A brief review of Axion Cosmology

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Abstract

The most elegant and promising solution of the strong CP problem in quantum chromodynamics was provided by Peccei and Quinn in 1977 [1, 2]. They postulated that the full Lagrangian of the standard model was invariant under an additional global $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry spontaneously broken at some energy scale f_A. The result is a new spineless particle named Axion. Axions can be copiously produced in the early universe both via thermal and non thermal processes. Thermal Axions contribute to the hot dark matter component of the universe while Cold Axions are natural candidates to solve the cold dark matter problem. In this work we first review and discuss the fundamental aspects of the axion theory explaining how axions solve the strong CP problem. Then we focus on the role of axions in cosmology trying to understand the way they affect and modify the standard cosmological scenario and why and how we can use cosmology to constrain the axion physics.

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Part I

Strong CP problem and Axions

In this part of the paper we give a brief overview of the strong CP problem in Quantum Chromodynamics (QCD) and of the Peccei Quinn solution. We introduce the Axions as the most elegant solution to the strong CP problem and we describe their most important proprieties.

1 The $U_A(1)$ Problem

Let us start our discussion on the strong CP problem describing another problem known as $U_A(1)$ Problem. It seems fun but, as we will see, the strong CP problem arises from the solution of the $U_A(1)$ Problem. In order to understand the nature of the $U_A(1)$ problem we need to briefly review the basic aspects of QCD and the chiral structure of the Dirac spinor fields.

1.1 Basics of QCD

Here we briefly review some of the most fundamental aspects of QCD. A complete description of QCD can be found for example in [3, 4].

As nuclei are formed from protons and neutrons, in the same way Hadrons (i.e. particles that undergo strong interactions) are made of bound states of charged quarks. Quarks can have three different colours (*red*, *blue*, *green*) and different flavors (i.e. *up*, *down*, *strange* ...). At the beginning , when quarks were first introduced, they were considered a convenient way to motivate the appearance of particular representations of the approximate symmetry group SU(3) and for this reason dynamical problems such as the absence of free quarks were neglected. Instead quarks can be described as interacting particles by a fundamental quantum field theory. QCD (i.e. the quantum theory of strong interactions) is a non abelian gauge field theory based on the gauge group SU(3)[5]. The particles linked to the gauge field are said gluons, and their role is to bind Hadrons together. A non abelian gauge theory in some sense is similar to an abelian one and in fact quarks in QCD appear in a very similar way that electrons in QED, while gluons are analogous to photons. Nevertheless there are some important differences like the fact that unlike electrons and photons, quarks and gluons never appear as physical particles. The fact that quarks and gluons are never observed as free particles is solved by the so called confinement requirement which asserts that the dynamics of QCD are such that only SU(3) singlet states are present in the space of finite energy physical states. Furthermore in QCD there are no mass-less states (as photons in QED) except the Pions and associated pseudo-scalar particles in the limit of vanishing quark masses. Note that a crucial consequence of the confinement requirement is that QCD cannot be described in the conventional perturbation theory since the starting point of a perturbation theory will require free quarks and free massless gluons. Nevertheless one can show that for non abelian gauge theories uniquely in four spacetime dimensions, perturbation theory has the property of asymptotic freedom. This justifies the application of the perturbation theory in order to calculate quantitatively measurable predictions.

The Lagrangian of a non abelian SU(3) gauge theory has 8 gauge fields A^a_μ ($a = 1...8$) corresponding to 8 different gluons. The strength tensor is:

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f_{abc} A^b_\mu A^c_\nu \tag{1}$$

where g is the coupling constant and the f_{abc} are the totally antisymmetric structure constants of the SU(3) group. In terms of the 3×3 Gell-Mann matrices λ_a we have:

$$\left[\frac{1}{2}\lambda_a, \frac{1}{2}\lambda_b\right] = i f_{abc} \frac{1}{2}.\lambda_c \tag{2}$$

The quark fields belong to the complex 3-dimensional representation of SU(3) defined by matrices λ_a. The QCD Lagrangian therefore is:

$$\boxed{\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} G^{\mu\nu}_a G^a_{\mu\nu} + \sum \overline{q}_f \left(i\gamma^\mu D_\mu - m_f\right) q_f} \tag{3}$$

where the sum is over the spinor indices (suppressed for brevity), the different quark colour indices (suppressed for brevity) and over the different flavor indices f. Note that each flavor quark has a given mass m_f. D_μ is the covariant derivative defined in such a way that:

$$D_\mu q_f = \partial_\mu q_f - i g A^a_\mu \frac{1}{2} \lambda_a q_f \tag{4}$$

Local infinitesimal SU(3) gauge transformation reads:

$$\begin{cases} \delta A_{\mu a} = \frac{1}{g}\left(\partial_\mu \xi_a + g f_{abc} A_{\mu b} \xi_c\right) \\ \delta q_f = i \xi_a \frac{1}{2} \lambda_a q_f \end{cases} \tag{5}$$

and it easy to show that the QCD Lagrangian (3) is invariant under (5).
We do not want to investigate further general details of the QCD Theory since this goes beyond our goals and we remand the interested reader to the literature dedicated.

1.2 The Chirial Structure of Dirac Spinors

As a preliminary to describe the $U_A(1)$ problem we examine the chiral structure of the Dirac fields. Such fields (that for example could be the electron field or the quark field) satisfy the Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) = 0, \tag{6}$$

that is obtained from the Lagrangian

$$\mathcal{L}(x) = \overline{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x). \tag{7}$$

where $\overline{\psi} = \psi^\dagger \gamma^0$ and the γ^μ matrices are defined by:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I. \tag{8}$$

We can define a further matrix γ^5

$$\gamma_5 \equiv \gamma^0 \gamma^1 \gamma^2 \gamma^3. \tag{9}$$

It is easy to check that $(\gamma^5)^2 = 1$ and that $\gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5$.
At this point let us define the Left handed and Right handed Dirac fields respectively as

$$\psi_L = \frac{1}{2}\left(1 - \gamma^5\right)\psi \tag{10}$$

$$\psi_R = \frac{1}{2}\left(1 + \gamma^5\right)\psi \tag{11}$$

so that:

$$\psi = \psi_L + \psi_R. \tag{12}$$

If we write the Dirac Equation in term of ψ_L and ψ_R, being

$$\overline{\psi}_R = \frac{1}{2}\left(1 - \gamma_5\right)\overline{\psi} \tag{13}$$

$$\overline{\psi}_L = \frac{1}{2}\left(1 + \gamma_5\right)\overline{\psi} \tag{14}$$

we find:

$$\boxed{\mathcal{L}(x) = \overline{\psi}_R i\gamma^\mu \partial_\mu \psi_R + \overline{\psi}_L i\gamma^\mu \partial_\mu \psi_L - m\left[\overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R\right]} \tag{15}$$

