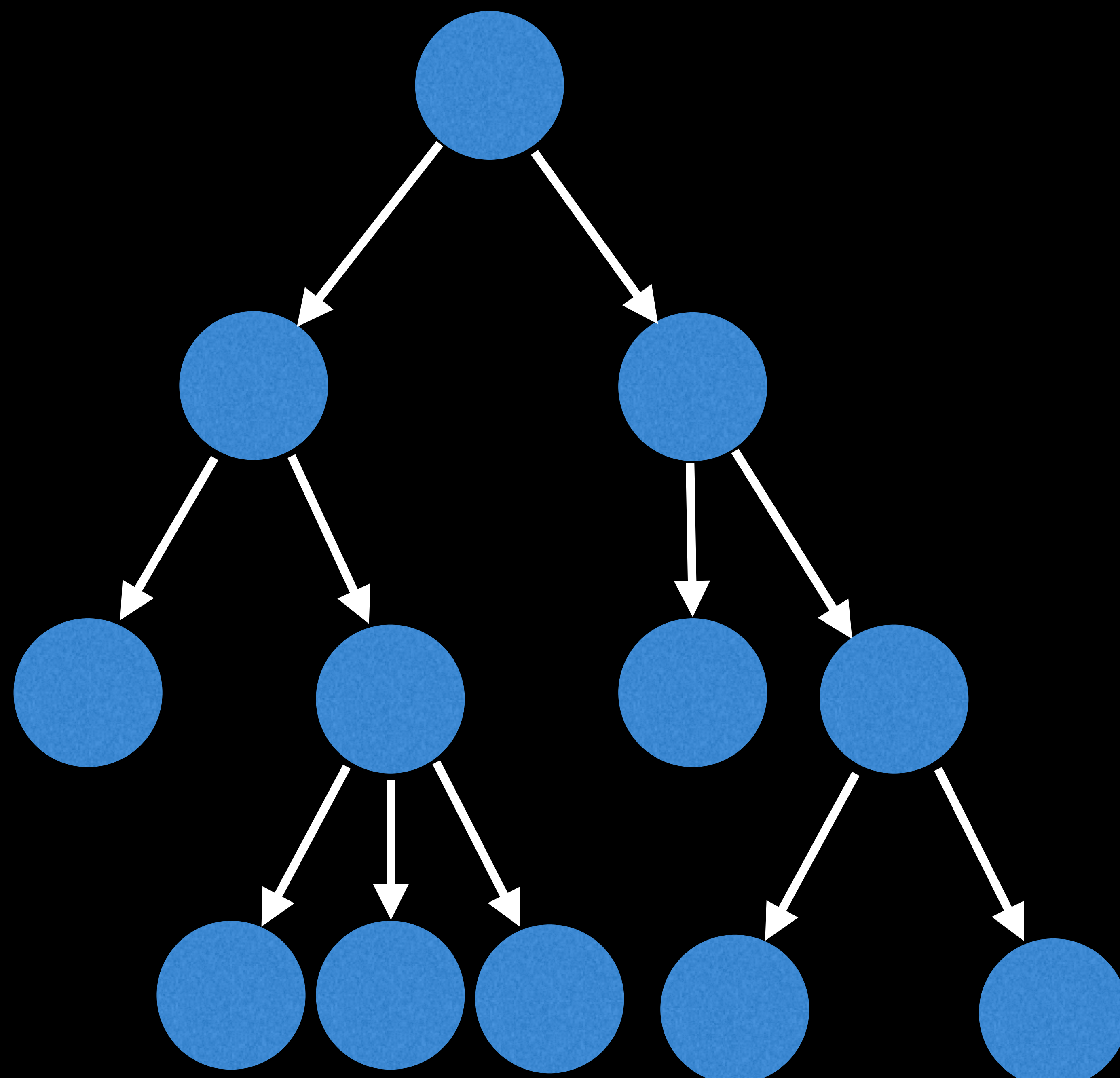
It's often useful to **require uniqueness** on the node values in your tree. Change the invariant to be strictly $<$ rather than $\leq$:

$$\texttt{x.left.value} < \texttt{x.value} < \texttt{x.right.value}$$



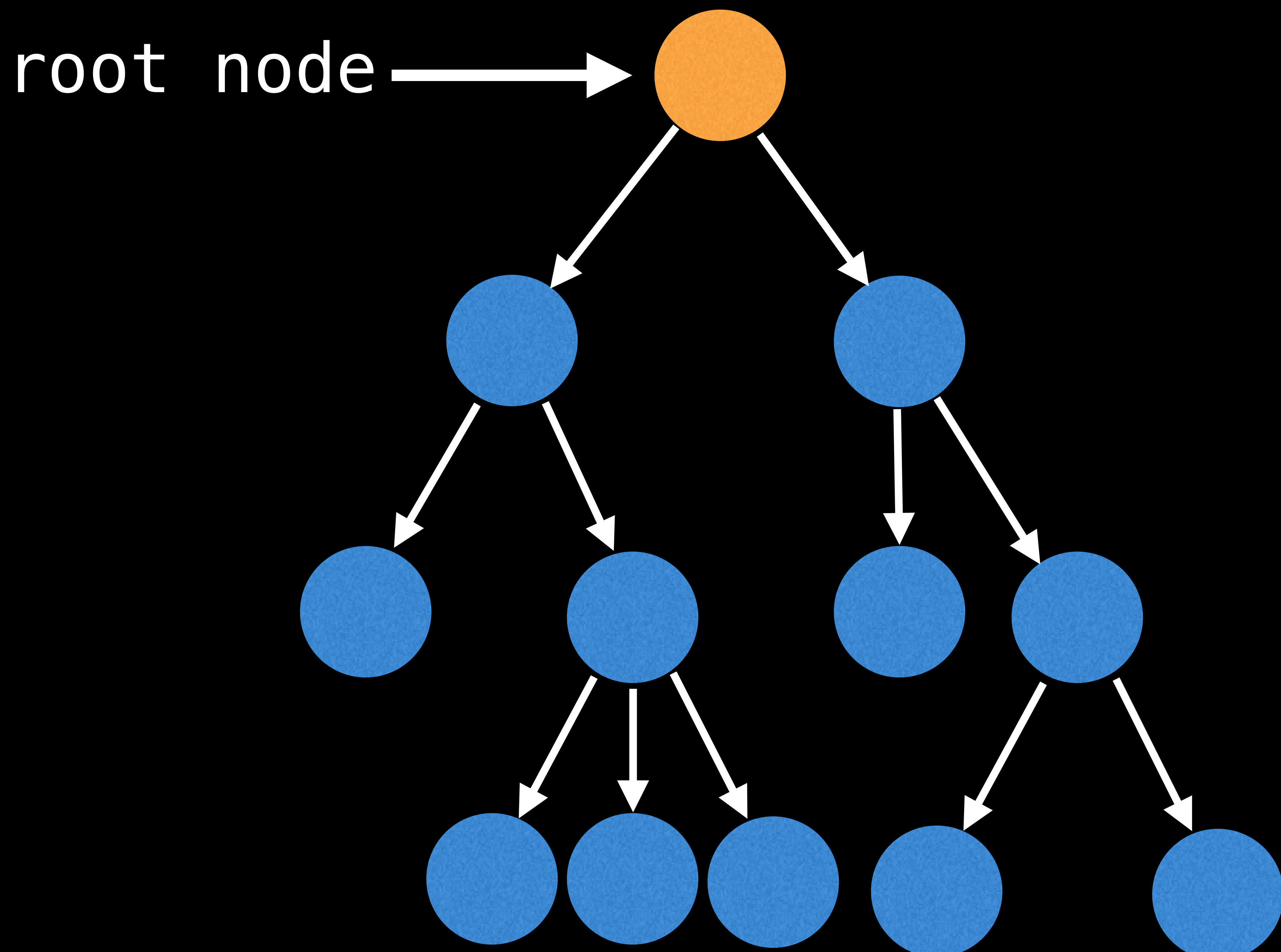
Storing rooted trees

Rooted trees are most naturally defined recursively in a top–down manner.



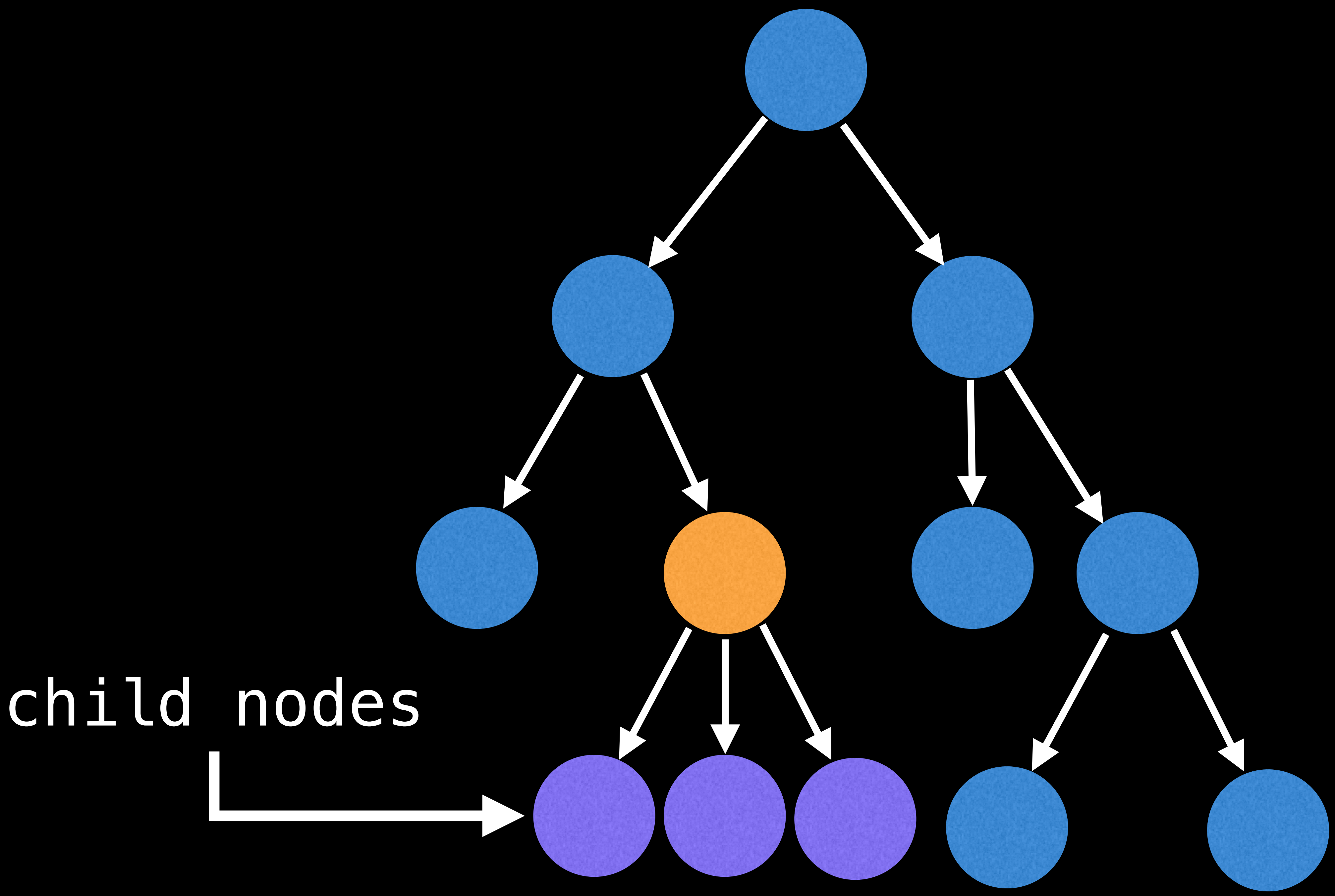
Storing rooted trees

In practice, you always maintain a pointer reference to the **root node** so that you can access the tree and its contents.



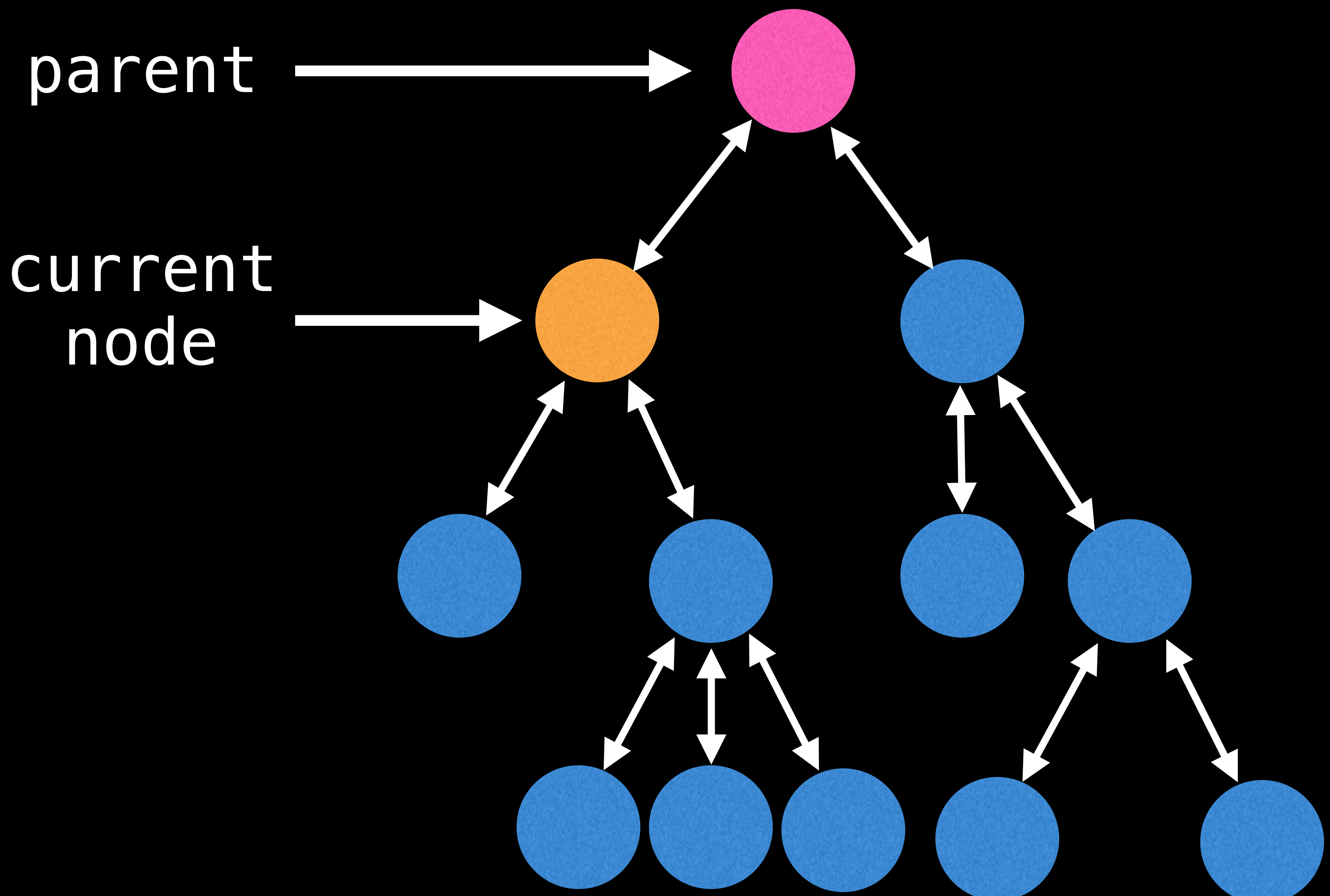
Storing rooted trees

Each node also has access to a list of all its **children**.



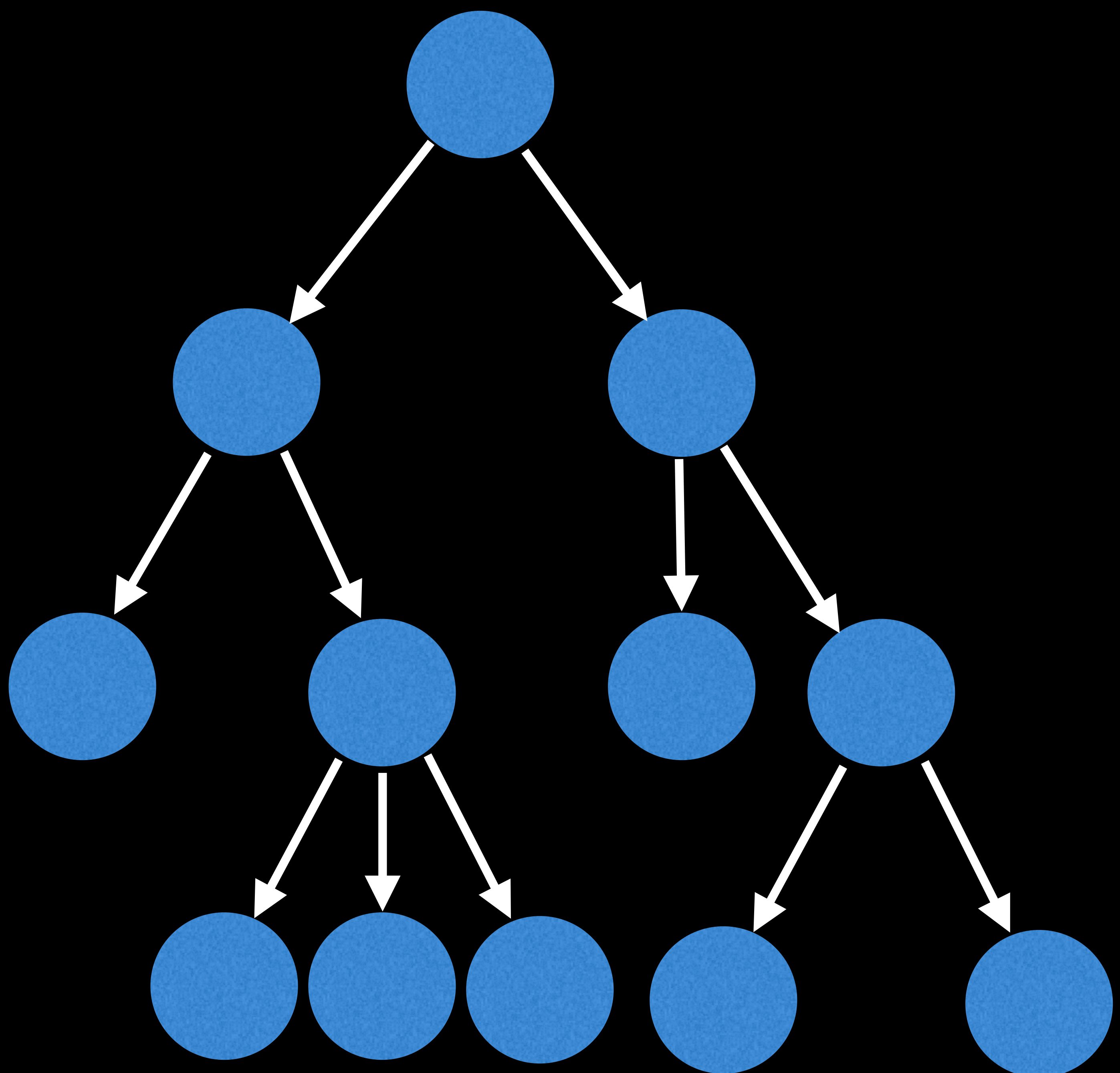
Storing rooted trees

Sometimes it's also useful to maintain a pointer to a node's **parent node** effectively making edges **bidirectional**.



Storing rooted trees

However, this isn't *usually* necessary because you can access a node's parent on a recursive function's callback.



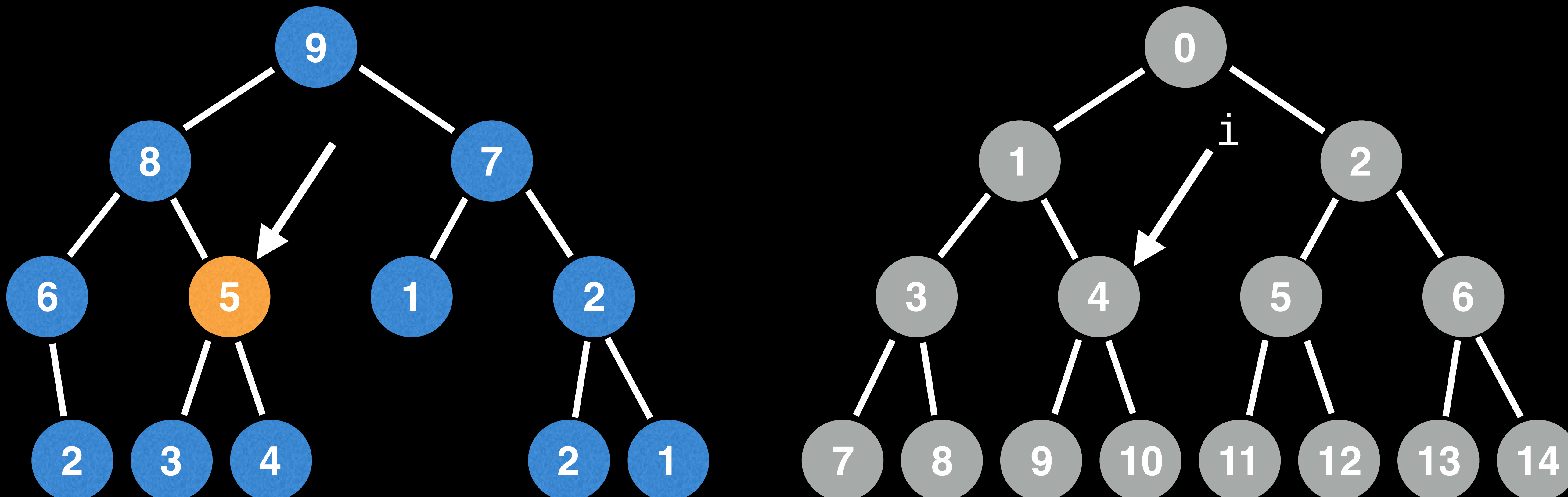
Storing rooted trees

If your tree is a **binary tree**, you can store it in a **flattened array**.

This trick also works for any n-ary tree

Storing rooted trees

In this flattened array representation, each node has an assigned index position based on where it is in the tree.

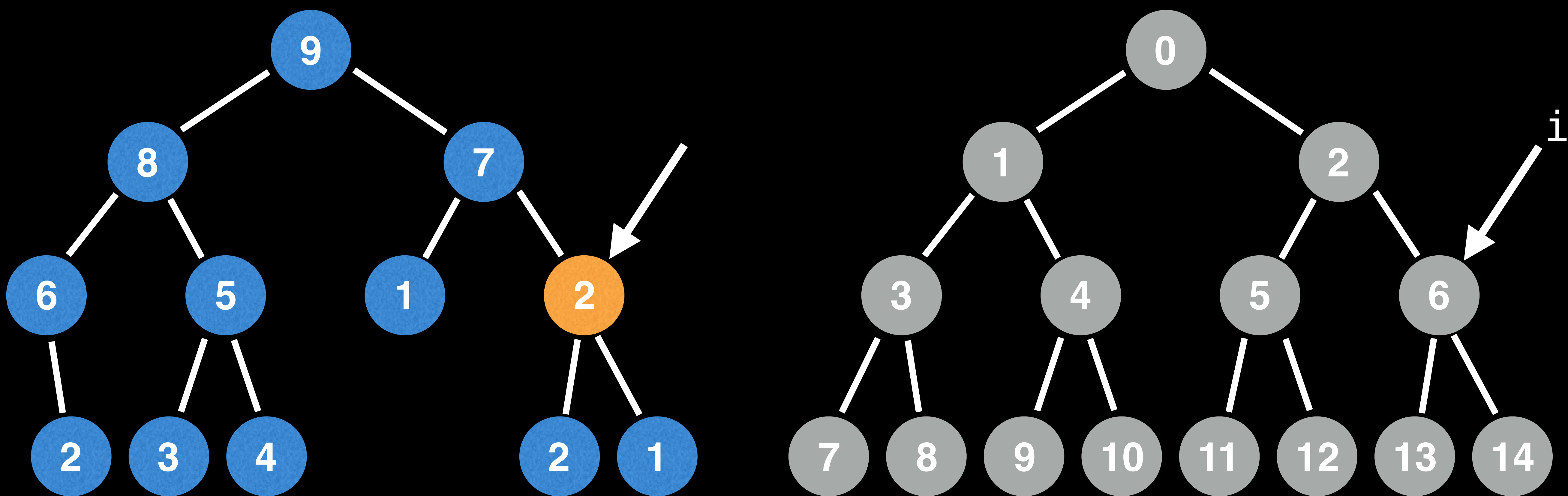


0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
9	8	7	6	5	1	2	∅	2	3	4	∅	∅	2	1

This trick also works for any n-ary tree

Storing rooted trees

In this flattened array representation, each node has an assigned index position based on where it is in the tree.

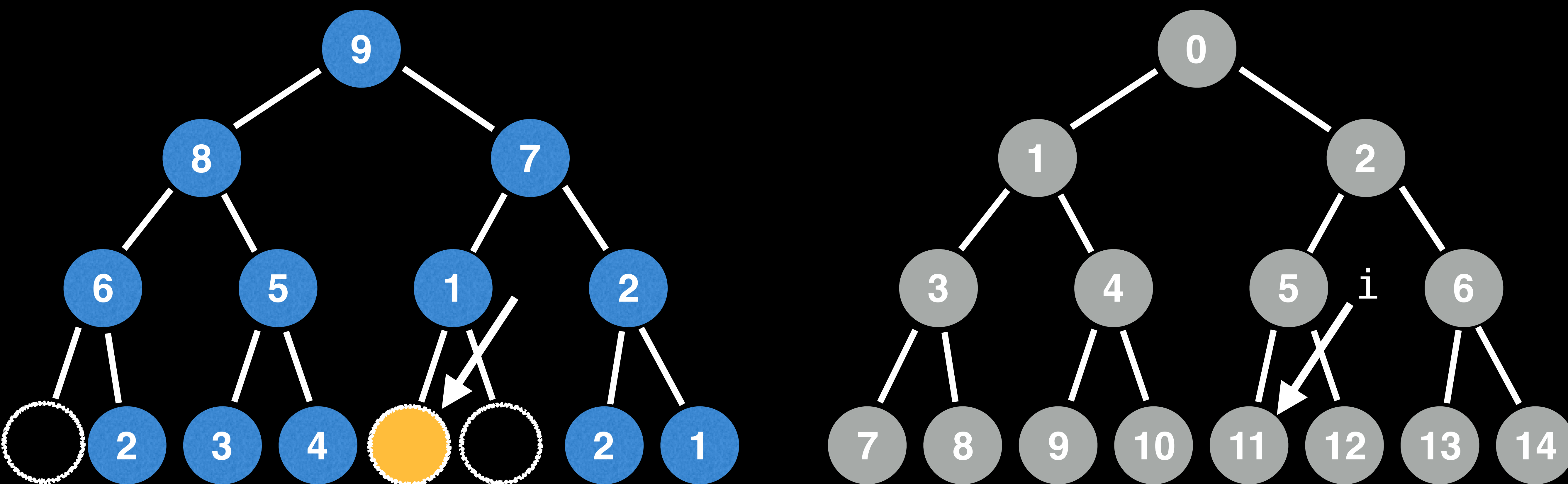


0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
9	8	7	6	5	1	2	∅	2	3	4	∅	∅	2	1

This trick also works for any n-ary tree

Storing rooted trees

Even nodes which aren't currently present have an index because they can be mapped back to a unique position in the "index tree" (gray tree).



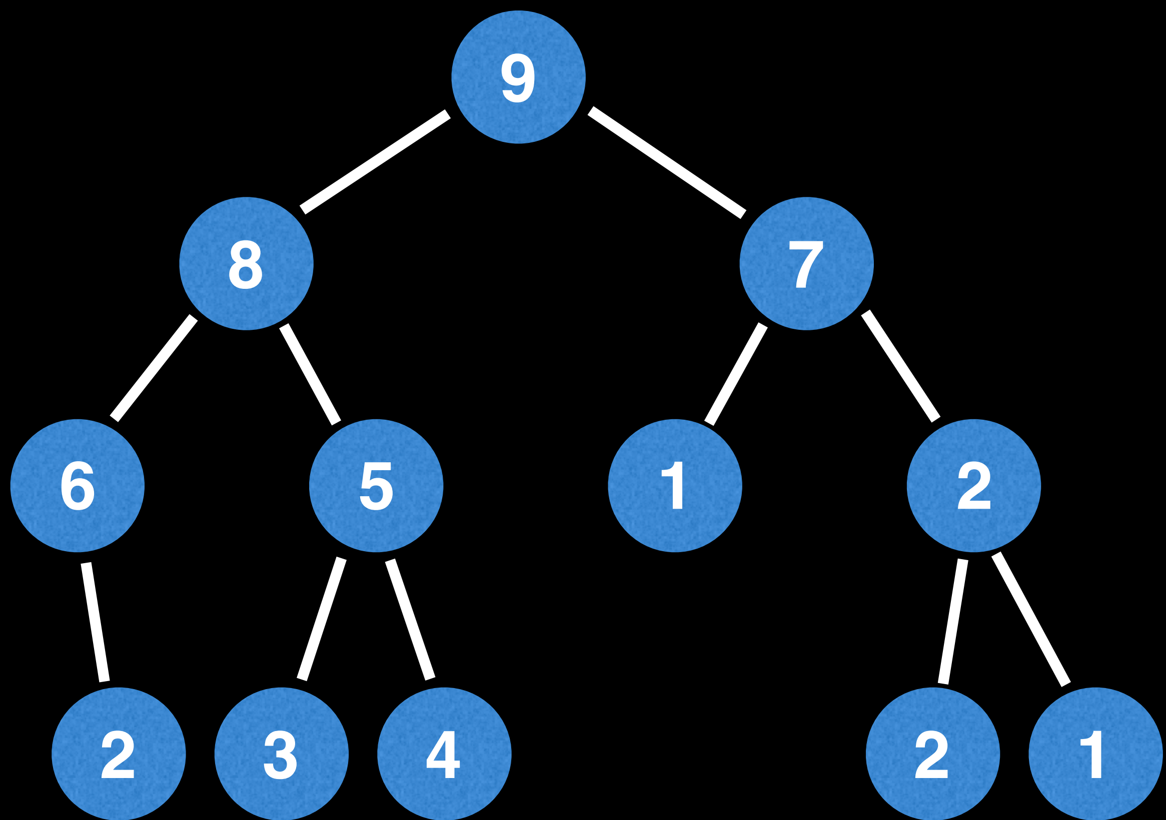
index tree

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
9	8	7	6	5	1	2	∅	2	3	4	∅	∅	2	1

This trick also works for any n-ary tree

Storing rooted trees

The root node is always at index 0 and the children of the current node *i* are accessed relative to position *i*.



Let *i* be the index of the current node

left node: 2*i + 1
right node: 2*i + 2

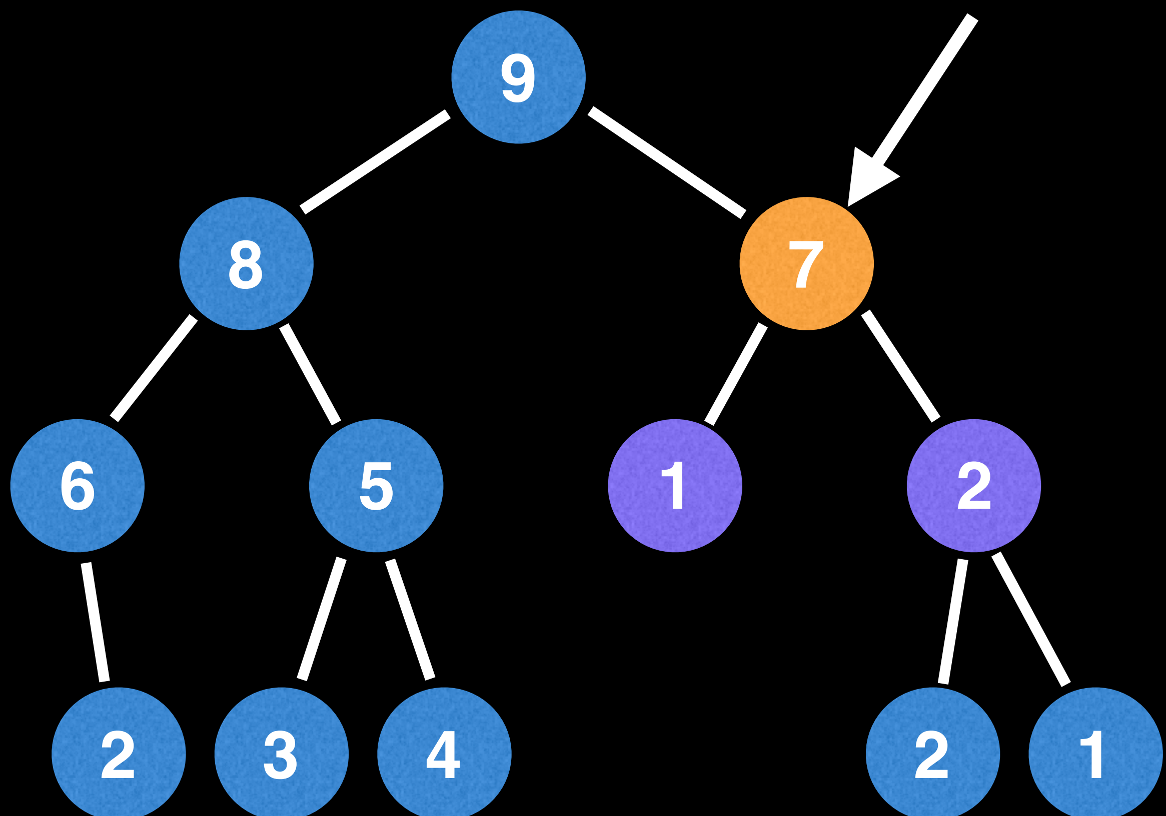
Reciprocally, the parent of node i is: floor((i-1)/2)

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
9	8	7	6	5	1	2	∅	2	3	4	∅	∅	2	1

This trick also works for any n-ary tree

Storing rooted trees

The root node is always at index 0 and the children of the current node *i* are accessed relative to position *i*.



Let *i* be the index of the current node

left node: 2*i + 1
right node: 2*i + 2

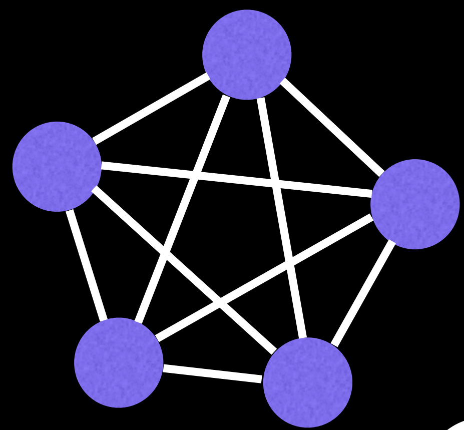
Reciprocally, the parent of node i is: floor((i-1)/2)

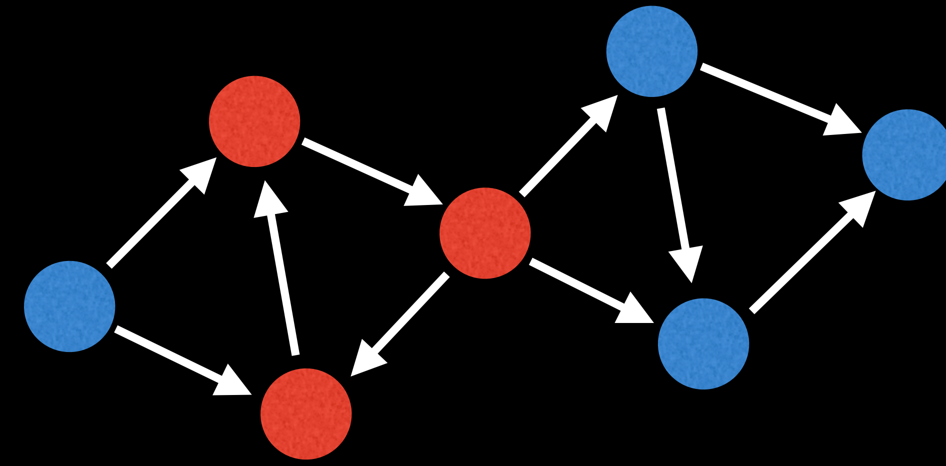
i

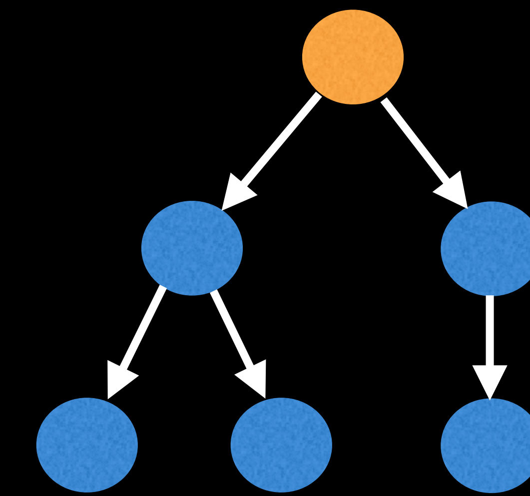
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
9	8	7	6	5	1	2	∅	2	3	4	∅	∅	2	1

This trick also works for any n-ary tree

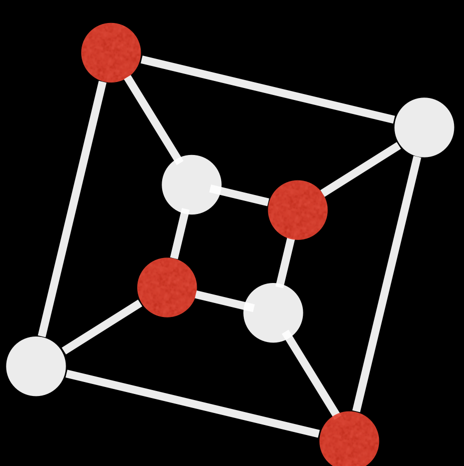
Next Video: beginner tree algorithms

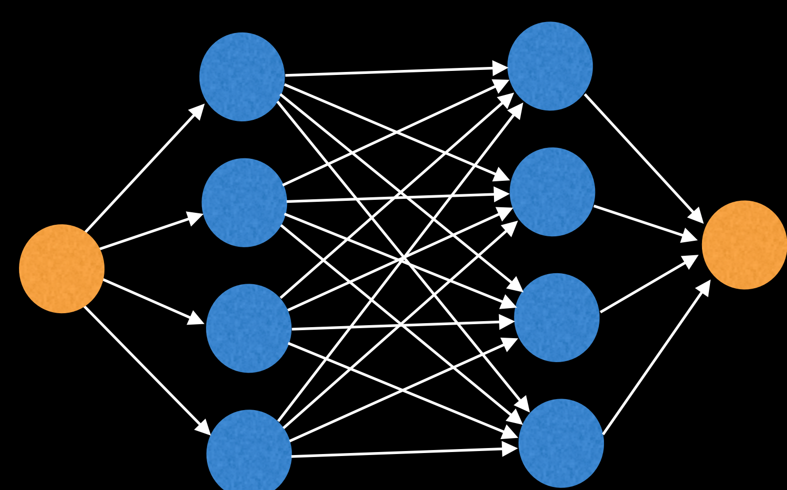


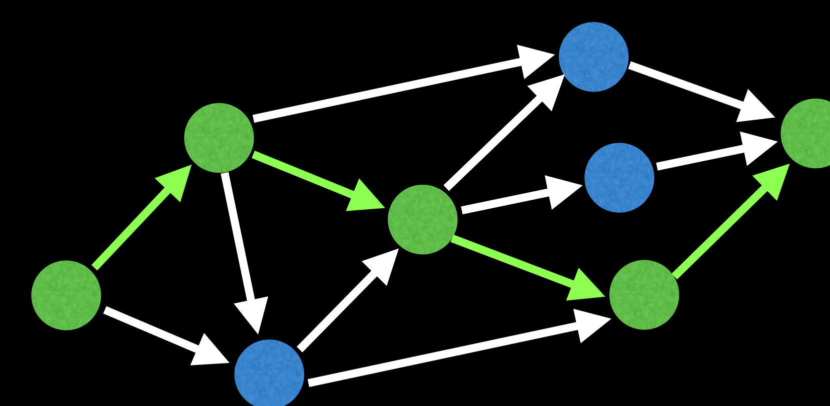




Graph Theory Video Series





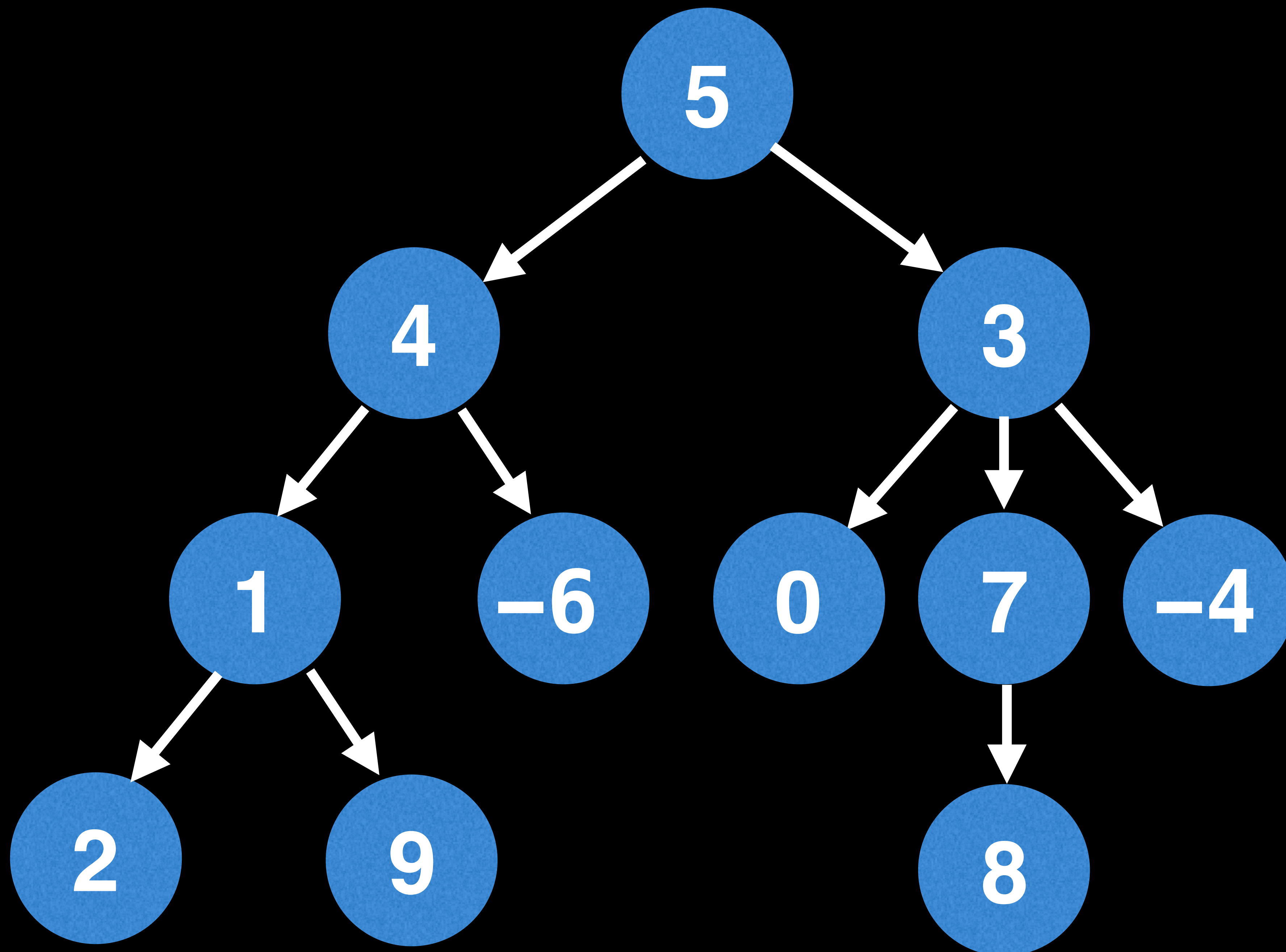


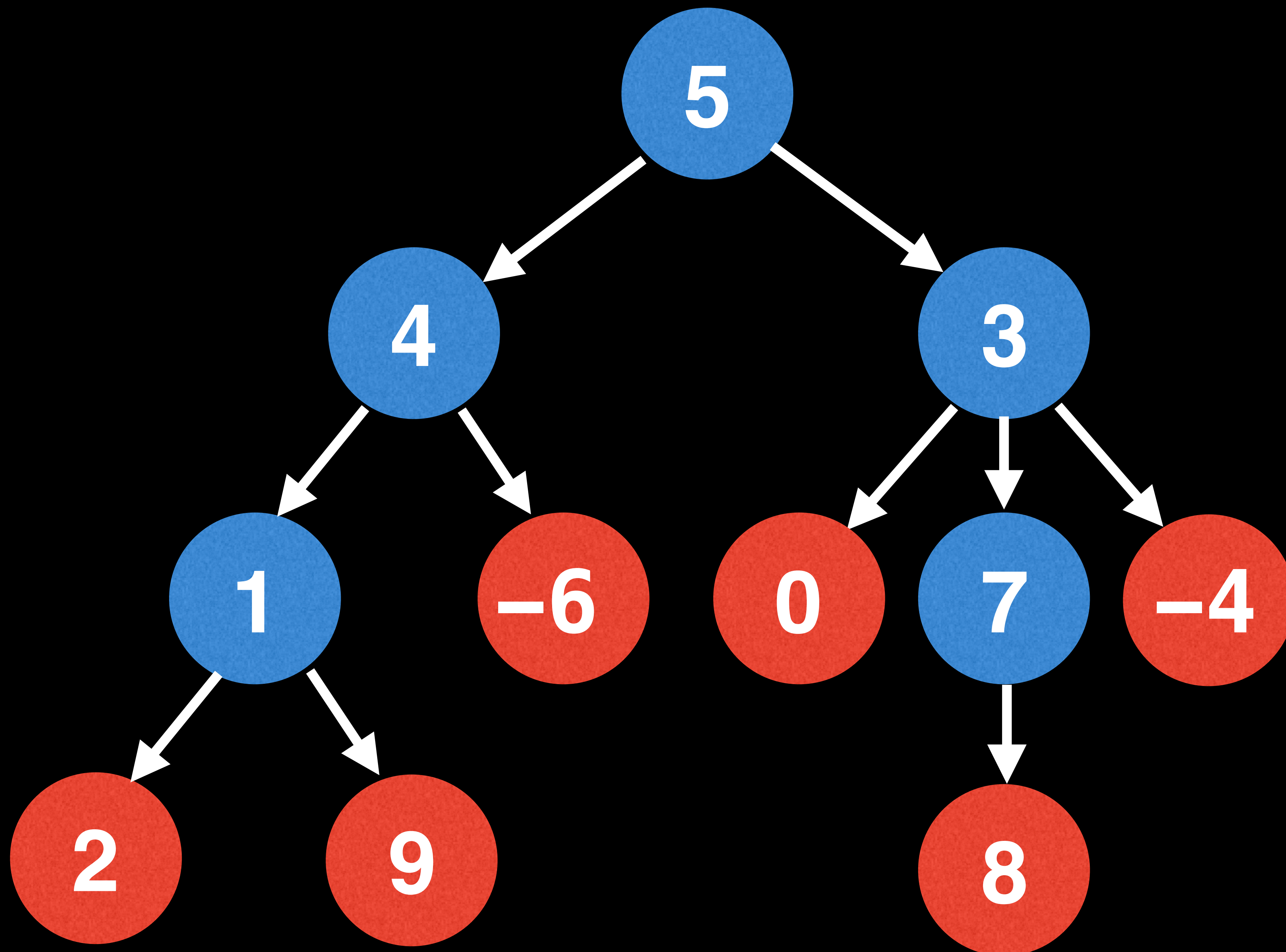
Beginner tree algorithms

William Fiset

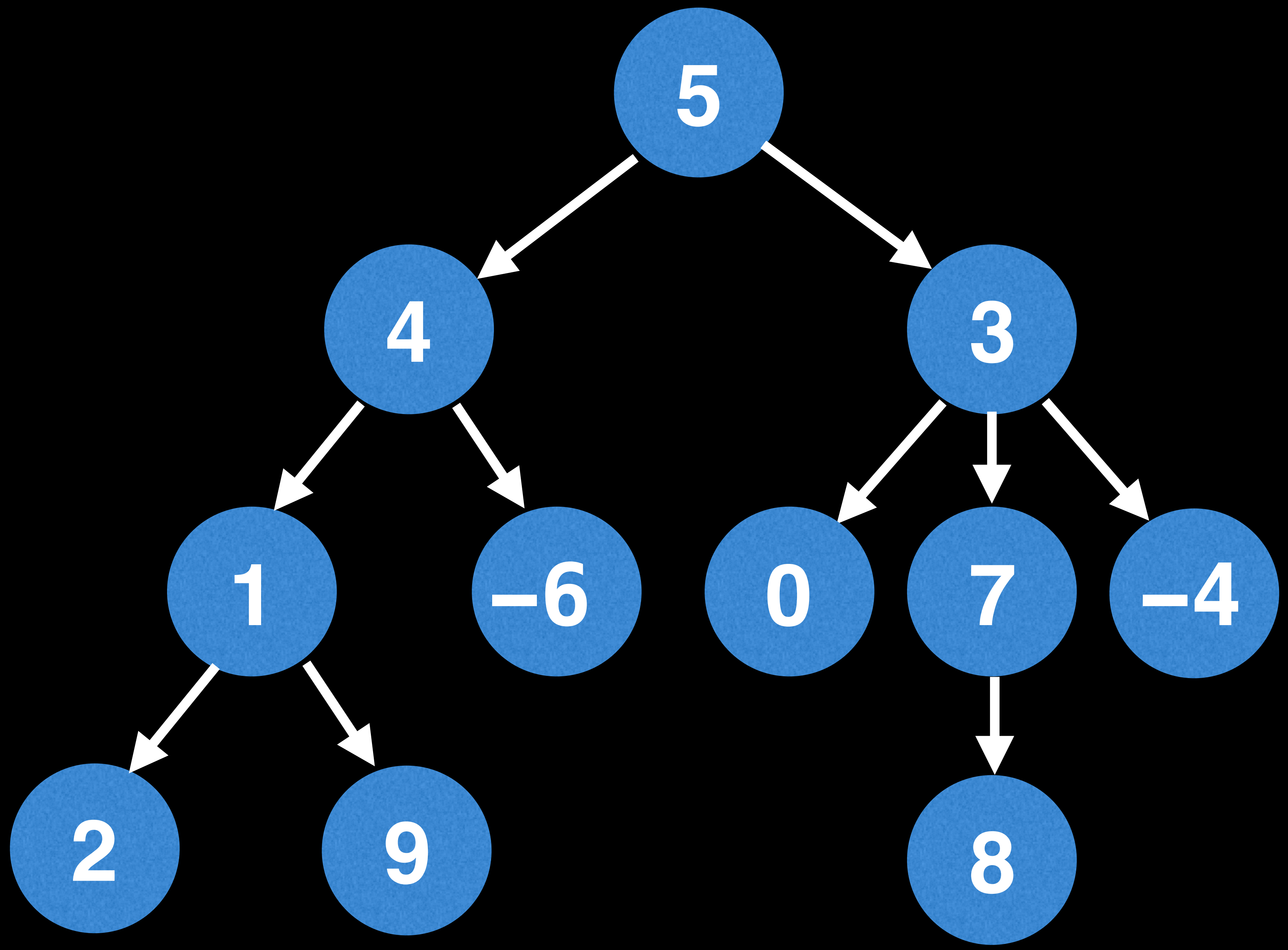
Problem 1: leaf node sum

What is the sum of all the leaf node values in a tree?


5
4
3
1
–6
0
7
–4
2
9
8



$$2 + 9 - 6 + 0 + 8 - 4 = 9$$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4
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0
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–4
2
9
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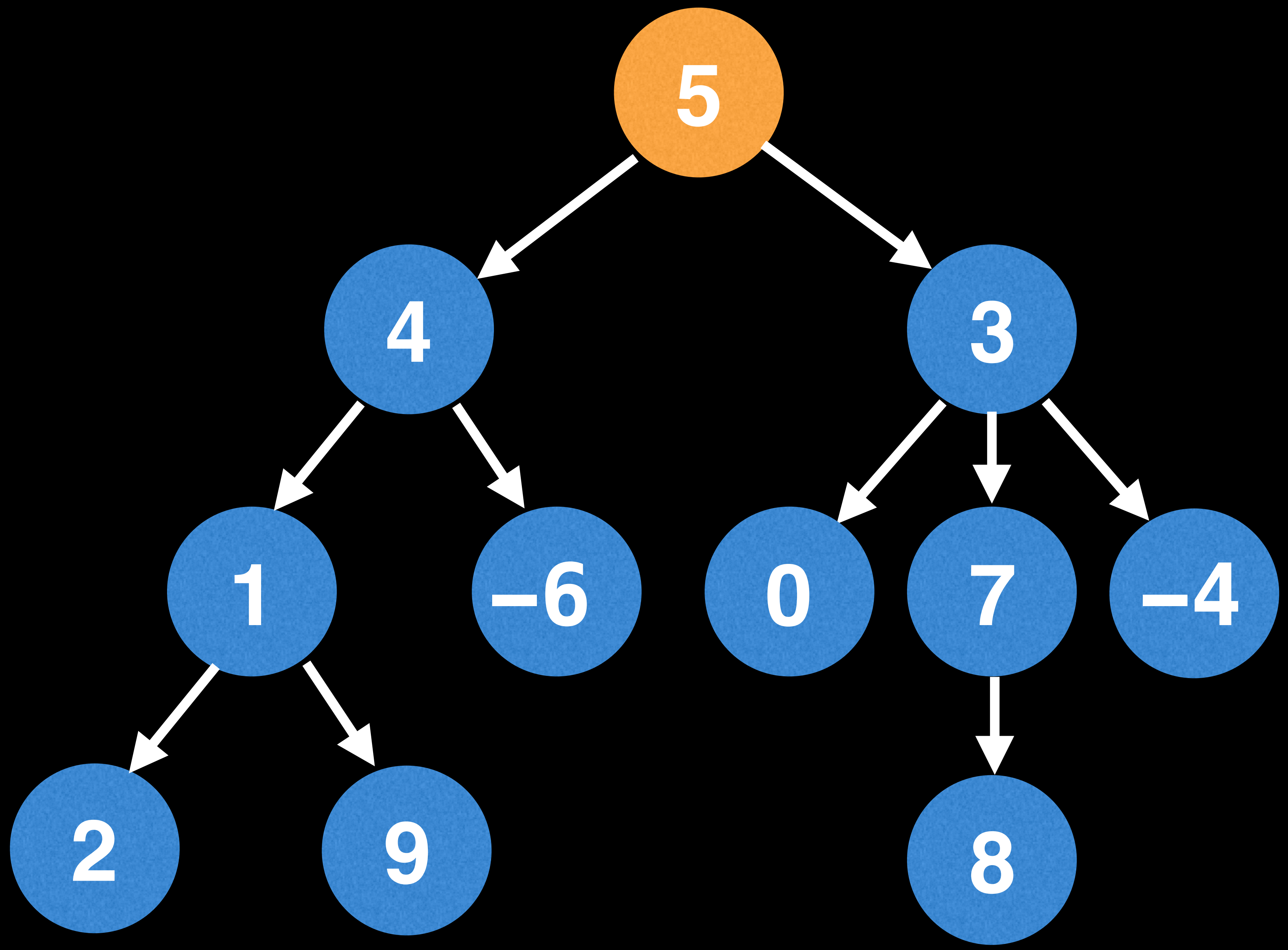
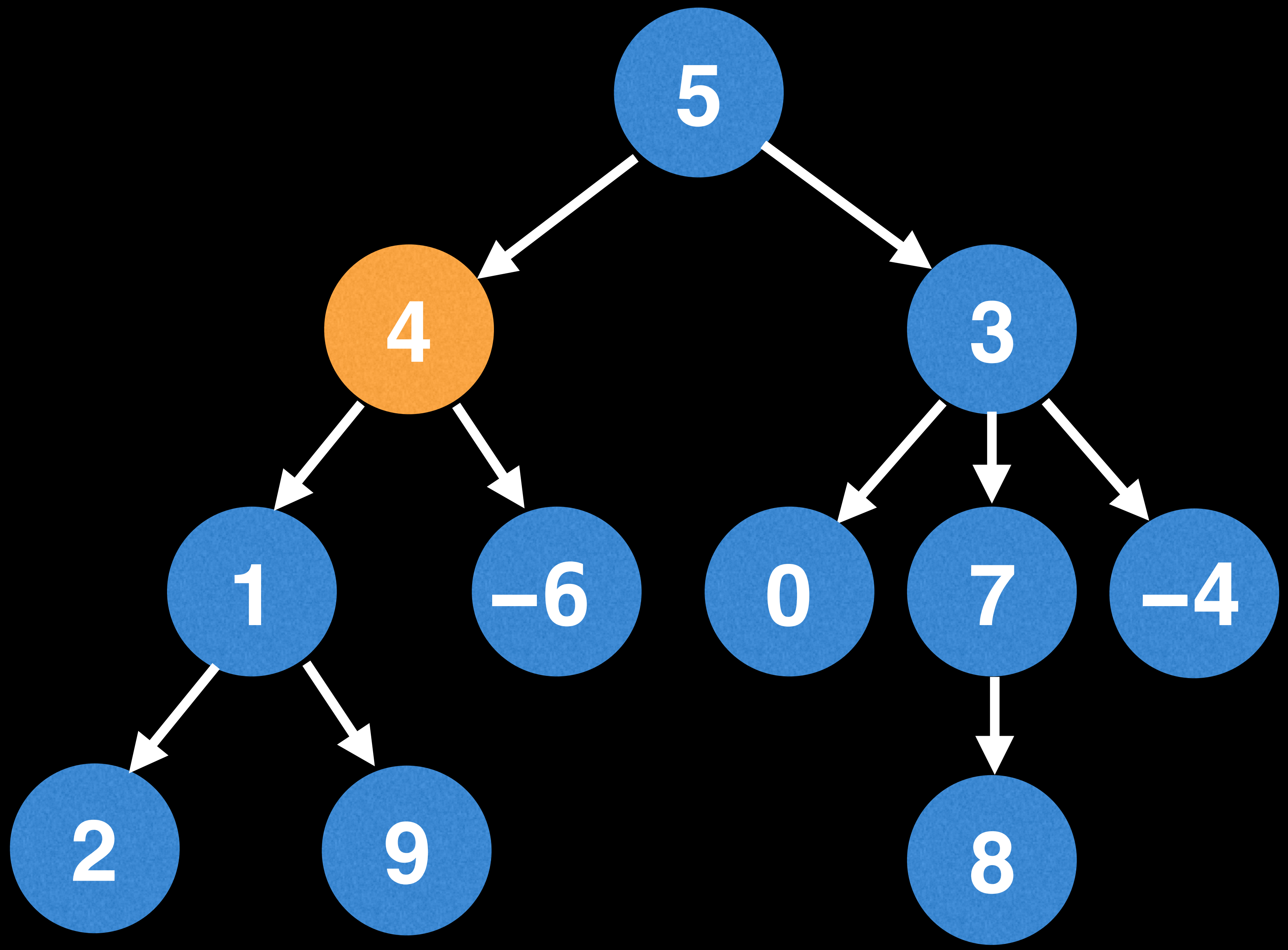
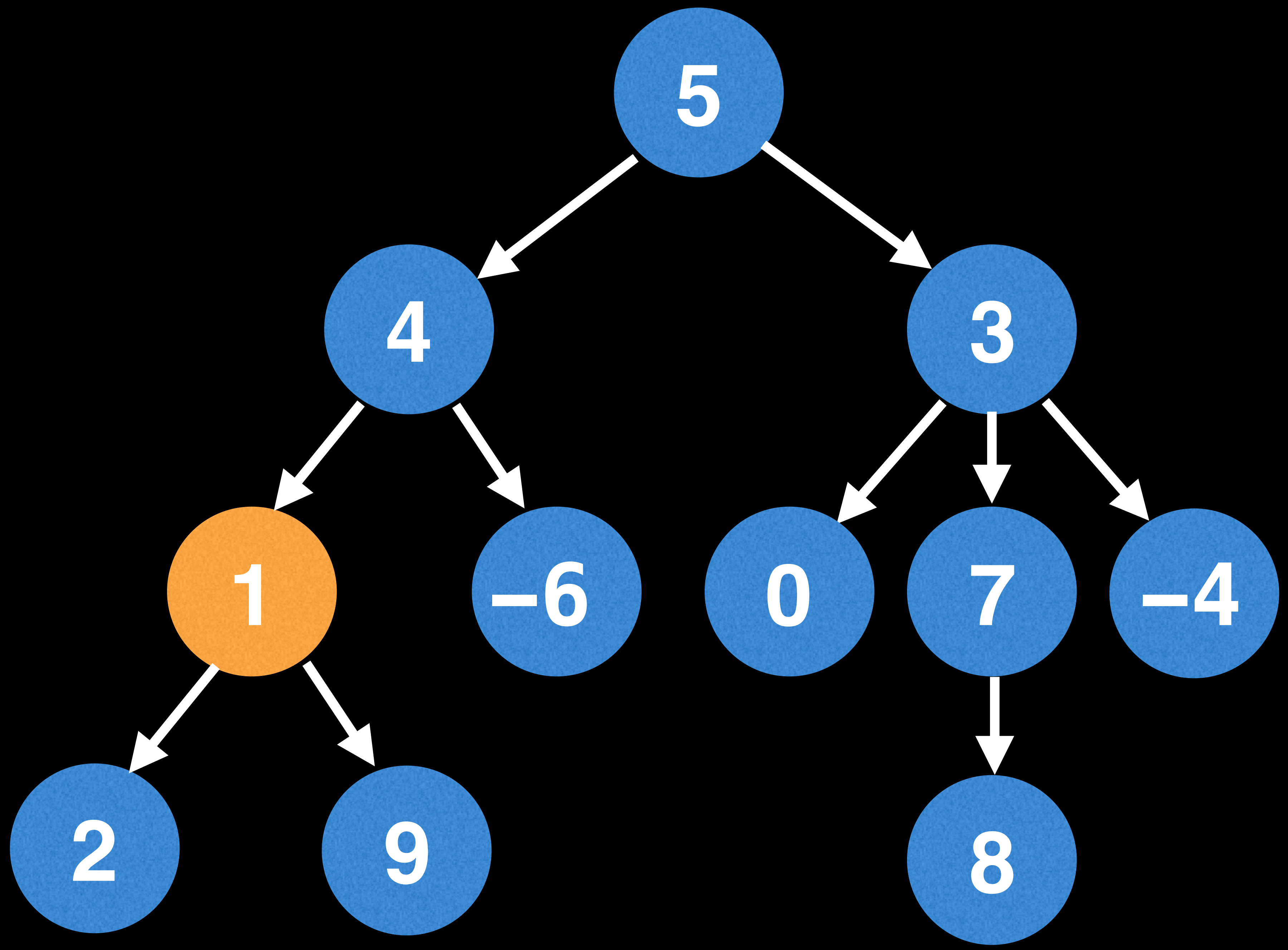
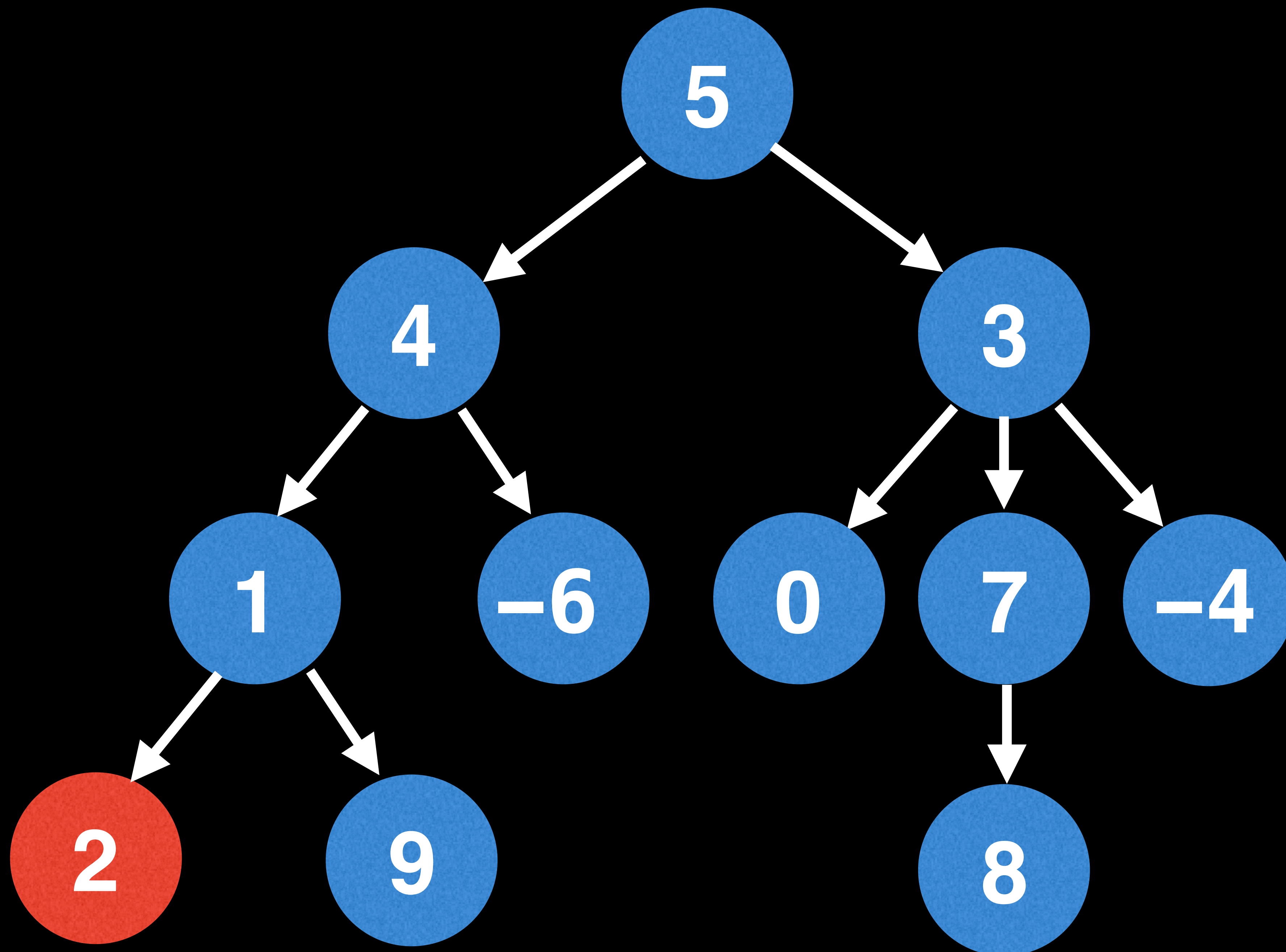
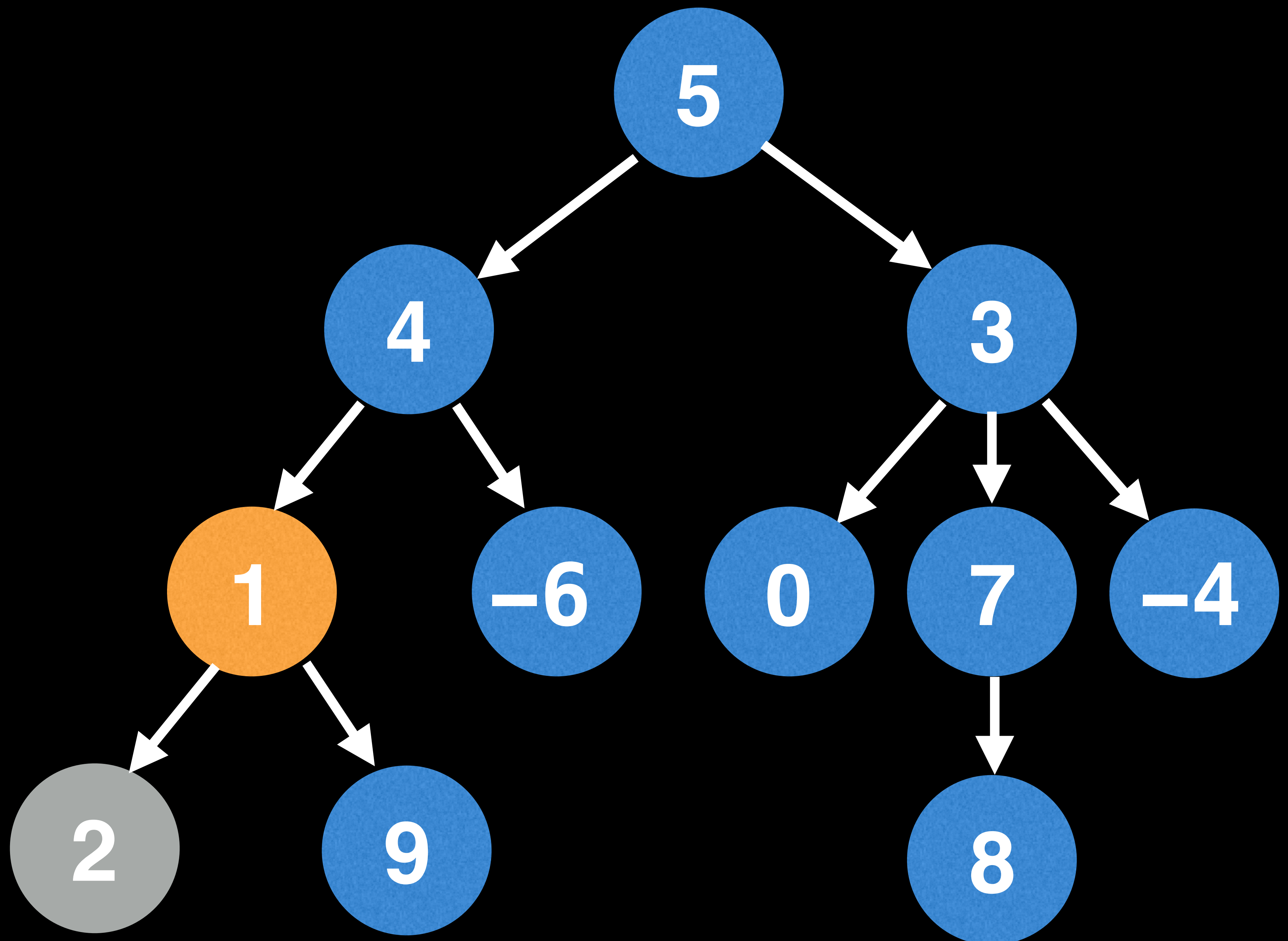


When dealing with rooted trees you begin with having a reference to the root node as a starting point for most algorithms.

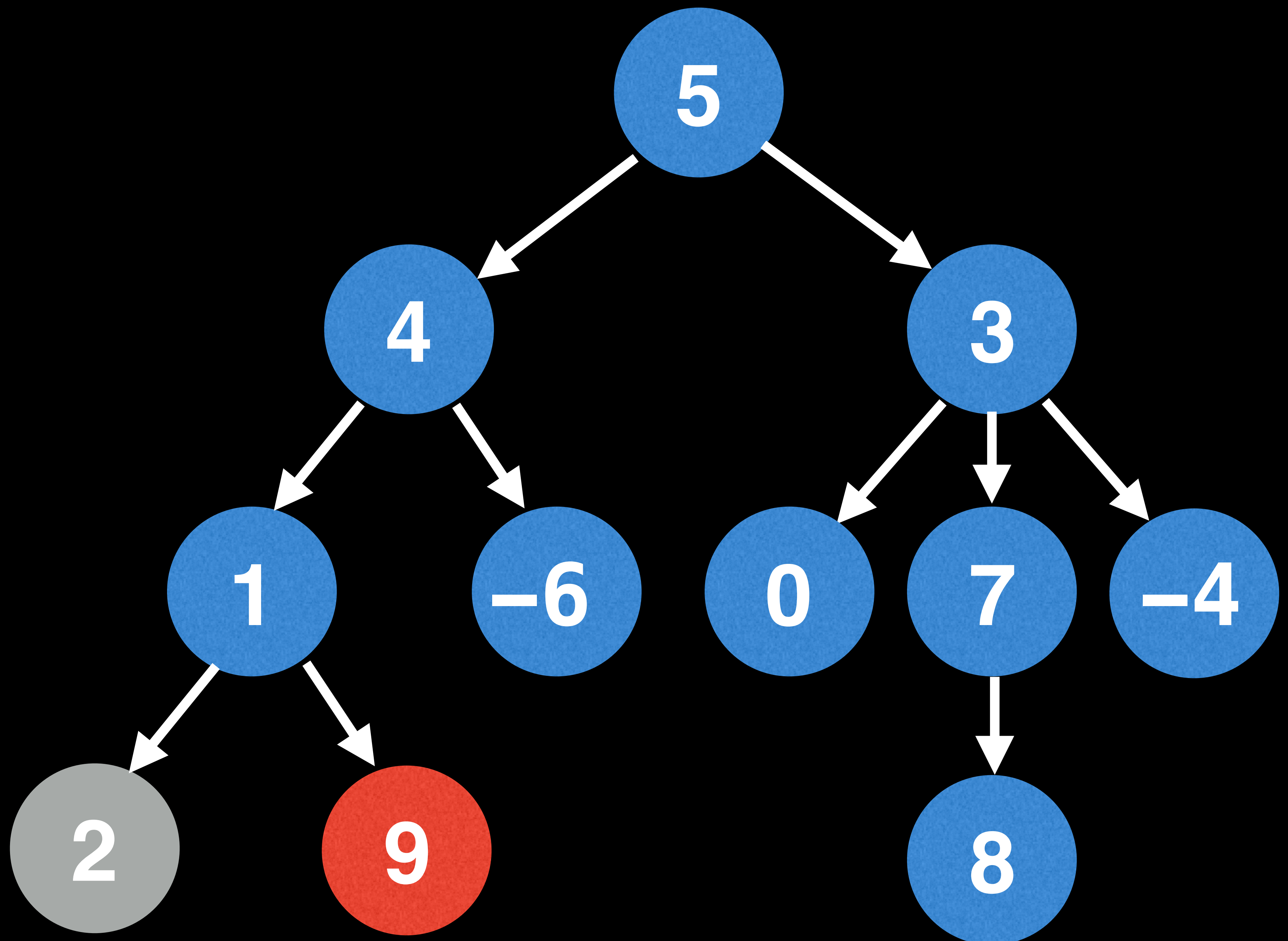

5
4
3
1
−6
0
7
−4
2
9
8


5
4
3
1
−6
0
7
−4
2
9
8


5
4
3
1
–6
0
7
–4
2
9
8



2



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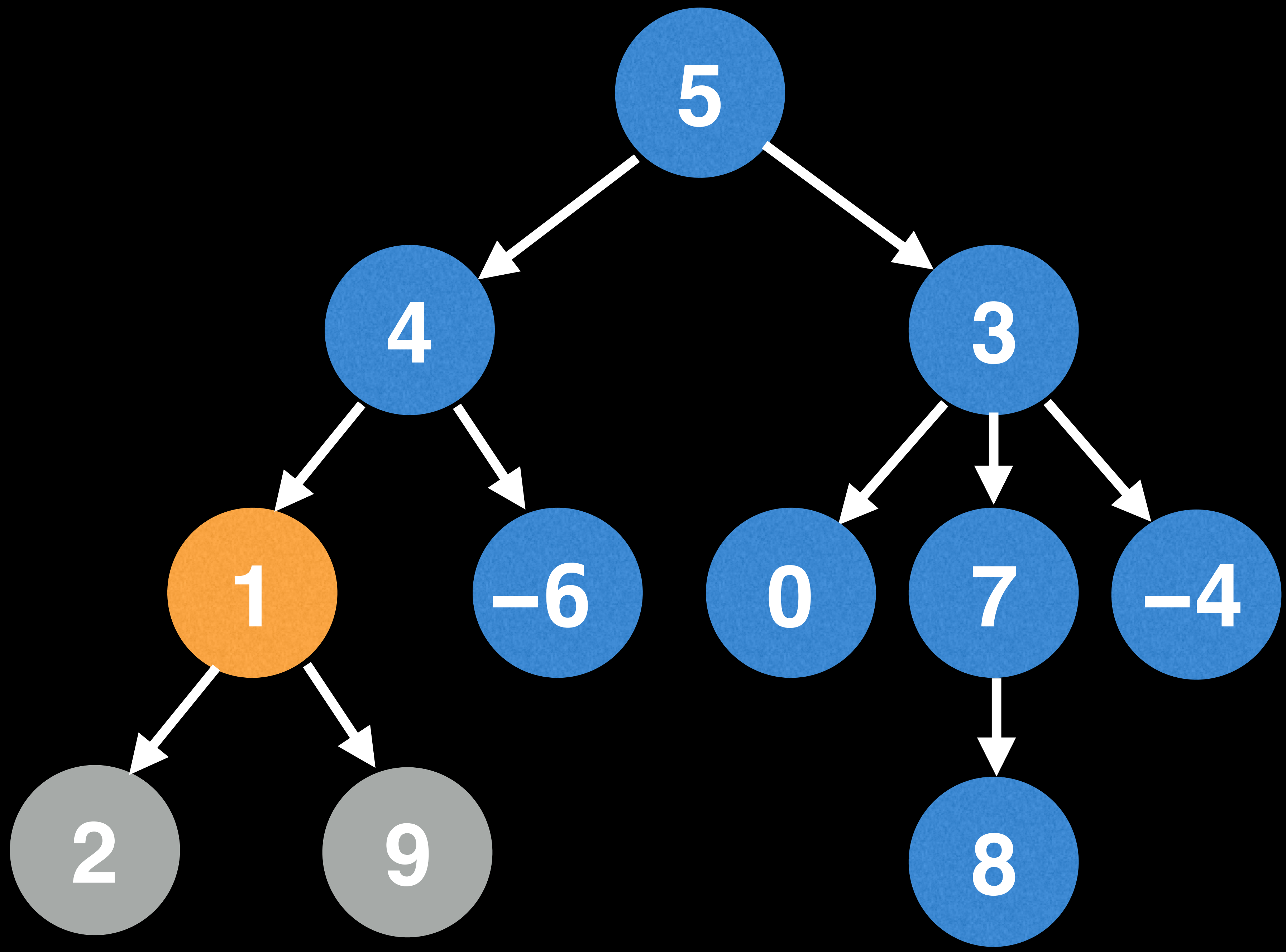
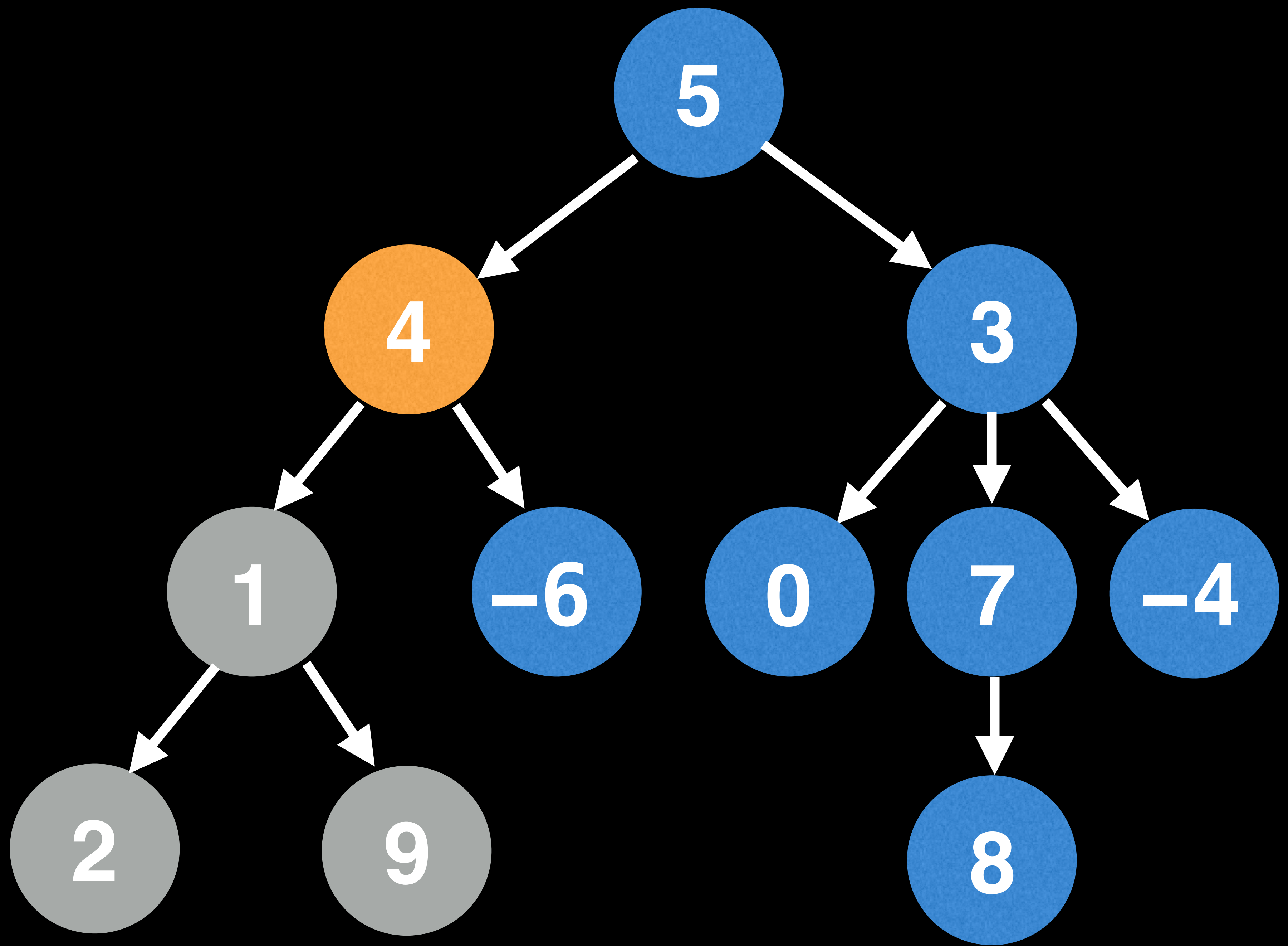
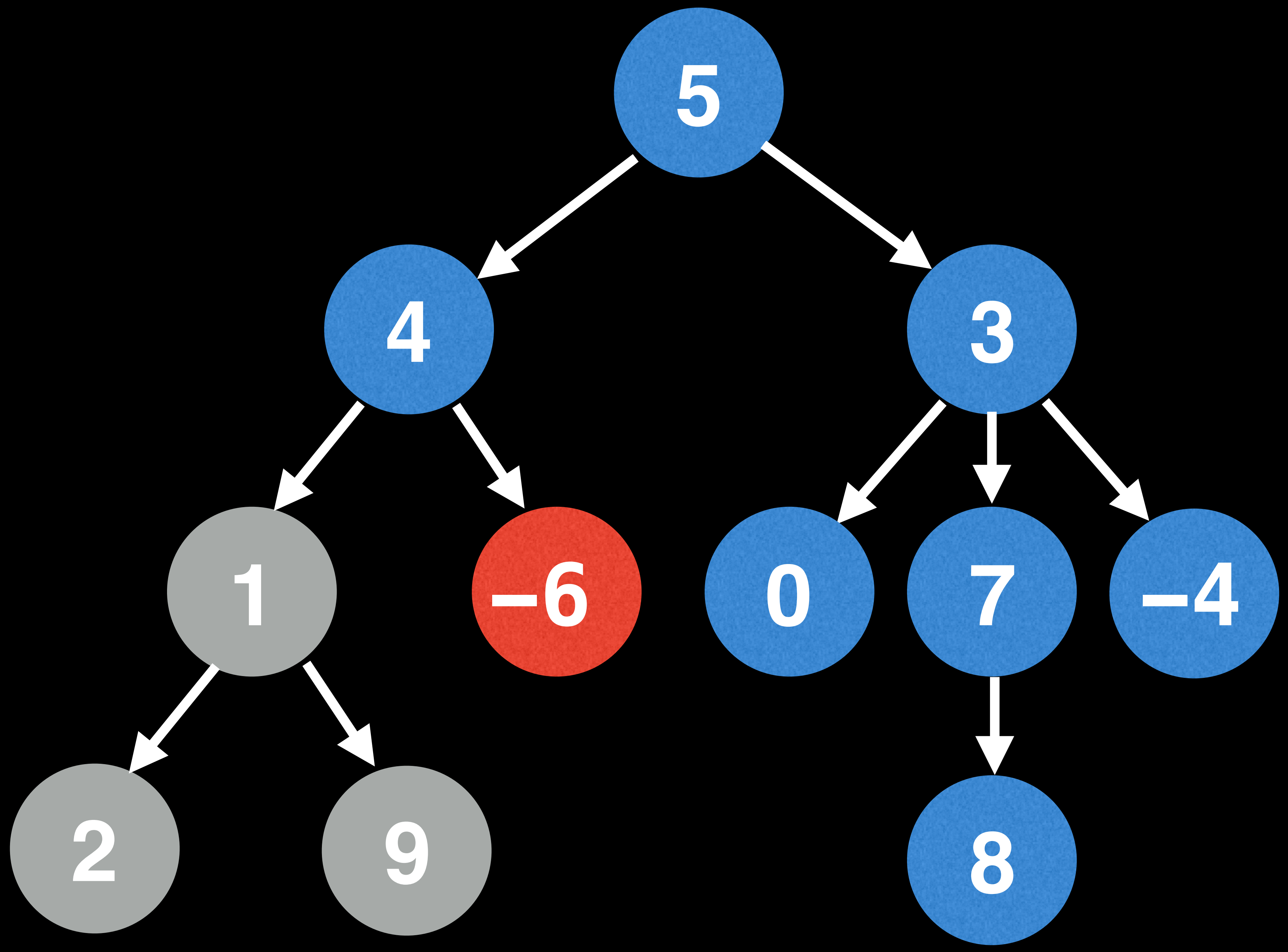