We see that the kinetic part of the Lagrangian is a sum of two terms involving the right and left chiral components separately, while the mass term couples right to left and left to right. From this point of view we see that the electron (or the appropriate Dirac particle) mass comes from an interaction that transforms left handed, or negative helicity, electrons into right handed, or positive helicity, electrons and vice versa. For example we also see that an (approximately) massless neutrino has no interaction inducing such a L-R flip so that it remains purely left-handed. Another interesting fact to show is that for massless particles $m = 0$, because of $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$, we have that $\gamma^5\left(i\gamma^\mu\partial_\mu\psi\right) = 0 = -\left(i\gamma^\mu\gamma^5\psi\right)$ and so that both ψ_L and ψ_R respect the Dirac equation

$$i\gamma^\mu \partial_\mu \psi_R = i\gamma^\mu \partial_\mu \psi_L = 0. \tag{16}$$

1.3 QCD Symmetries

Let us consider the QCD Lagrangian (3) and let us focus on the quark Dirac field term $\sum \overline{q}_f \left(i\gamma^\mu D_\mu - m_f\right) q_f$. As we have shown, if we consider the limit $m_f \to 0$, $\forall f$, both the Right and the Left handed Dirac fields respect the Dirac Equation. So in this limit, for f flavors of quarks, we have a large global symmetry

$$\mathrm{U_R(f)} \times \mathrm{U_L(f)} \tag{17}$$

that corresponds to the freedom of arbitrary chiral rotations of the f flavor of quarks into each other. We can use the chirial rotation invariance in order to define the Vectorial component (V=R+L) and the Axial component (A=R-L) so that, in the quark vanishing masses, the exact symmetry of the theory is

$$\mathrm{U_V(f)} \times \mathrm{U_A(f)} \tag{18}$$

If we call Λ_{QCD} the dynamical energy scale of the QCD, since the masses of the up and down quarks are m_u , $m_d \ll \Lambda_{\mathrm{QCD}}$ we have that

$$\mathrm{U_V(2)} \times \mathrm{U_A(2)} \tag{19}$$

is a very good approximate symmetry of the theory that can be further decomposed as:

$$\boxed{\underbrace{\mathrm{SU_V(2)} \times \mathrm{U_V(1)}}_{\mathrm{U_V(2)}} \times \underbrace{\mathrm{SU_A(2)} \times \mathrm{U_A(1)}}_{\mathrm{U_A(2)}}} \tag{20}$$

Let us focus on the vectorial part and on the axial part separately.

1.3.1 $\mathbf{SU_V(2) \times U_V(1)}$

The $\mathrm{SU_V(2)}$ part is the symmetry group associated with the quark field transformations:

$$q_i \to \left[e^{i\frac{\vec{\alpha}\vec{\tau}}{2}}\right]_{ij} q_j \tag{21}$$

and the Noether current:

$$\vec{J}^\mu_V = \overline{q}\gamma^\mu \frac{\vec{\tau}}{2} q \tag{22}$$

This symmetry is exact if $m_u = m_d$. However since m_u , $m_d \ll \Lambda_{\mathrm{QCD}}$ so $\mathrm{SU_V(2)}$ must be a good approximated symmetry. The conservation quantity associated with this symmetry is the **Isospin**.

On the other hand it is easy to show that $\mathrm{U_V(1)}$ is always an exact symmetry of the theory (i.e. it is not important the value of the quark masses) corresponding to the quark field transformations:

$$q_i \to e^{i\alpha} q_i \tag{23}$$

and the Noether current

$$J^\mu_V = \overline{q}\gamma^\mu q. \tag{24}$$

The conservation of the Noether current is nothing else that the **Barion Number**.

1.3.2 $\mathbf{SU_A(2) \times U_A(1)}$

The situation is very different if we consider the Axial $\mathrm{SU_A(2)} \times \mathrm{U_A(1)}$ symmetry. The reason is that in QCD the dynamical formation of quark and anti-quark condensation $\langle \overline{q}q \rangle$ breaks both the global $\mathrm{SU_A(2)}$ and $\mathrm{U_A(1)}$ symmetries.

The $\mathrm{SU_A(2)}$ symmetry is associated with the quark field transformations

$$q_i \to \left[e^{i\frac{\vec{\alpha}\vec{\tau}\gamma_5}{2}}\right]_{ij} q_j \tag{25}$$

and the Noether Axial current

$$\vec{J}^\mu_A = \overline{q}\gamma^\mu\gamma_5 \frac{\vec{\tau}}{2} q. \tag{26}$$

As to the Broken symmetries correspond Goldstone bosons, the break of the $\mathrm{SU_A(2)}$ symmetry is associated with the π-triplet with masses $m_\pi \approx 0$.
On the other hand the $\mathrm{U_A(1)}$ symmetry corresponds to the quark field transformations

$$q_i \to e^{i\alpha\gamma_5/2} q_i \tag{27}$$

and it is associated with the Noether axial current

$$J_5^\mu = \frac{1}{2}\bar{q}\gamma^\mu\gamma_5 q. \tag{28}$$

The very interesting fact is that the broken $\mathrm{U_A}(1)$ symmetry (in the limit of finite but small quark masses) would imply a Goldstone boson essentially degenerate with the pions triplet that instead is not observed in hadron spectrum. Why? This is what is known as the $\mathrm{U_A}(1)$ problem. In order to solve this problem we have to review briefly the structure of the vacuum state of the Gauge Theories and in particular the structure of the QCD vacuum.