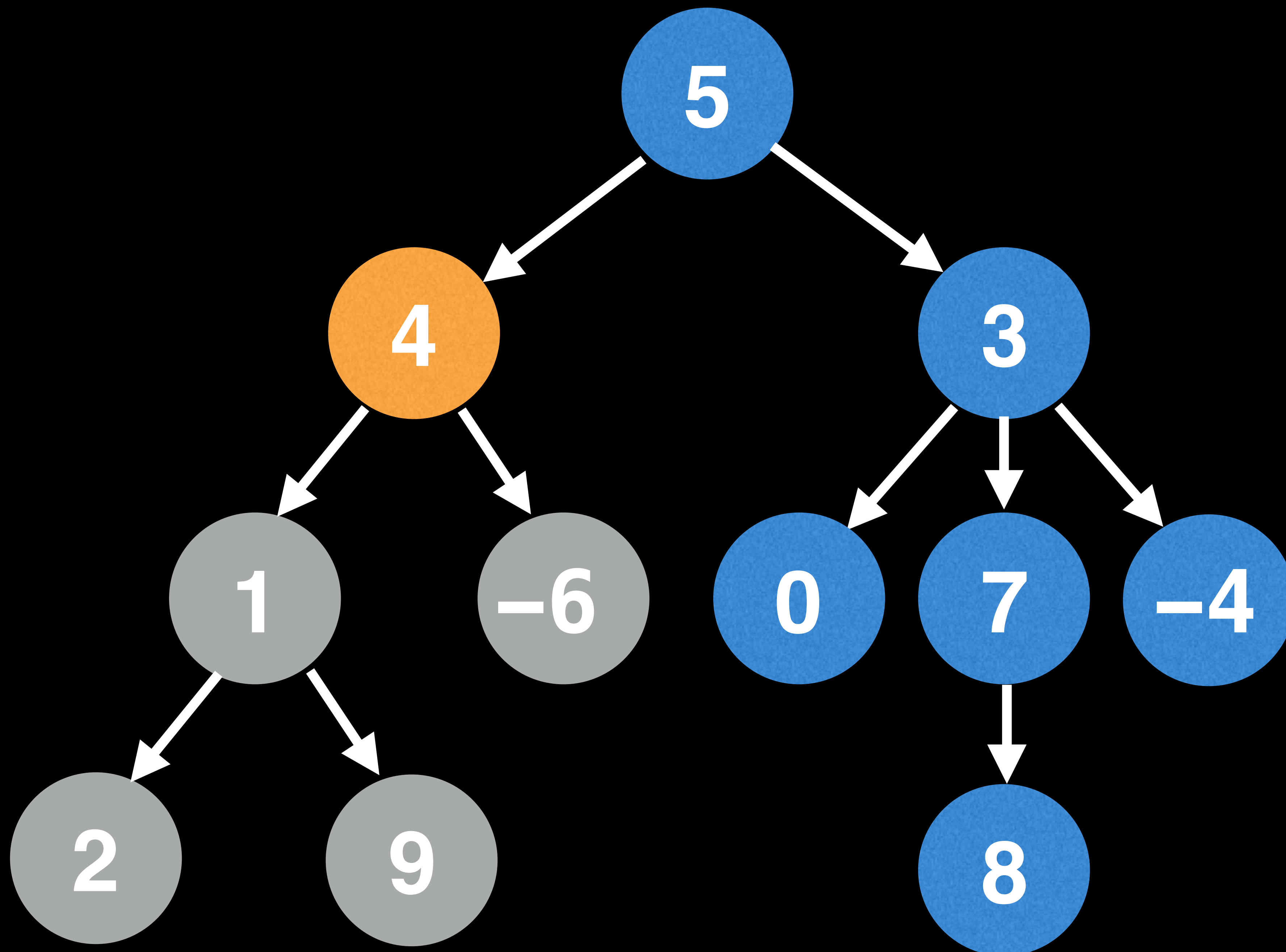
2 + 9



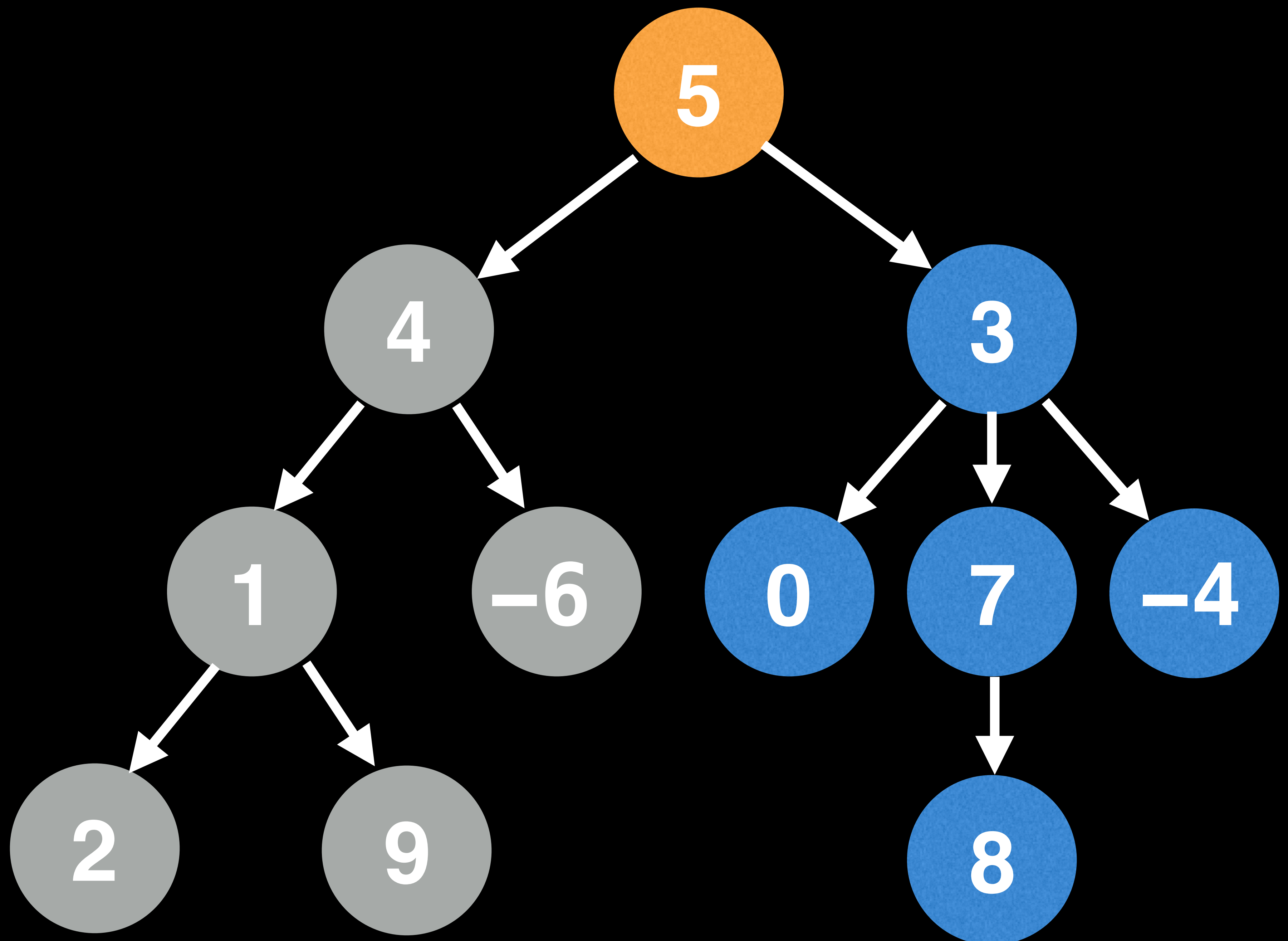
2 + 9



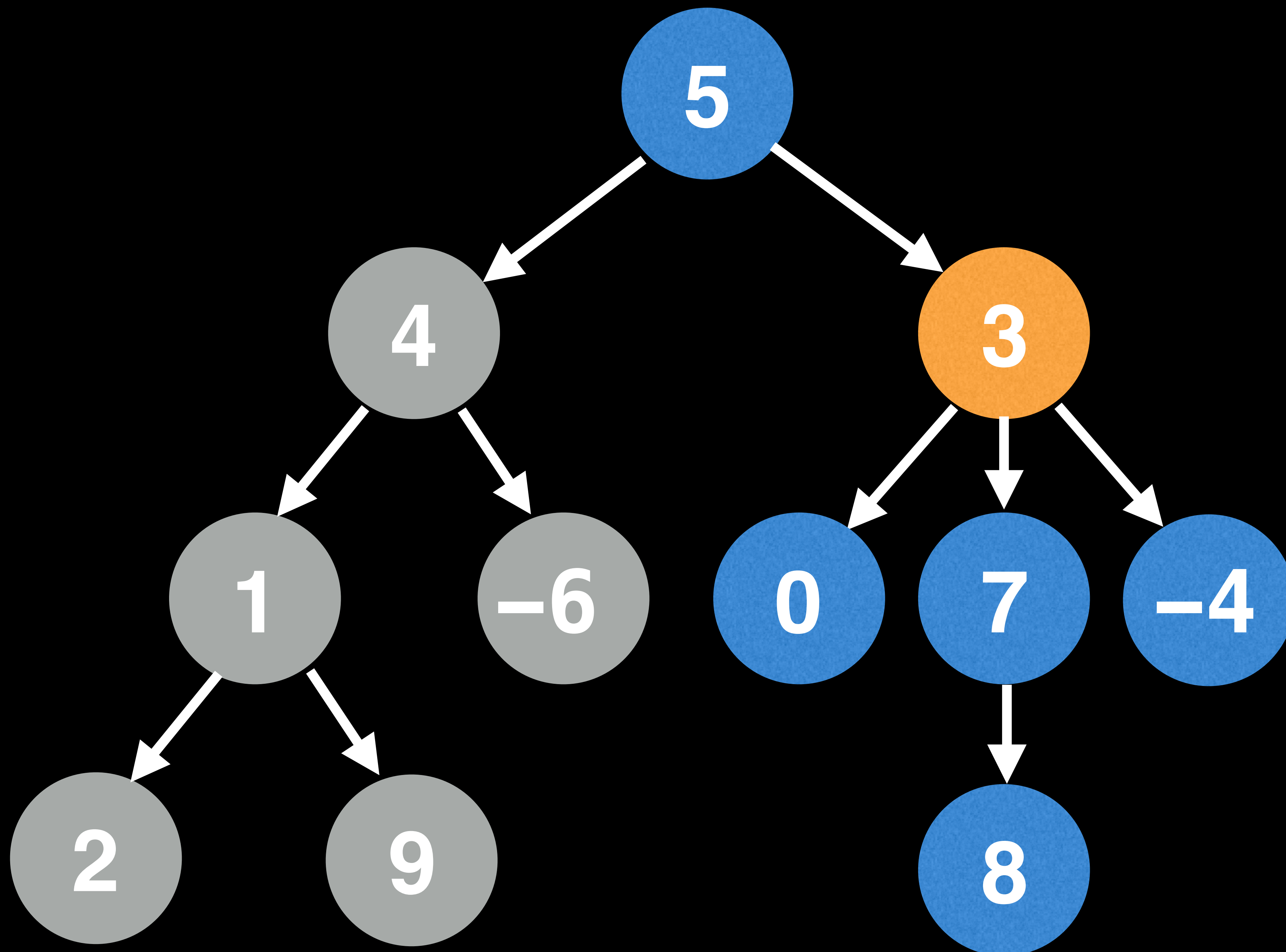
2 + 9



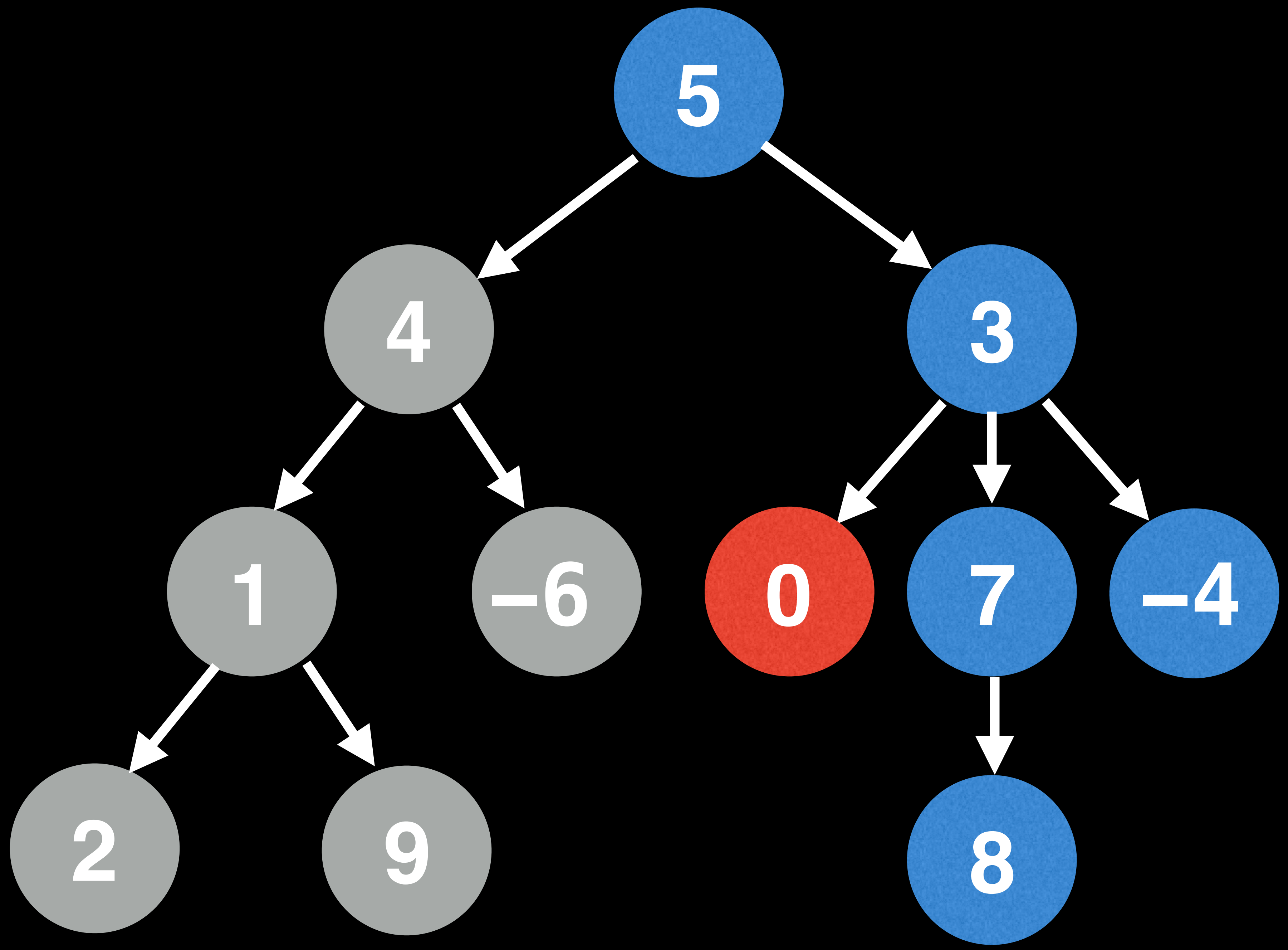
$2 + 9 - 6$



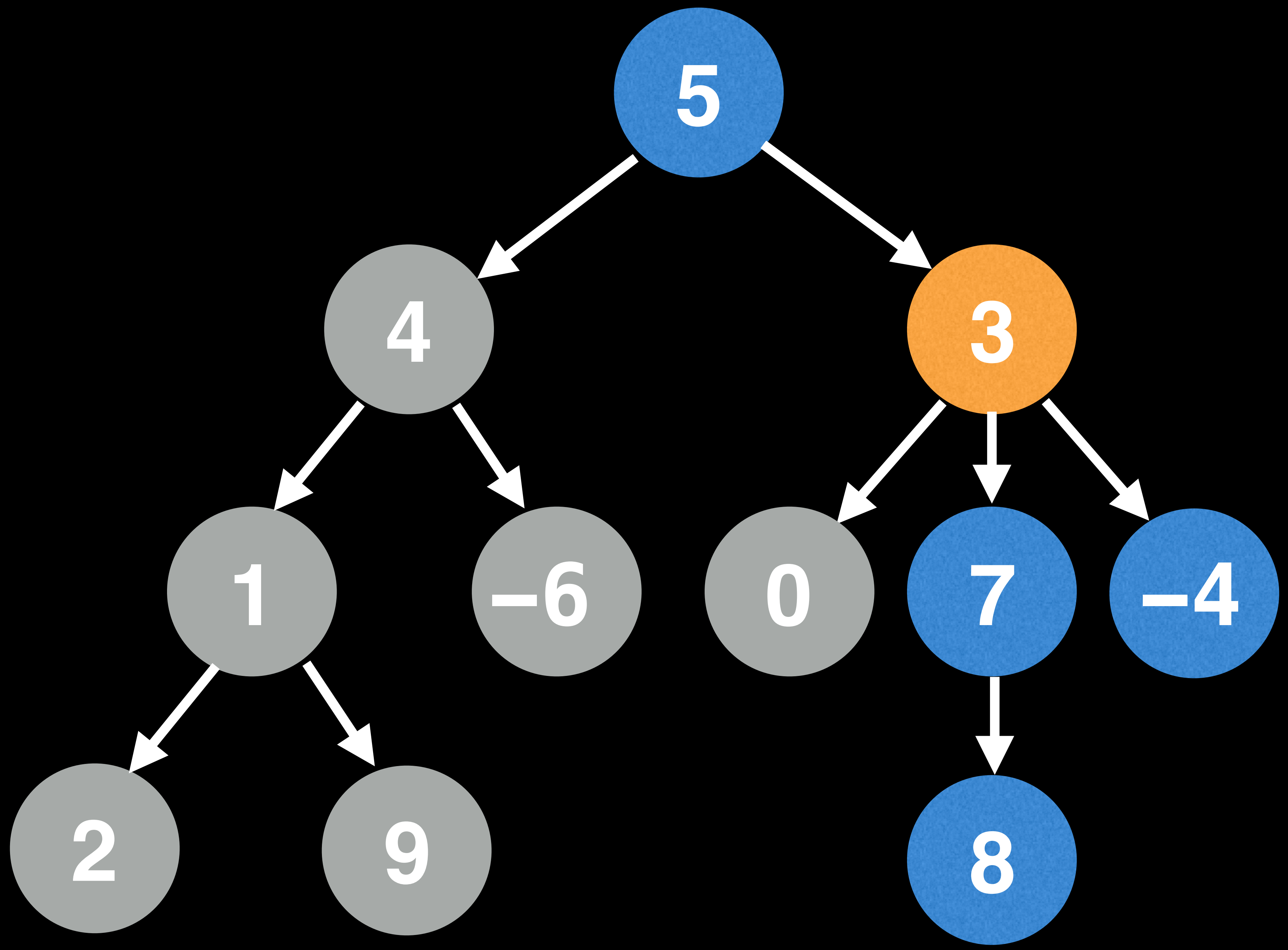
$2 + 9 - 6$



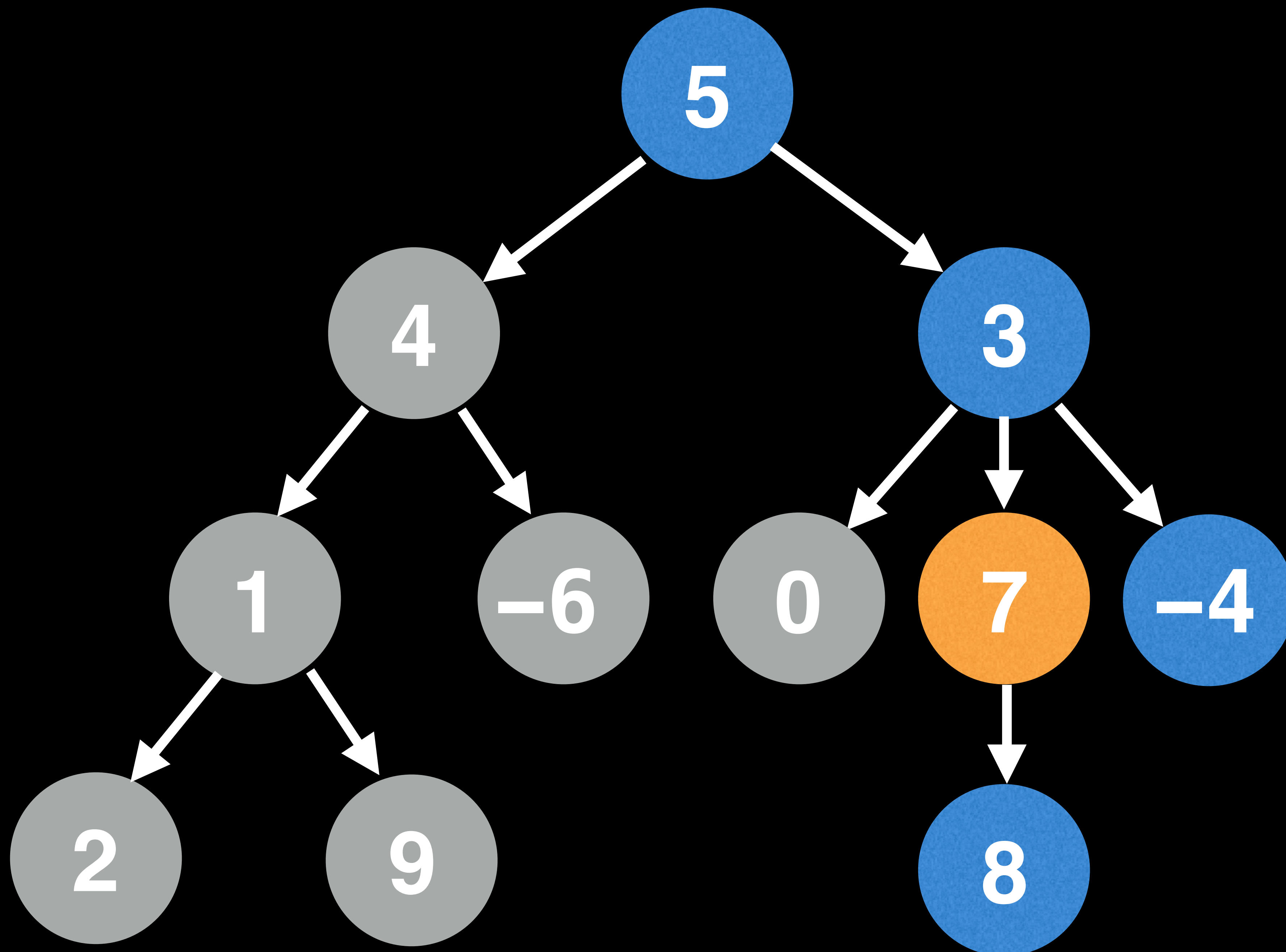
$2 + 9 - 6$



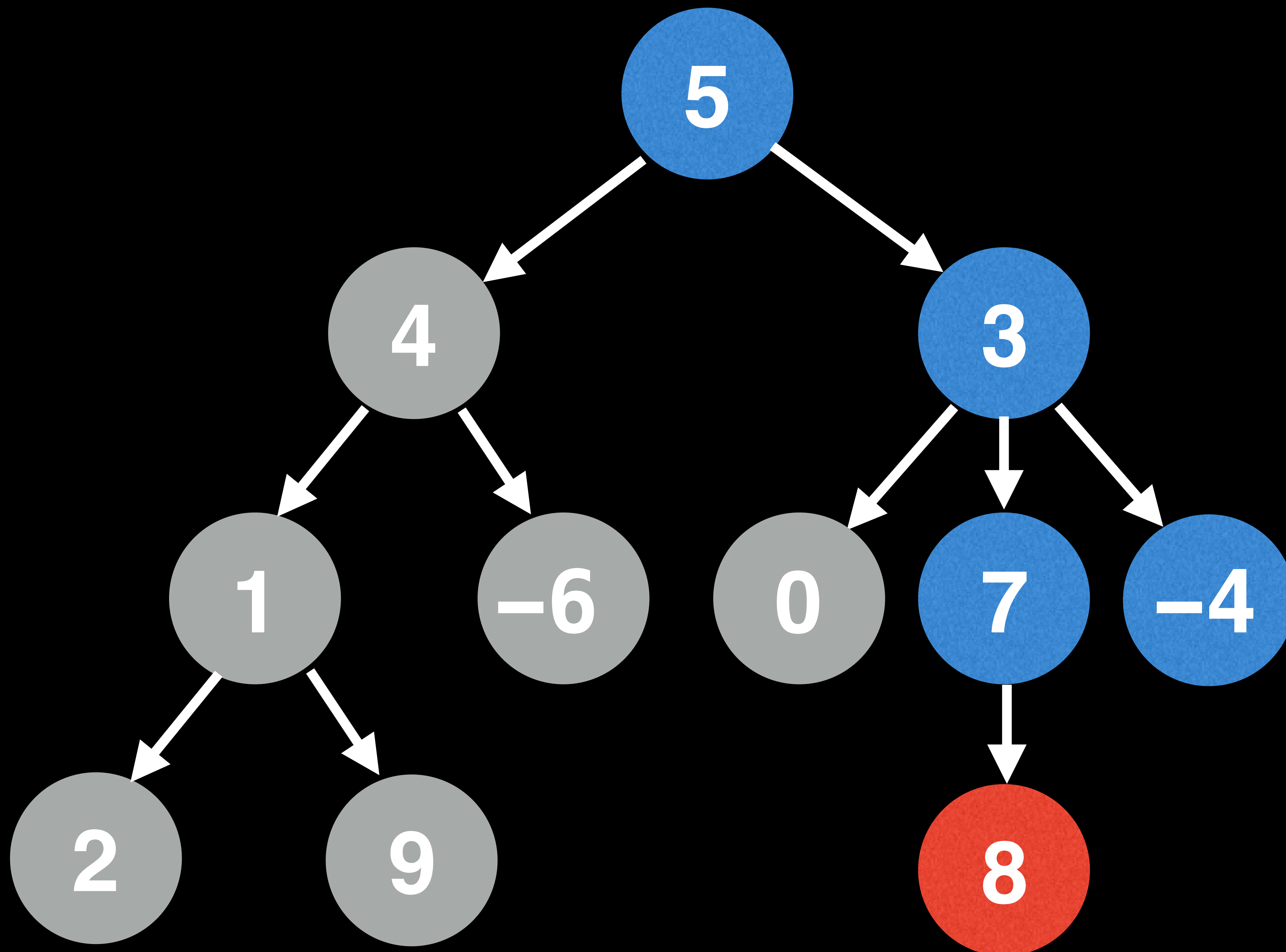
2 + 9 − 6



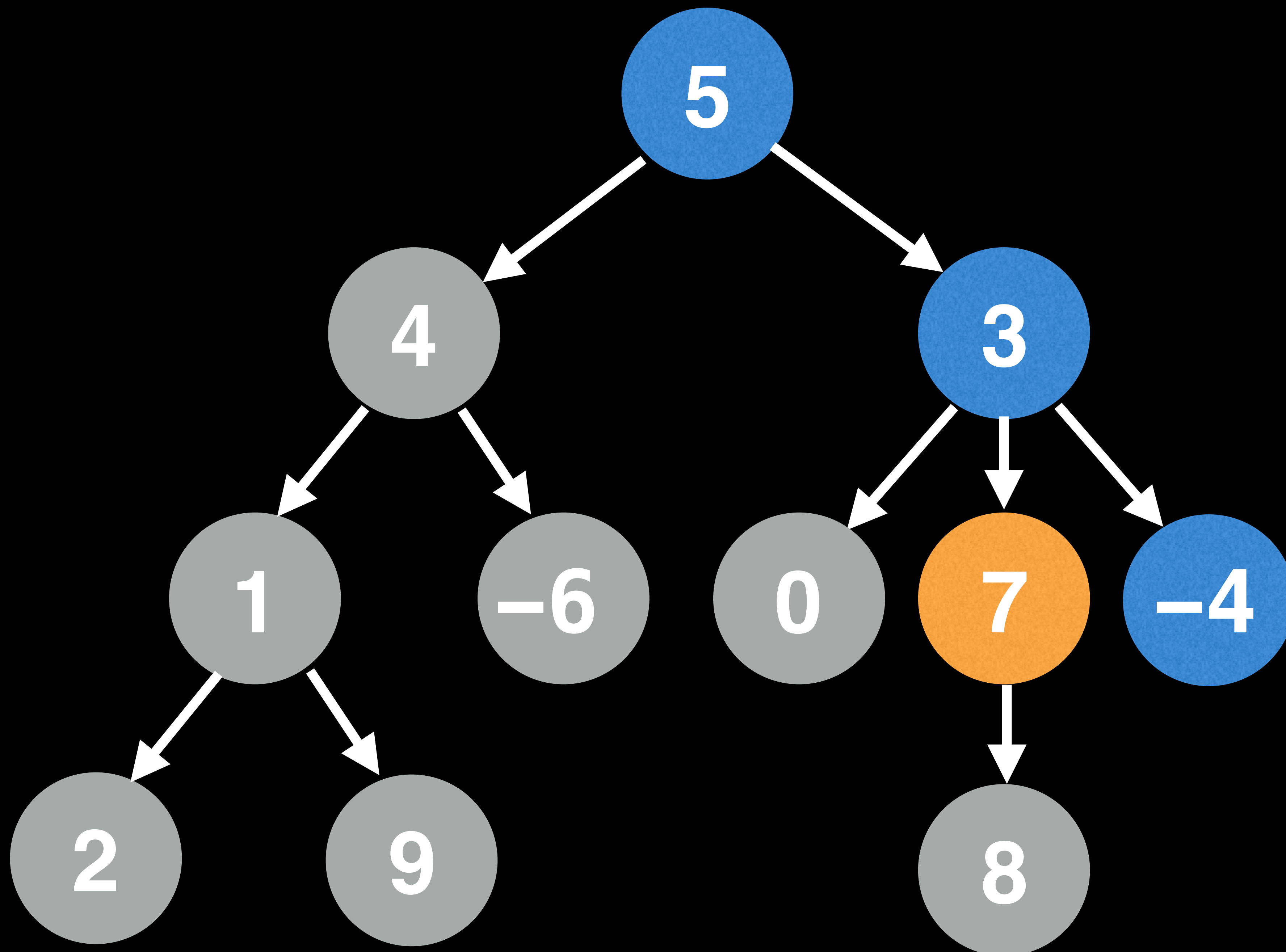
$2 + 9 - 6 + 0$



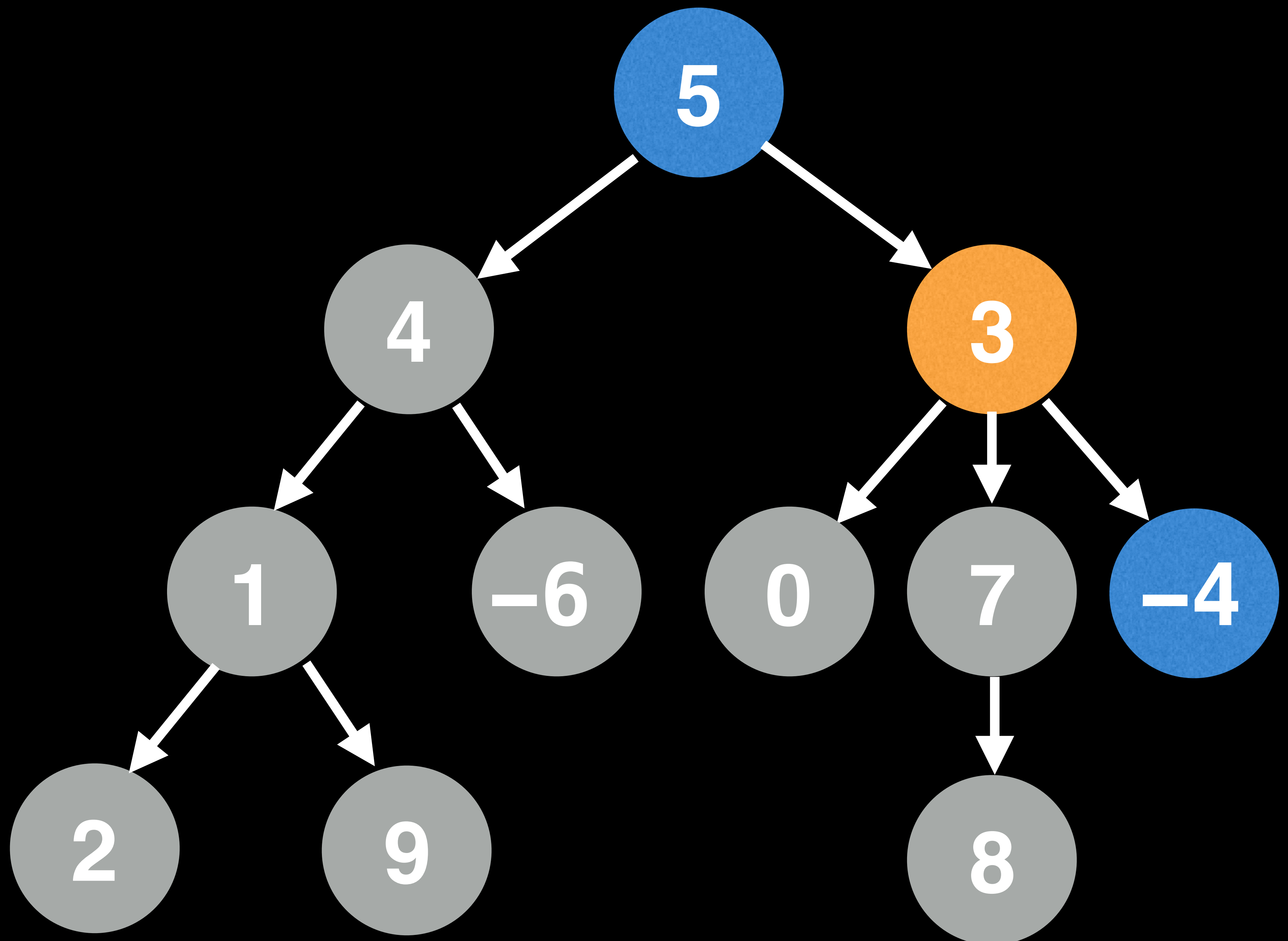
2 + 9 − 6 + 0



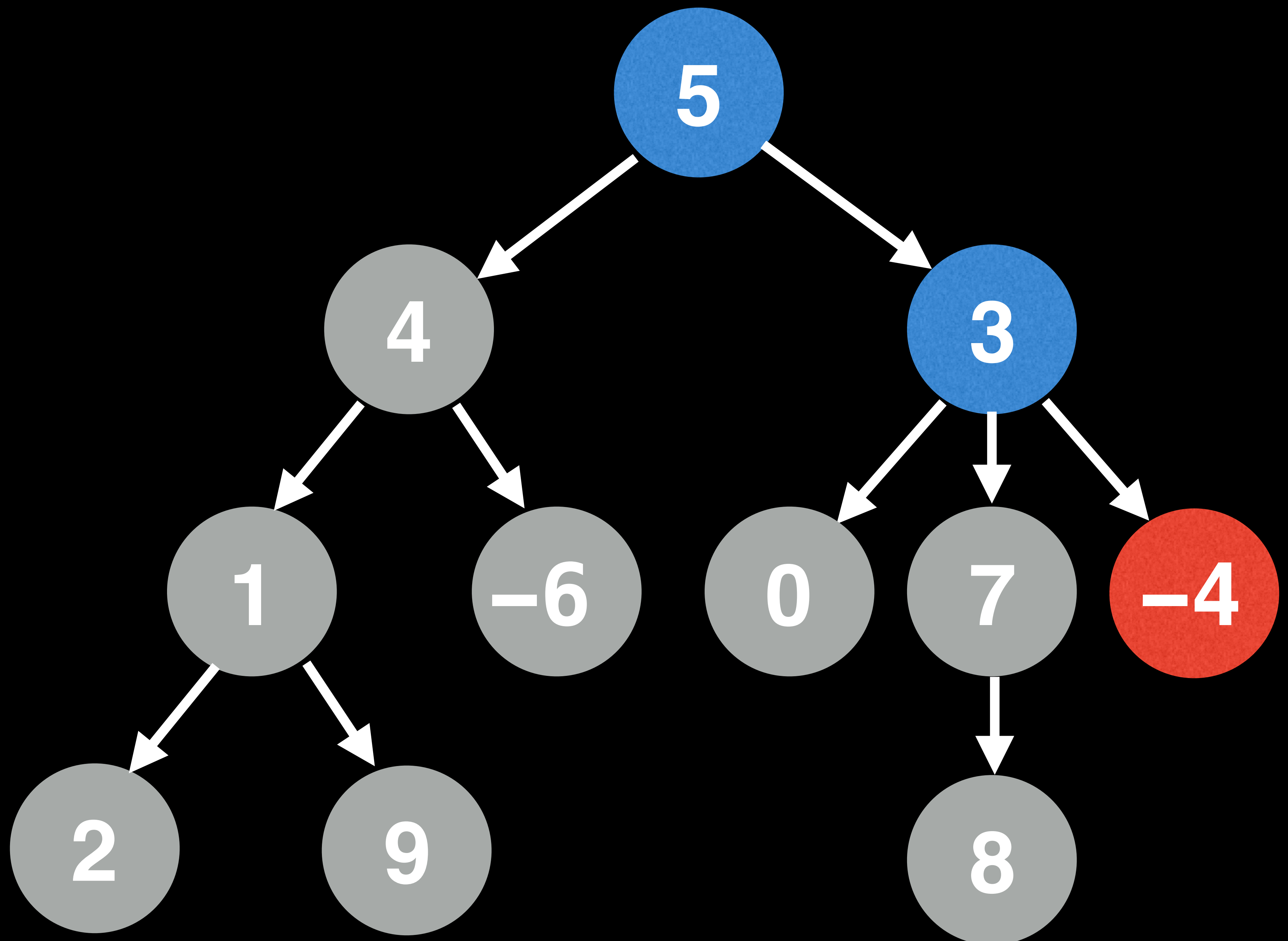
2 + 9 − 6 + 0



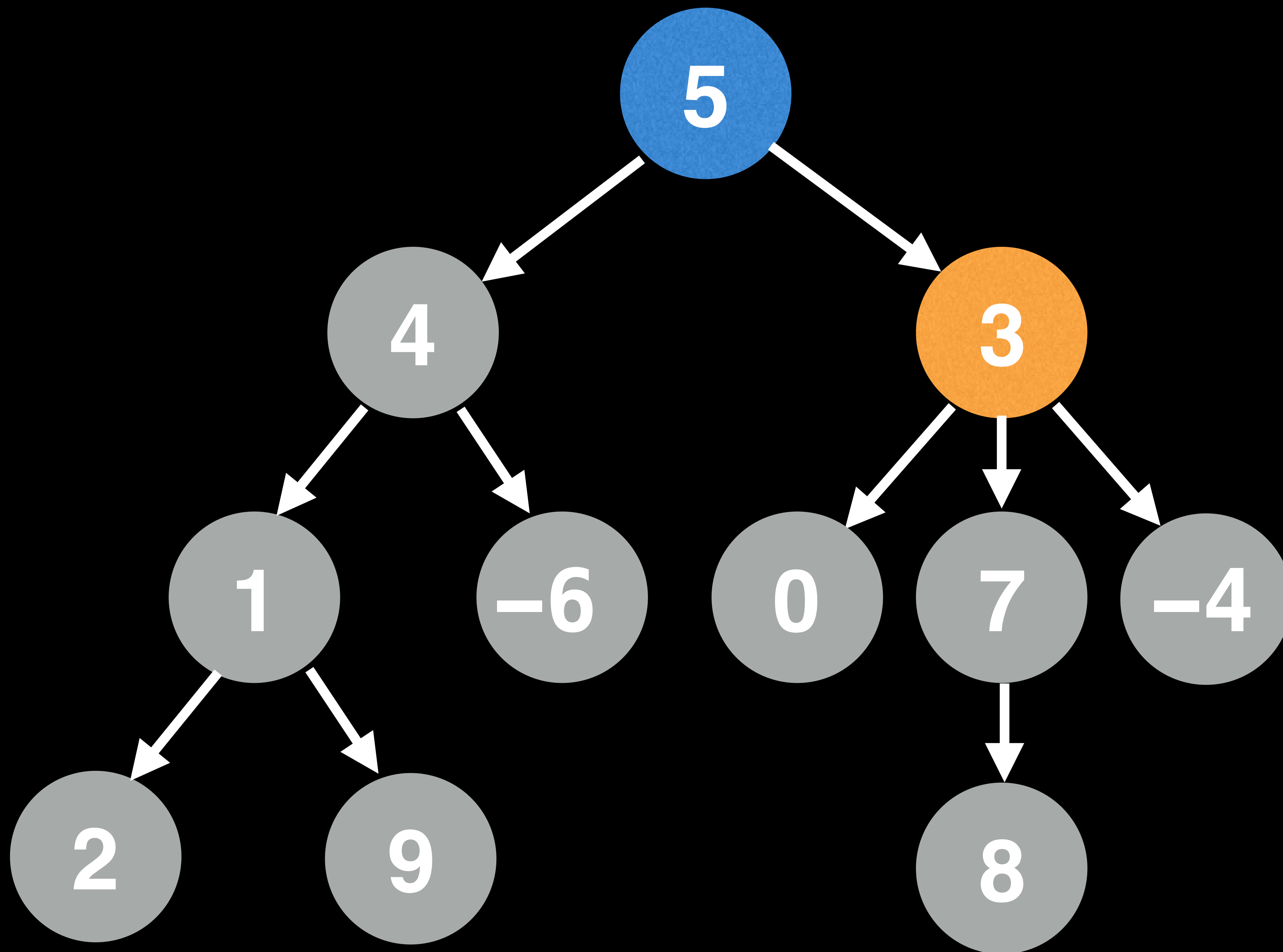
2 + 9 − 6 + 0 + 8



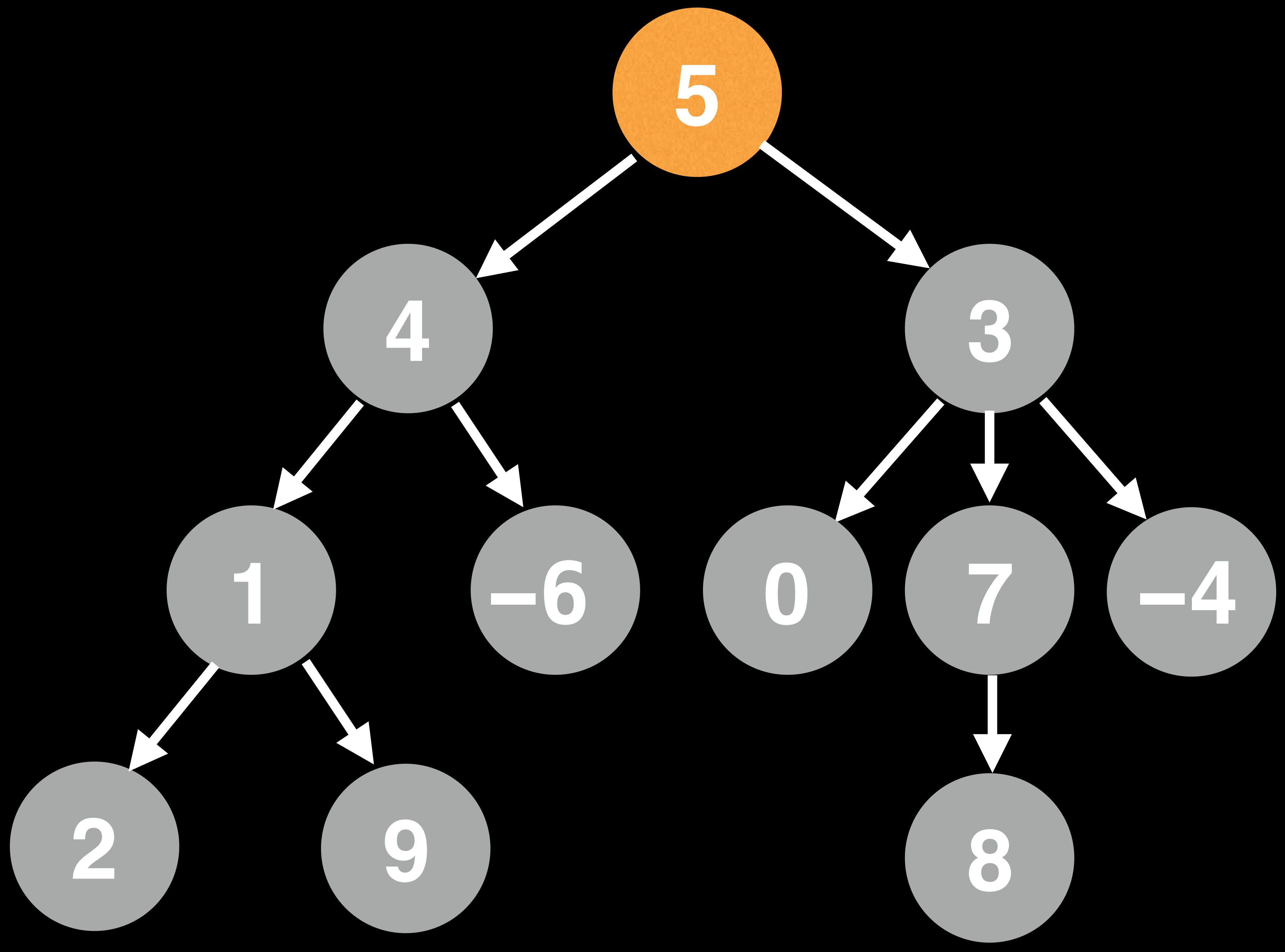
2 + 9 - 6 + 0 + 8



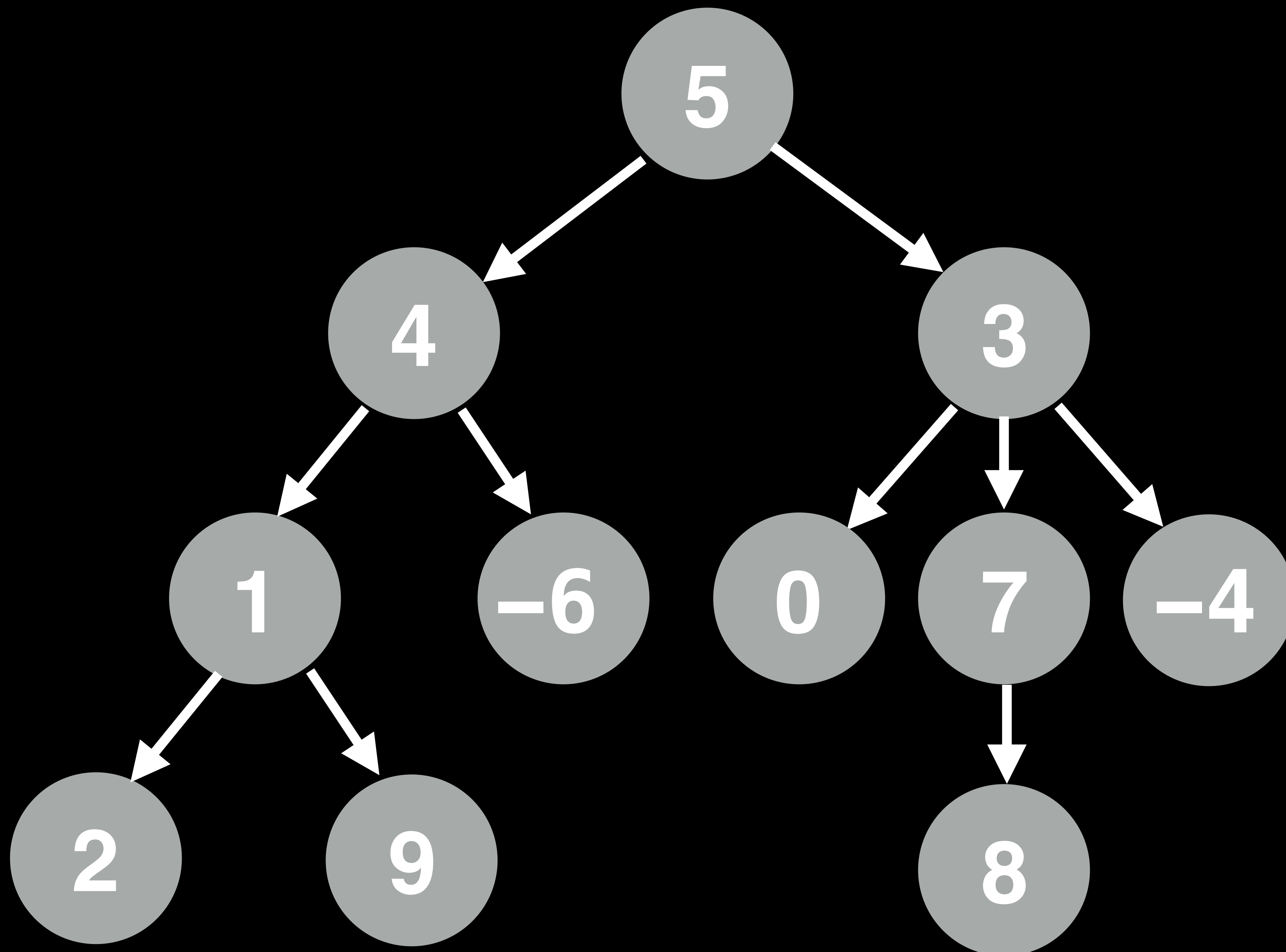
$$2 + 9 - 6 + 0 + 8$$



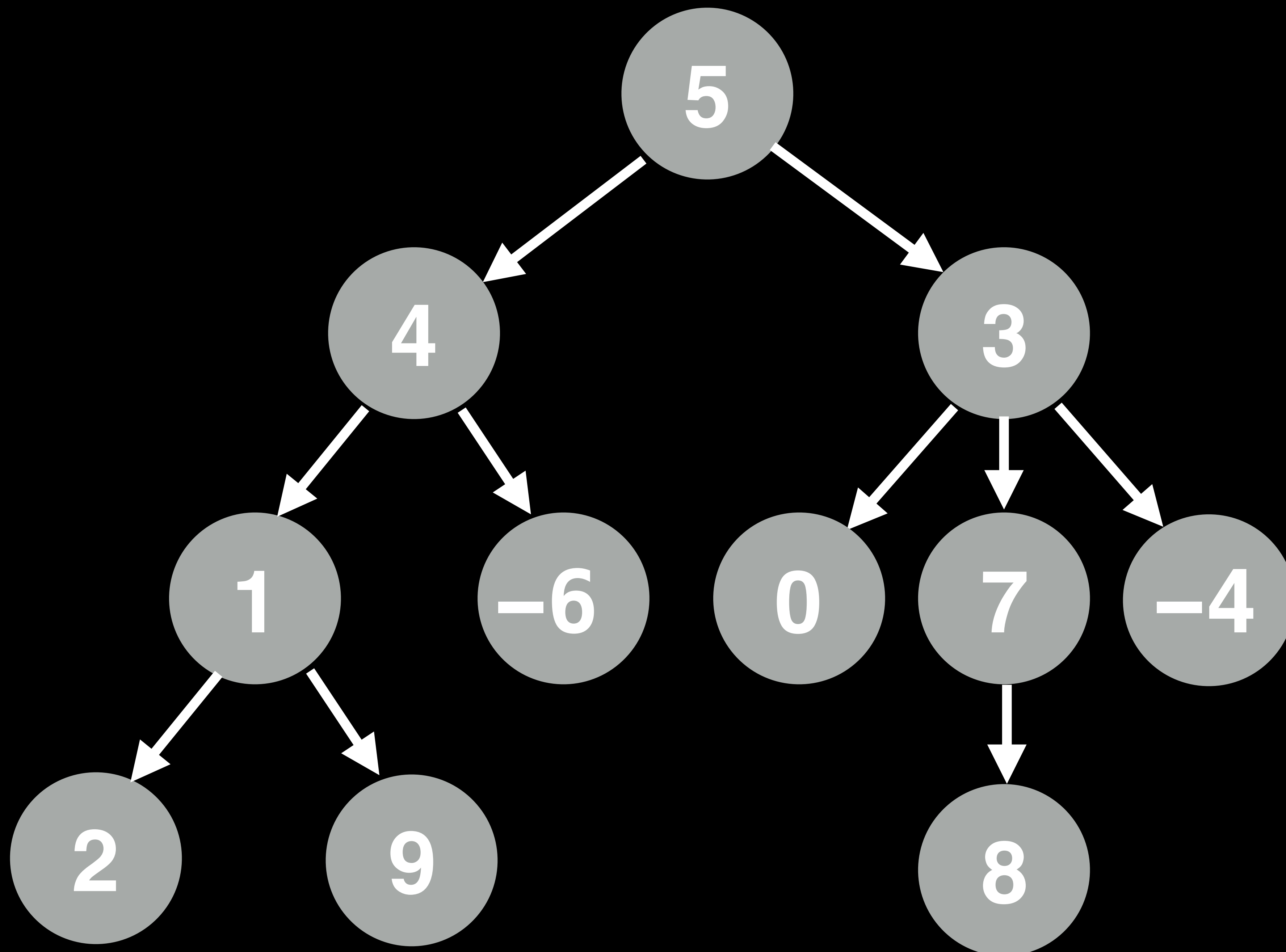
2 + 9 − 6 + 0 + 8 − 4



2 + 9 − 6 + 0 + 8 − 4



2 + 9 − 6 + 0 + 8 − 4



2 + 9 − 6 + 0 + 8 − 4 = 9

```
# Sums up leaf node values in a tree.
# Call function like: leafSum(root)
function leafSum(node):
  # Handle empty tree case
  if node == null:
    return 0
  if isLeaf(node):
    return node.getValue()
  total = 0
  for child in node.getChildNodes():
    total += leafSum(child)
  return total

function isLeaf(node):
  return node.getChildNodes().size() == 0
```

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function isLeaf(node):
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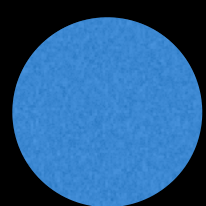
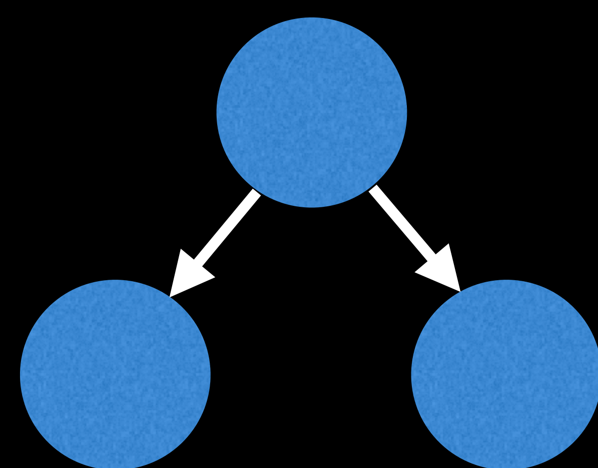
```

Problem 2: Tree Height

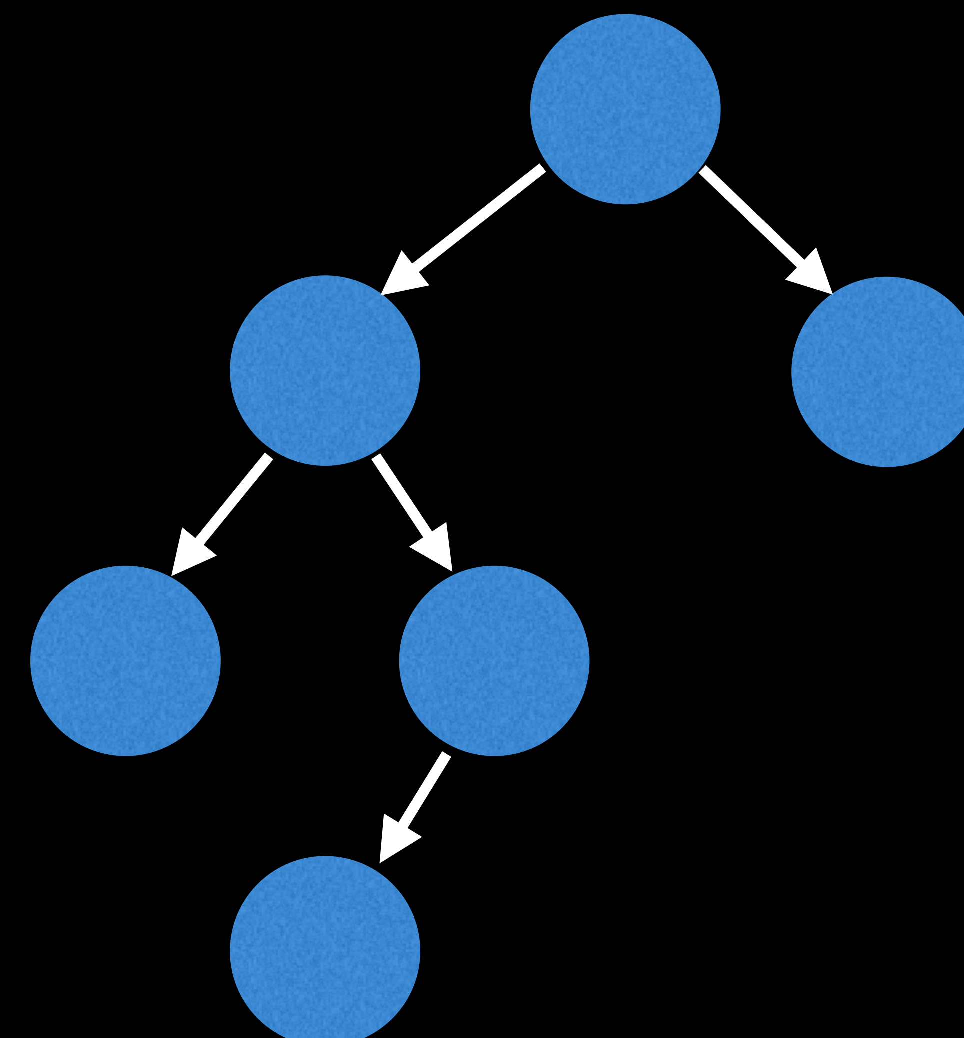
Find the **height** of a **binary tree**. The **height** of a tree is the number of edges from the root to the lowest leaf.



height 0

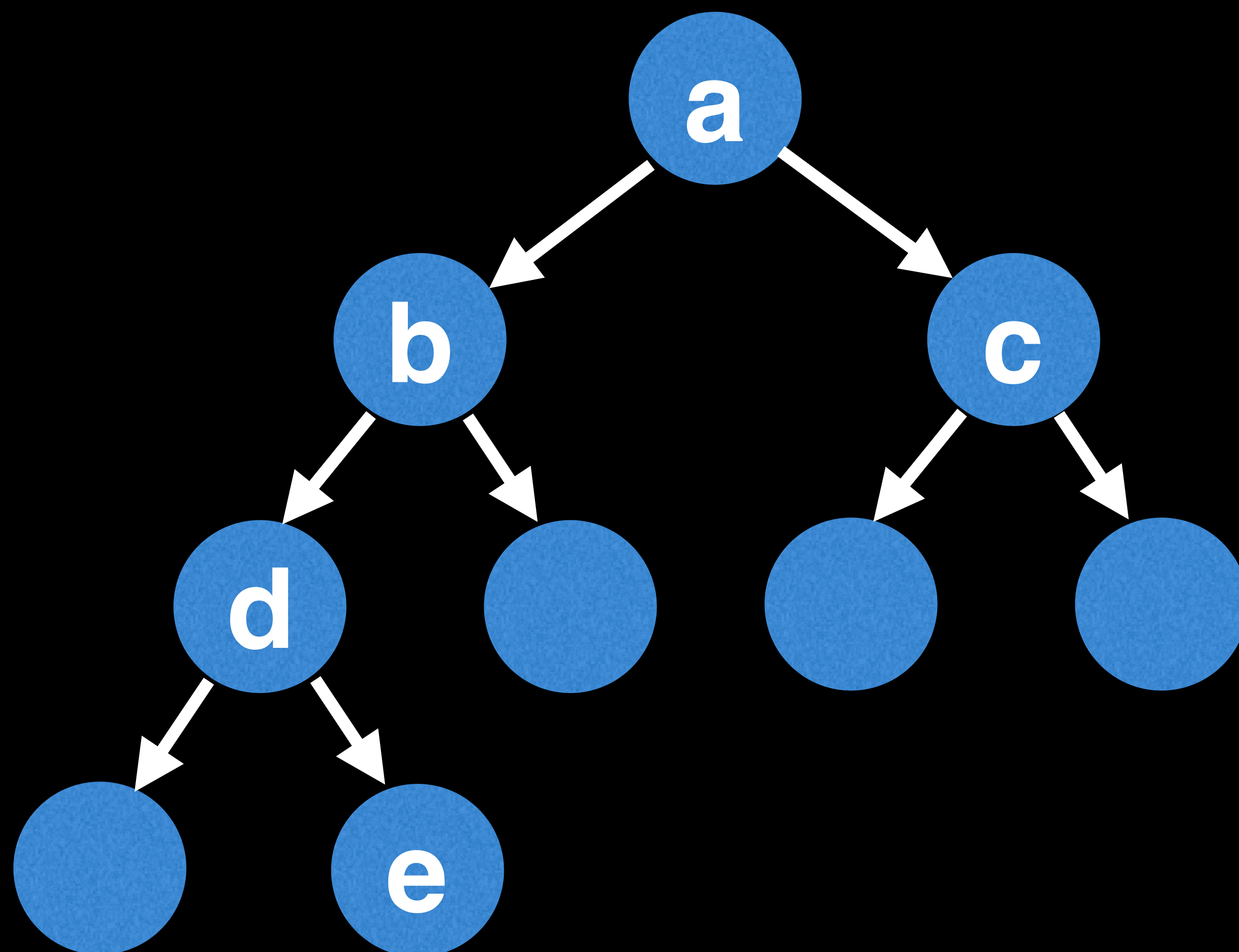


height 1

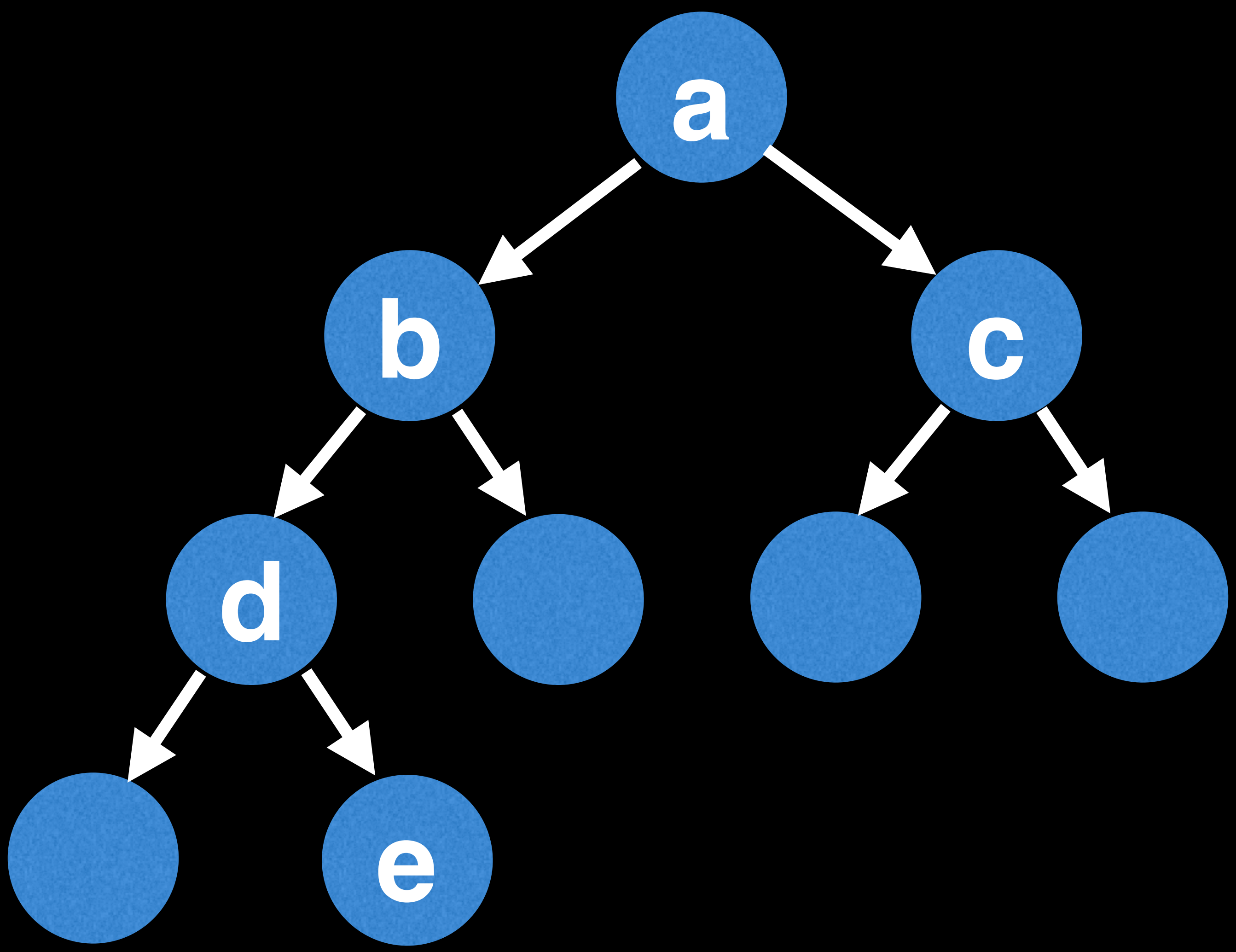


height 3

Let **h**(x) be the height of the subtree rooted at node x.

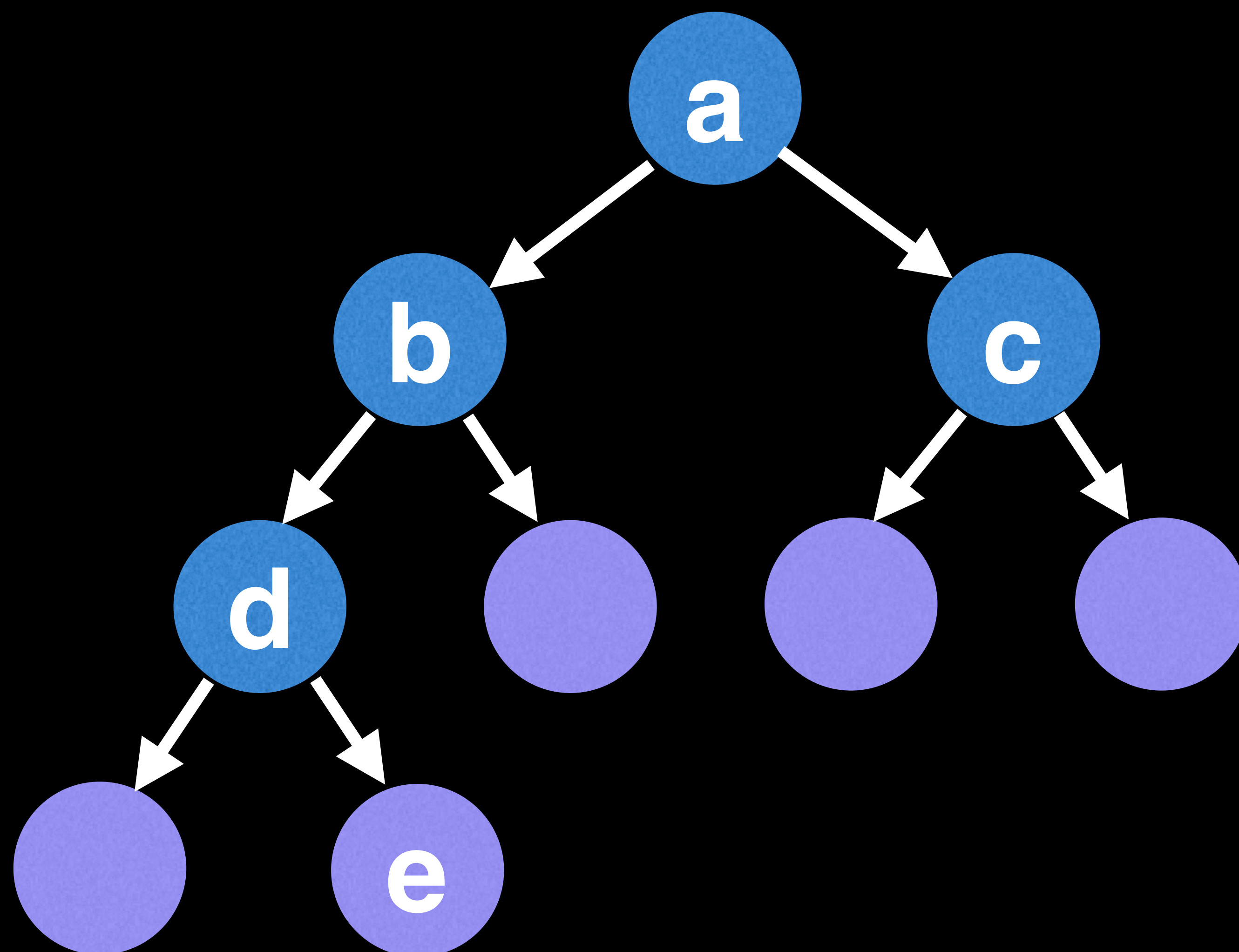


Let **h**(x) be the height of the subtree rooted at node x.



h(a) = 3, **h**(b) = 2, **h**(c) = 1, **h**(d) = 1, **h**(e) = 0

By themselves, leaf nodes such as node **e** don't have children, so they don't add any additional height to the tree.

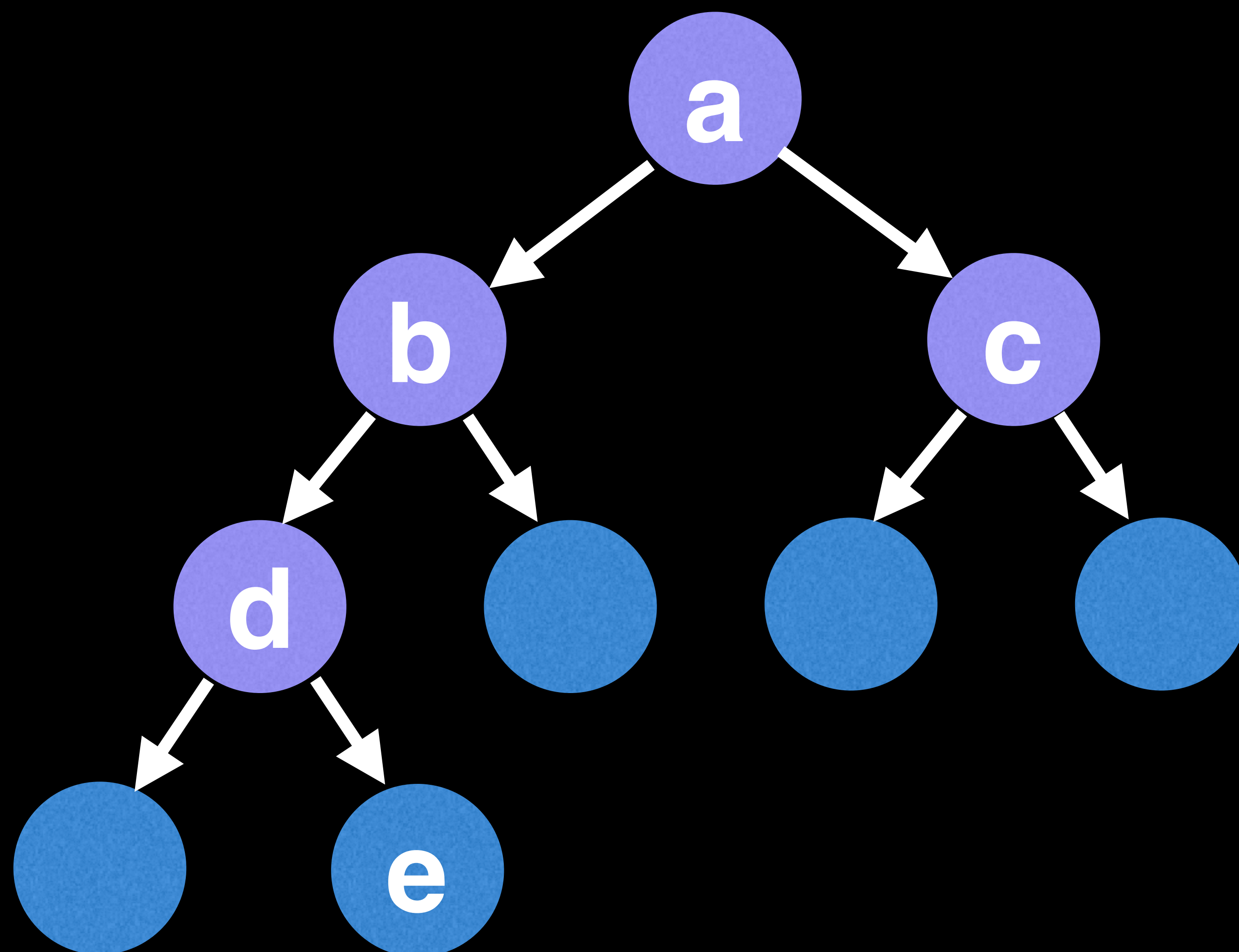


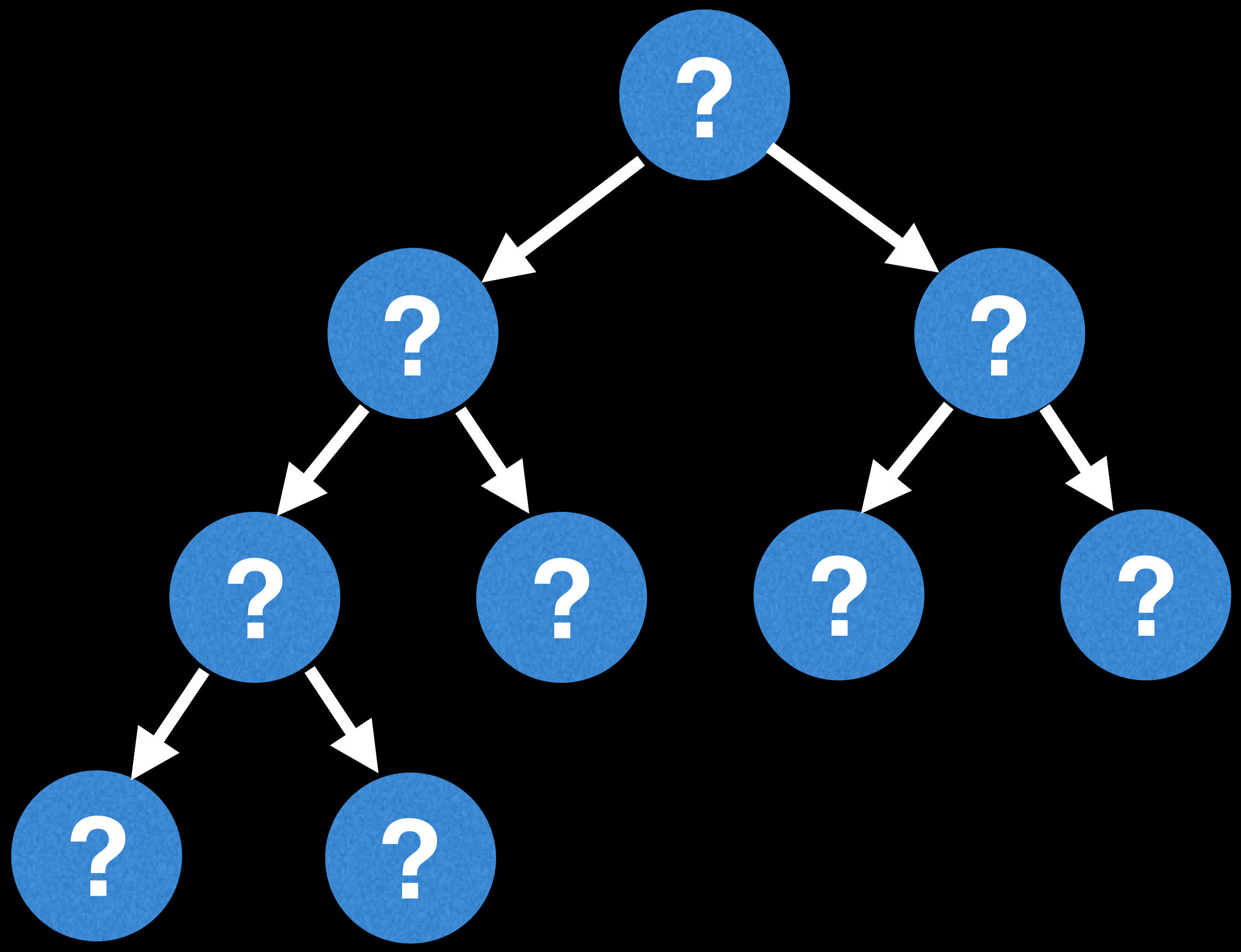
As a base case we can conclude that:

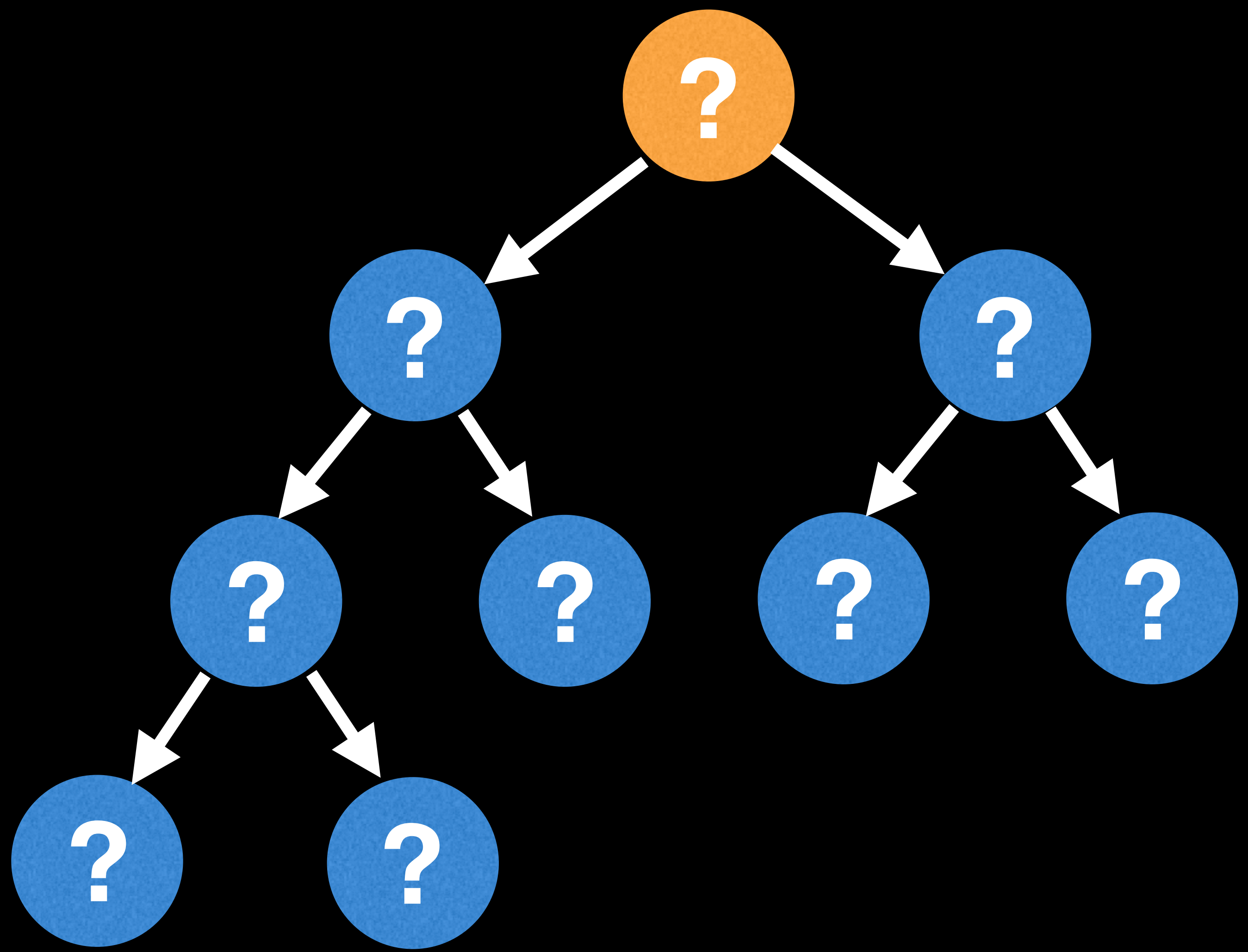
h(leaf node) = 0

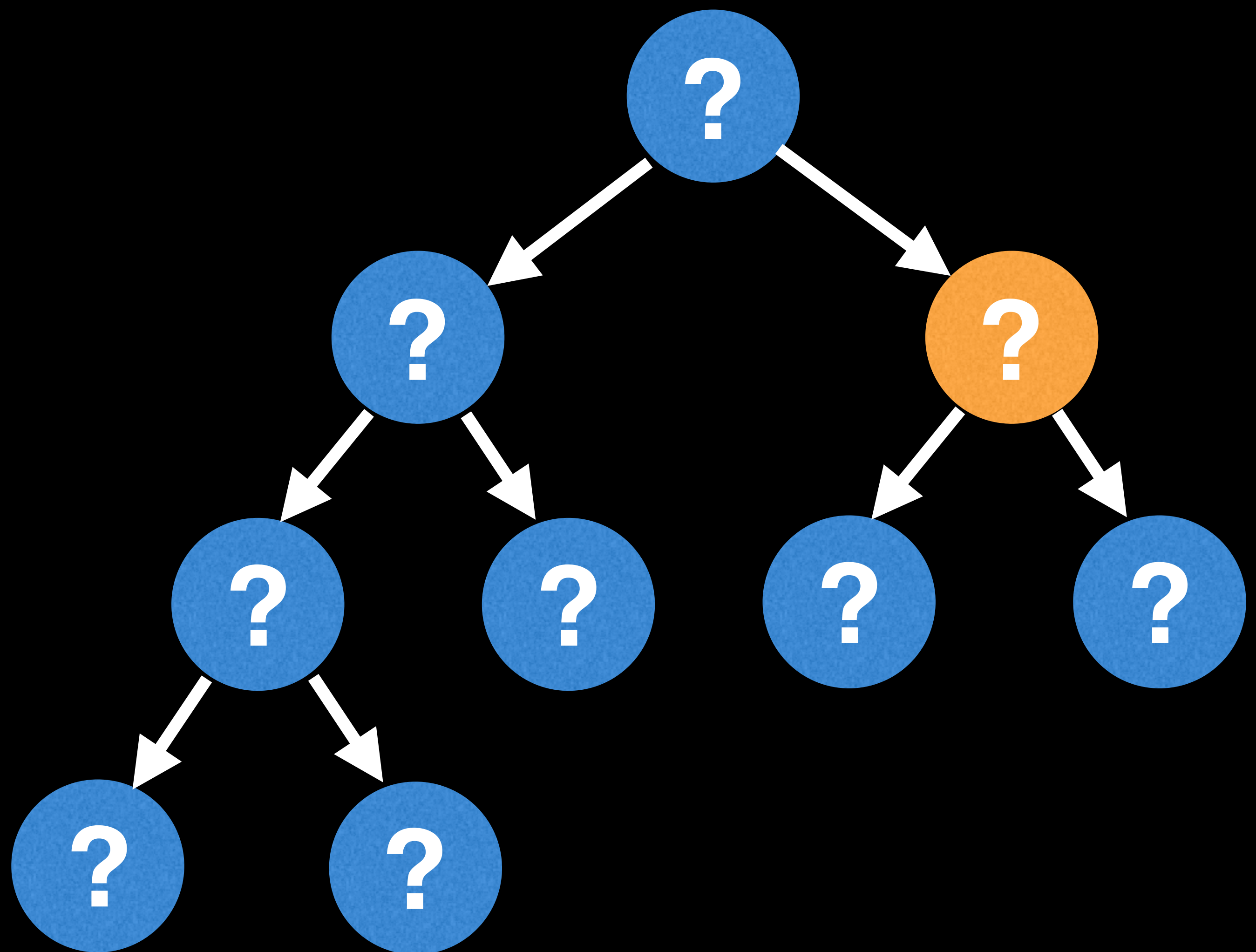
Assuming node x is not a leaf node, we're able to formulate a recurrence for the height:

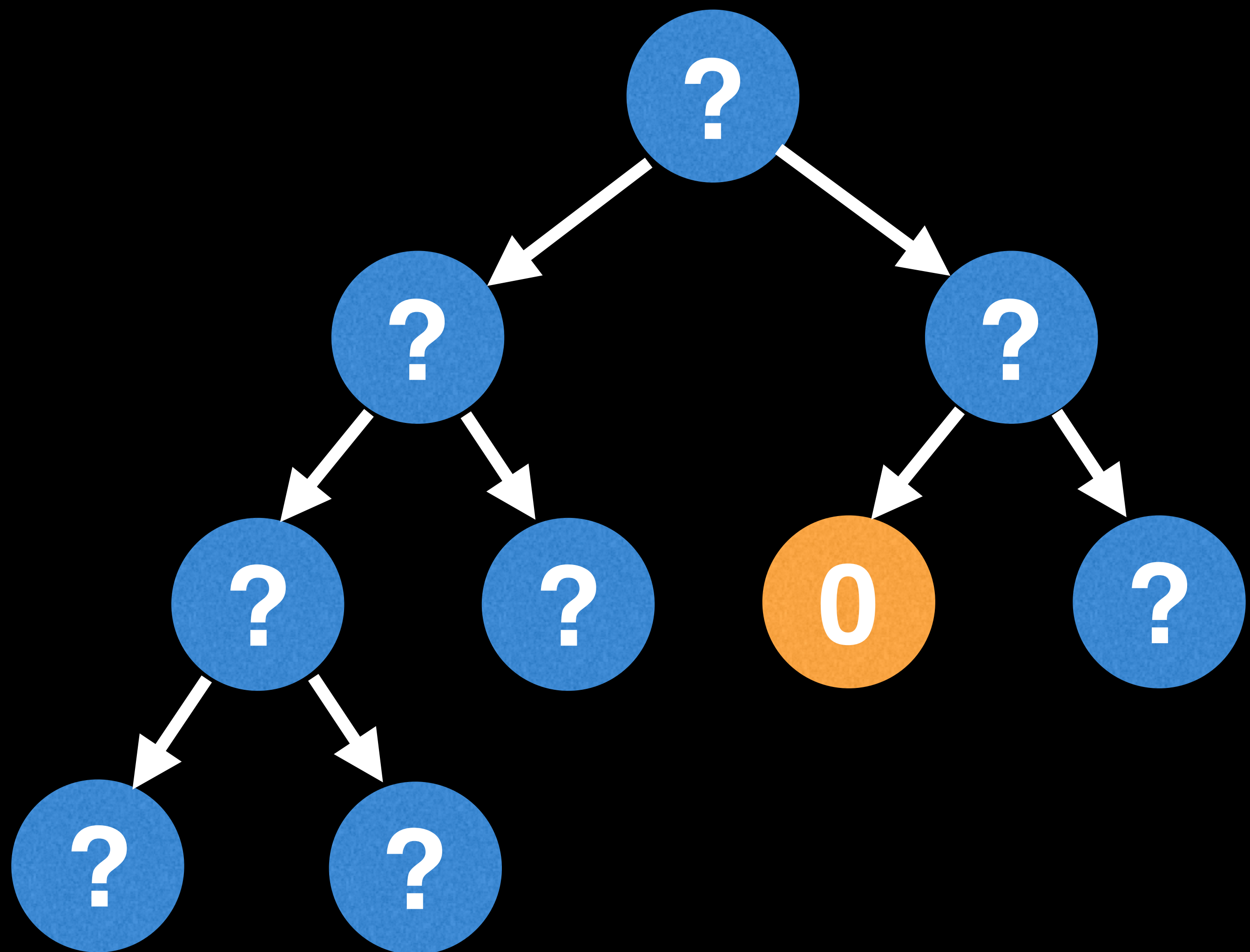
h(x) = **max**(**h**(x.left), **h**(x.right)) + 1



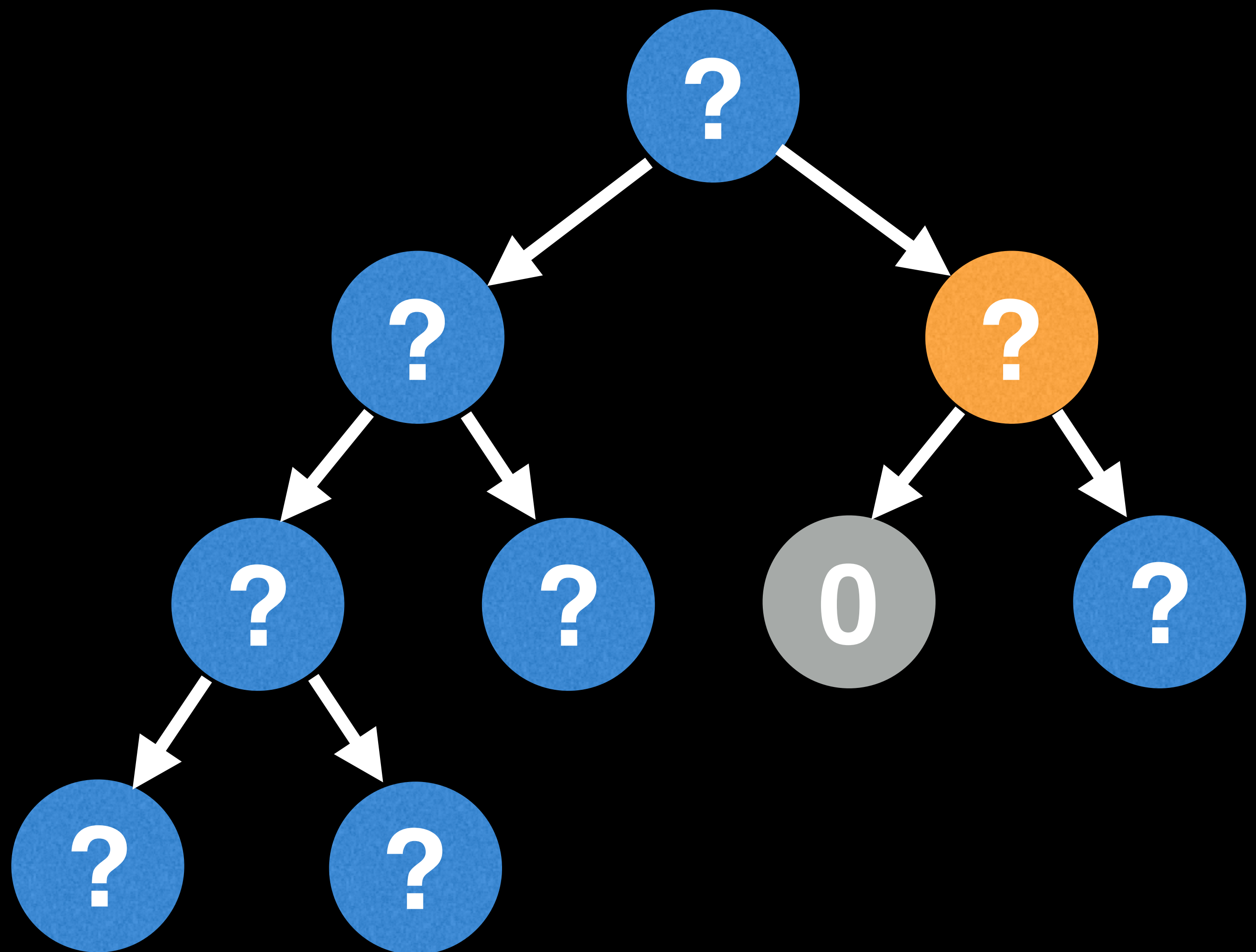








Leaf node has a height of 0


?
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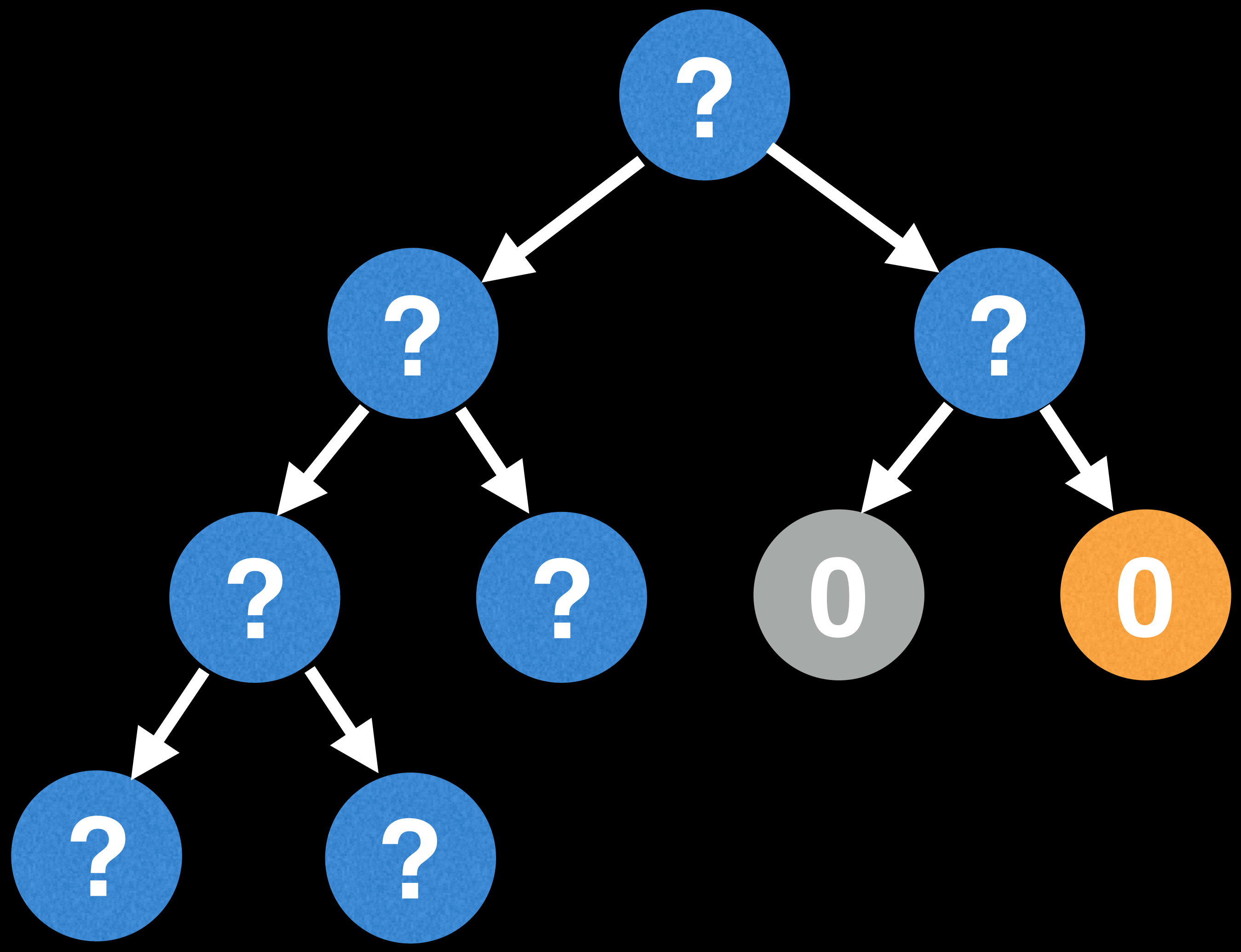
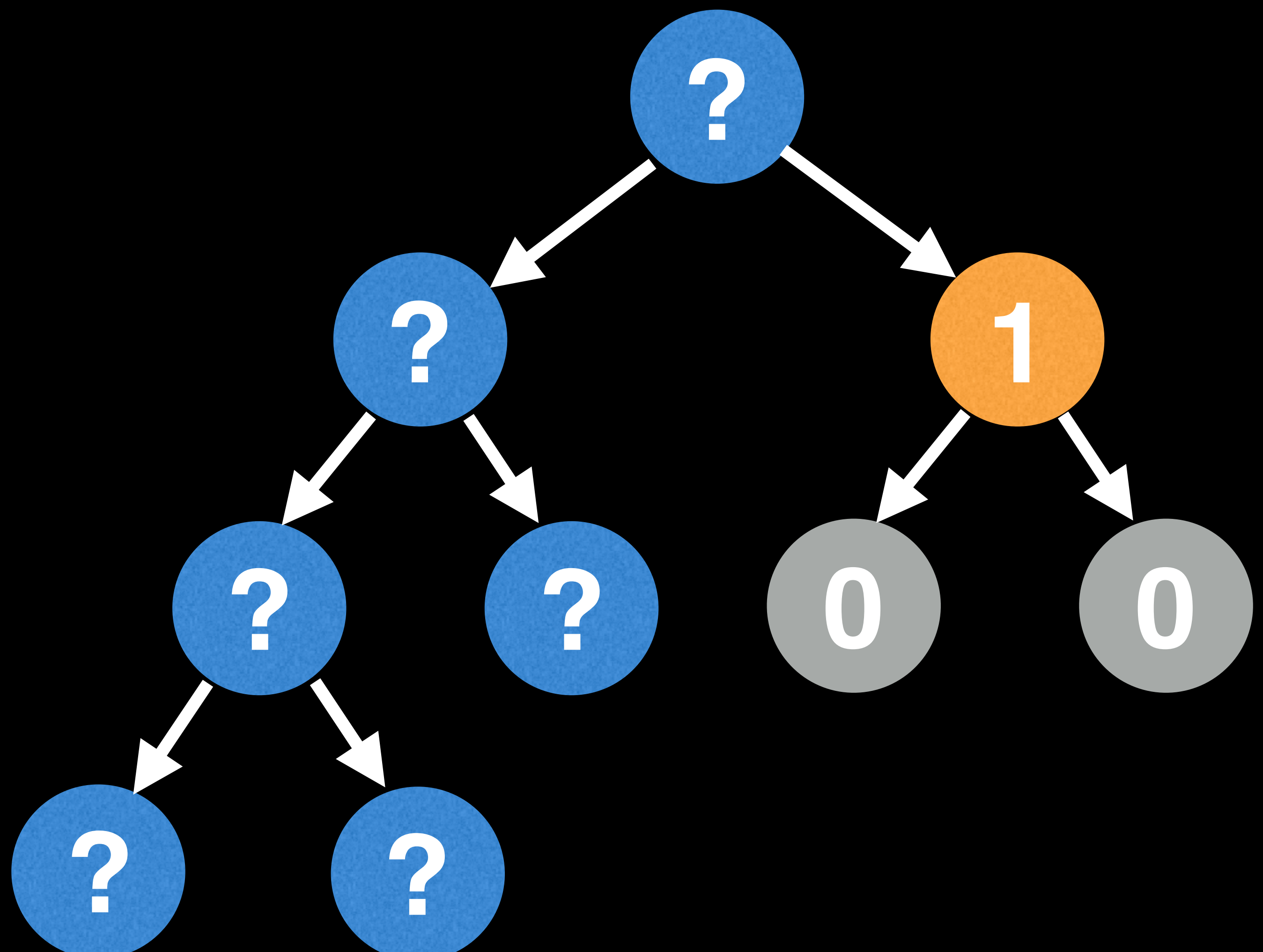
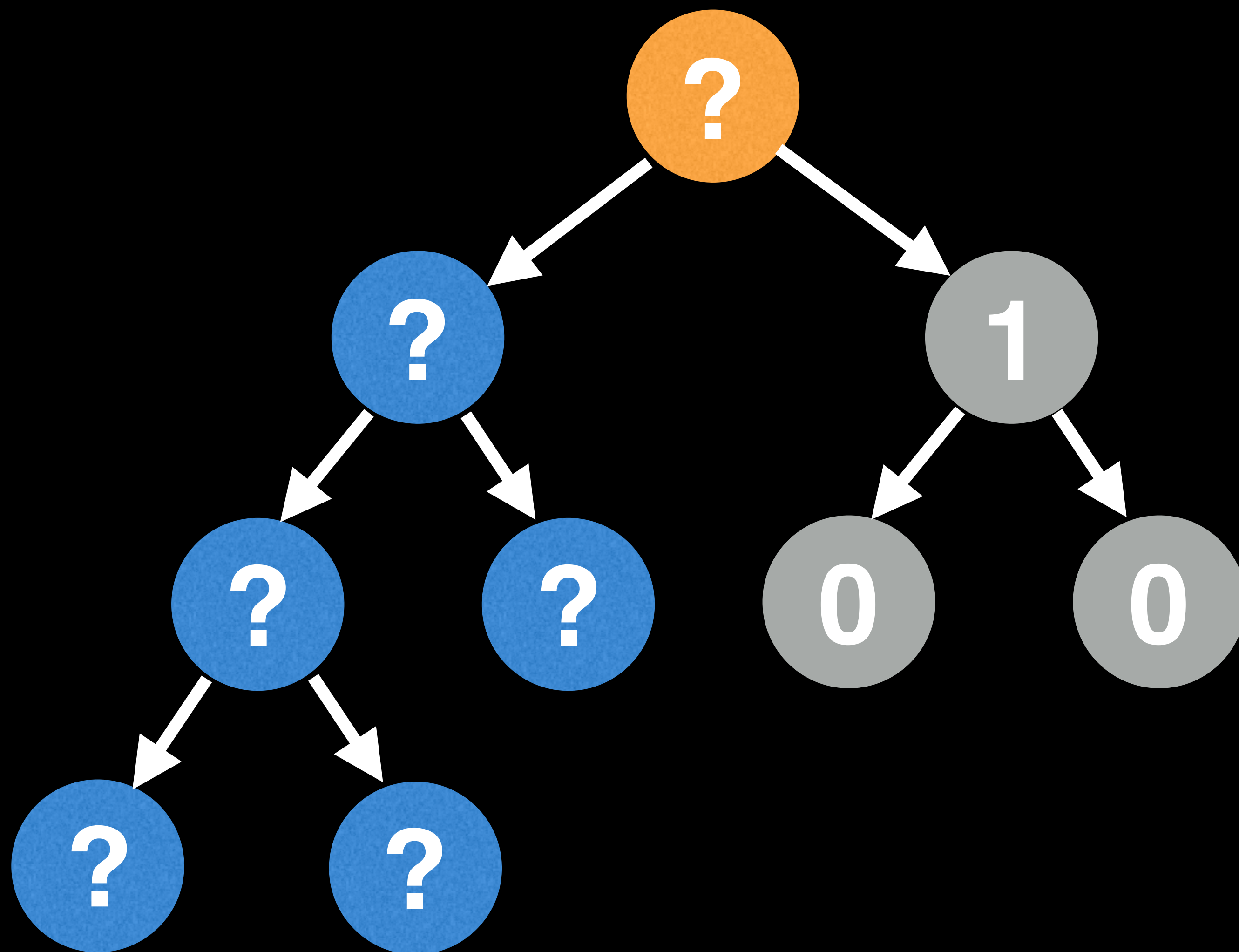


Leaf node has a height of 0