1.3.3 Chiral Anomaly

One can think that a possible solution of the $\mathrm{U_A}(1)$ problem may be represented by the chiral anomaly of the $\mathrm{U_A}(1)$ Axial current J_5^μ. In other words the divergence of J_5^μ gets quantum correction:

$$\partial_\mu J_5^\mu = n_f \frac{g_s^2}{32\pi^2} G_a^{\mu\nu}\tilde{G}_{\mu\nu}^a \tag{29}$$

where n_f is the number of different flavors f and $\tilde{G}_{\mu\nu}^a$ is the dual strength energy tensor

$$\tilde{G}_{\mu\nu}^a \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G_a^{\alpha\beta}. \tag{30}$$

In this way, even if the QCD is formally invariant under the $\mathrm{U_A}(1)$ transformations, the chiral anomaly will change the action introducing a term

$$\delta S \propto \int d^4x \partial_\mu j_5^\mu \propto \int d^4x G_a^{\mu\nu}\tilde{G}_{\mu\nu}^a \tag{31}$$

So it seems that the $\mathrm{U_A}(1)$ is no more a symmetry at the quantum level. Unfortunately one can show that defining

$$K^\mu = \epsilon^{\mu\alpha\beta\gamma}A_{a\alpha}\left[G_{a\beta\gamma} - \frac{g_s}{3}f_{abc}A_{b\beta}A_{c\gamma}\right] \tag{32}$$

the quantity $G_a^{\mu\nu}\tilde{G}_{\mu\nu}^a$ can be expressed as a total divergence of K^μ

$$G_a^{\mu\nu}\tilde{G}_{\mu\nu}^a = \partial_\mu K^\mu \tag{33}$$

and so choosing the usual condition $A_a^\mu = 0$ at the spatial infinity the chiral term (31) vanishes

$$\delta S \propto \int d^4x G_a^{\mu\nu}\tilde{G}_{\mu\nu}^a \propto \int d^4x \partial_\mu K^\mu \propto \int d\Sigma_\mu K^\mu = 0 \tag{34}$$

and the $\mathrm{U_A}(1)$ is a good symmetry of the theory again. The problem is not solved. The real solution of the $\mathrm{U_A}(1)$ problem comes from the understanding of the structure of the QCD vacuum state that implies different boundary conditions for the gauge field A_a^μ that must be a pure gauge field at the spatial Infinity.

1.4 The structure of QCD vacuum

The vacuum state of non abelian gauge field theories (like QCD) is more complex and structured than one may expect. Let us try to understand why. In a gauge theory we have some freedom that we use to require that the gauge fields (i.e. the Gluon fields) A_a^μ respect the condition

$$A_a^0 = 0. \tag{35}$$

This s known as **Temporal Gauge**. In this gauge the fields are time independent and, under a gauge transformation $\Omega(\vec{r})$, transform as

$$\frac{\tau_a}{2}A_a^i(\vec{r}) \equiv A^i(\vec{r}) \to \Omega(\vec{r})A^i(\vec{r})\Omega(\vec{r})^{-1} + \frac{i}{g}\Omega(\vec{r})\nabla^i.\Omega^{-1}(\vec{r}) \tag{36}$$

The vacuum state is of course defined to be the state where the vector potential is either zero or in a gauge equivalent configuration to zero. From eq. (36), one can see that in this case the vacuum gauge equivalent configurations correspond to the term $\frac{i}{g}\Omega(\vec{r})\nabla^i.\Omega^{-1}(\vec{r})$. If we require that at the spatial infinity $\Omega(\vec{r} \to \infty) \to 1$, we can classify all these gauge equivalent vacuum states by how $\Omega(\vec{r})$ goes to unity at spatial infinity. In other words

$$\lim_{\vec{r}\to\infty} \Omega(\vec{r}) \to e^{i\,2n\pi} \tag{37}$$

where the integer number n is called **winding number** (or sometimes **topological charge** or even **instanton number**) and it is related to the Jacobian of a $S_3 \to S_3$ transformation that maps the physical space onto the group space. It can be shown that [7]

$$n = \frac{ig_s^3}{24\pi^2} \int d^3r\, \mathrm{Tr}\, \epsilon_{ijk} A_n^i(\vec{r}) A_n^j(\vec{r}) A_n^k(\vec{r}) \tag{38}$$

where A_n is the transformed gauge field under the transformation Ω_n. We can so use the winding number n in order to classify the different vacuum states $|n\rangle$ corresponding to the different ways of how $\Omega(\vec{r}) \to 1$ at the spatial infinity. Moreover we can note that, because of eq. (37), we can obtain the gauge transformation Ω_n using n-times Ω_1 and so that the vacuum states are not really gauge invariant since for example the action of the gauge transformation Ω_1 on n-vacuum state gives a different $(n+1)$-vacuum state

$$\Omega_1 |n\rangle = |n+1\rangle. \tag{39}$$

However it is possible to show that superimposing n different vacuum states, one can obtain a gauge invariant vacuum, the so called θ-vacuum [8]

$$|\theta\rangle \equiv \sum_n e^{-in\theta} |n\rangle. \tag{40}$$

This is of course a gauge invariant vacuum state, but the very interesting fact of the θ-vacuum is that the vacuum to vacuum transition is non zero:

$$_+\langle\theta|\theta\rangle_- = \sum_{m,n} e^{im\theta} e^{-in\theta} {}_+\langle m|n\rangle_- = \sum_\nu e^{i\nu\theta} \left[\sum_n {}_+\langle n+\nu|n\rangle_- \right] \tag{41}$$

where $\nu = m - n$ is the difference in the winding number. One can show that ν is given by:

$$\nu = \frac{g_s^2}{32\pi^2} \int d^4x\, G_{a\mu\nu} \tilde{G}_a^{\mu\nu} = \frac{g_s^2}{32\pi^2} \int d\Sigma_\mu K^\mu \neq 0 \tag{42}$$

Thus the complex structure of the QCD θ-vacuum provides a very interesting fact: the term that comes from the chiral anomaly $\int d^4x\, G_{a\mu\nu} \tilde{G}_a^{\mu\nu}$ is not zero but it is equal to the difference in winding numbers ν. Using the path Integrals formalism in order to describe the vacuum to vacuum transition, one finds:

$$_+\langle\theta|\theta\rangle_- = \sum_\nu \int \delta A^\mu e^{iS_{\mathrm{eff}}[A]} \delta \left[\nu - \frac{g_s^2}{32\pi^2} \int d^4x G_a^{\mu\nu} \tilde{G}_{a\mu\nu} \right] \tag{43}$$

where the effective action S_{eff} is given by:

$$\boxed{S_{\mathrm{eff}} = S_{\mathrm{QCD}}[A] + \theta \frac{g_s^2}{32\pi^2} \int d^4x\, G_a^{\mu\nu} \tilde{G}_{a\mu\nu}} \tag{44}$$

Therefore the θ-term that arises from the non trivial vacuum structure of the gauge theories can be interpreted as a further term in the action of the theory. This additional term violates both P and CP symmetries (while of course it preserves the CPT symmetry). By analogy to the electromagnetic theory, we may say that this term roughly corresponds to the $\vec{E}_a \cdot \vec{B}_a$ interactions.
The solution to the $\mathrm{U_A}(1)$ problem, should be now clear. We have to consider that the boundary condition to impose on the gauge transformation (i.e. their behavior at the spatial infinity) and the consequent non trivial structure of the vacuum state connect the chiral anomaly with the sectors of vacuum to vacuum transition with $\nu \neq 0$. In other words we have to consider that there are sectors of the theory where $\nu \neq 0$. Since the QCD Perturbation theory is connected to the $\nu = 0$ sector where the $G\tilde{G}$ term vanishes, the effects of the $\nu \neq 0$ sectors are necessarily non-perturbative but of course they have to be considered. In these sectors the chiral anomaly plays a crucial role since the charge Q_5 associated with the Noether current J_5^μ, eq.(28), becomes

$$\Delta Q_5 = \int d^4x \partial_\mu J_5^\mu = n_f \frac{g_s^2}{32\pi^2} \int d^4x G_a^{\mu\nu} \tilde{G}_{a\mu\nu} = n_f \nu \tag{45}$$

and so it is never conserved if $\nu \neq 0$. Thus, if we include the $\nu \neq 0$ sectors in the QCD theory (as we have to do), $\mathrm{U_A}(1)$ **is never a symmetry** and this solves the problem.