```
height = max(0, 0) + 1 = 1
```


?
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1
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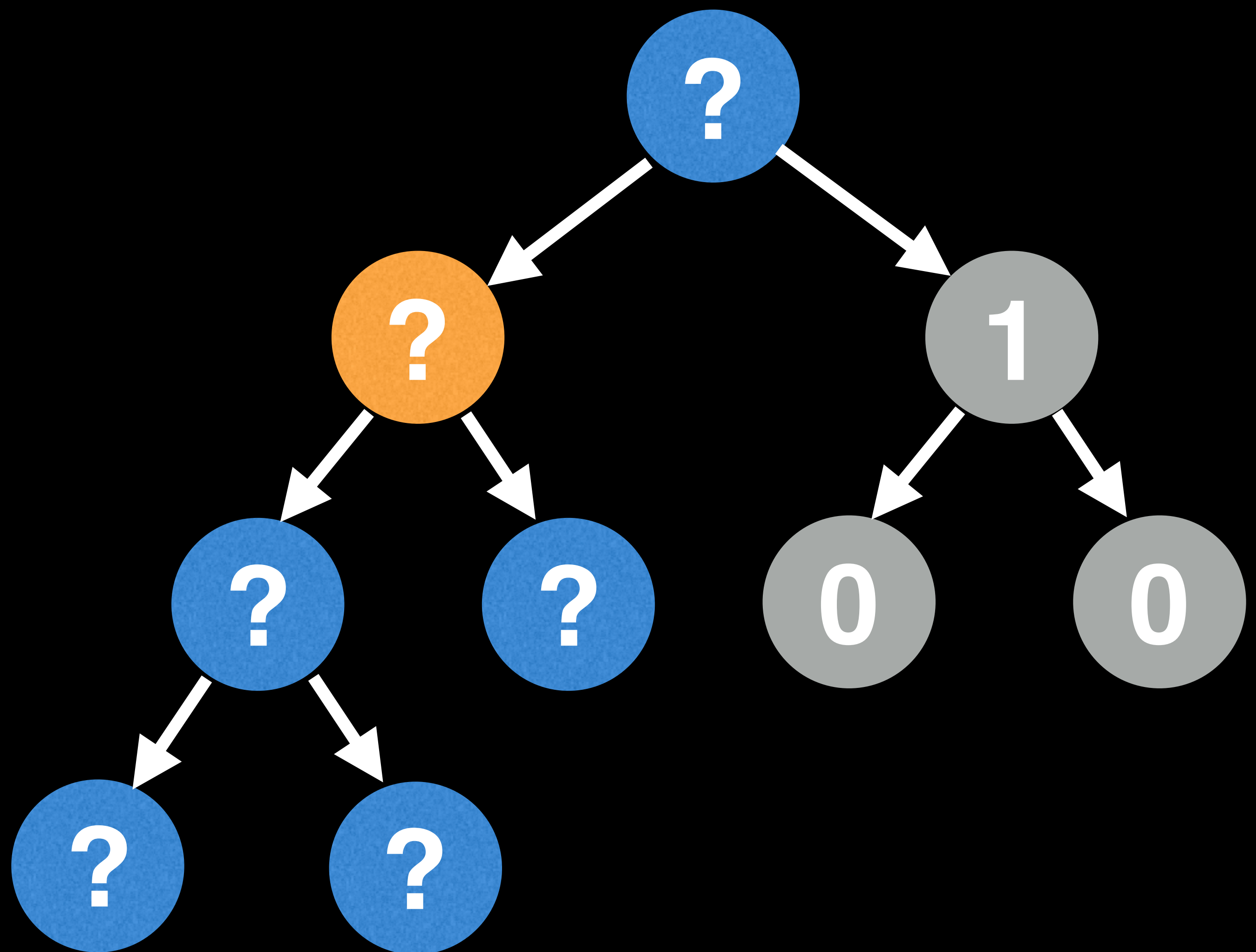
?
0
0
?
?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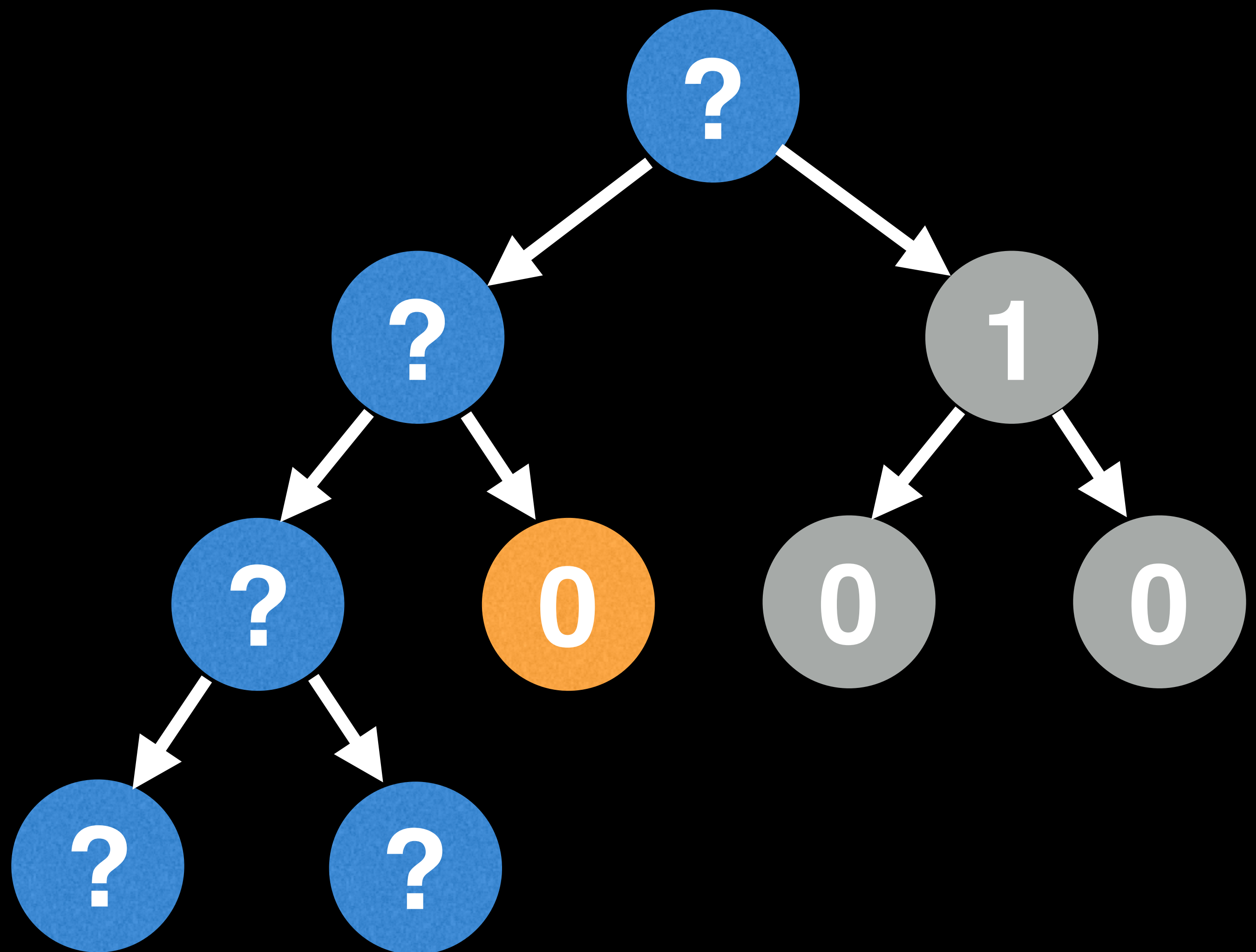
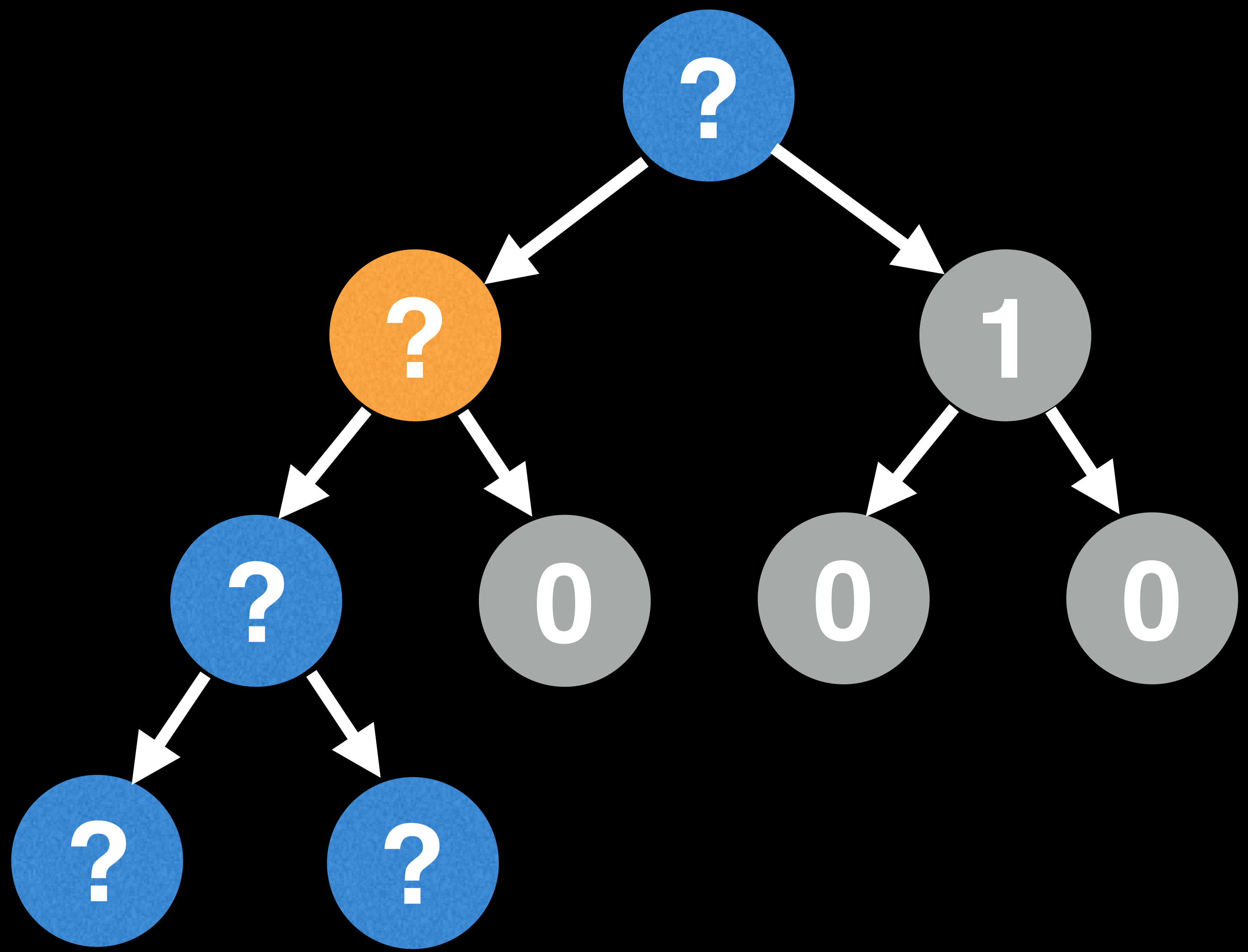


Leaf node has a height of 0


?
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1
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0
0
0
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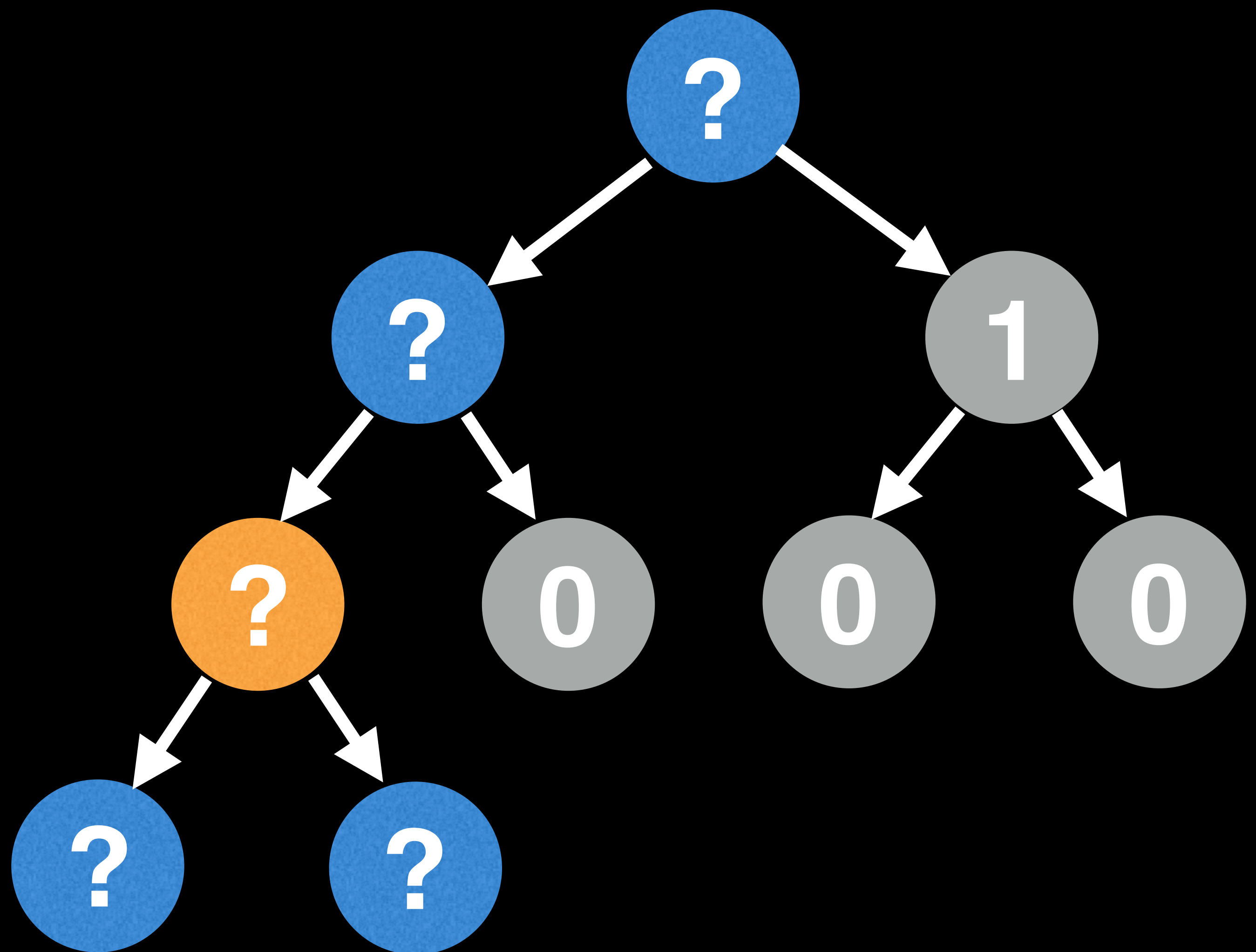
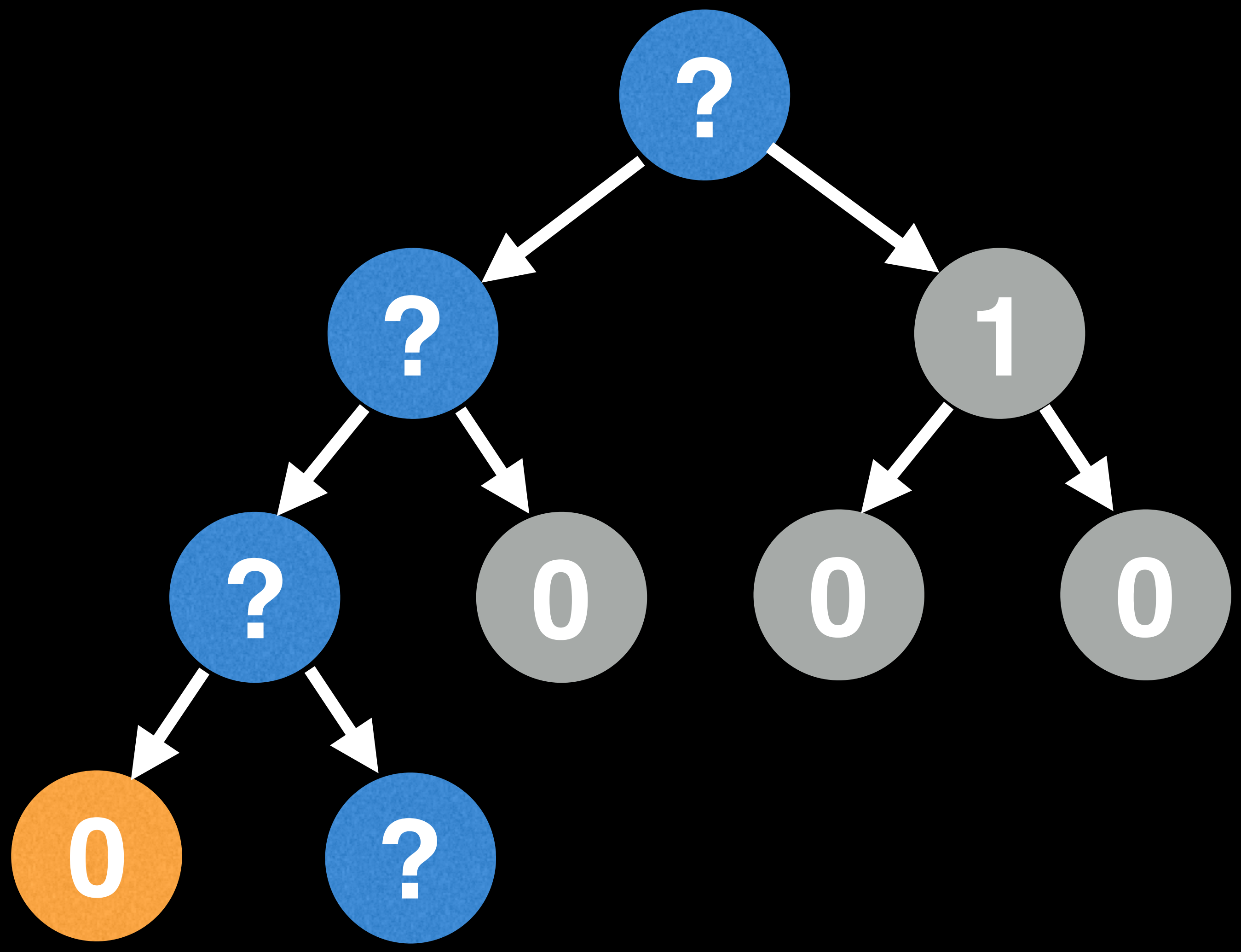
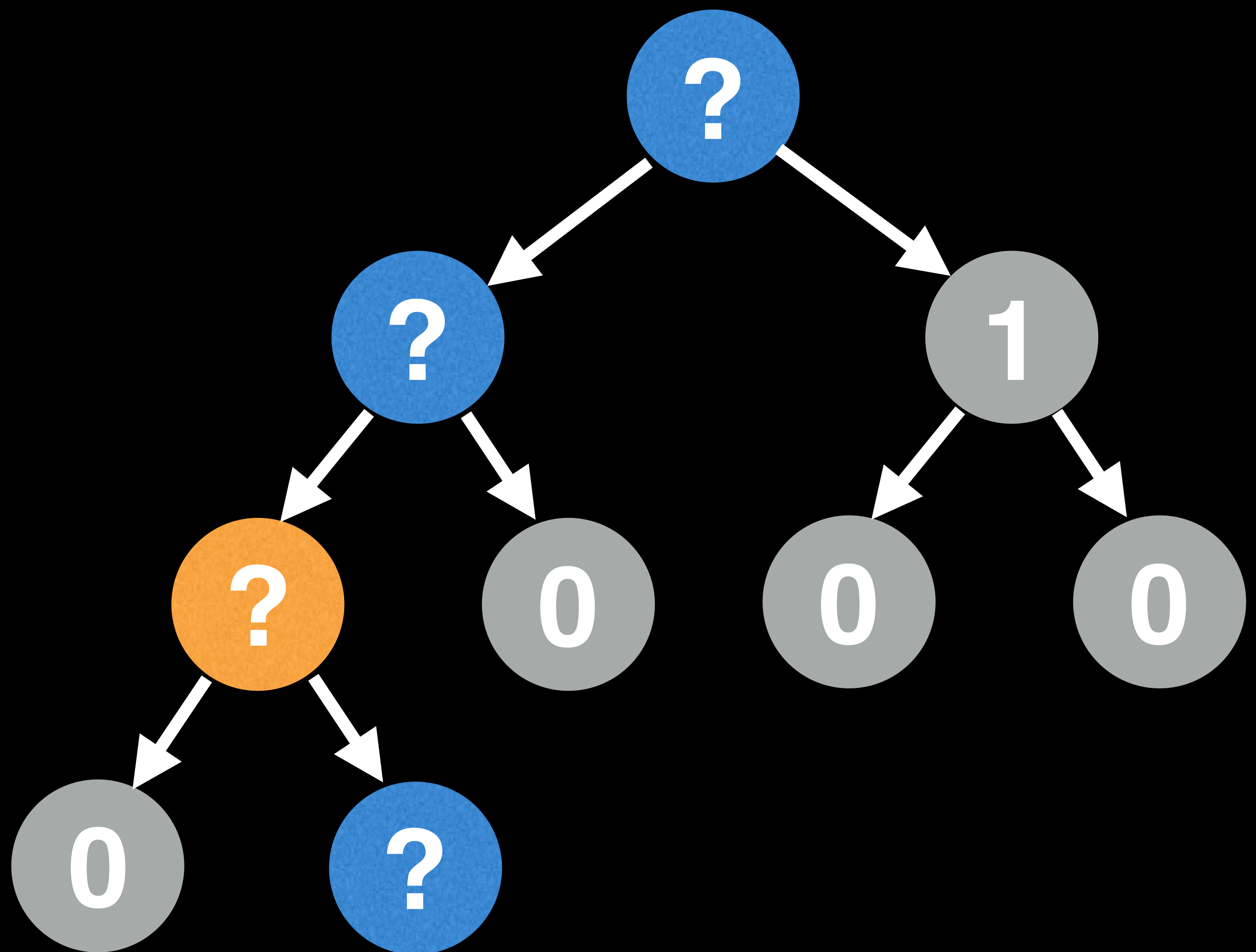
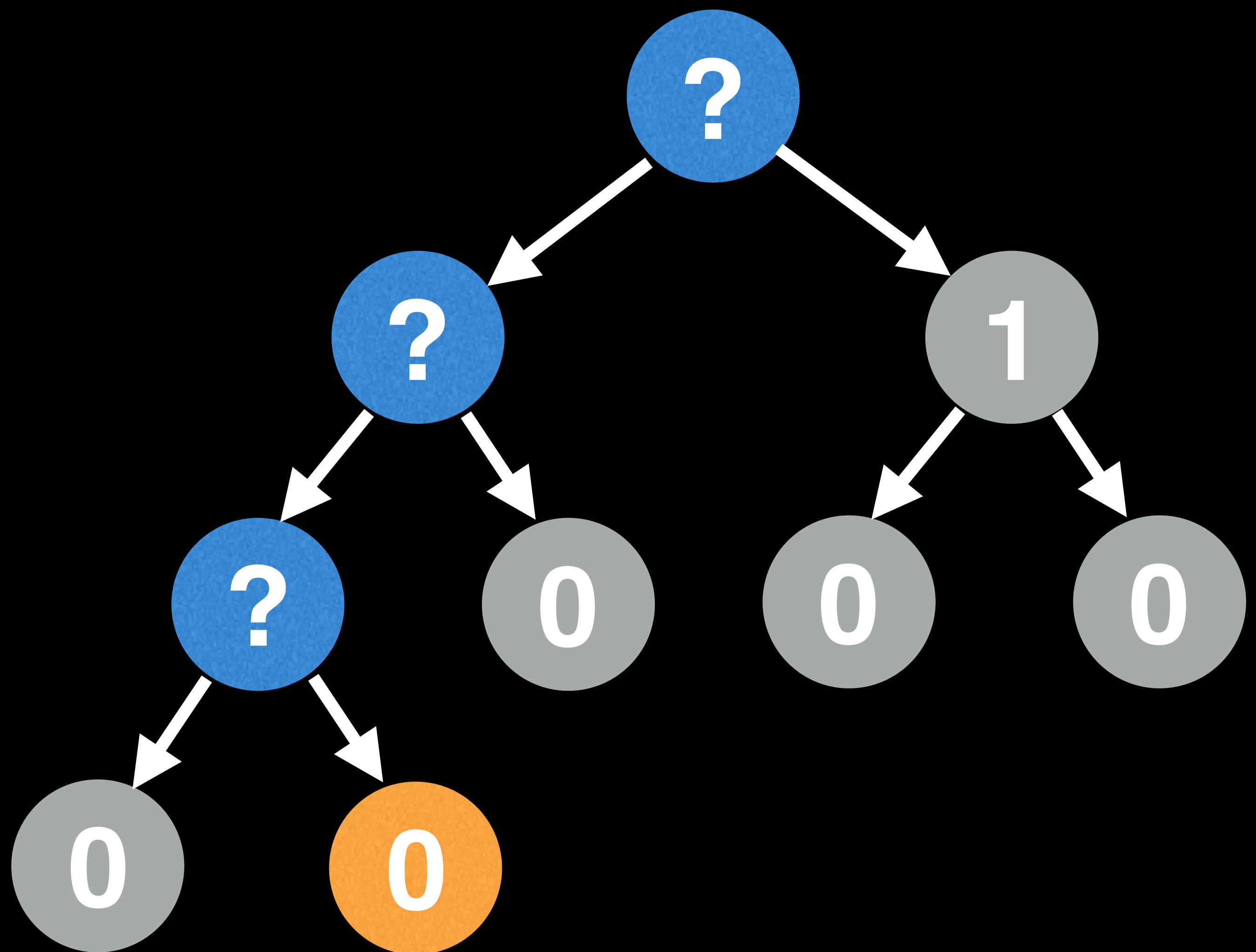
?


?
?
1
?
0
0
0
?
?

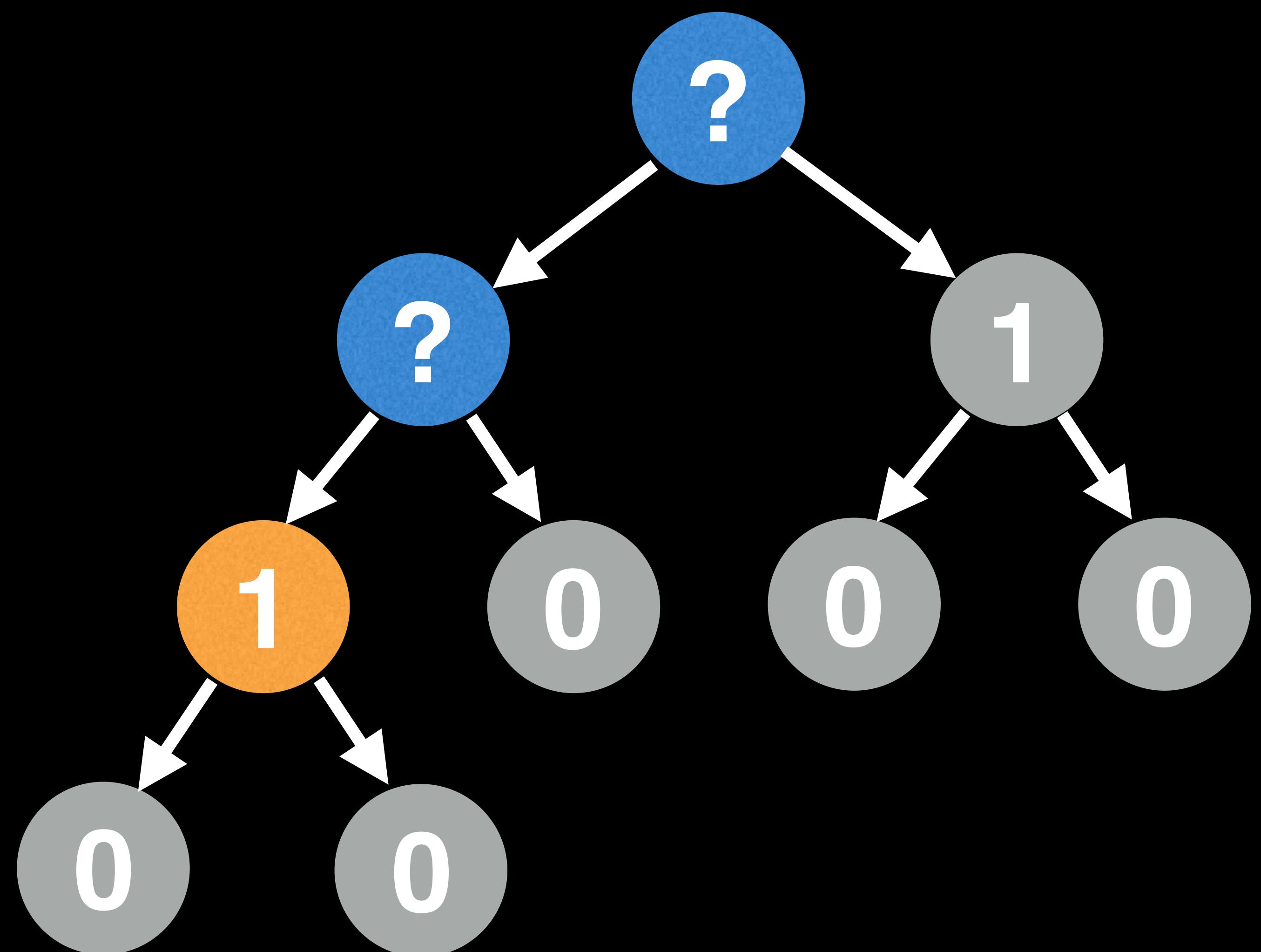


Leaf node has a height of 0

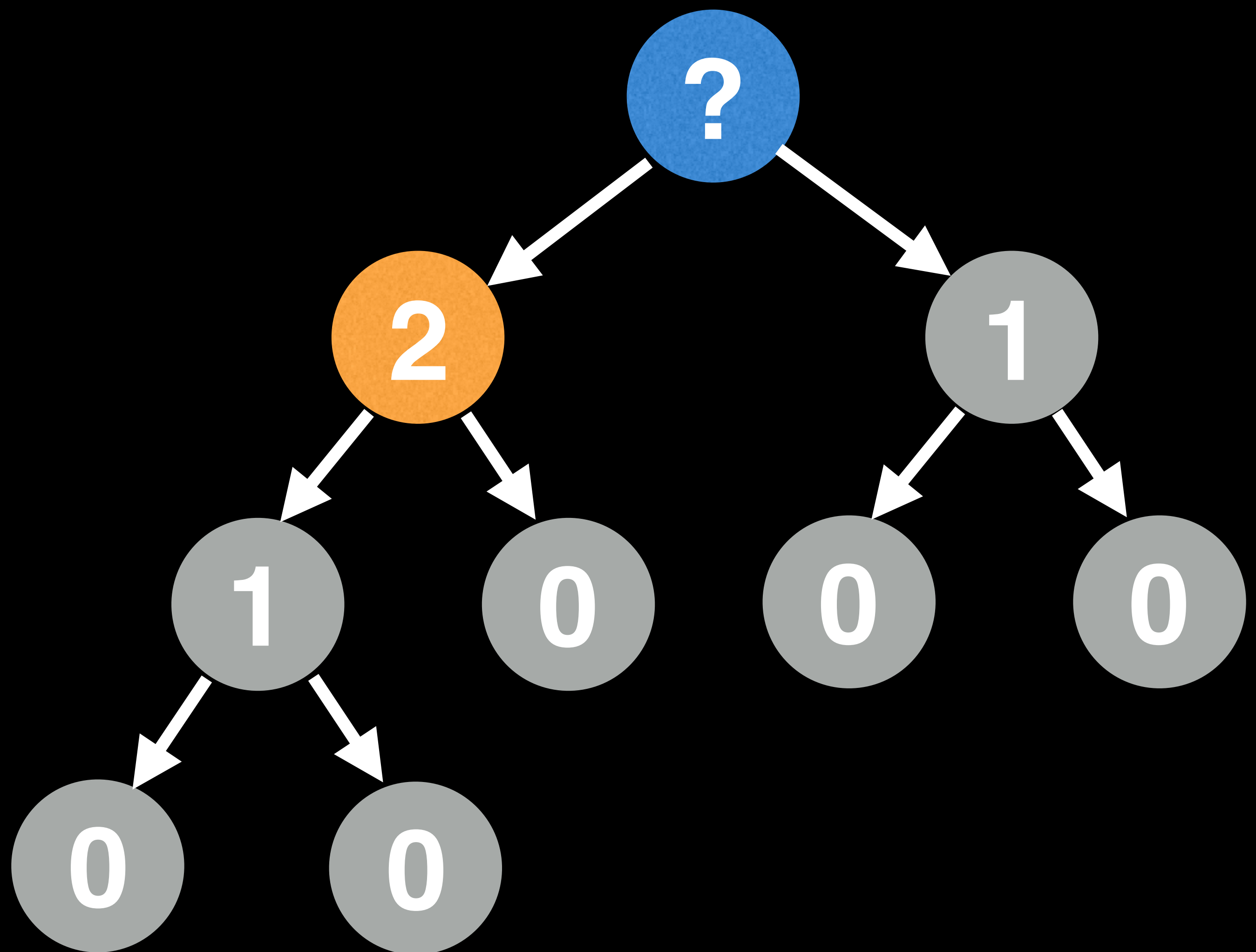

?
?
1
?
0
0
0
0
?



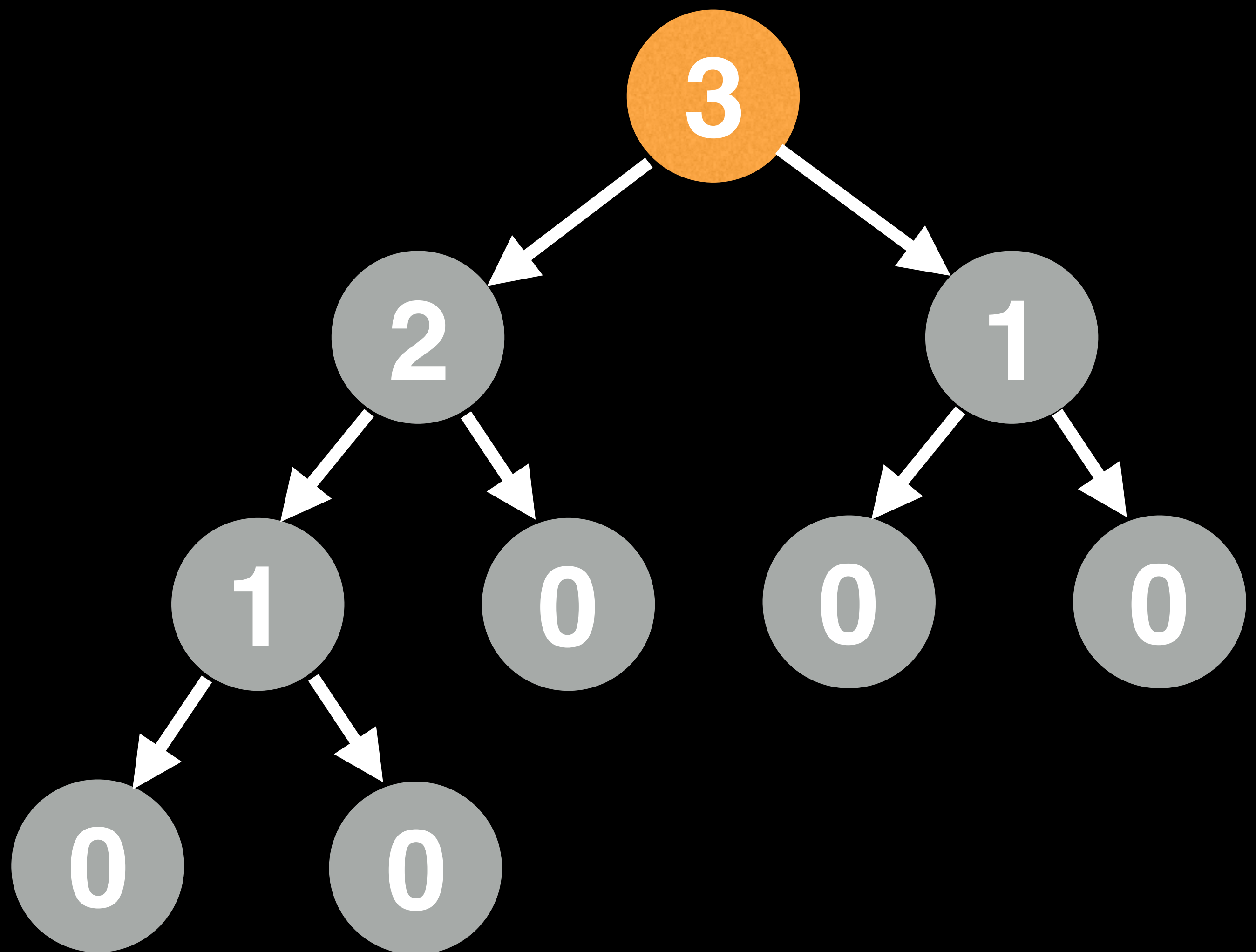
Leaf node has a height of 0



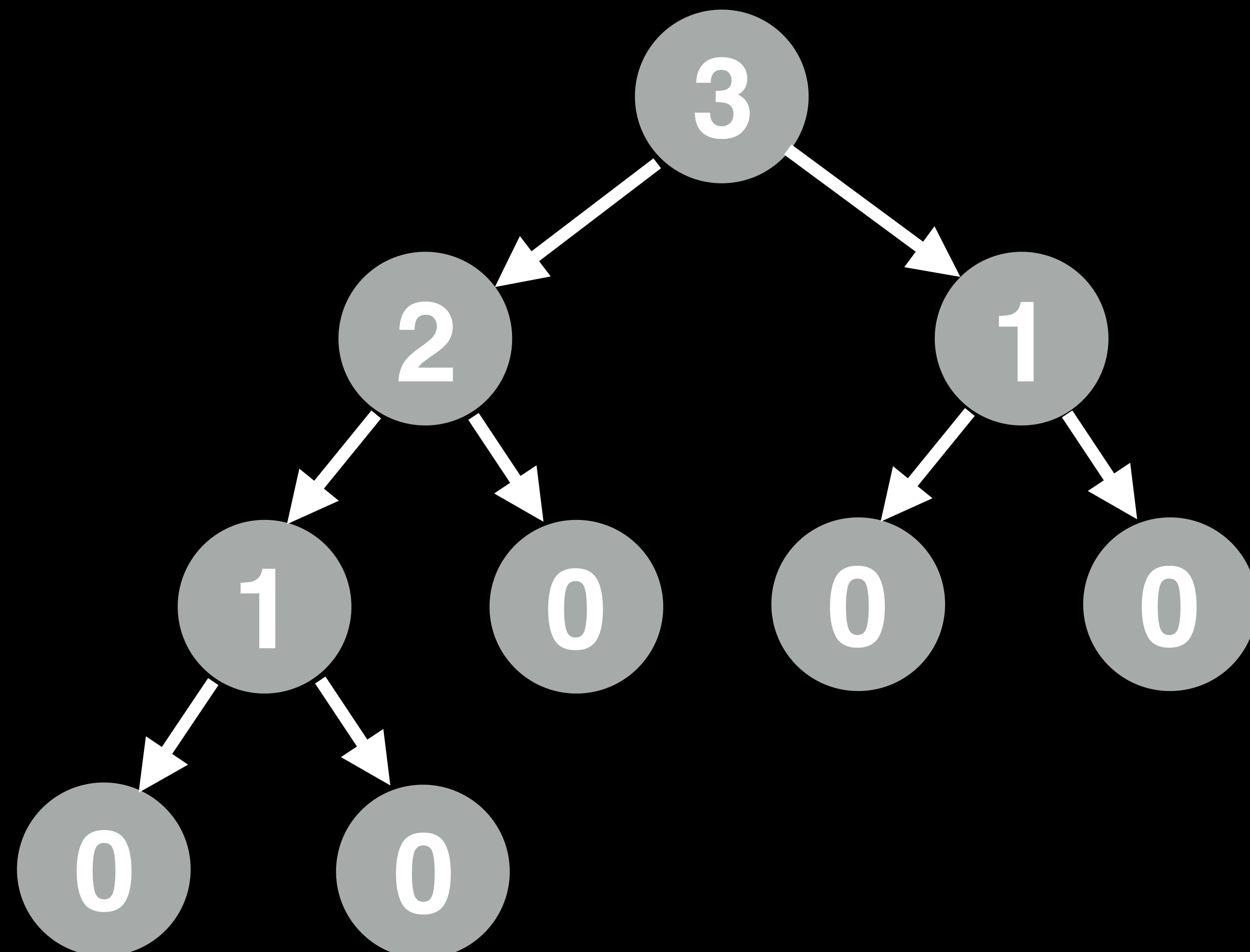
height = **max**(0, 0) + 1 = 1




```
height = max(1, 0) + 1 = 2
```



height = max(2, 1) + 1 = 3


3
2
1
1
0
0
0
0
0

```
# The height of a tree is the number of
# edges from the root to the lowest leaf.
function treeHeight(node):
  # Handle empty tree case
  if node == null:
     return -1

  # Identify leaf nodes and return zero
  if node.left == null and node.right == null:
    return 0

  return max(treeHeight(node.left),
             treeHeight(node.right)) + 1
```

```
# The height of a tree is the number of
# edges from the root to the lowest leaf.
function treeHeight(node):
  # Handle empty tree case
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     return -1

  # Identify leaf nodes and return zero
  if node.left == null and node.right == null:
    return 0

  return max(treeHeight(node.left),
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function treeHeight(node):
  # Handle empty tree case
  if node == null:
     return -1

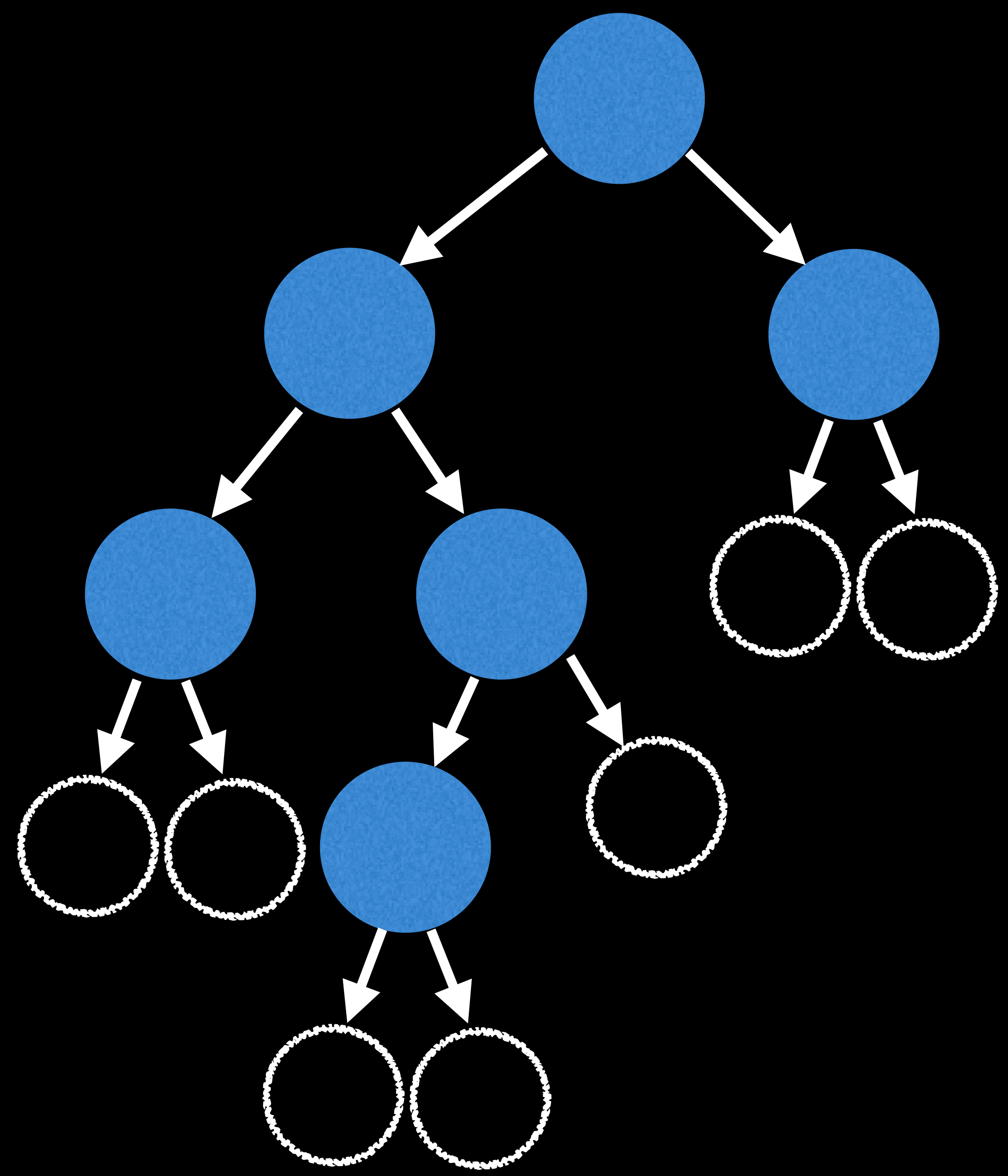
  # Identify leaf nodes and return zero
  if node.left == null and node.right == null:
    return 0

  return max(treeHeight(node.left),
             treeHeight(node.right)) + 1
```

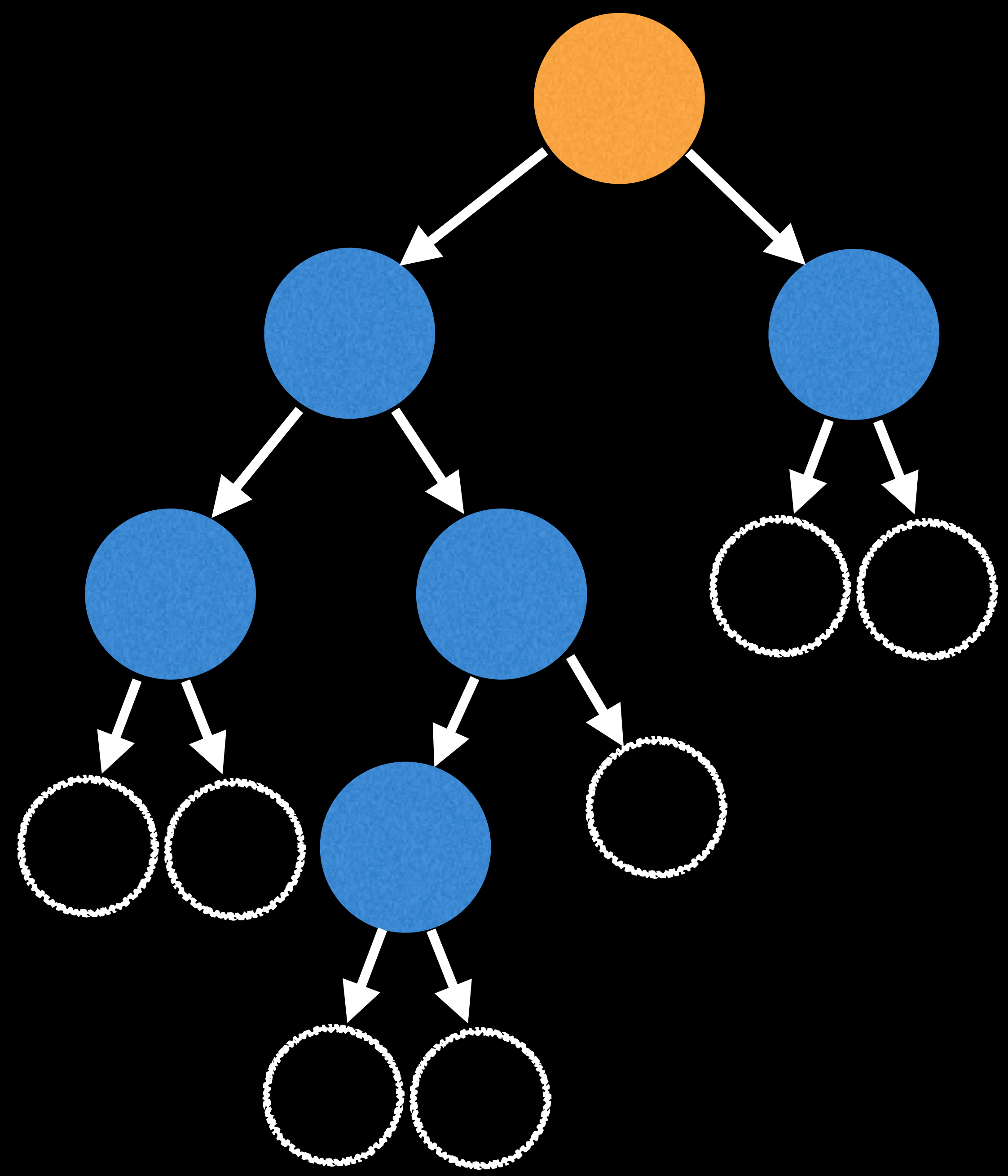
```
# The height of a tree is the number of
# edges from the root to the lowest leaf.
function treeHeight(node):
  # Return -1 when we hit a null node
  # to correct for the right height.
  if node == null:
     return -1

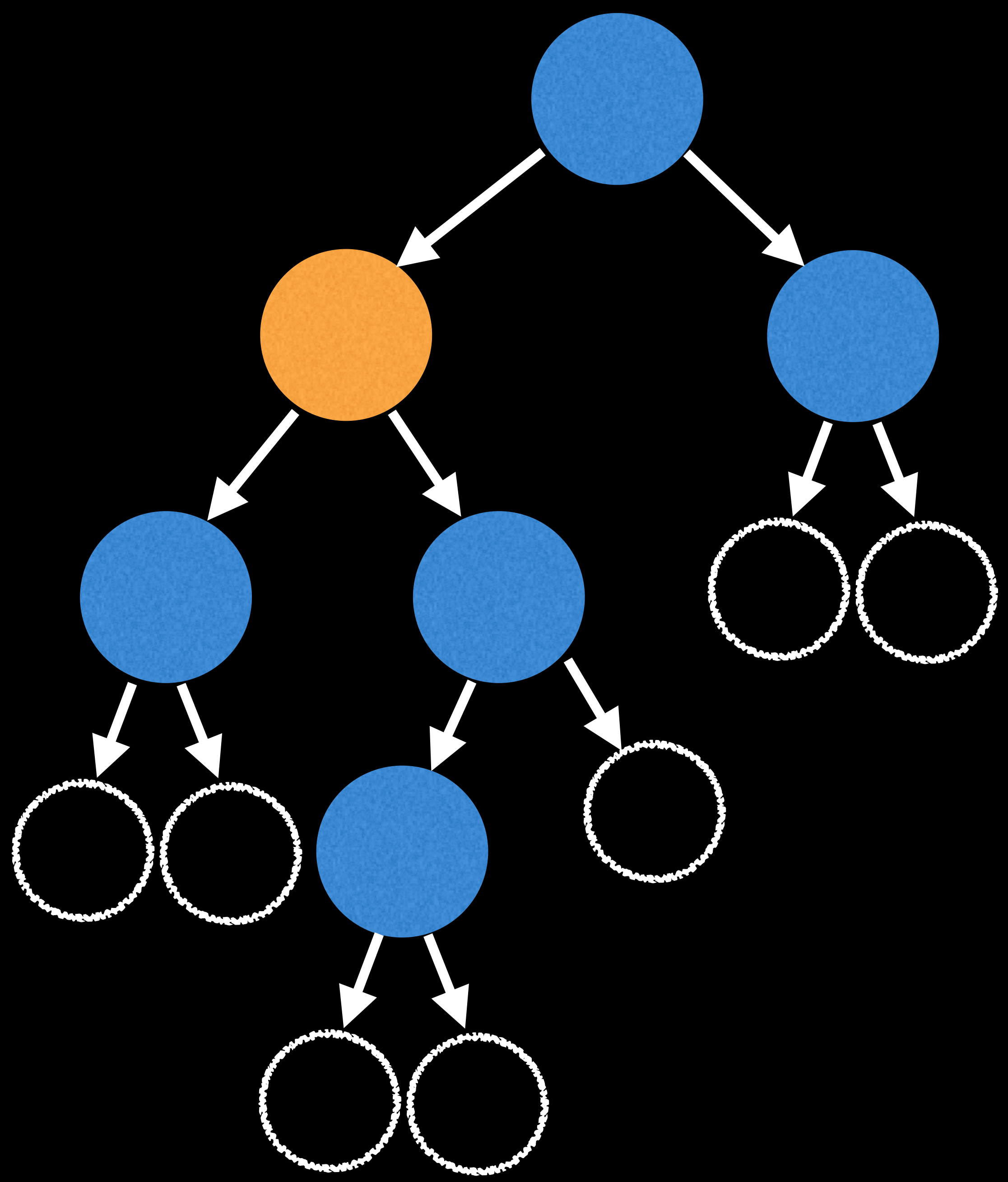
  return max(treeHeight(node.left),
             treeHeight(node.right)) + 1
```

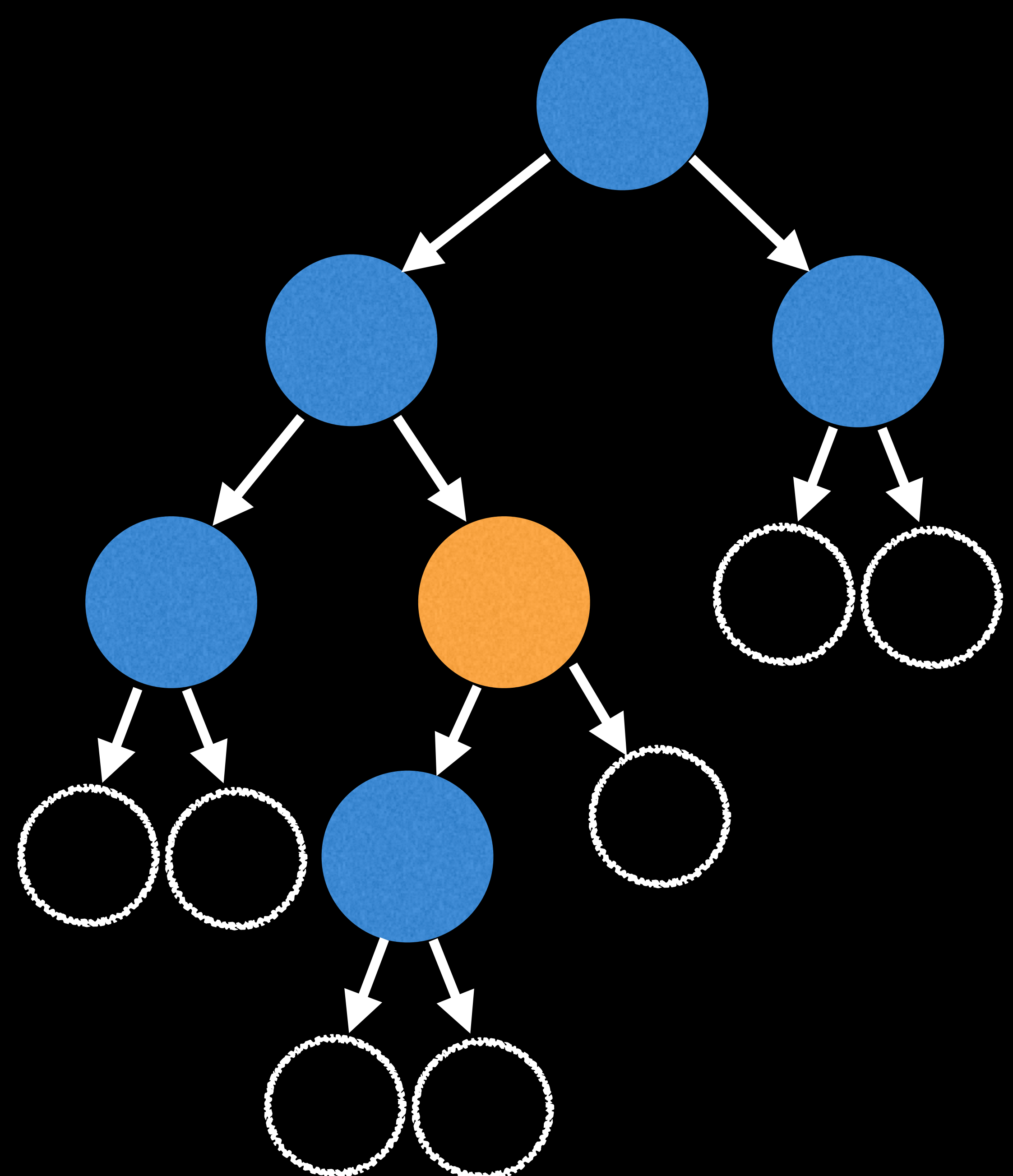
Notice that if we visit the null nodes our tree is one unit taller.

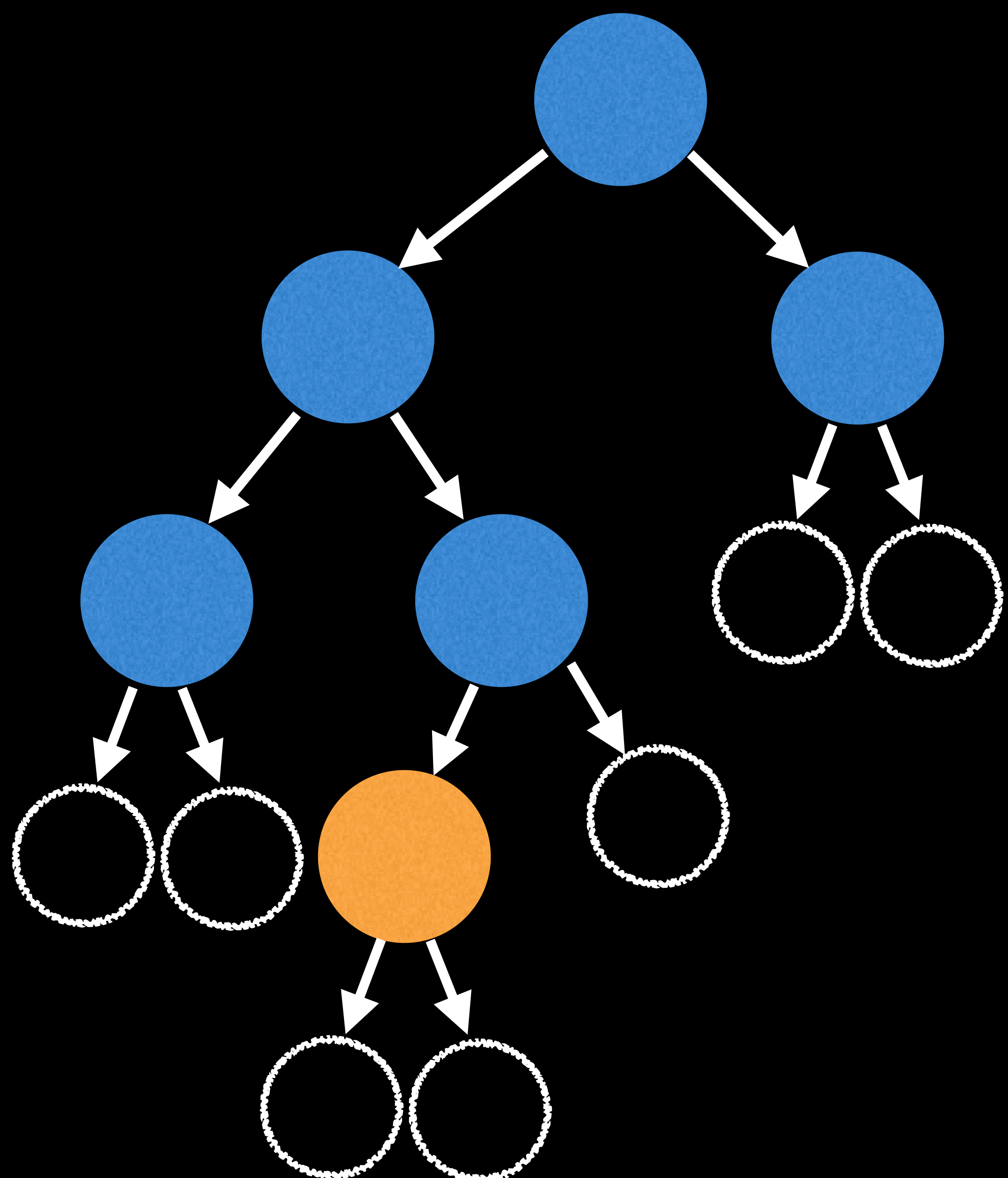


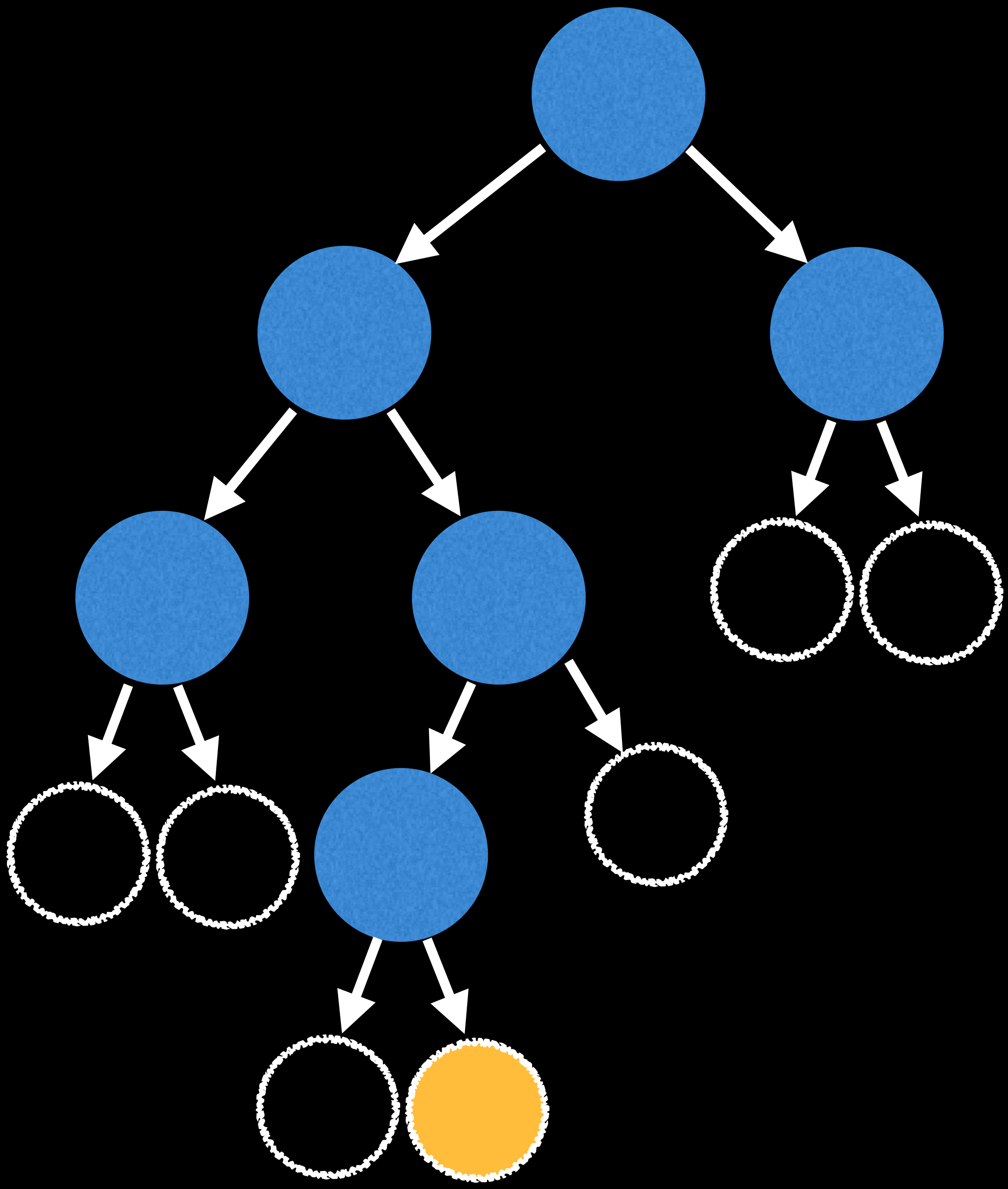
When we go down the tree we need to correct for the height added by the null nodes.

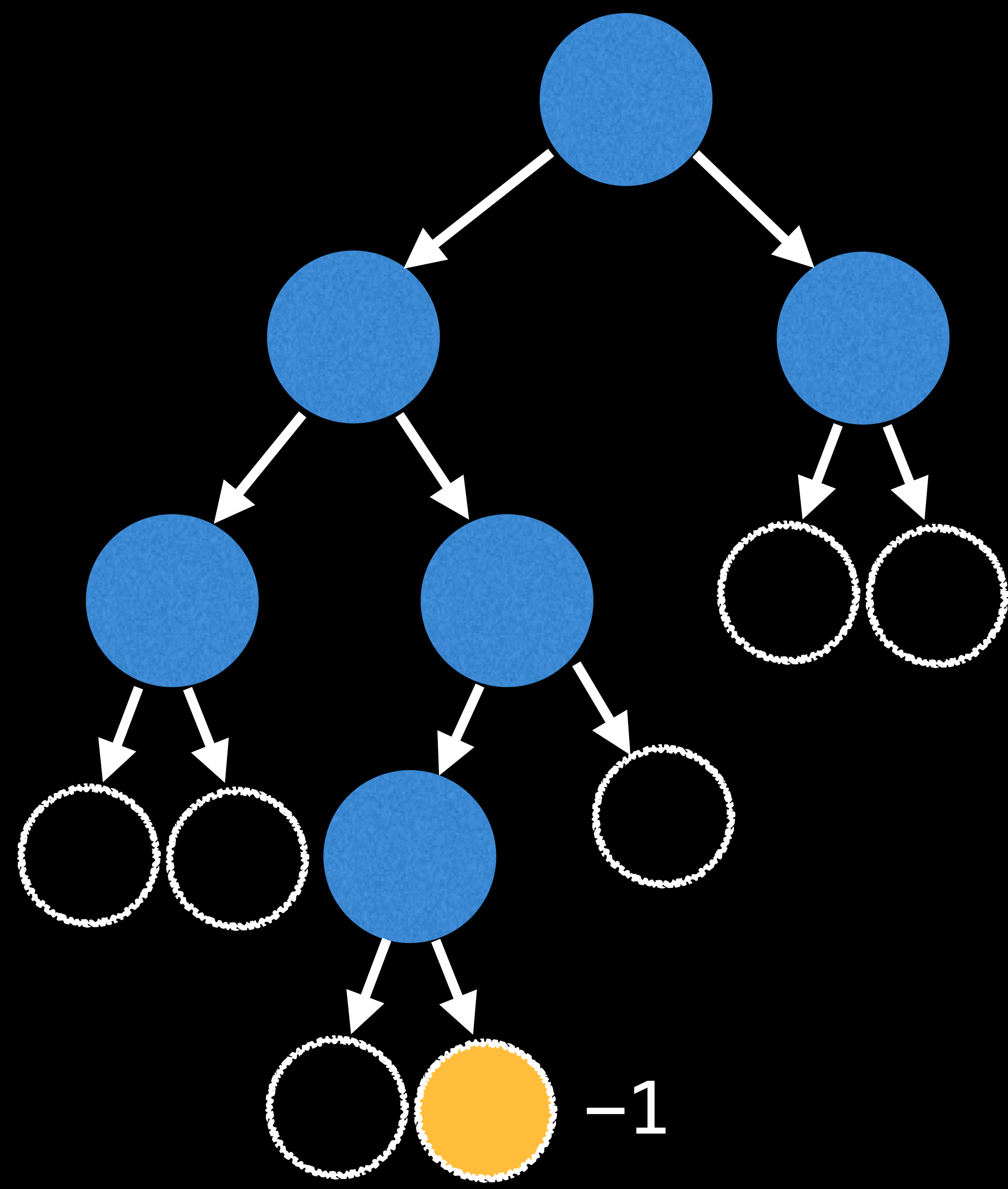


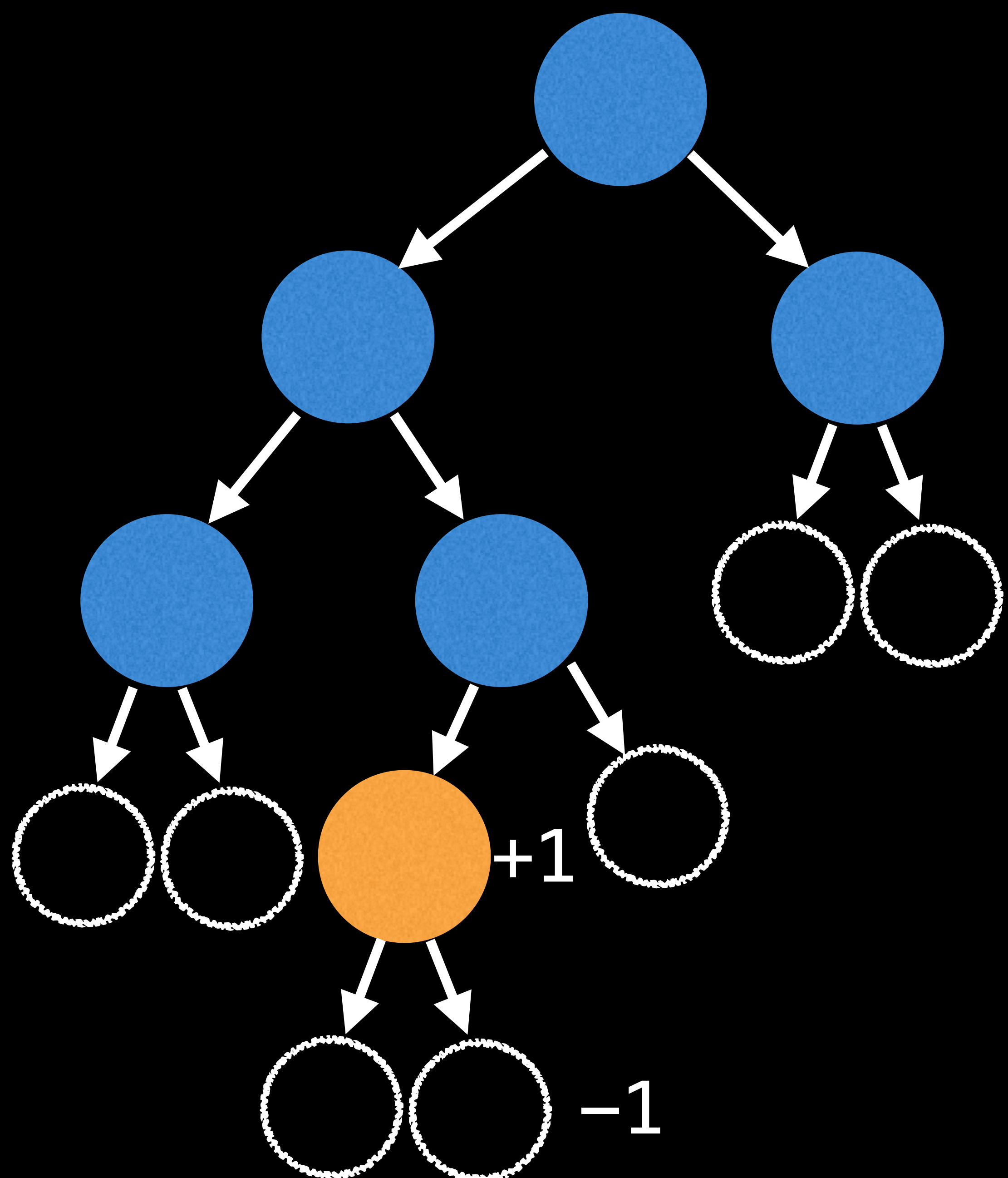




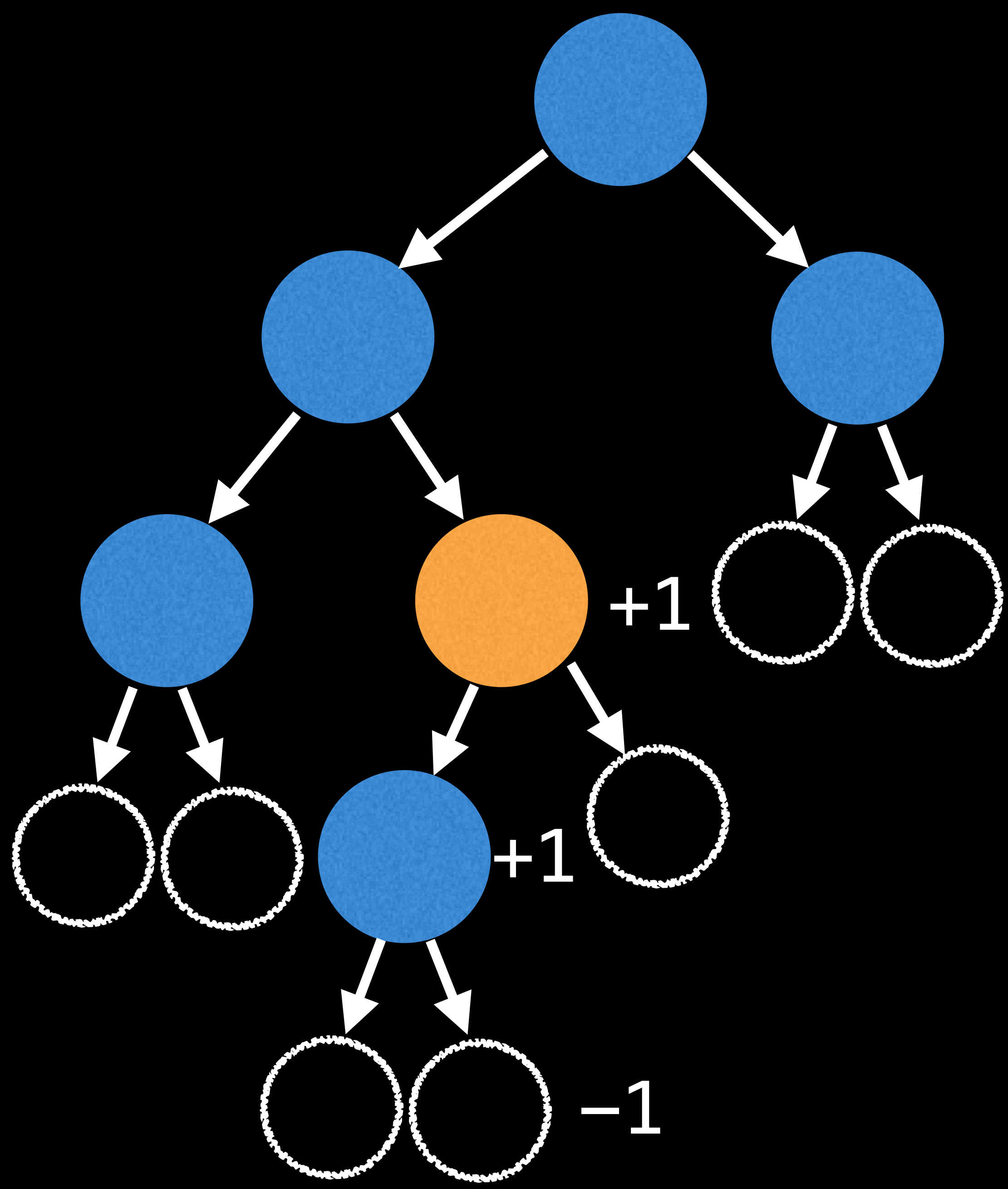





−1



+1
−1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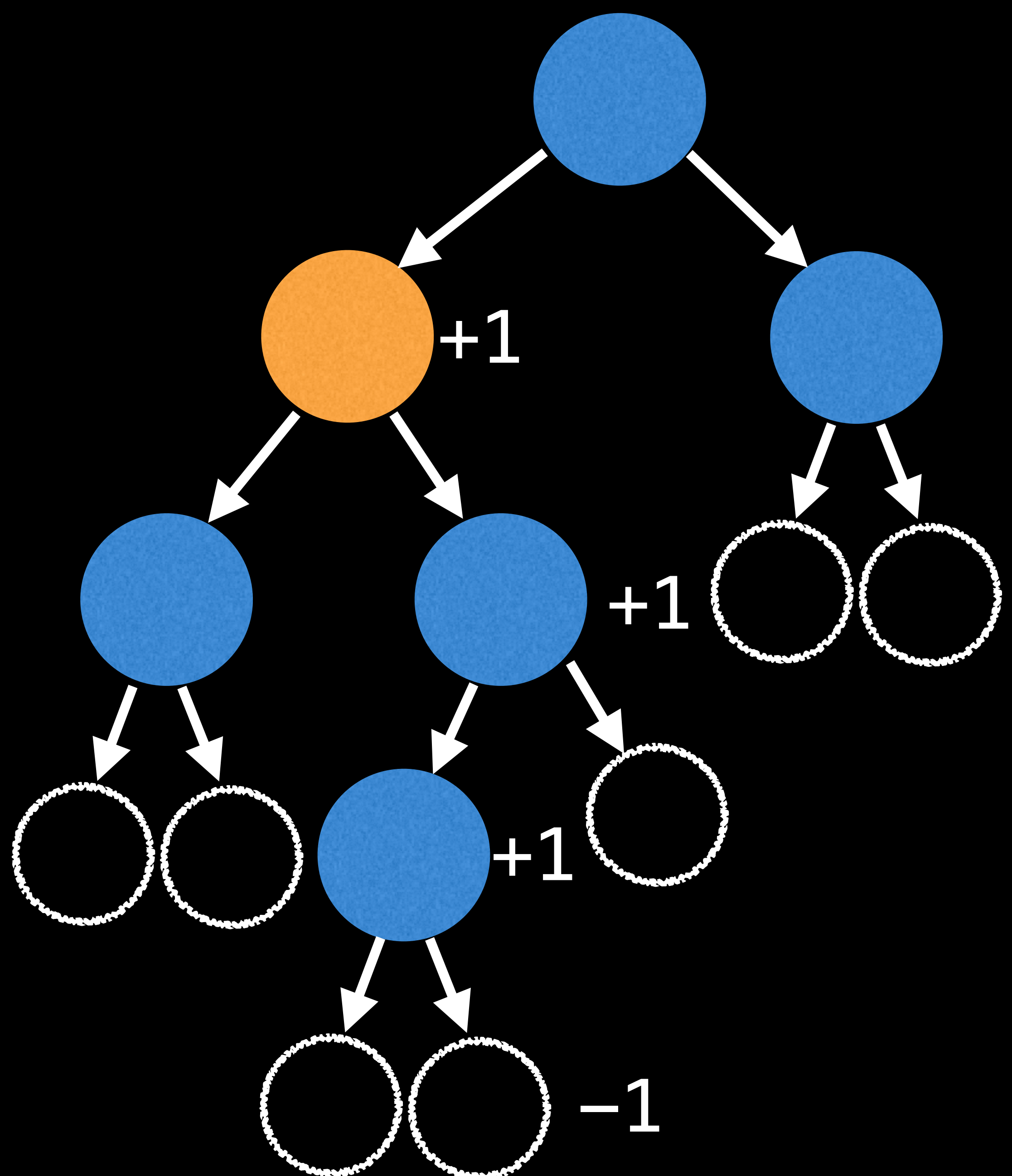
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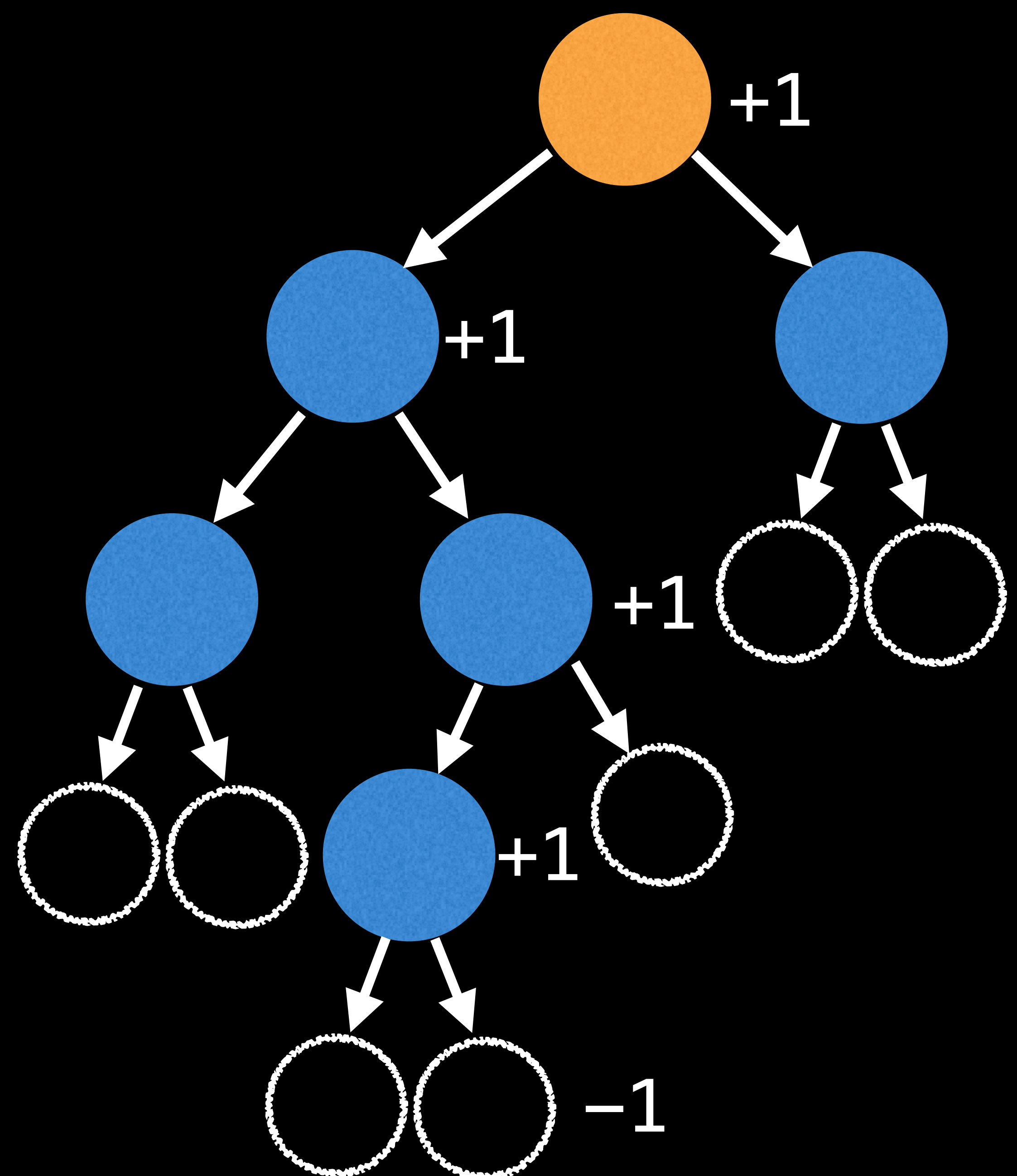
+1
+1
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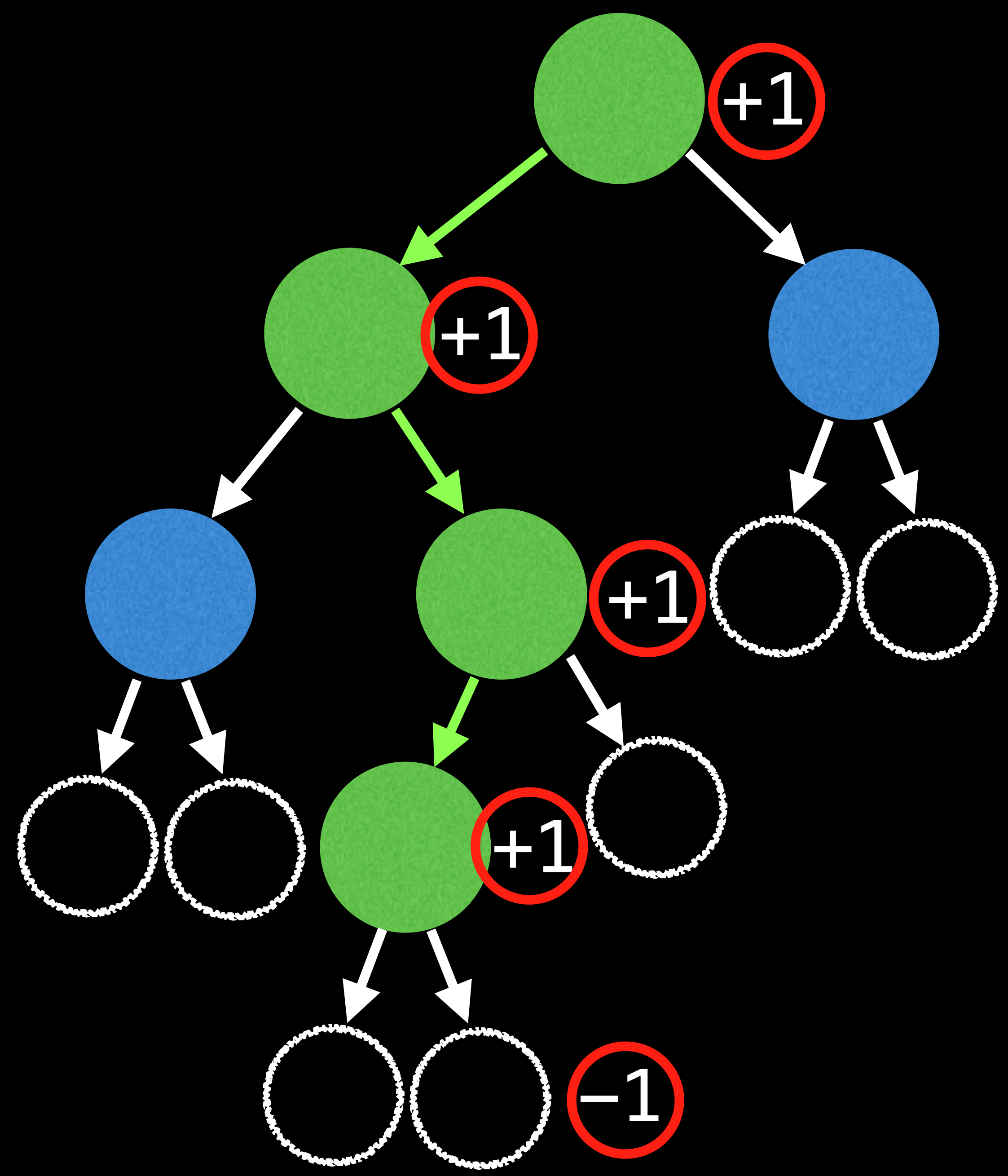
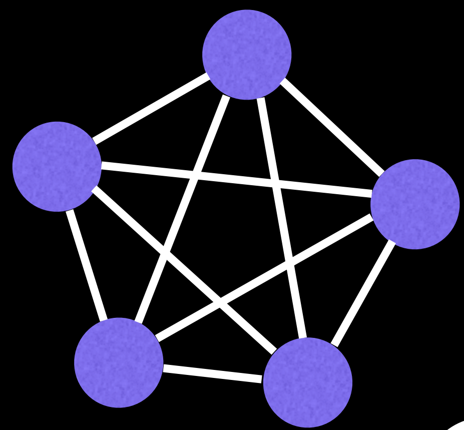


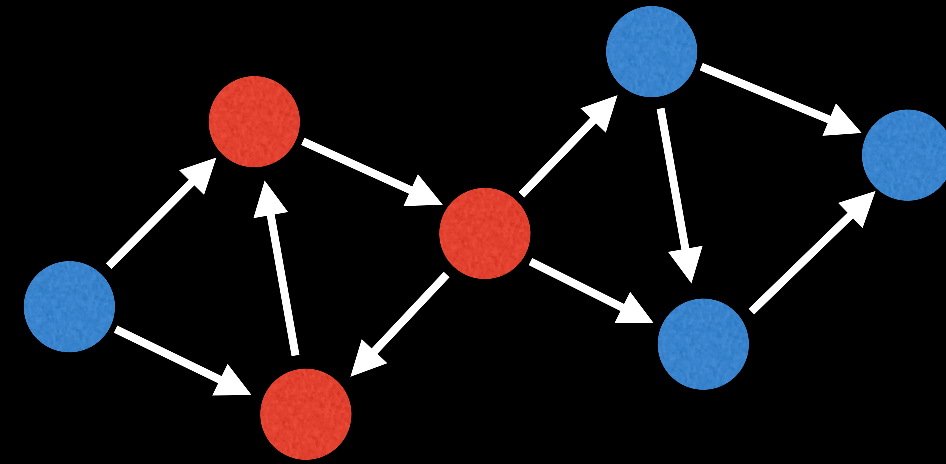
$$1 + 1 + 1 + 1 - 1 = 3$$

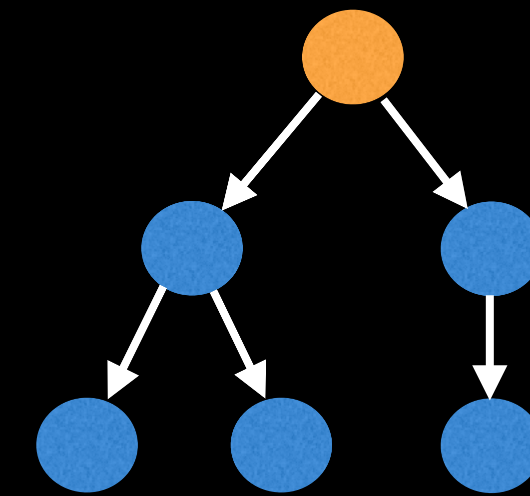
```
# The height of a tree is the number of
# edges from the root to the lowest leaf.
function treeHeight(node):
  # Return -1 when we hit a null node
  # to correct for the right height.
  if node == null:
     return -1

  return max(treeHeight(node.left),
             treeHeight(node.right)) + 1
```

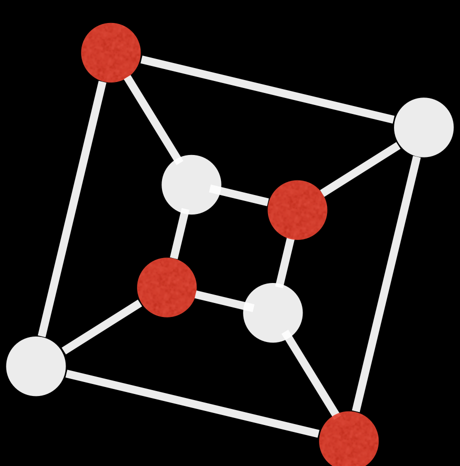
Next Video: rooting a tree

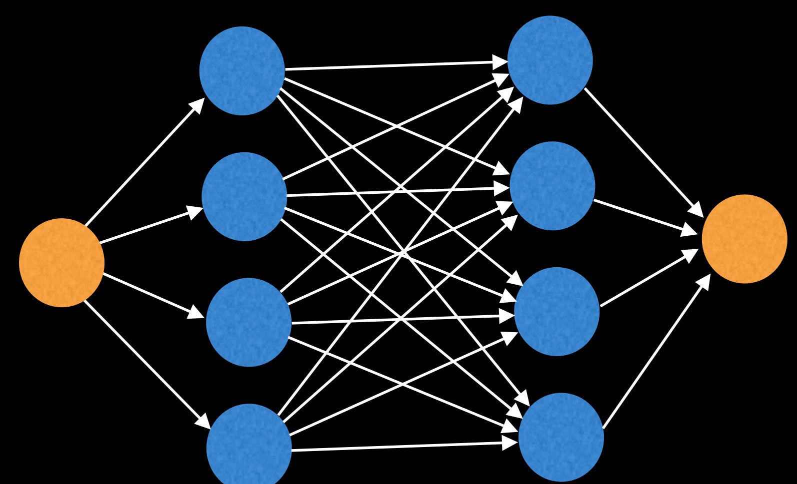


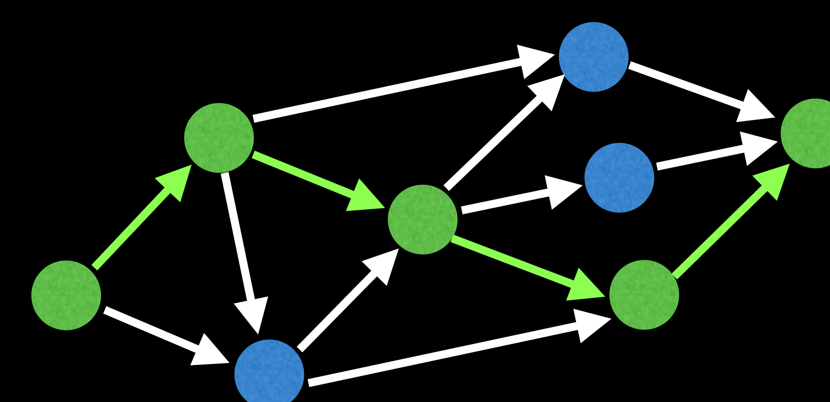




Graph Theory Video Series





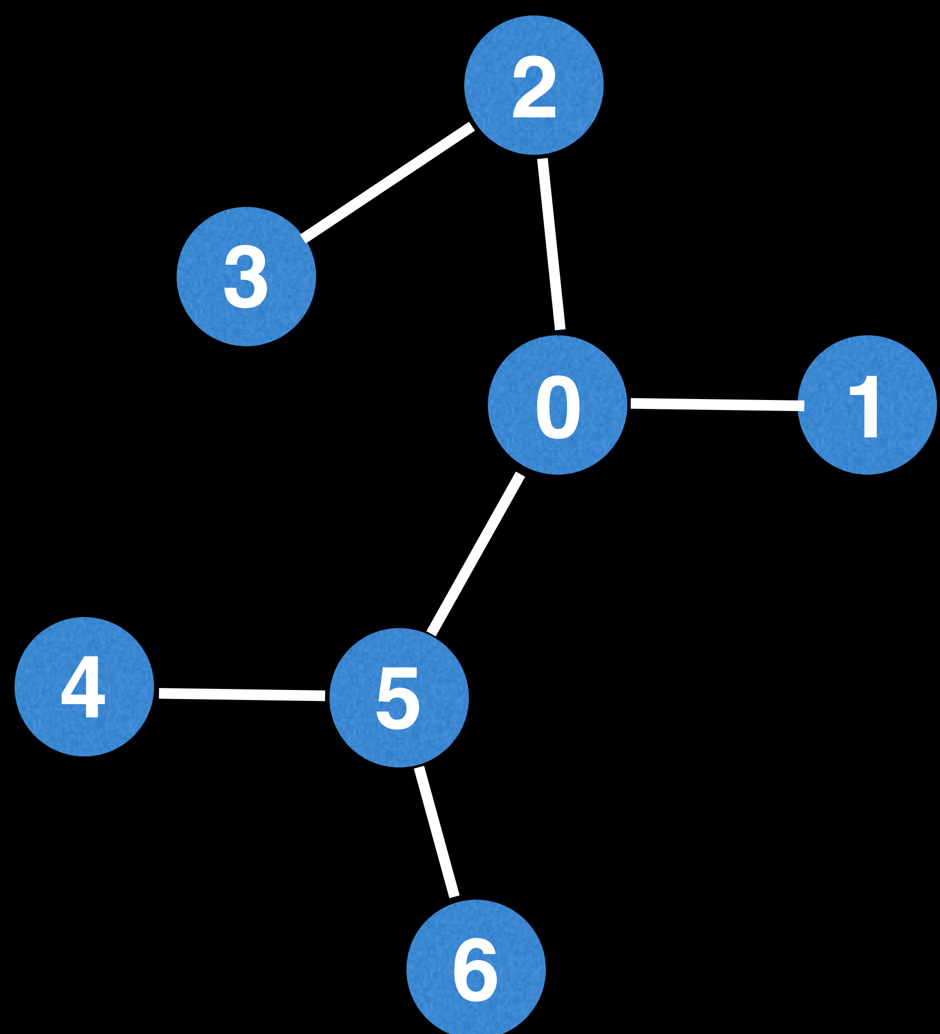


Rooting a tree

William Fiset

Rooting a tree

Sometimes it's useful to root an undirected tree to add structure to the problem you're trying to solve.

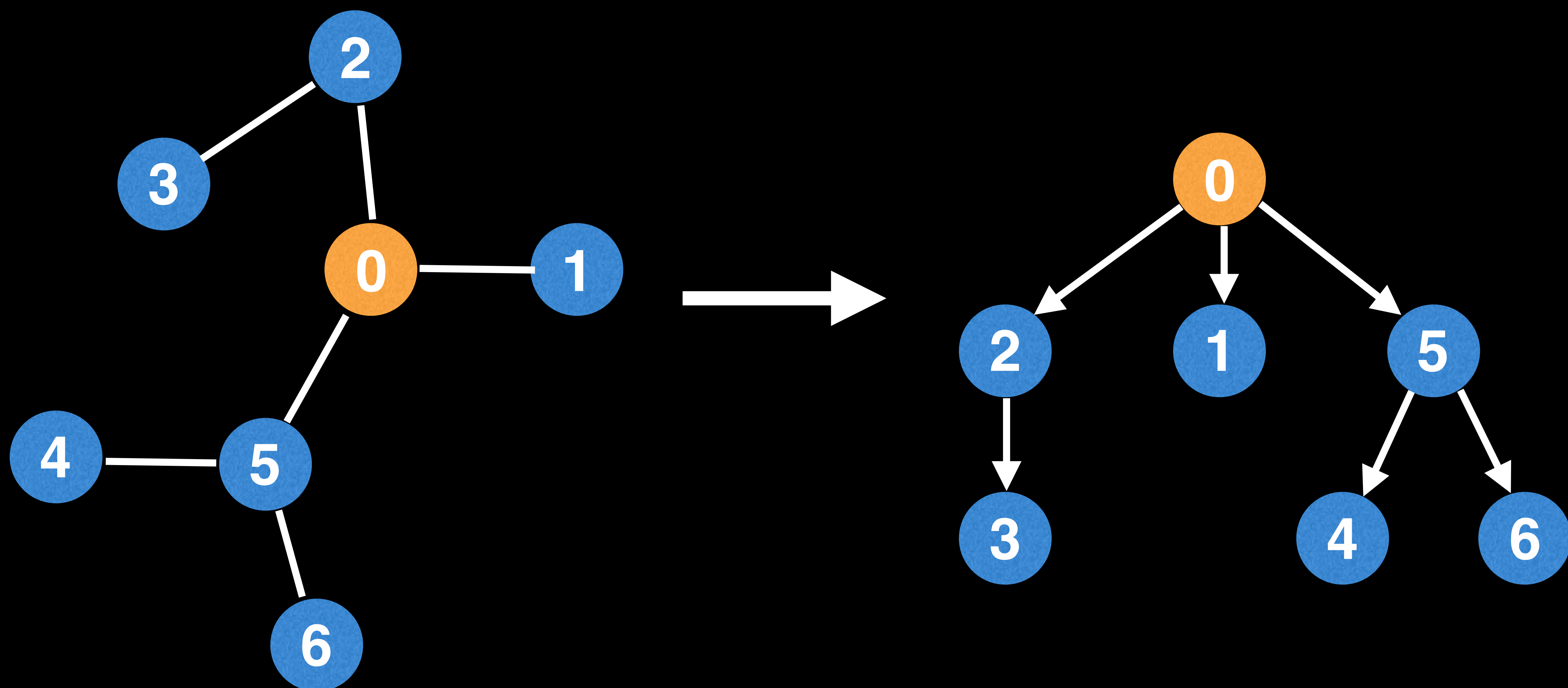


Undirected graph adjacency list:

```
0 -> [2, 1, 5]
1 -> [0]
2 -> [3, 0]
3 -> [2]
4 -> [5]
5 -> [4, 6, 0]
6 -> [5]
```

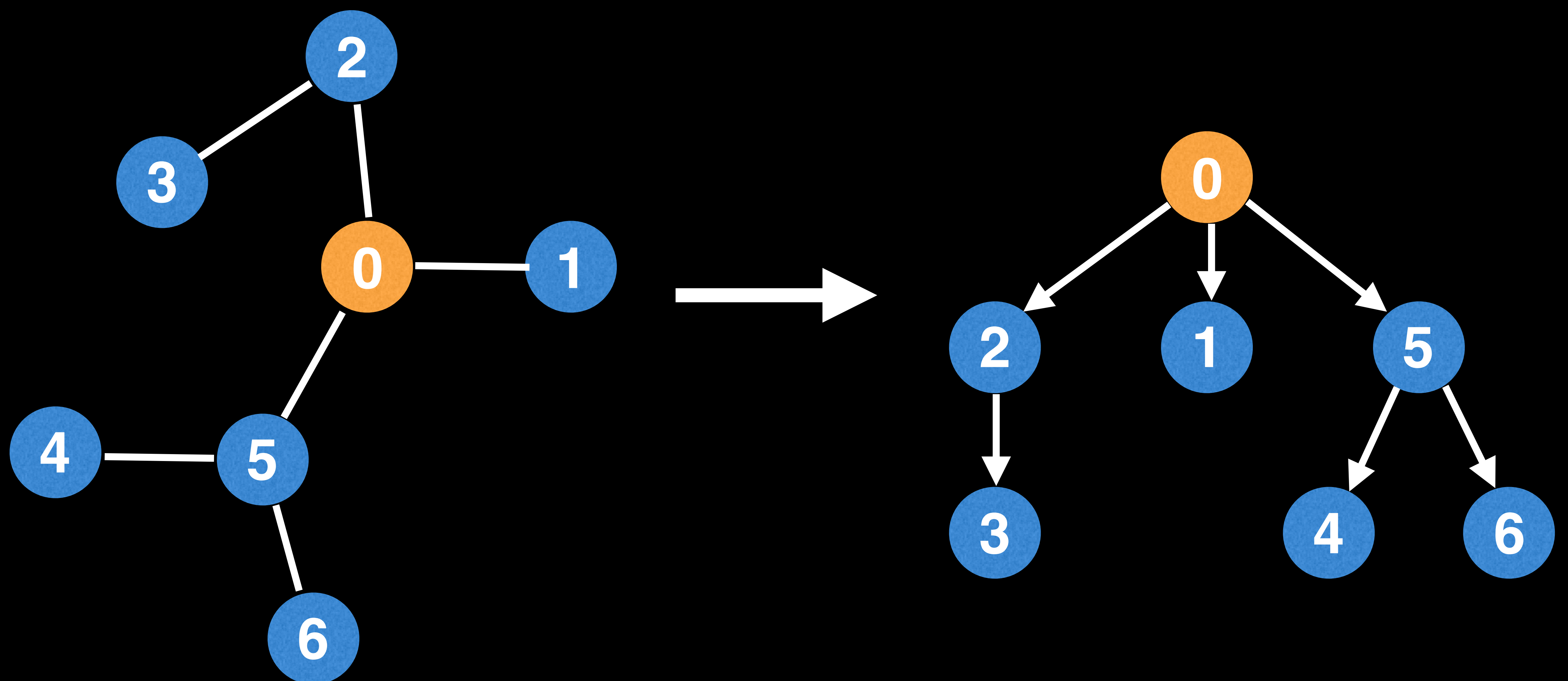
Rooting a tree

Sometimes it’s useful to root an undirected tree to add structure to the problem you’re trying to solve.



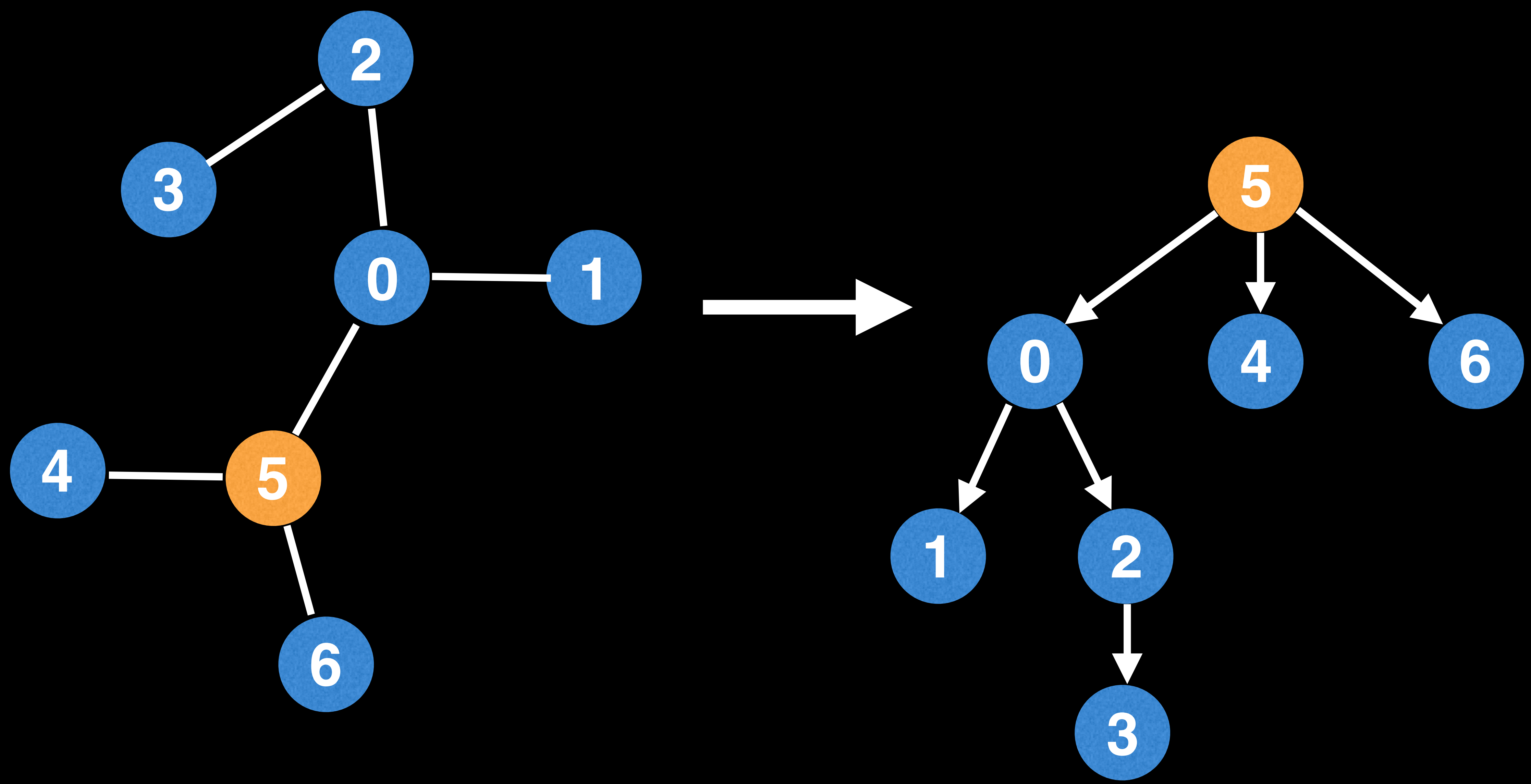
Rooting a tree

Conceptually this is like "picking up" the tree by a specific node and having all the edges point downwards.

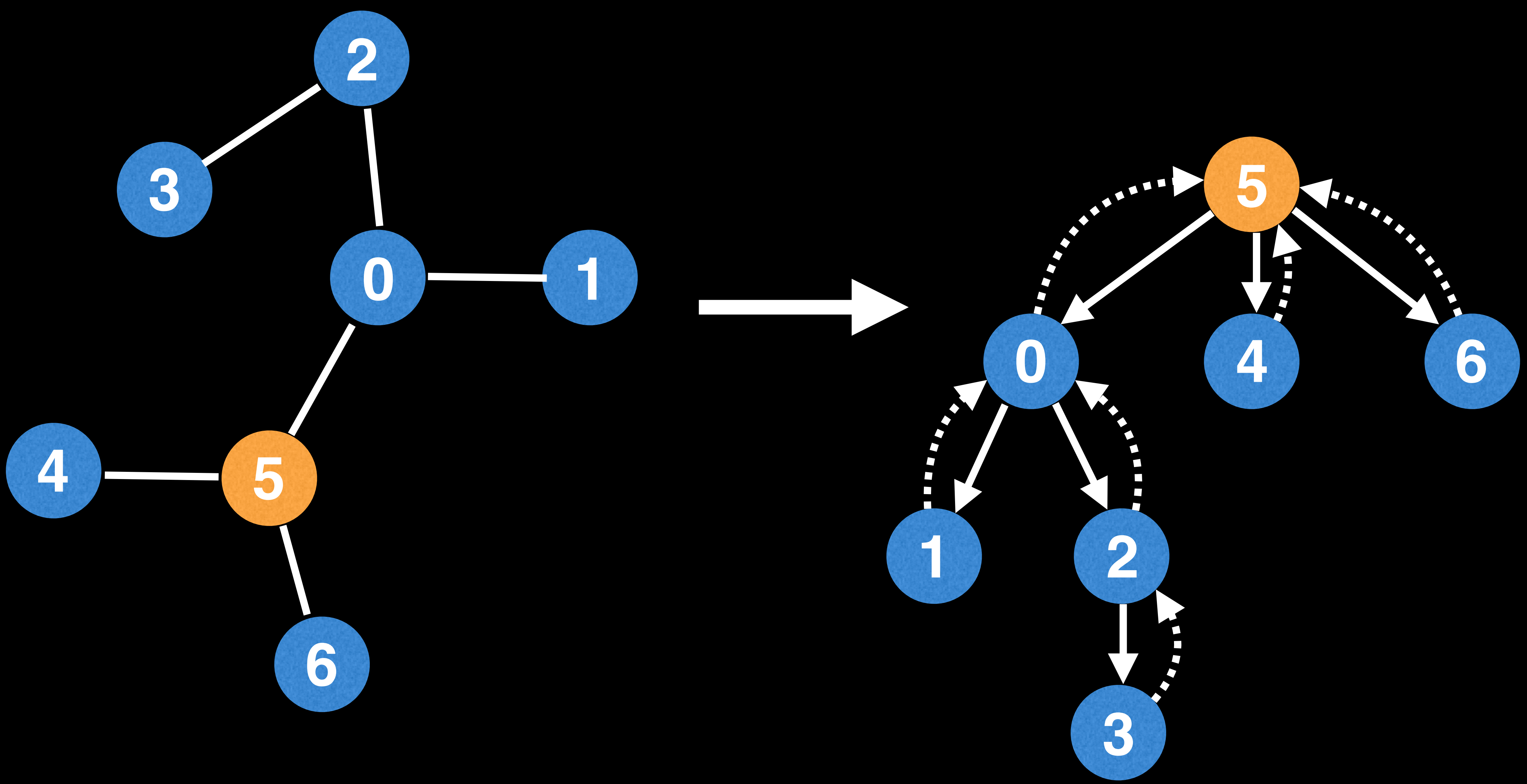


Rooting a tree

You can root a tree using any of its nodes.



In some situations it’s also useful to keep have a reference to the parent node in order to walk up the tree.



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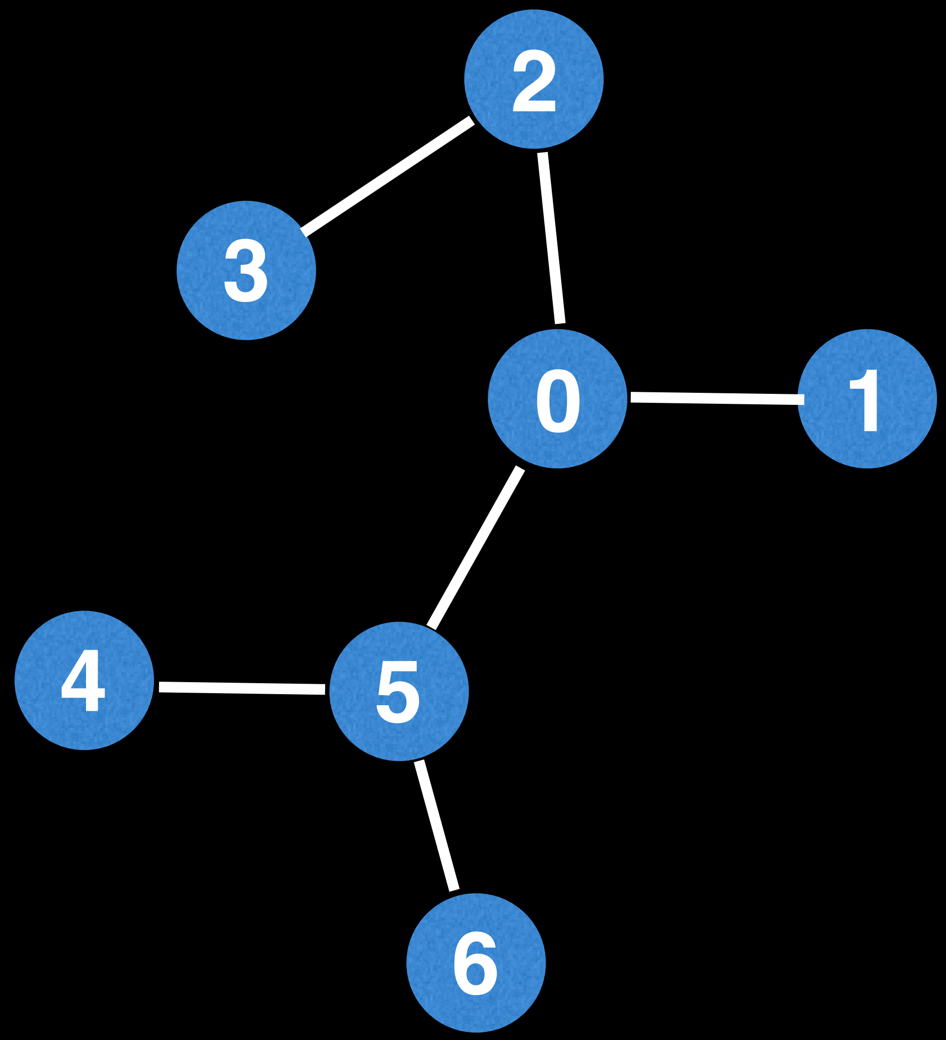

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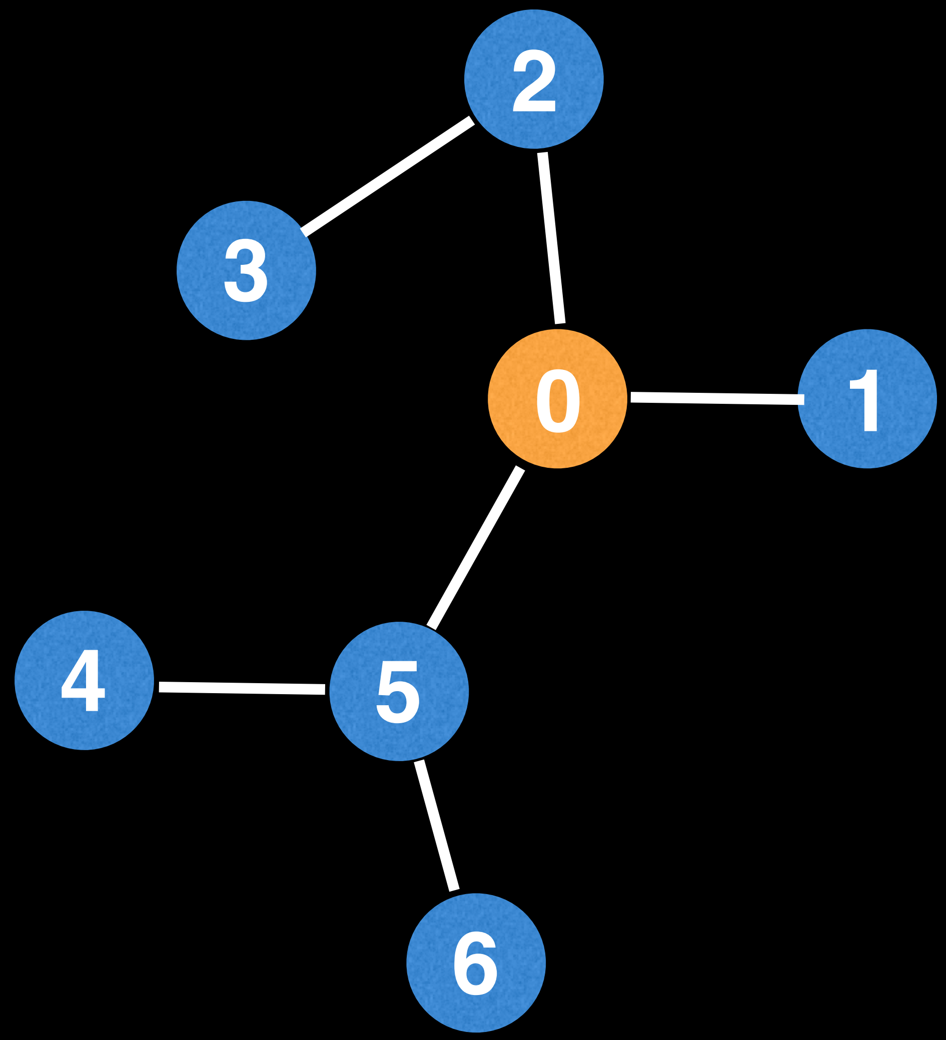

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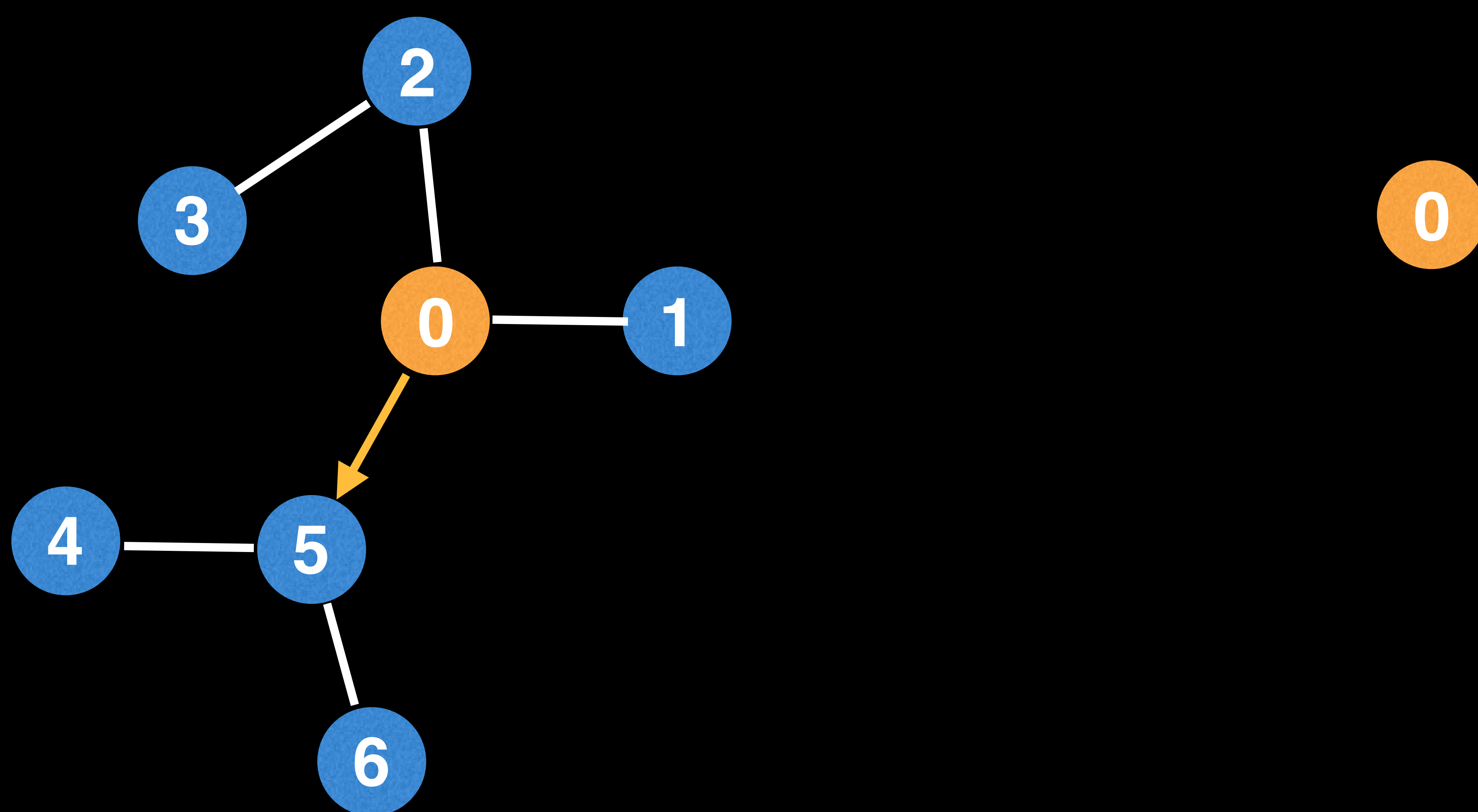

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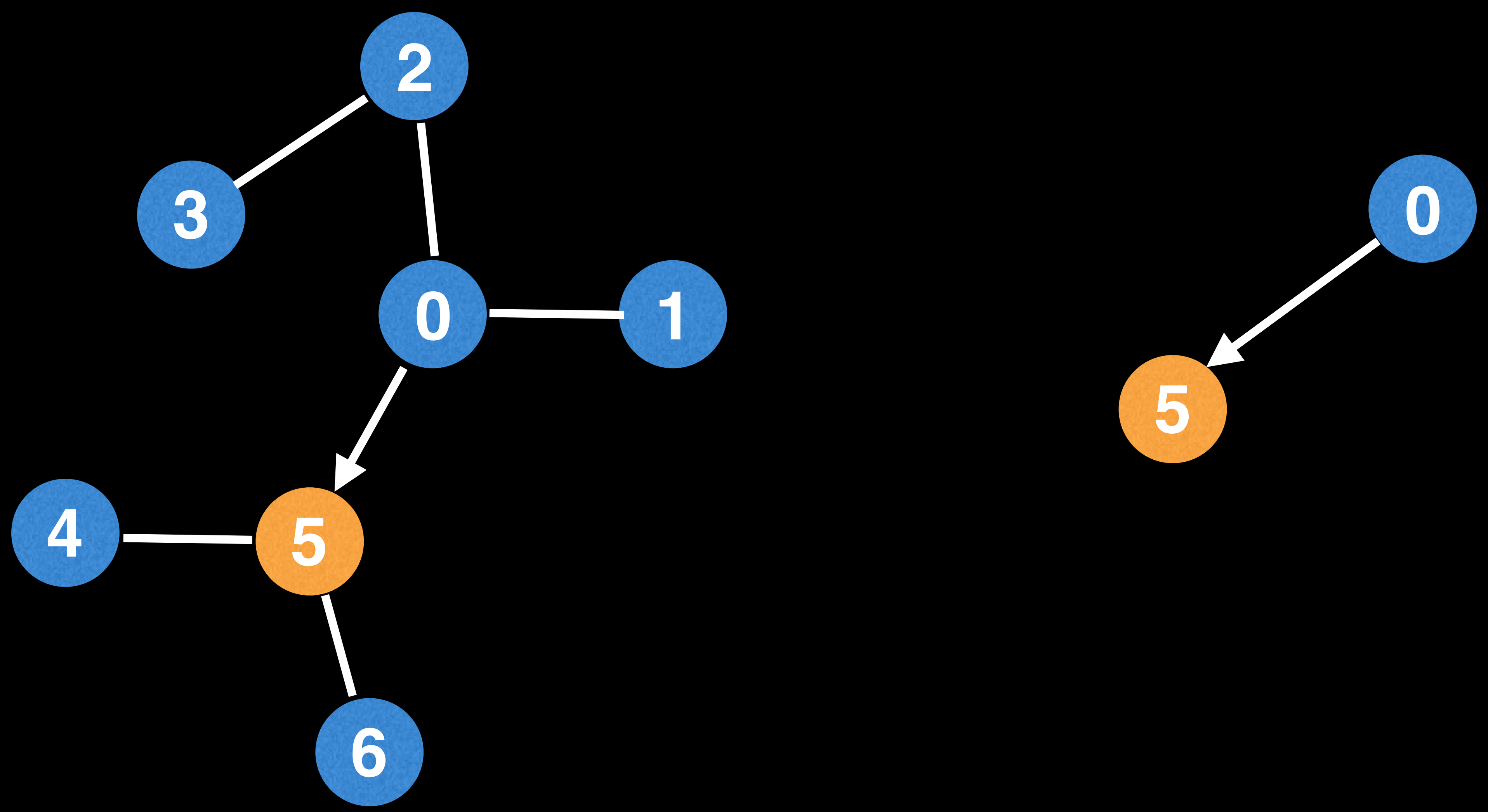

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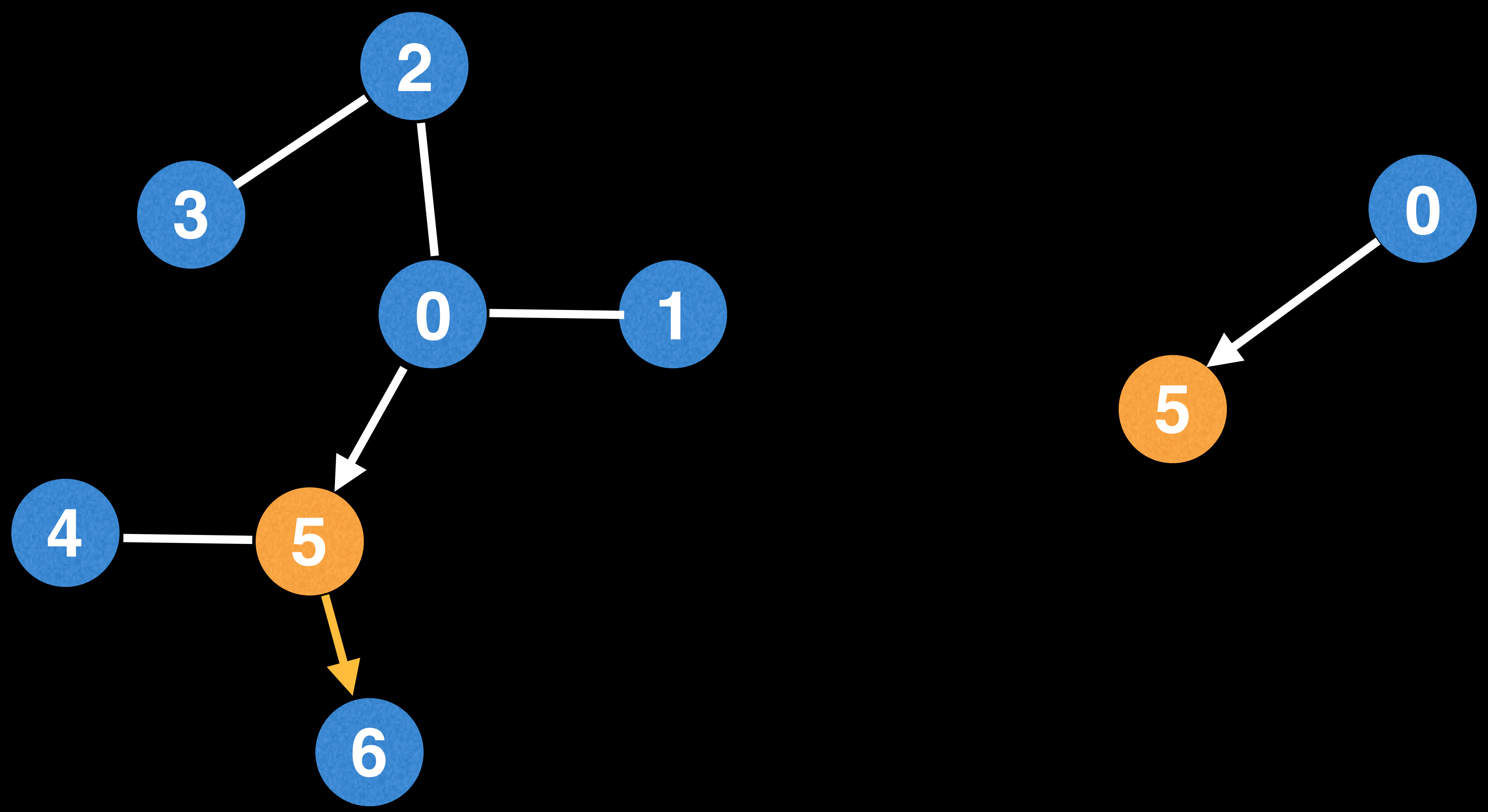

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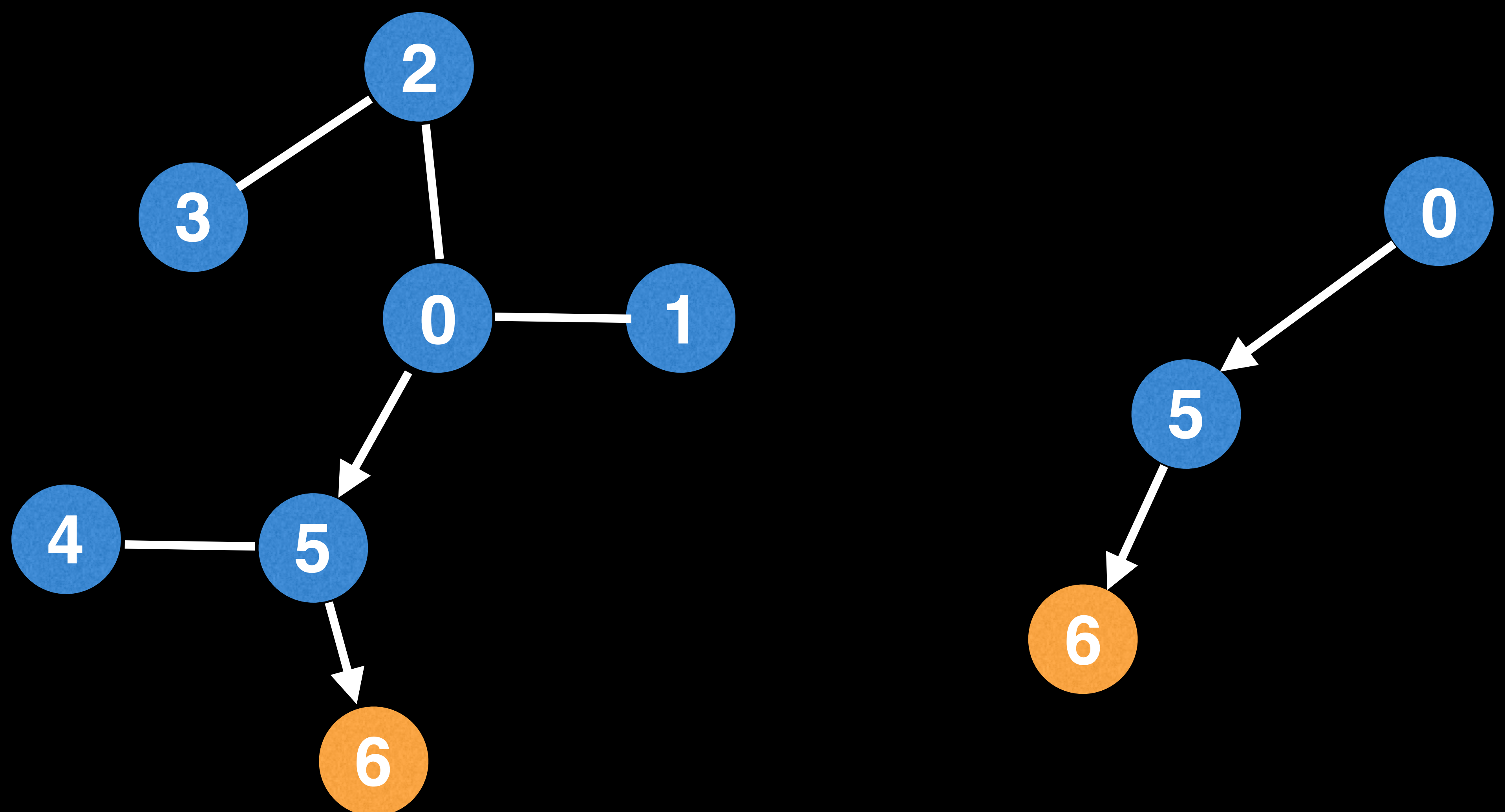

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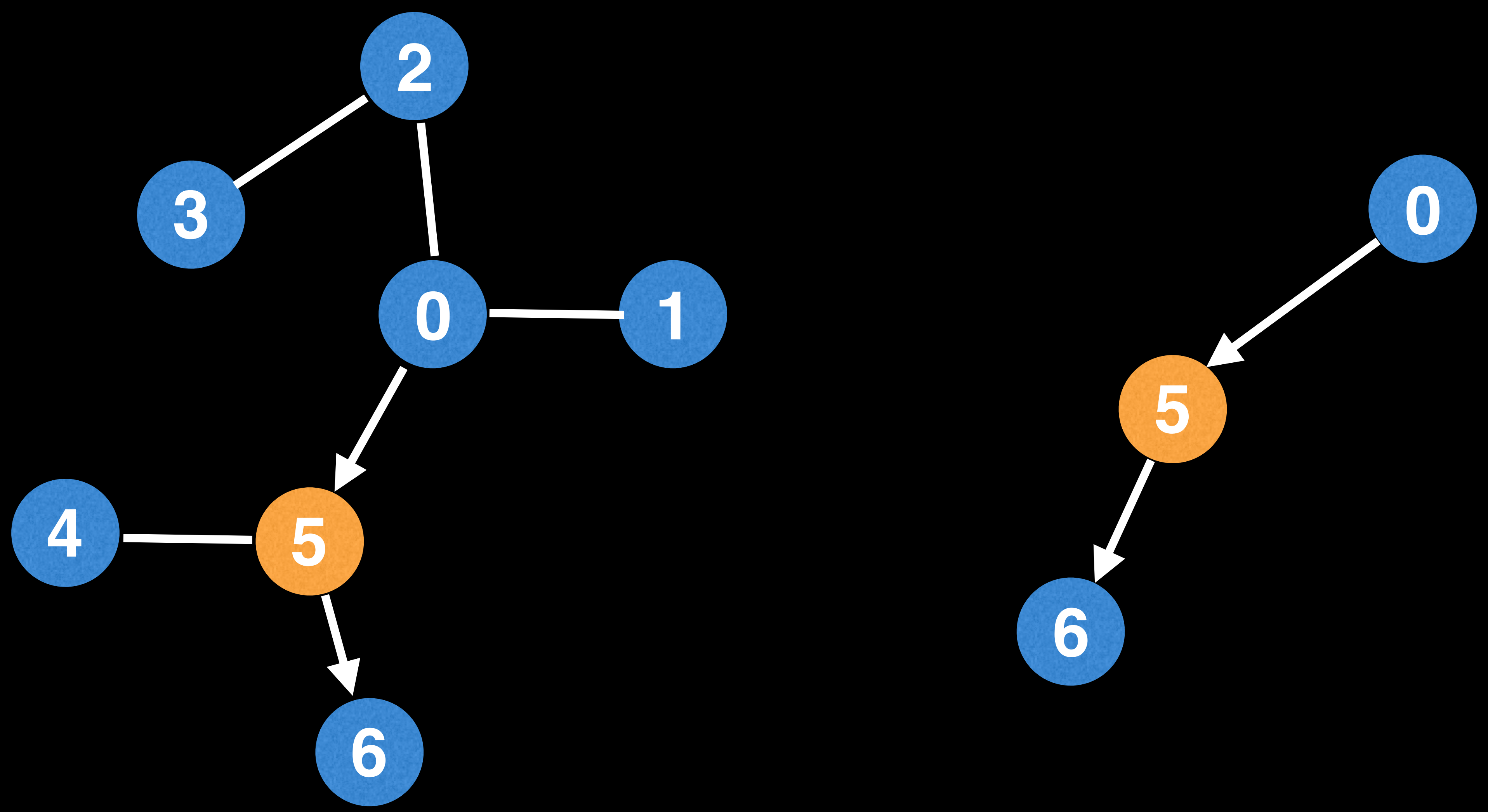

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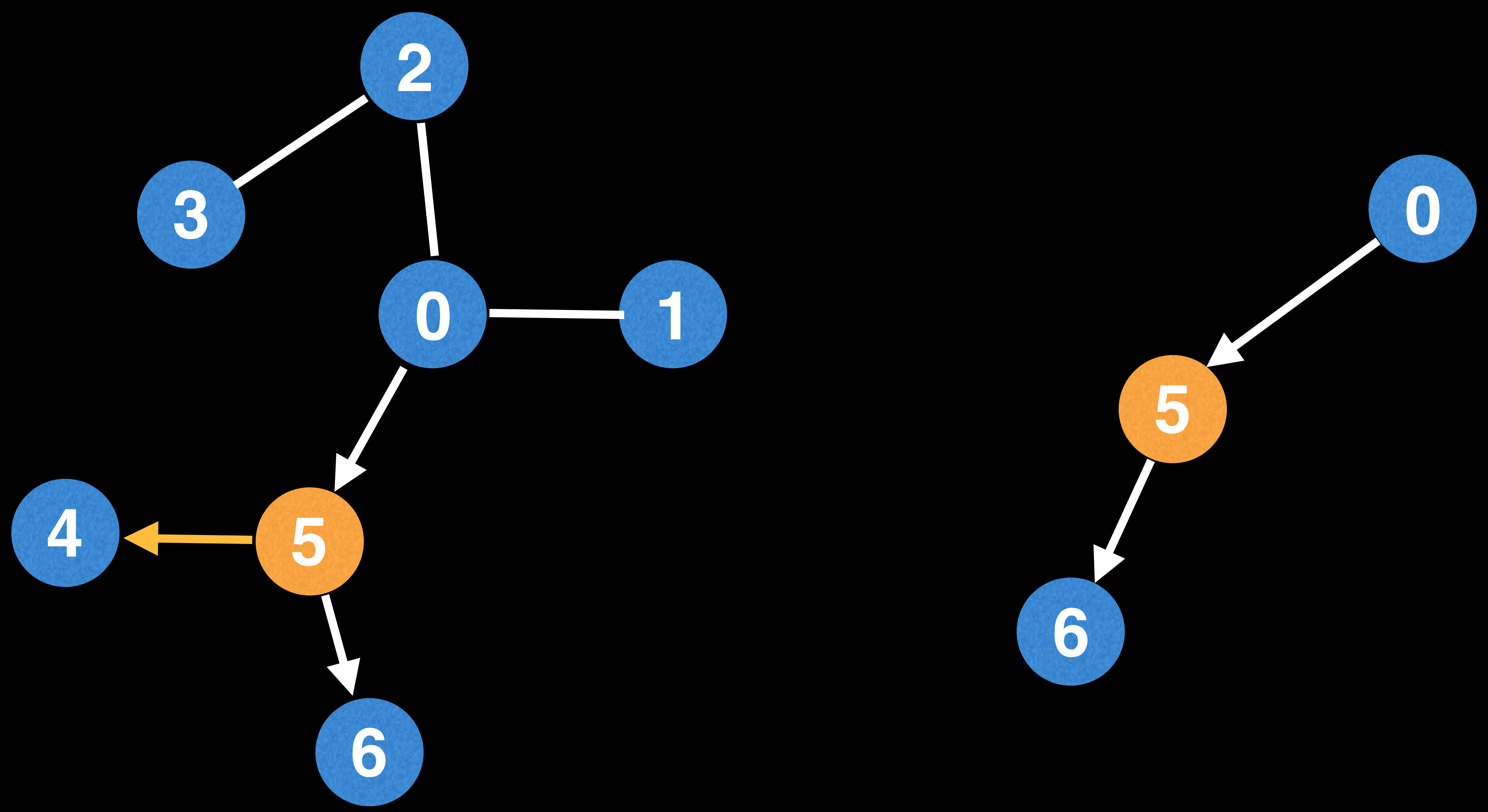

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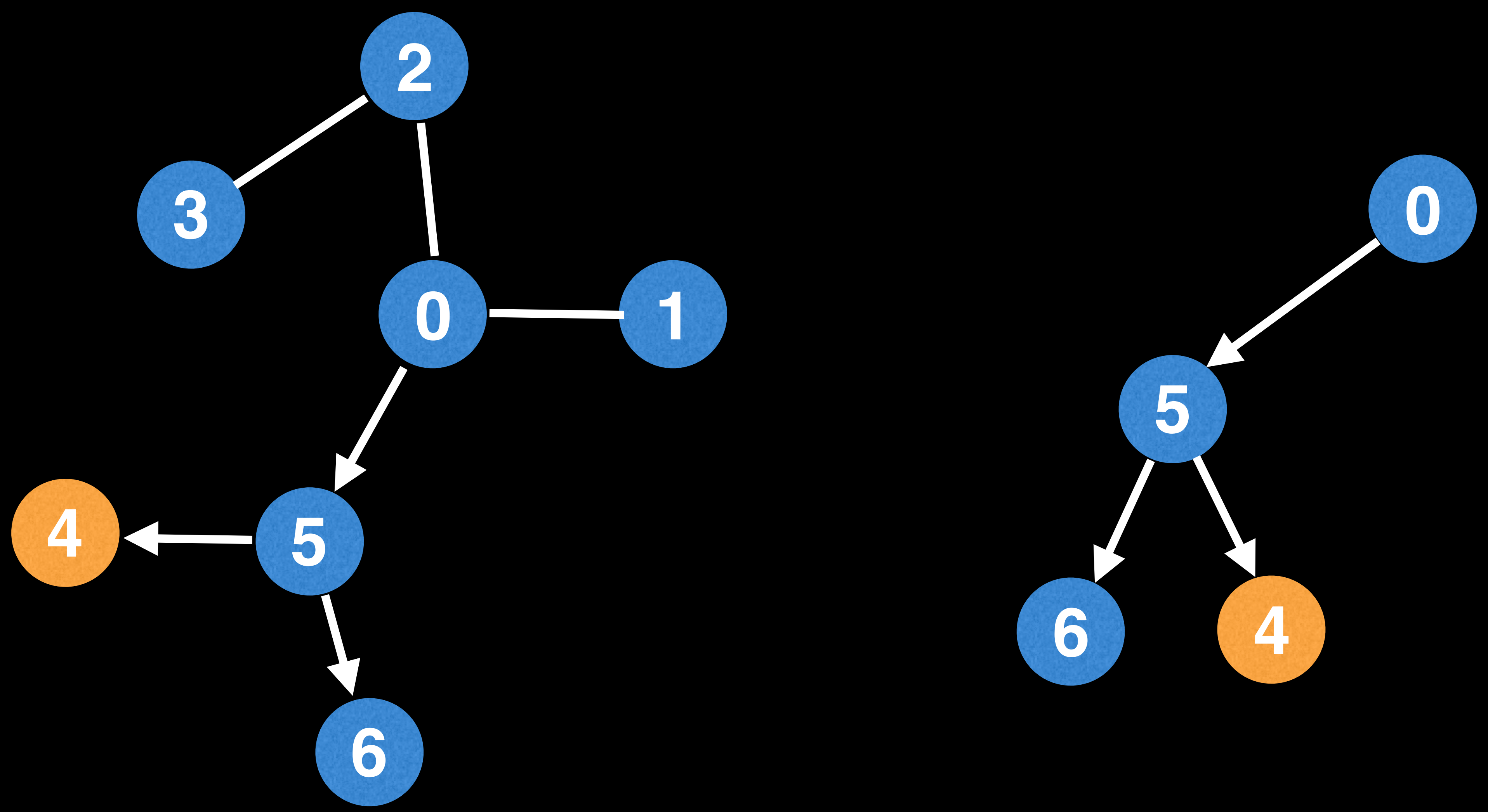

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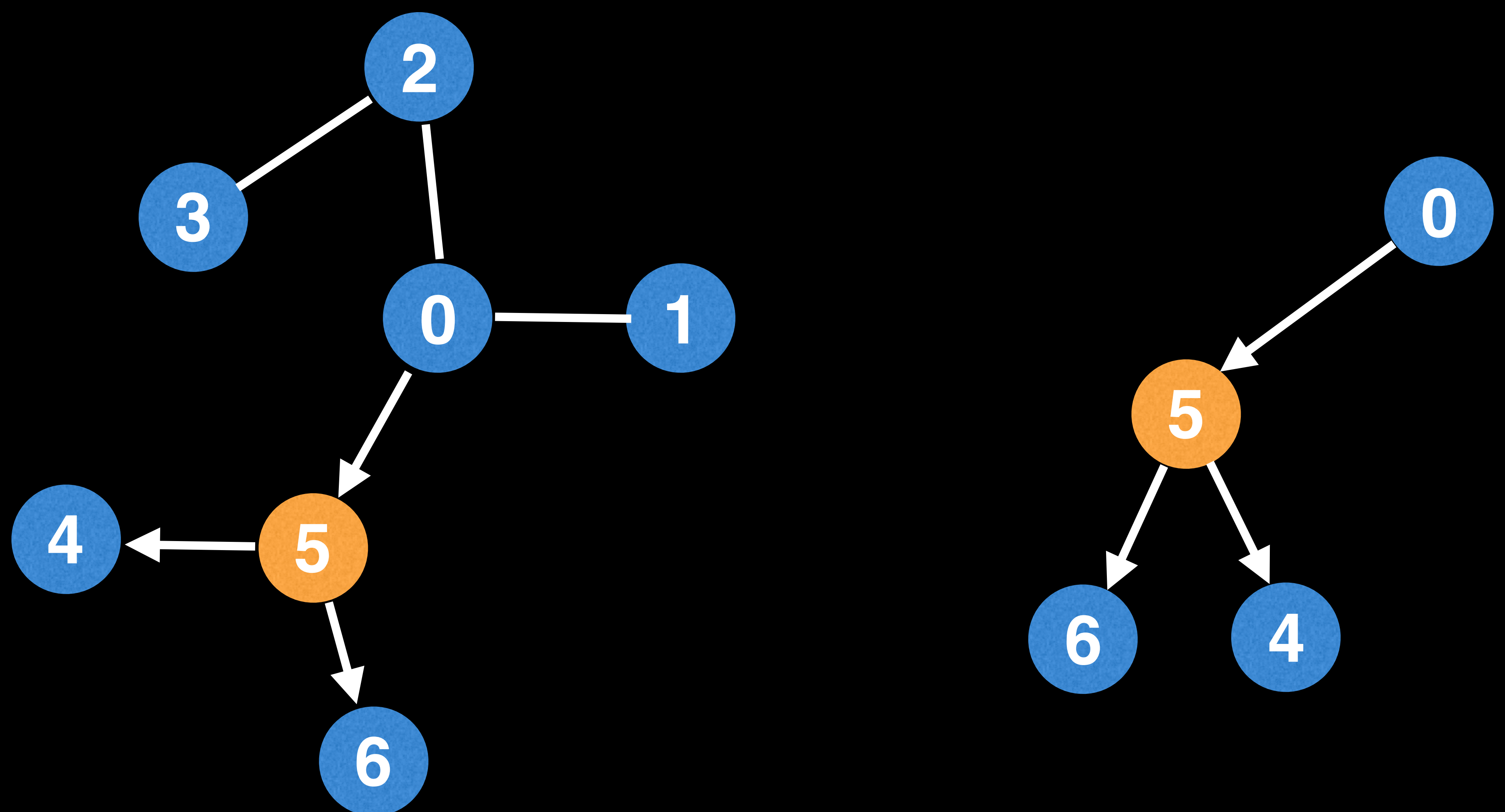

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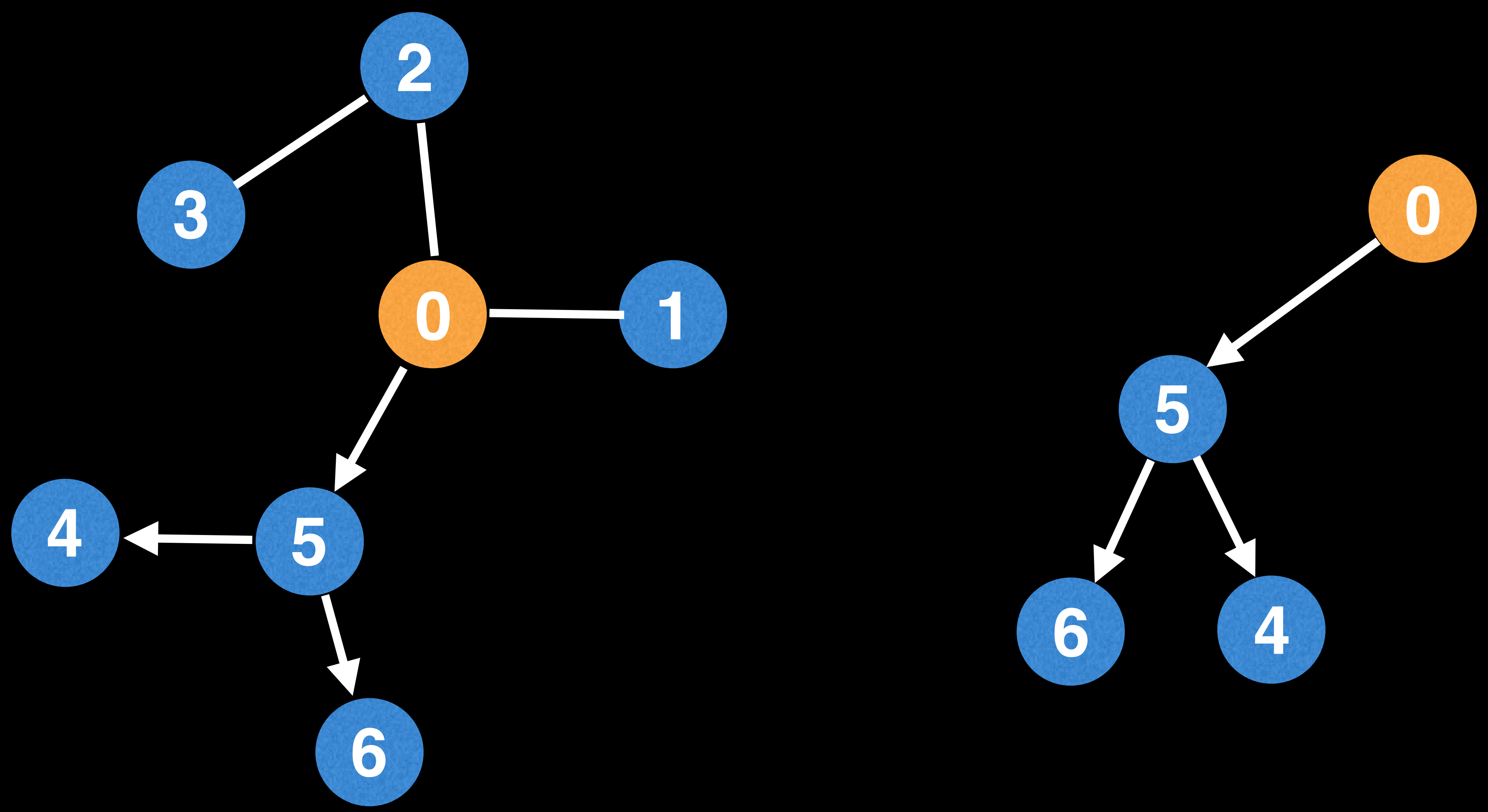

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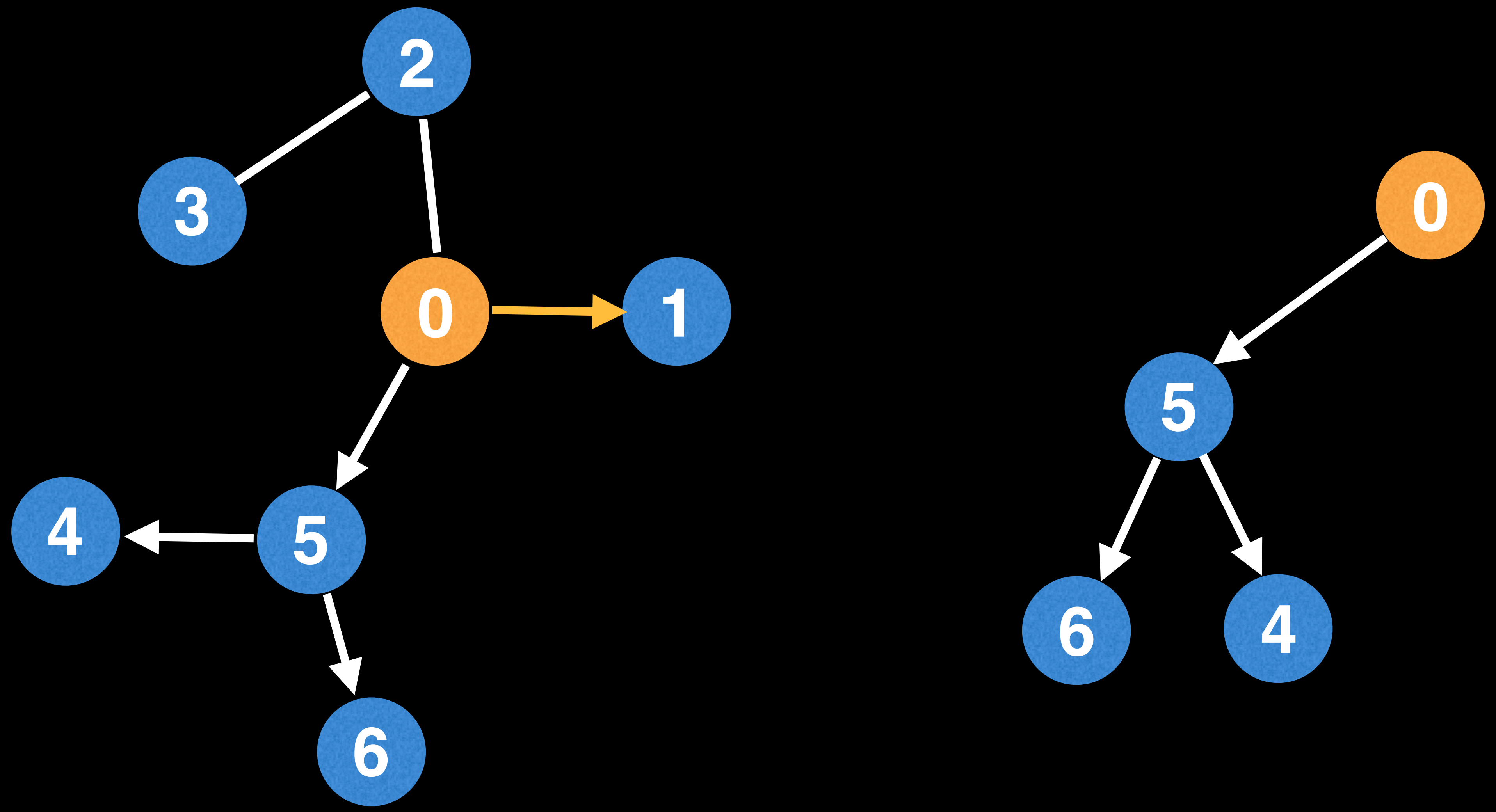

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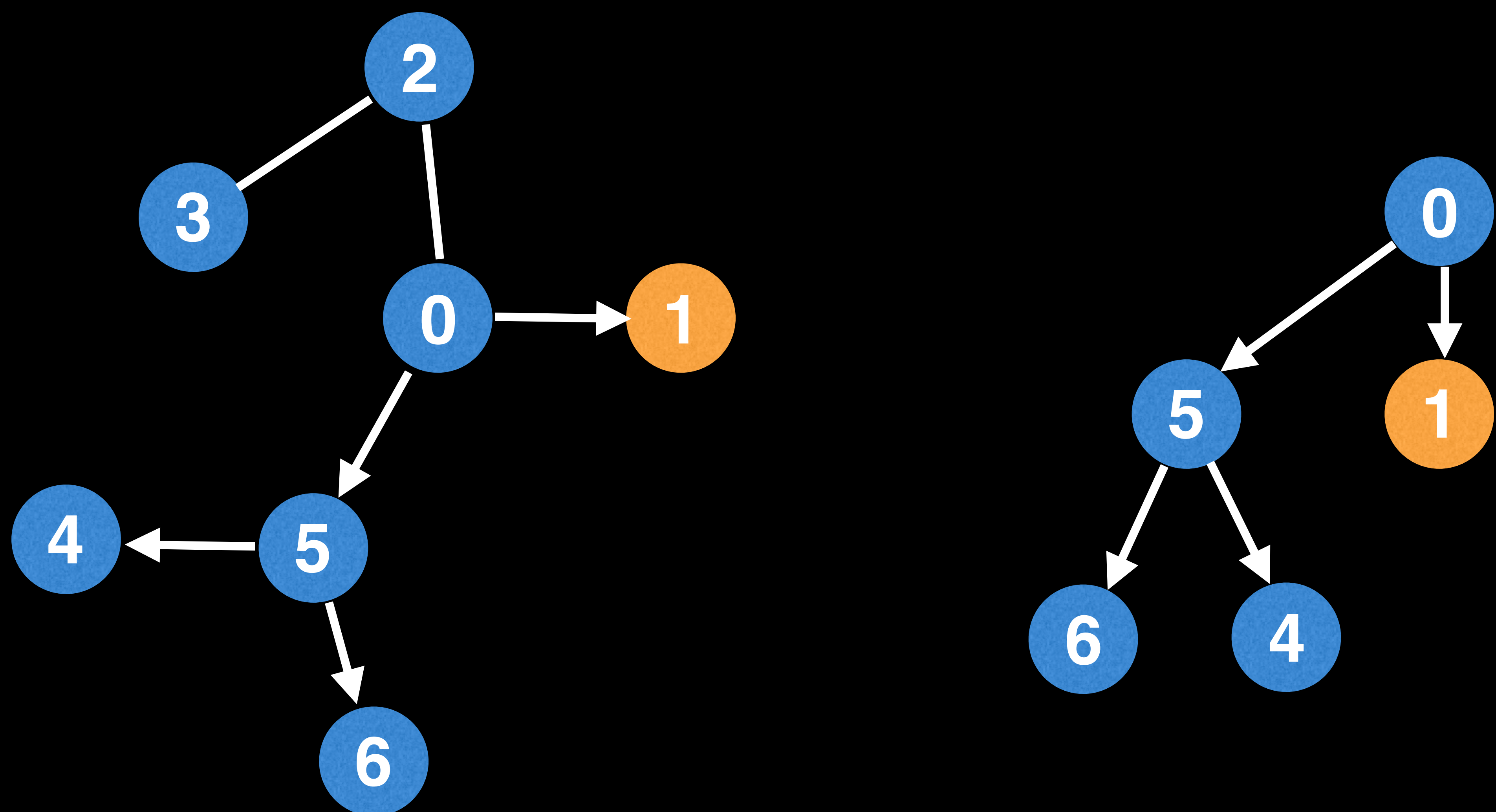

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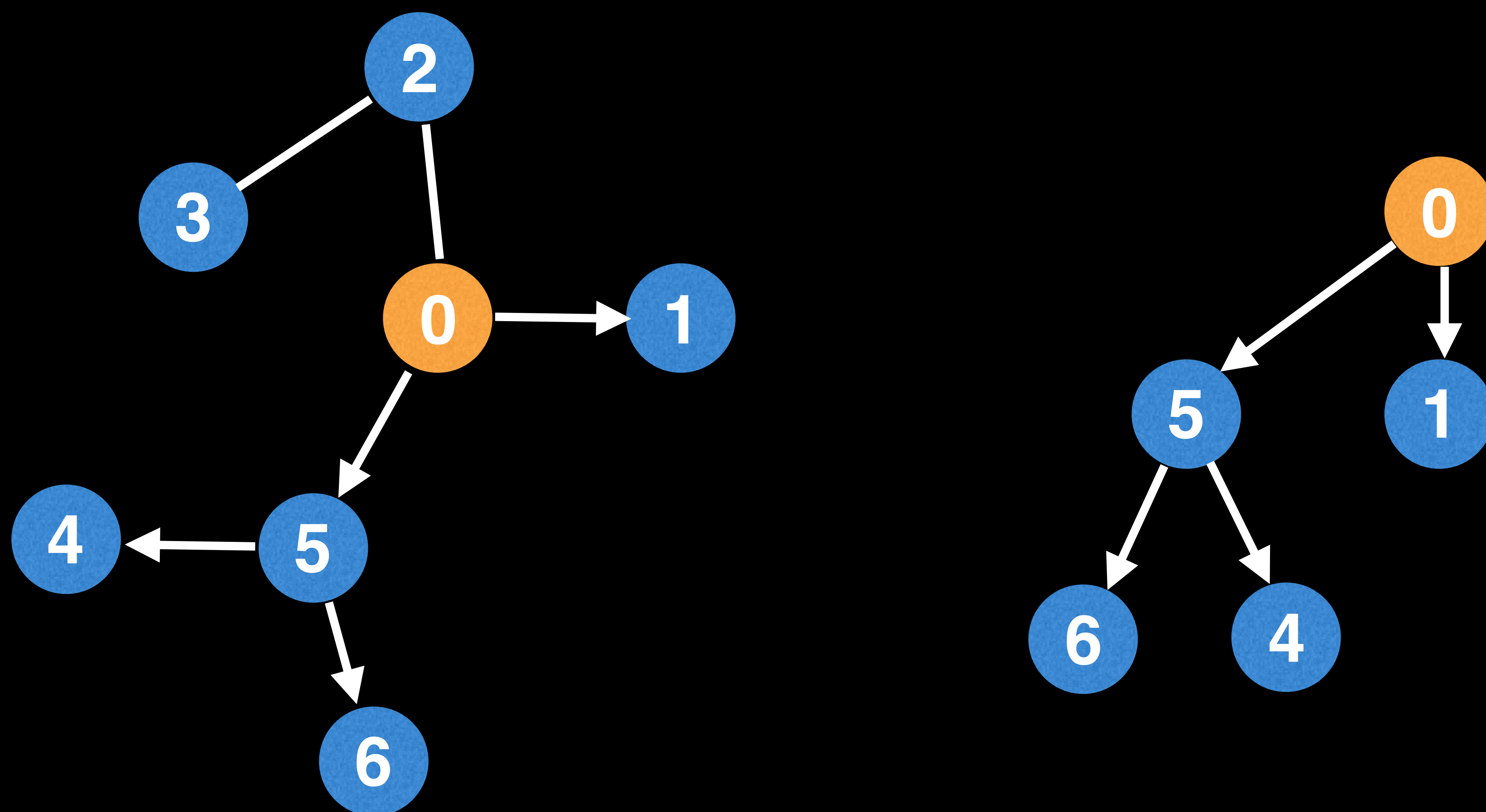

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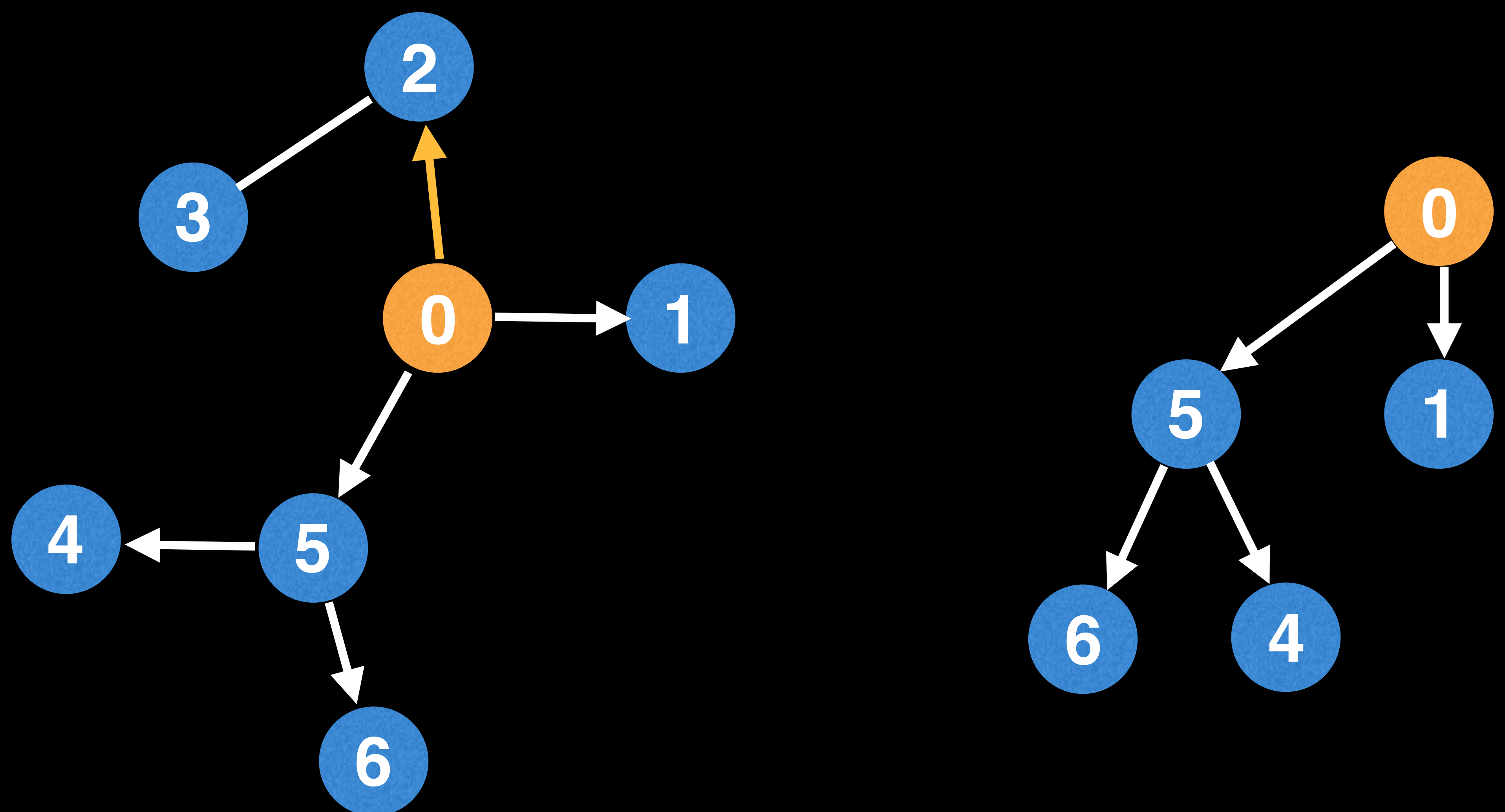

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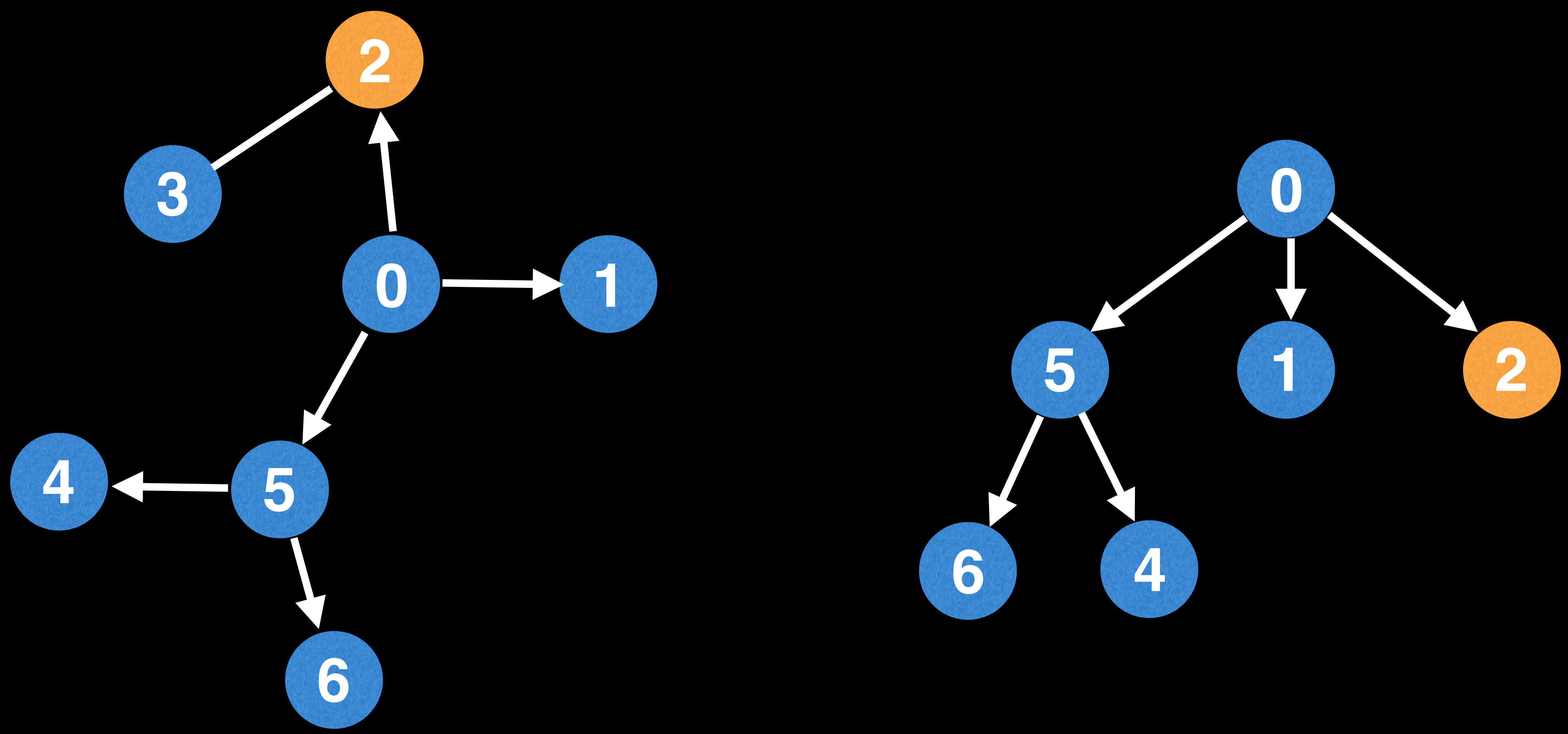

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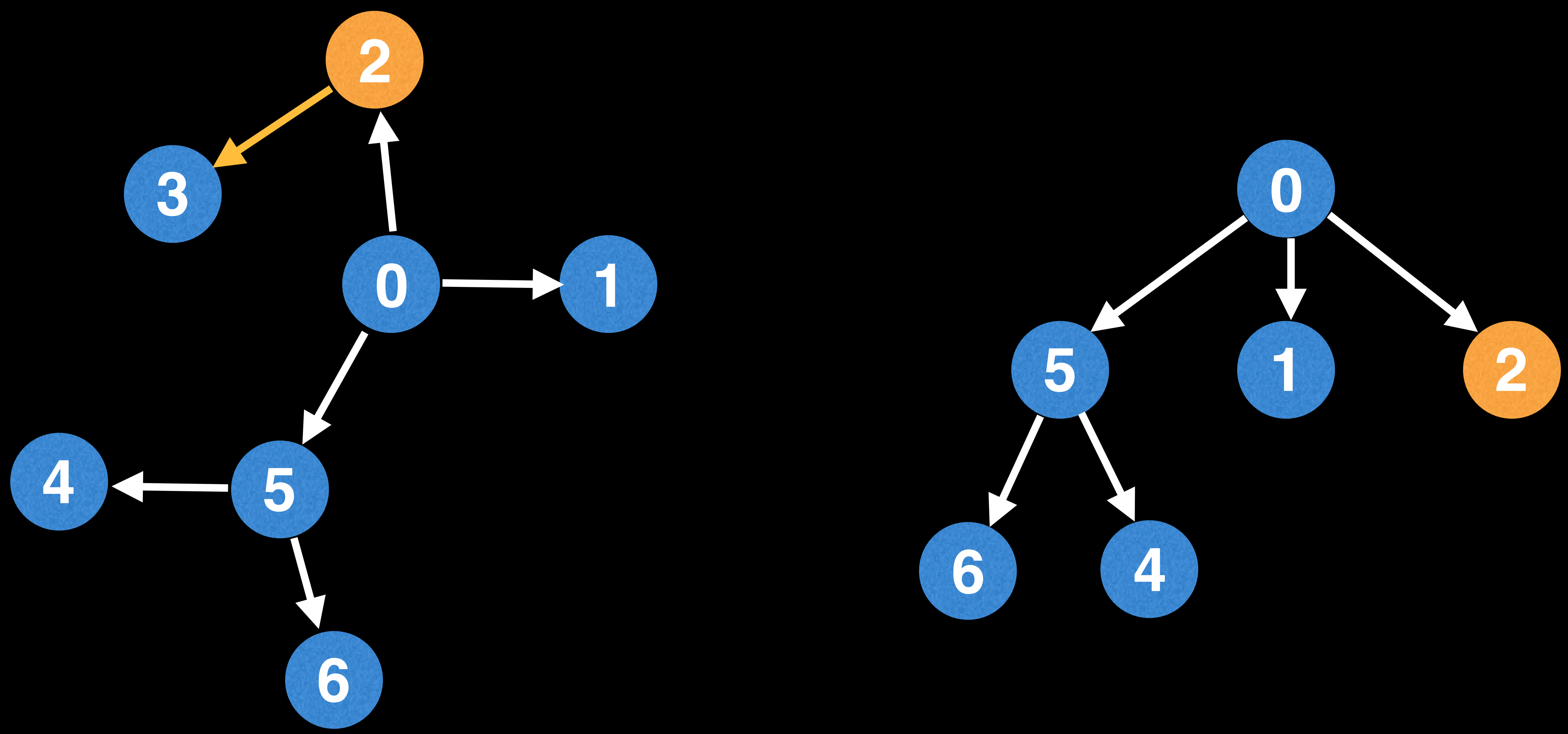

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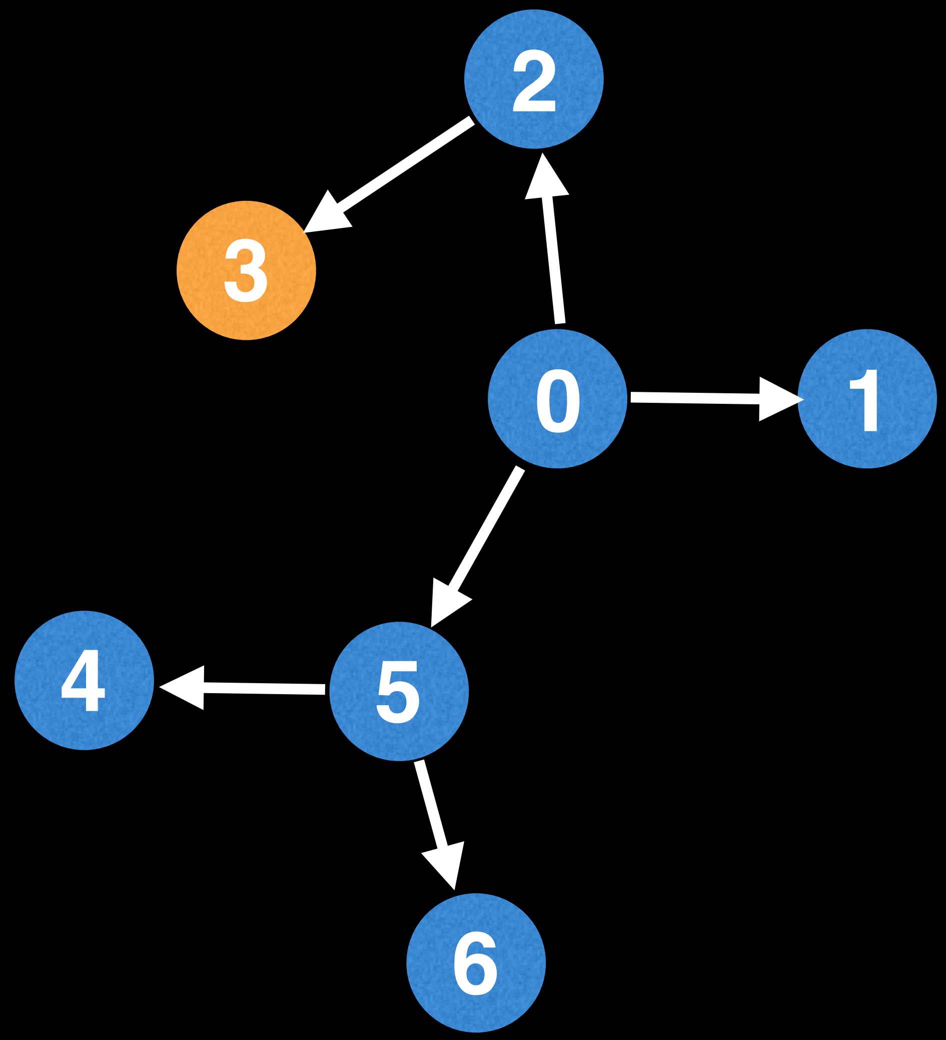

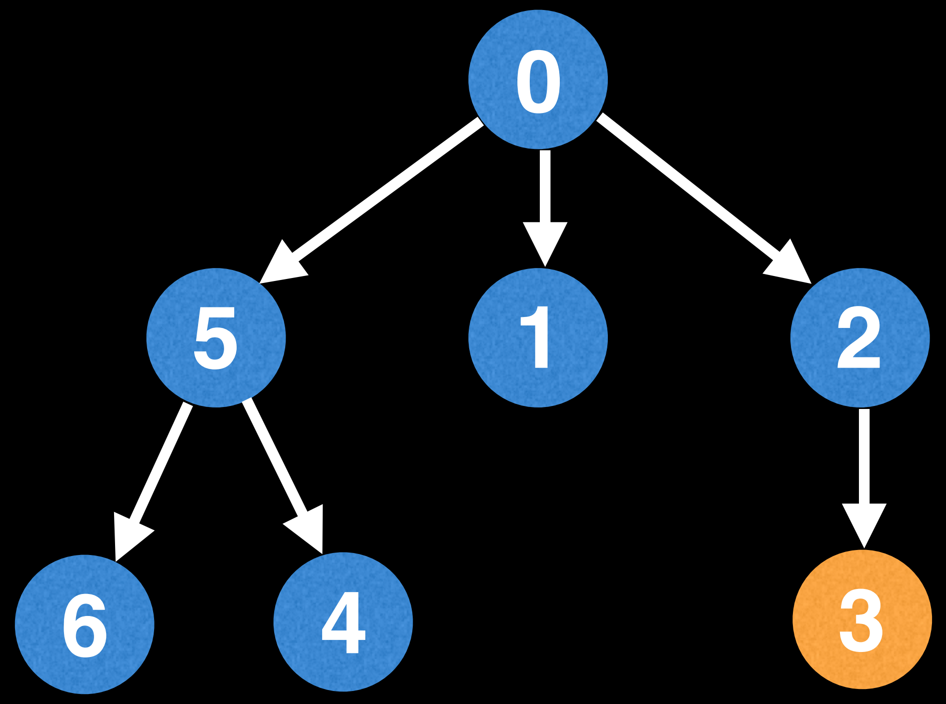

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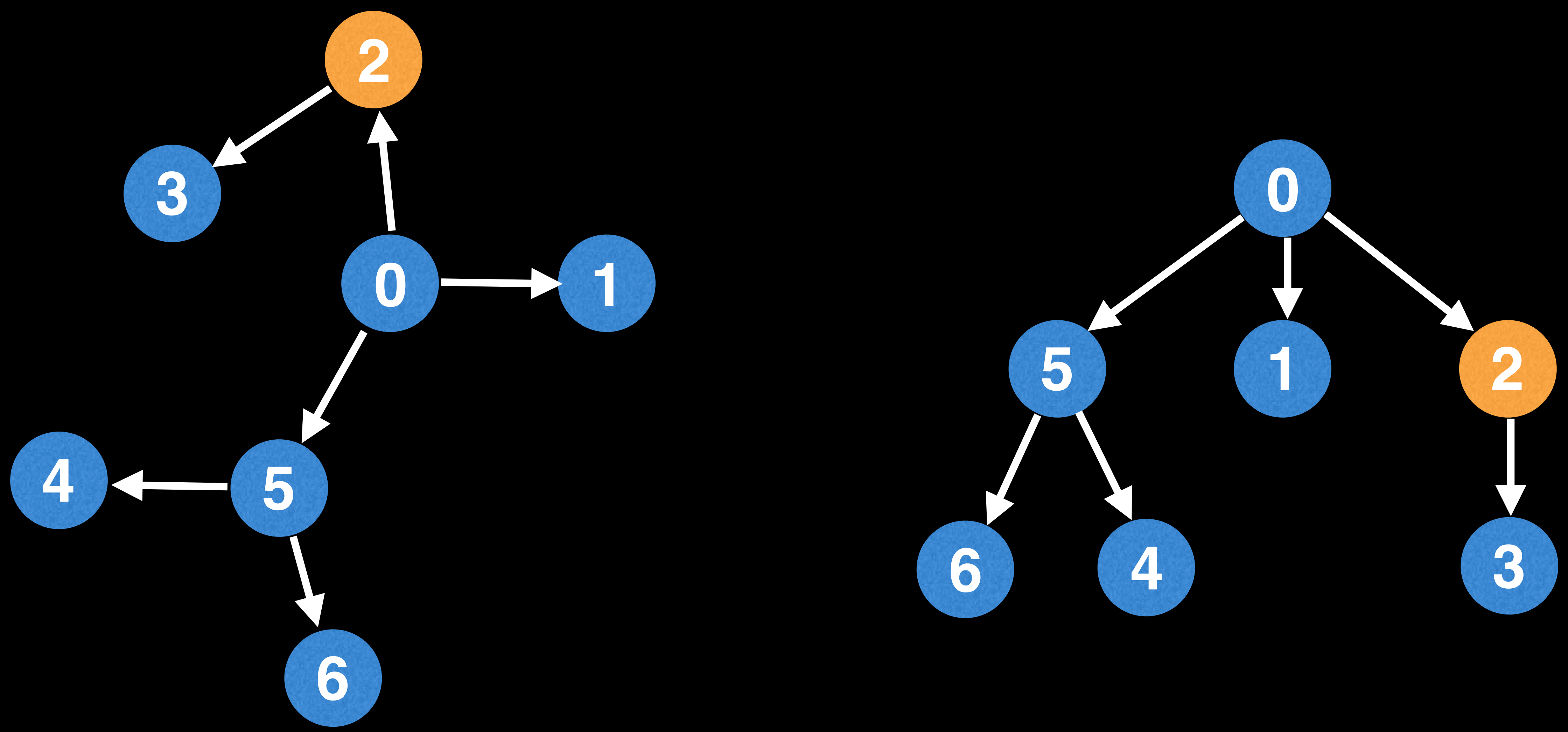

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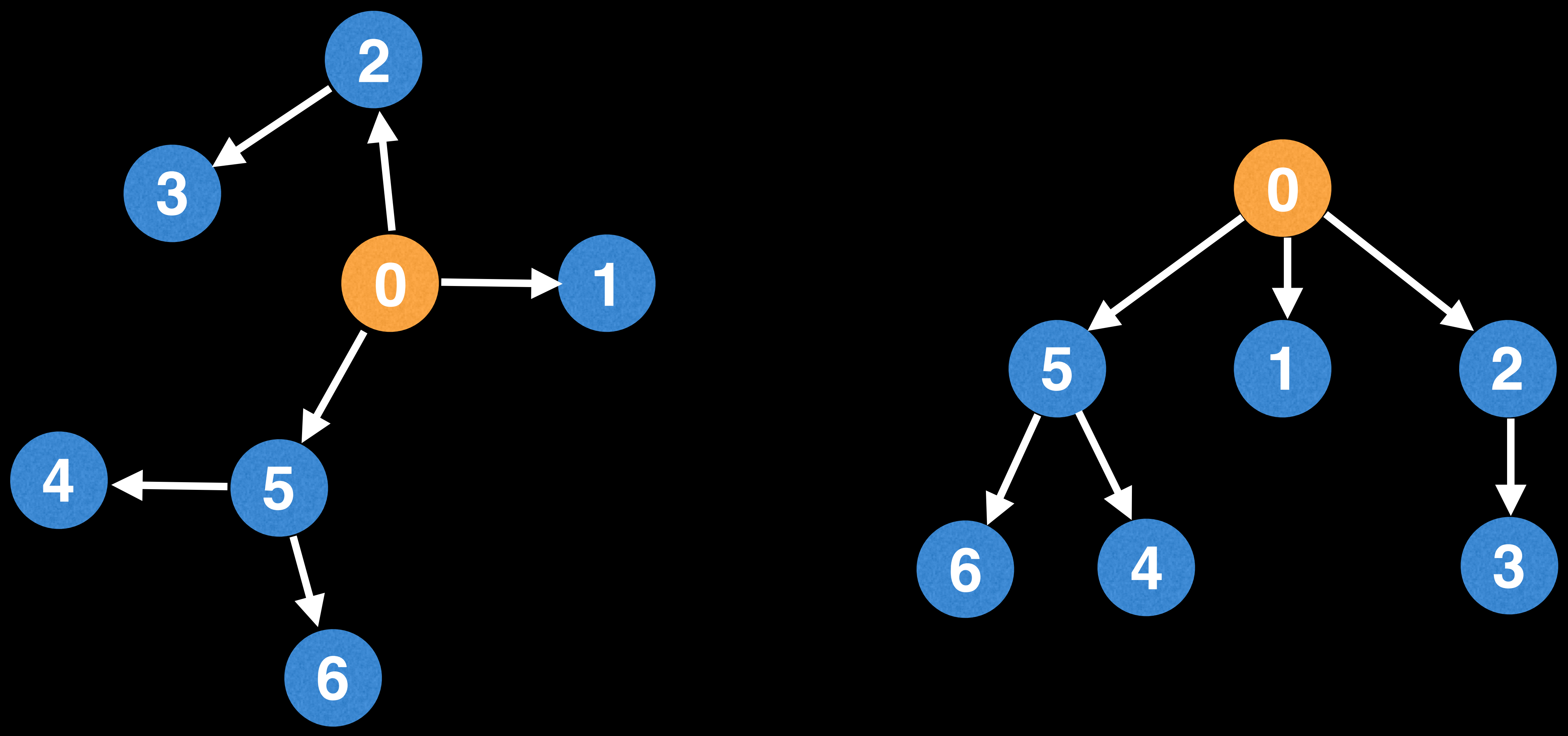

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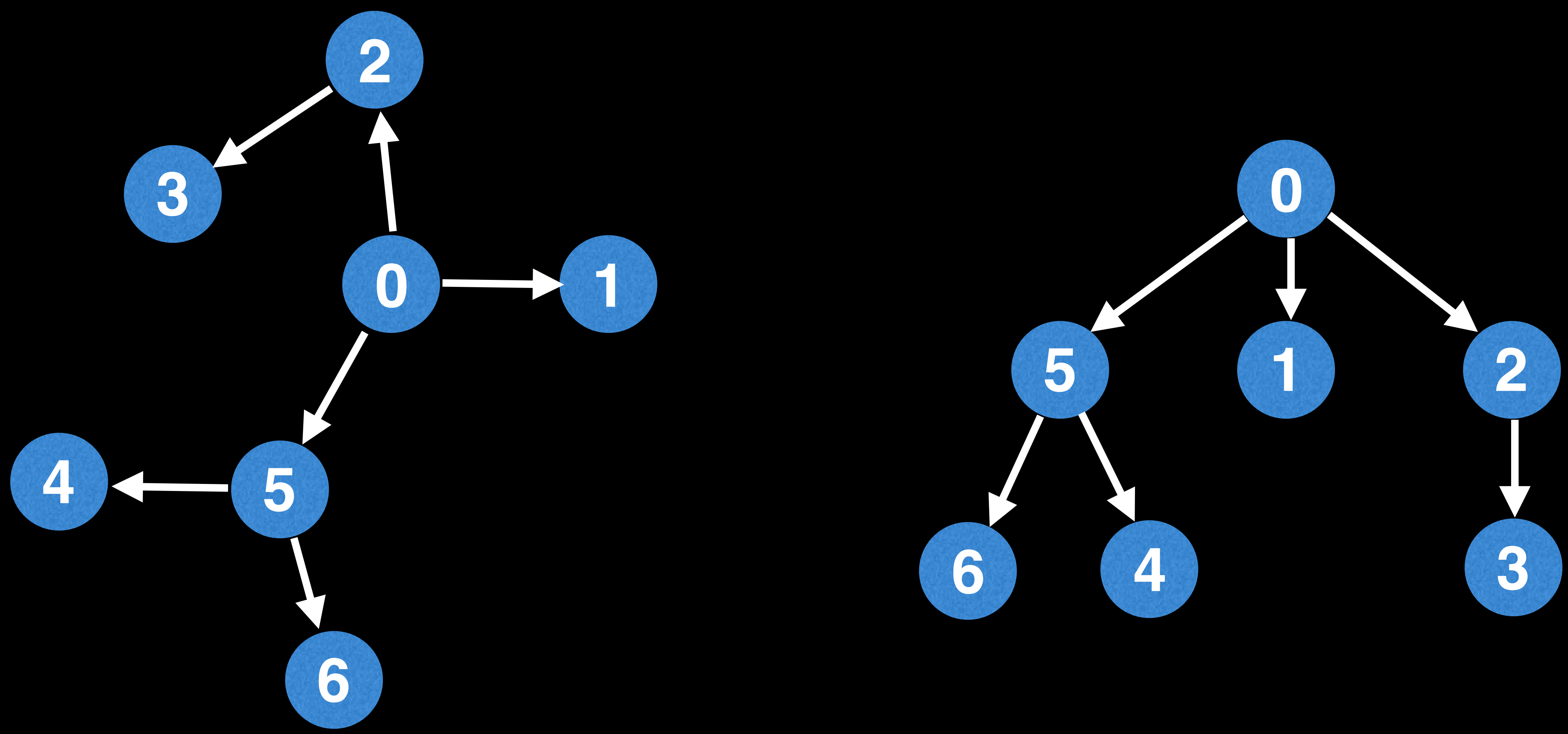

Rooting tree pseudocode

```
# TreeNode object structure.
class TreeNode:
  # Unique integer id to identify this node.
  int id;

  # Pointer to parent TreeNode reference. Only the
  # root node has a null parent TreeNode reference.
  TreeNode parent;

  # List of pointers to child TreeNodes.
  TreeNode[] children;
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# g is the graph/tree represented as an adjacency
# list with undirected edges. If there’s an edge between
# (u, v) there’s also an edge between (v, u).
# rootId is the id of the node to root the tree from.
function rootTree(g, rootId = 0):
  root = TreeNode(rootId, null, [])
  return buildTree(g, root, null)

# Build tree recursively depth first.
function buildTree(g, node, parent):
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root node has no parent!

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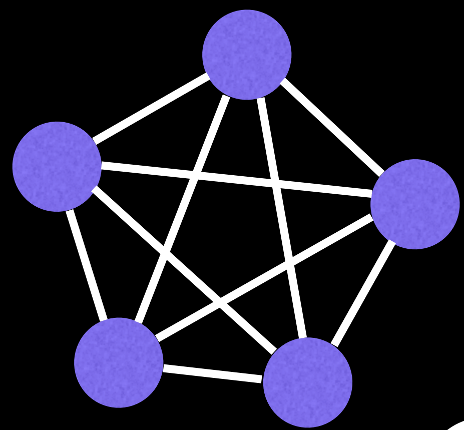
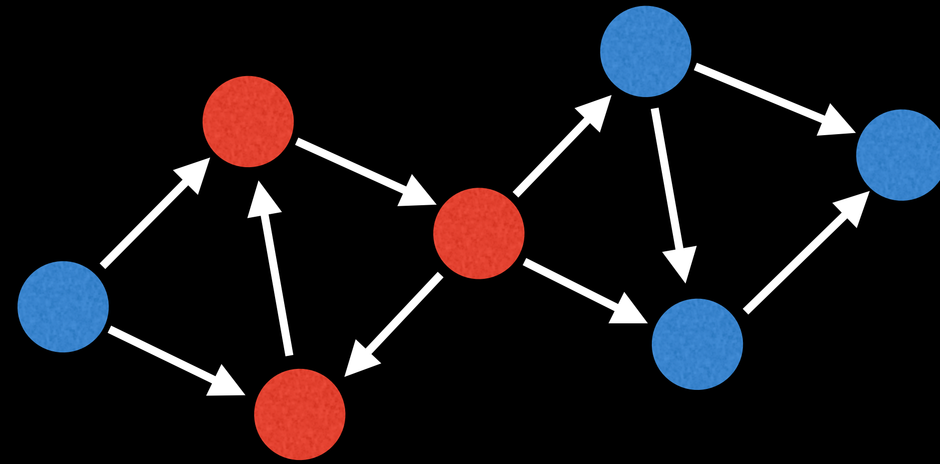
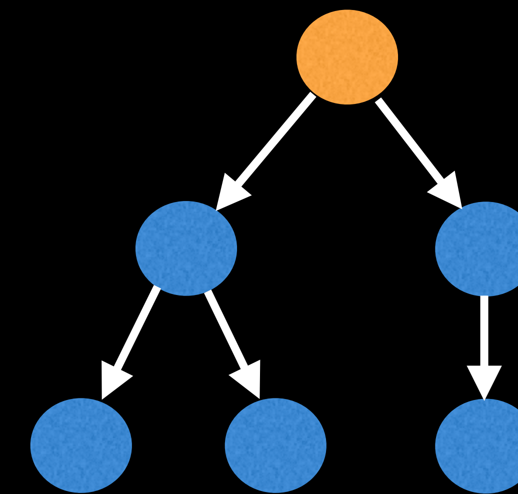
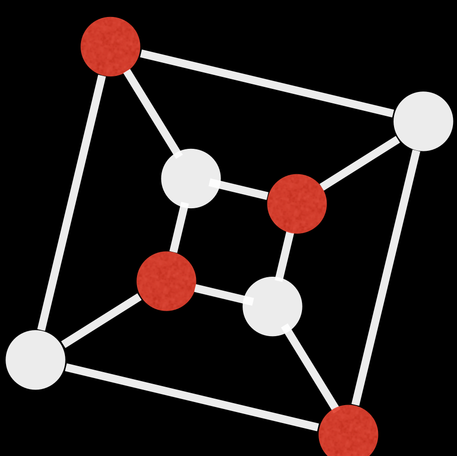
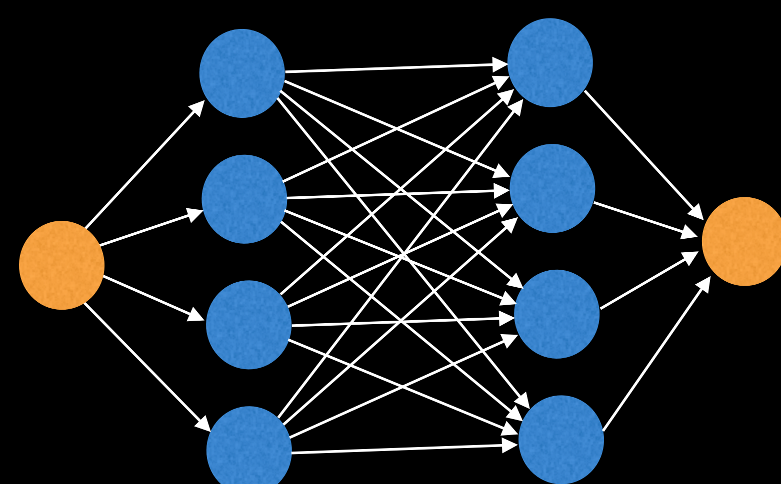


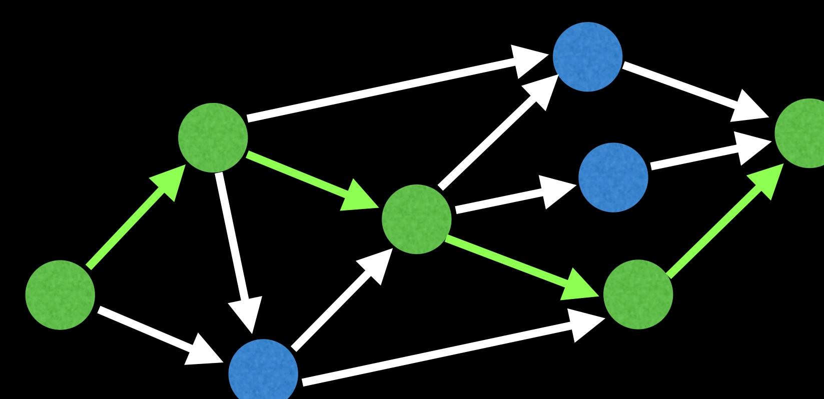




Graph Theory Video Series







Center(s) of a tree

🌿William Fiset🌿

Center(s) of undirected tree

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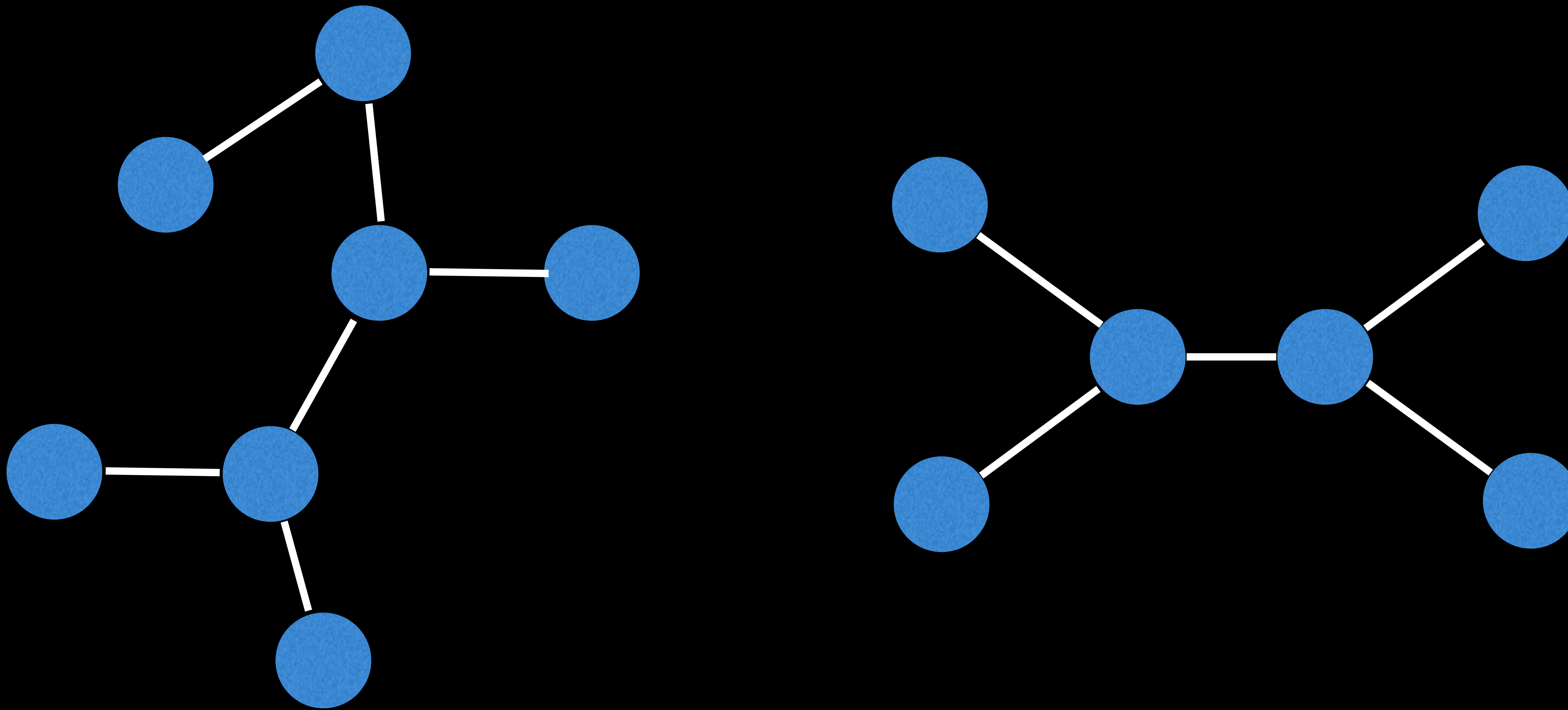

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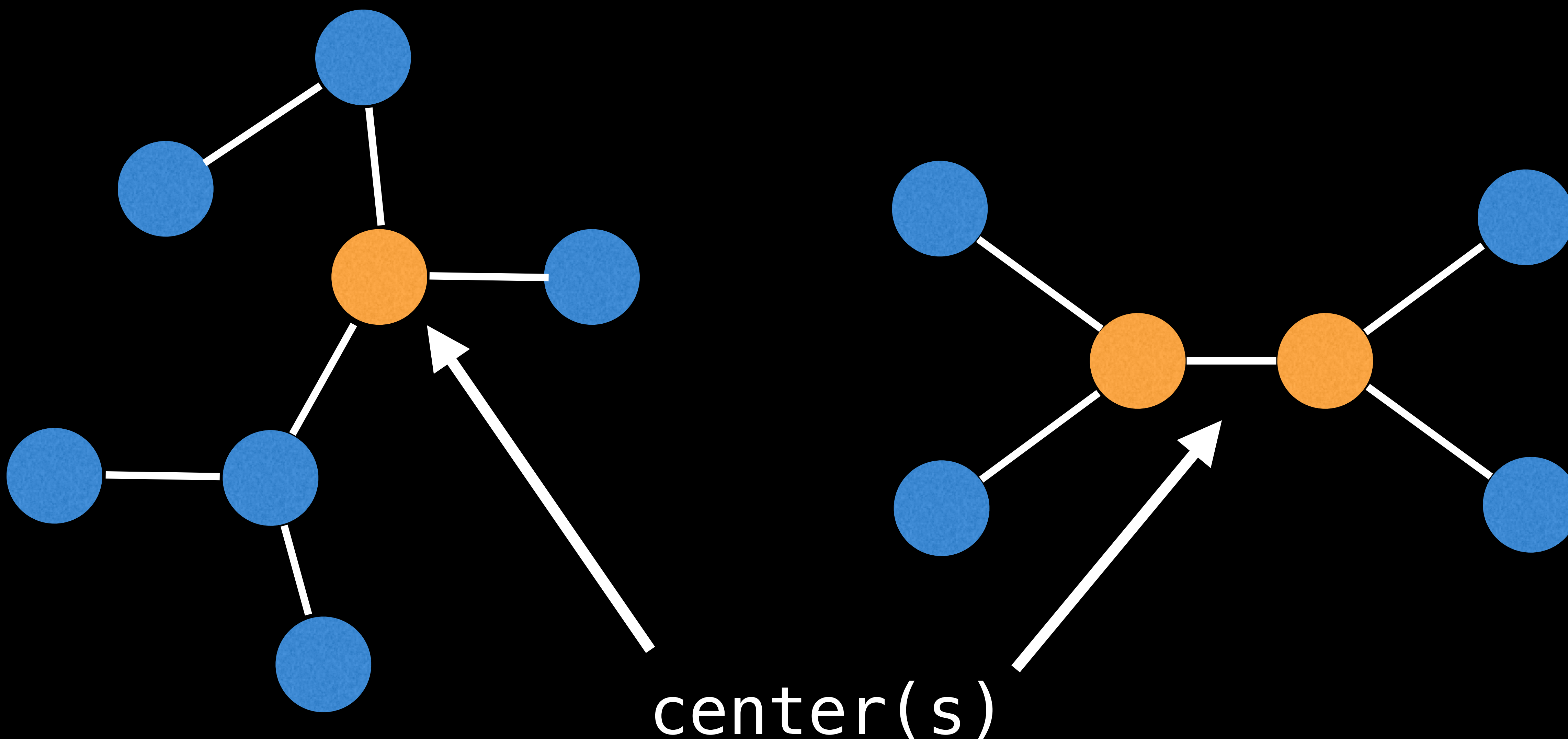

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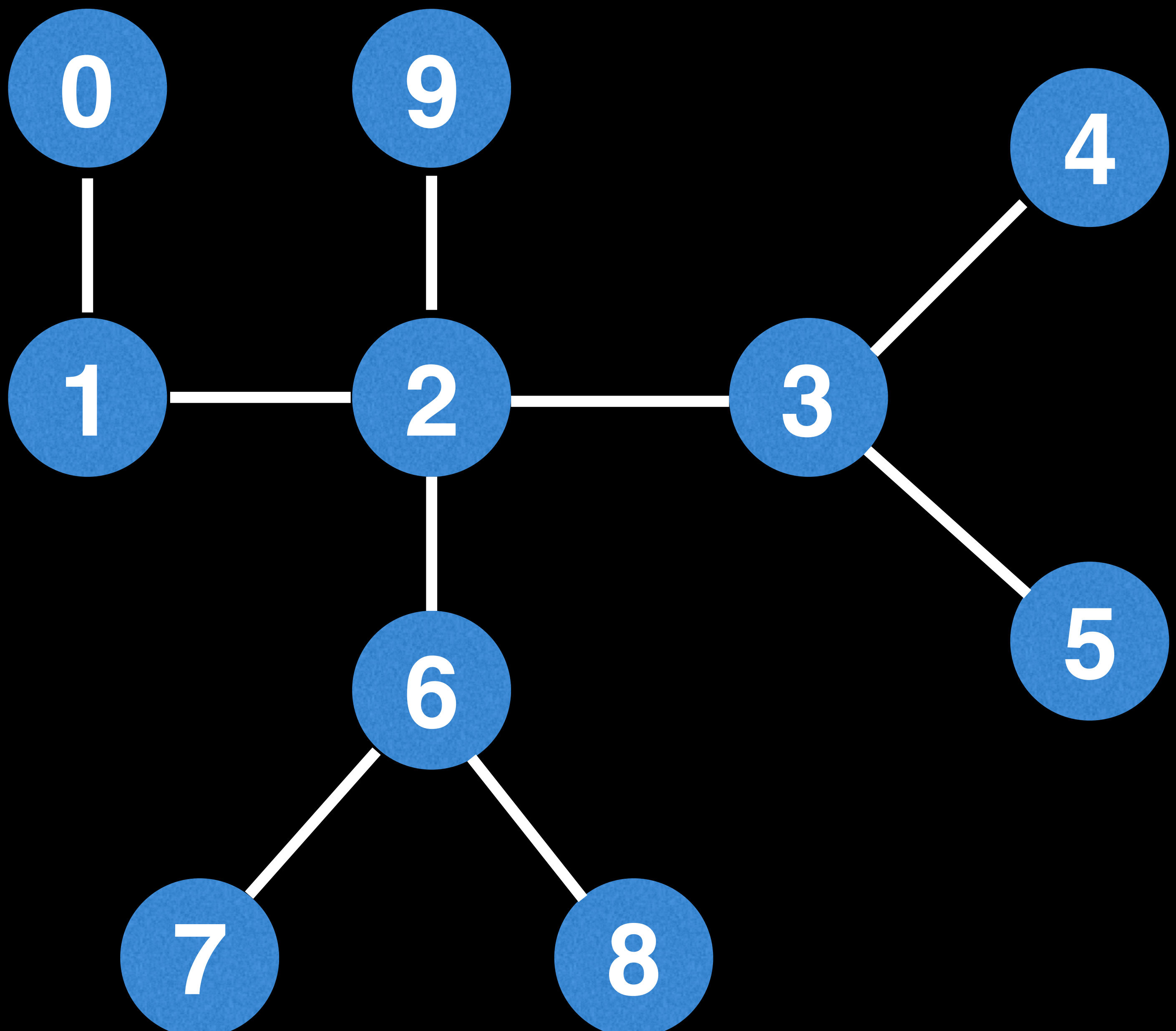
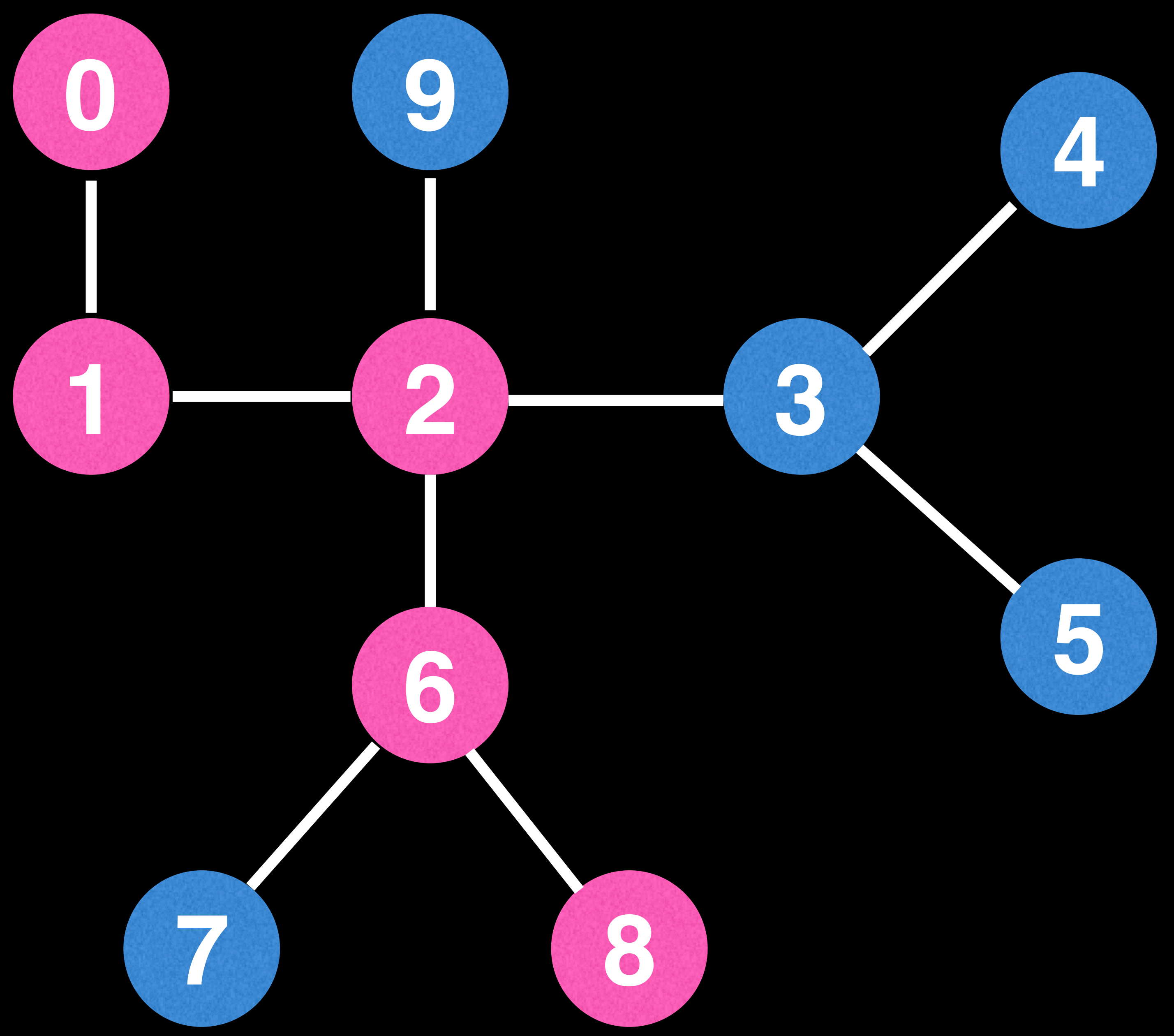


Notice that the center is always the middle vertex or middle two vertices in every longest path along the tree.