2 Strong CP problem

So far we have discussed and solved the so called $U_A(1)$ problem. It is interesting and maybe singular that, the solution of the $U_A(1)$ problem will generate another problem, the so called strong CP problem from which, as we will see, Axions arise. If together with the QCD we consider also the weak interactions another term similar to those obtained from the $U_A(1)$ problem solution turns out. In particular the origin of this additional term is due to the mass matrix of quarks which emerges from the spontaneous breakdown of the electroweak gauge symmetry. The mass matrix of quarks is neither Hermitian nor diagonal and in general it is complex so that the respective terms in the Lagrangian are

$$\mathcal{L}_{\text{ mass}} = -\bar{q}_{\text{R}_i} M_{ij} q_{\text{L}_j} - \bar{q}_{\text{L}_i} \left(M^{\dagger}\right)_{ij} q_{\text{R}_j} \tag{46}$$

If one wants to go to a physical basis then the matrix must be diagonalized by separate unitary transformations of the chiral quark fields that encompass also the $U_A(1)$ chiral rotations

$$\begin{cases} q_{\text{R}} \to e^{i\frac{\alpha}{2}} q_{\text{R}} \\ q_{\text{L}} \to e^{-i\frac{\alpha}{2}} q_{\text{L}} \end{cases} \tag{47}$$

where α is

$$\alpha = \frac{1}{n_f} \text{ Arg det } M. \tag{48}$$

With analogous considerations to those discussed in the previous section one can show that such chiral rotation alters the vacuum angle in such a way that, if we define

$$\tilde{Q}_5 \equiv \int d^3x \tilde{J}_5^o \tag{49}$$

the effect of a chiral rotation on the vacuum state is

$$e^{i\alpha\tilde{Q}_5} |\theta\rangle = |\theta + n_f \alpha\rangle \tag{50}$$

Therefore, using again the path integral formalism to describe the vacuum to vacuum transition as in the previous section, but considering also this effect coming from the electroweak interactions, one finds that the full additional term to include in the Lagrangian (or equivalently in the action) is

$$\mathcal{L}_{\text{CP}} = (\theta + \text{ Arg det } M) \frac{g_s^2}{32\pi^2} G_{a\mu\nu} \tilde{G}_a^{\mu\nu}. \tag{51}$$

It is so useful to define the parameter $\bar{\theta}$ as

$$\bar{\theta} \equiv \theta + \text{ Arg det } M \tag{52}$$

so that we finally have

$$\boxed{\mathcal{L}_{\text{CP}} = \bar{\theta} \frac{g_s^2}{32\pi^2} G_{a\mu\nu} \tilde{G}_a^{\mu\nu}.} \tag{53}$$

The Lagrangian (53) contains all the terms coming from QCD and EW interactions that violate the CP symmetry. The parameter $\bar{\theta}$, that is a combination of QCD and electroweak parameters, gives us a measure of the CP violation. This is a free parameter of the theory and so it can assume all the values with the same probability. The parameter $\bar{\theta}$ is strongly constrained from the electric dipole momentum of the neutron d_n that reads [9, 10, 11]

$$d_n \approx \bar{\theta} \left(\frac{e\, m_q}{m_N^2} \right) < 3^{-26} \; [e\, \text{cm}] \tag{54}$$

For $\bar{\theta}$ this implies

$$\boxed{\boxed{\bar{\theta} < 10^{-9}.}} \tag{55}$$

Why is it so small? How is it possible that completely different contributes of order $\mathcal{O}(1)$, coming from completely different physics (QCD and EW) compensate each other with this incredible high level of precision? Even if all the values of $\bar{\theta}$ have the same probability we would like to find a dynamical mechanism able to explain a value of $\bar{\theta}$ so small. This is known as **Strong CP problem**.

3 Axions

How can we solve the standard CP problem? The easiest solution is to require a new additional chiral symmetry so that we can always rotate the $\bar{\theta}$ in such a way that it can be chosen to be zero. One can see that this can always be done if the mass of one of the quarks, say m_u, is zero. In this way, in fact, with a chiral rotation, one can choose $\theta = 0$ so that $\bar{\theta} \propto \det M \propto m_u = 0$. However, in general, both from the theoretical side and from the experimental side, the quarks are regarded to be massive $m_u \neq 0$ and so this is not a good way to solve the problem. Axions arise from the solution of the strong CP problem proposed by Peccei and Quinn [1] that today it is still the most elegant and fascinating solution to this problem.

3.1 $\mathbf{U_{PQ}(1)}$ Symmetry

What Peccei and Quinn postulated [1, 2] was that the **full Lagrangian** of the standard model was invariant under an additional global chiral U(1) symmetry, today known as $\mathrm{U_{PQ}}(1)$ symmetry. If the $\mathrm{U_{PQ}}(1)$ symmetry exists and is exact, so the strong CP problem would be trivially solved as performing a chirial rotation we can always set $\bar{\theta} = 0$. However such a symmetry cannot be exact but it can be spontaneously broken at some energy scale f_A. Even if $\mathrm{U_{PQ}}(1)$ is spontaneously broken, $\bar{\theta}$ is dynamically driven to zero. In this case there will be an associated pseudo Goldstone boson in the theory, that is named **Axion** [12, 13, 14]. However Axions are not massless since, as we are going to see, the chiral $\mathrm{U_{PQ}}(1)$ symmetry is anomalous and for this reason Axions get a mass of order $\sim \frac{\Lambda^2_{\mathrm{QCD}}}{f_A}$. So let us try to understand in more details which are the implications coming from the Peccei Quinn $\mathrm{U_{PQ}}(1)$ symmetry. First of all we notice that since the Axion field ϕ_A is the filed of the Goldstone boson associated with the spontaneously broken symmetry $\mathrm{U_{PQ}}(1)$, this field translates under the $\mathrm{U_{PQ}}(1)$ transformations