Center(s) of undirected tree



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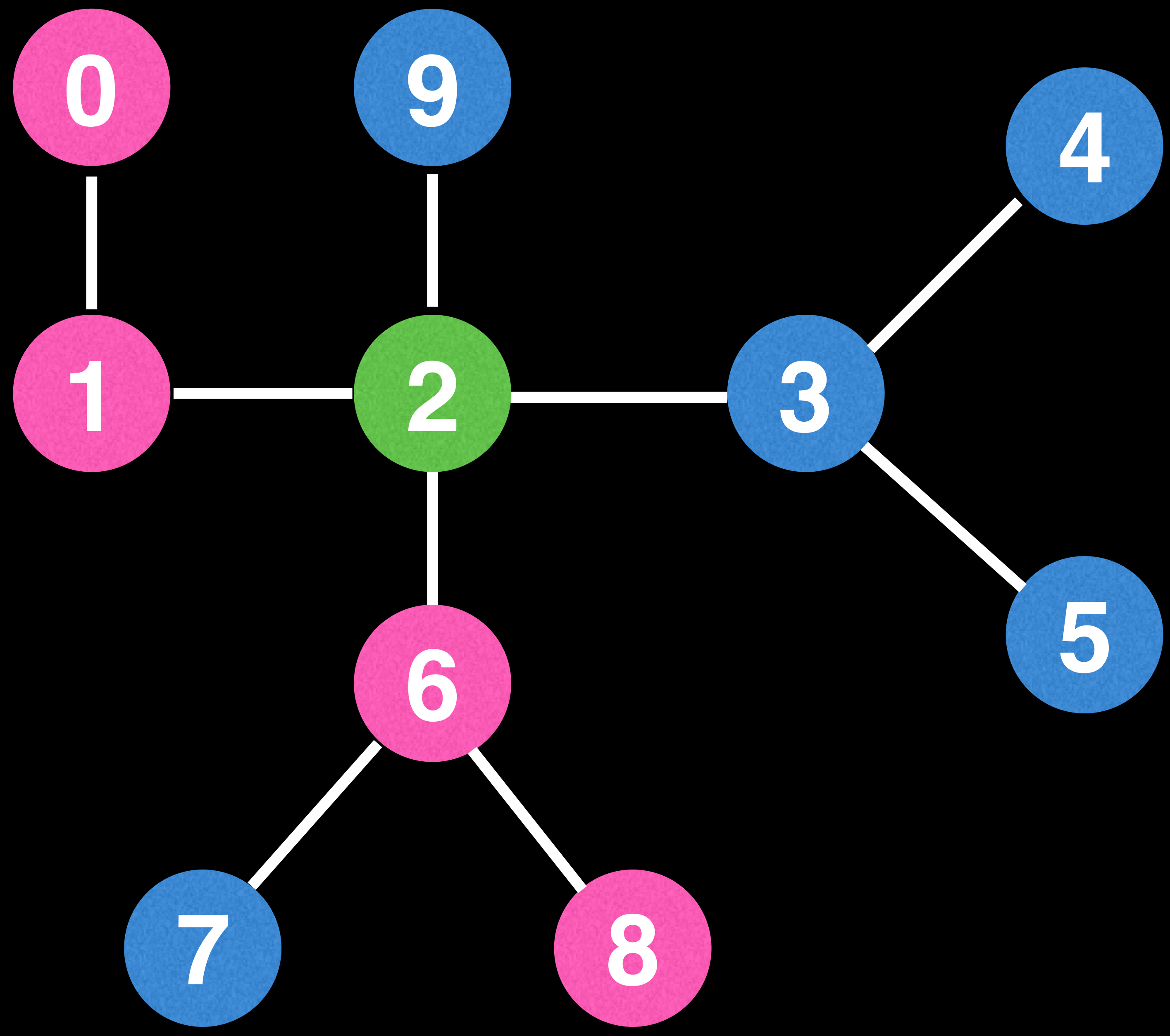

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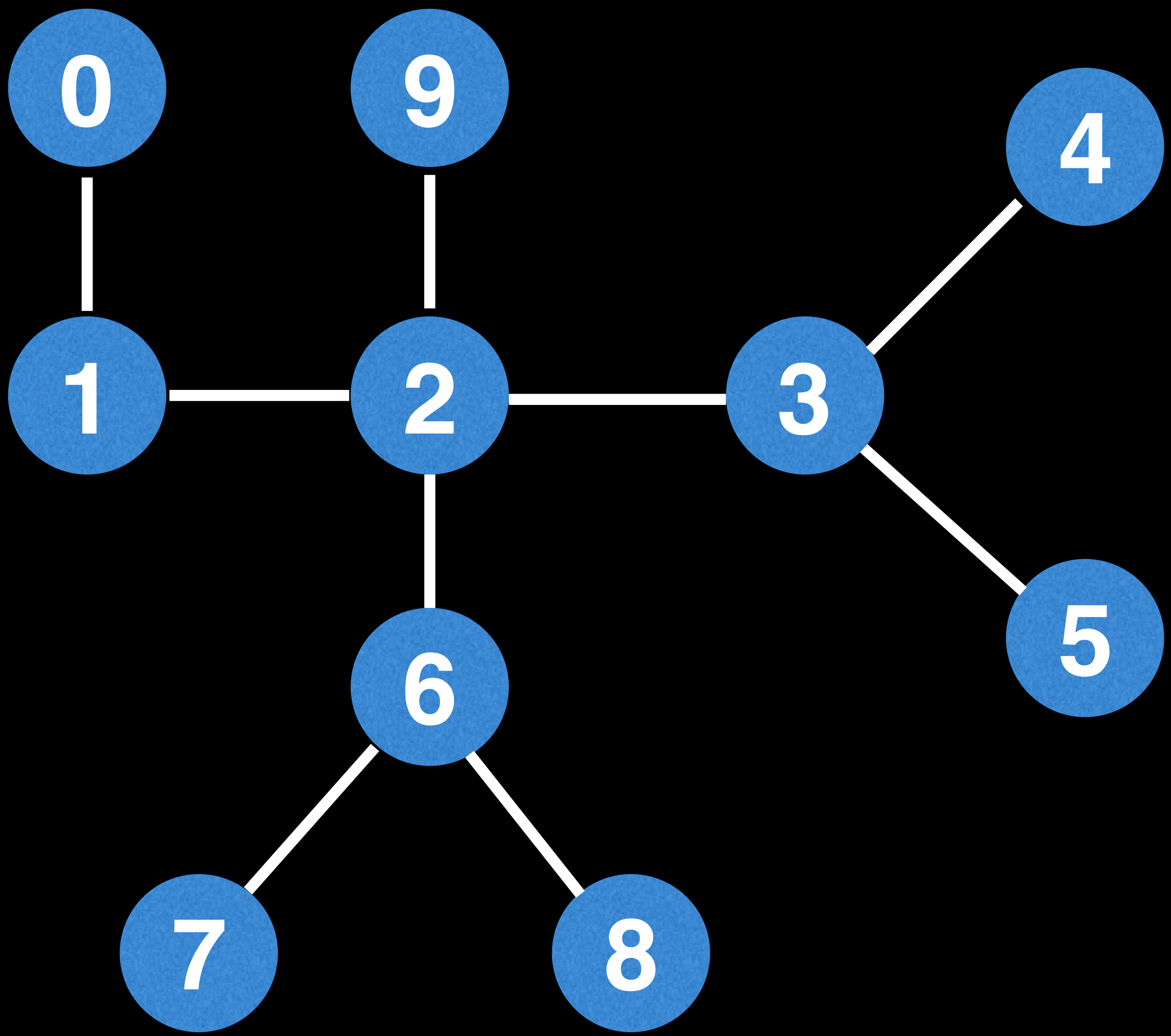

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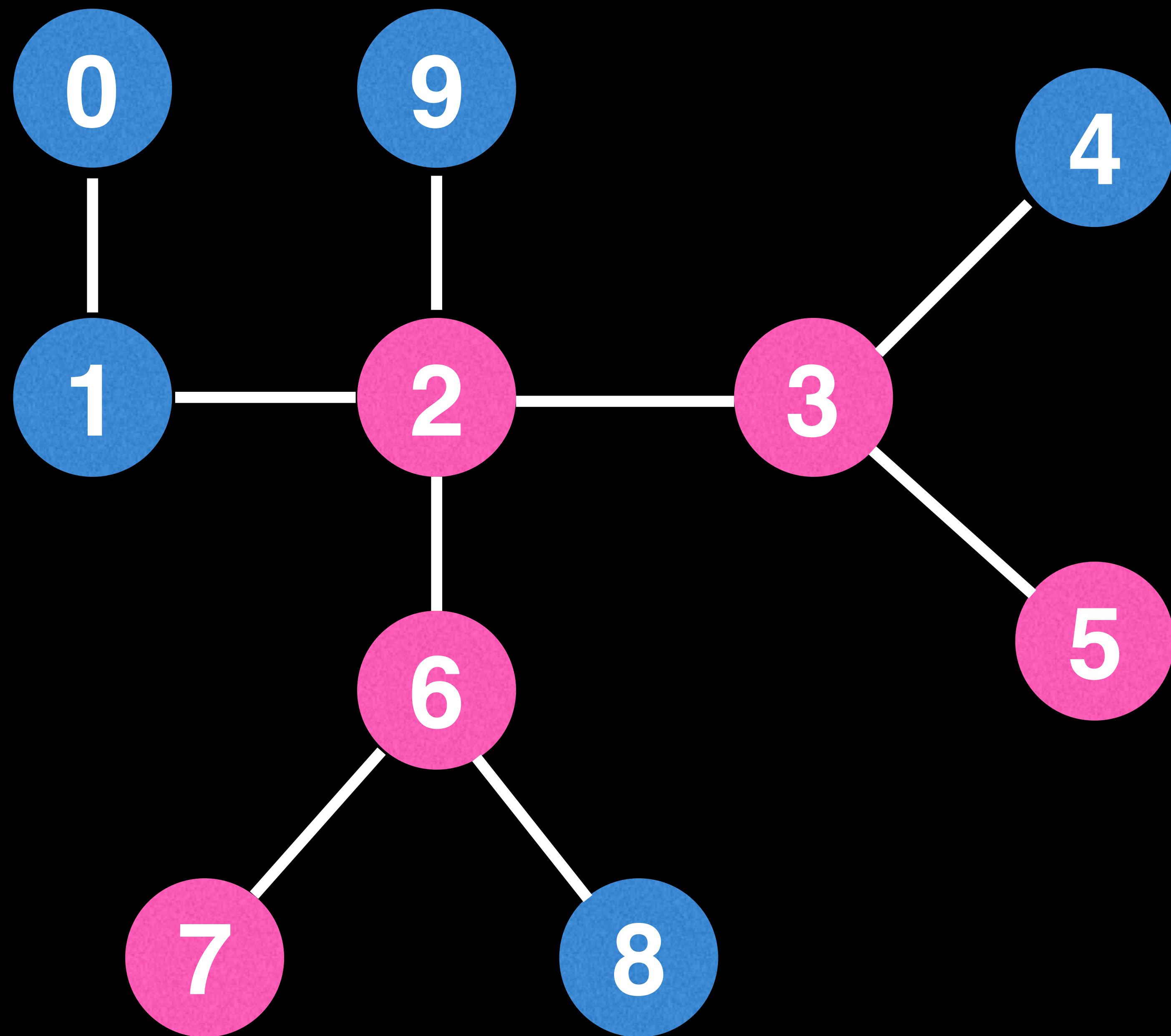

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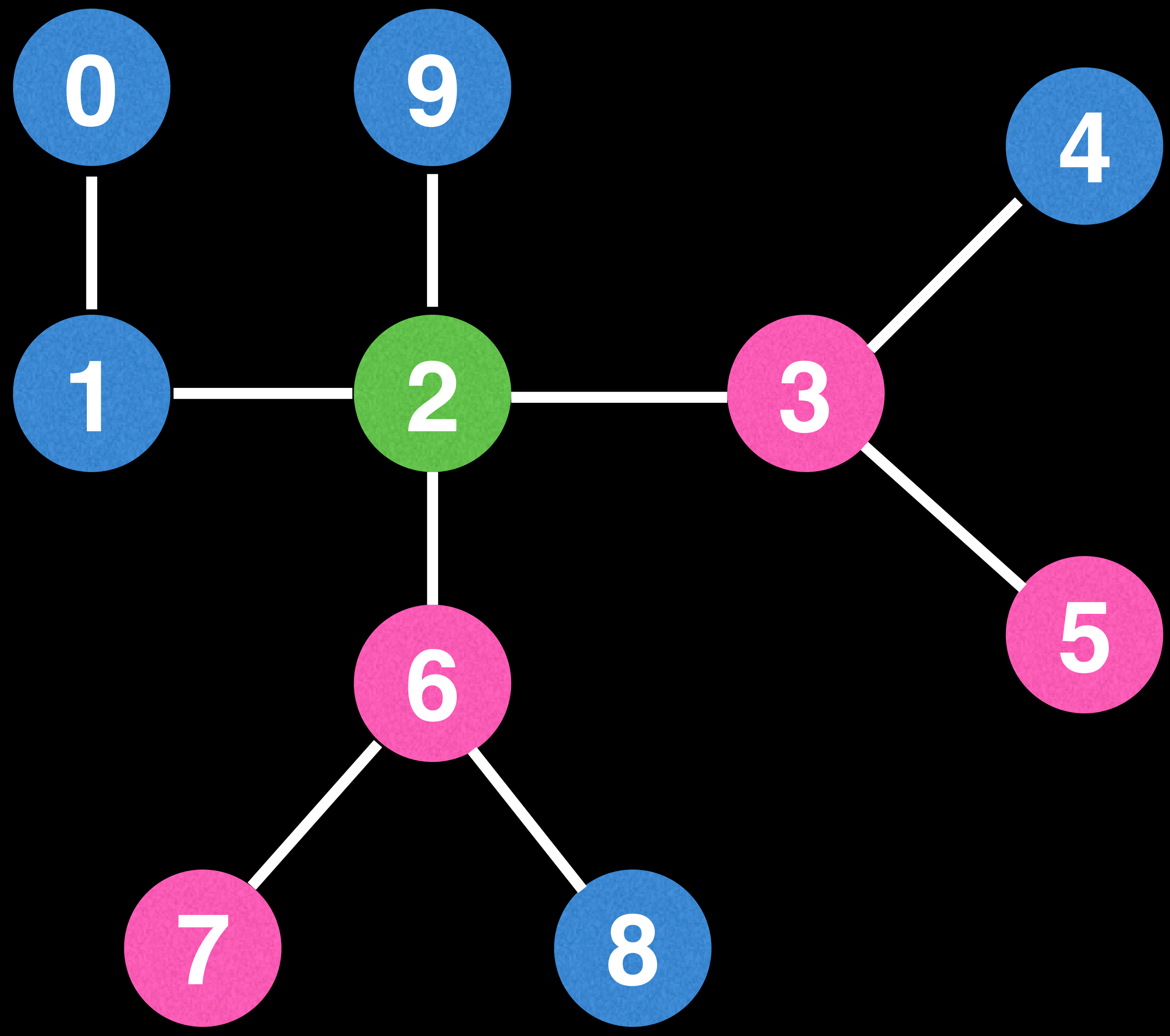

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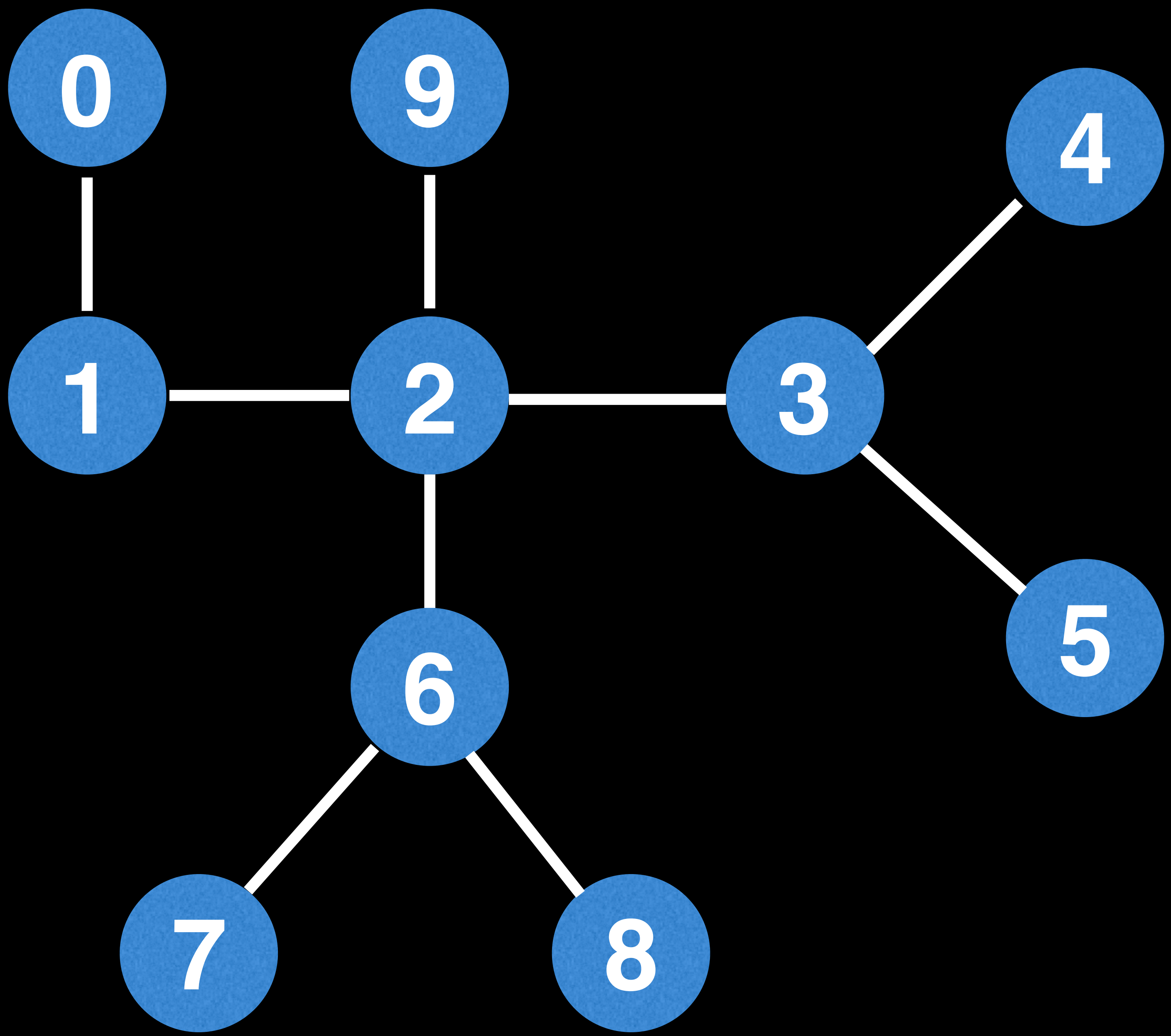
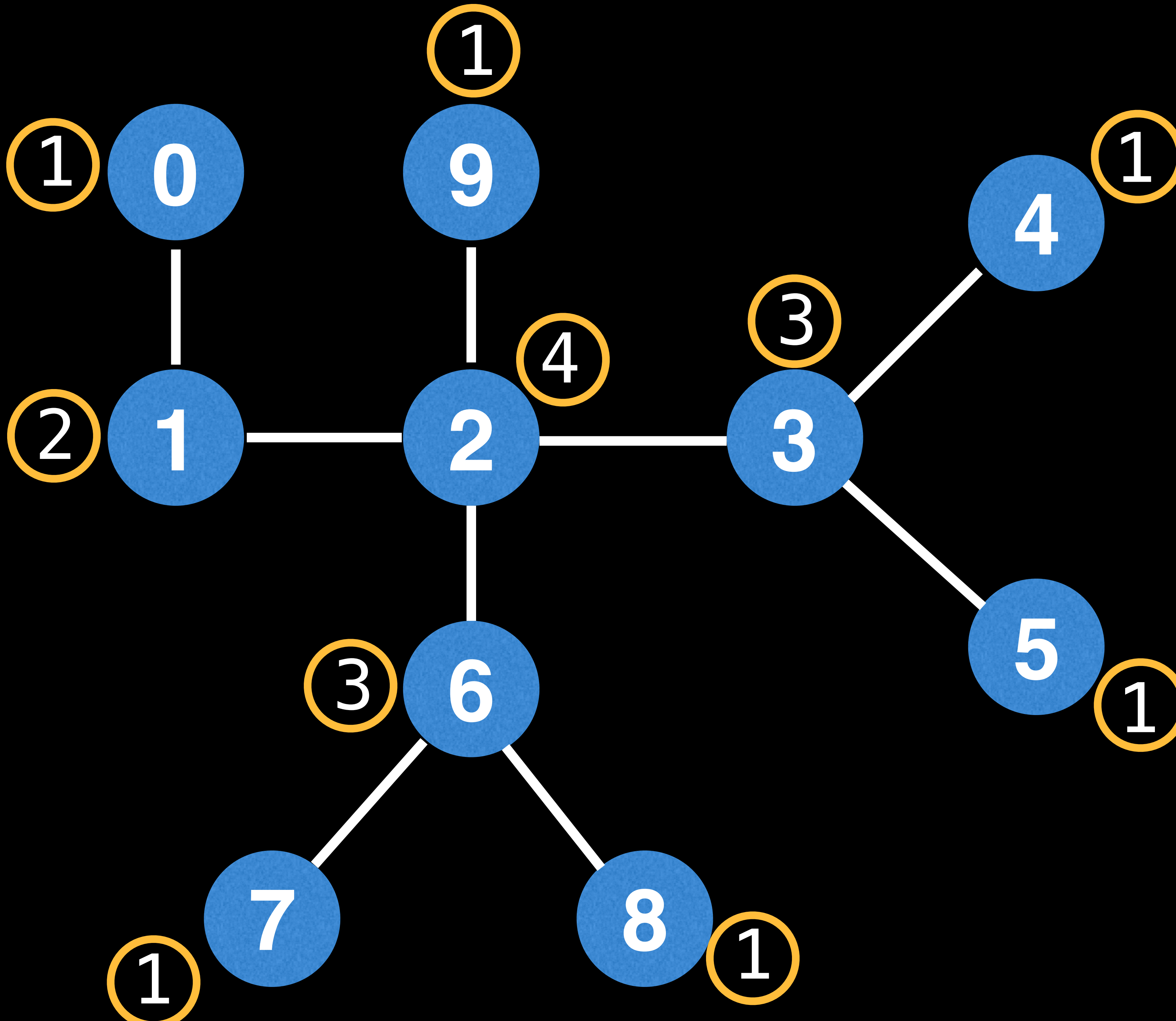


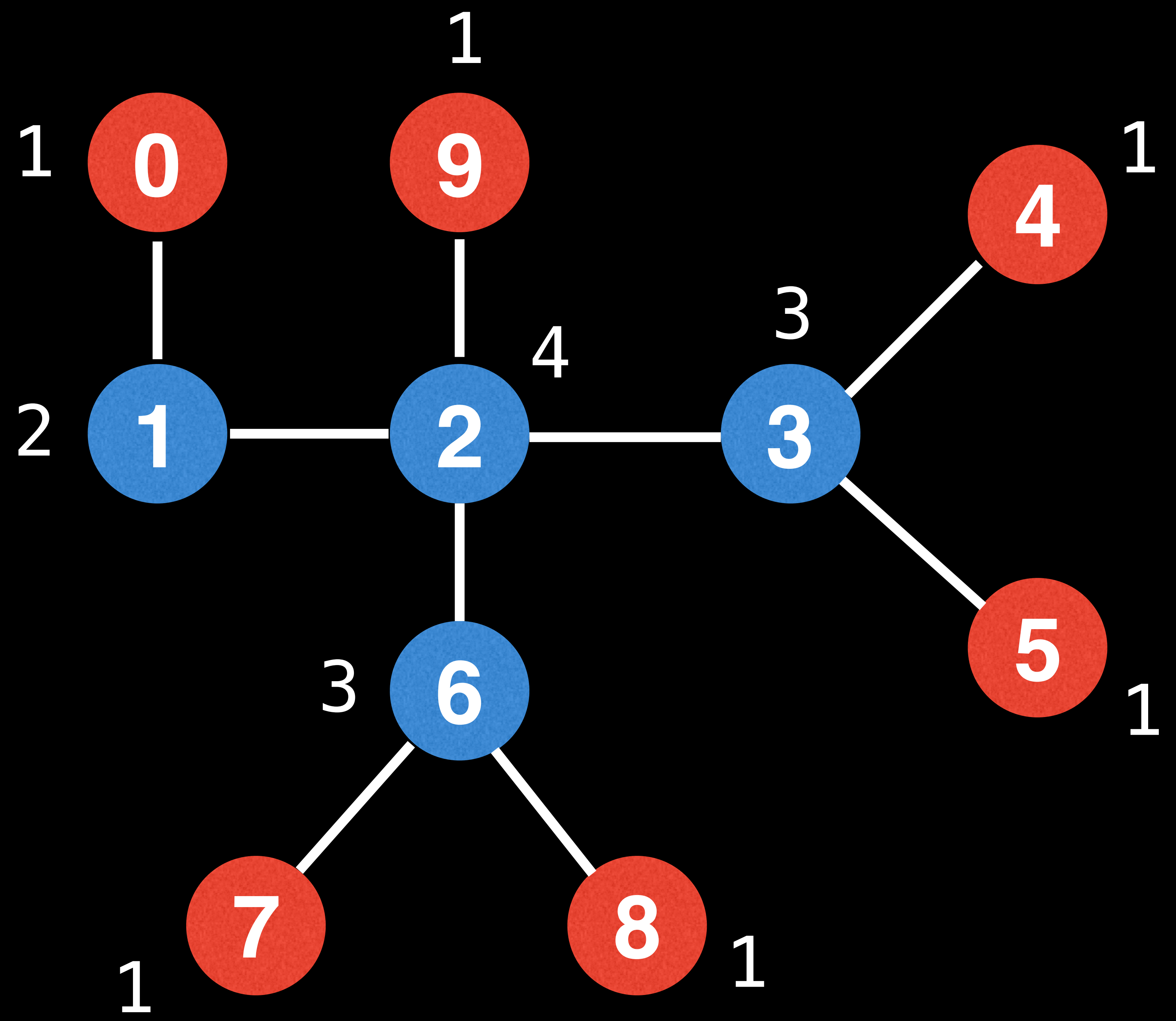
Another approach to find the center is to iteratively pick off each leaf node layer like we were peeling an onion.

Center(s) of undirected tree



The orange circles represent the **degree** of each node. Observe that each leaf node will have a degree of 1.

Center(s) of undirected tree



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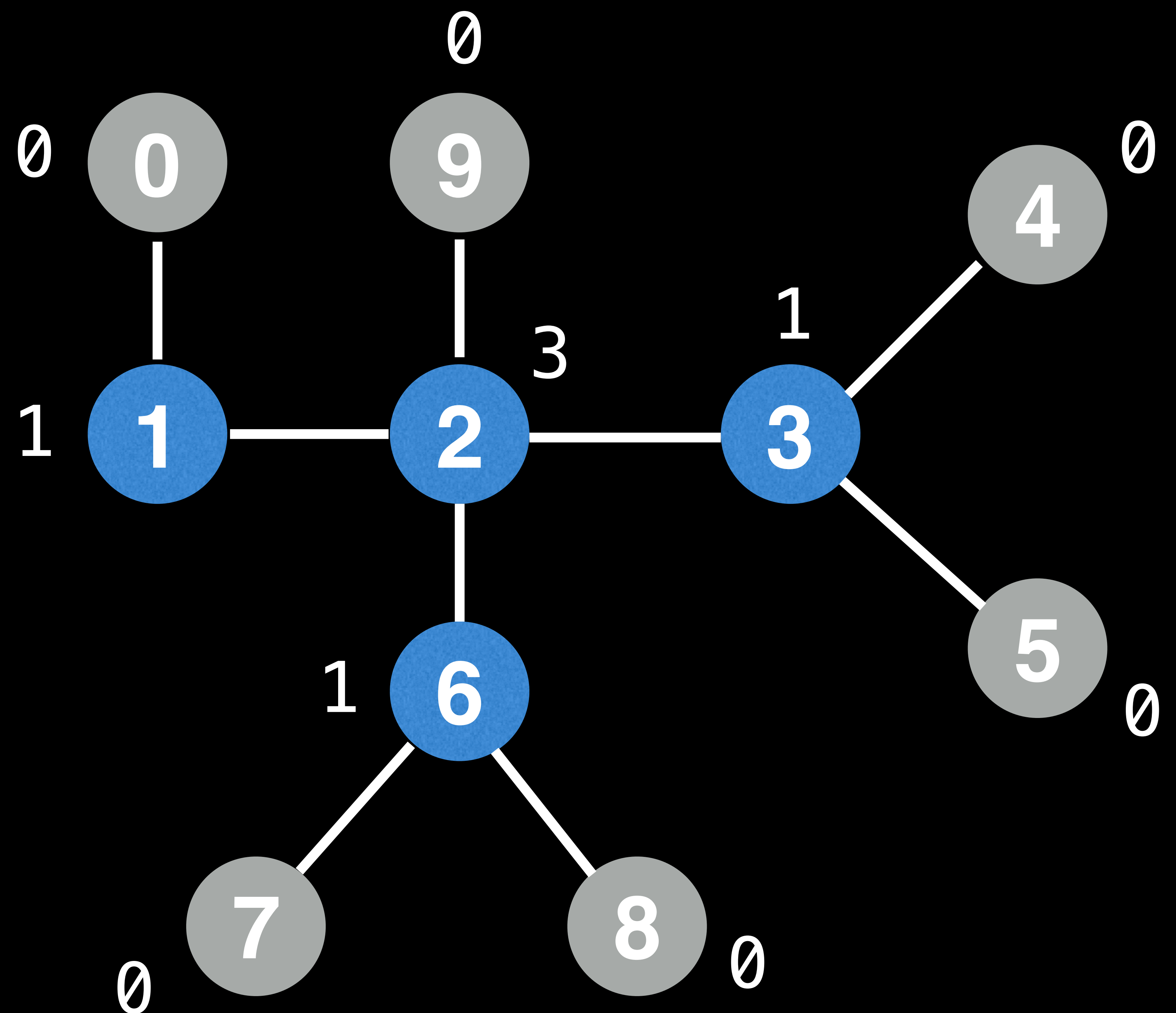
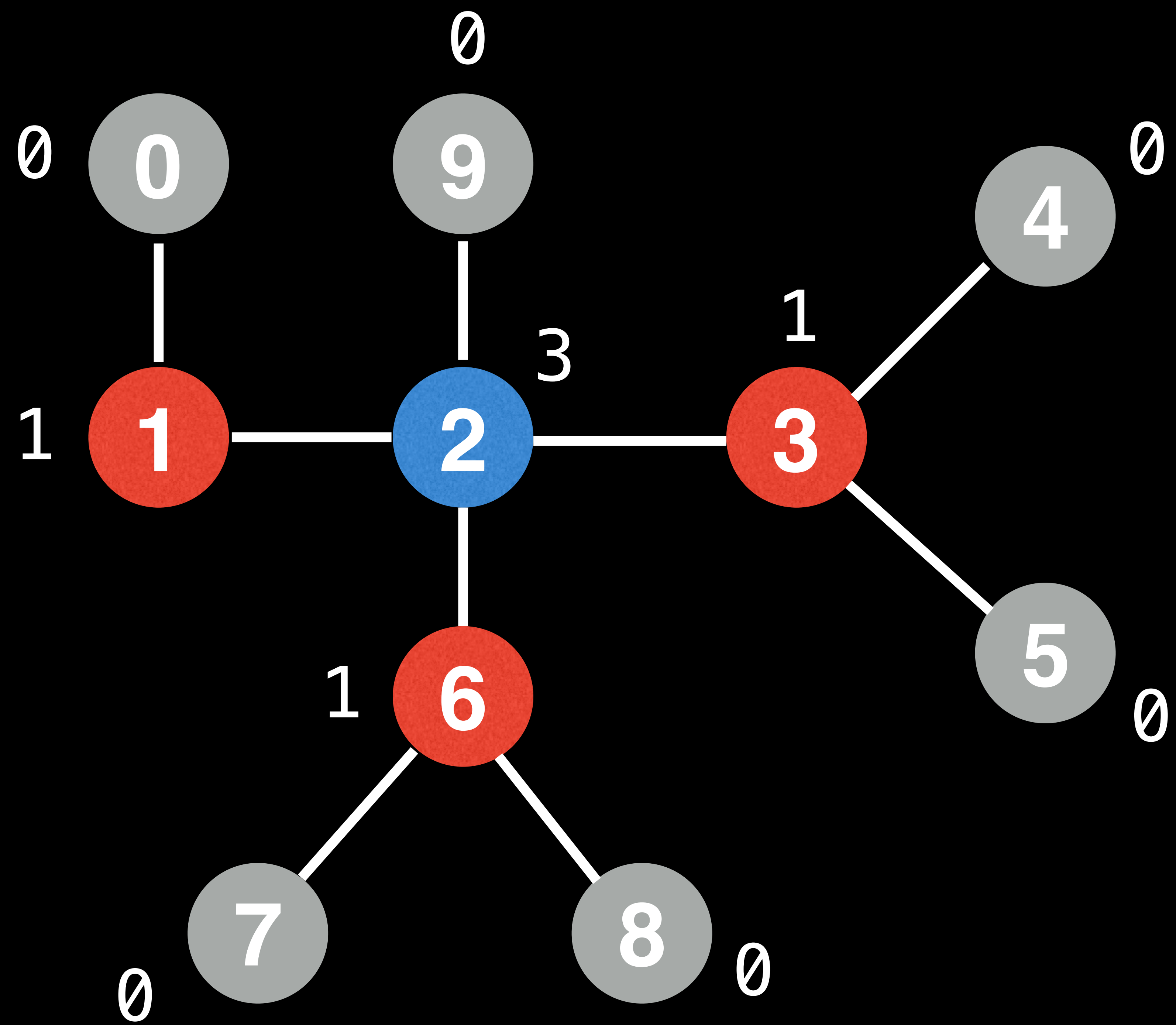


As we prune nodes also reduce the node degree values.

Center(s) of undirected tree



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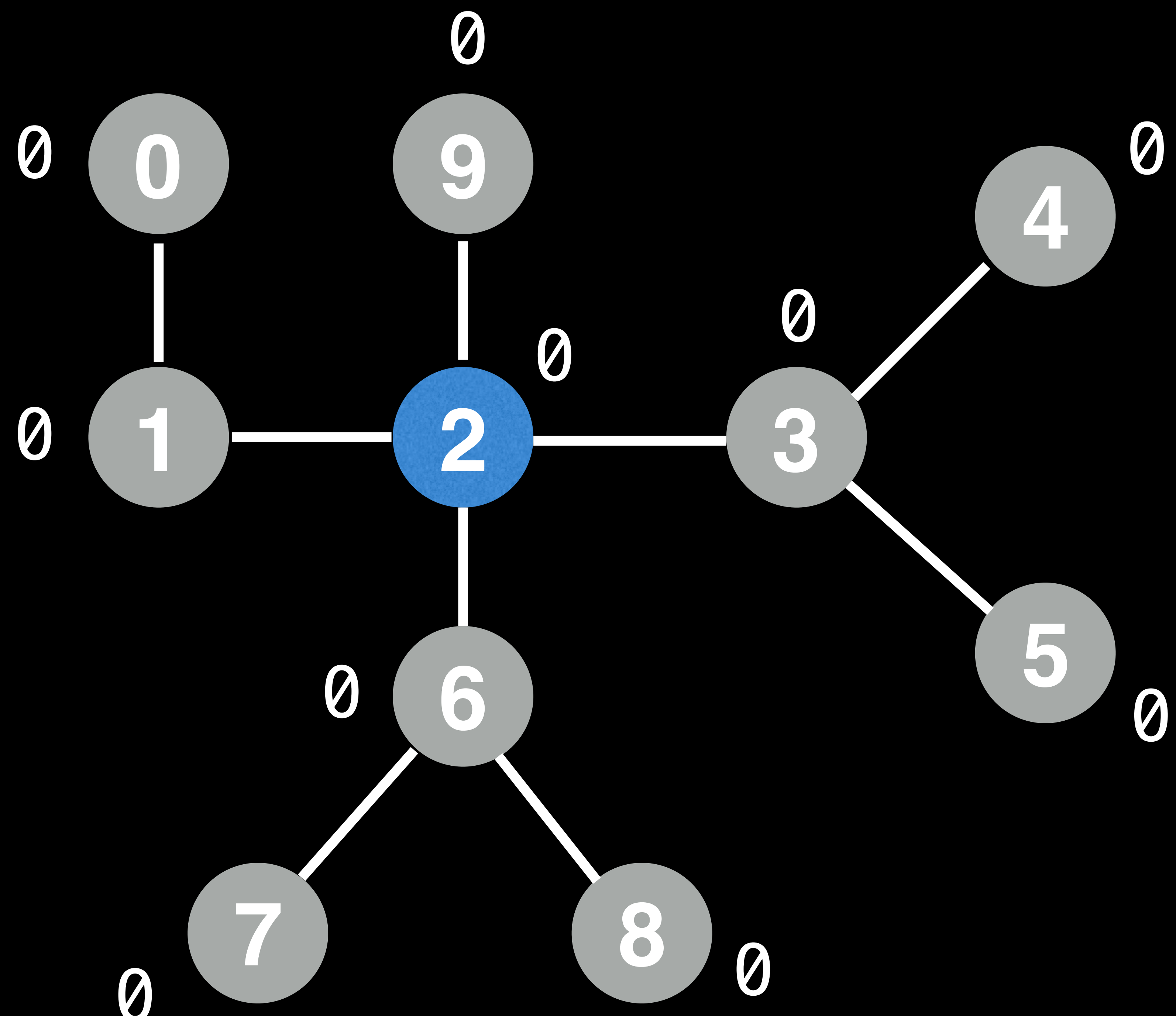

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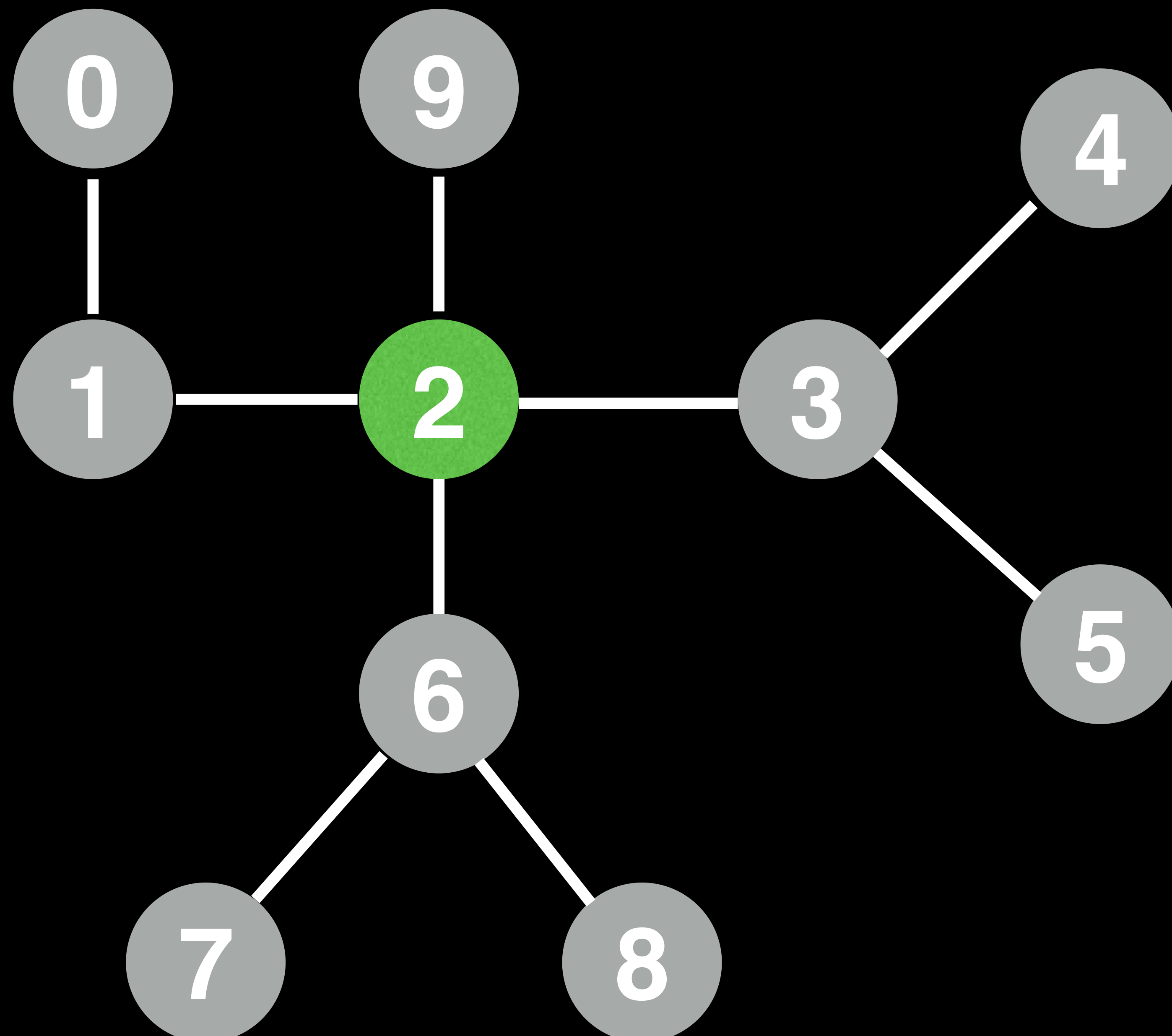

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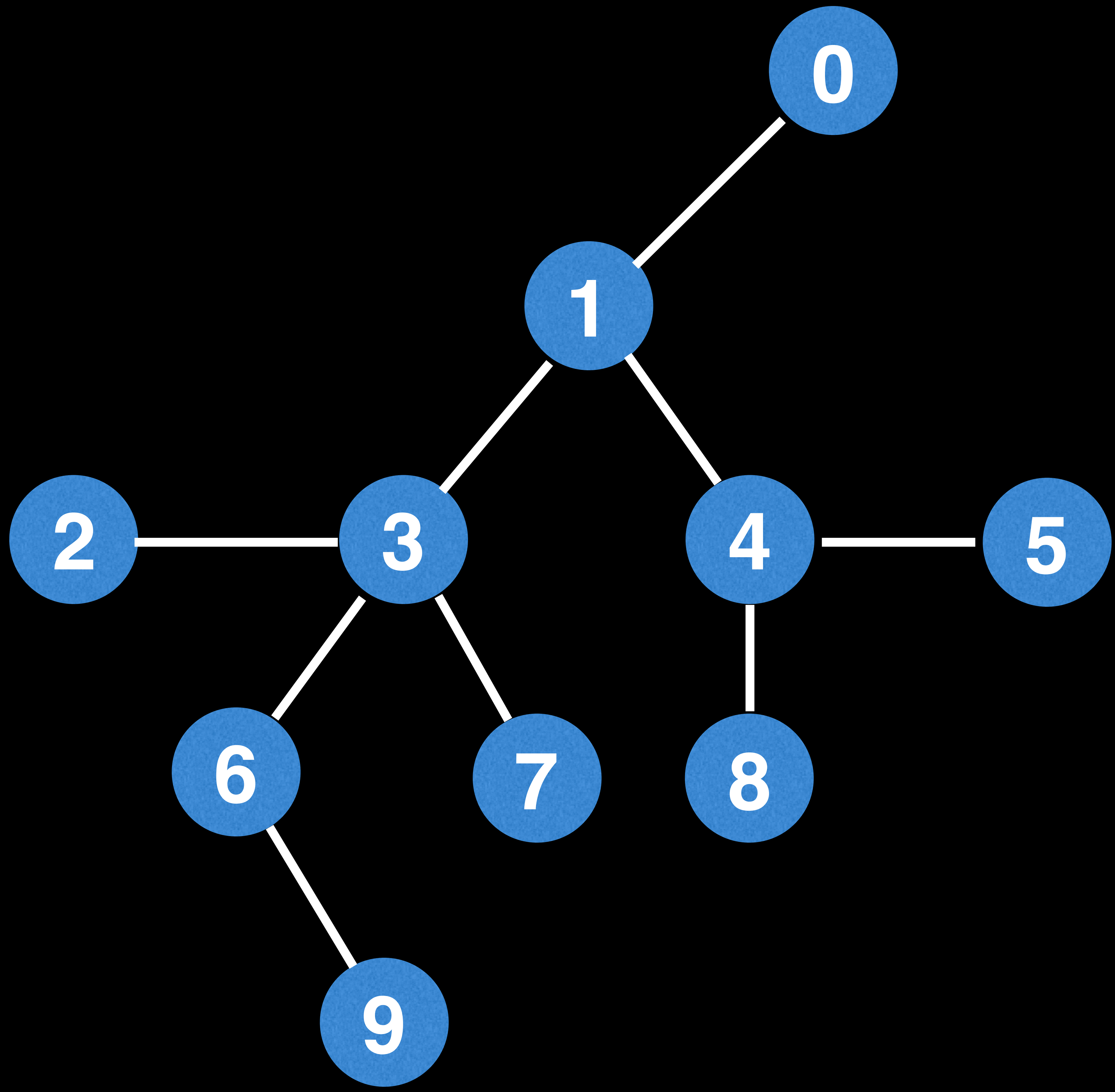

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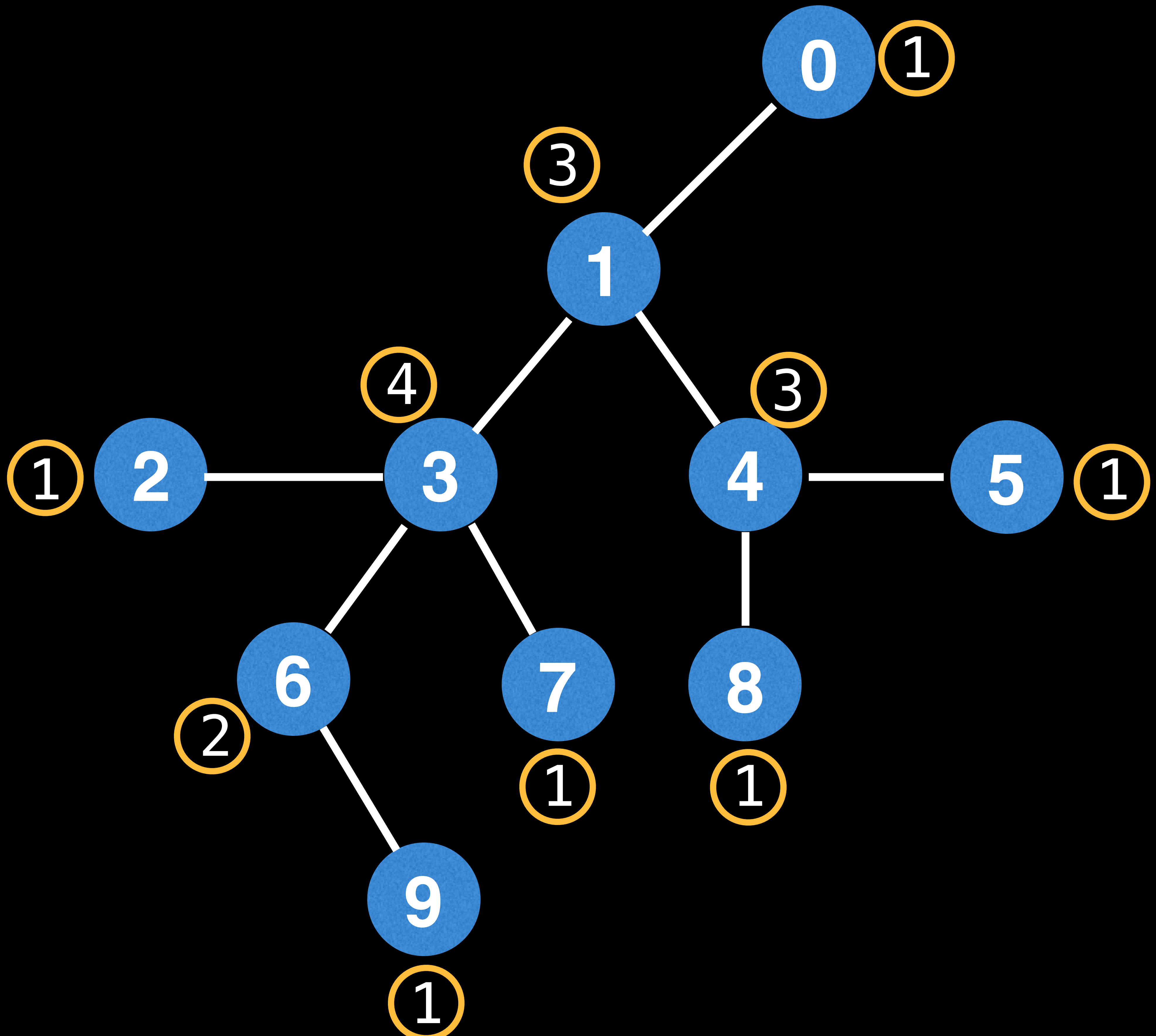

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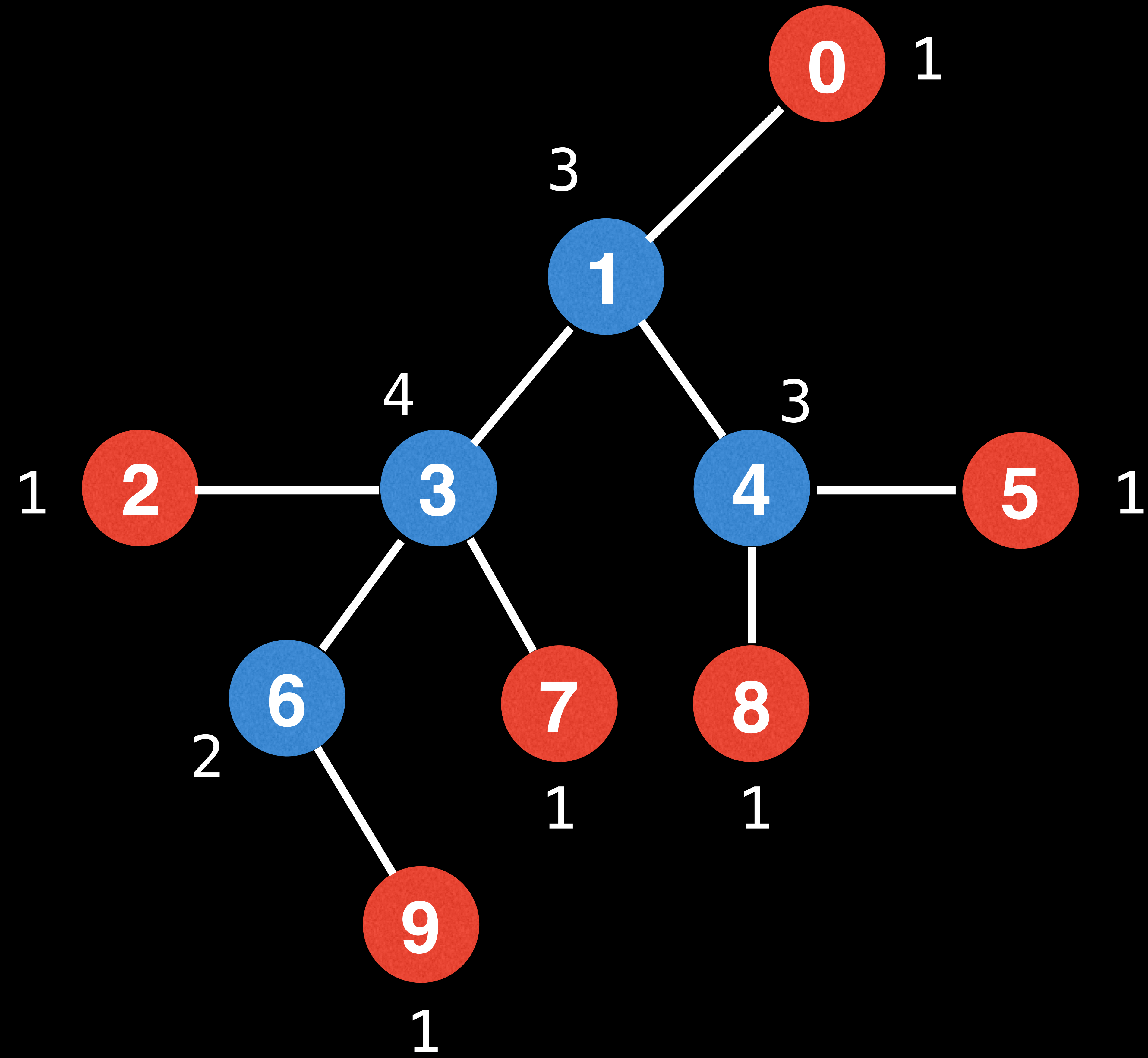

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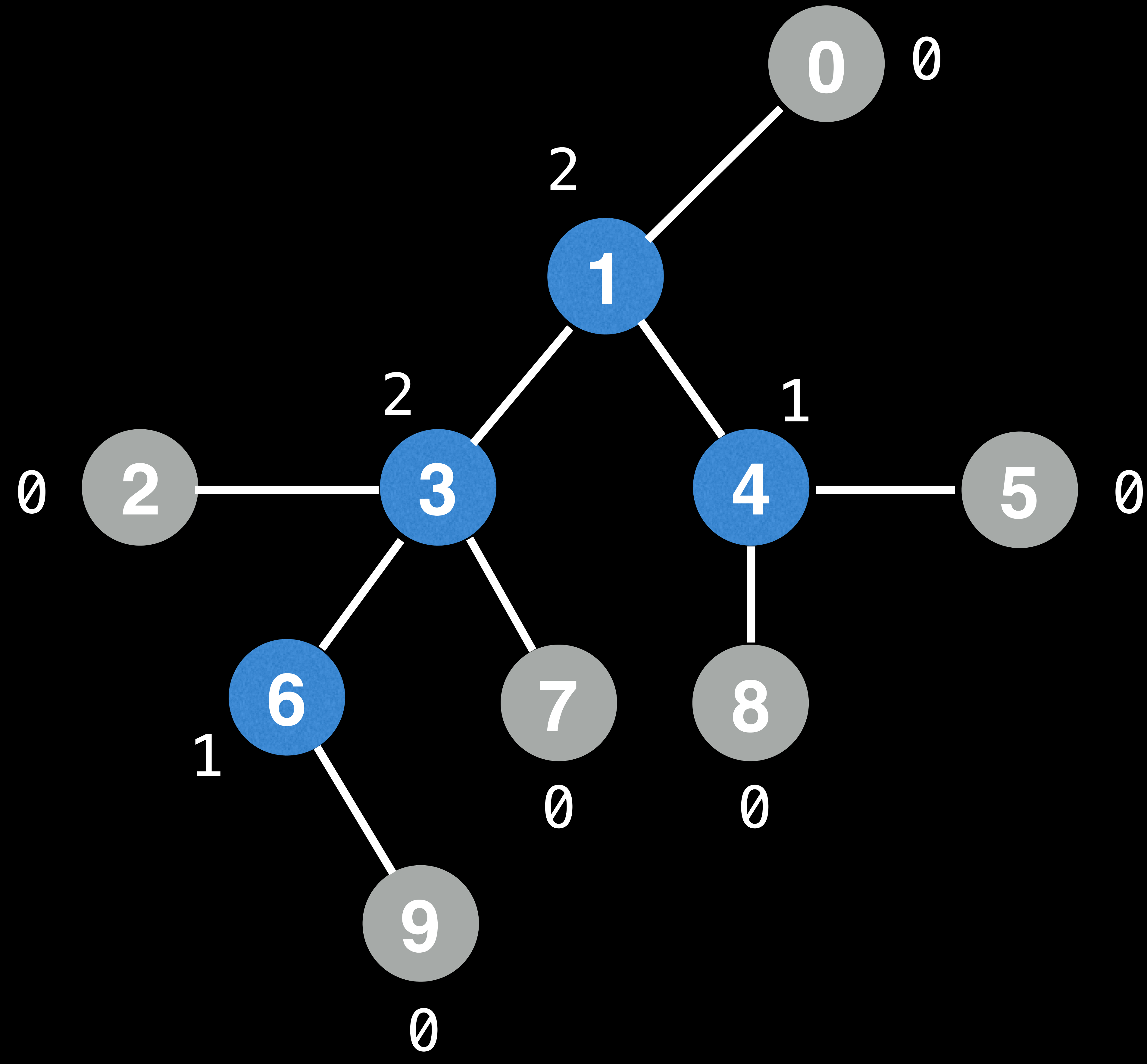

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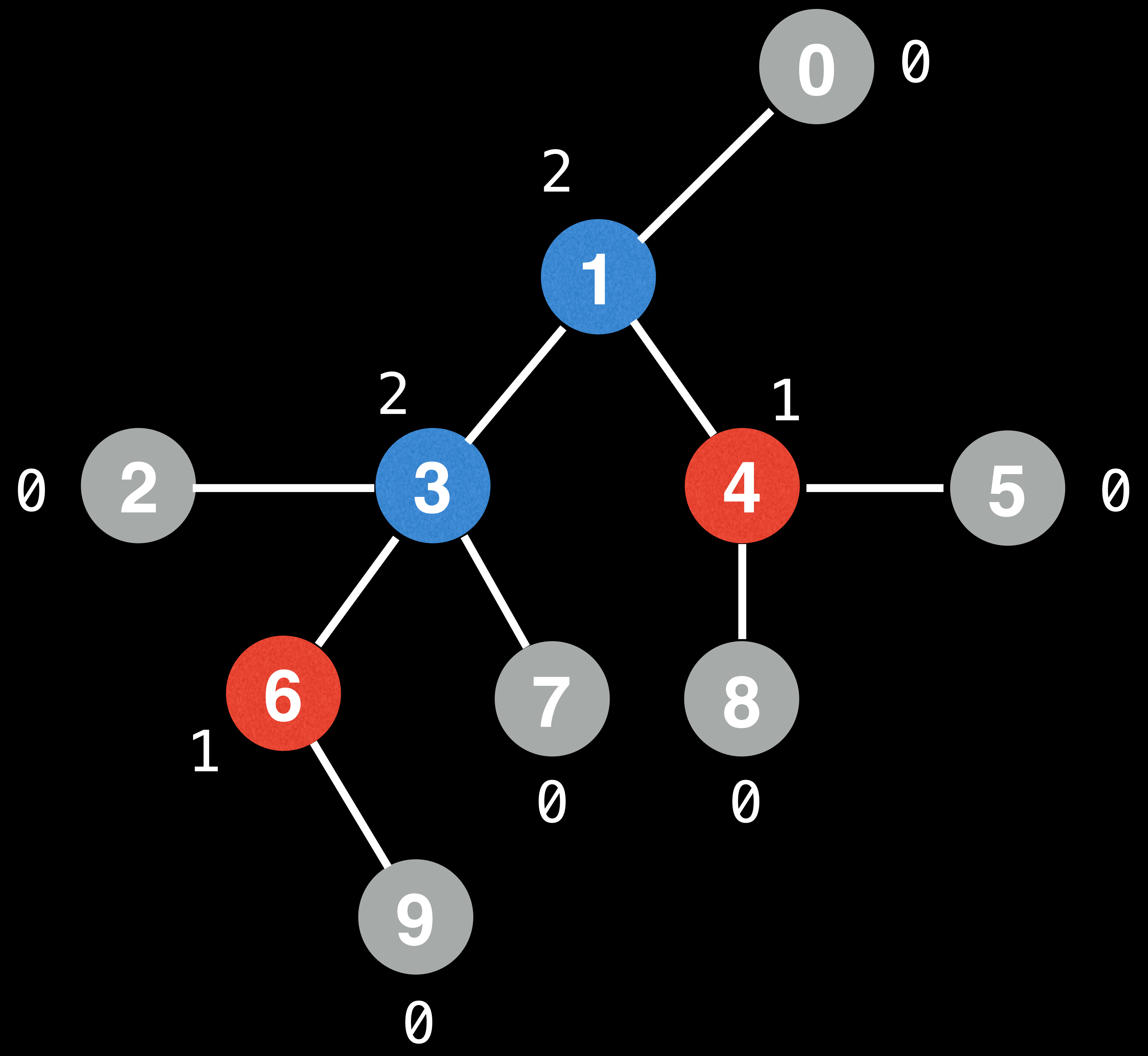

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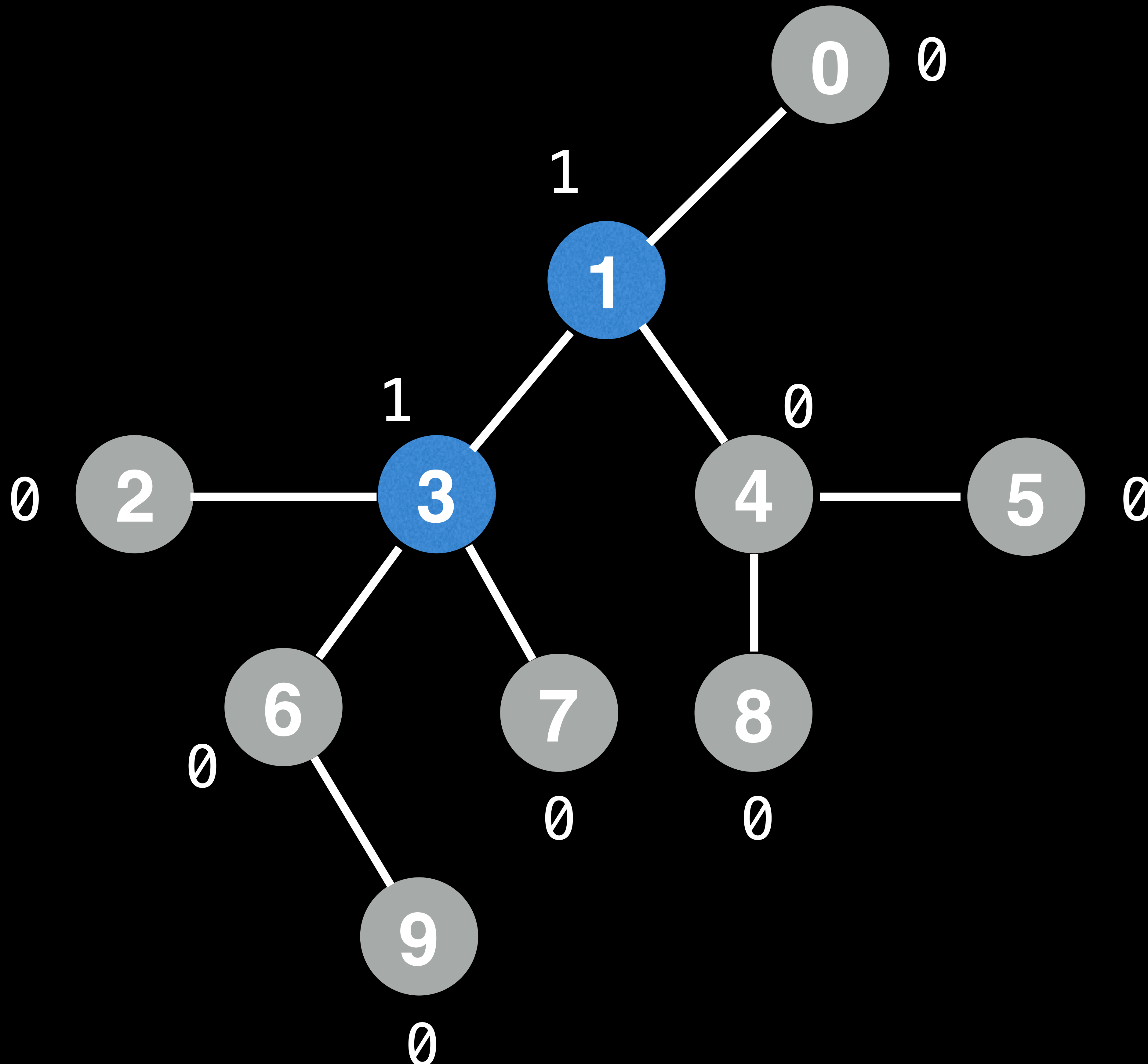

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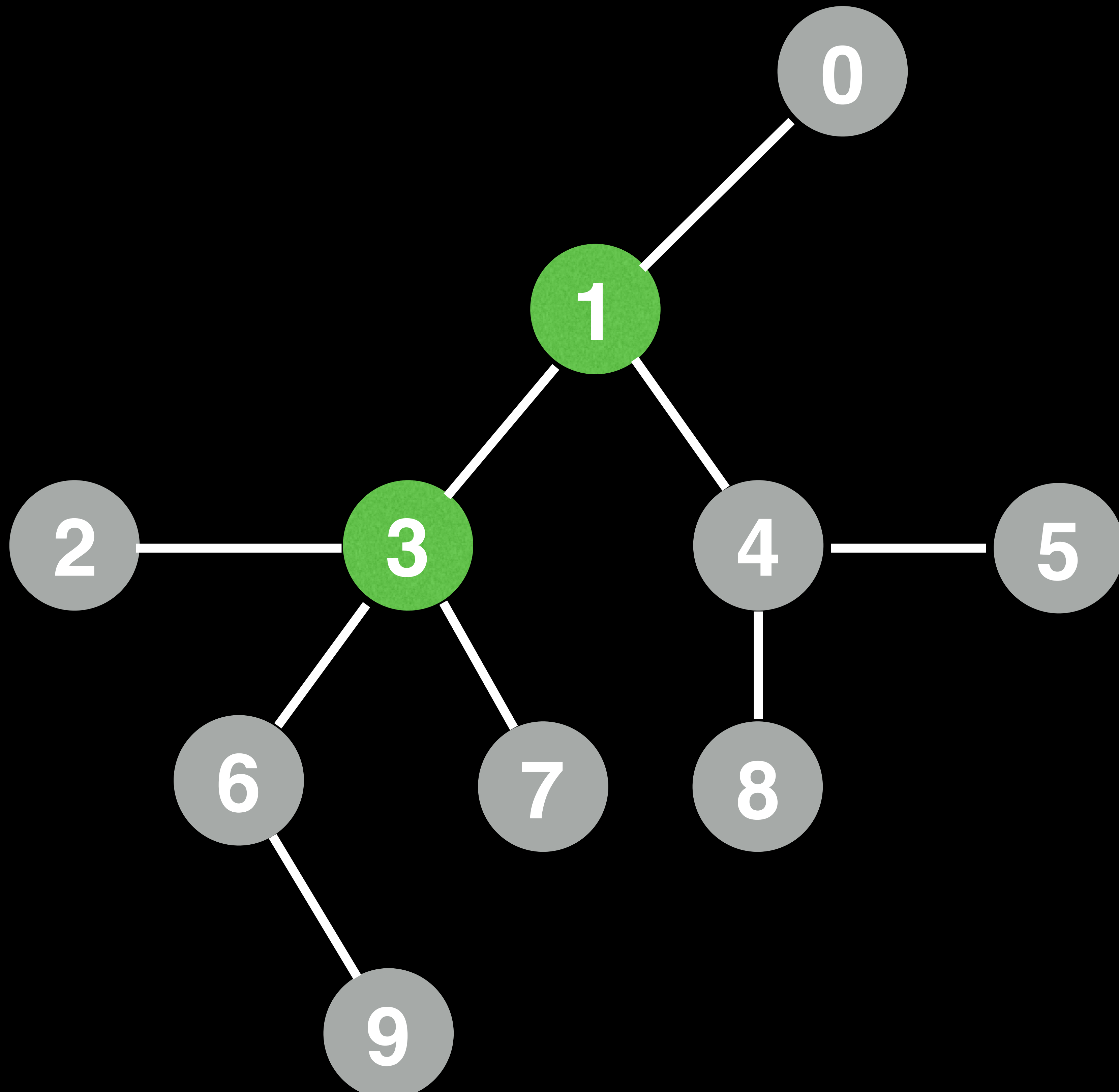