$$\phi_A(x) \underset{U_{\mathrm{PQ}}(1)}{\rightarrow} \phi_A(x) + \alpha\, f_A \tag{56}$$

where α is the phase parameter of the transformation $e^{i\frac{\alpha}{2}}$ and f_A is again the breakdown symmetry scale. Then if we want the Lagrangian that describes the full theory to be invariant under the $\mathrm{U_{PQ}}(1)$ symmetry we need to insert the axion field only derivatively coupled so that when ϕ_A transforms according to eq. (56) the term $\alpha\, f_A$ vanishes being constant. Moreover we have to consider also the chirial anomaly that forces us to introduce also a direct coupling between the axion field and the Gluon field. All these considerations completely fix the form of the additional terms in the effective Lagrangian $\mathcal{L}_{\mathrm{eff}}$ with respect to the Standard Model Lagrangian $\mathcal{L}_{\mathrm{SM}}$:

$$\boxed{\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{SM}} + \underbrace{\bar{\theta}\frac{g_s^2}{32\pi^2}G_a^{\mu\nu}\tilde{G}^a_{\mu\nu}}_{\text{CP violating}} \underbrace{-\frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A + \frac{\phi_A}{f_A}\xi\frac{g_s^2}{32\pi^2}G_a^{\mu\nu}\tilde{G}^a_{\mu\nu}}_{\text{Free Axion Lagrangian } \mathcal{L}_A} + \underbrace{\mathcal{L}_{\mathrm{int.}}\left[\frac{\partial^\mu\phi_A}{f_A};\{\psi\}\right]}_{\text{interaction Lagrangian}}} \tag{57}$$

where $\{\psi\}$ are all the other fields in the theory and ξ is a model dependent parameter defined by the chiral anomaly of the $\mathrm{U_{PQ}}(1)$ current J^μ_{PQ} as

$$\boxed{\partial_\mu J^\mu_{\mathrm{PQ}} = \xi\frac{g_s^2}{32\pi^2}G_a^{\mu\nu}\tilde{G}^a_{\mu\nu}} \tag{58}$$

Let us focus on the free Axion Lagrangian $\mathcal{L}_A$ in (57):

$$\mathcal{L}_A = -\frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A + \frac{\phi_A}{f_A}\xi\frac{g_s^2}{32\pi^2}G_a^{\mu\nu}\tilde{G}^a_{\mu\nu} \tag{59}$$

we can recognize the axion potential

$$V_A = -\frac{\phi_A}{f_A}\xi\frac{g_s^2}{32\pi^2}G_a^{\mu\nu}\tilde{G}^a_{\mu\nu} \tag{60}$$

If we do not take into account the QCD effects so, minimizing the vacuum expectation value (VEV) of the potential $\langle V_A\rangle = -\frac{\xi}{f_A}\frac{g_s^2}{32\pi^2}\langle G_a^{\mu\nu}\tilde{G}^a_{\mu\nu}\rangle$, one can see that all the values $0 \leq \langle\phi_A\rangle \leq 2\pi\frac{f_A}{\xi}$ are permitted. On the contrary, taking into account the CP violating term in (57) (i.e. considering the QCD anomalies), because of the periodicity of $\langle G_a^{\mu\nu}\tilde{G}^a_{\mu\nu}\rangle$, we have en effective potential V_A^{eff} in the effective $\bar{\theta} + \langle\phi_A\rangle\frac{\xi}{f_A}$ vacuum angle

$$V_A^{\mathrm{eff}} \sim \cos\left(\bar{\theta} + \frac{f_A}{\xi}\langle\phi_A\rangle\right) \tag{61}$$

that can be minimized to obtain the Axion VEV:

$$\langle\phi_A\rangle = -\frac{f_A}{\xi}\,\bar{\theta} \tag{62}$$

so that

$$\left\langle \frac{\partial V_A}{\partial \phi_A} \right\rangle = -\frac{\xi}{f_A}\frac{g_s^2}{32\pi^2}\left\langle G_a^{\mu\nu}\tilde{G}^a_{\mu\nu}\right\rangle\Big|_{\langle\phi_A\rangle=-\frac{f_A}{\xi}\bar{\theta}} = 0. \tag{63}$$

Note also that expanding the potential around the minimum, Axions acquire a mass given by:

$$m_{\phi_A} = \left\langle \frac{\partial^2 V_A}{\partial \phi_A^2} \right\rangle = -\frac{\xi}{f_A}\frac{g_s^2}{32\pi^2}\frac{\partial}{\partial\phi_A}\left\langle G_a^{\mu\nu}\tilde{G}_{a\mu\nu}\right\rangle\Big|_{\langle\phi_A\rangle=-\frac{f_A}{\xi}\bar{\theta}} \tag{64}$$

So, being the Axion field massive, the mass term must be introduced in the Lagrangian. We want to show that, with all these efforts, the strong CP problem is automatically solved. In fact if we define the physical Axion field ϕ_A^{phys}

$$\phi_A^{\text{phys}} \equiv \phi_A - \langle\phi_A\rangle = \phi_A + \frac{f_A}{\xi}\bar{\theta} \tag{65}$$

rewriting the Lagrangian (57) in terms of ϕ_A^{phys} (and including the previous mentioned mass term) we obtain:

$$\boxed{\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SM}} - \frac{1}{2}\partial_\mu\phi_A^{\text{phys}}\partial^\mu\phi_A^{\text{phys}} - \frac{1}{2}m_{\phi_A}^2(\phi_A^{\text{phys}})^2 + \frac{\xi}{f_A}\phi_A^{\text{phys}}\frac{g_s^2}{32\pi^2}G_a^{\mu\nu}\tilde{G}^a_{\mu\nu} + \mathcal{L}_{\text{int.}}\left[\frac{\partial^\mu\phi_A}{f_A};\{\psi\}\right]} \tag{66}$$