Some trees have two centers

```
# g = tree represented as an undirected graph
function treeCenters(g):
  n = g.numberOfNodes()
  degree = [0, 0, …, 0] # size n
  leaves = []
  for (i = 0; i < n; i++):
    degree[i] = g[i].size()
    if degree[i] == 0 or degree[i] == 1:
      leaves.add(i)
      degree[i] = 0
  count = leaves.size()
  while count < n:
    new_leaves = []
    for (node : leaves):
      for (neighbor : g[node]):
        degree[neighbor] = degree[neighbor] - 1
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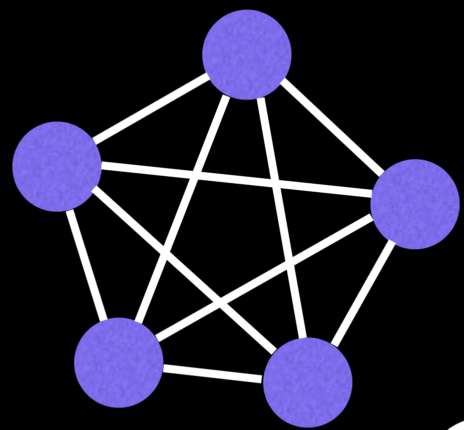
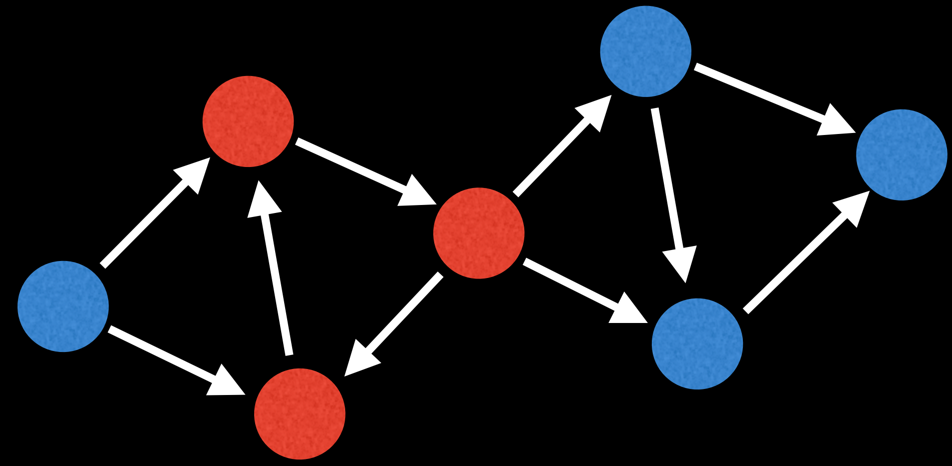
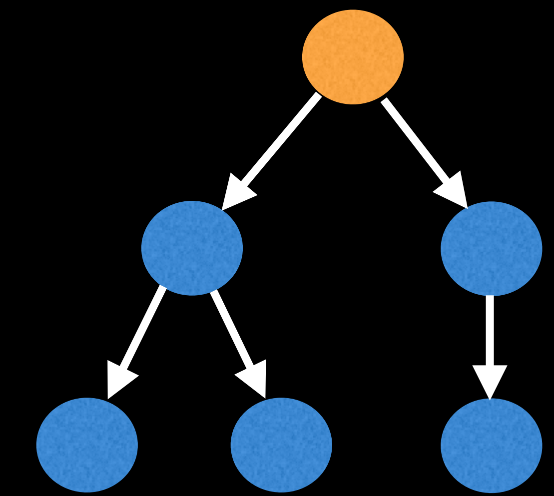
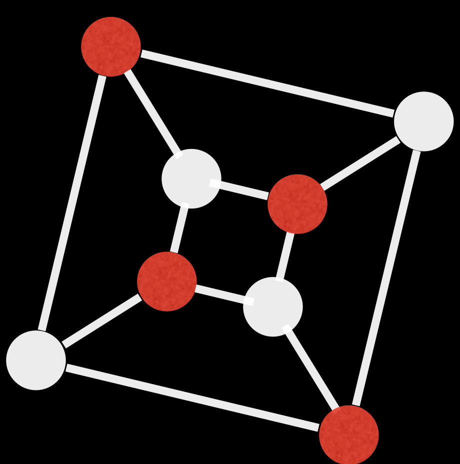
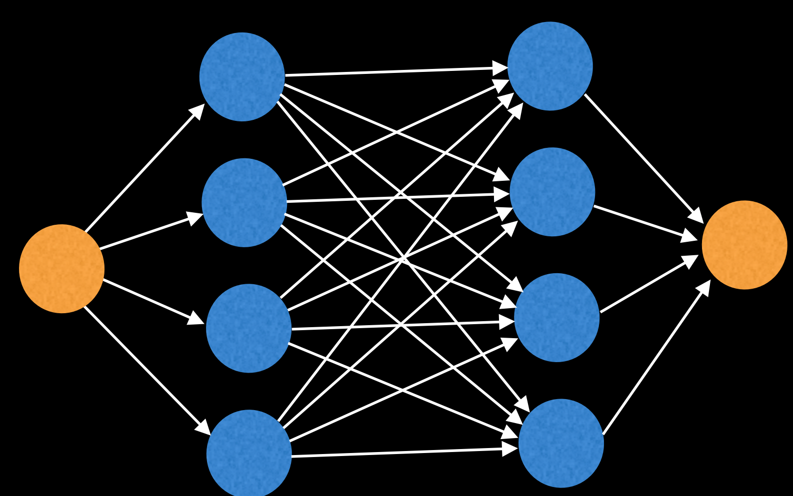


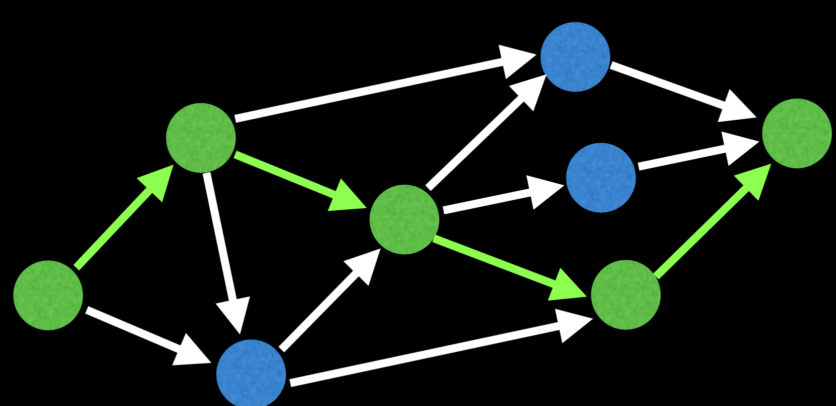




Graph Theory Video Series







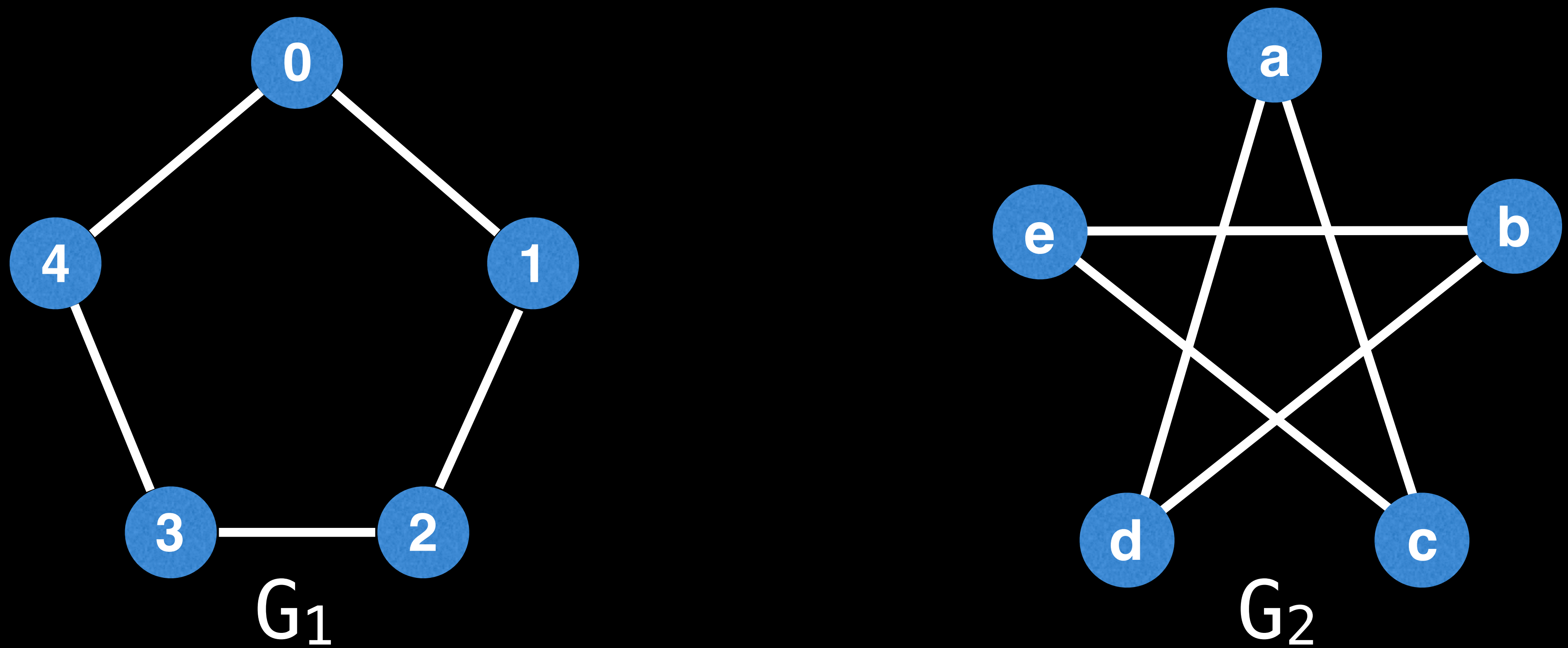
Isomorphisms in trees

A question of equality

William Fiset

Graph Isomorphism

The question of asking whether two graphs G_1 and G_2 are **isomorphic** is asking whether they are *structurally* the same.



Even though G_1 and G_2 are labelled differently and may appear different they are structurally the same graph.

Graph Isomorphism

We can also define the notion of a graph isomorphism more rigorously:

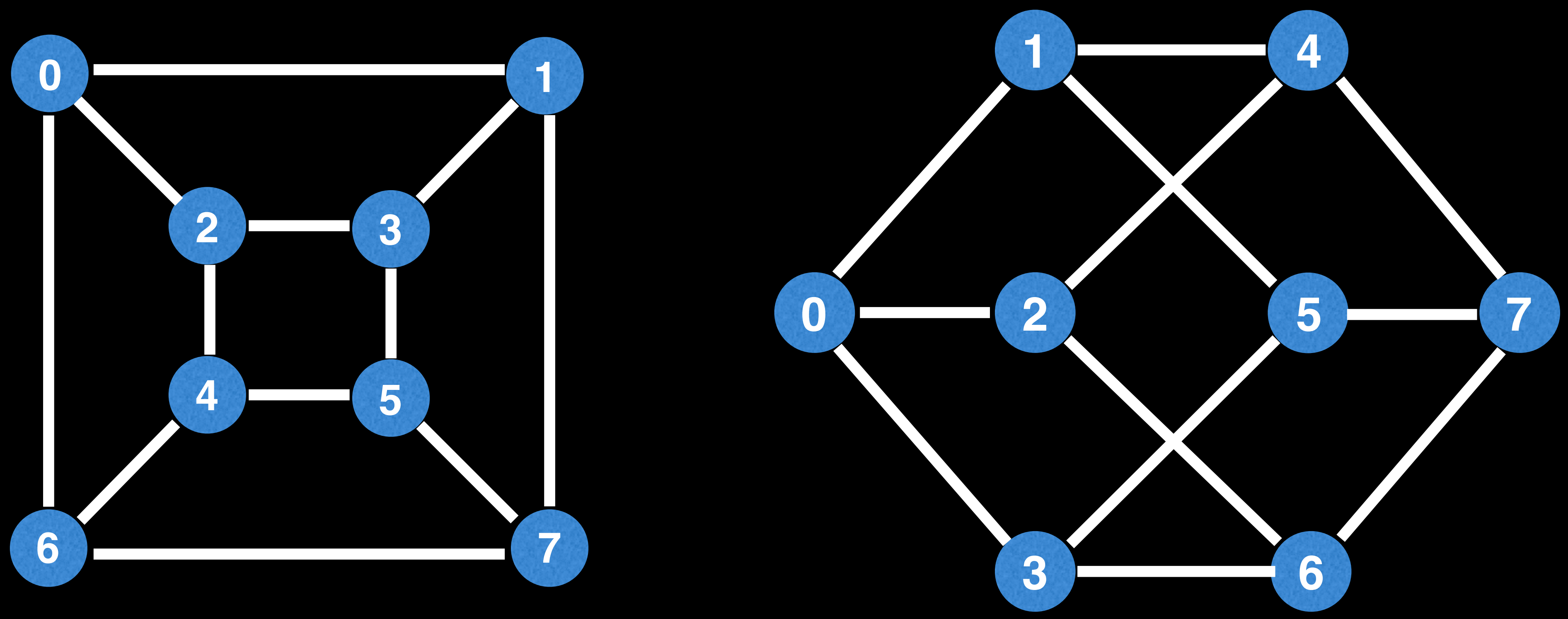
$G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic if there exists a **bijection** φ between the sets $V_1 \rightarrow V_2$ such that:

$$\forall\ u,v \in V_1,\ (u,v) \in E_1 \iff (\varphi(u),\varphi(v)) \in E_2$$

In simple terms, for an isomorphism to exist there needs to be a function φ which can map all the nodes/edges in G_1 to G_2 and vice-versa.

Graph Isomorphism

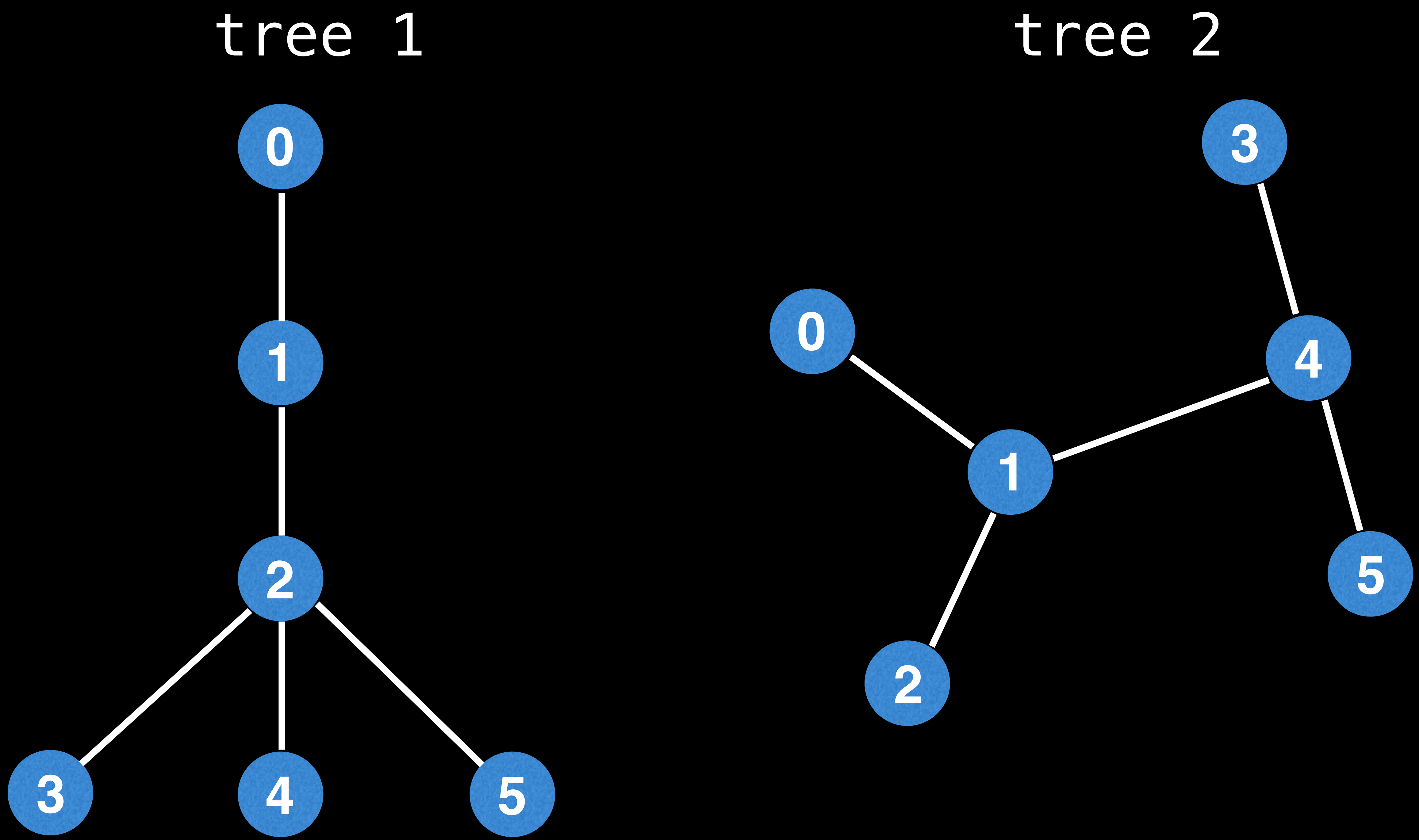
Determining if two graphs are isomorphic is not only not obvious to the human eye, but also a difficult problem for computers.



It is still an open question as to whether the graph isomorphism problem is NP complete. However, many polynomial time isomorphism algorithms exist for graph subclasses such as trees.

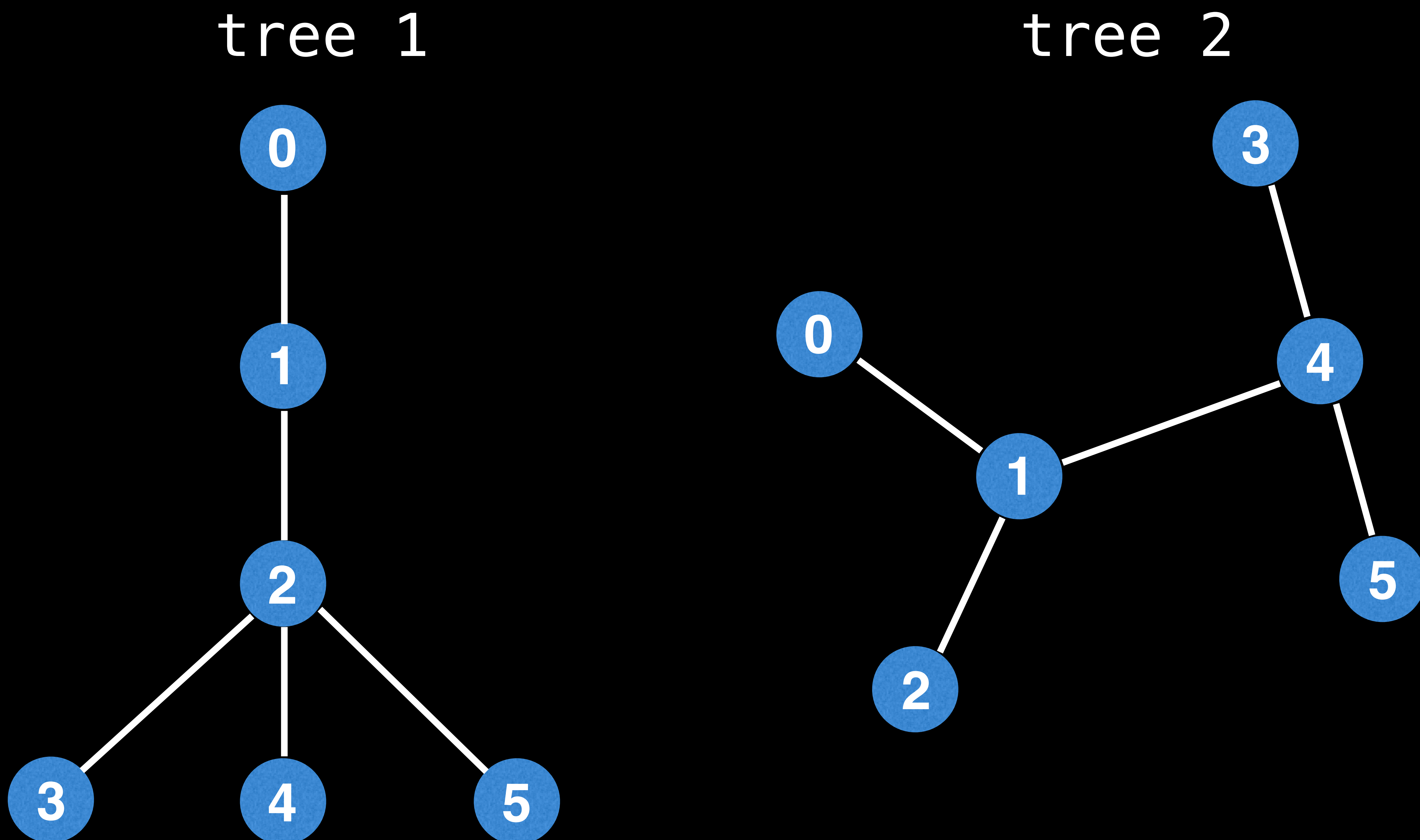
Isomorphic Trees

Isomorphic Trees



Q: Are these trees isomorphic?

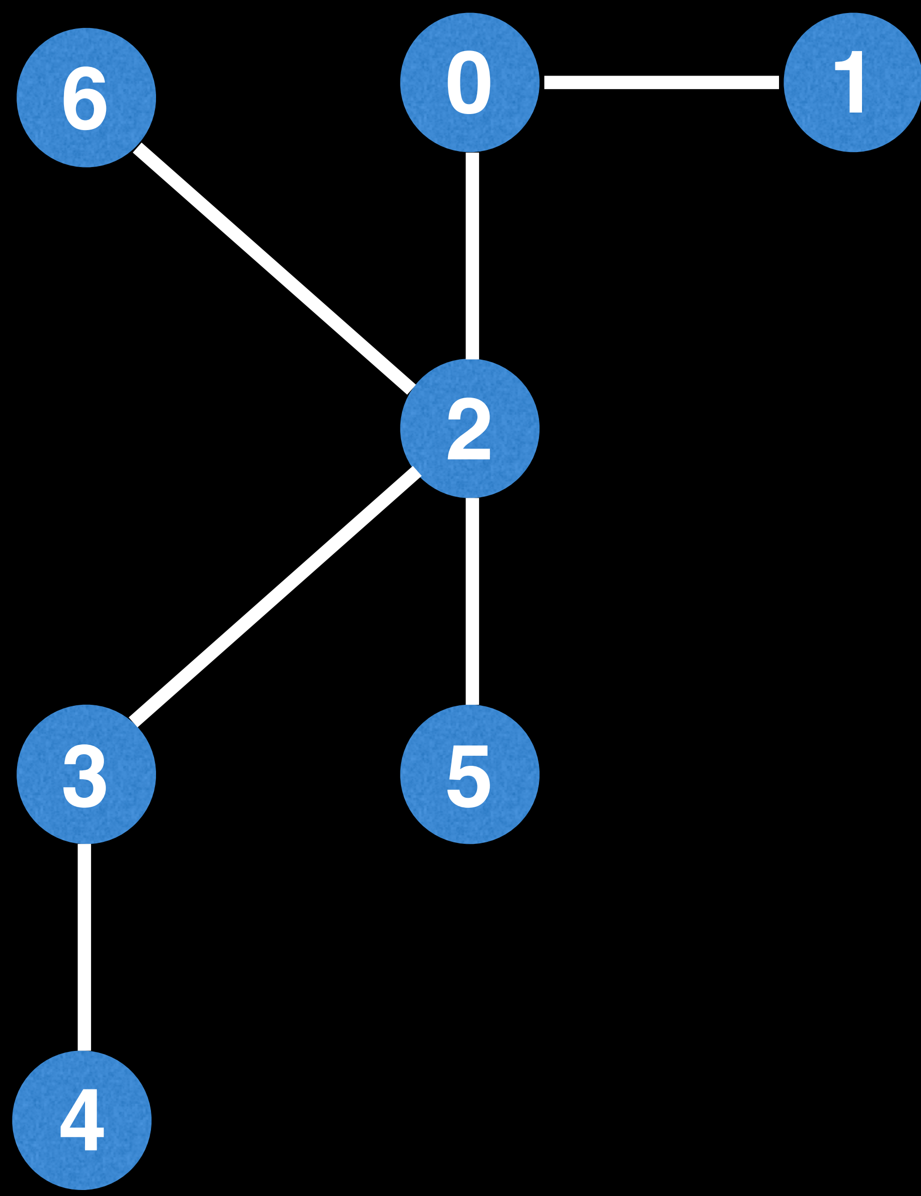
Isomorphic Trees



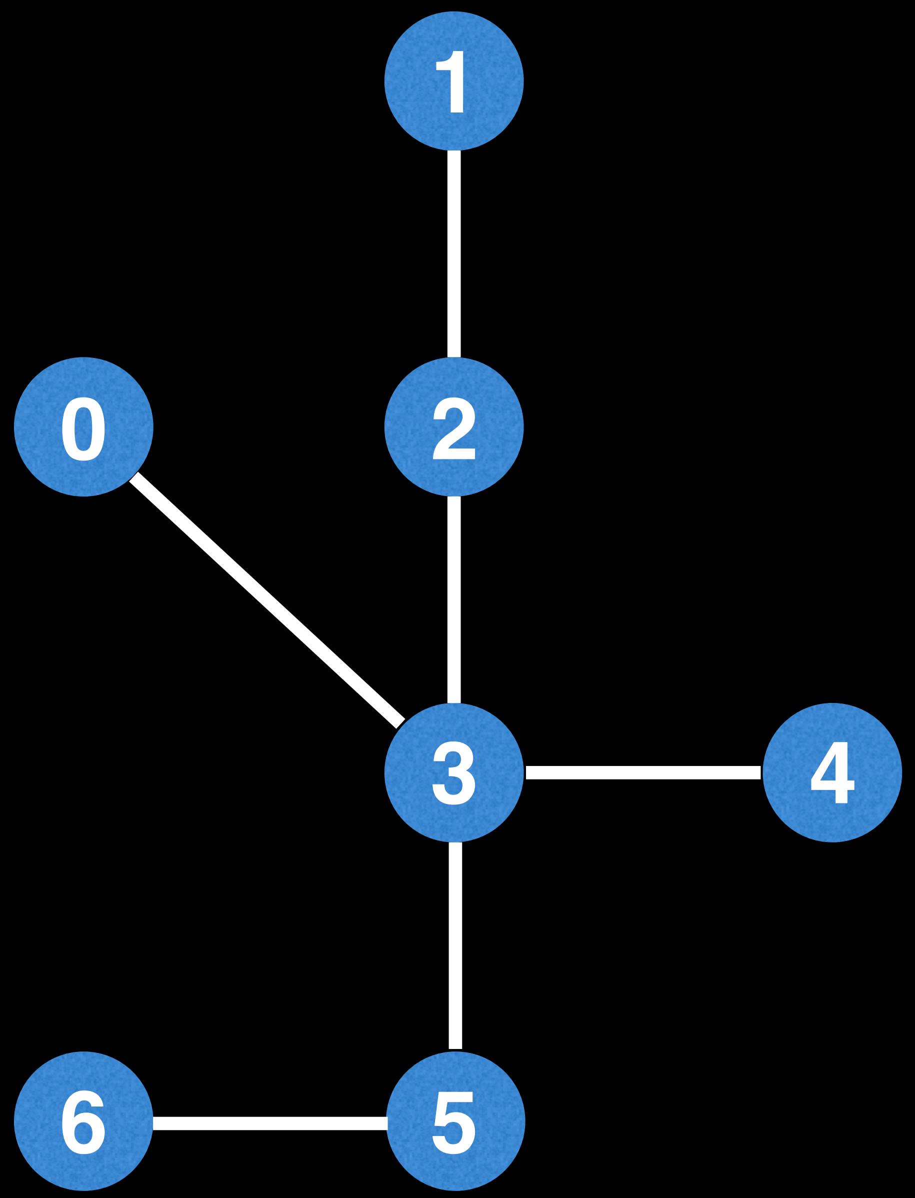
A: no, these trees are structurally different.

Isomorphic Trees

tree 3

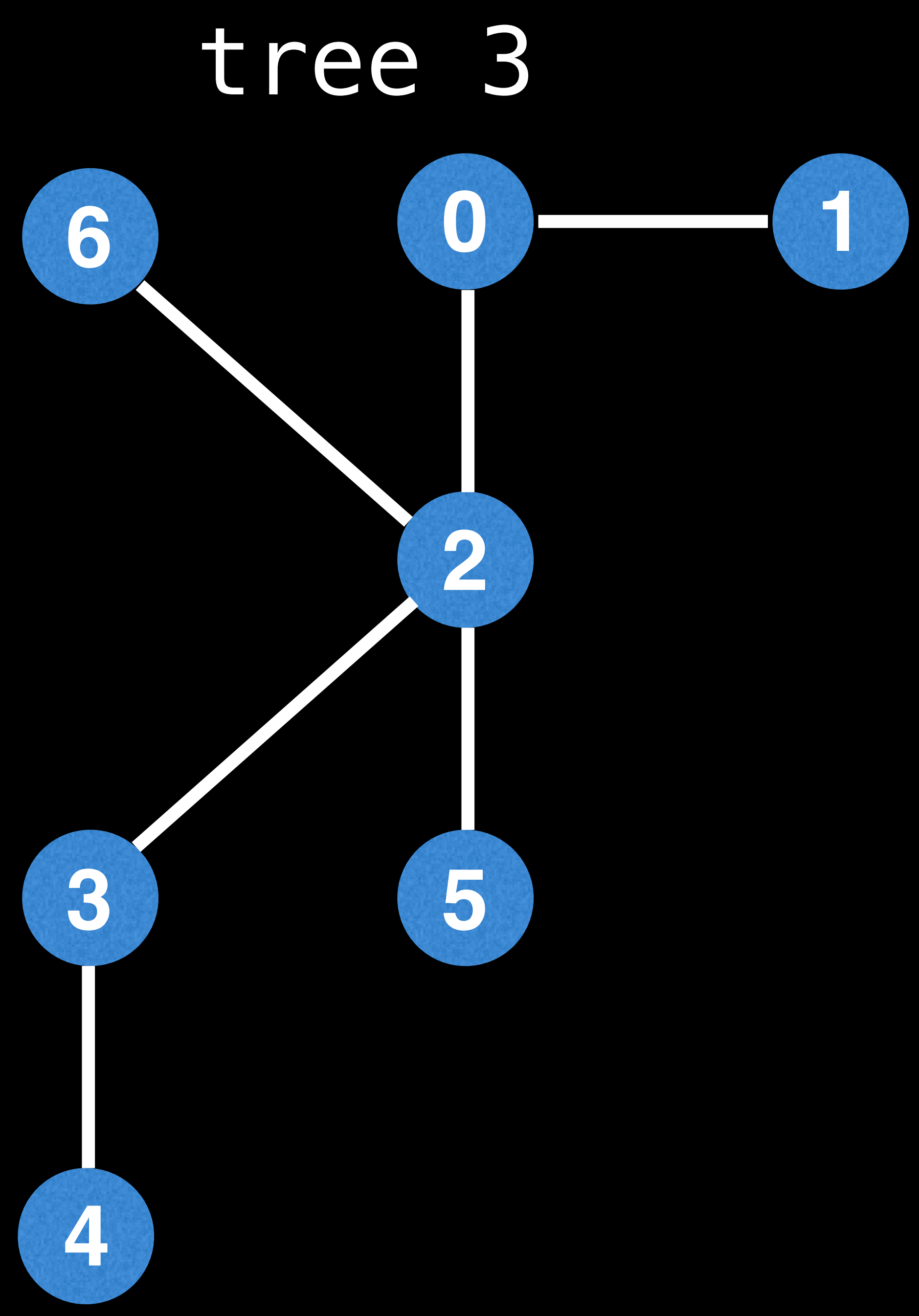


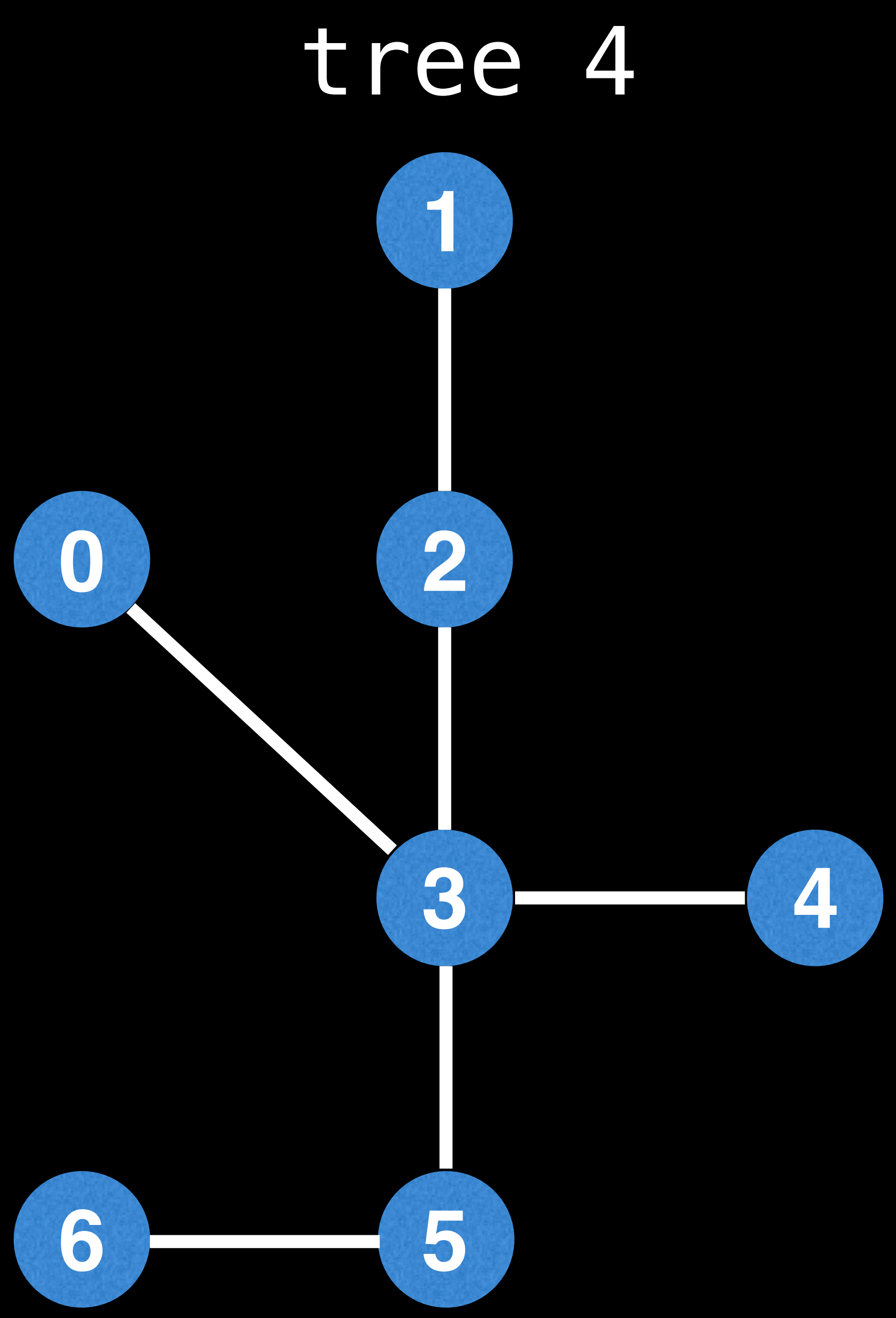
tree 4



Q: Are these trees isomorphic?

Isomorphic Trees





Yes, one possible label mapping is:
6->0, 1->1, 0->2, 2->3, 5->4, 3->5, 4->6

Identifying Isomorphic Trees

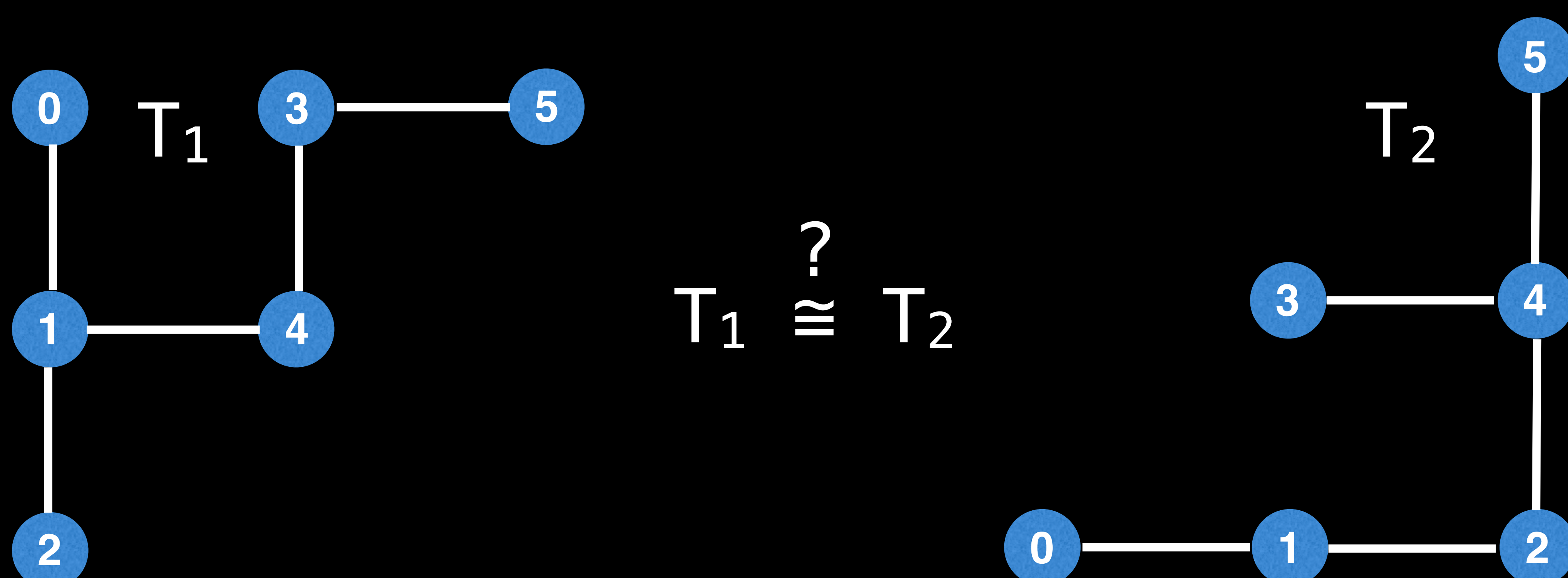
There are several very quick **probabilistic** (usually hash or heuristic based) algorithms for identifying isomorphic trees. These tend to be fast, but also error prone due to hash collisions in a limited integer space.

The method we'll be looking at today involves **serializing** a tree into a **unique encoding**. This unique encoding is simply a unique string that represents a tree, if another tree has the same encoding then they are isomorphic.

Identifying Isomorphic Trees

We can directly serialize an unrooted tree, but in practice serializing a rooted tree is typically easier code wise.

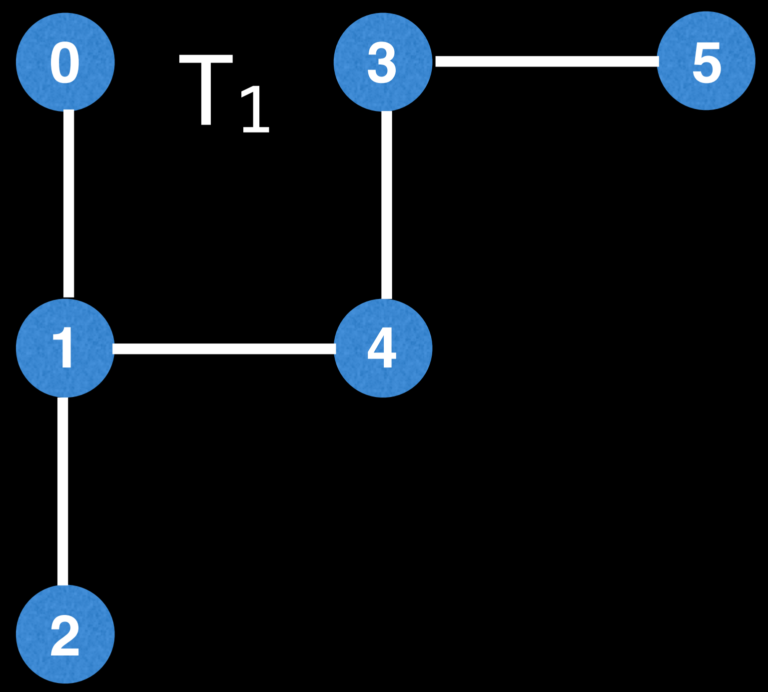
However, one caveat to watch out for if we're going to root our two trees T_1 and T_2 to check if they're isomorphic is to ensure that the same root node is selected in both trees before serializing/encoding the trees.

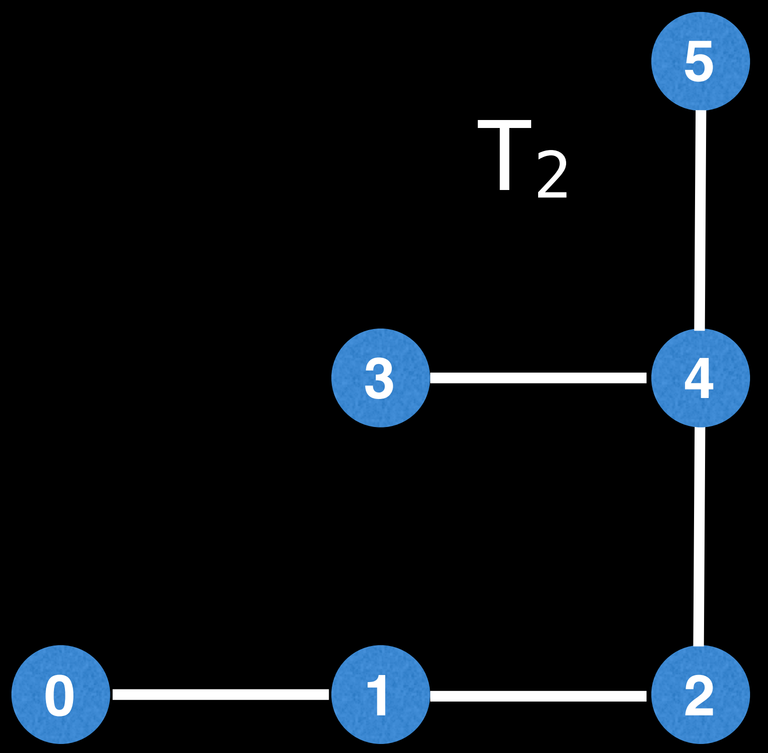


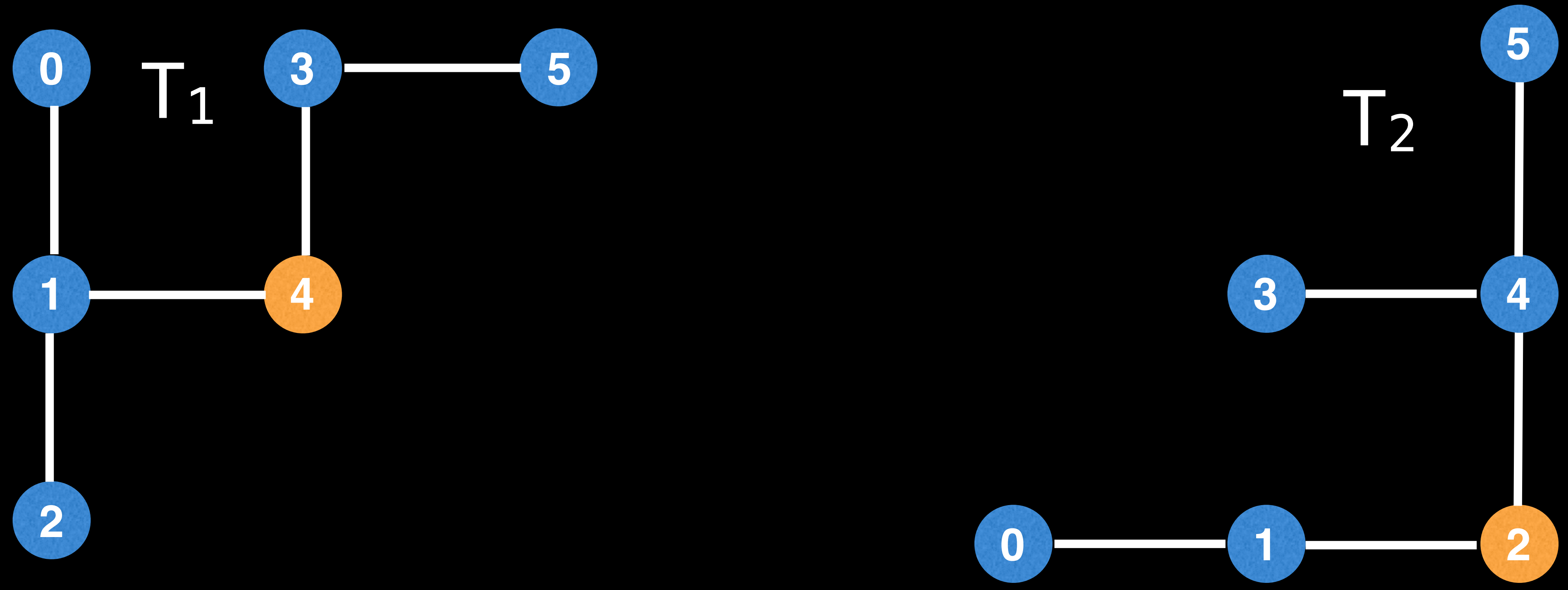
Identifying Isomorphic Trees

To select a common node between both trees we can use what we learned from finding the center(s) of a tree to help ourselves.

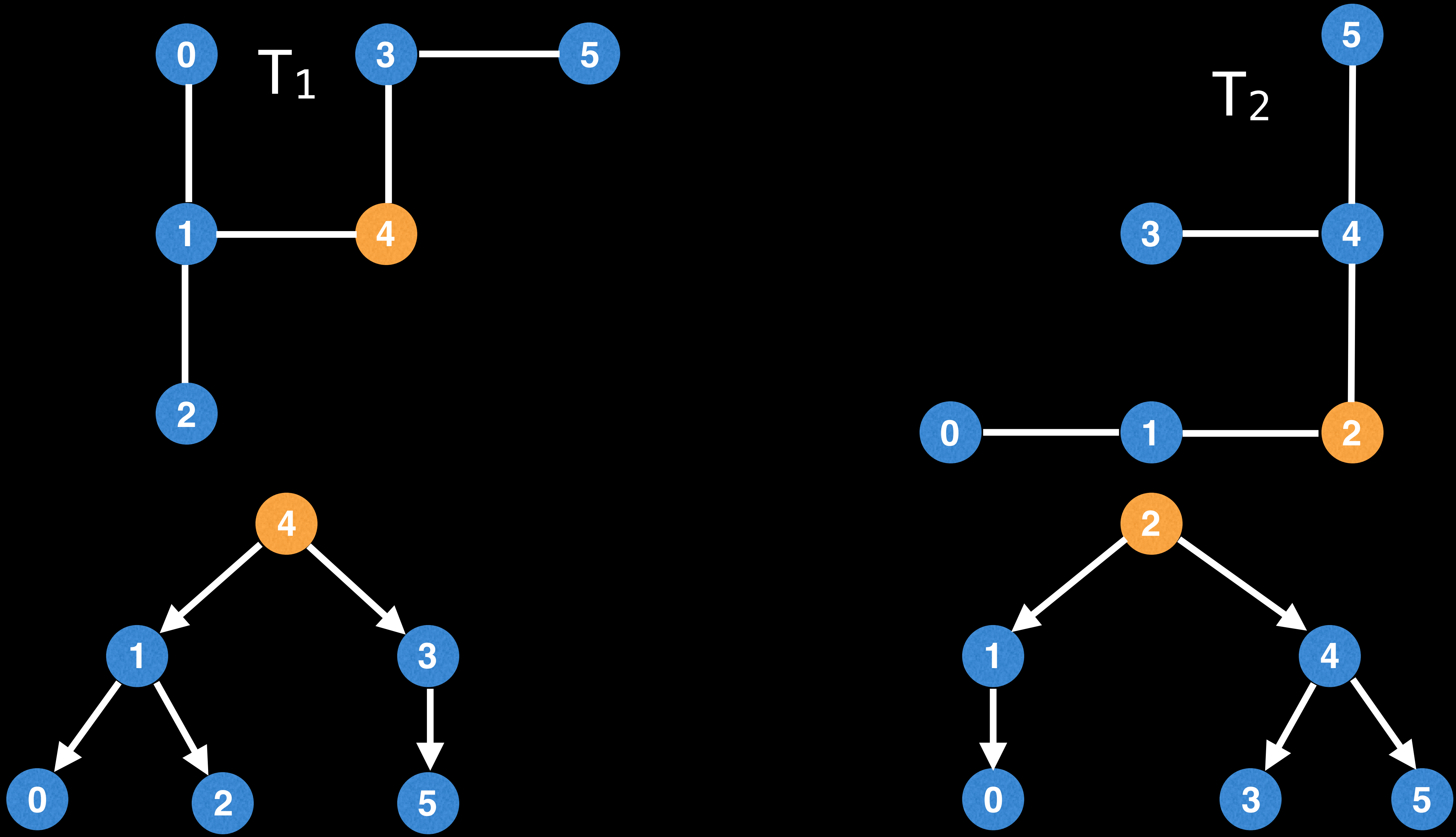
<insert video frame>


T1
0
1
2
3
4
5

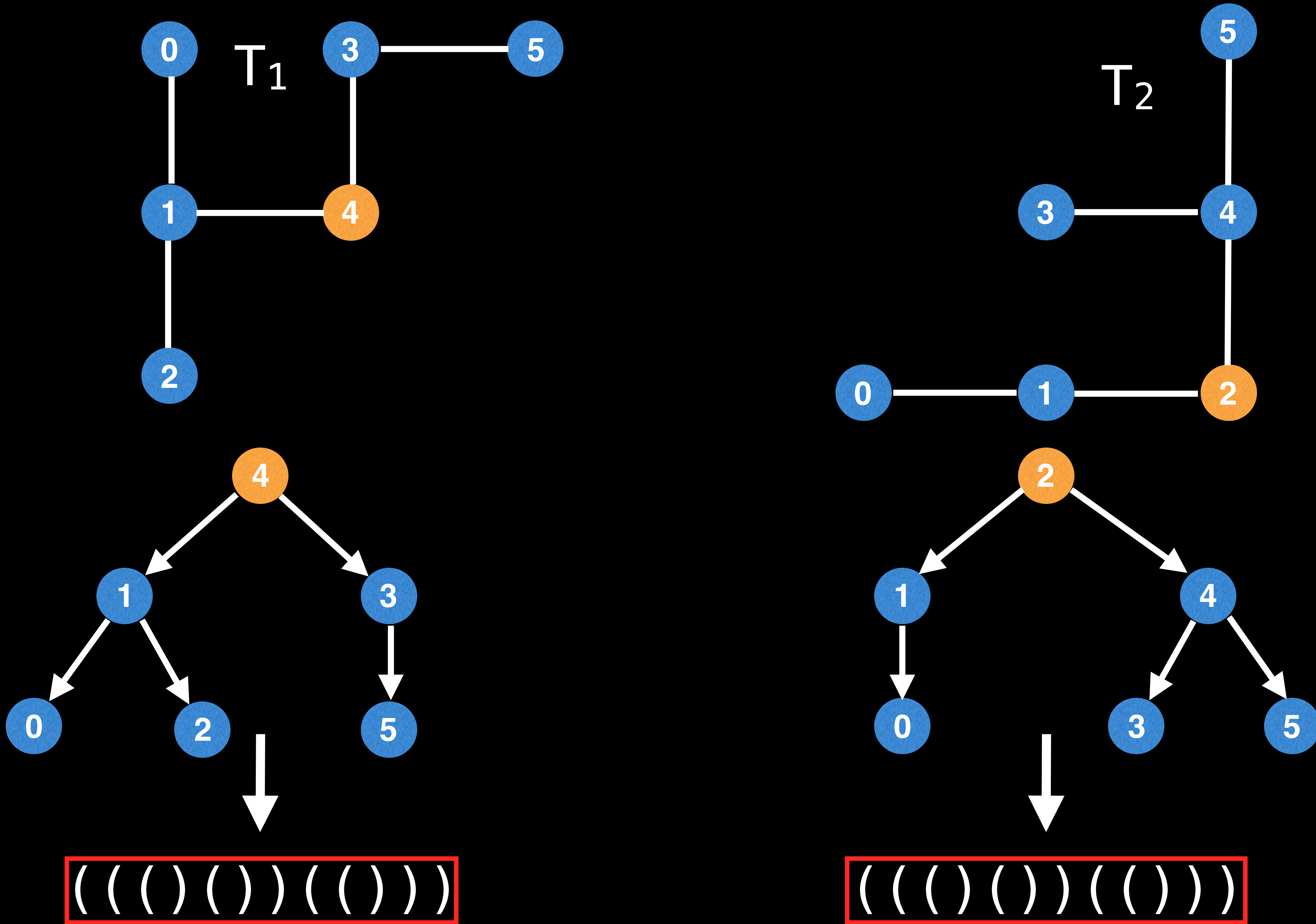

T2
0
1
2
3
4
5



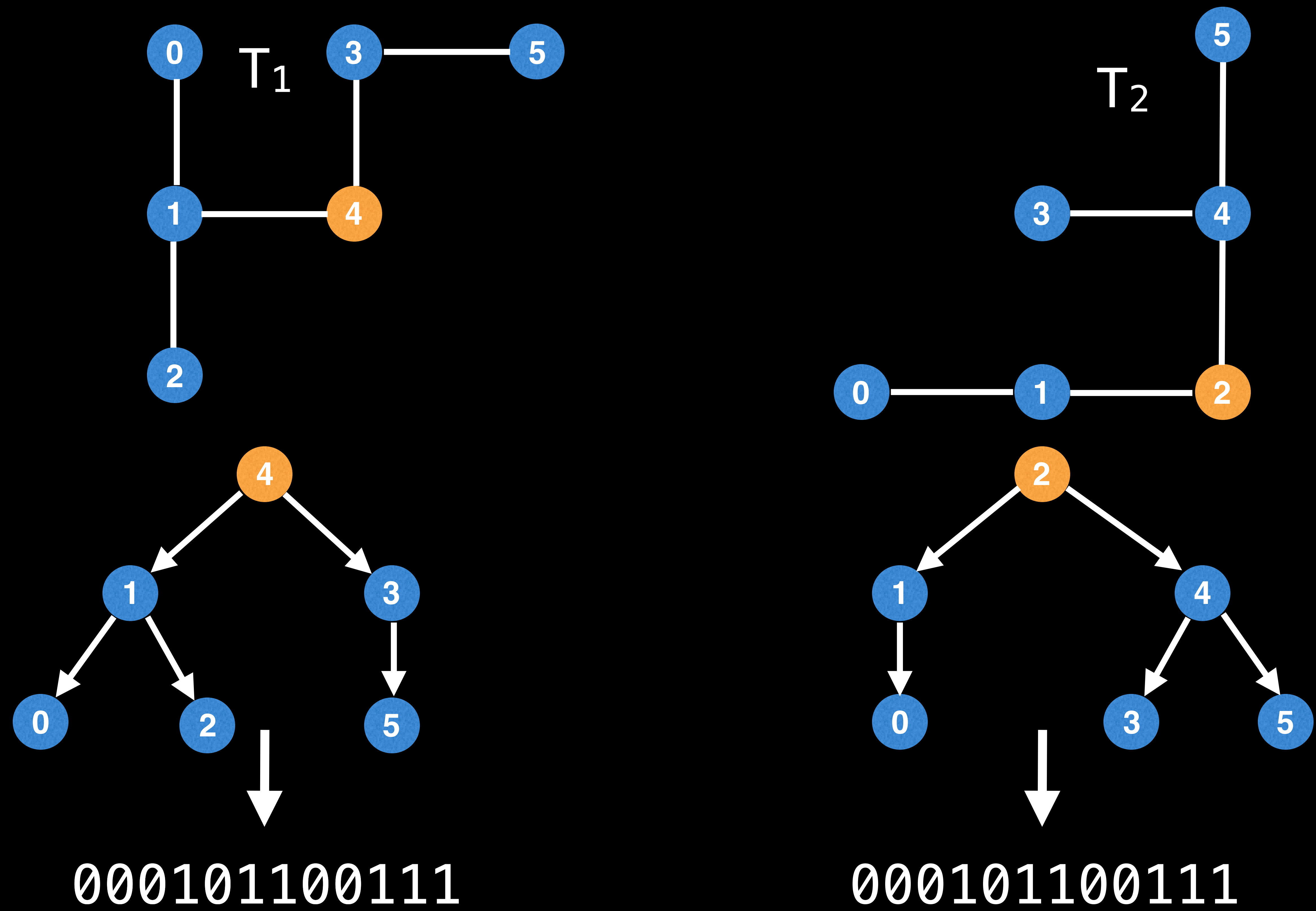
Find the center(s) of the original tree. We’ll see how to handle the case where either tree can have more than 1 center shortly.



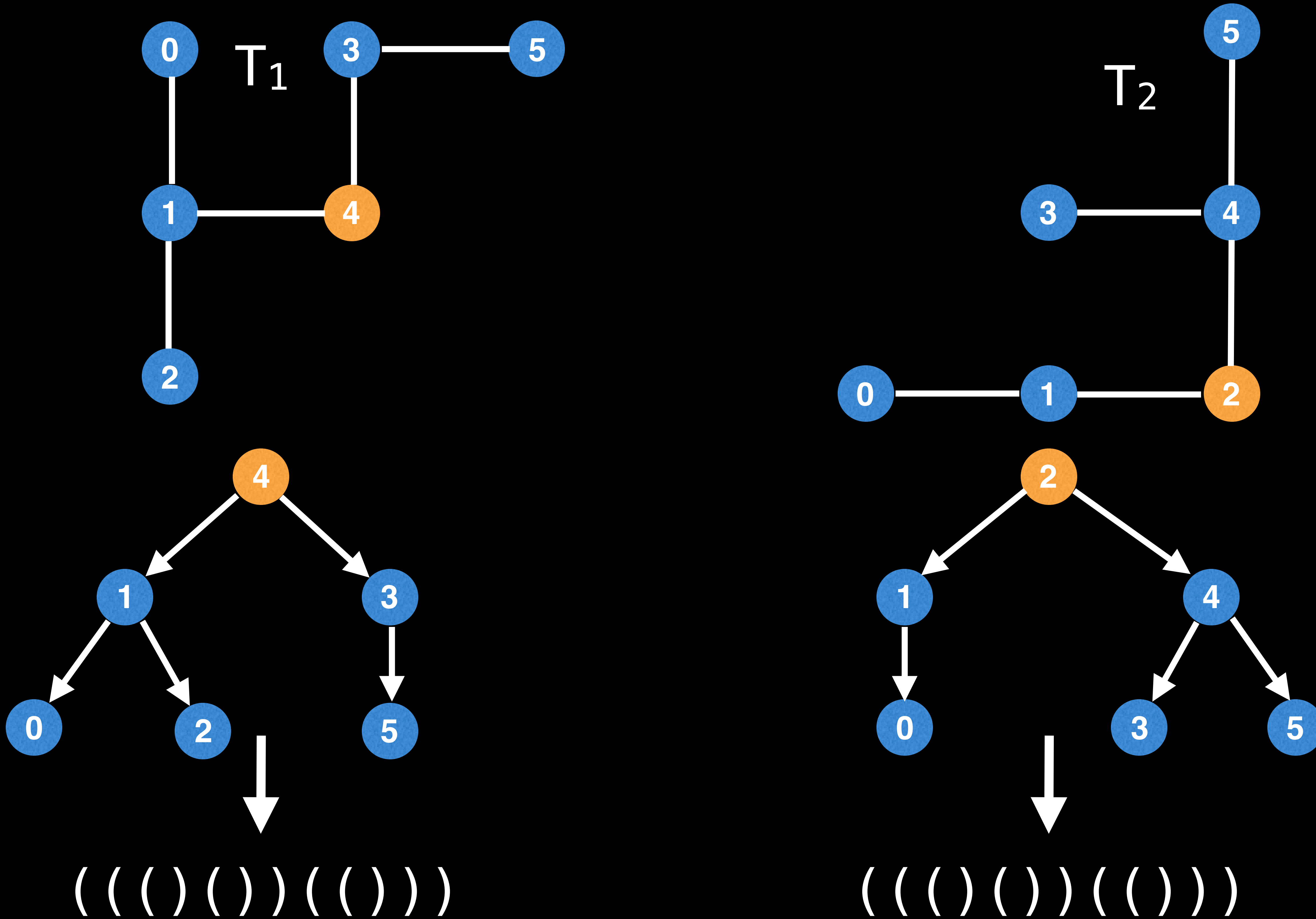
Root the tree at the center node.



Generate the encoding for each tree and compare the serialized trees for equality.



The tree encoding is simply a sequence of left '(' and right ')' brackets. However, you can also think of them as 1’s and 0’s (i.e a large number) if you prefer.



It should also be possible to reconstruct the tree solely from the encoding. This is left as an exercise to the reader… 😛

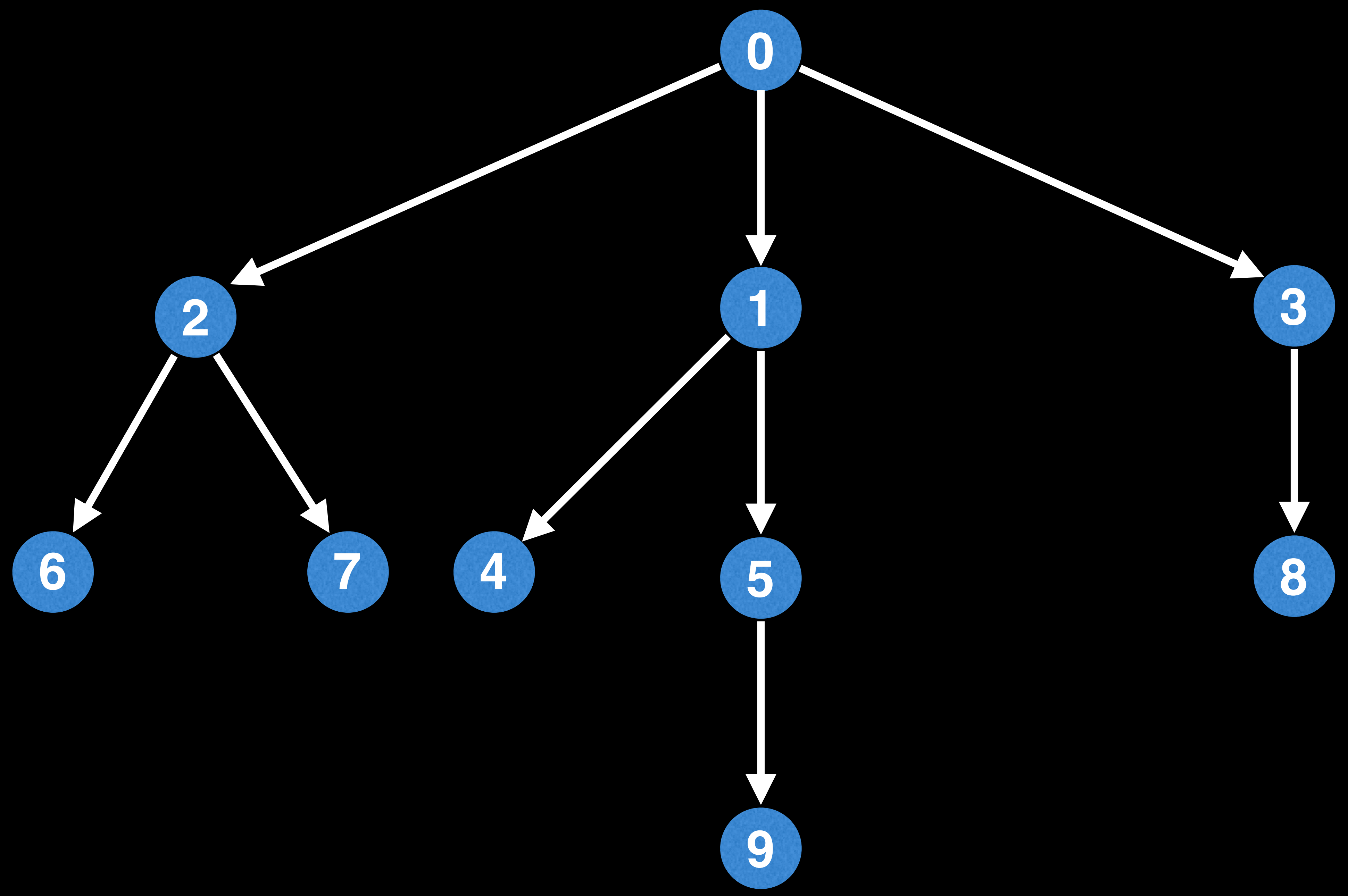
Generating the tree encoding

The AHU (Aho, Hopcroft, Ullman) algorithm is a clever serialization technique for representing a tree as a unique string.

Unlike many tree isomorphism invariants and heuristics, AHU is able to capture a **complete history** of a tree’s **degree spectrum** and structure ensuring a deterministic method of checking for tree isomorphisms.

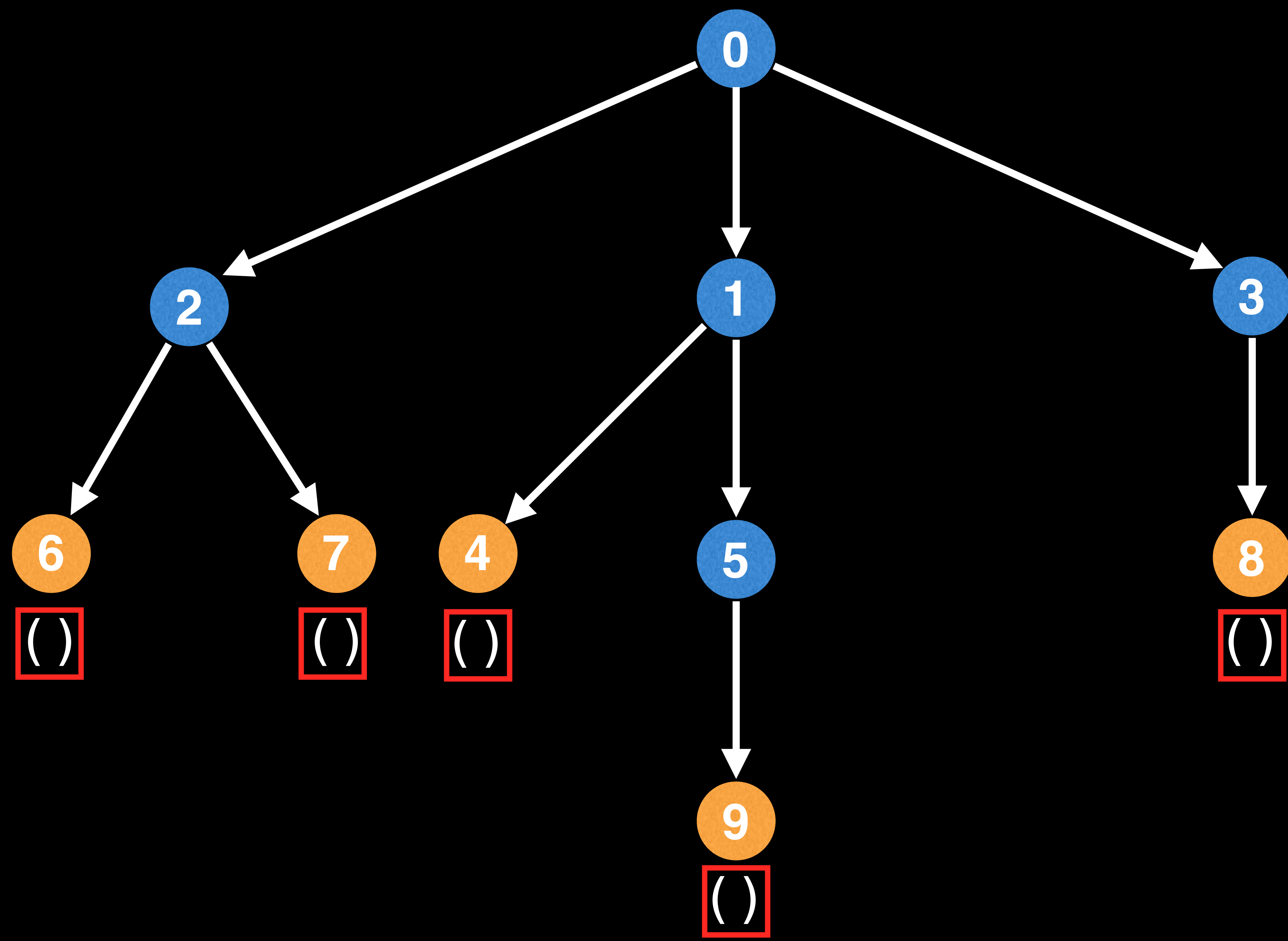
Let’s have a closer look…

Tree Encoding



Tree Encoding

Start by assigning all leaf nodes Knuth tuples: '()'



Tree Encoding

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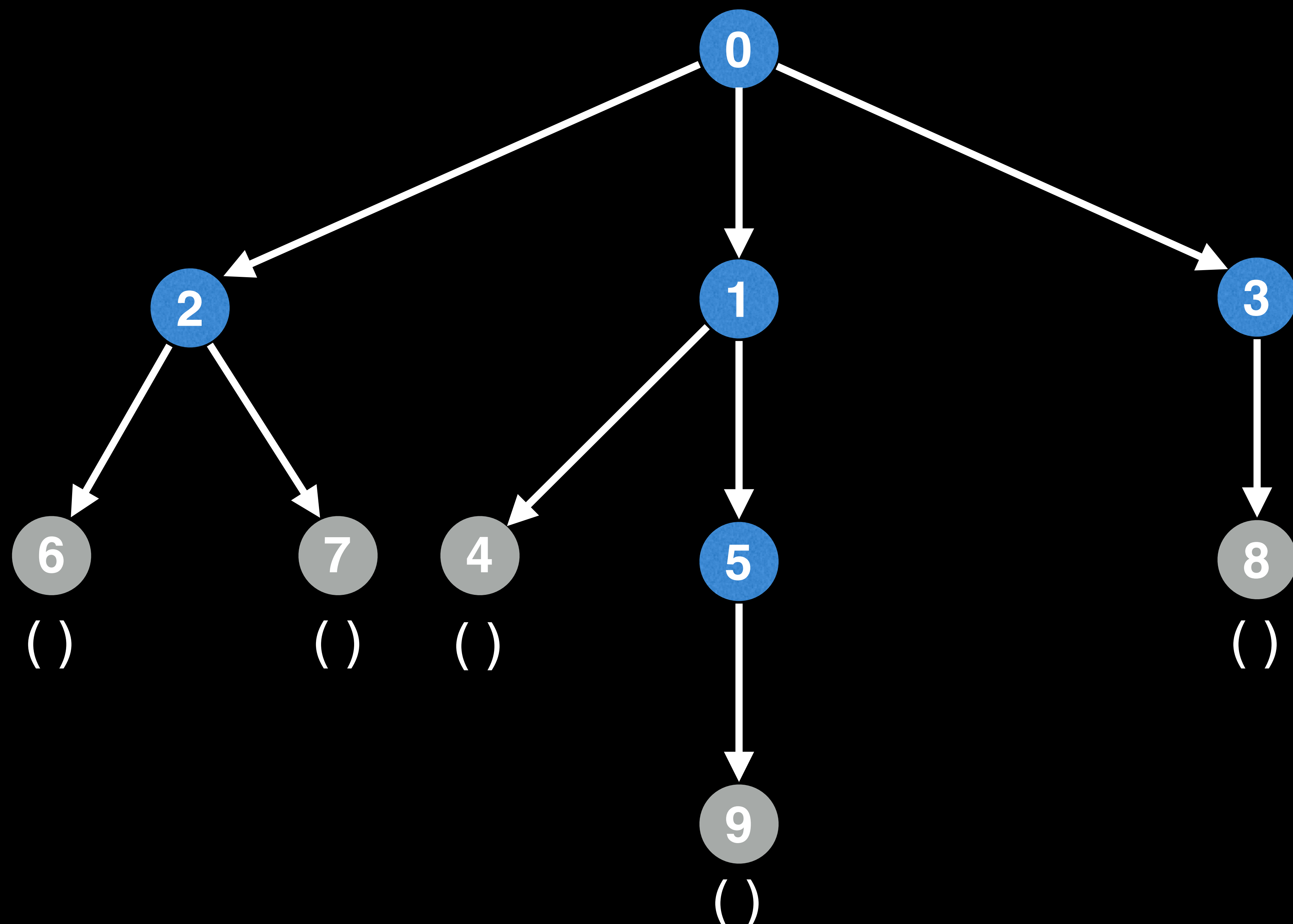
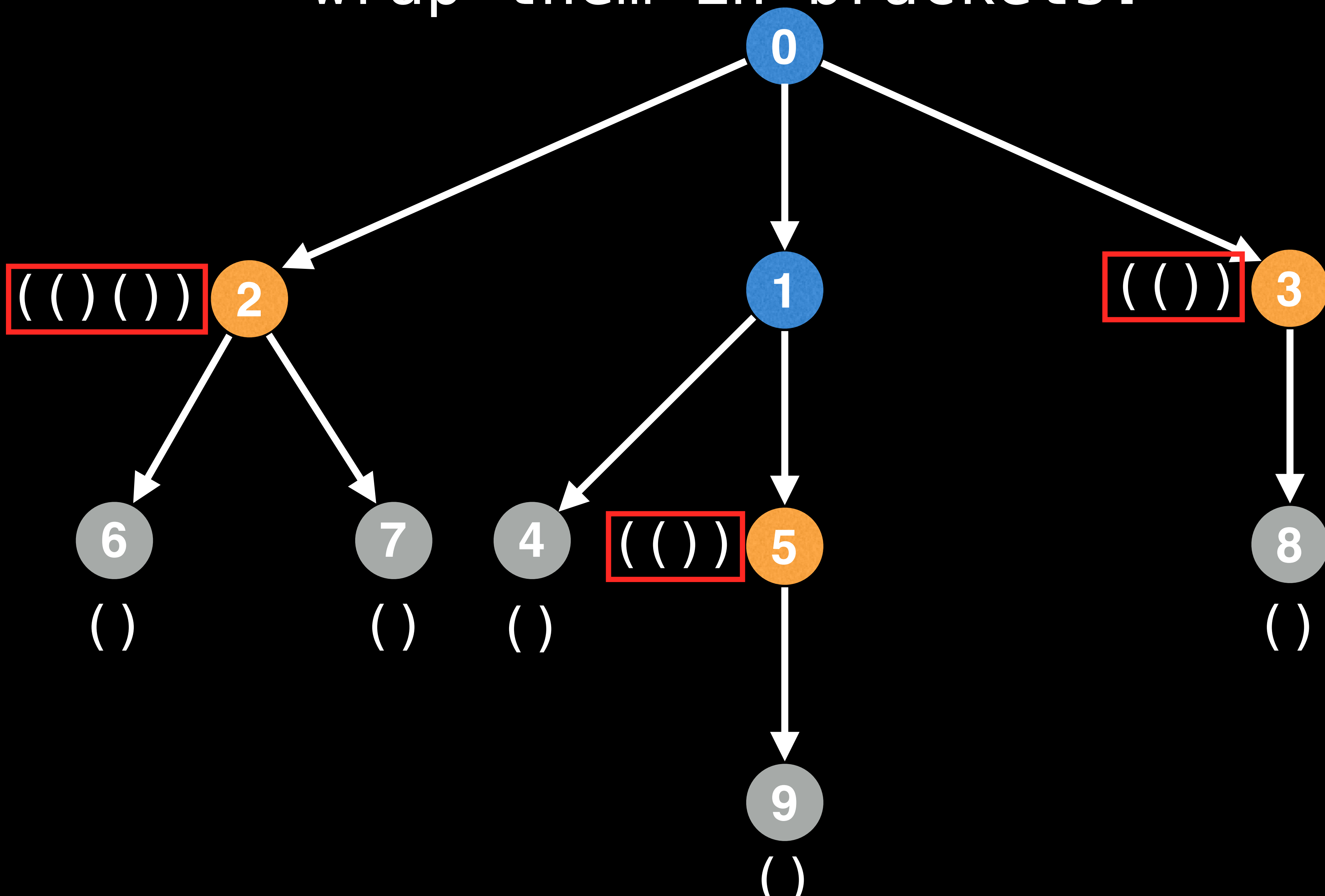


Tree Encoding

Process all nodes with grayed out children and combine the labels of their child nodes and wrap them in brackets.



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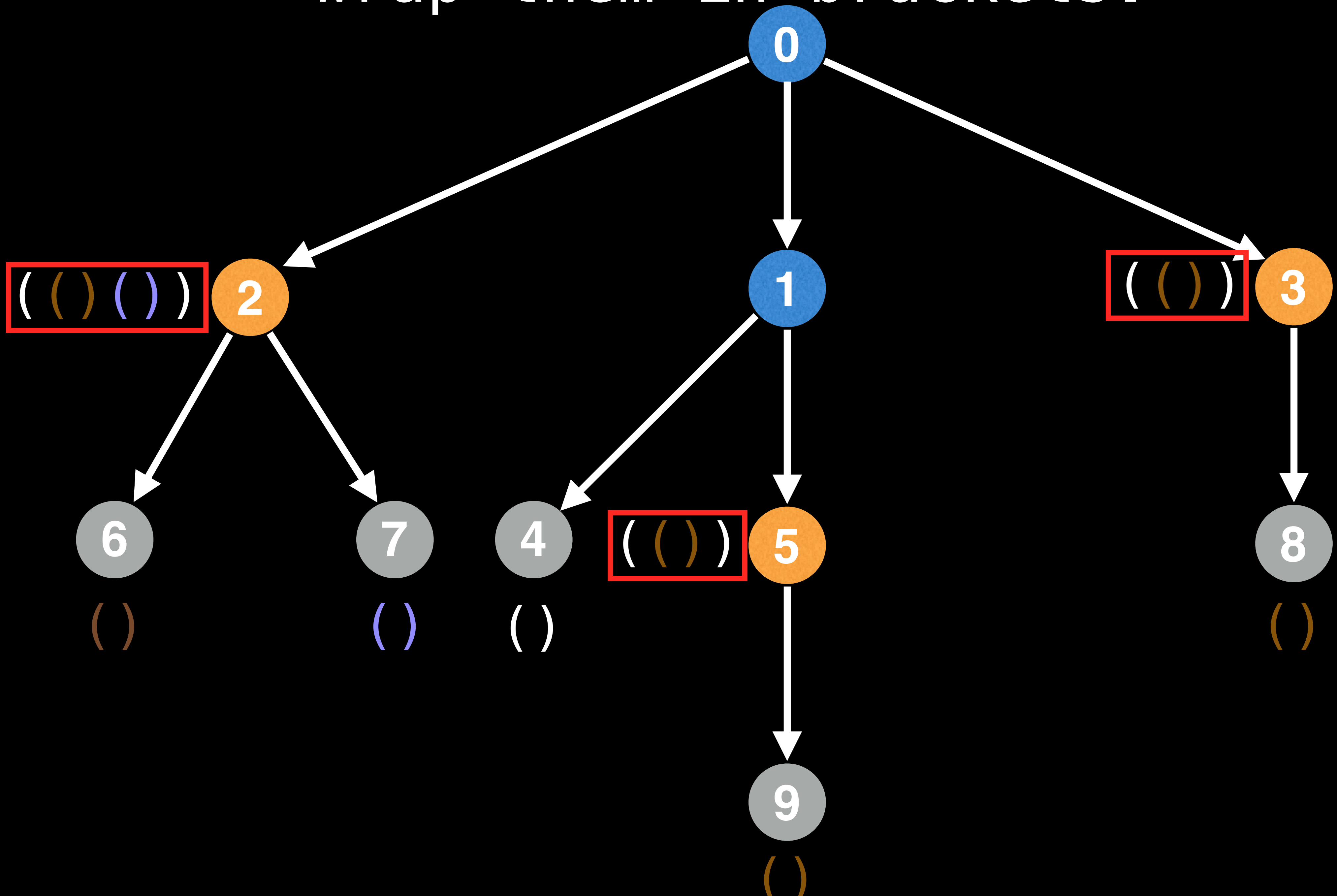
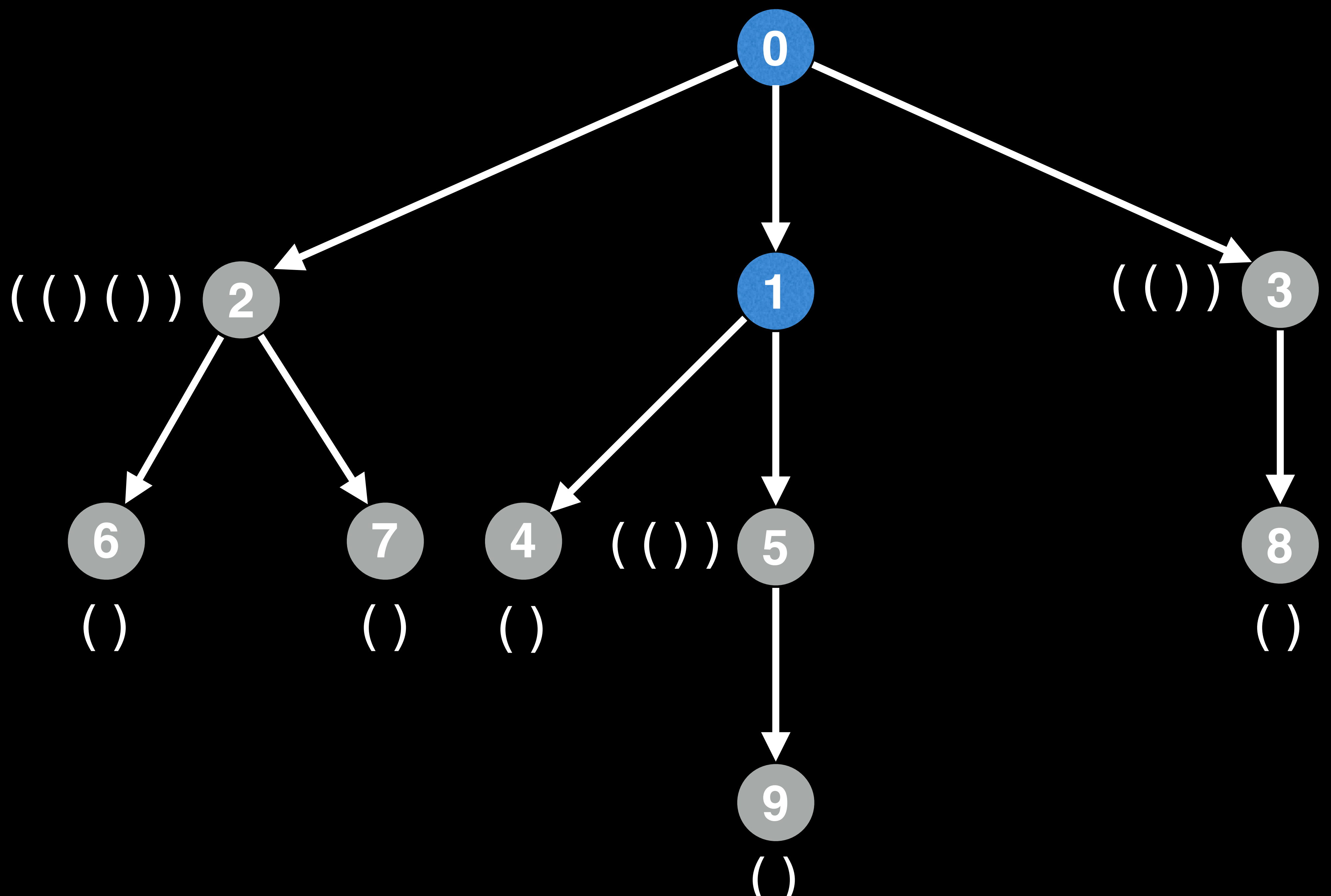
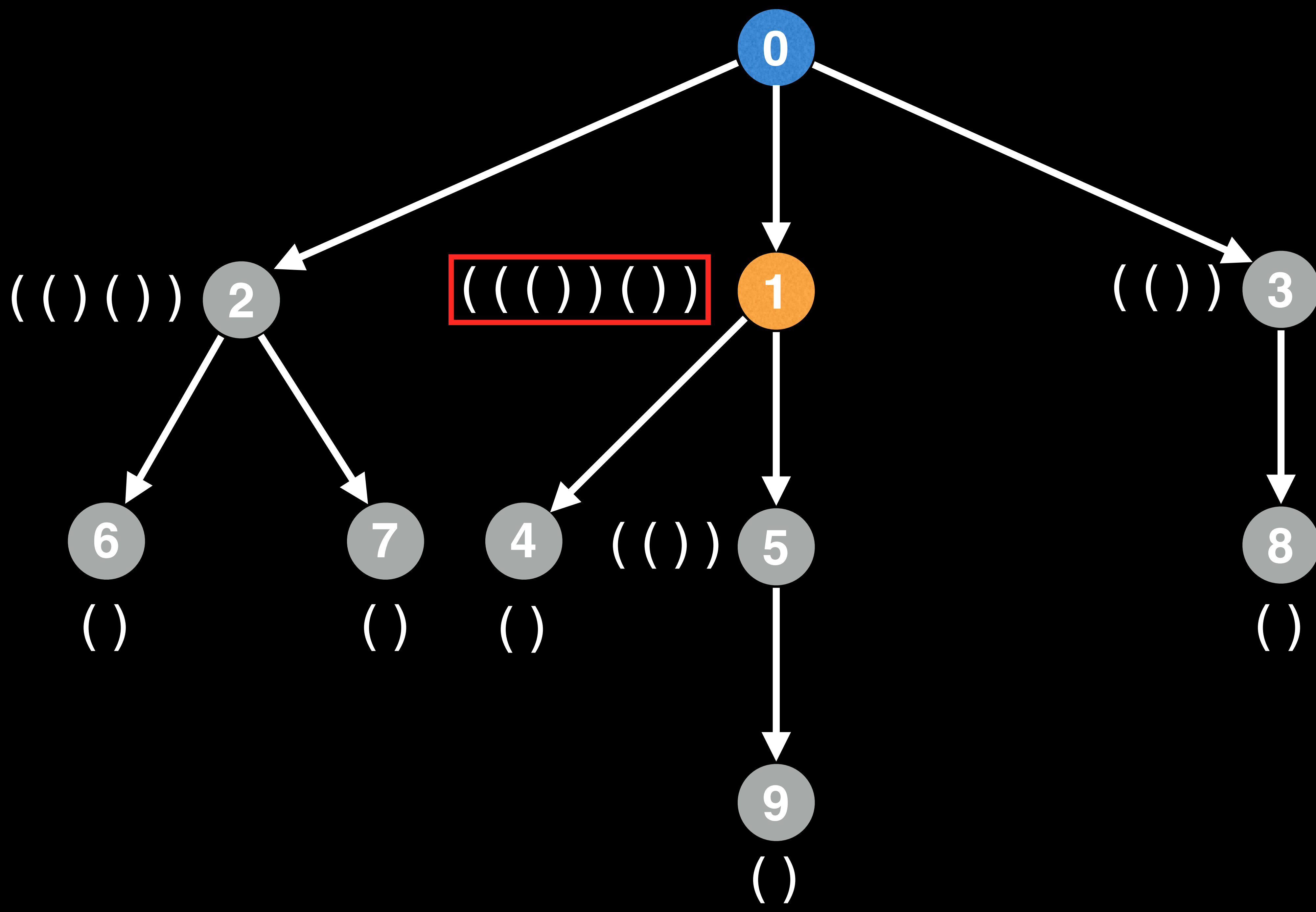


Tree Encoding

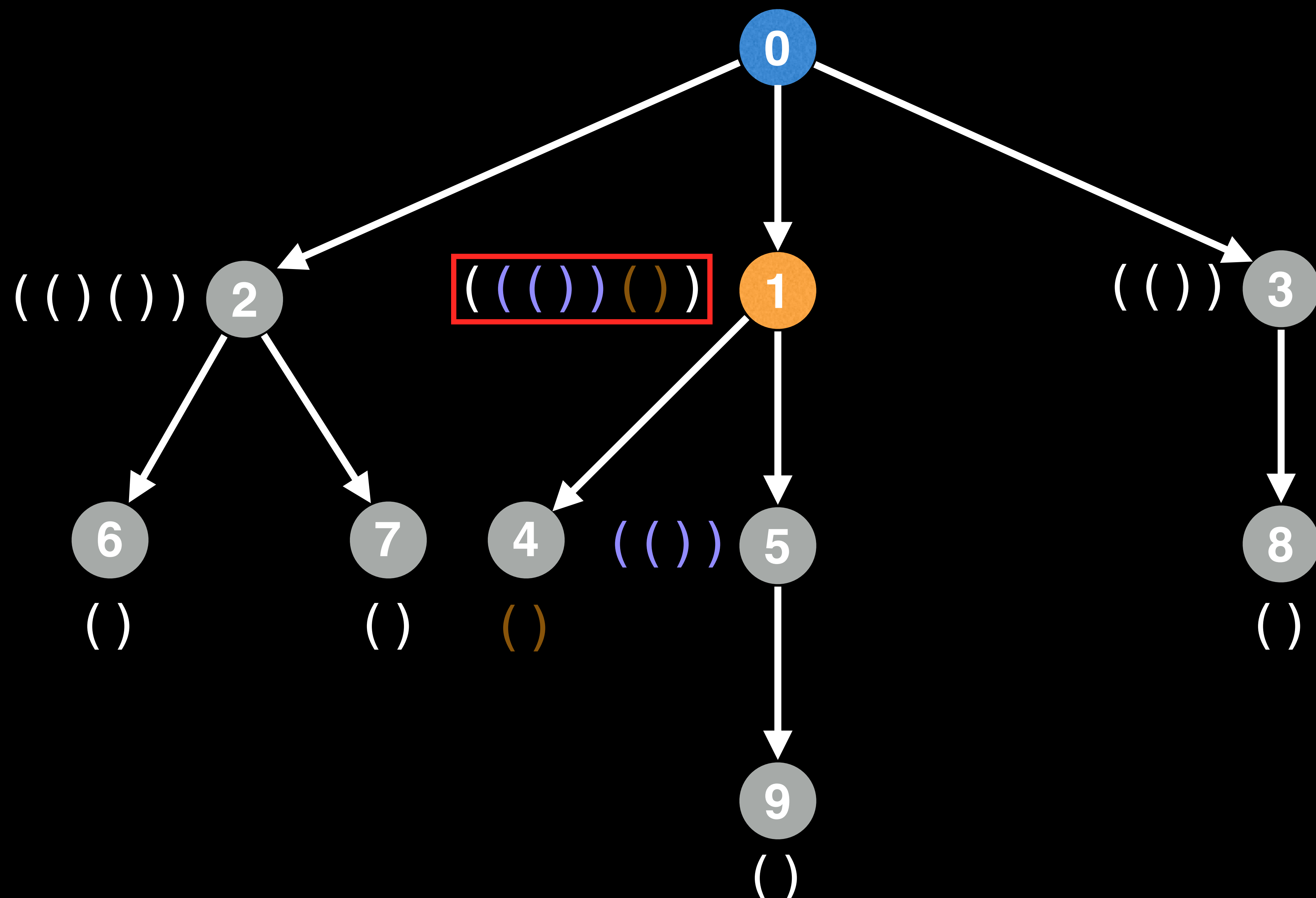


Tree Encoding



Tree Encoding

Notice that the labels get ***sorted*** when combined, this is important.



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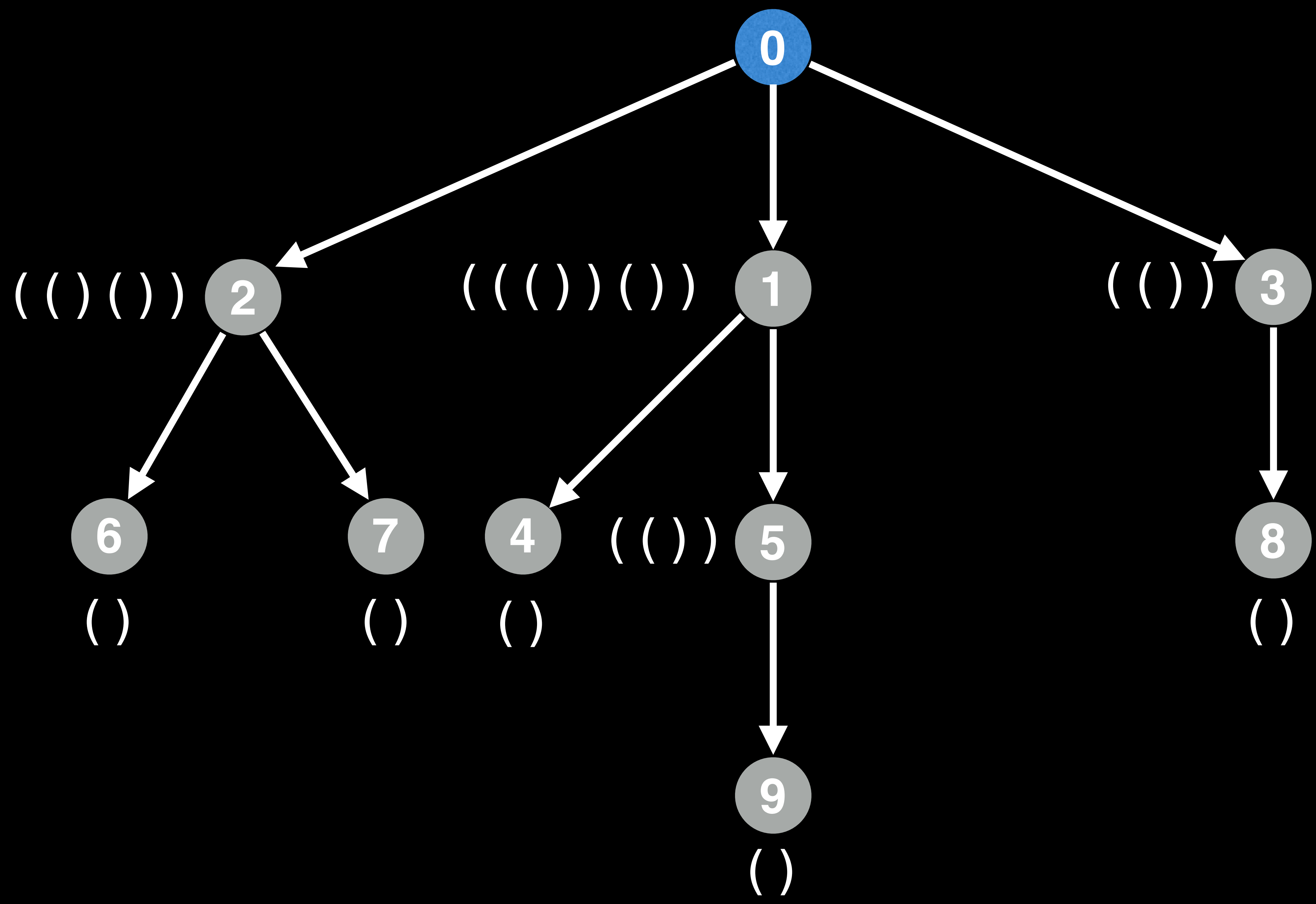

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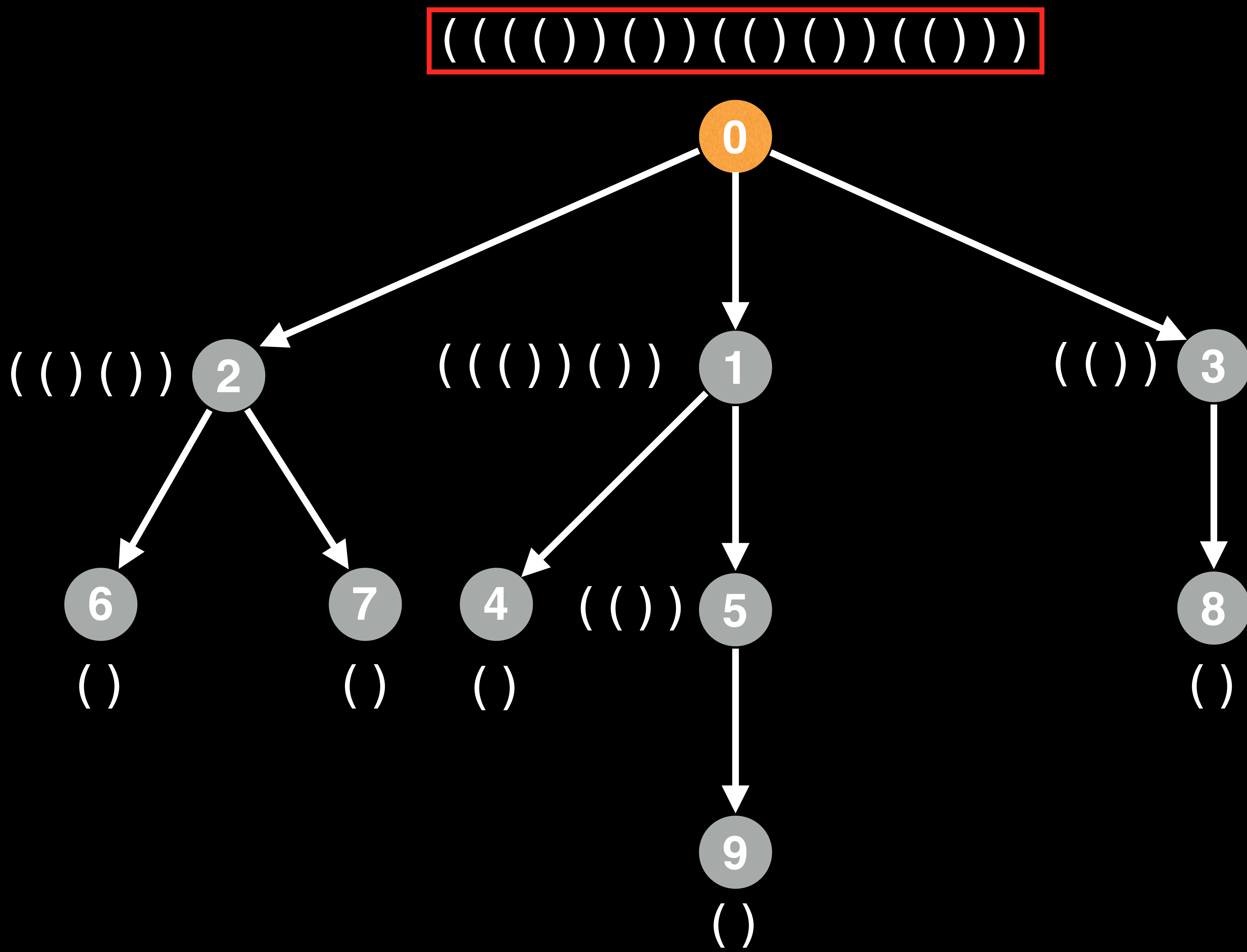
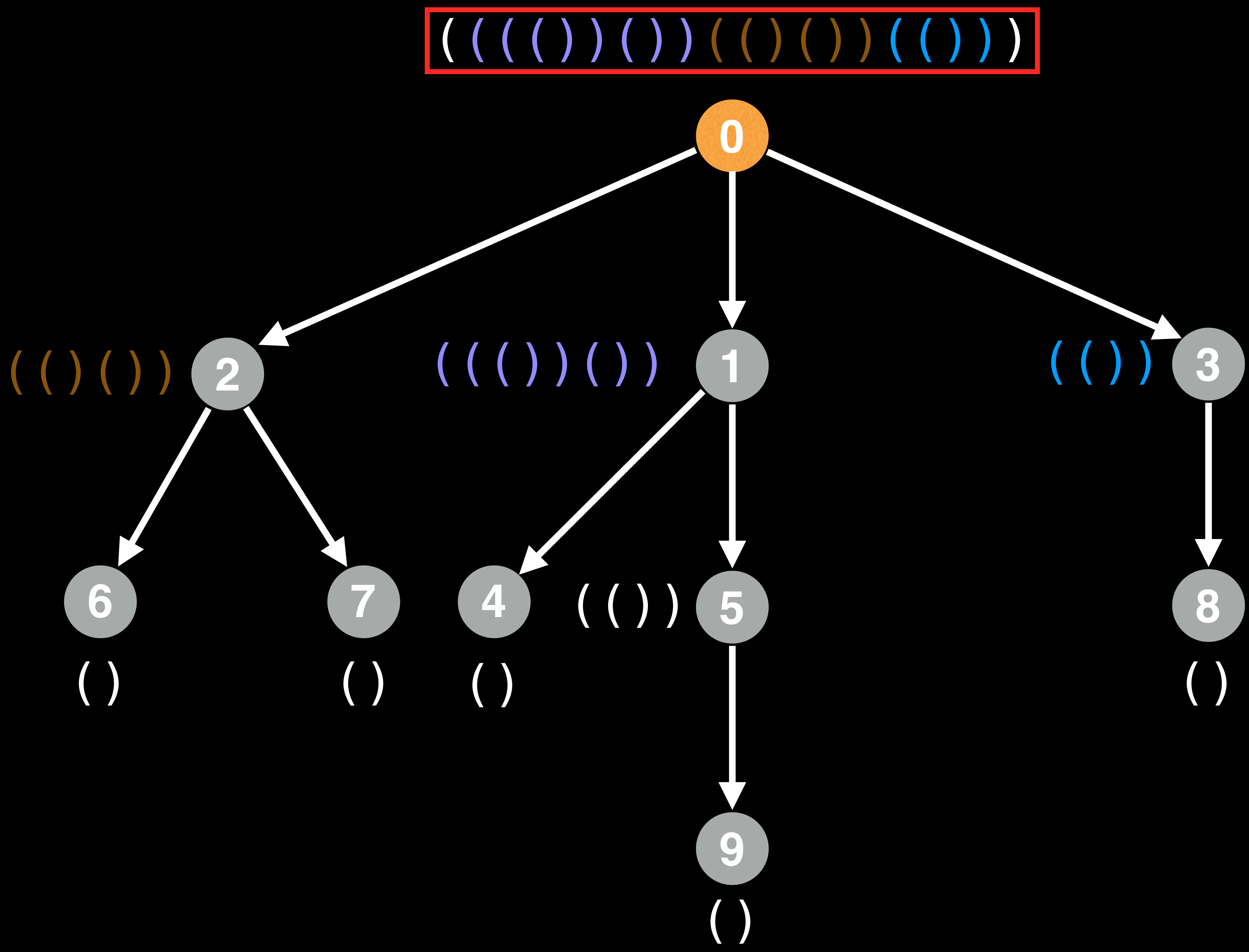


Tree Encoding



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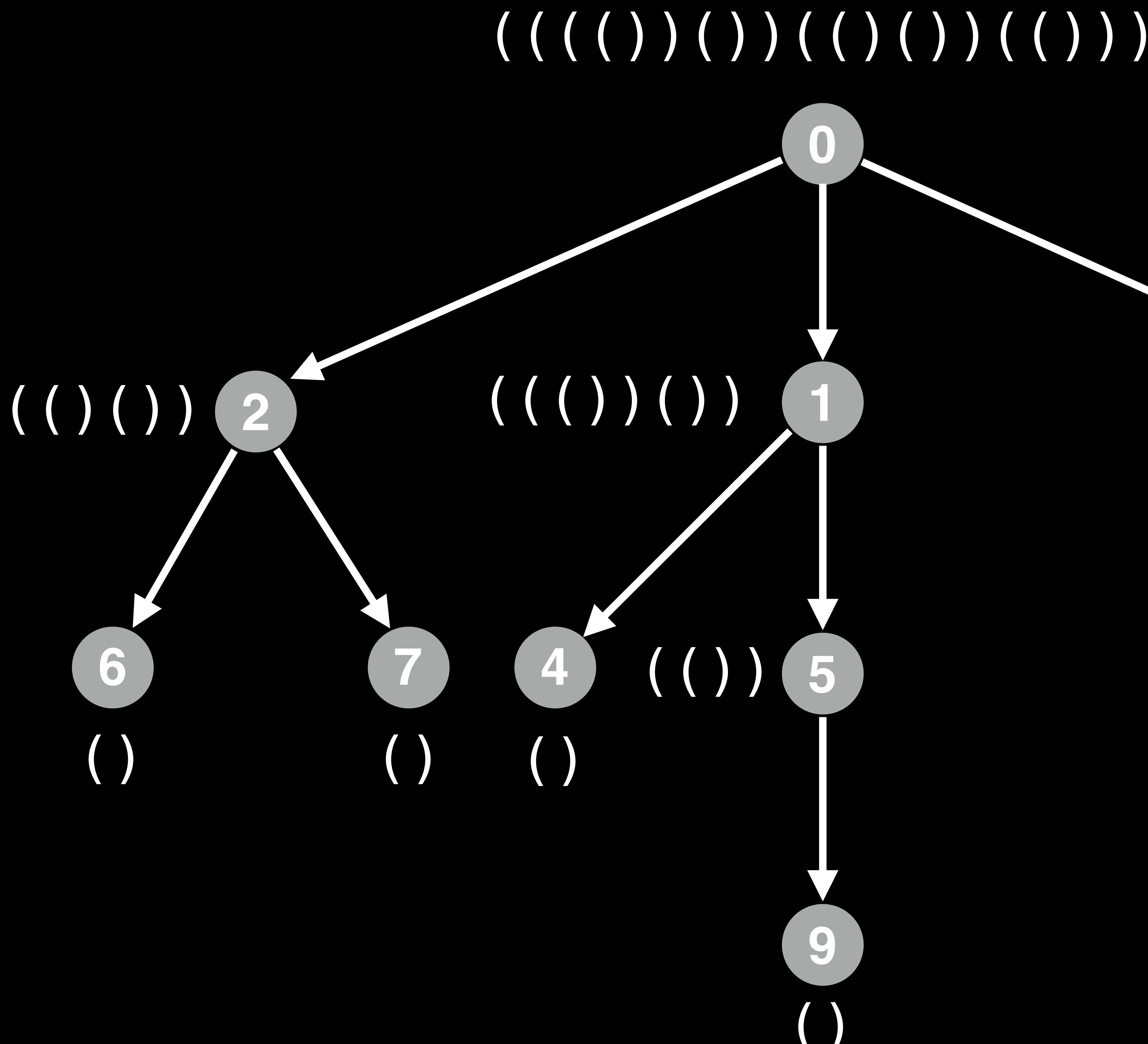

Tree Encoding Summary

In summary of what we did for AHU:

- Leaf nodes are assigned Knuth tuples '()' to begin with.
- Every time you move up a layer the labels of the previous subtrees get sorted lexicographically and wrapped in brackets.
- You cannot process a node until you have processed all its children.

Unrooted tree encoding pseudocode

```
# Returns whether two trees are isomorphic.
# Parameters tree1 and tree2 are undirected trees
# stored as adjacency lists.
function treesAreIsomorphic(tree1, tree2):
  tree1_centers = treeCenters(tree1)
  tree2_centers = treeCenters(tree2)

  tree1_rooted = rootTree(tree1, tree1_centers[0])
  tree1_encoded = encode(tree1_rooted)

  for center in tree2_centers:
    tree2_rooted = rootTree(tree2, center)
    tree2_encoded = encode(tree2_rooted)
    # Two trees are isomorphic if their encoded
    # canonical forms are equal.
    if tree1_encoded == tree2_encoded:
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Unrooted tree encoding pseudocode

Rooted trees are stored recursively in TreeNode objects:

```
# TreeNode object structure.
class TreeNode:
  # Unique integer id to identify this node.
  int id;

  # Pointer to parent TreeNode reference. Only the
  # root node has a null parent TreeNode reference.
  TreeNode parent;

  # List of pointers to child TreeNodes.
  TreeNode[] children;
```

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Unrooted tree encoding pseudocode

```
function encode(node):
  if node == null:
    return ""

  labels = []
  for child in node.children():
    labels.add(encode(child))

  # Regular lexicographic sort
  sort(labels)

  result = ""
  for label in labels:
    result += label

  return "(" + result + ")"
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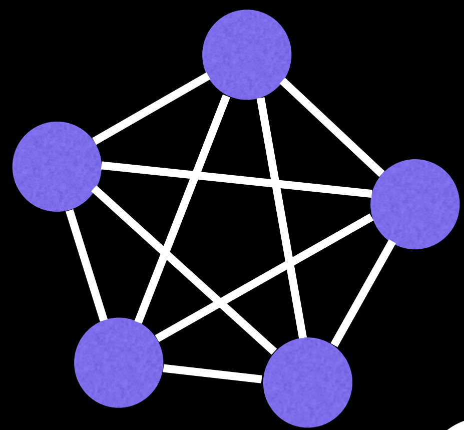
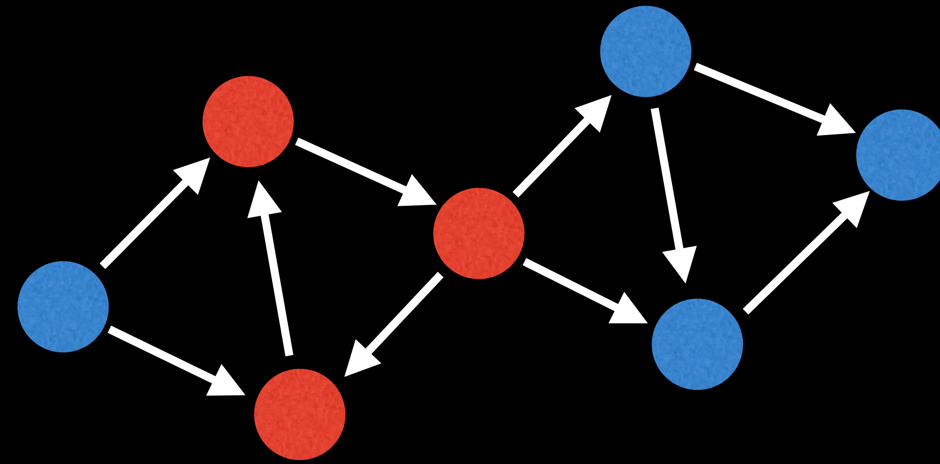
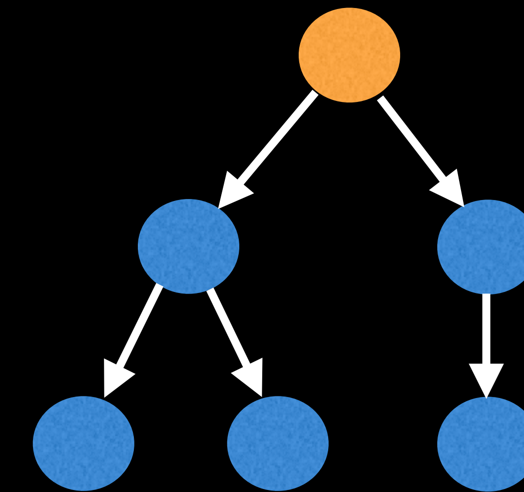
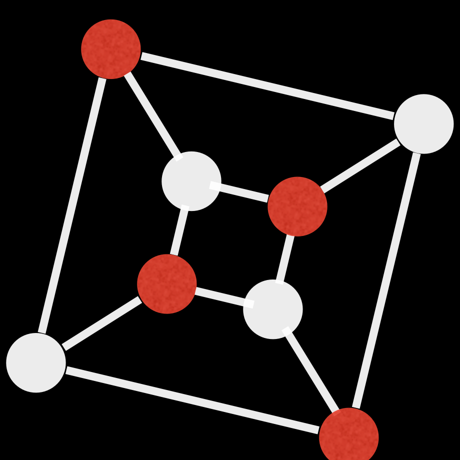
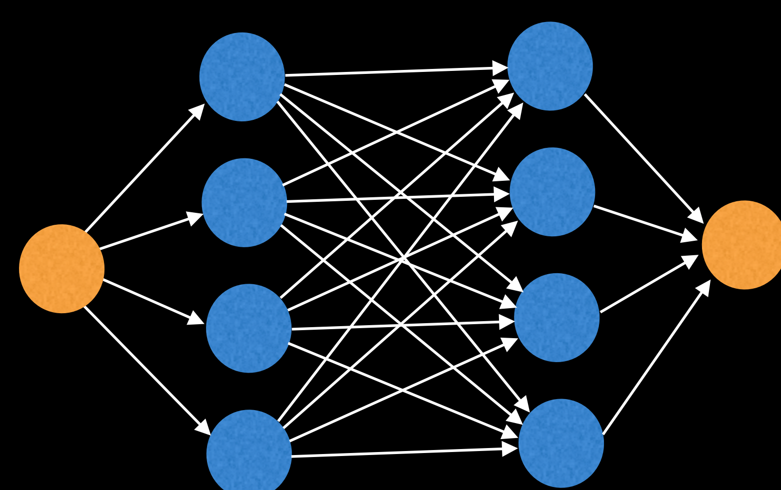
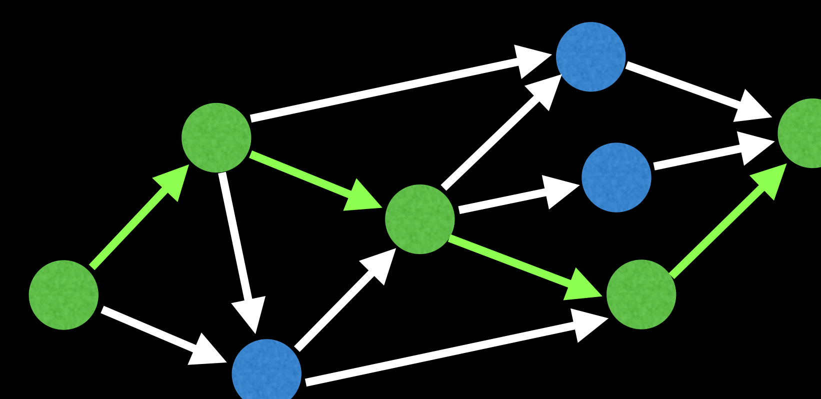




Graph Theory Video Series

Isomorphisms in trees source code

A question of equality

William Fiset

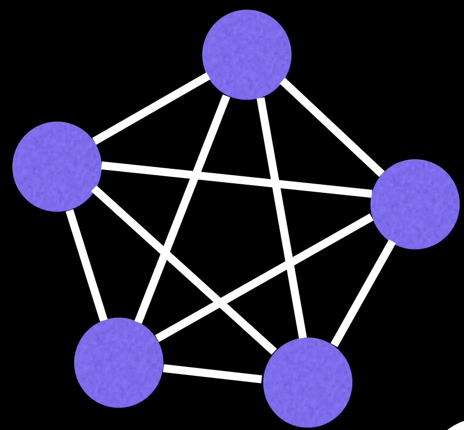
Previous video explaining identifying isomorphic trees:

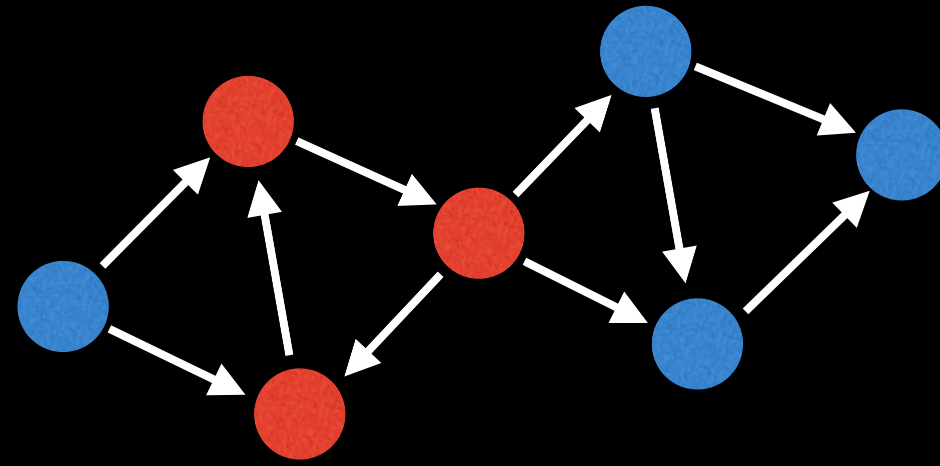
Source Code Link

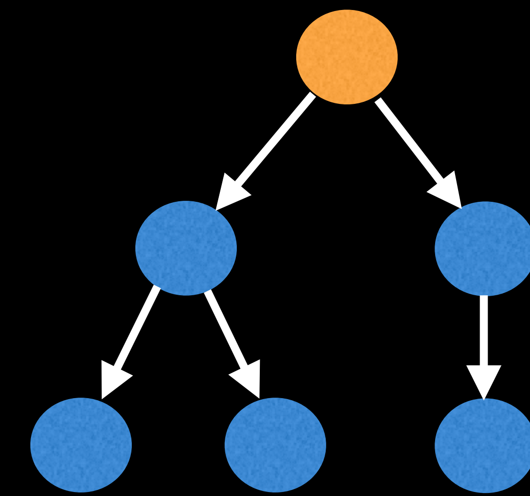
Implementation source code can be found at the following link:

github.com/williamfiset/algorithms

Link in the description below:







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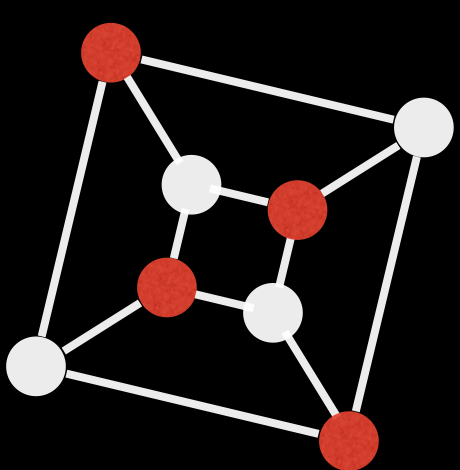
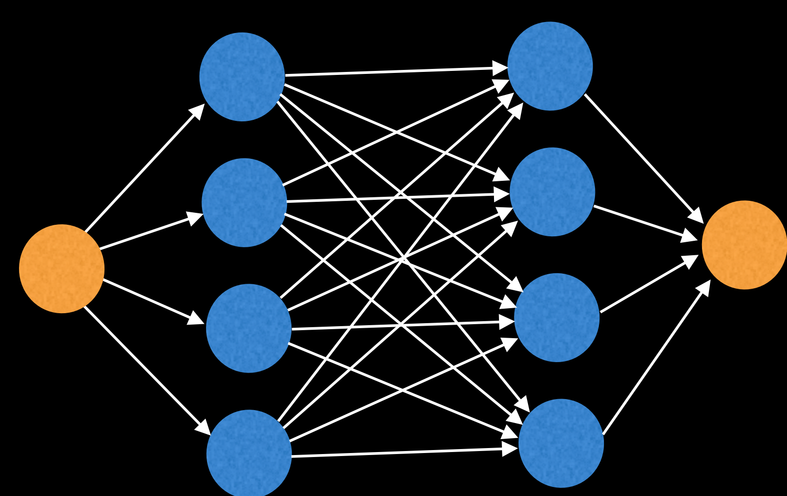
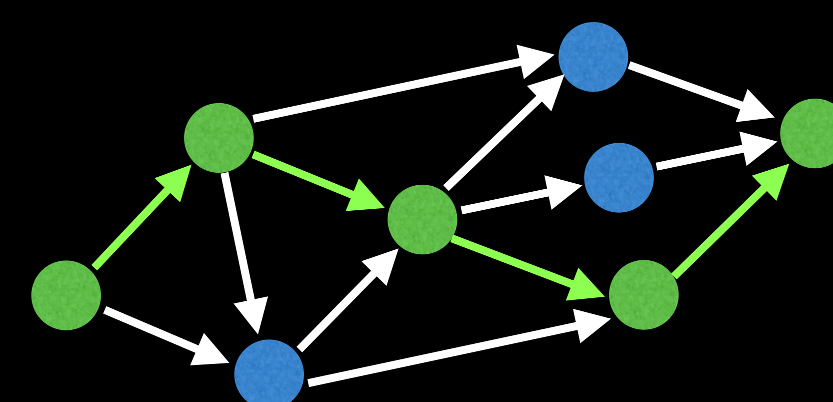






Lowest Common Ancestor

🌿William Fiset🌿

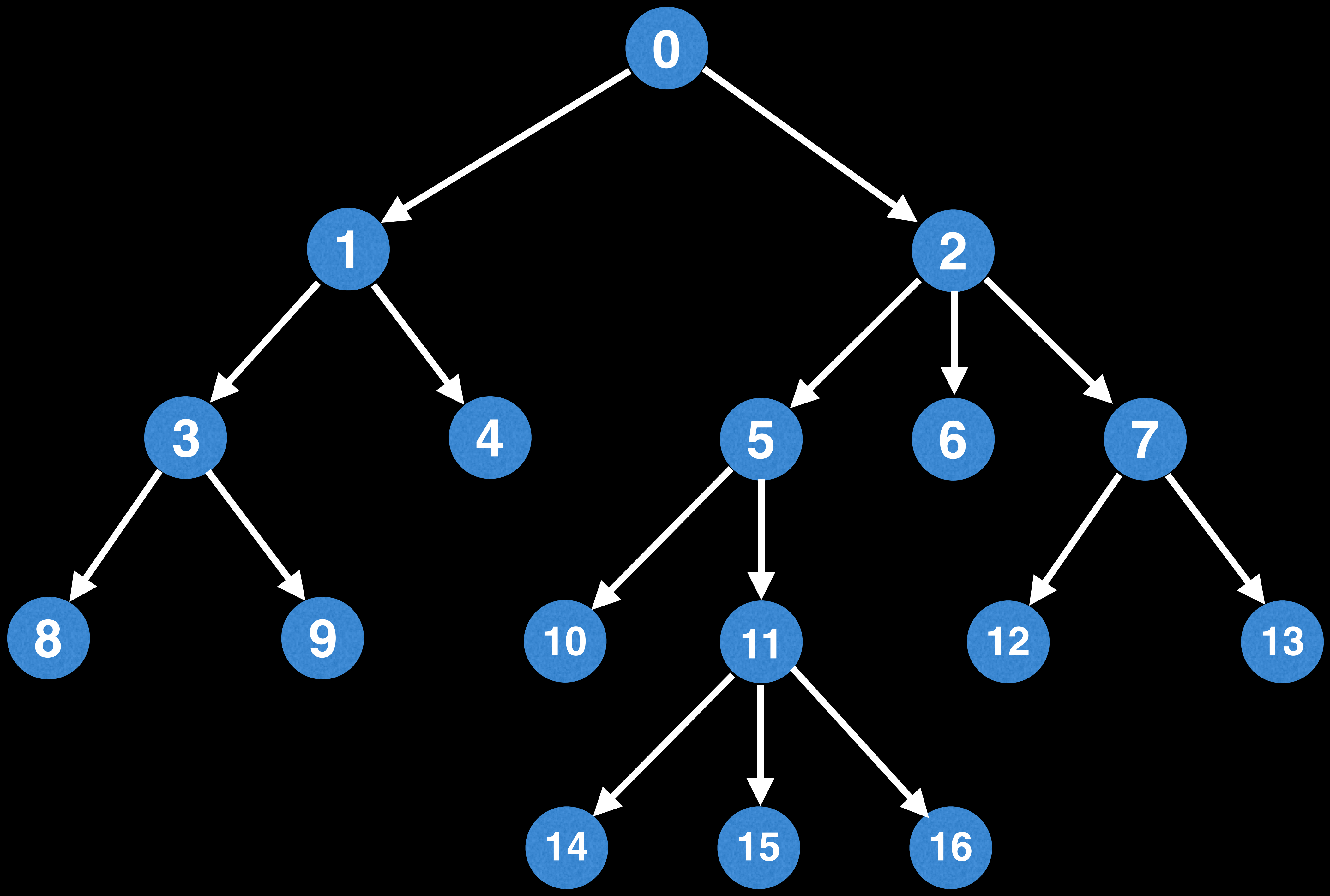
Definition

What is a LCA?

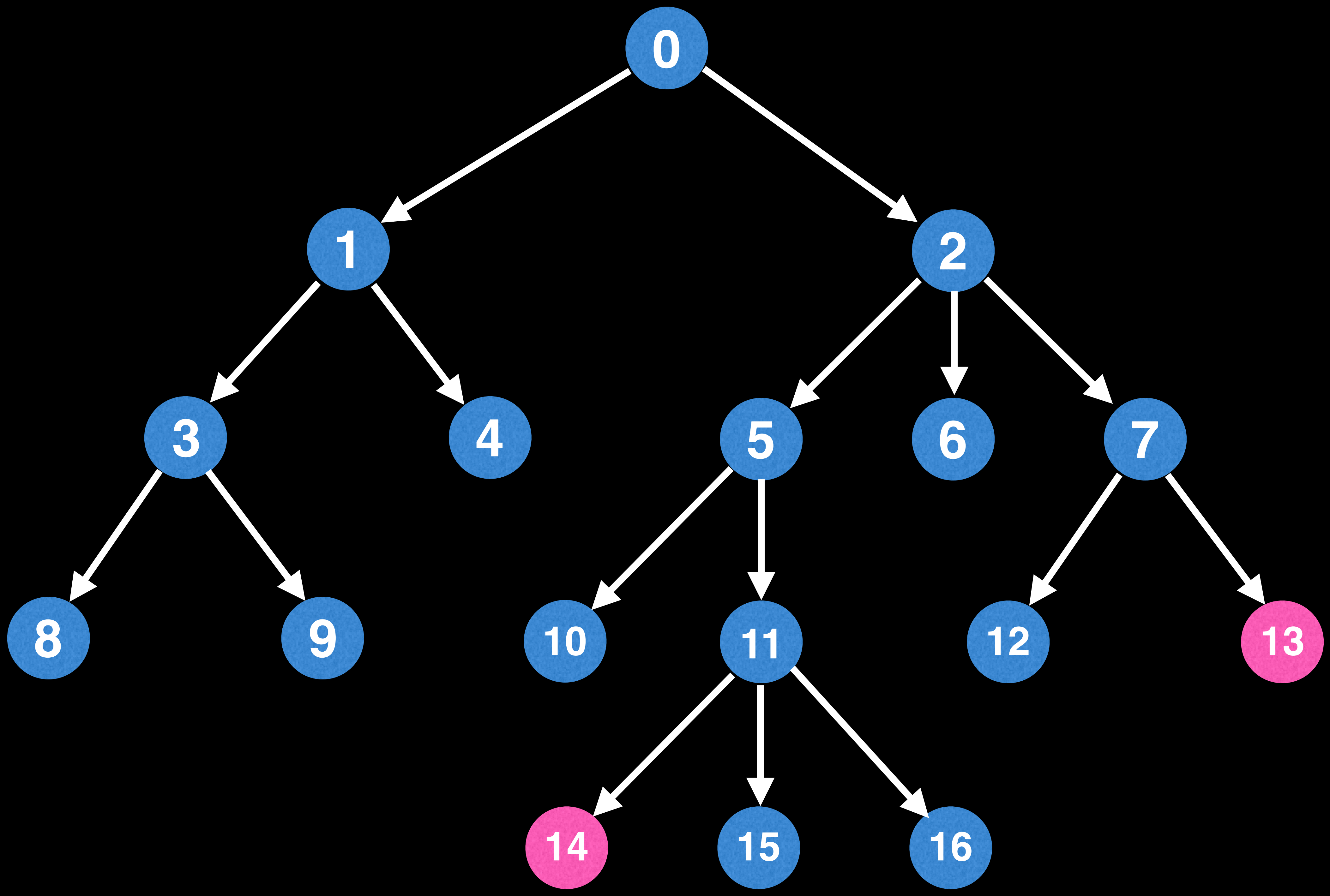
Where is it used/found?

Give examples, maybe 2 or 3? Family tree,

Understanding LCA

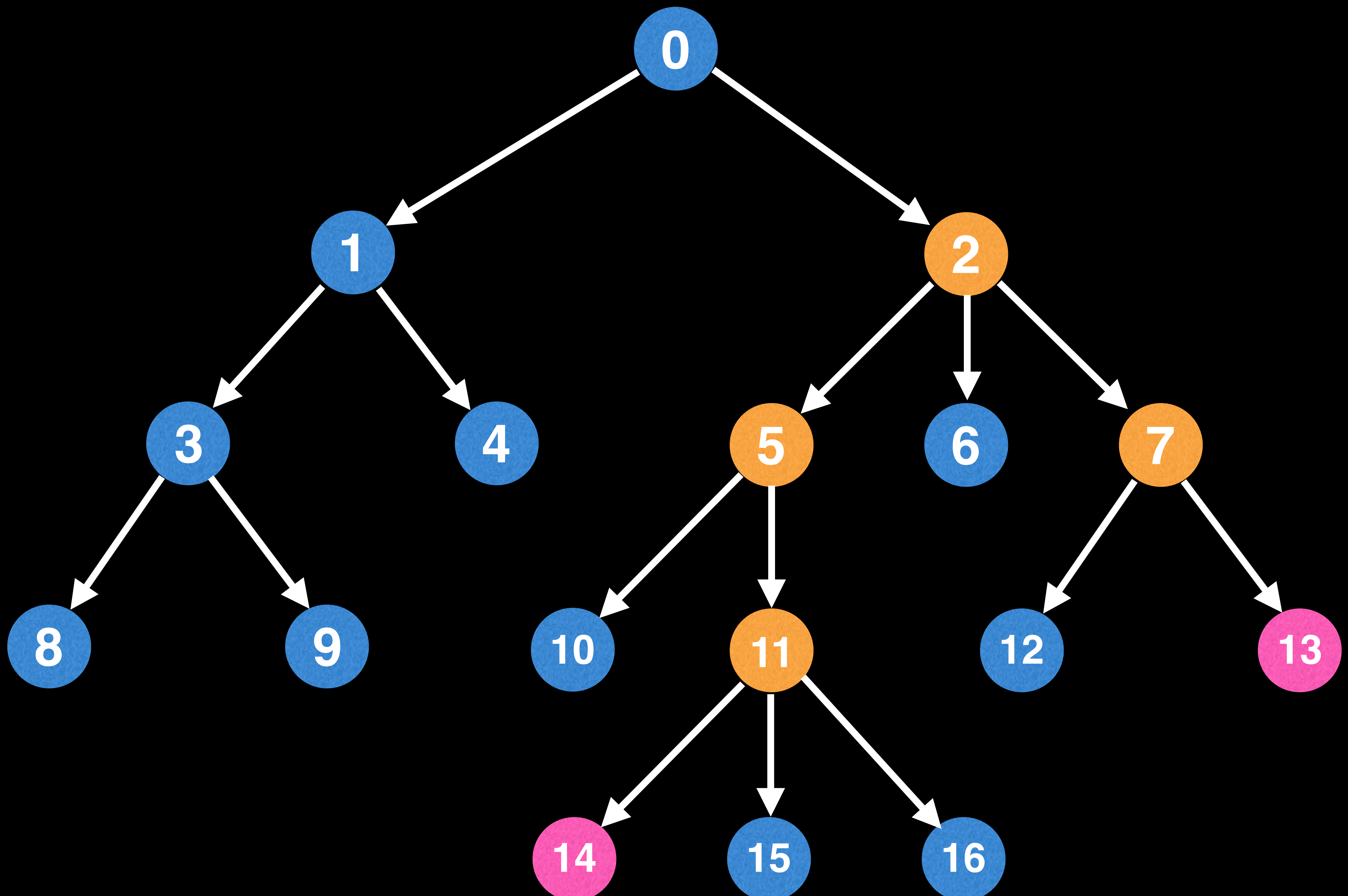


Understanding LCA



Understanding LCA

LCA(13, 14) = 2



Understanding LCA

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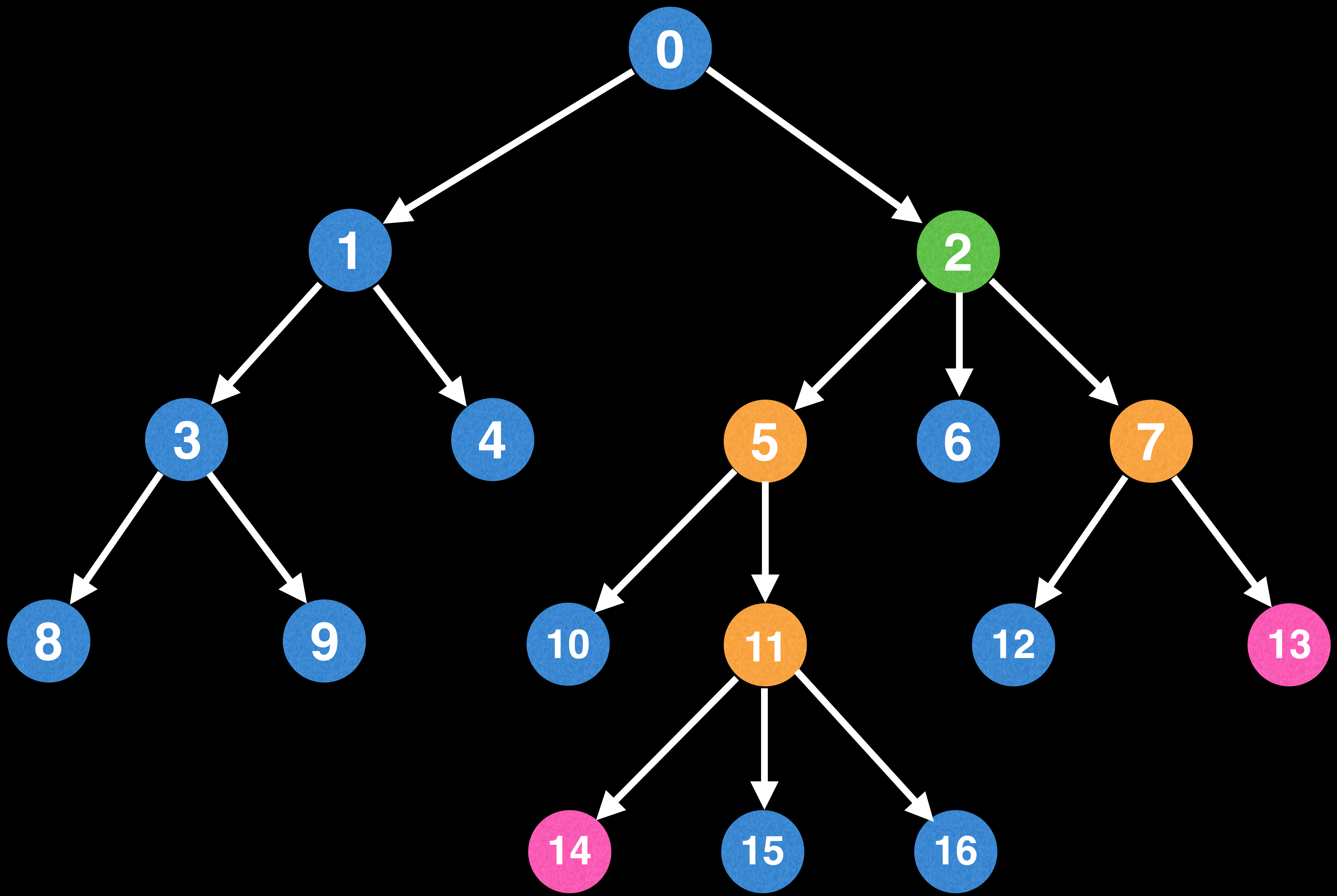

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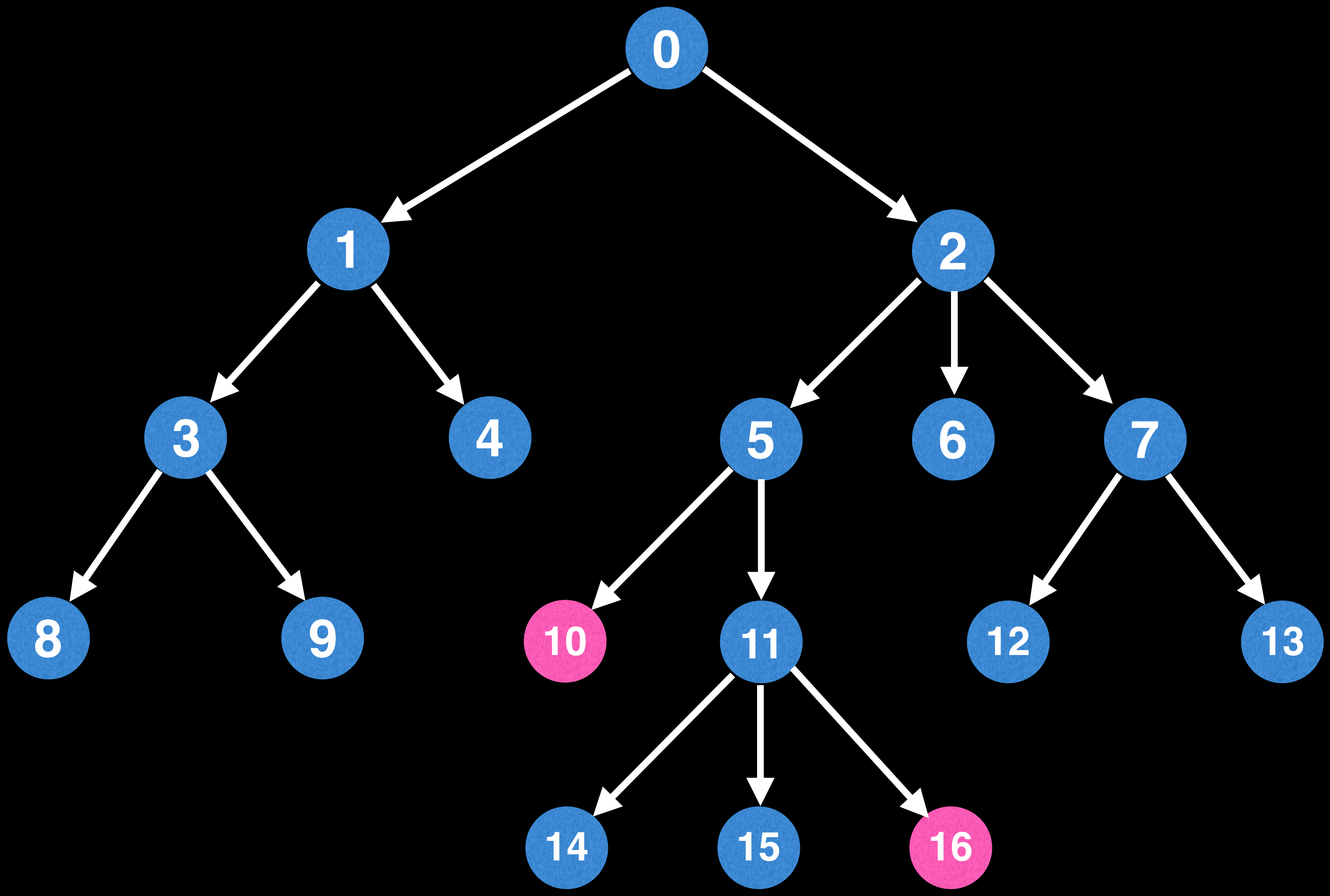
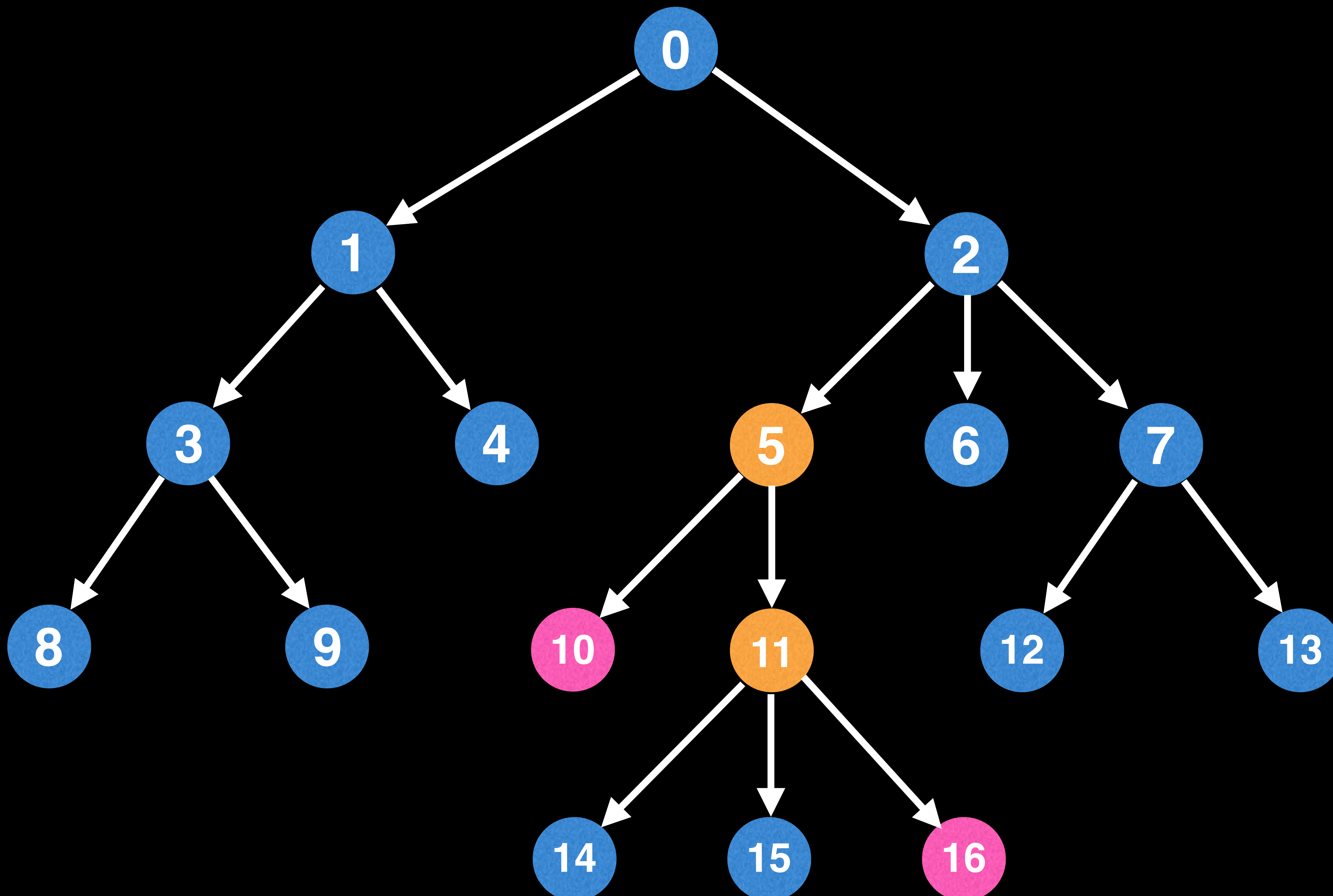


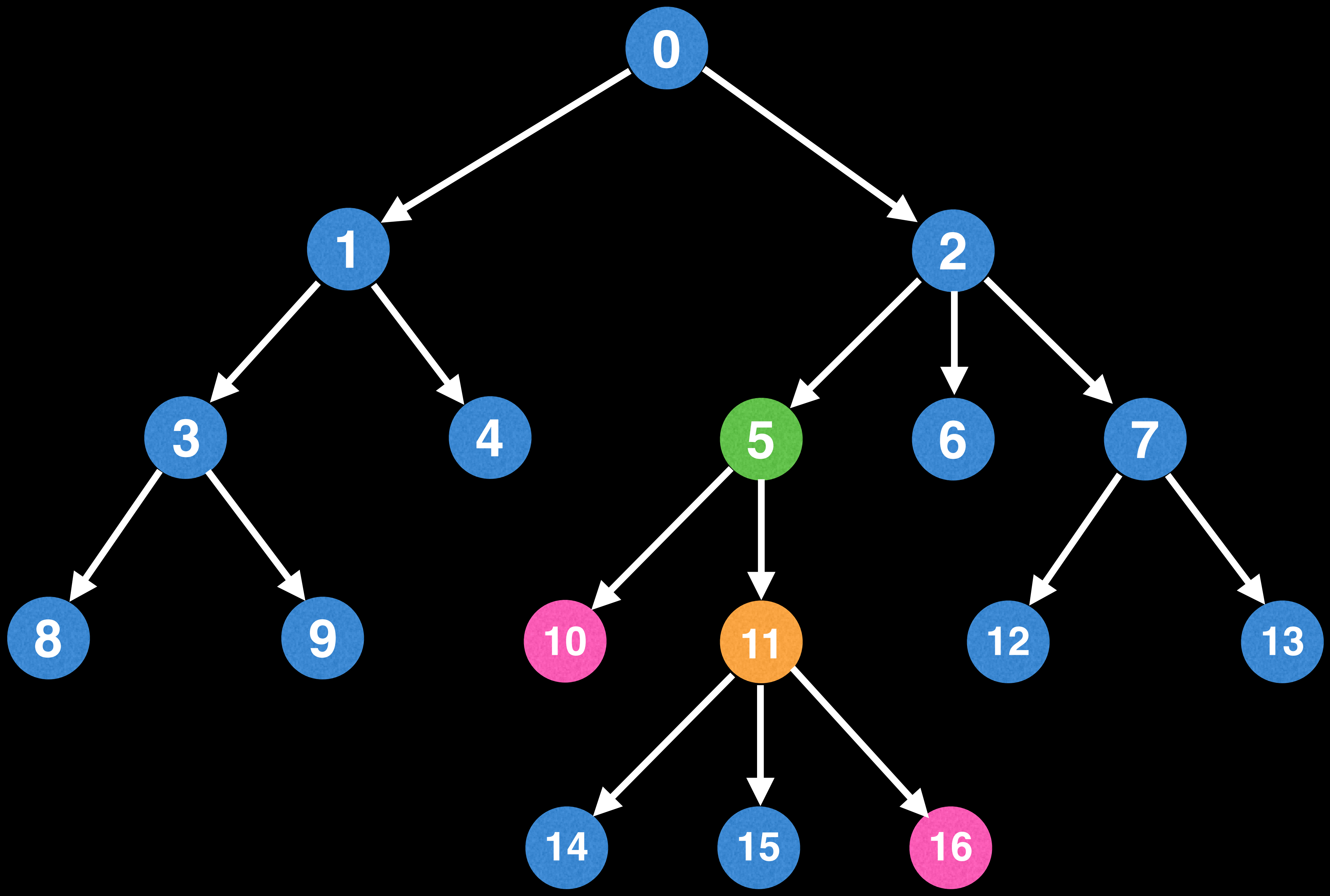
Understanding LCA

LCA(10, 16) = 5

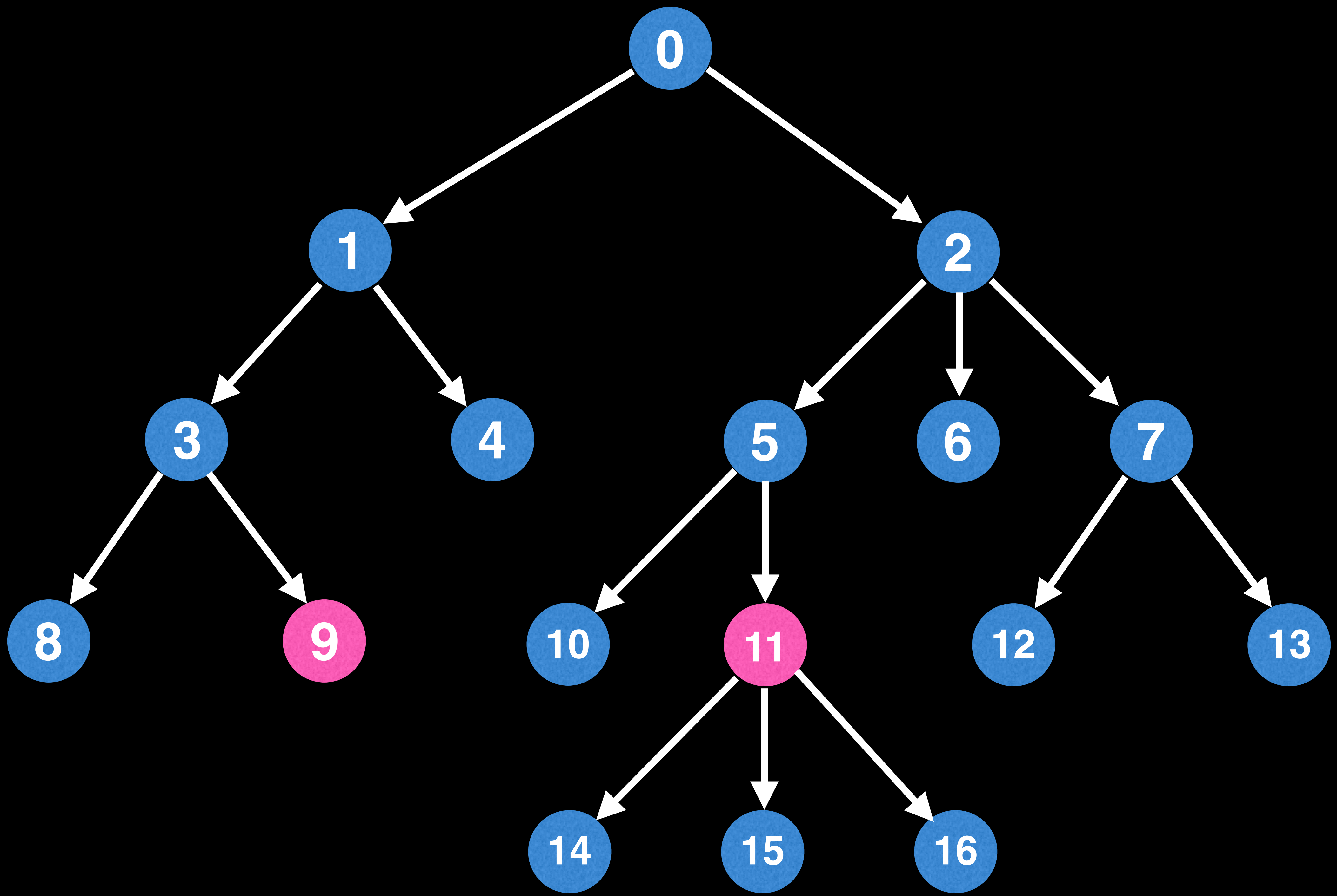


Understanding LCA

LCA(10, 16) = 5

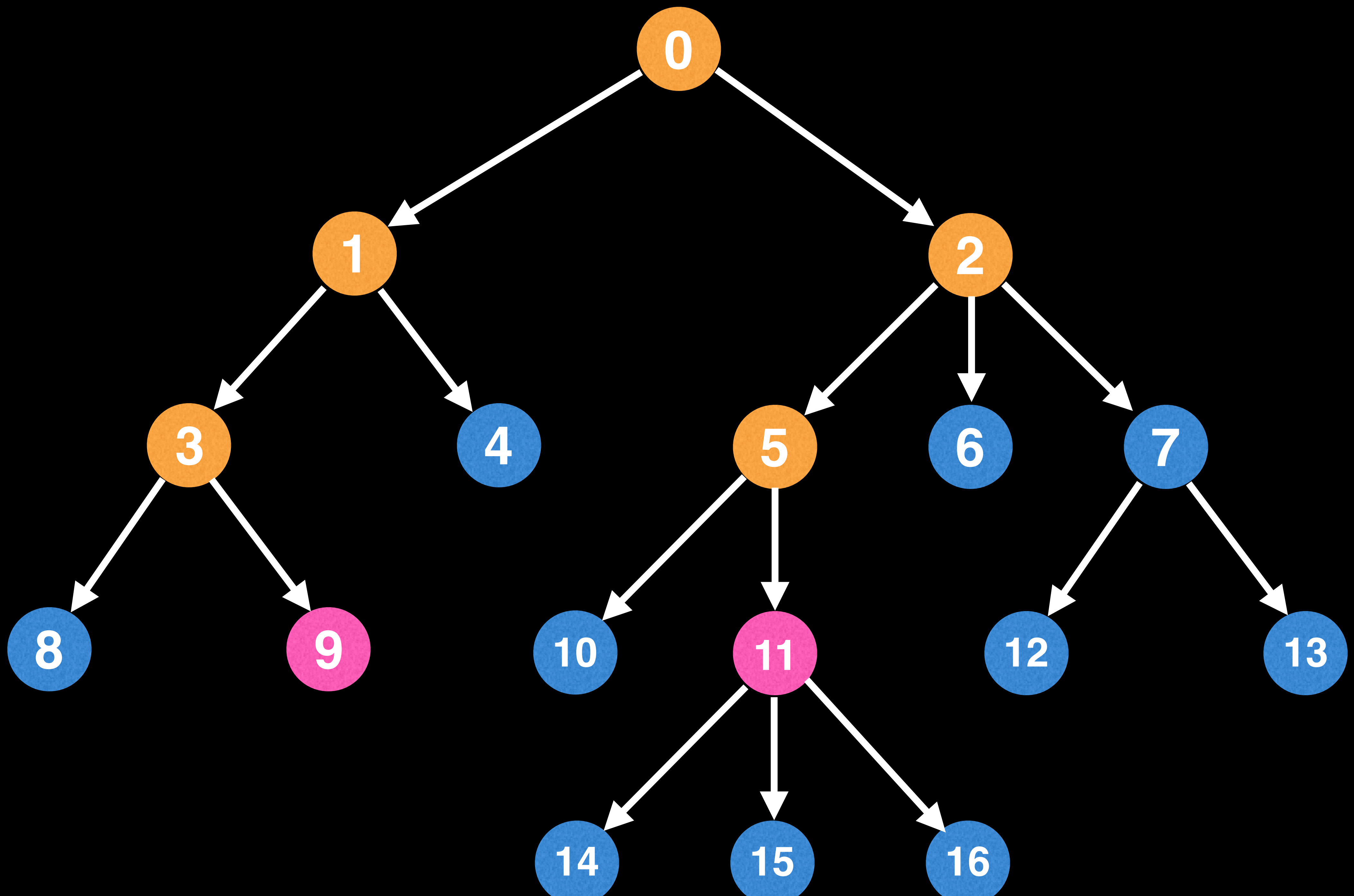


Understanding LCA



Understanding LCA

LCA(9, 11) = 0



Understanding LCA

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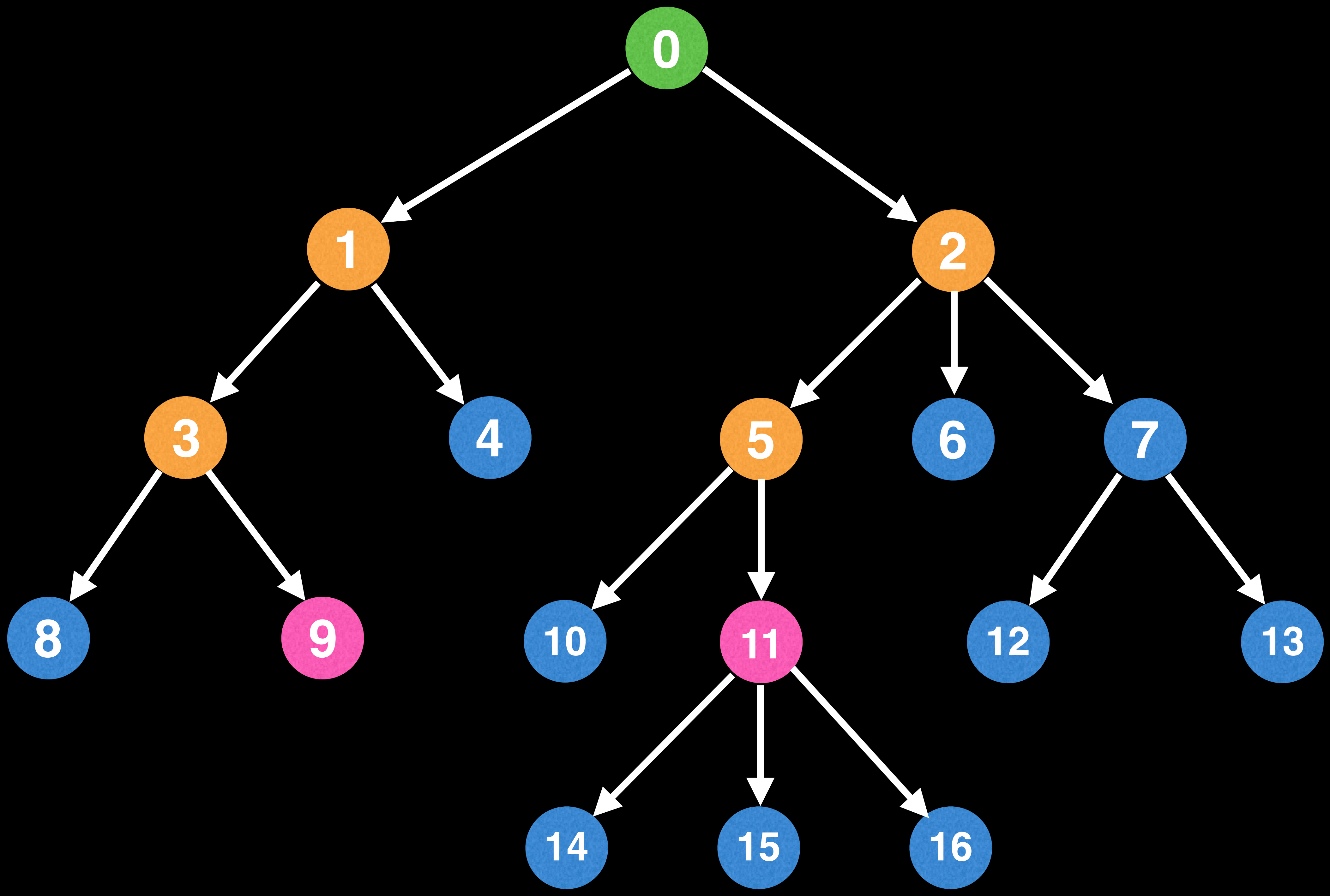
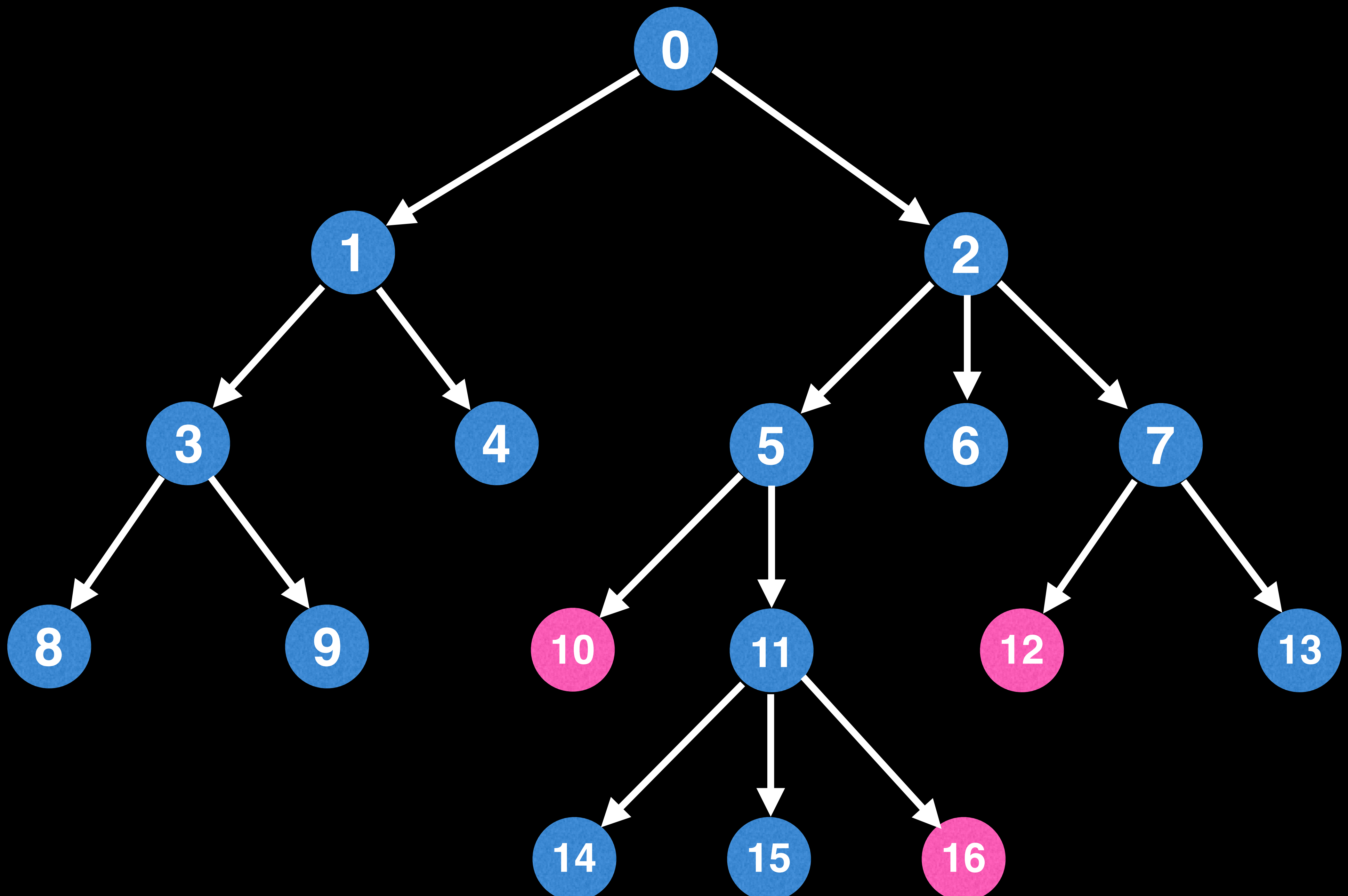


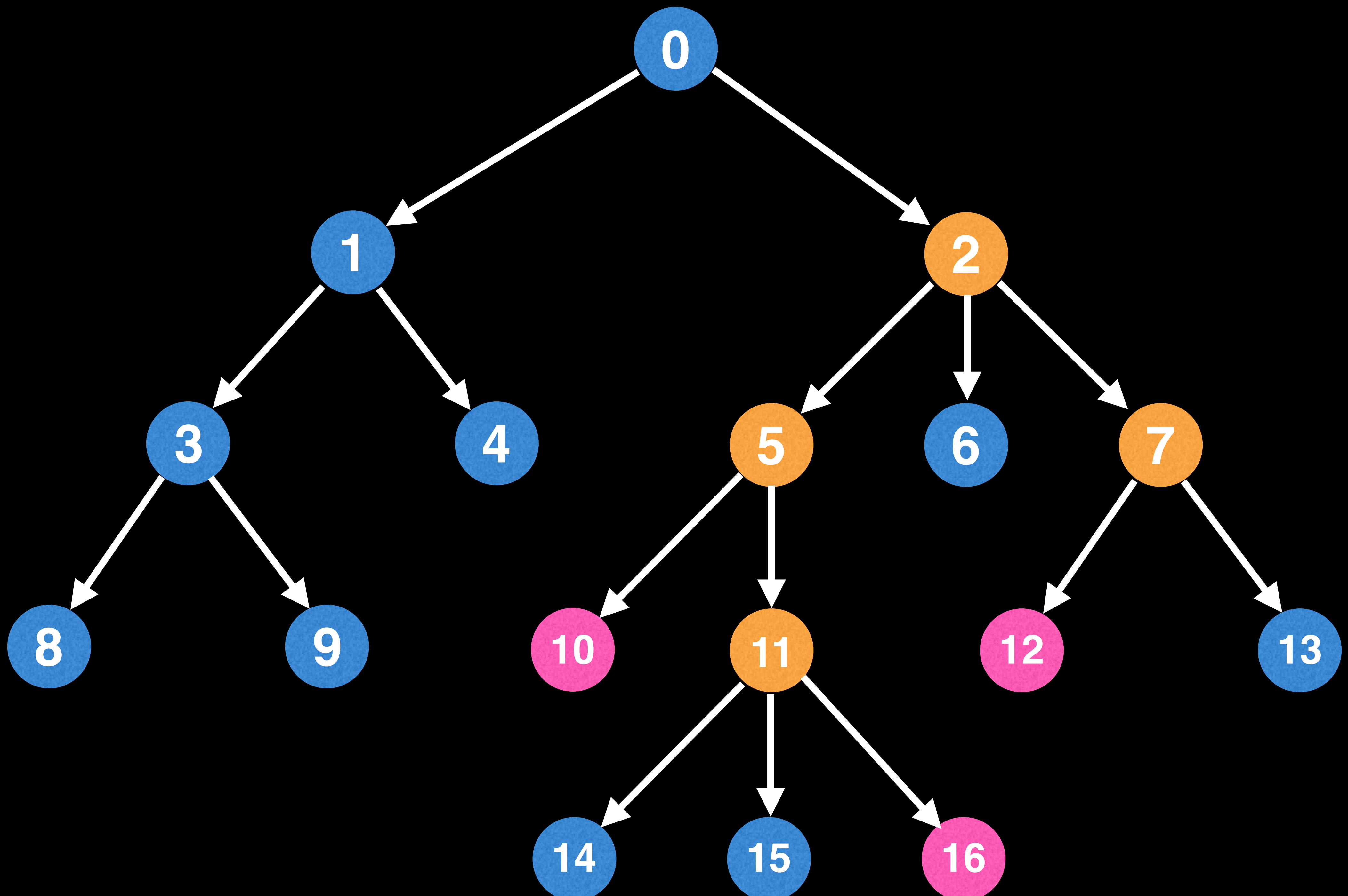
Understanding LCA

You can also find the LCA of more than 2 nodes



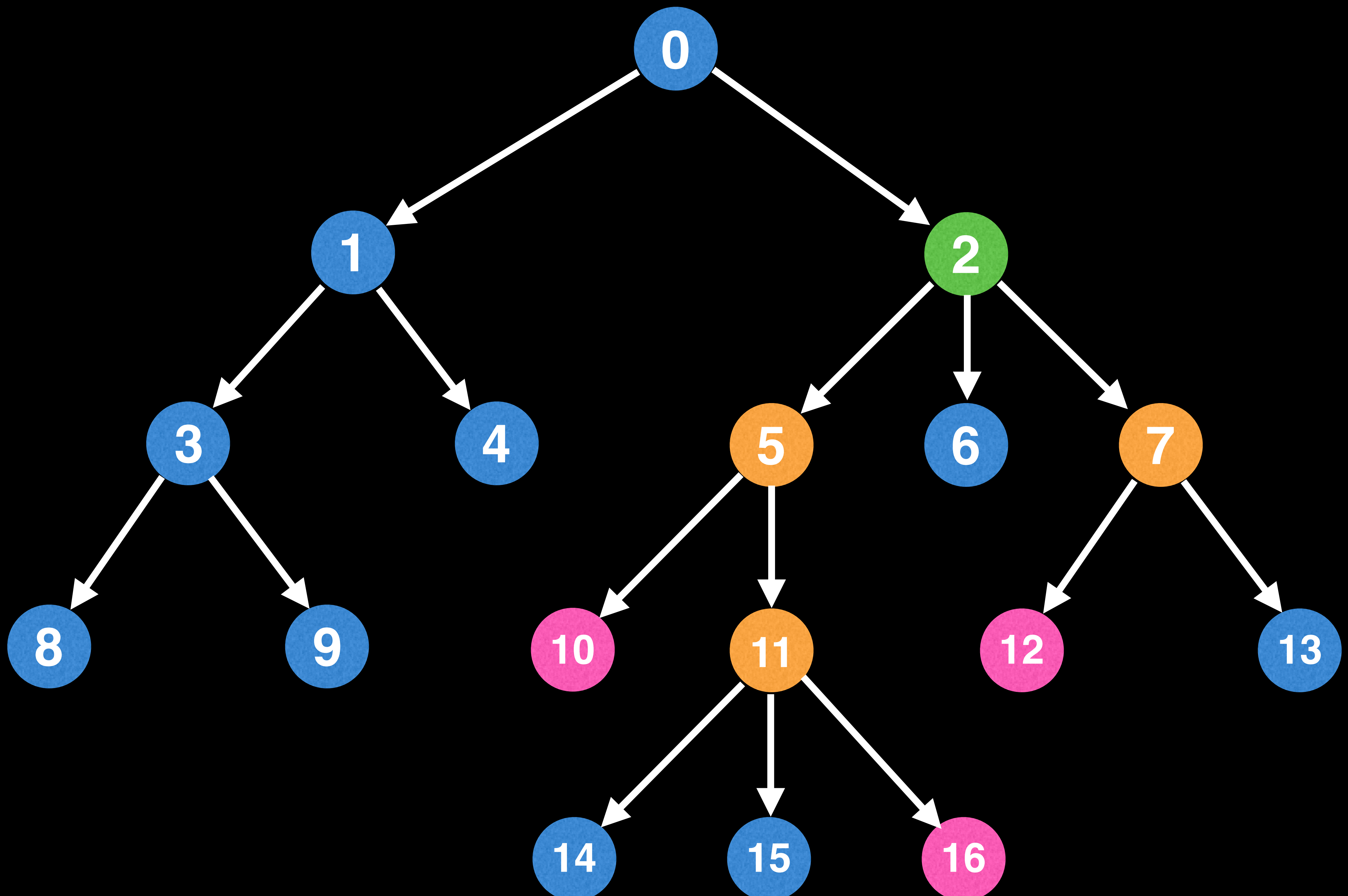
Understanding LCA

LCA(10, LCA(12, 16)) = 2



Understanding LCA

LCA(10, LCA(12, 16)) = 2



Situation matters

There are lots of considerations to take into account when solving the LCA problem which should change how you choose to tackle the problem.

Situation matters

Question 1: How is the tree stored?

First of all, you're going to need a rooted tree to be able to find the LCA of two nodes.

Is the tree a regular rooted tree, or a binary tree? A binary search tree? Does the tree have parent pointer references?

The only reason you’d care about whether your tree is a BST is because it makes it easier to search.

Situation matters

Question 2:
Do you already have a reference to the nodes in the tree or do you need to find them?

Often, you can set yourself up in a position where you would know the reference/id of the nodes to query, otherwise you need to search the tree which in the worst of scenarios is **$O(n)$**.

Situation matters

Question 3:
Is the tree static or dynamic? I.e, can nodes be added or removed?

Static trees are typically easier to work with, but there are algorithms which can handle dynamic trees quite well.

Situation matters

Question 4:
Do you need an online or offline algorithm?

That is, do you know all the LCA queries in advance? Or do you need to be able to dynamically find the LCA?

Various LCA methods

todo(william): investigate Euler tour method with minimum range query

Tarjan’s offline method with the union find seems interesting

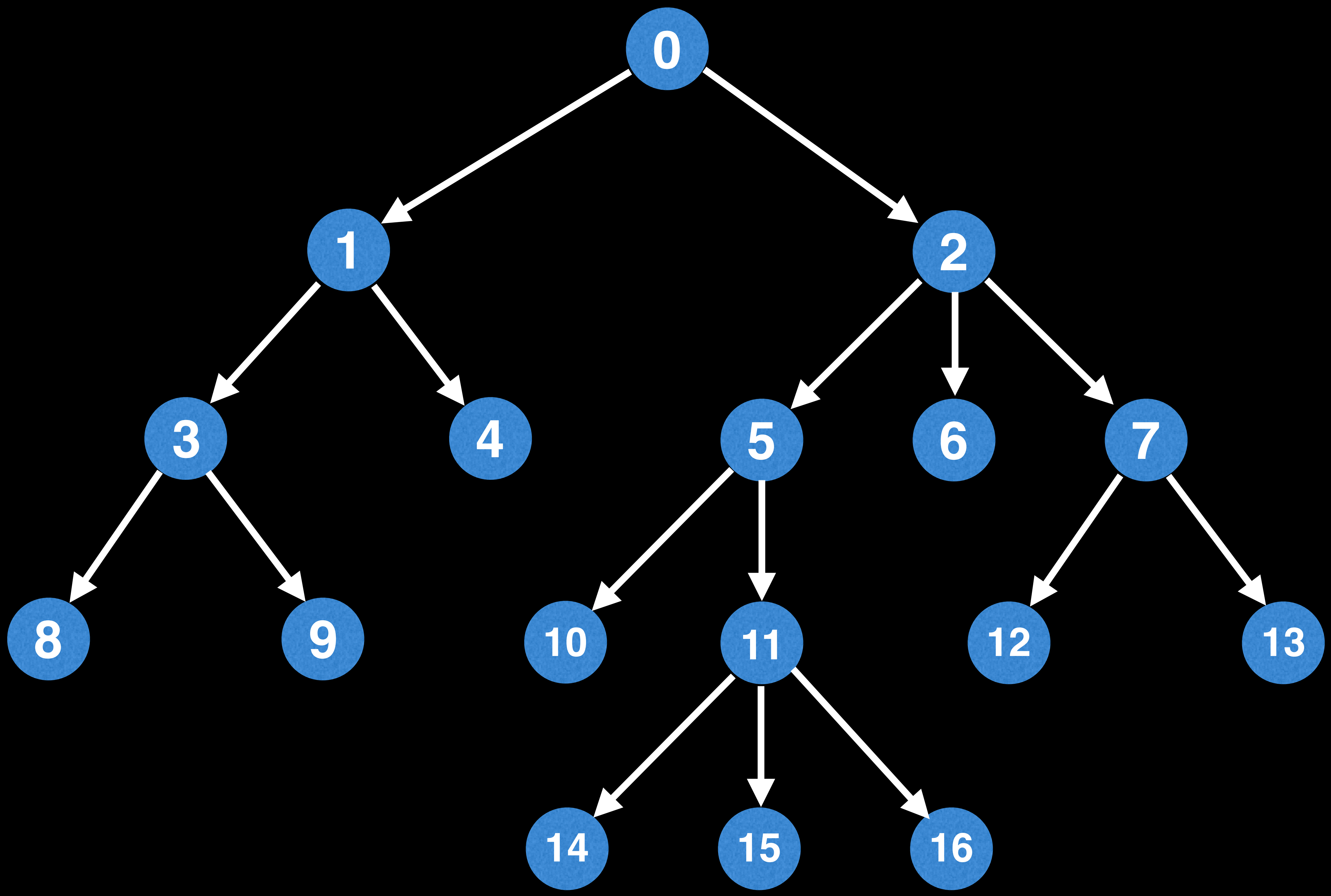
todo(william): implement light heavy decomposition and verify that dynamically adding nodes can be done in O(1) or even O(logn)

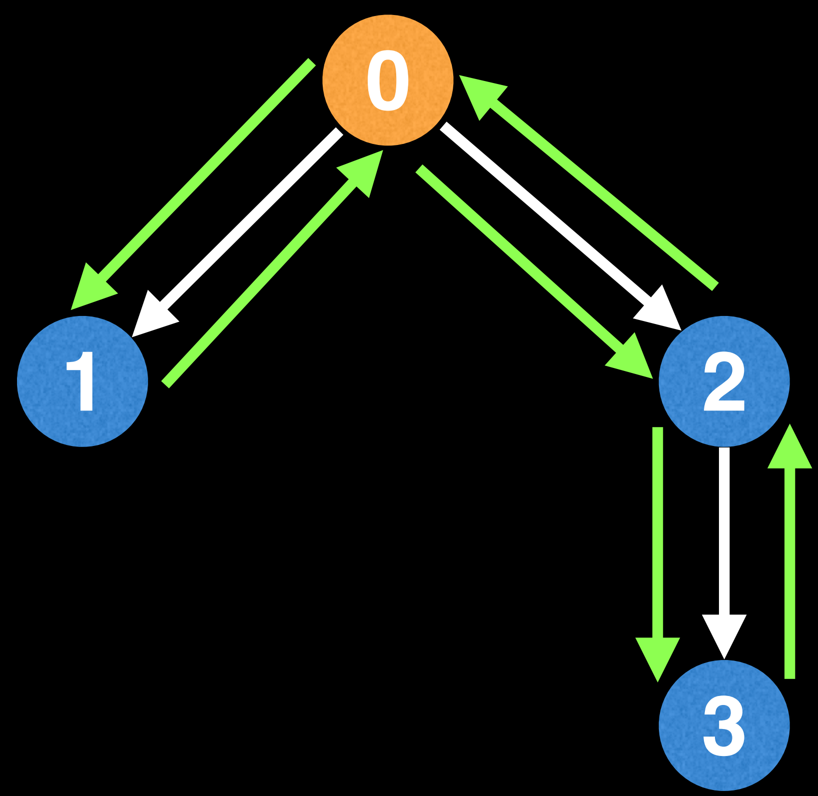
Various LCA methods

Mention this next method (Euler tour + sparse table LCA method) works best with static trees

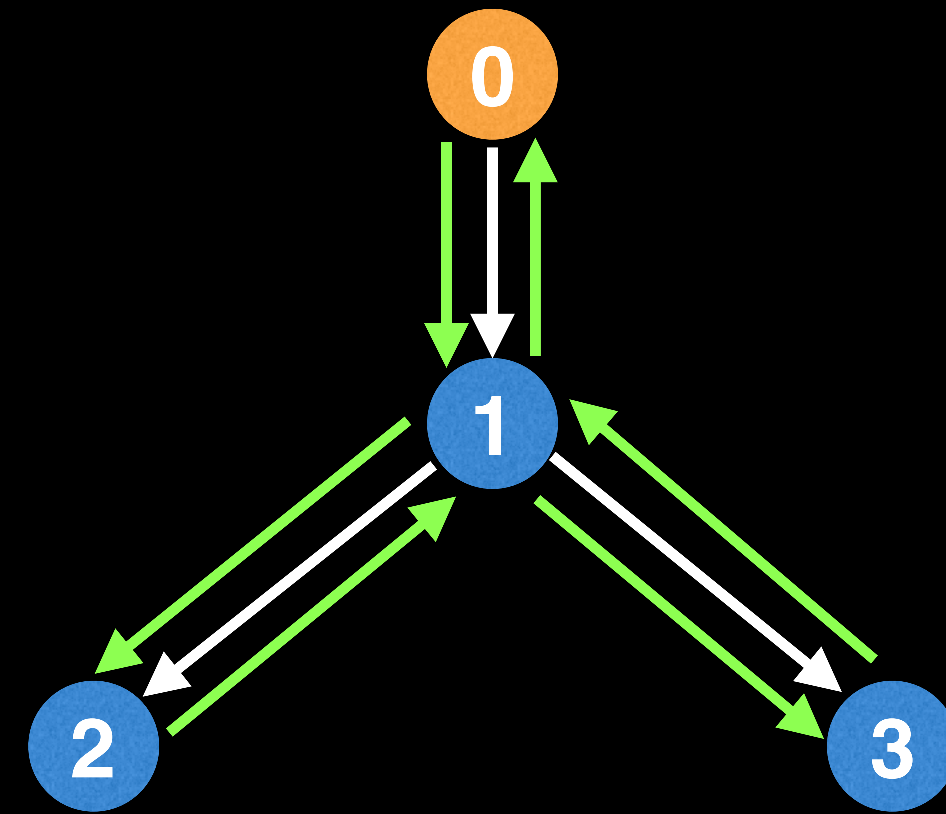
Rooted tree, Static tree, have node references, Online algorithm

LCA on a static tree





tour: [0,1,0,2,3,2,0]

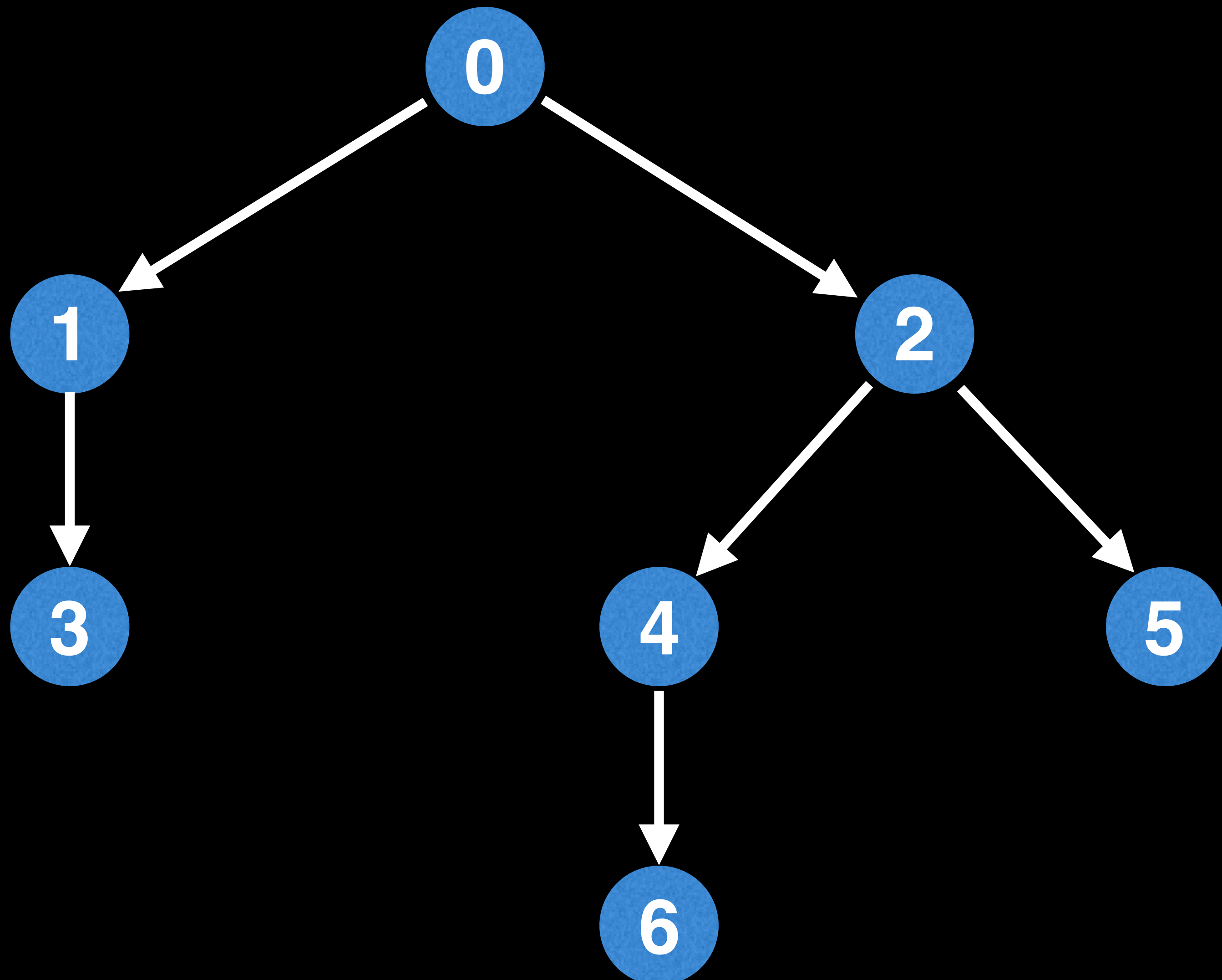


tour: [0,1,2,1,3,1,0]

Start the **Eulerian tour** (Eulerian circuit) at the root node, visits all edges (assume white edges are undirected), and finally return to the root node.

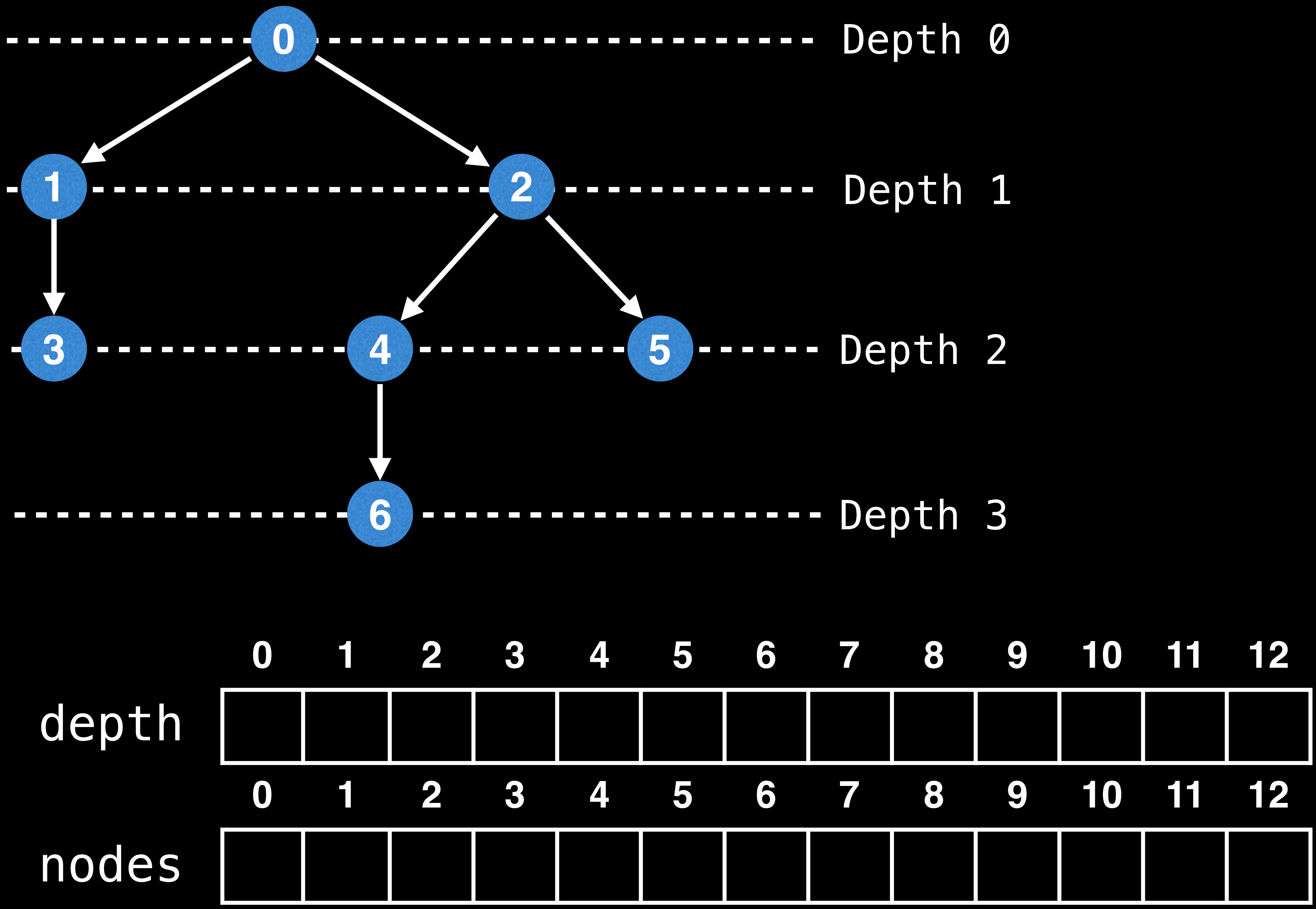
An Euler tour on a tree requires visiting 2 nodes per edge, plus 1 additional visit for the start node:

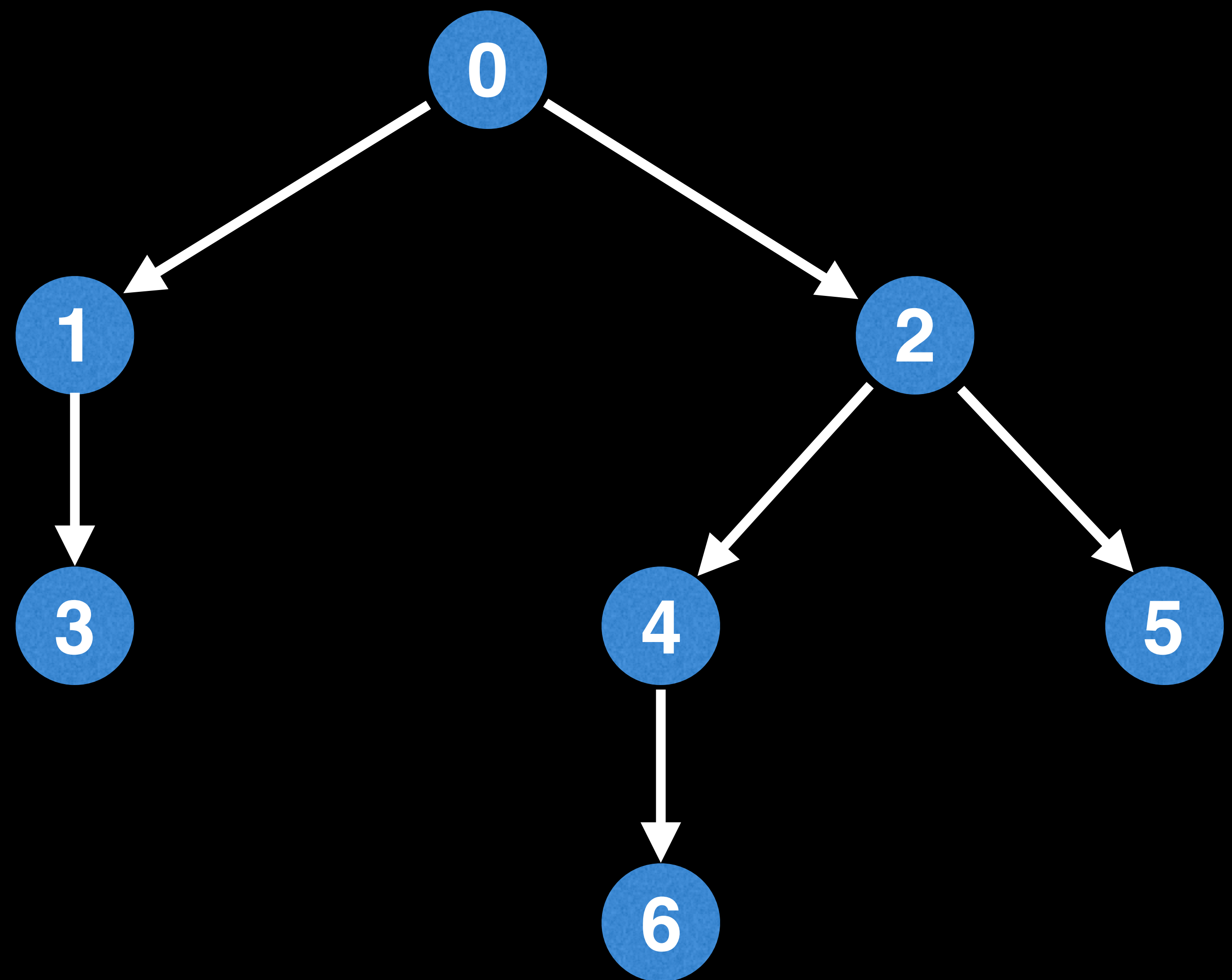
$$2*(n-1) + 1 = 2n - 2 + 1 = 2n - 1 \text{ nodes}$$



	0	1	2	3	4	5	6	7	8	9	10	11	12
depth													

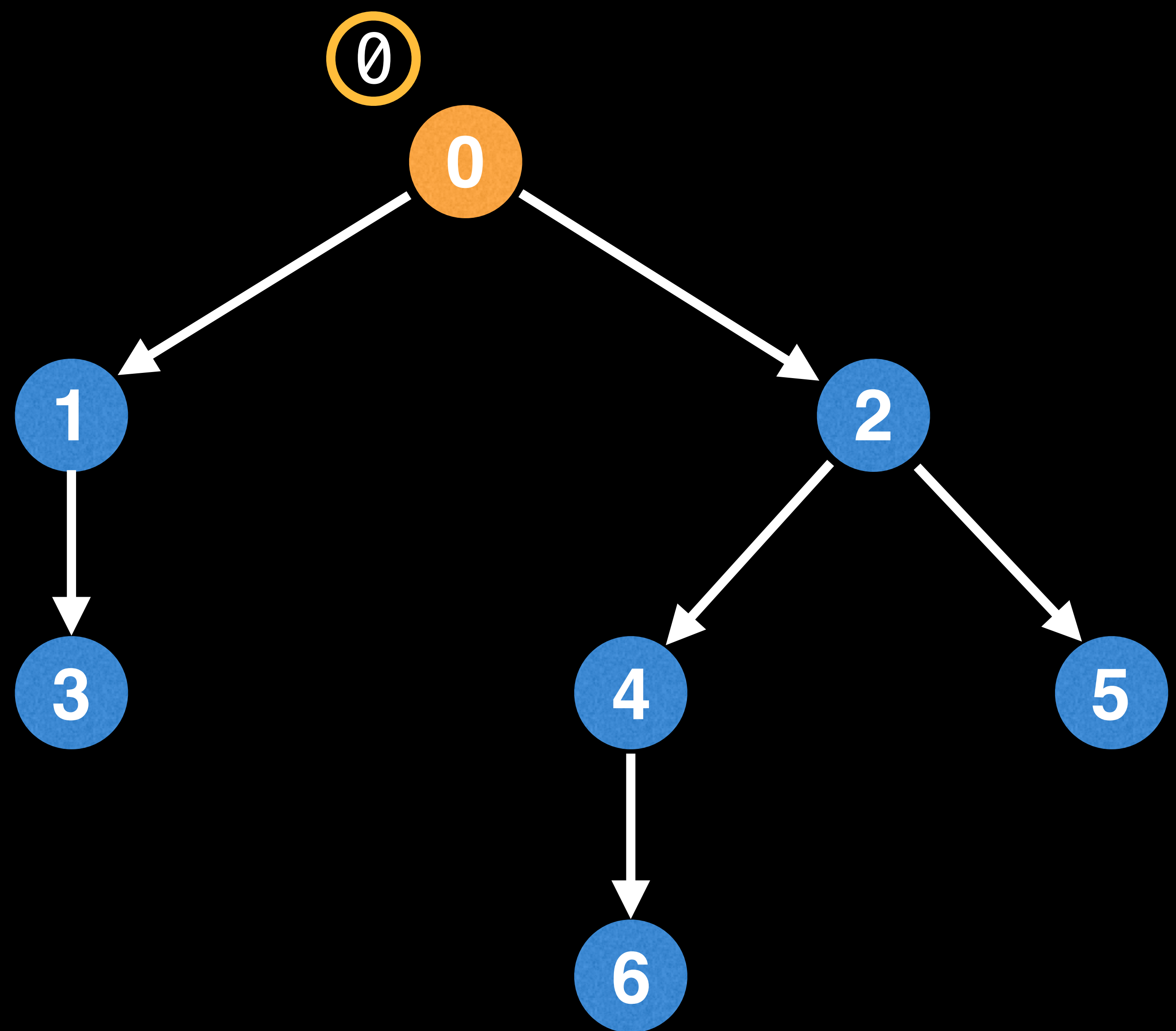
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes													


0
1
2
3
4
5
6
Depth 0
Depth 1
Depth 2
Depth 3
0 1 2 3 4 5 6 7 8 9 10 11 12
depth
0 1 2 3 4 5 6 7 8 9 10 11 12
nodes



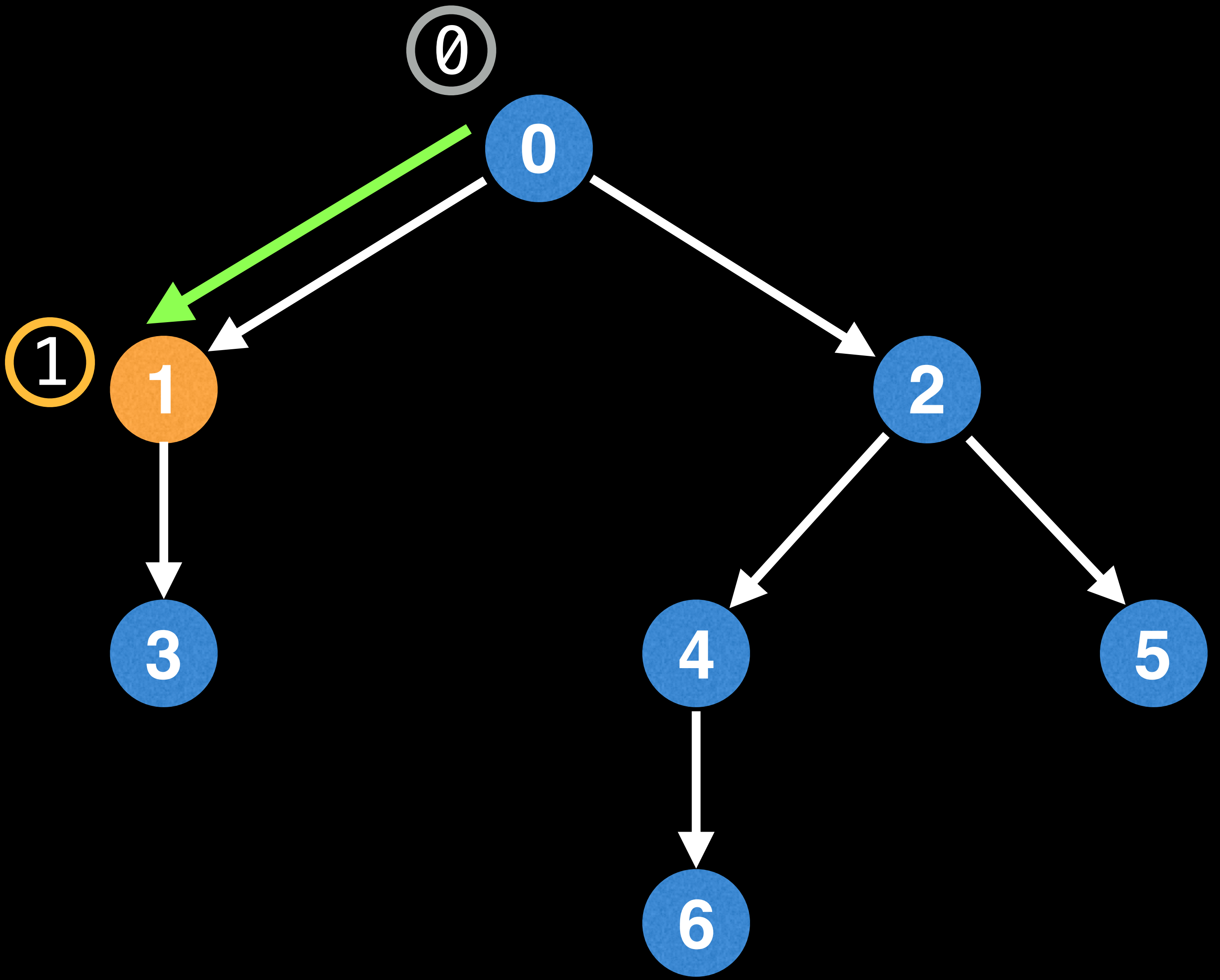
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth													

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes													



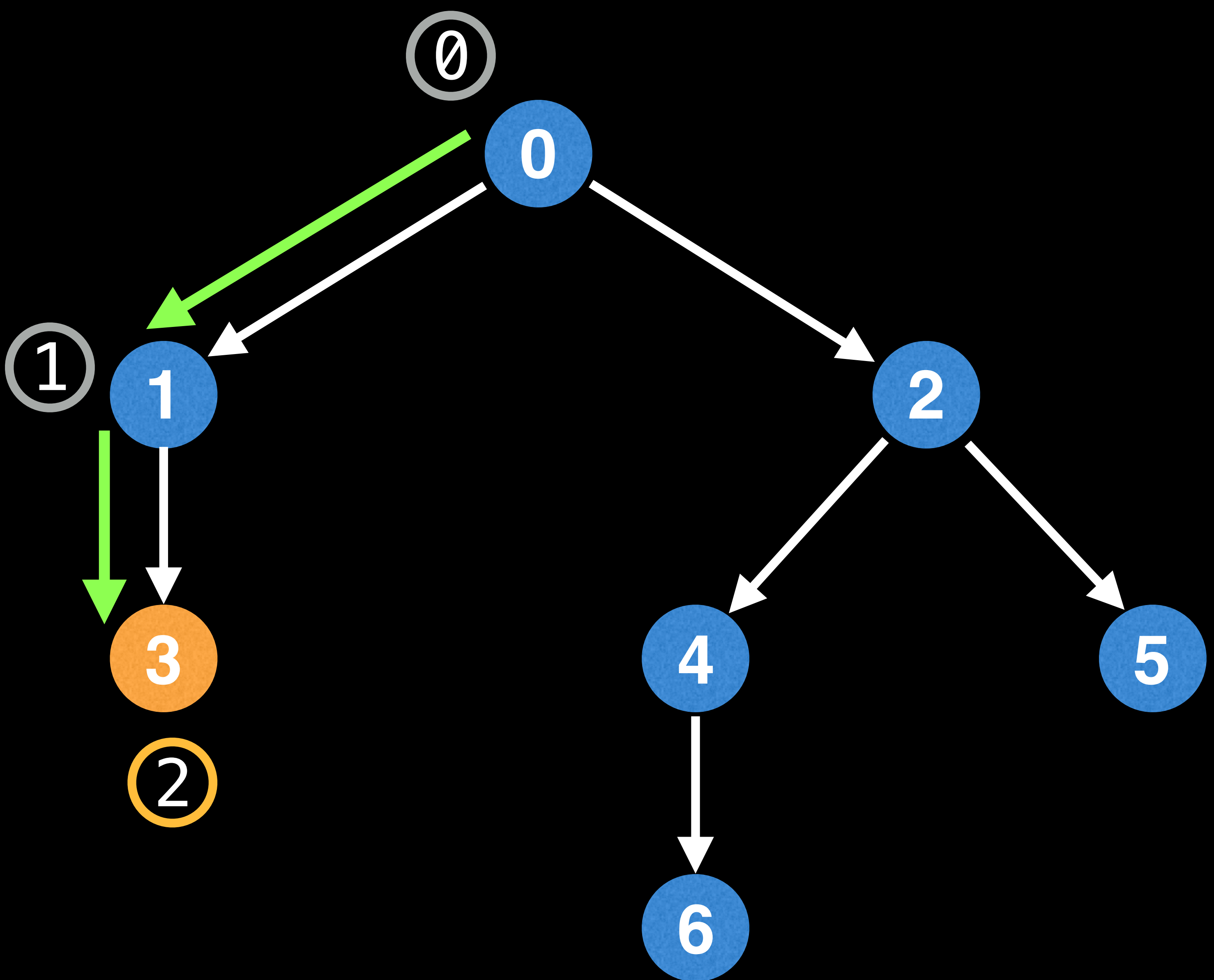
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0												

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0												



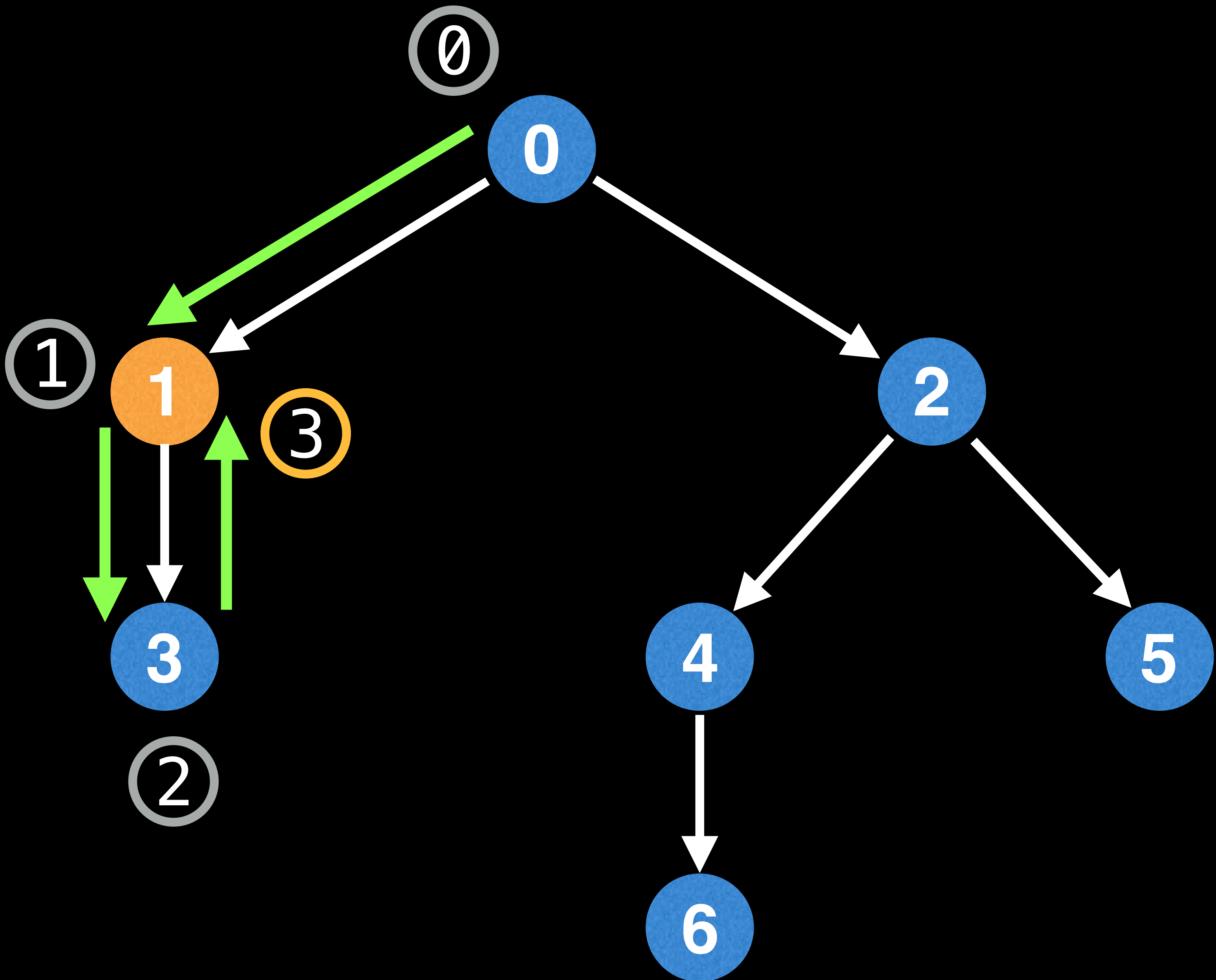
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1											

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1											



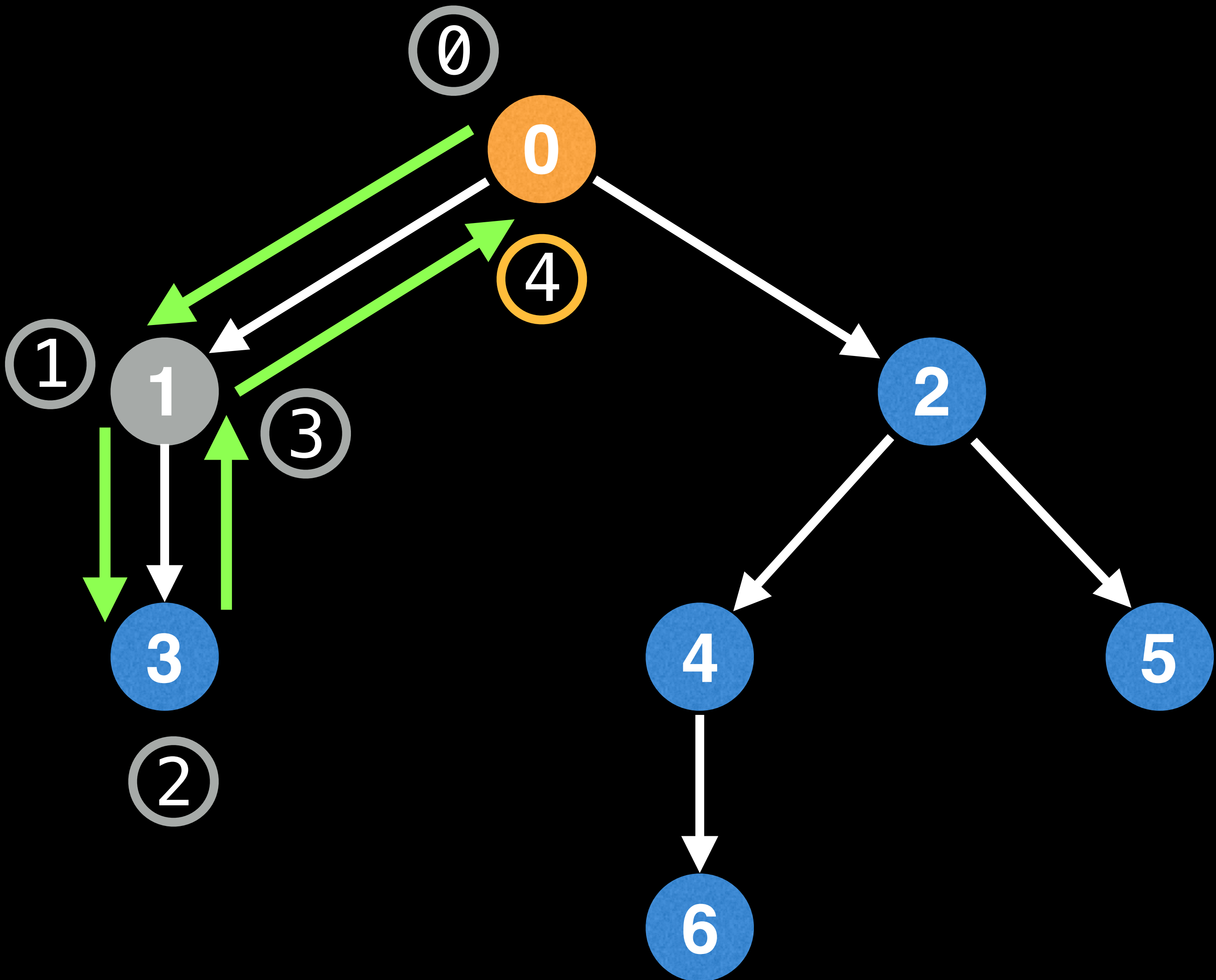
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2										

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3										



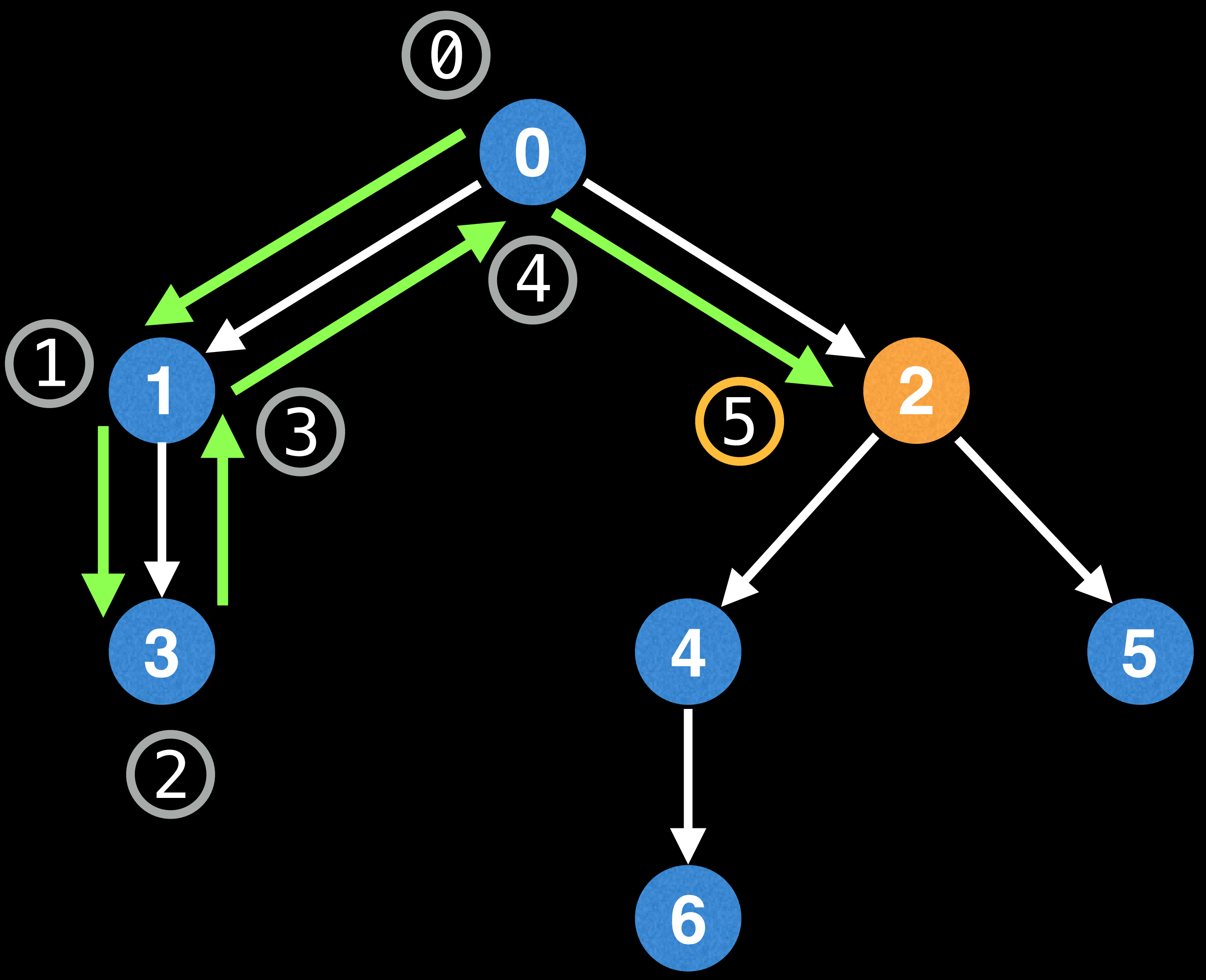
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1									

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1									



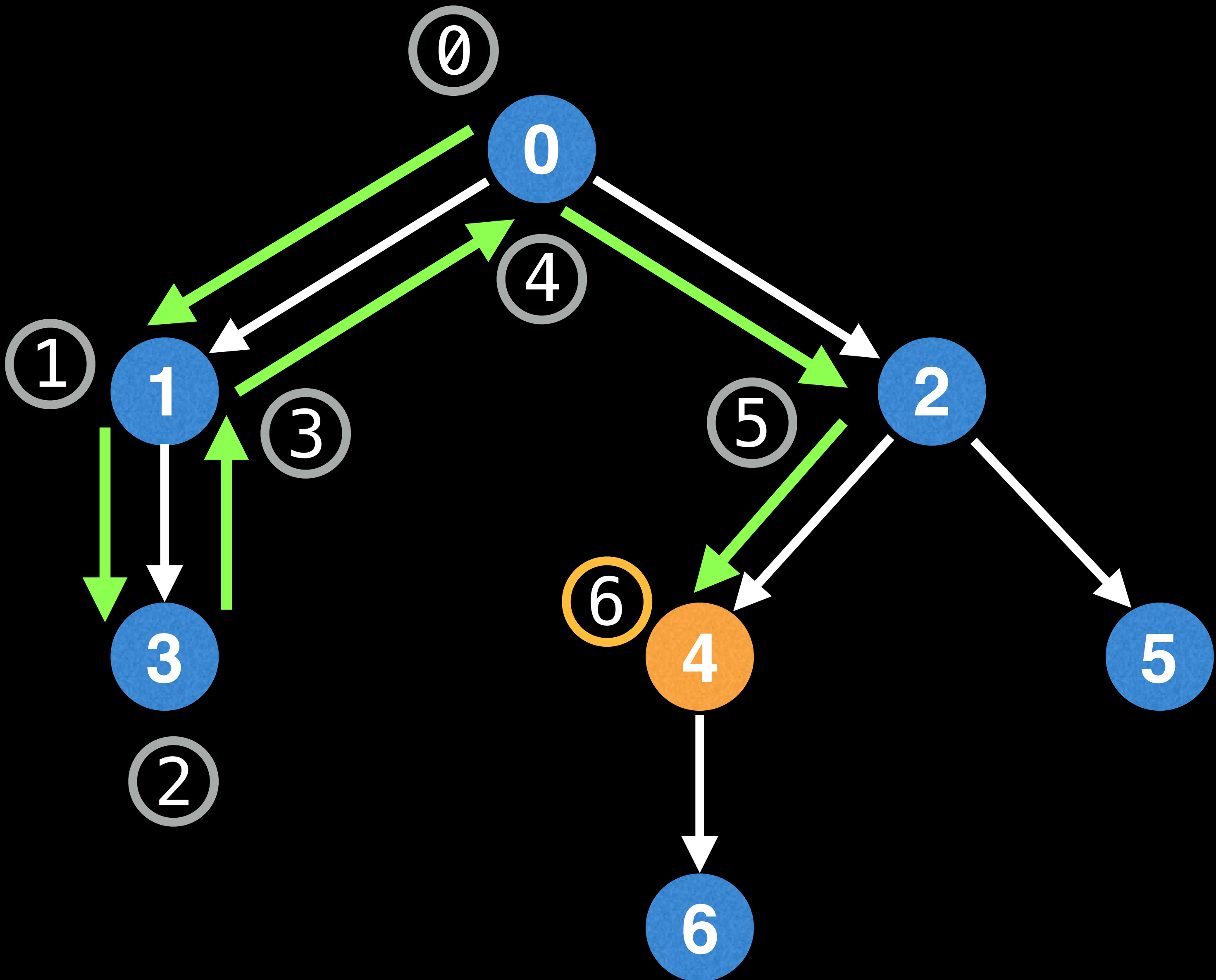
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0								

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0								



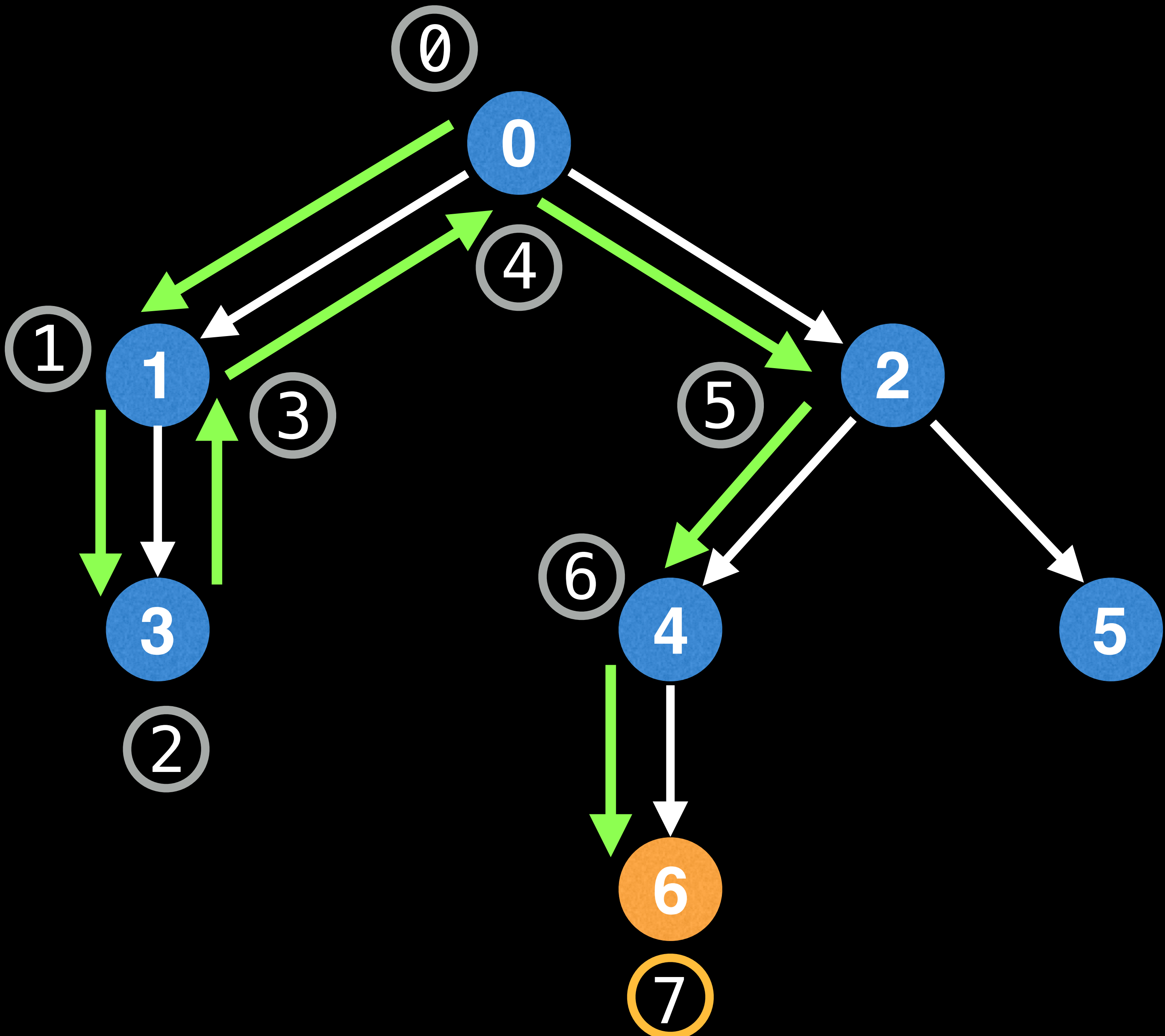
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1							

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2							



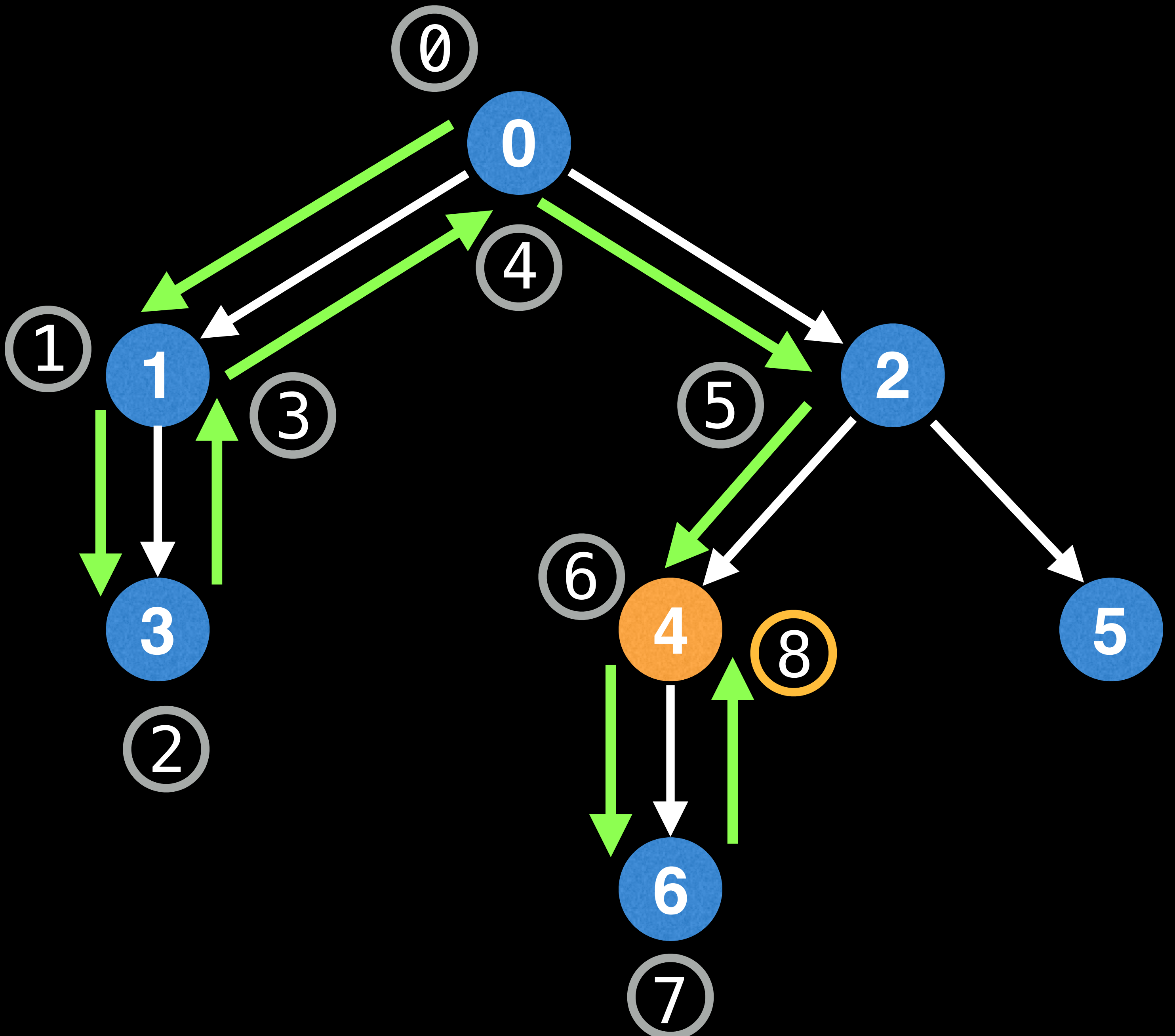
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2						

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4						



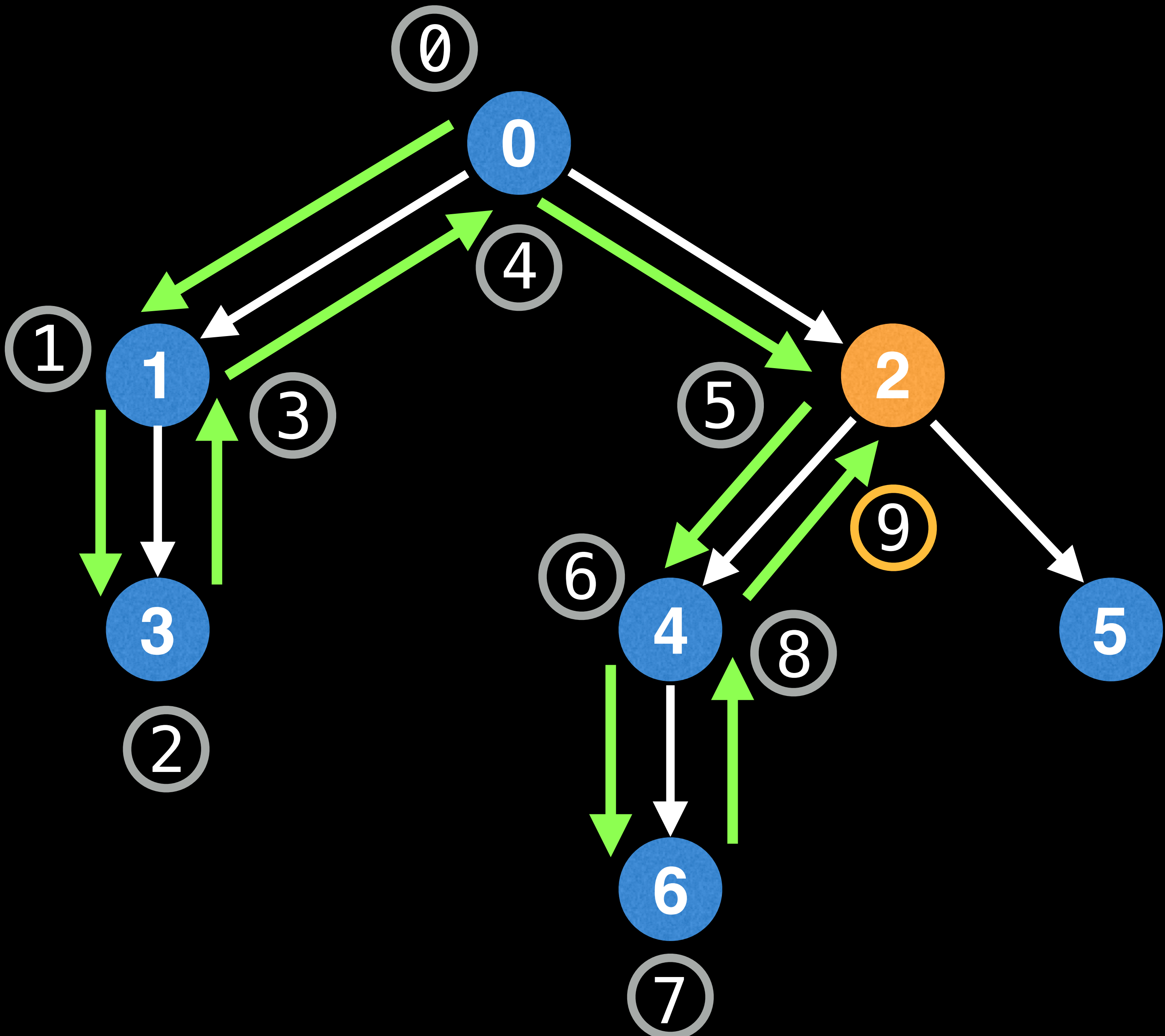
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3					

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6					

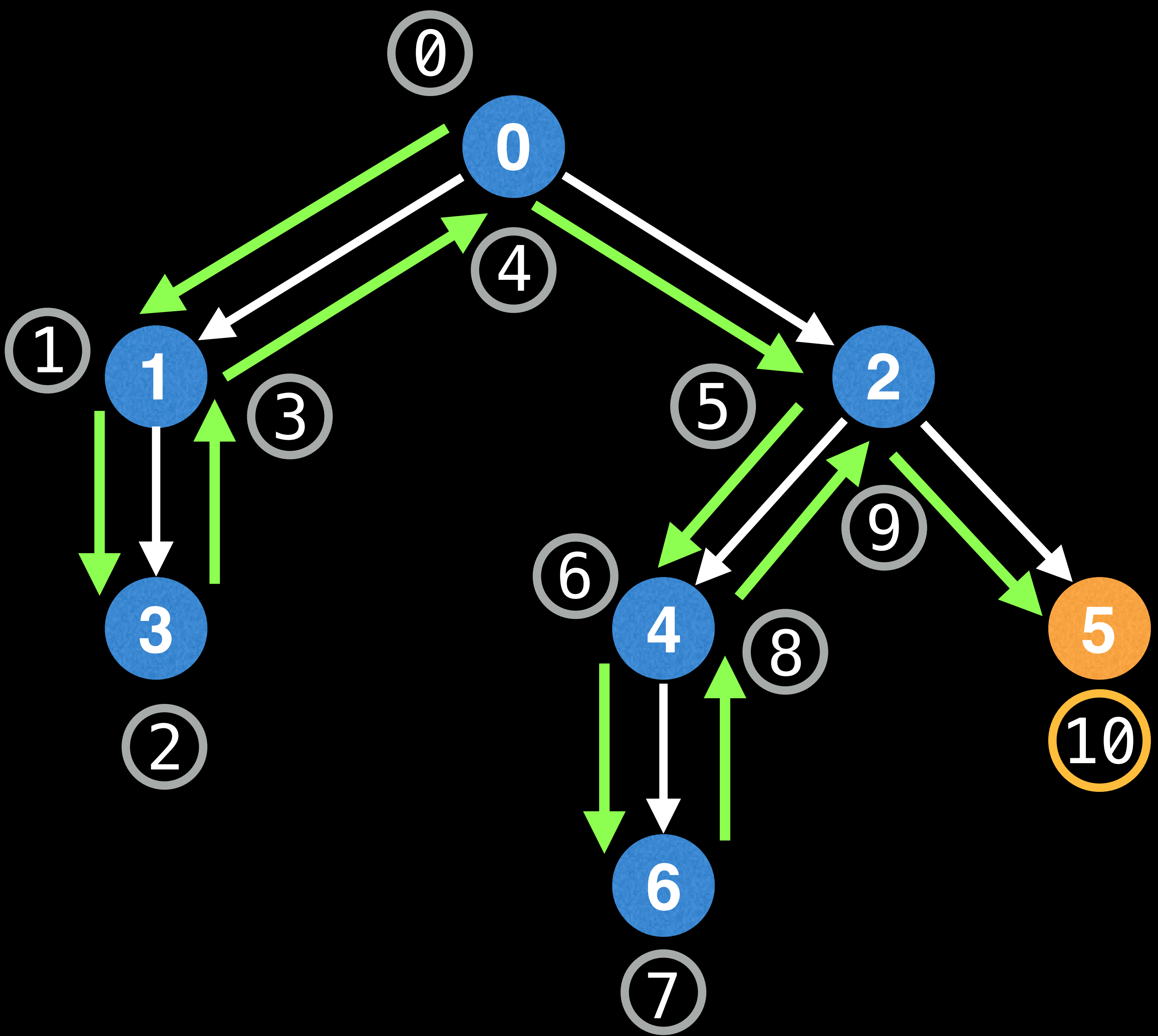


	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2				

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4				

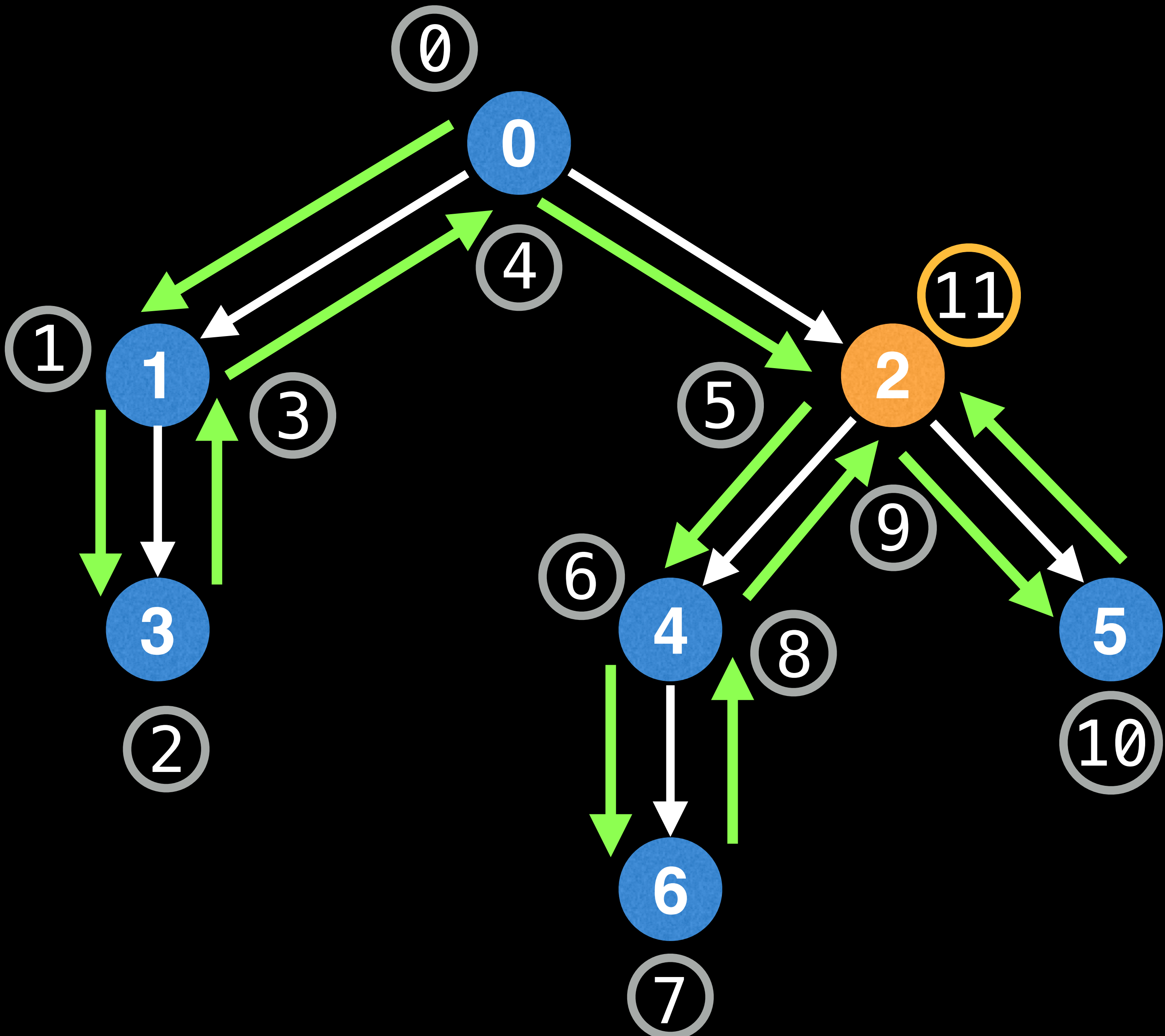


	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1			