We can see that the CP violating term with $\bar{\theta}$ is cancelled. The effect of the Peccei-Quinn $\mathrm{U_{PQ}}(1)$ symmetry must be clear now: it allows us to replace the static free parameter $\bar{\theta}$ of the theory, with a dynamical field, the Axion field, that drives the vacuum angle to zero solving the strong CP problem. The original model of the axion was proposed by Weinberg and Wilczek, based on the idea of Peccei and Quinn. This is called the Peccei-Quinn-Weinberg-Wilczek (PQWW) model, or the *visible axion model.* In this model, the axion field is identified as a phase direction of the standard model Higgs field. It is necessary to introduce two (or more) Higgs doublets, since the axion degree of freedom does not exist in the theory with single Higgs doublet. Let us denote two Higgs doublets as ϕ_1 and ϕ_2. The PQWW axion is visible, in the sense that it predicts observable signatures in the laboratory experiments. However, the theoretical predictions of the PQWW axion contradict with experimental limits. The problem of the original visible axion model can be avoided if the PQ symmetry is broken at some energy scale higher than the electroweak scale since the couplings of axions with other particles are suppressed as $\propto \frac{1}{f_A}$. This fact motivates the "invisible" axion model. In this model, the axion is not the phase direction of the standard model Higgs doublet. We must introduce a complex singlet scalar field, whose phase would be identified as the axion. Such models are called *Invisible axion model.*

3.2 Axion Potential

So far we have discussed the strong CP problem and the Peccei Quinn solution from which the presence of a new boson named Axion derives. As we have said the $\mathrm{U_{PQ}}(1)$ symmetry is broken at some high scale f_A and Axions arise. The role of non perturbative physics is crucial for Axions and it becomes important at some temperature $T_{\text{NP}} \ll f$ providing a periodic potential for the Axions. If we define Λ the scale of non perturbative physics the potential can therefore be put in the form

$$V_A = \Lambda_A^4\, U(x) \tag{67}$$

where $U(x)$ has at least one minimum and one maximum on the interval $x \in [-\pi\,,\,\pi]$. In fact, even if for a more detailed discussion on the structure of the QCD vacuum and its energy we remand to [16], we have already explained that we expect a periodic contribute coming from the CP violating term. A particularly simple choice for the potential therefore is

$$V_A(\phi_A) = \Lambda_A^4\left[1 - \cos\left(\frac{N_{\text{DW}}\phi_A}{f_A}\right)\right] \tag{68}$$

where N_{DW} is a integer number called *Domain Wall number* that, unless otherwise stated, we will set $N_{\text{DM}} = 1$ To understand what the Domain Wall number is, we remand to section 5.2. As we will see, the domain walls are disastrous from a cosmological point of view [15].

It is important to remark that the potential (68) is not unique since we cannot predict it exactly without a full knowledge of the non perturbative physics. For example in the potential one can include the so called higher order instanton corrections that will add the terms $\sim \cos^n\left(\frac{\phi_A}{f_A}\right)$ to the potential (68). However the potential (68) is of course a good starting point to study (for example) the Axion self-interactions. Moreover note that if $\phi_A \ll f_A$ so the potential can be expanded as a Taylor series and, defining $m_A \equiv \frac{\Lambda_A^4}{f_A^2}$ it reads:

$$V_A(\phi_A) \approx \frac{1}{2}m_A^2\phi^2 \tag{69}$$

Since the symmetry breaking scale f_A is typically rather high, while the non perturbative scale λ_A is typically lower the Axion mass is small.

Part II
Axion Cosmology

In this section we are going to review different aspects of cosmological Axions. The basic idea is that, because of the spontaneous break of the PQ symmetry, Axions can be copiously produced in the early stages of universe, both via thermal and non-thermal processes. Thermal axions with sub-eV masses contribute to the hot dark matter component of the universe while non thermal Axions are natural candidates for the cold dark matter component. We are going to review both the formation processes.

4 Thermal Axions

We call Thermal Axions the population of Axions created and annihilated during interactions among particles in the primordial universe. In this section we are going to review the main aspects of thermal axion cosmology. Let us consider the processes

$$A + i \leftrightarrow f_1 + f_2 \tag{70}$$

where A are axions, f_1 and f_2 are the final product of the interactions and i is a given specie with number density n_i and cross section σ_i. If we call Γ the rate at which axions are created and annihilated in these processes, so we have:

$$\Gamma = \sum_i n_i \left\langle \sigma_i v \right\rangle \tag{71}$$

where with $\langle \ldots \rangle$ we indicate the averaging process over the momentum distributions of the particles involved. Let us call $n_A^{\rm th}(t)$ the thermal axion number density and $n_A^{\rm eq}$ the Axion number density at the thermal equilibrium:

$$n_A^{\rm eq} = \frac{\zeta(3)}{\pi^2} T^3 \tag{72}$$

where $\zeta(3) = 1.202\ldots$ is the Riemann zeta function of argument 3. The number density of thermal axions solves the Boltzmann equation

$$\frac{\mathrm{d}n_A^{\rm th}}{\mathrm{d}t} + 3Hn_A^{\rm th} = \Gamma\left(n_A^{\rm eq} - n_A^{\rm th}\right) \tag{73}$$

Since in an expanding universe the temperature $T \propto \frac{1}{a(t)}$ we can write the time derivative in term of the temperature simply as :

$$\frac{d}{dt} = -H\,T\,\frac{d}{dT} \tag{74}$$

so that taking the derivative of the equation (72), we have

$$\frac{\mathrm{d}n_A^{\rm eq}}{\mathrm{d}t} = -3Hn_A^{\rm eq}. \tag{75}$$

Since in a radiation era the Hubble rate $H(t) \propto \frac{1}{a(t)}$ where $a(t)$ is the scale factor, combining equations (73) and (75) we finally obtain

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left[a^3(t)\left(n_A^{\rm th} - n_A^{\rm eq}\right)\right] = -\Gamma\, a^3(t)\left(n_A^{\rm th} - n_A^{\rm eq}\right).} \tag{76}$$

Therefore a thermal distribution of Axions is reached exponentially fast if

$$\Gamma \gg H \tag{77}$$

Principally, Axions can interact with Quark, Gluons, Photons, Electrons and Hadrons. Interactions are model dependent, but the Axion mass and couplings are inversely proportional to the axion coupling constant f_A

$$m_A = \frac{f_\pi m_\pi}{f_A}\frac{\sqrt{R}}{1+R} \approx 0.6\mathrm{eV}\,\frac{10^7\mathrm{GeV}}{f_A} \tag{78}$$

with $f_\pi = 93\,\mathrm{MeV}$ is the pion decay constant and $R = 0.553 \pm 0.043$ being the up-to-down quark masses ratio.