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4	2			



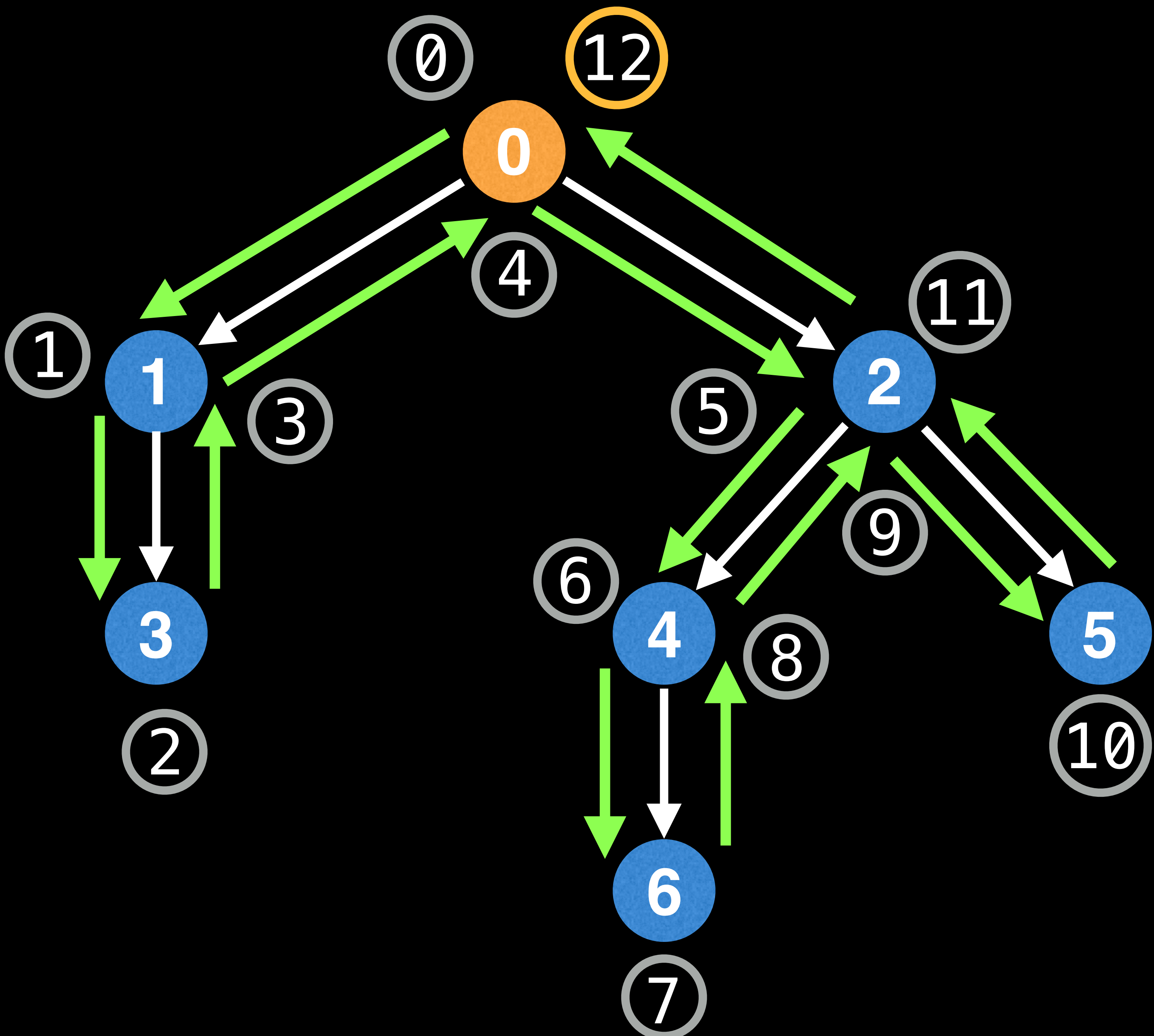
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2		

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4	2	5		



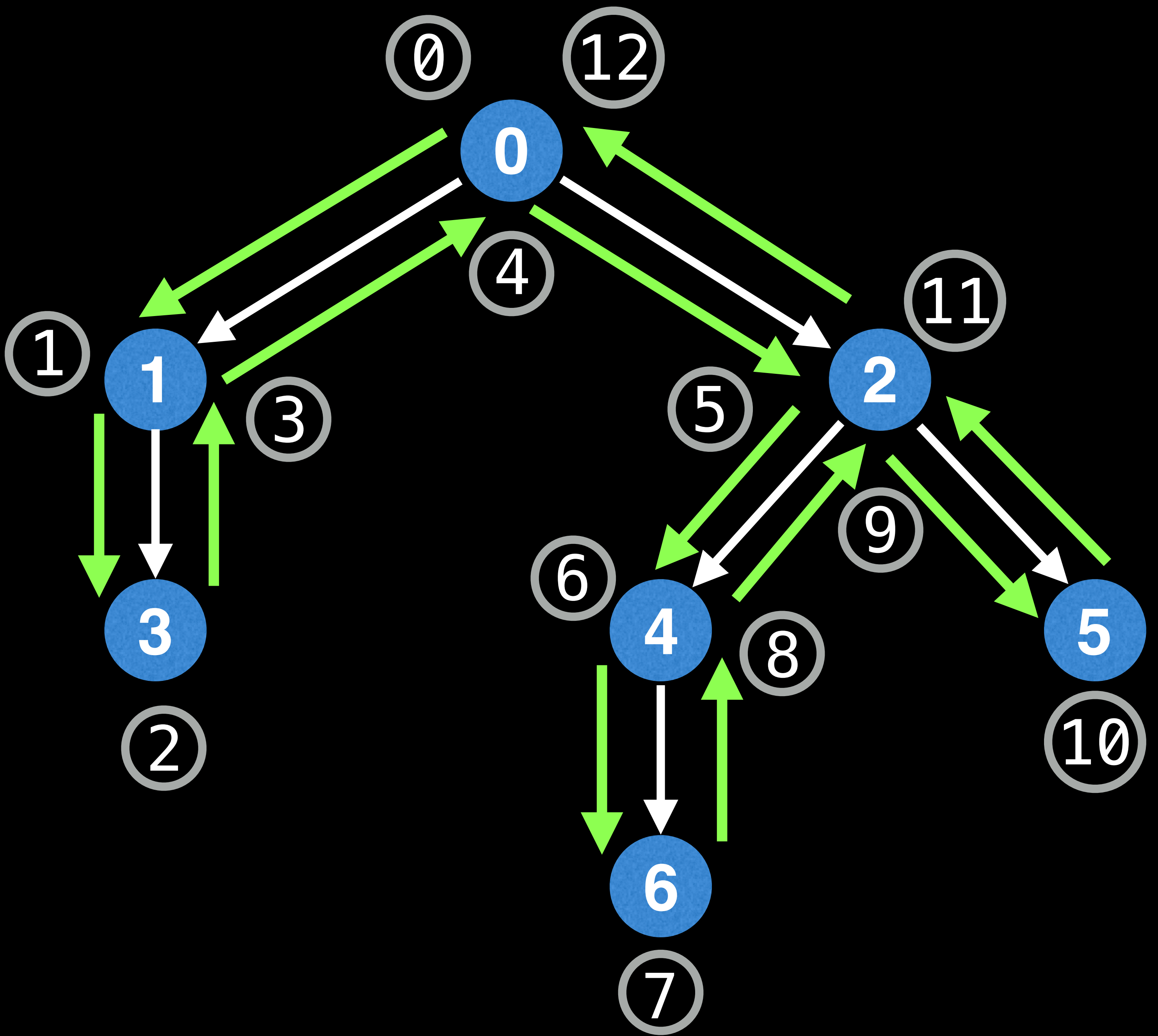
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4	2	5	2	



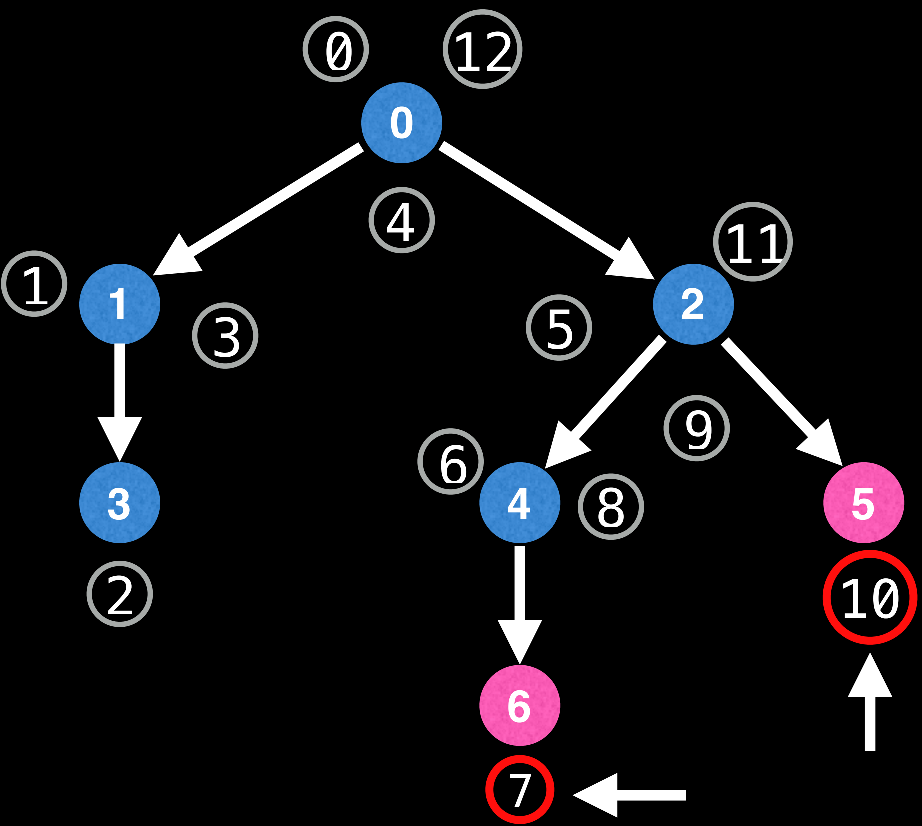
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	0

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4	2	5	2	0



	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	0

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4	2	5	2	0



Q: What is **LCA**(6, 5)?

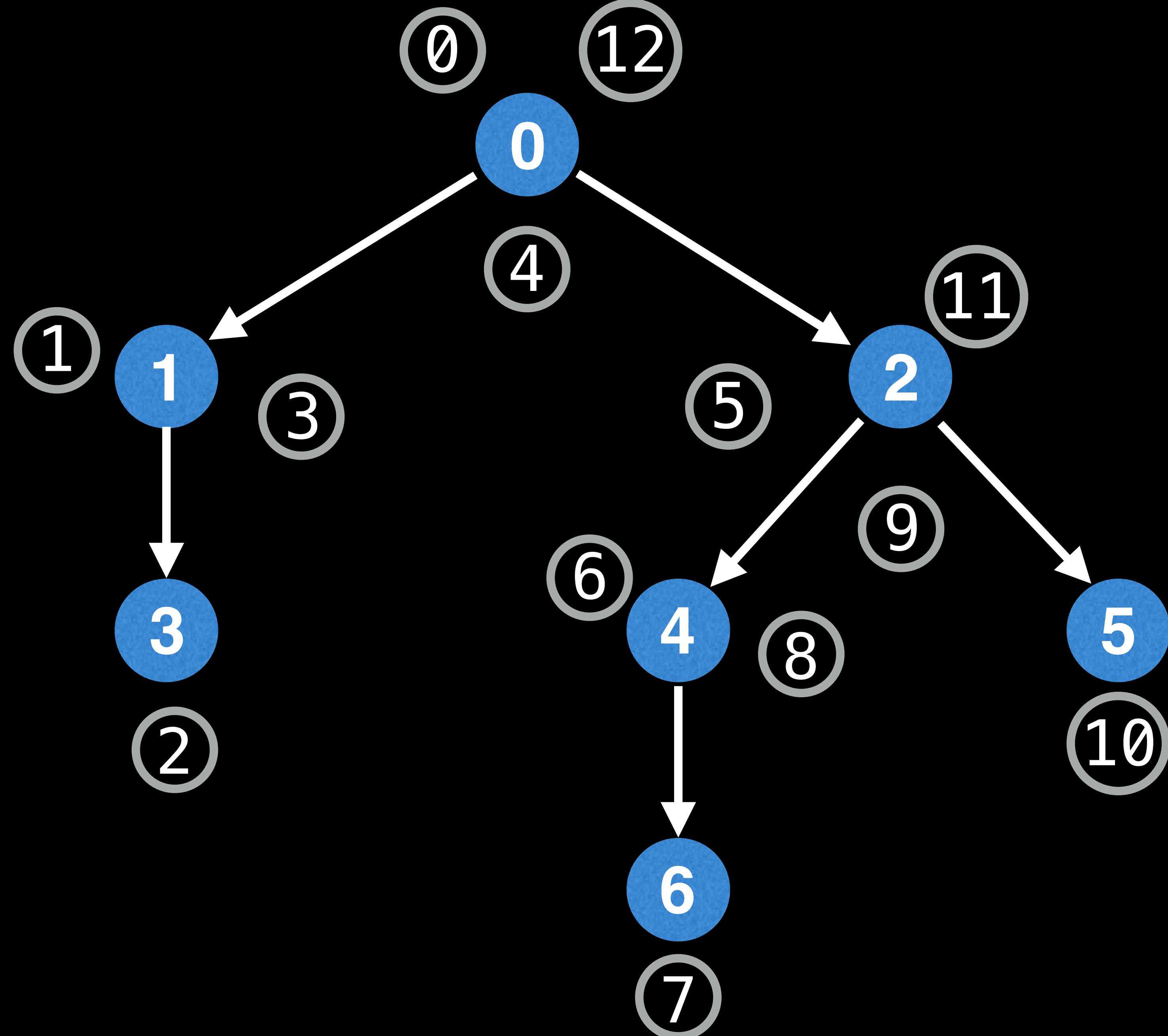
1. Find the index position value for the nodes \`a\` and \`b\` (5 and 6 respectably)

2. Using the depth array, find the index of the minimum value in the range of the indices obtained in step 1

3. Using the index obtained in step 2, index into the \`nodes\` array containing the tree node data.

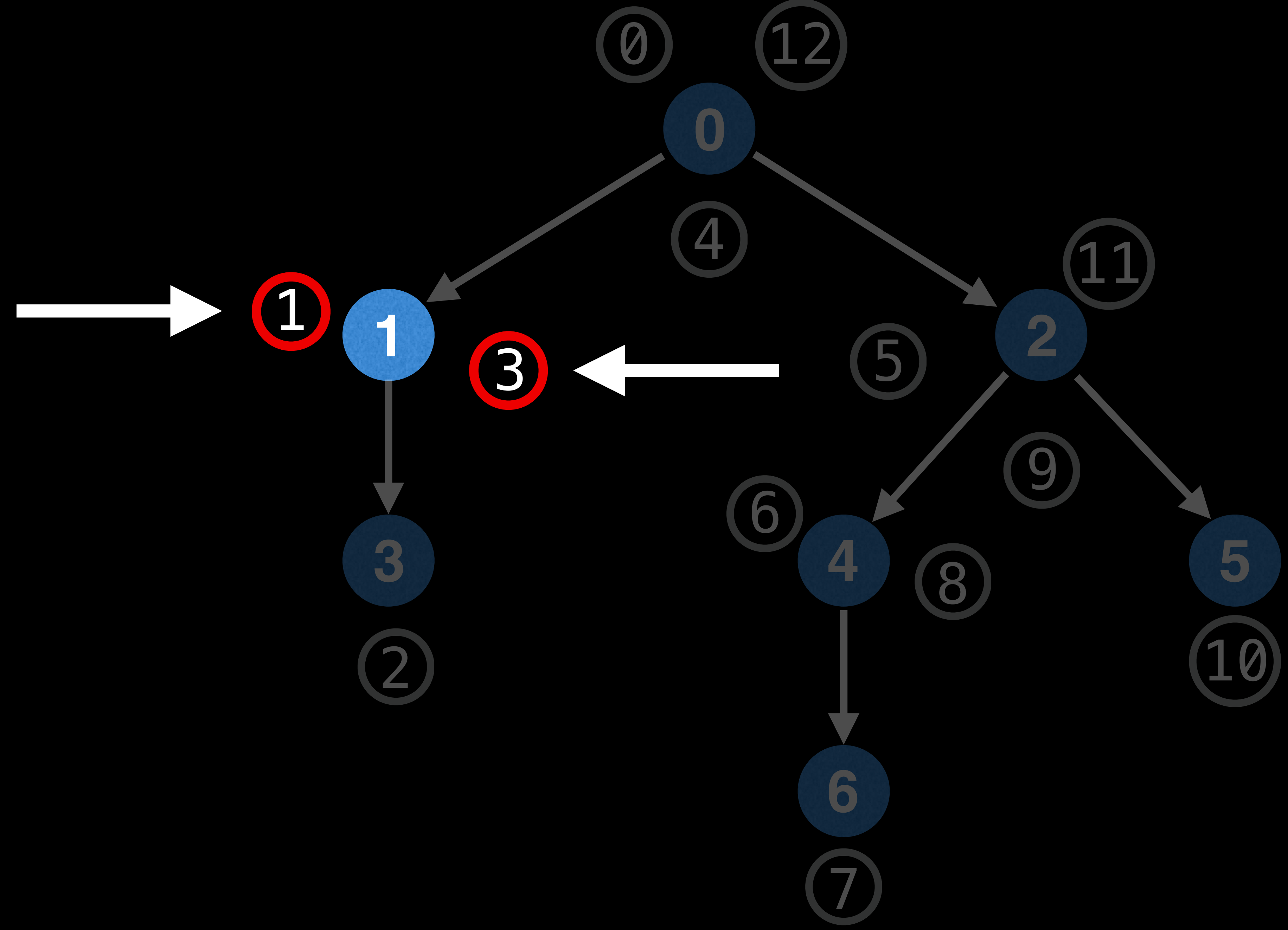
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	0

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4	2	5	2	0

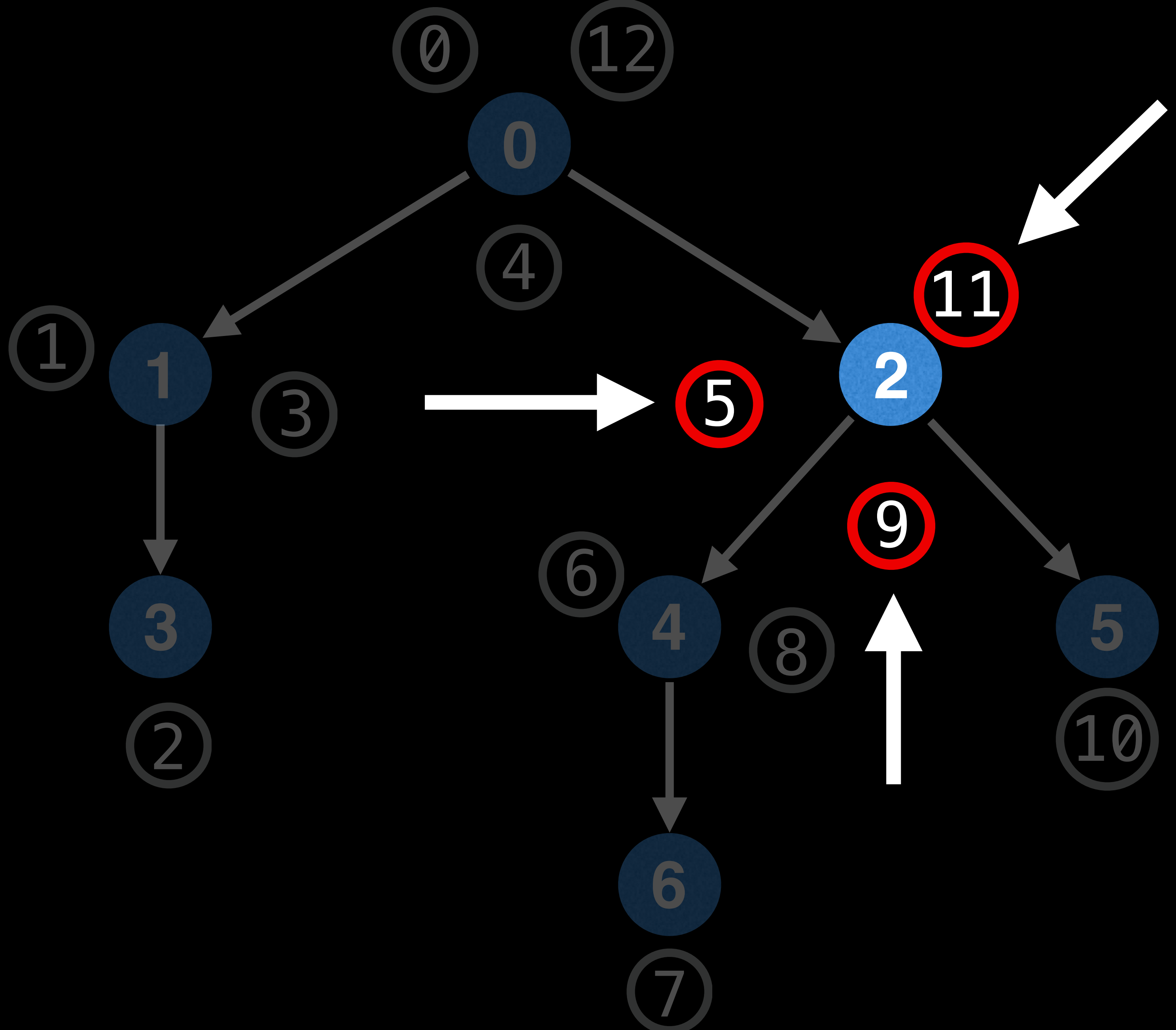


If you recall, step 1 was to find the index position value for the nodes the two nodes with ids `a` and `b`.

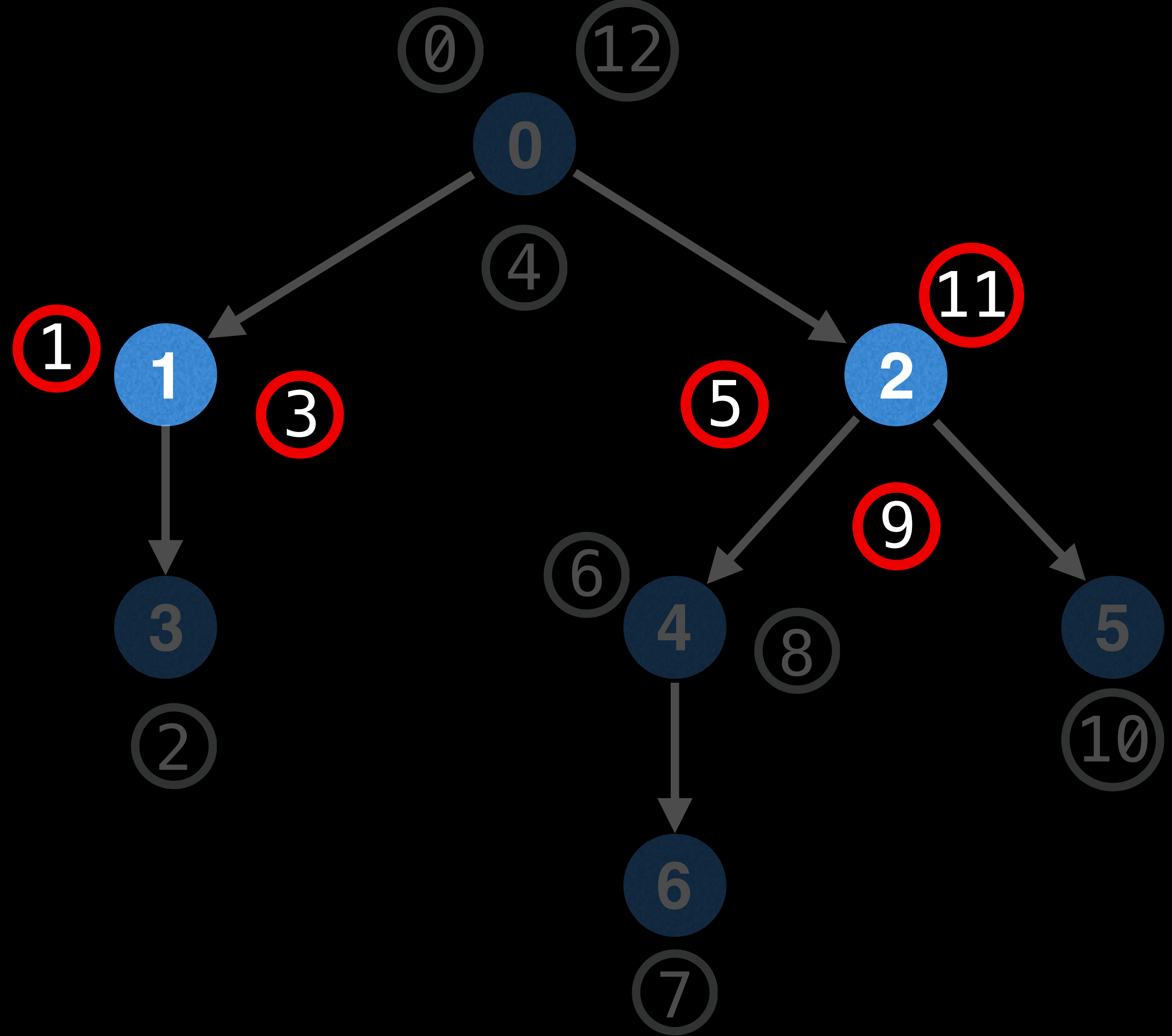
However, an issue we soon run into is that there are 2n - 1 nodes index positions in the Euler tour, and only n nodes in total, so a perfect 1 to 1 reverse mapping isn't possible.



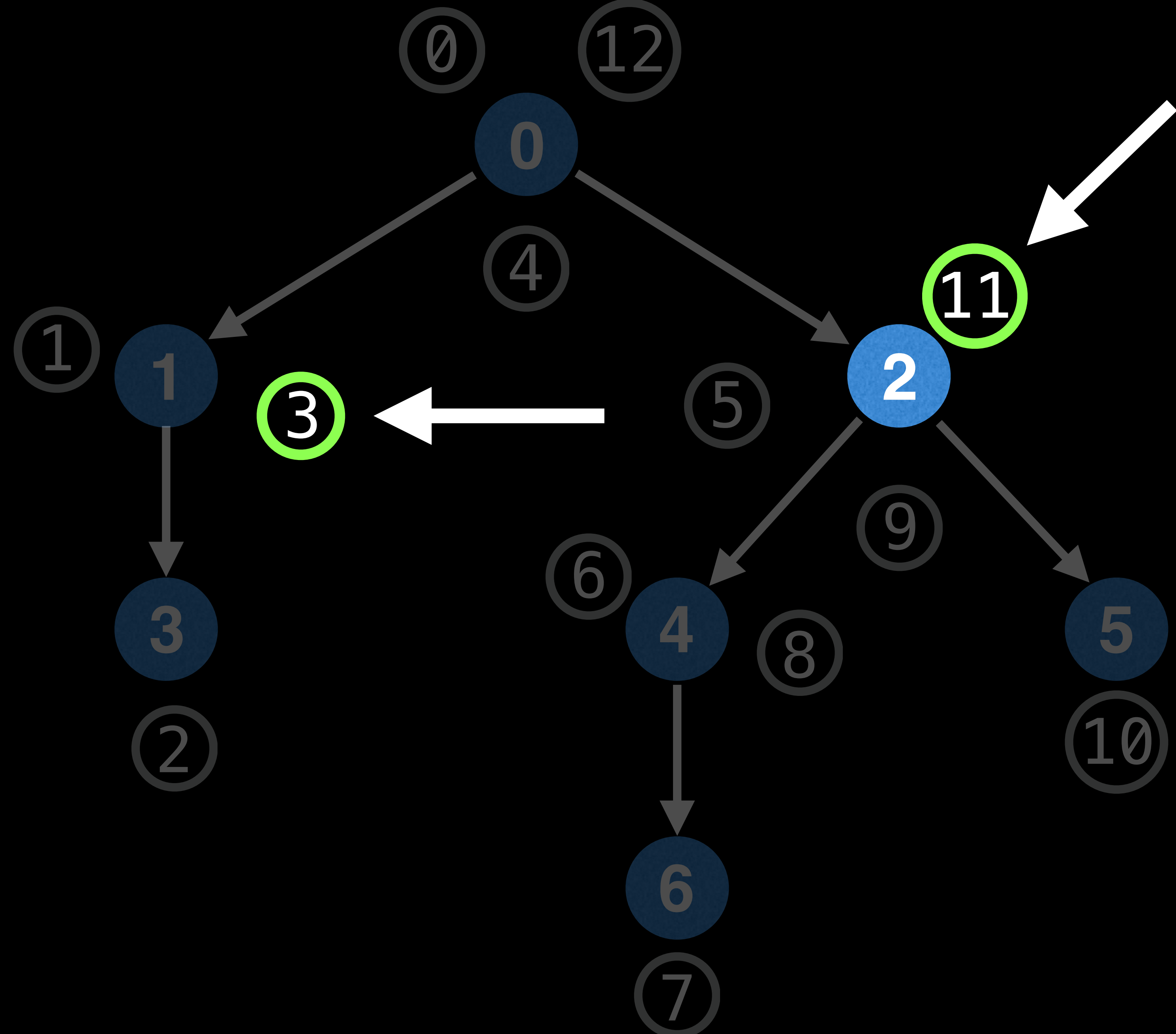
For example, the inverse mapping of node 1 could map to either index 1 or index 3 in the Euler tour.



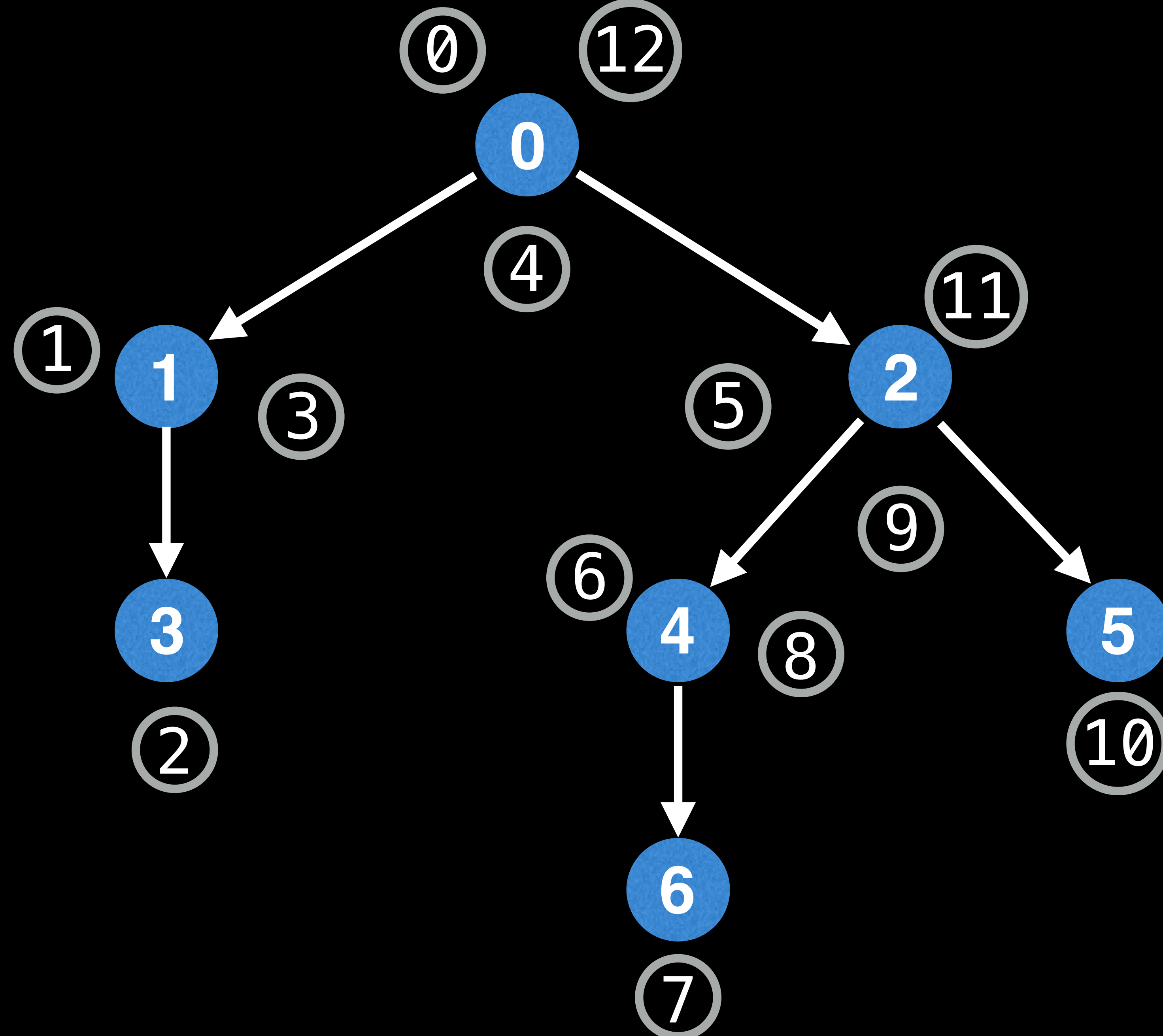
Similarly, the inverse mapping for node 2 could map to either index 5, 9 or 11 in the Euler tour.



Now, suppose we wanted to query the lowest common ancestor for nodes 1 and 2, which index values would want to choose for each node?

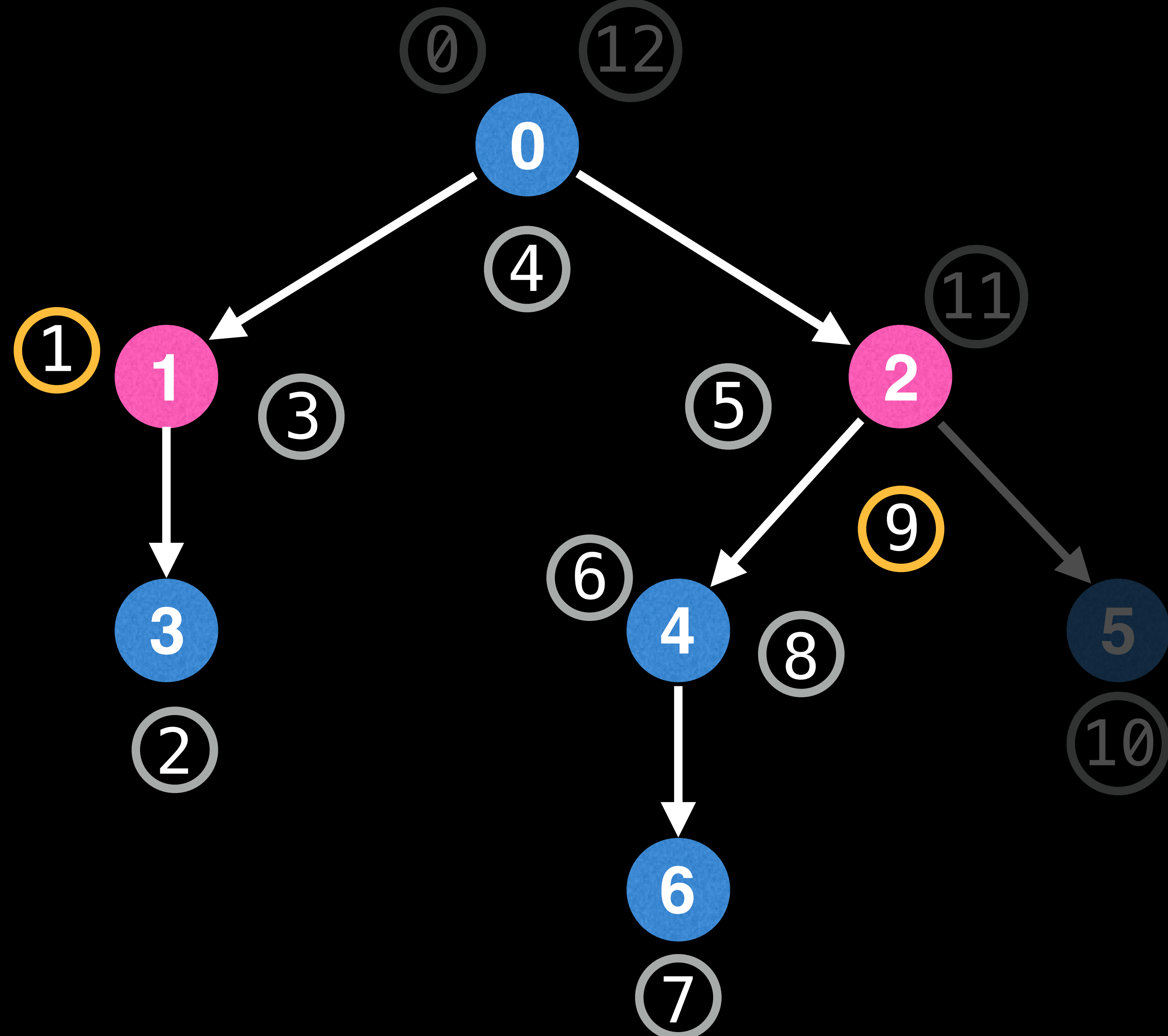


The answer is that it doesn't matter, any of the inverse index values will do. However, in practice I find that it is easiest to select the **last encountered index** while doing the Euler tour.



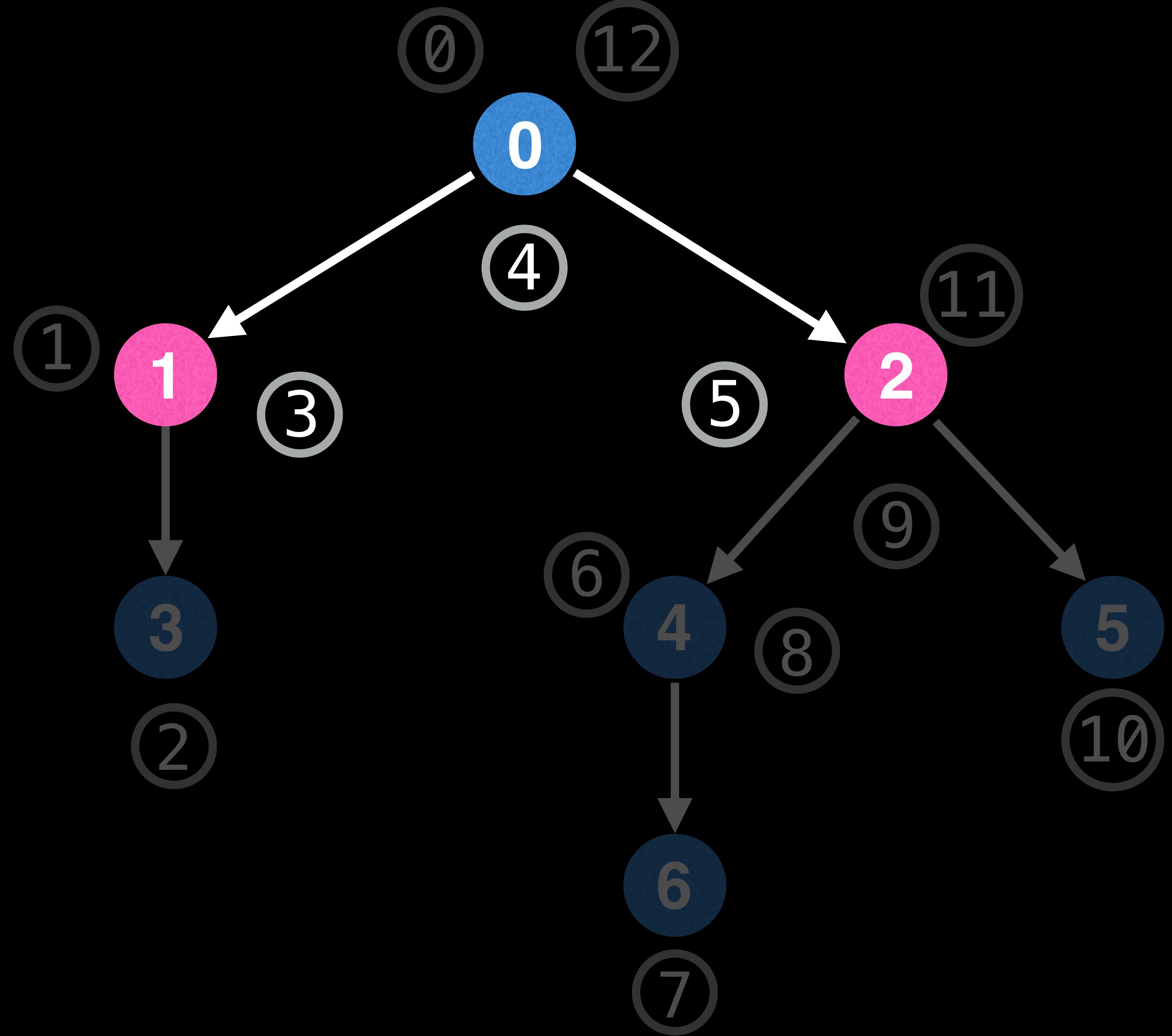
The reason the selection of the inverse index mapping doesn't matter is that it will not affect the value obtained from the **Range Minimum Query (RMQ)** in step 2

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	0



Suppose that for the LCA(1, 2) we had selected index 1 for node 1 and index 9 for node 2, meaning the range [1, 9] in the depth array for the RMQ.

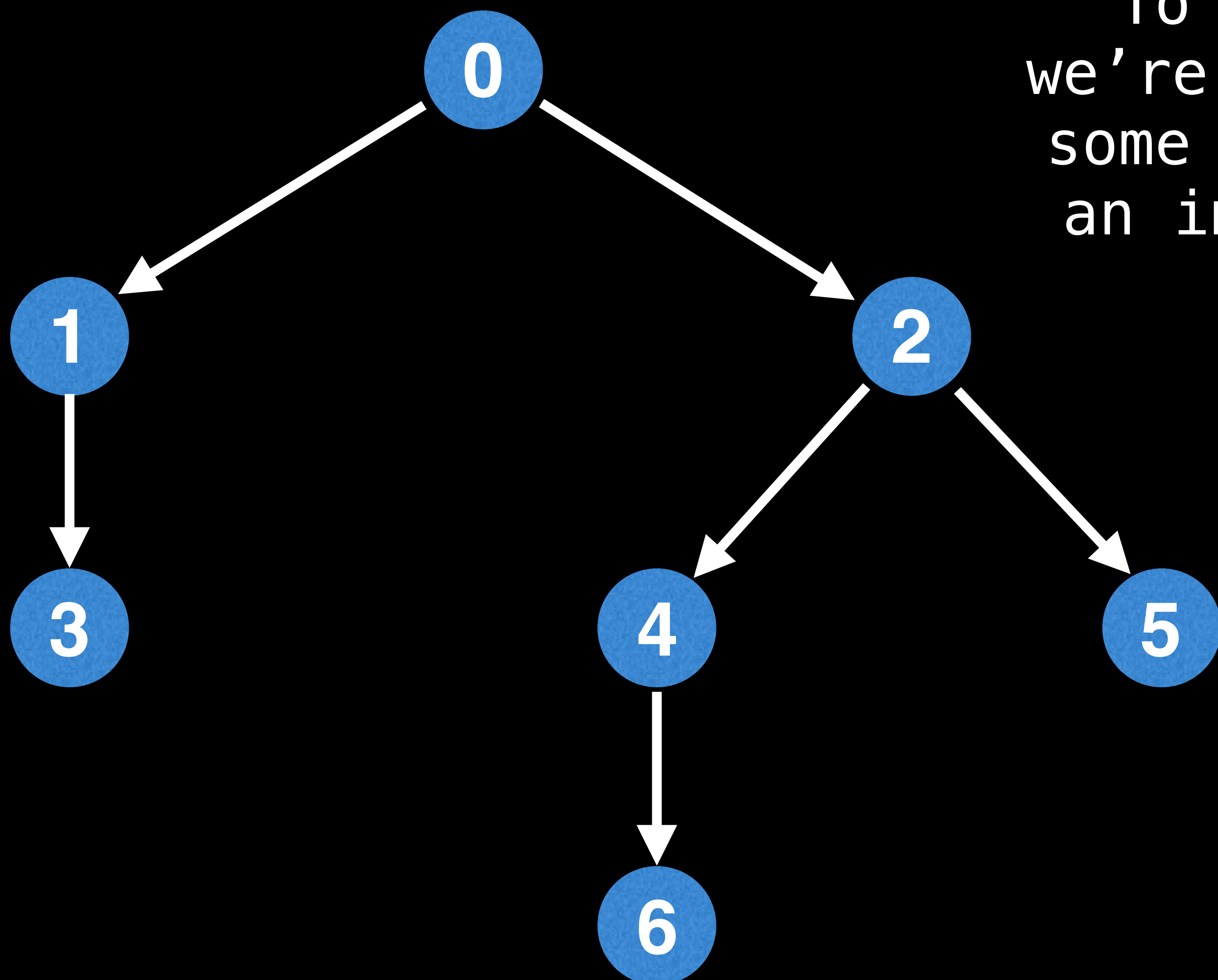
	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	0



You may think the range [3, 5] would be better since the interval is smaller. However, this doesn't matter since RMQ take O(1) with a sparse table.

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	0

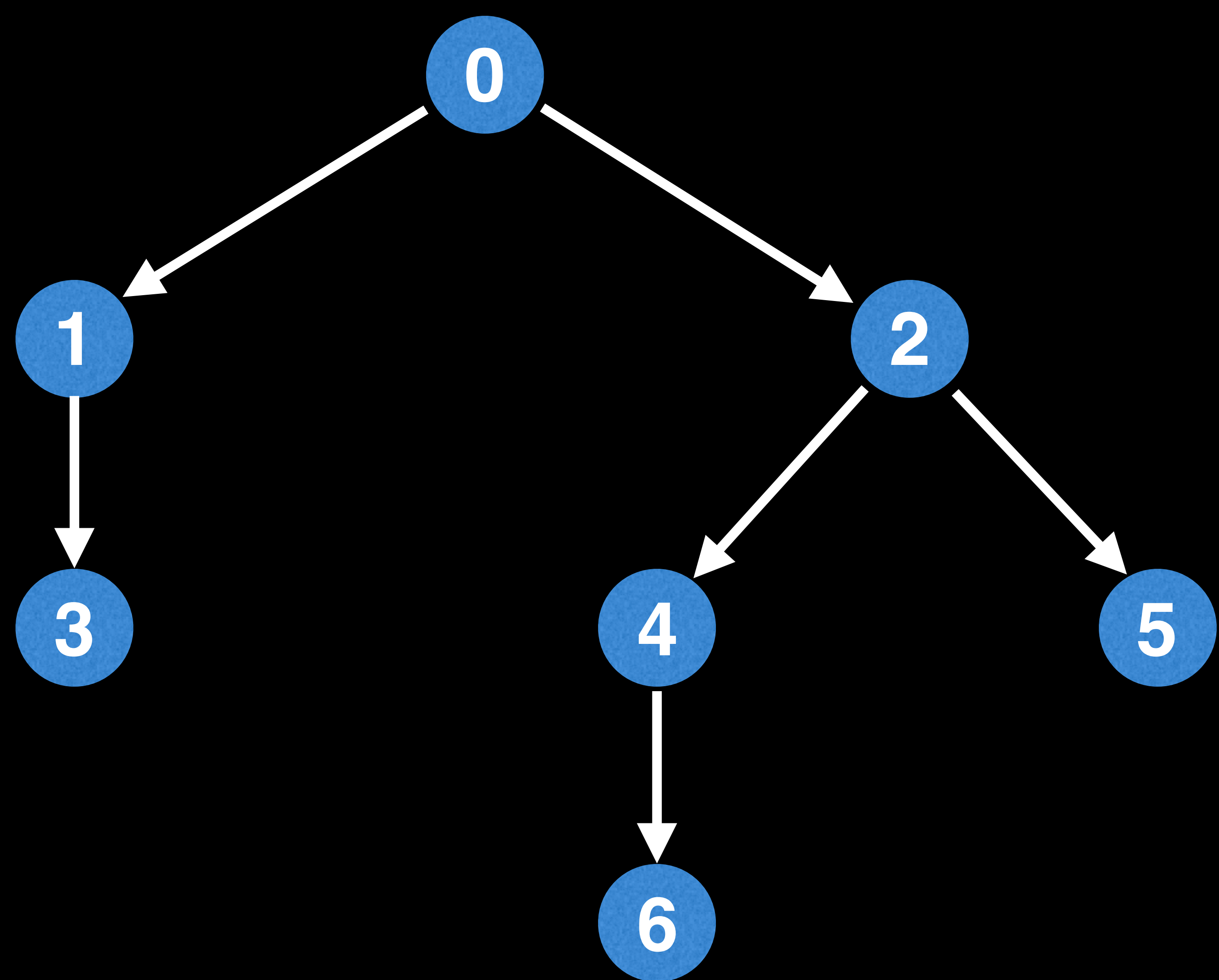
To maintain an inverse mapping, we're going to need to keep track of some additional information, namely an inverse map I will call `last`.



	0	1	2	3	4	5	6
last							

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth													

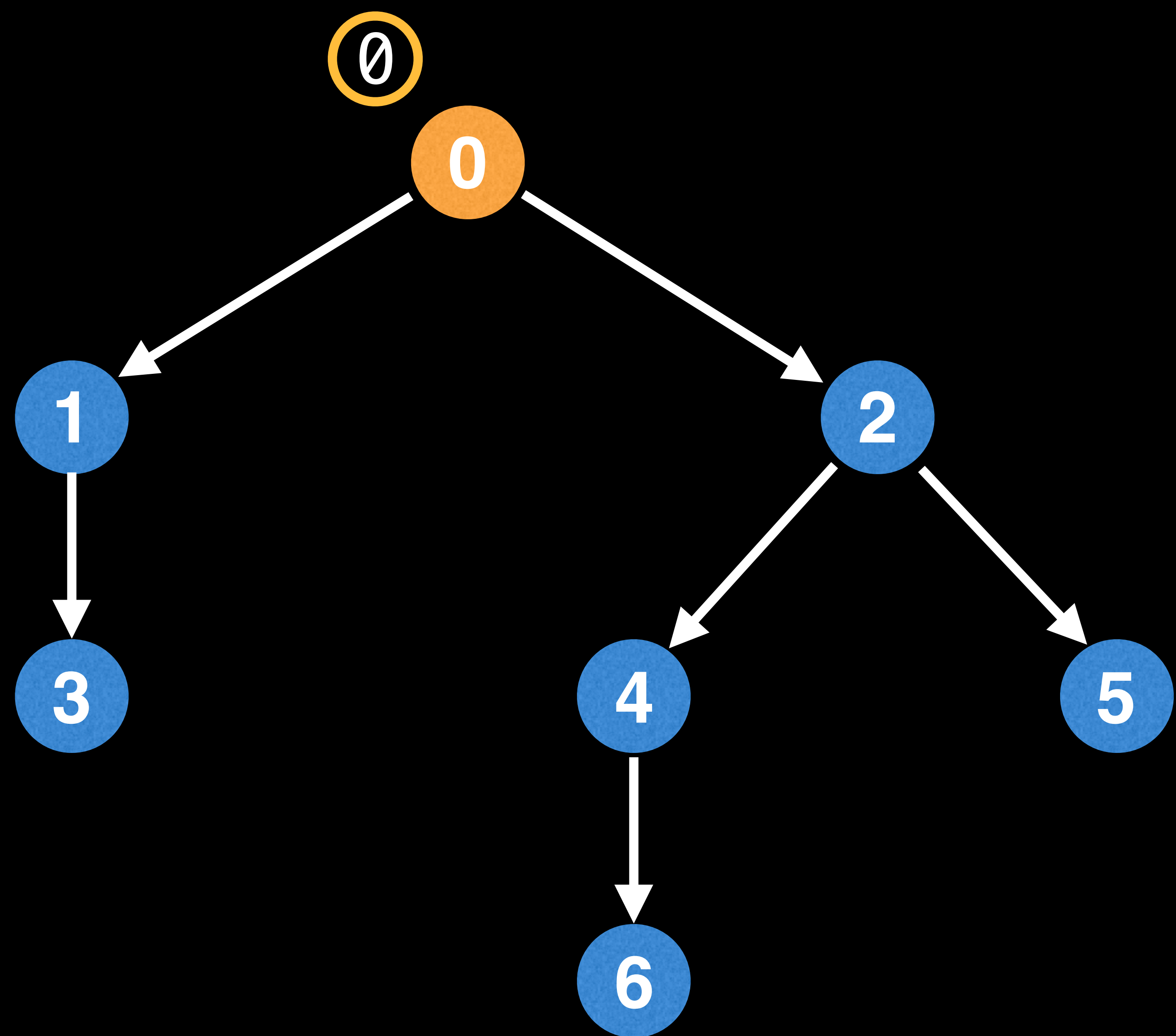
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes													



	0	1	2	3	4	5	6
last							

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth													

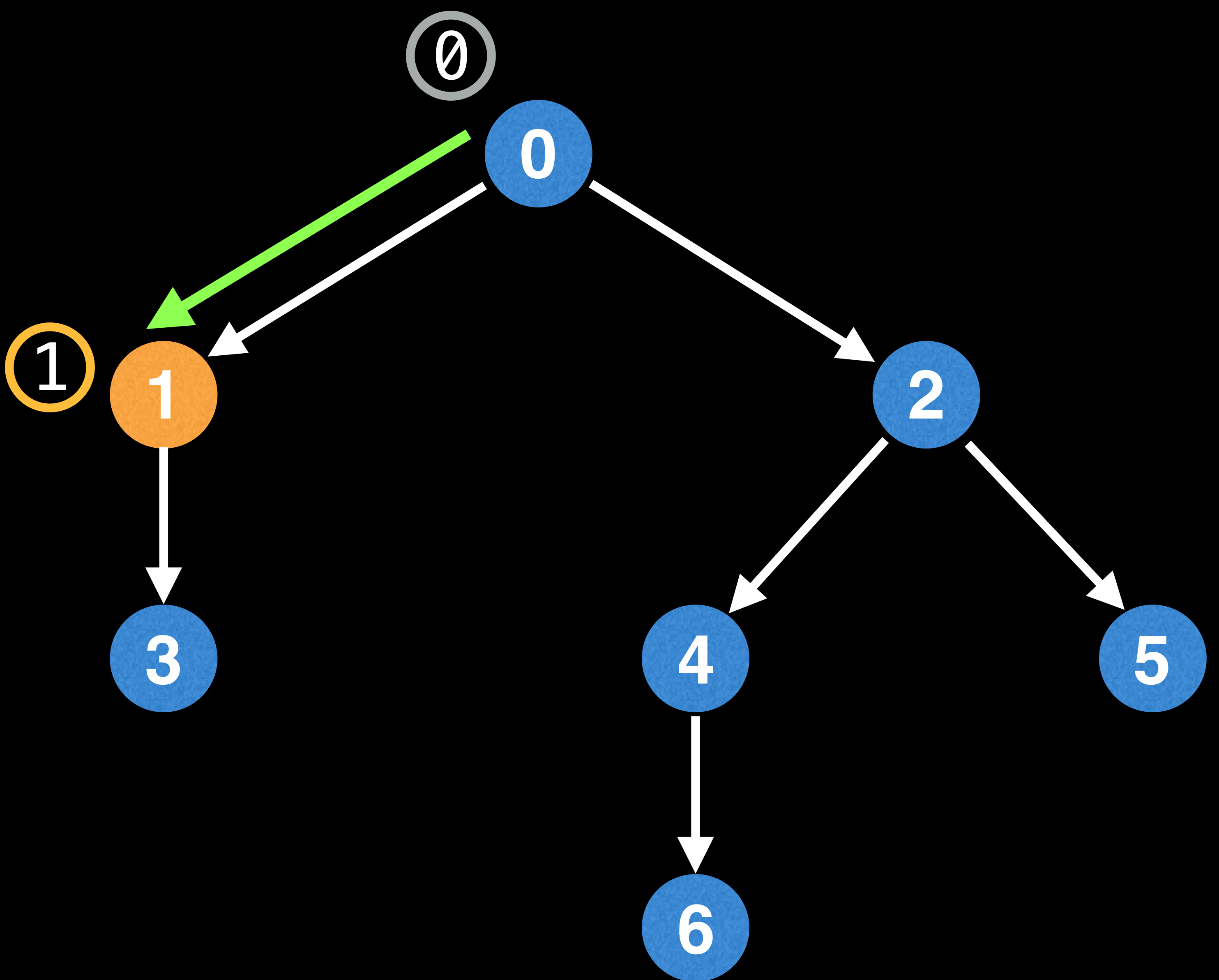
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes													



	0	1	2	3	4	5	6
last	0						

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0												

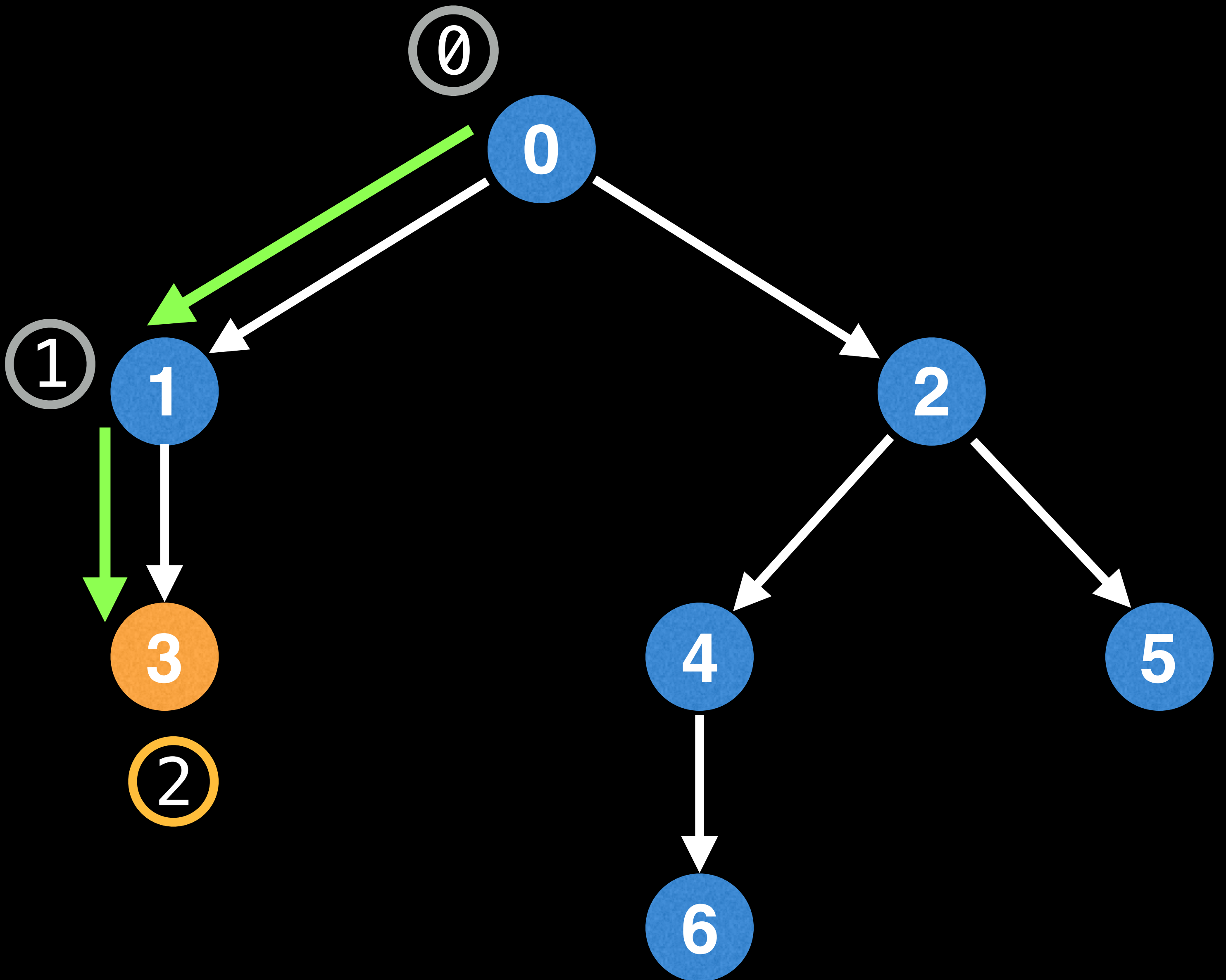
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0												



0	1	2	3	4	5	6
0	1					

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1											

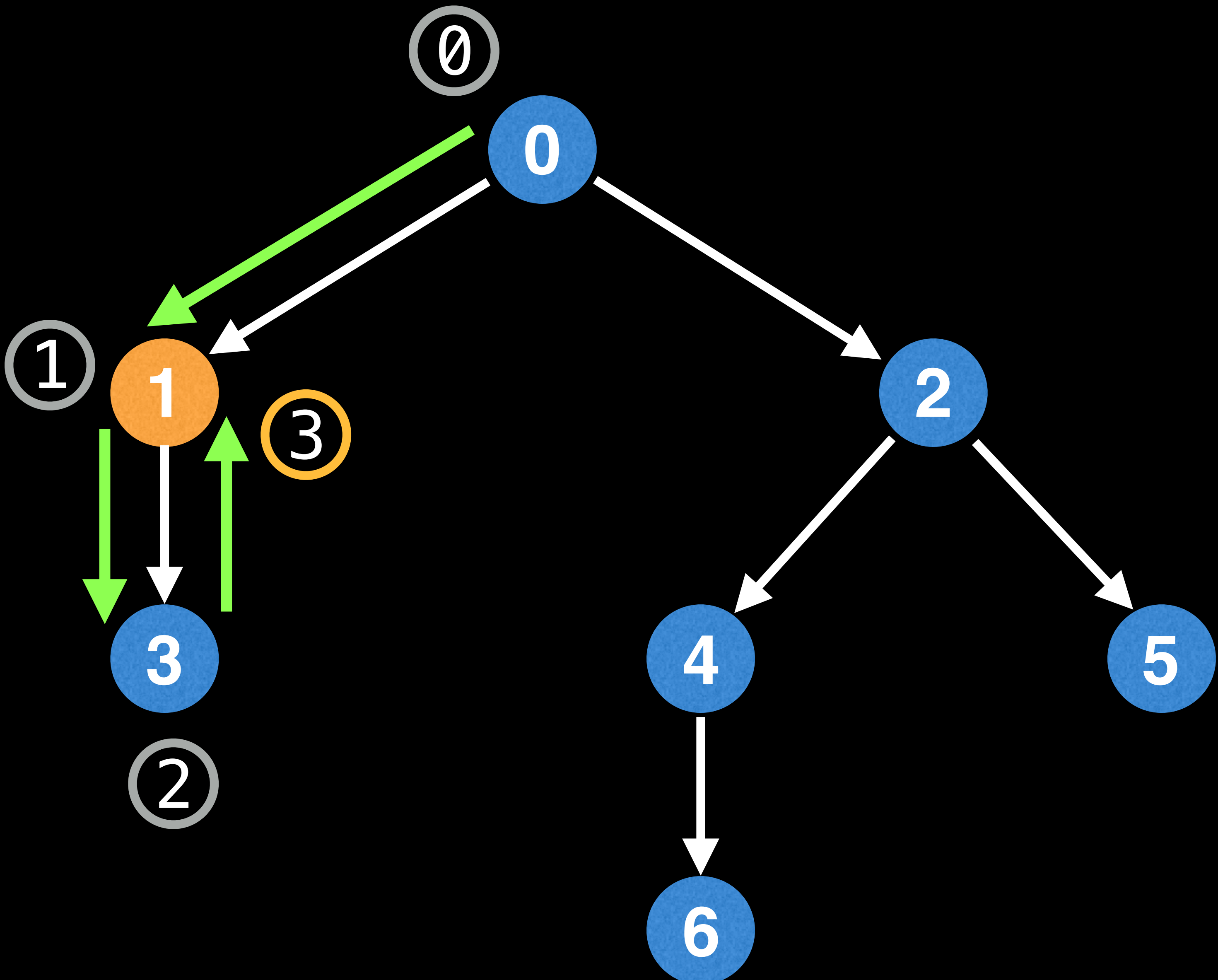
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1											



0	1	2	3	4	5	6
0	1		2			

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2										

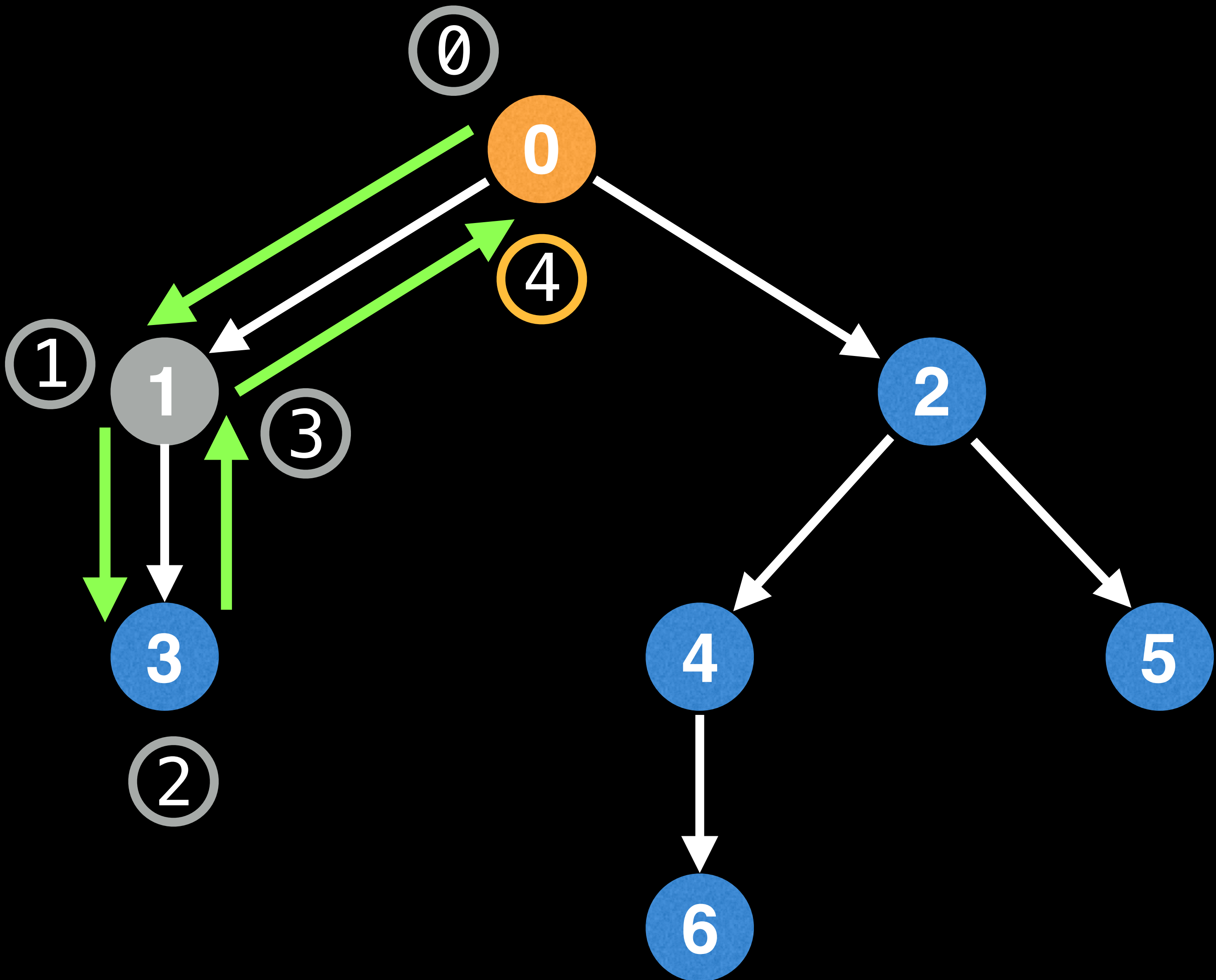
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3										



0	1	2	3	4	5	6
0	3		2			

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1									

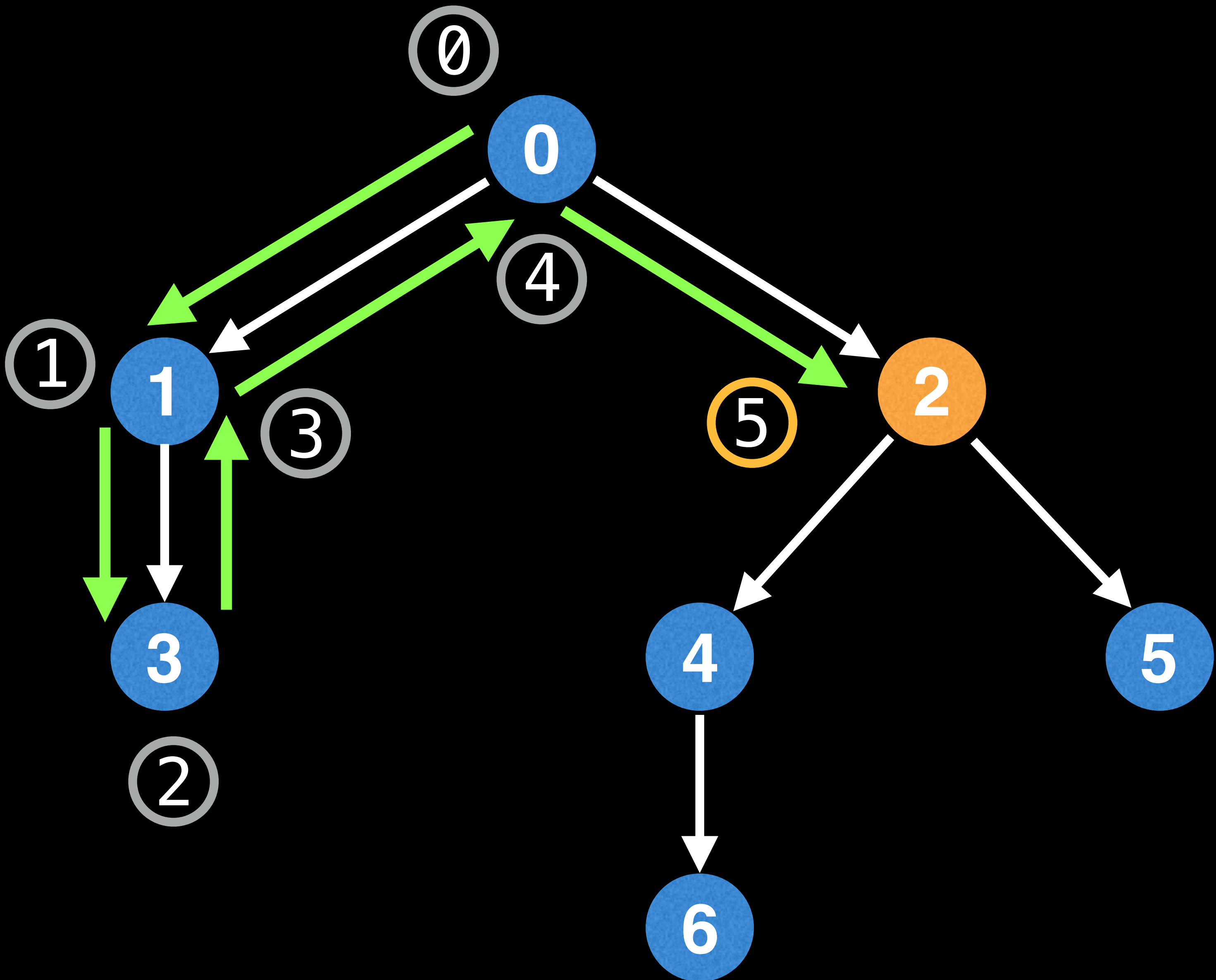
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1									



0	1	2	3	4	5	6
4	3		2			

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0								

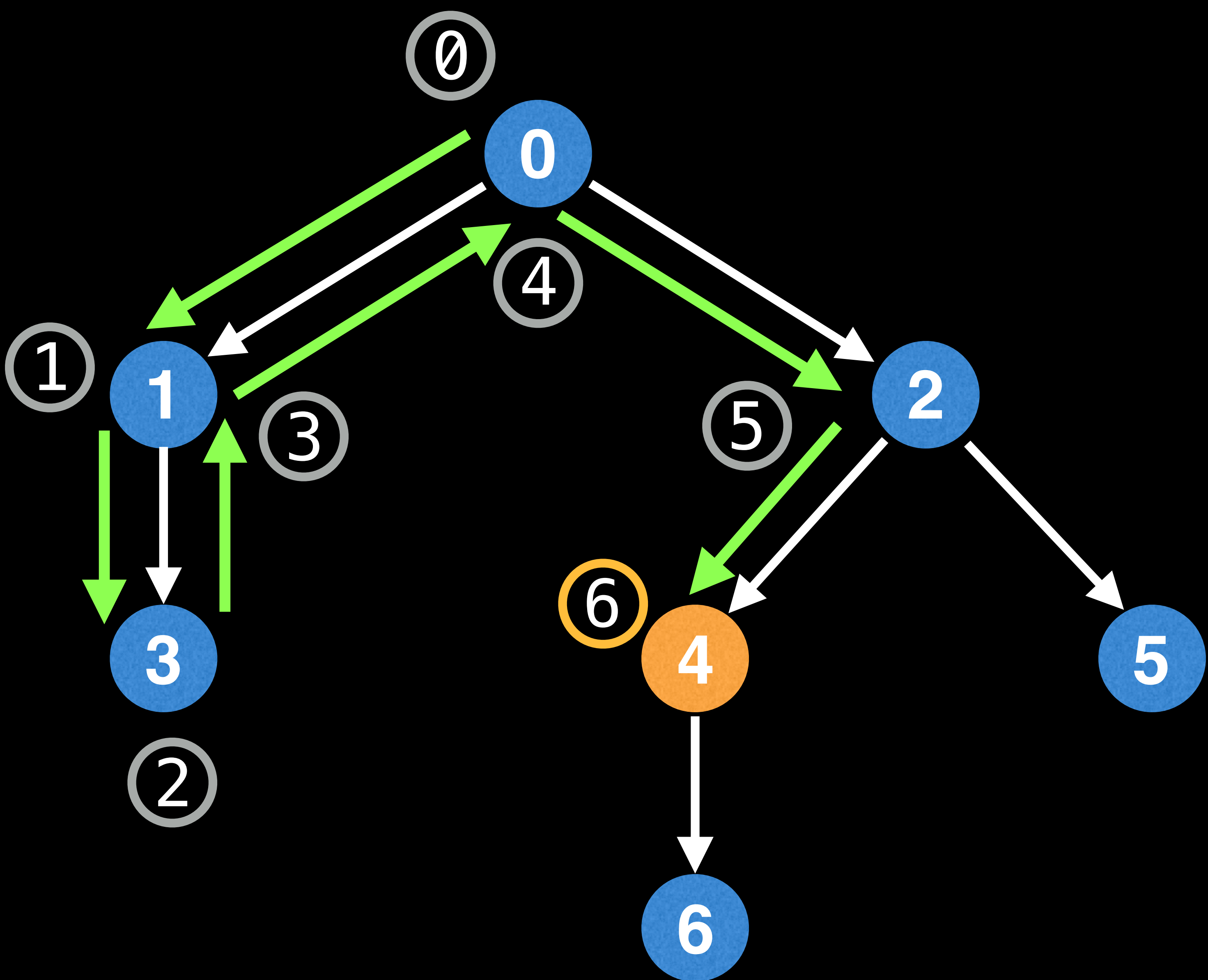
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0								



0	1	2	3	4	5	6
4	3	5	2			

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1							

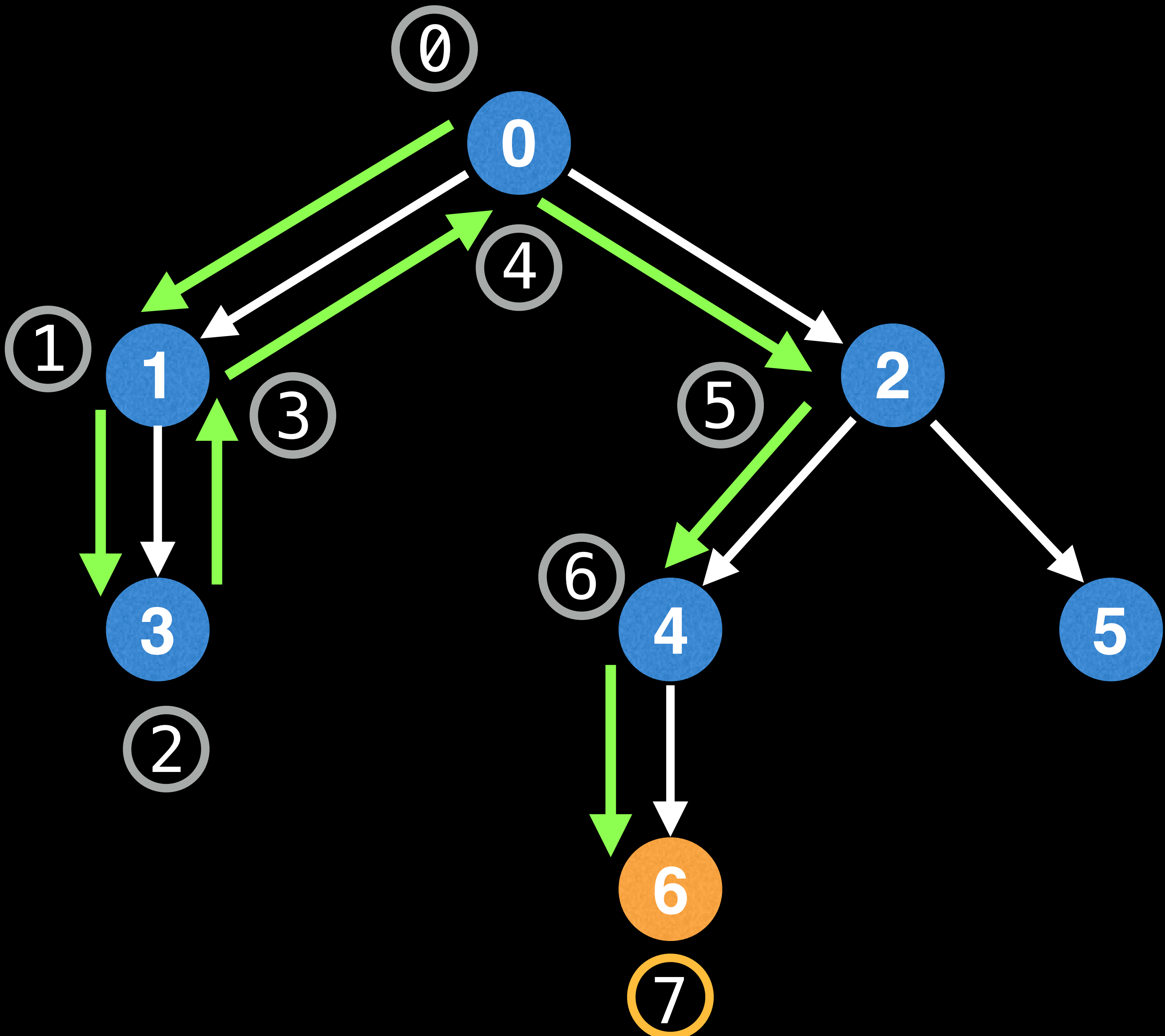
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2							



0	1	2	3	4	5	6
4	3	5	2	6		

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2						

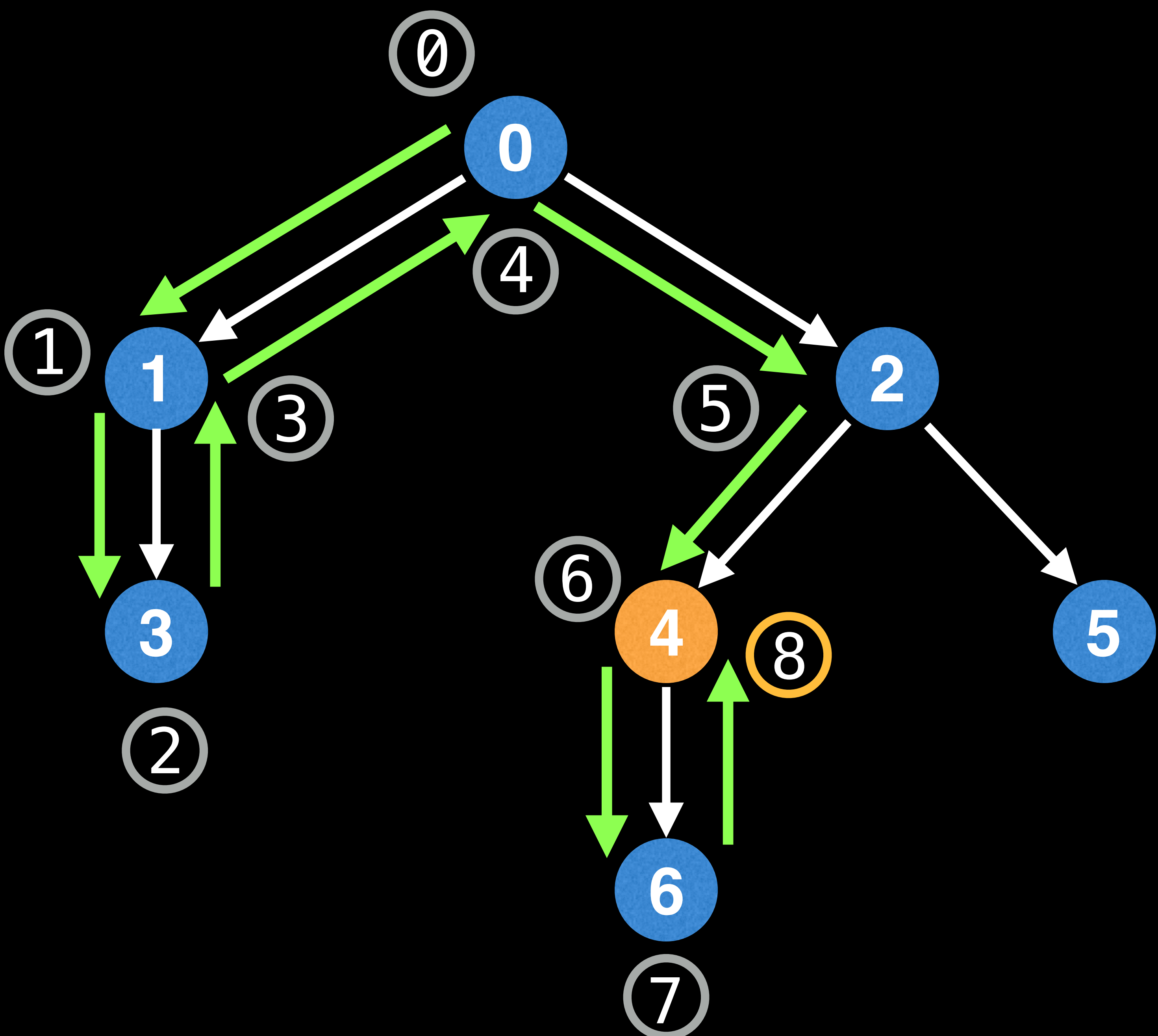
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4						



0	1	2	3	4	5	6
4	3	5	2	6		7

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3					

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6					



0	1	2	3	4	5	6
4	3	5	2	8		7

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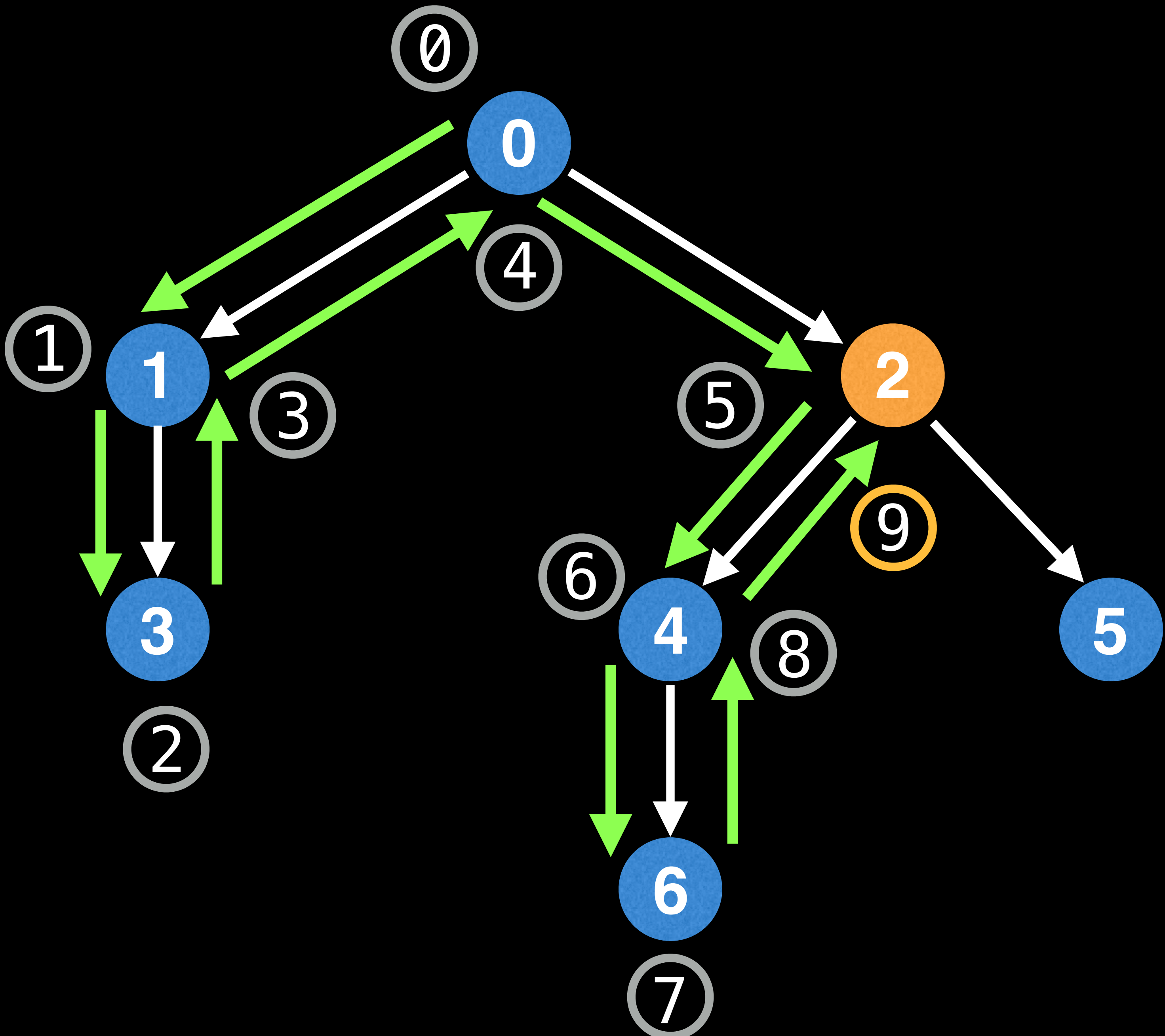
depth	0	1	2	1	0	1	2	3	2				

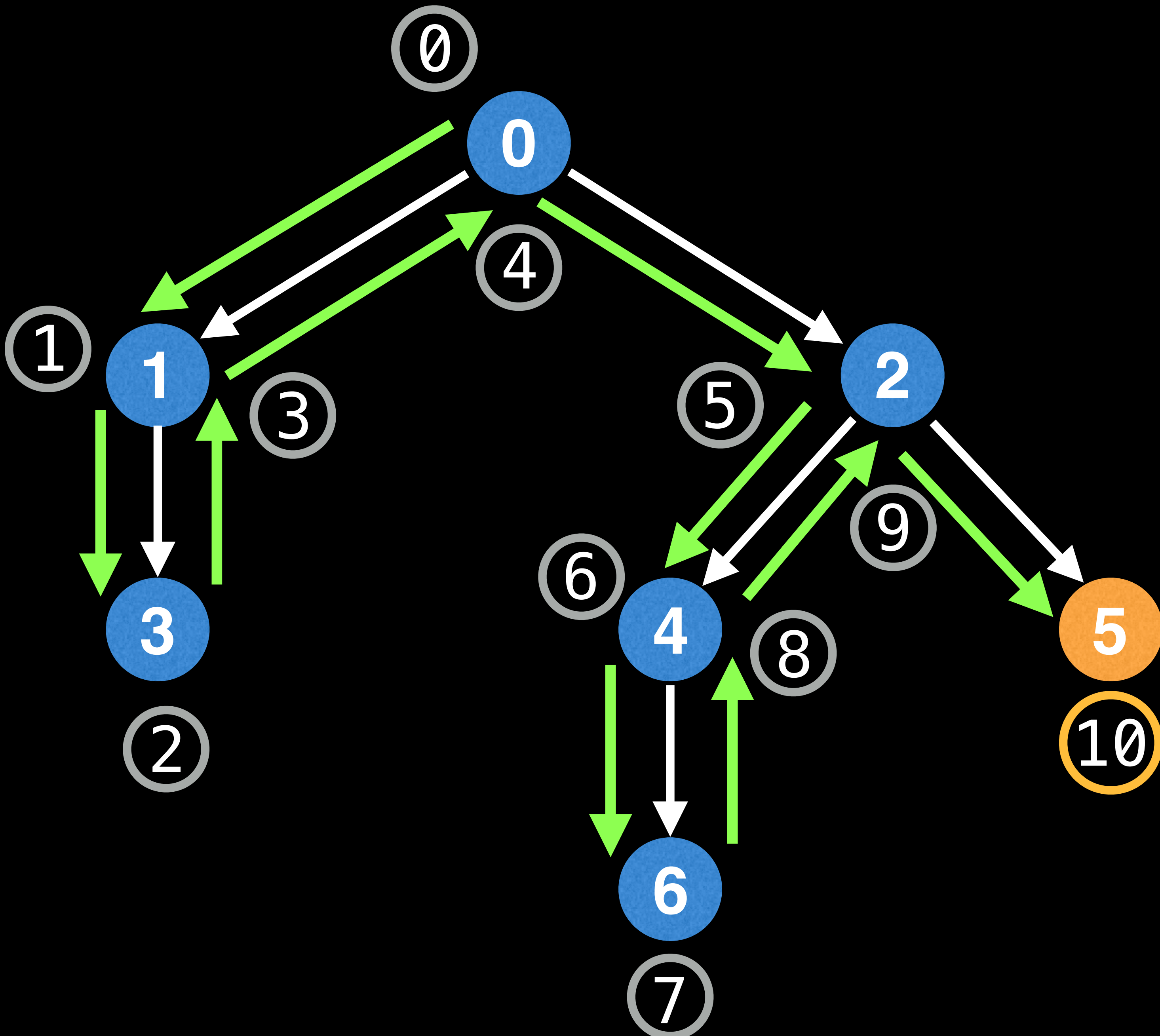
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4				



0	1	2	3	4	5	6
4	3	9	2	8		7

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1			

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4	2			



0	1	2	3	4	5	6
4	3	9	2	8	10	7

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2		

	0	1	2	3	4	5	6	7	8	9	10	11	12
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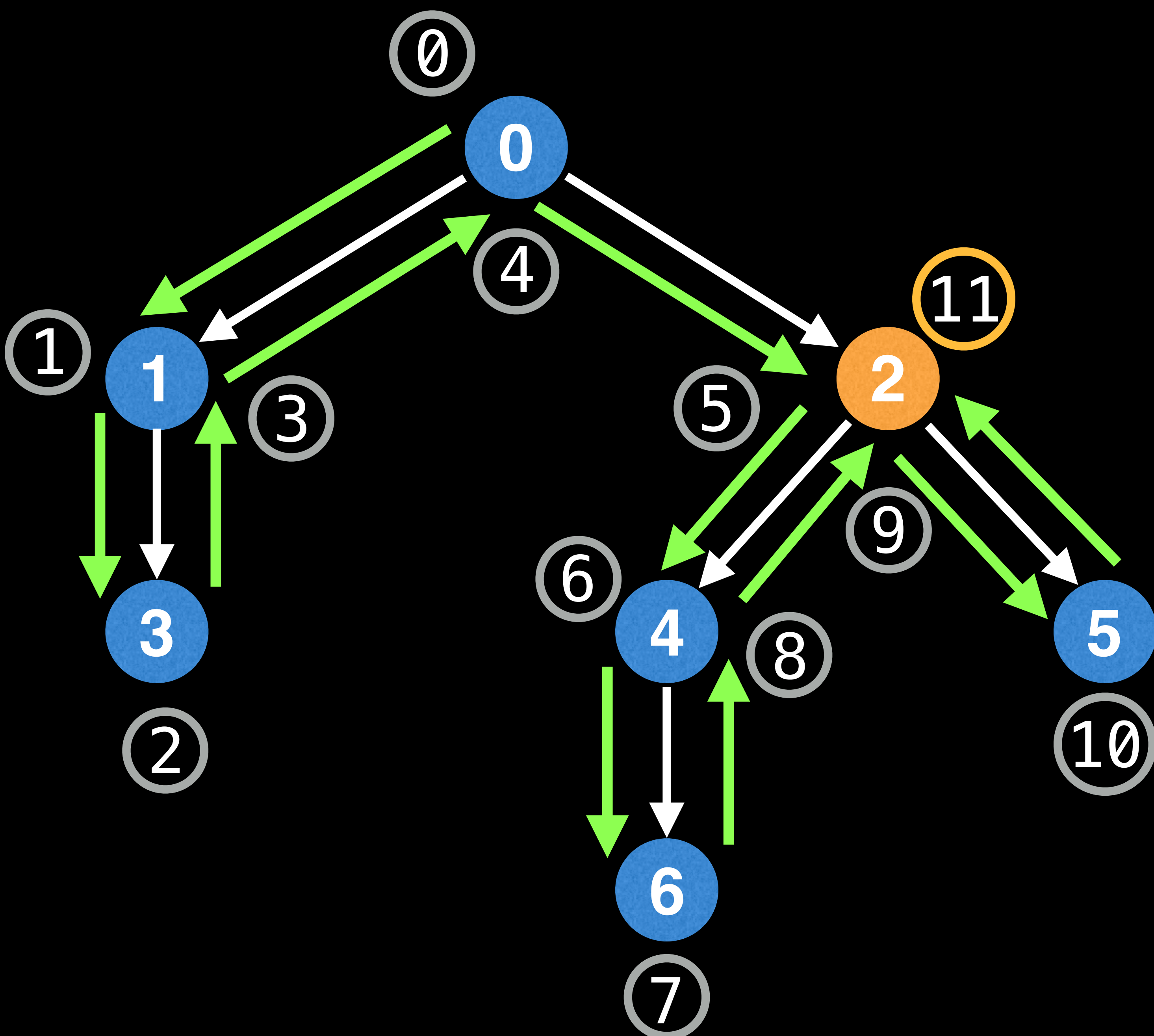
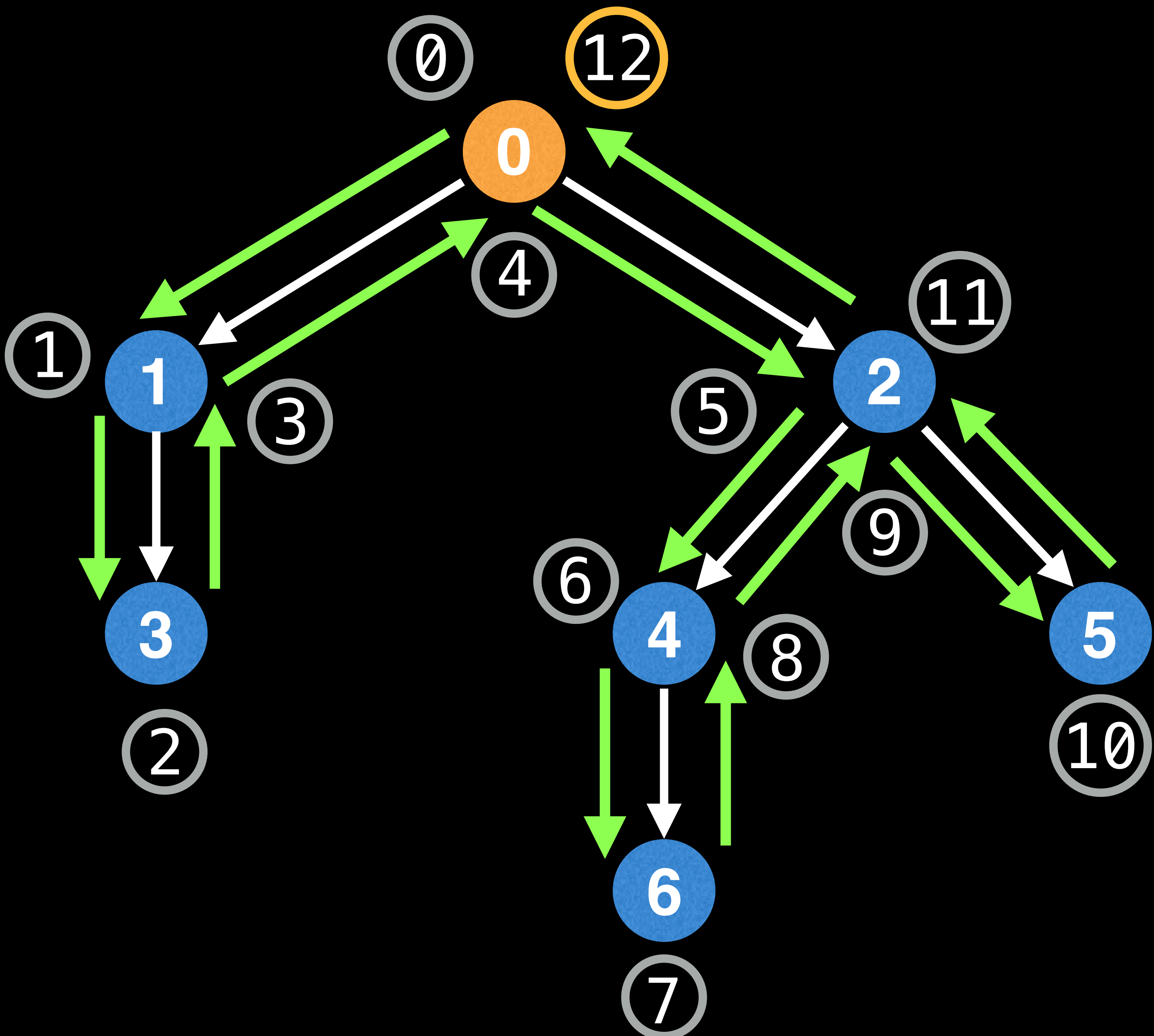


0	1	2	3	4	5	6
4	3	11	2	8	10	7

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	

	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4	2	5	2	



0	1	2	3	4	5	6
12	3	11	2	8	10	7

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	0

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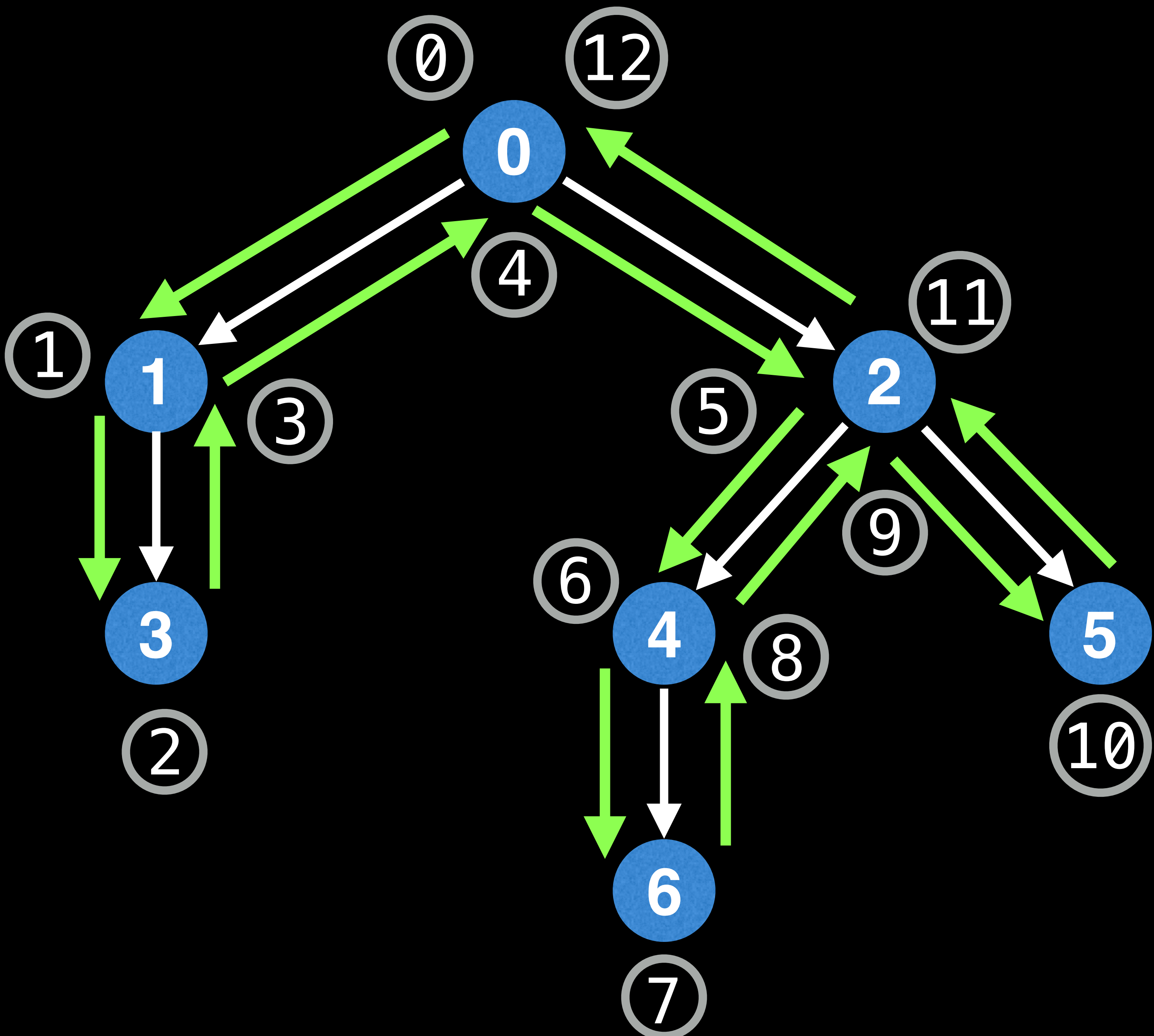
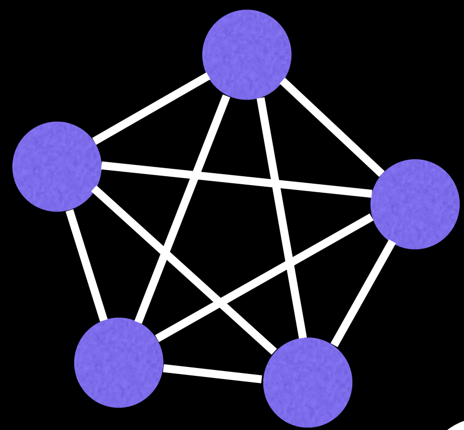
nodes	0	1	3	1	0	2	4	6	4	2	5	2	0

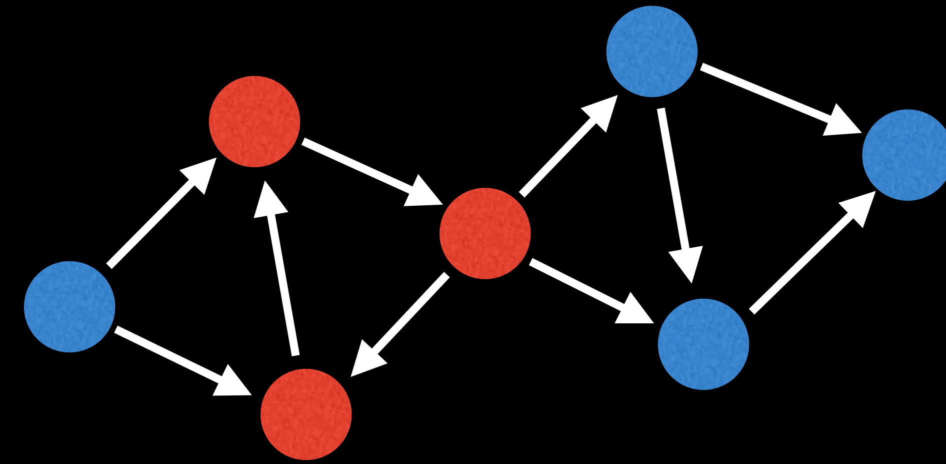


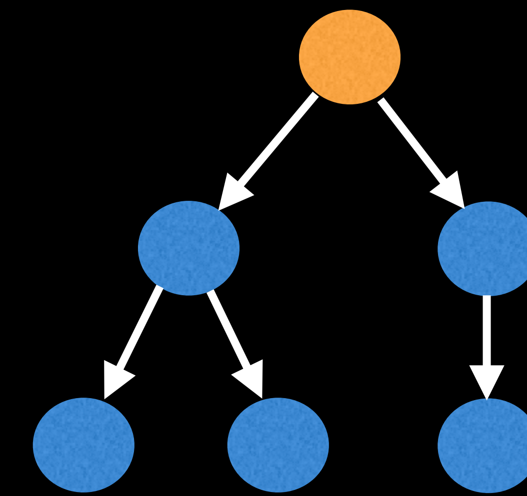
0	1	2	3	4	5	6
12	3	11	2	8	10	7

	0	1	2	3	4	5	6	7	8	9	10	11	12
depth	0	1	2	1	0	1	2	3	2	1	2	1	0

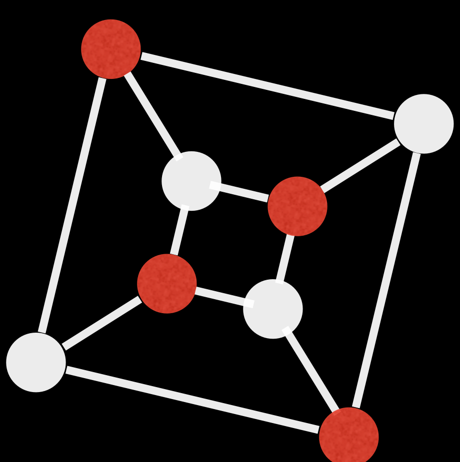
	0	1	2	3	4	5	6	7	8	9	10	11	12
nodes	0	1	3	1	0	2	4	6	4	2	5	2	0

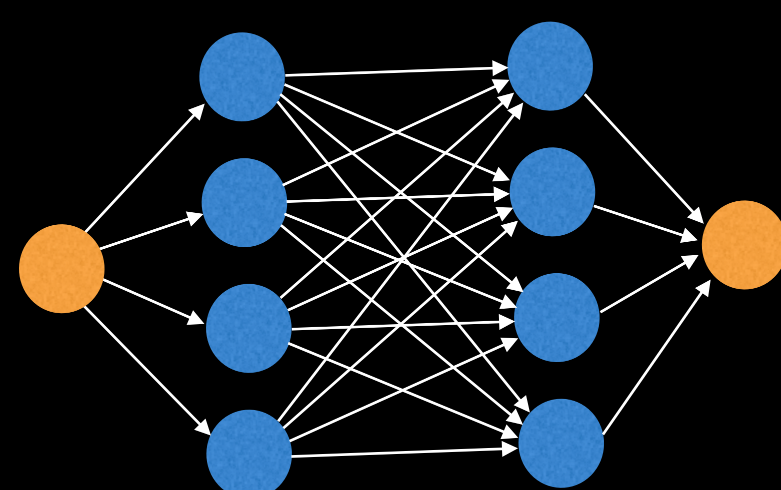


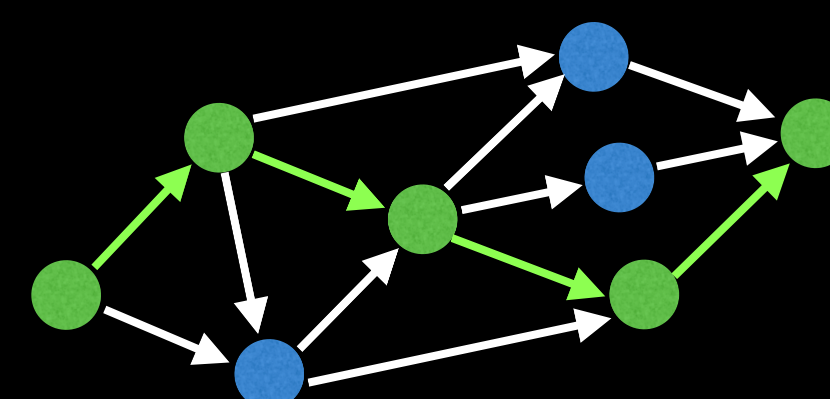




Graph Theory Video Series



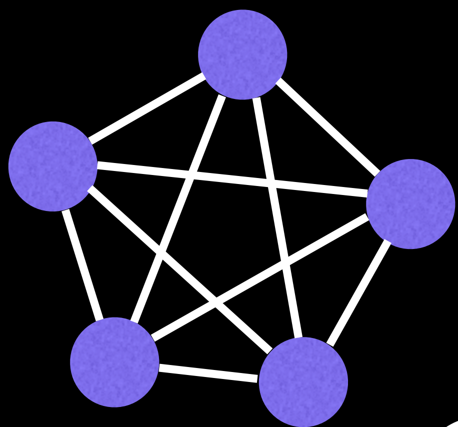


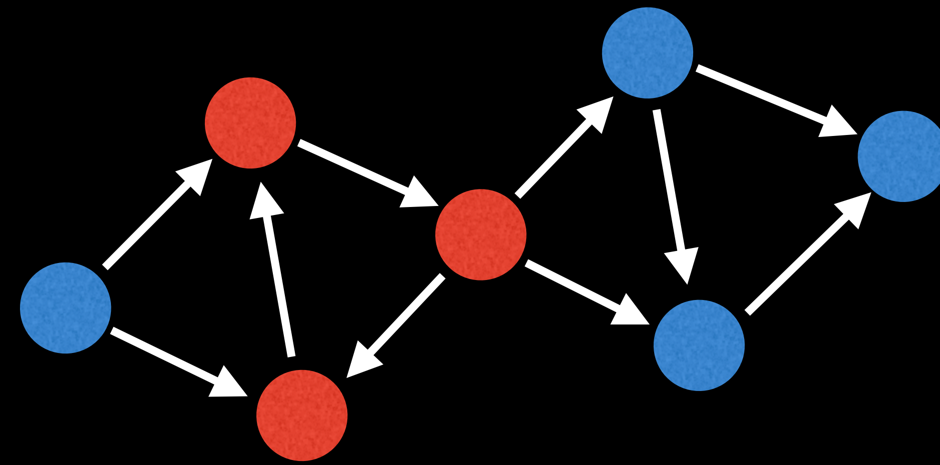


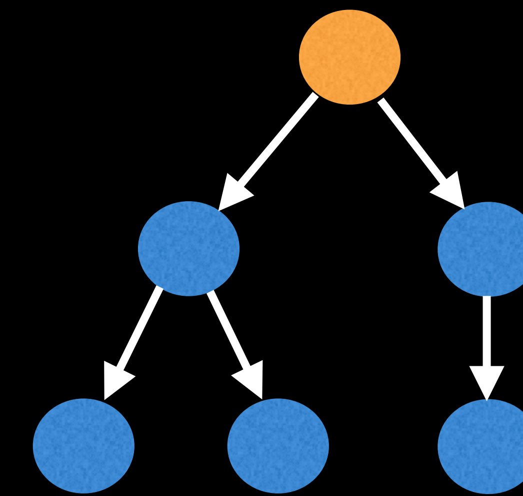
Heavy-Light Decomposition

(Heavy path decomposition)

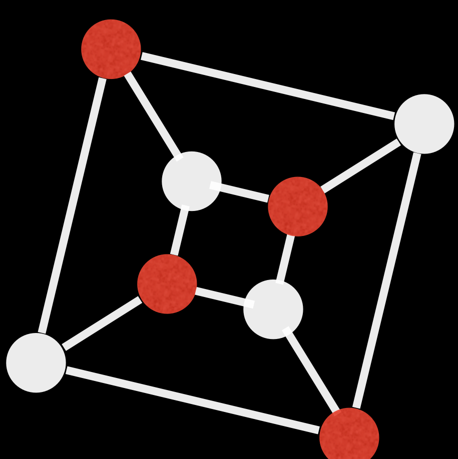
Micah Stairs | William Fiset

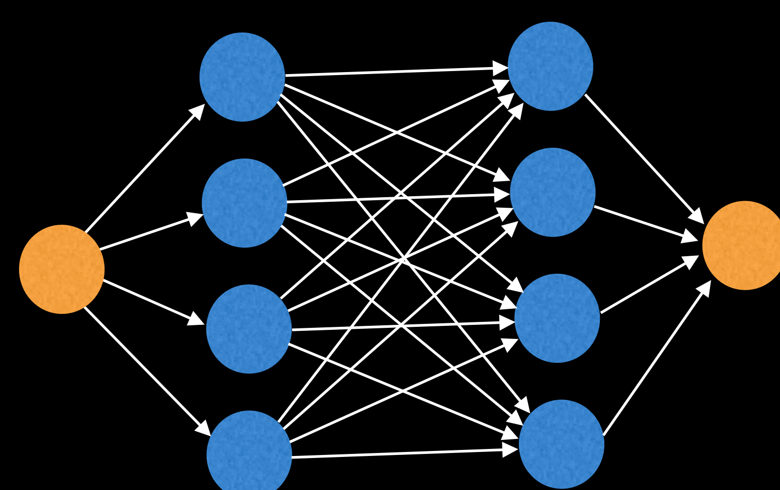


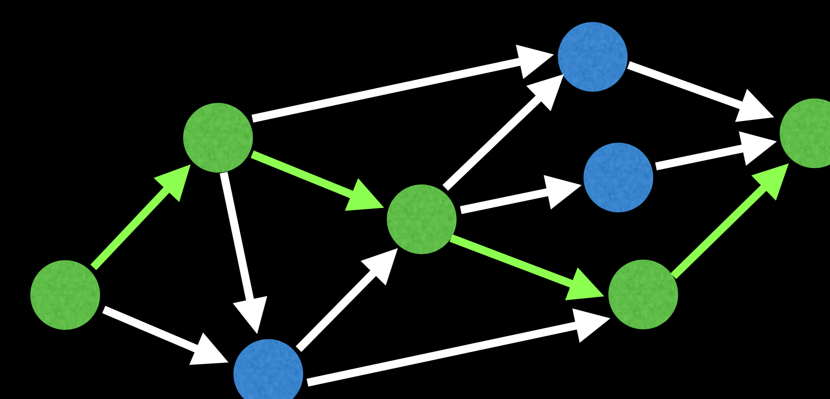




Graph Theory Video Series







Tree Centroid Decomposition

TODO: maybe add some nifty divide and conquer quote

William Fiset