4.1 Interactions with Quarks and Gluons

In order to thermalize axions in the early universe there are a lot of processes that may be more or less model dependent. Some almost model independent processes are given by the interactions between Axions and Gluons. Such interactions are present in every extended particle model that involves Axions. For a detailed description of these processes we remand the interested reader to [17]. However a rough estimation of their cross section is

$$\sigma \sim \frac{g_s^6}{512\pi^5}\frac{1}{f_A^2} \tag{79}$$

where f_A is the PQ symmetry breaking scale. For $T > 1\,\mathrm{TeV}$ the number density of Quarks and Gluons are

$$n_q = n_{\overline{q}} = 27\frac{\zeta(3)}{\pi^2}T^3 \quad \text{and} \quad n_g = 16\frac{\zeta(3)}{\pi^2}T^3 \tag{80}$$

while

$$H^2 = \frac{8\pi G}{3}\frac{g_{\rm tot}\pi^2}{30}T^4 \tag{81}$$

with

$$g_{\rm tot} = \sum_b g_b + \sum_f \frac{7}{8}g_f \tag{82}$$

being the total number of thermally excited spin degrees of freedom, (bosonic (b) + fermionic (f)). $g_{\rm tot} \approx 107.75$ at the temperature we are interested in. In this way one finds that, thanks to the reactions with quarks and gluons, quarks are in thermal equilibrium with the primordial soup until the temperature:

$$T_{\rm D} \sim 5\times 10^{11}\mathrm{GeV}\left(\frac{f_A}{10^{12}\mathrm{GeV}}\right)^2 \tag{83}$$

Therefore this argument suggests that a population of thermal axions can be produced in the early universe. However we have to keep in mind that such a thermal axion population can be cancelled out from the primordial inflation. As one can see from the above relations, the quarks/gluons - axions reactions thermalize at very high temperatures $T_D \gg T_{\rm RH}$ with $T_{\rm RH}$ being the reheating temperature after the inflation.

4.2 Interactions with Hadrons

Another interesting possibility is the coupling among Axions and Hadrons. Among axions and hadrons couplings we are really interested in the axion-nucleon couplings responsible for the processes

$$N + N \leftrightarrow N + N + A \tag{84}$$

$$N + \pi \leftrightarrow N + N + A \tag{85}$$

and the axion-pion couplings responsible for the processes

$$\pi + \pi \leftrightarrow \pi + A \tag{86}$$

In the early universe, nucleons are so rare respect to pions, that only the axion-pion interactions (86) will be relevant for the thermalization purposes. The reaction (86) is almost model independent since Axions necessarily mix with the π^0 at relatively small temperatures $T \approx 200$ MeV after the QCD phase transition and before the pions annihilation. Therefore inflation cannot remove this population of thermal axions since, if the inflation occurred so late, it would also wipe out the baryons.
The axion-pion couplings are described by the Lagrangian

$$\mathcal{L}_{A\pi} = C_{A\pi}\frac{\partial_\mu A}{f_A f_\pi}\left(\pi^0\pi^+\partial_\mu\pi^- + \pi^0\pi^-\partial_\mu\pi^+ - 2\pi^+\pi^-\partial_\mu\pi^0\right) \tag{87}$$

where

$$C_{A\pi} = \frac{1-R}{3(1+R)} \tag{88}$$

with $R = 0.553 \pm 0.043$ being the up-to-down quark masses ratio.

4.3 Cosmological consequences

We have seen that a population of relic thermal axions can be produced in the early universe. If $f_A > 10^9$ GeV, the axion lifetime exceeds the age of the universe by many orders so that they survive to the present time. Between their last decoupling, at temperature T_D defined by

$$\Gamma\left(T_D\right) = H\left(T_D\right) \tag{89}$$

and today, thermal axions are diluted and redshifted by the expansion of the universe. Their present number density can be calculated simply imposing the total number conservation from the axion decoupling (t_D) and today (t_0), namely: $a(t_D)^3\, n_A^{\rm th}(t_D) = a(t_0)^3\, n_A^{\rm th}(t_0)$. Therefore:

$$n_A^{\rm th}\left(t_0\right) = n_A^{\rm th}(t_D)\left(\frac{a(t_D)}{a(t_0)}\right)^3 = \frac{\zeta(3)}{\pi^2}T_{\rm D}^3\left(\frac{a(t_D)}{a(t_0)}\right)^3 \tag{90}$$

On the other hand their average momentum is given by:

$$\left\langle p_A^{\rm th}\left(t_0\right)\right\rangle = \frac{\pi^4}{30\zeta(3)}T_{\rm D}\left(\frac{a(t_D)}{a(t_0)}\right) \approx 2.701\, T_{\rm D}\left(\frac{a(t_D)}{a(t_0)}\right) \tag{91}$$

If $\left\langle p_A^{\rm th}\left(t_0\right)\right\rangle \gg m_A$ the axion energy distribution is thermal with temperature

$$T_{A,0} = T_{\rm D}\left(\frac{a(t_D)}{a(t_0)}\right) \tag{92}$$

If between t_D and today (t_0) there is no entropy release, because of the entropy conservation, $T_{A,0}$ can be related to the temperature of the CMB photons $T_{\gamma,0}$:

$$T_{A,0} = \left(\frac{10.75}{g_D}\frac{4}{11}\right)^{1/3} T_{\gamma 0} \tag{93}$$

where g_D are the degrees of freedom at temperature T_D. From the axion decoupling temperature T_D, we can also compute the current axion number density n_A

$$n_A = \left(\frac{g_{\star S}\left(T_0\right)}{g_{\star S}\left(T_D\right)}\right)\frac{n_\gamma}{2} \tag{94}$$

where $n_\gamma \approx 412\,{\rm cm}^{-3}$ is the present photon density and $g_{\star S}$ refers to the number of *entropic* degrees of freedom that before the electron-positron annihilation are simply $g_{\star S} = g_{\rm tot}(T)$ (since all the relativistic particles are at the same temperature) while today are $g_{\star S}\left(T_0\right) = 3.91$ [18].

Thermal axions, as well as other light species (for example neutrinos), behave as extra dark radiation so that they contribute to the effective number of relativistic degrees of freedom $N_{\rm eff}$, defined as

$$\rho_{\rm rad} = \left[1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{\rm eff}\right]\rho_\gamma \tag{95}$$

where ρ_γ is the present energy density of the CMB. The canonical value $N_{\rm eff} = 3.046$ corresponds to the three active neutrino contribution plus an extra 0.046 correction coming from the heating after the $e^+\, e^-$ annihilation. The presence of thermal axions will increase the value of the effective number of relativistic degrees of freedom adding an amount $\Delta N_{\rm eff}$ given by

$$\Delta N_{\rm eff} = \frac{4}{7}\left(\frac{3}{2}\frac{n_A}{n_\nu}\right)^{4/3} \tag{96}$$

where n_ν is the present neutrino plus anti-neutrino number density per flavor.
Note that the expansion rate of the universe at the Big Bang Nucleosynthesis (BBN) period strongly depends on the effective number of relativistic degrees of freedom $N_{\rm eff}$, so if there are extra light species at the BBN epoch, the expansion rate of the universe will be higher, leading to a higher freeze out temperature for the weak interactions which translates into a higher primordial helium fraction. Therefore also the abundance of the primordial elements si sensitive to thermal axions. However, since axion couplings scale inversely with f_A, only thermal axions with low f_A (i.e. higher mass) can give a significant contribution to the energy of the Universe. Thermal populations are significant for $m_A \in [0.15\,,\,20]\,{\rm eV}$ as you can see from the blue line in figure 1 adopted from [19]. The blue line represents the thermal axion energy density as a function of the Peccei Quinn symmetry breaking or equivalently the axion mass. For $m_A \sim 0.15\,{\rm eV}$ the axion-pion coupling is the main mechanism production. For that values of the mass, the axion decoupling temperature approaches the QCD temperature

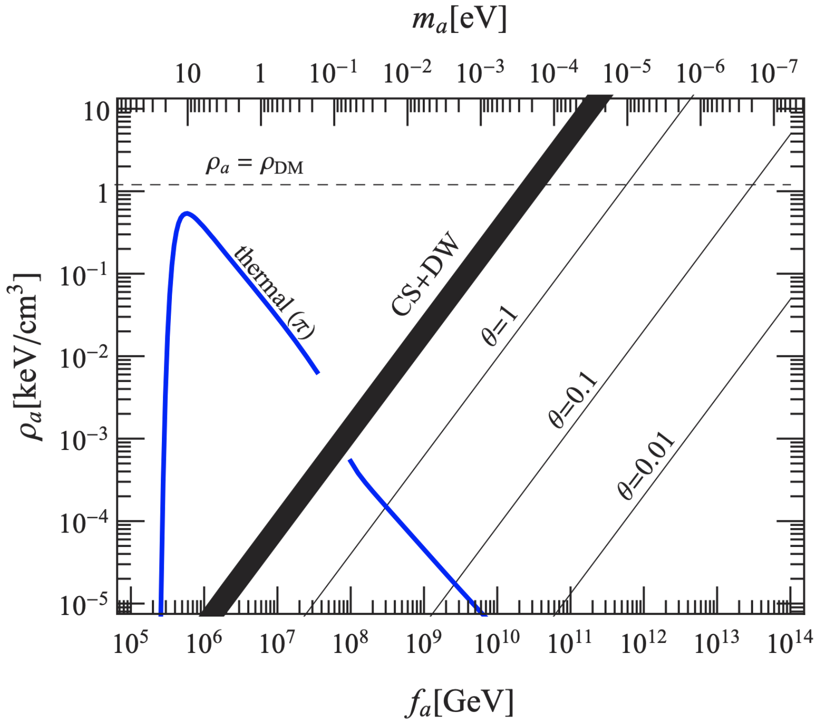

Figure 1: The present-day axion dark matter energy density as a function of f_A or equivalently m_A. The blue line represents the thermal axion energy density which contributes to the hot dark matter. The black lines denote the cold axion energy density for a given fixed value of the initial misalignment angle in the scenario in which the PQ symmetry breaking occurs before the inflation. The thick black region denotes the cold axion population in the scenario in which the PQ symmetry breaking occurs after inflation and takes into account both contributions from the re-alignment mechanism and from topological defects i.e. the cosmic string (CS) and the domain-wall (DW) decay. Image adopted from [19].

and the freeze out occurs after the QCD phase transition. If we consider masses smaller than 0.15 eV so the freeze out will occur before the QCD phase transition and the sharp drop in the thermal axion energy density is due to the dilution caused by the entropy release during the QCD epoch. In fact the number of entropic degrees of freedom $g_{\star S}$ reduces dramatically after the QCD phase transition, diluting the abundance of particles produced before it. Instead if we consider masses bigger than approximately 20 eV we can see that the thermal axion population disappears because of the axion decay

$$A \to \gamma\gamma \tag{97}$$

Therefore, if we want a non negligible population of thermal axions we have to require the axion mass to be constrained in the range $m_A \in [0.15\,,\,20]$ eV or equivalently $f_A \in [\sim 10^5\,,\,\sim 10^8]$ GeV. Thermal axions produced in this way are relativistic as long as $T_D > m_A$ becoming non-relativistic when $T_A < m_A$. As concerns the structure formation, thermal axions behave cosmologically in a manner similar to massive neutrinos, and contribute as hot Dark Matter, suppressing cosmological structure formation below the free-streaming scale.

5 Axion Cold Dark Matter

In the previous section we have studied the possibility that a population of thermal axions may be generated in the early universe. Here we focus on non-thermally produced axions, natural candidates for the cold dark matter component of the Universe. We give an overview of non thermal axion production, studying the evolution of the cosmological Axion field from the moment when the $U(1)_{\rm PQ}$ is spontaneously broken during the PQ phase transition to the moment when axions acquire mass during the QCD phase transition. As we will see, the phenomenology of non thermal axions is maybe larger than that of thermal axions since a lot of non standard scenarios may happen such as the formation of topological defects like cosmic strings or domain walls.

5.1 Vacuum Realignment

The equation of motion of the axion field ϕ_A can be derived assuming in the early universe the usual F.R.W. flat metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx_i dx_j. \tag{98}$$

and assuming that the axion field is minimal coupled to gravity. So we write the action:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_A\partial_\nu\phi_A - V_A(\phi_A)\right]. \tag{99}$$

where $M_p = \frac{1}{\sqrt{8\pi G}}$ is the reduced Planck Mass. Variating the action with respect to the Axion field ϕ_A we can find the equation of motion that reads:

$$\left[\partial_t^2 + 3H\partial_t - \frac{1}{a^2}\nabla_{\mathrm{x}}^2\right]\phi_A(x) + V_A'[\phi_A(x)] = 0 \tag{100}$$

where prime indicates a derivative with respect to ϕ_A and where V_A is the effective potential for the axion field that, as we have discussed in section 3.2, is periodic and turns out from non-perturbative QCD effects associated with instantons and can be qualitatively written as

$$V_A = f_A^2\, m_A^2(t)\left[1 - \cos\left(\frac{\phi_A}{f_A}\right)\right] \tag{101}$$

The axion mass is a function of temperature and hence of time: $m_A(t) = m_A[T(t)]$. High temperature effects ($T \approx 1$ GeV) of QCD instantons give [14, 20, 21, 22]:

$$m_A(T) \simeq 4 \times 10^{-9}\mathrm{eV}\left(\frac{10^{12}\mathrm{GeV}}{f_A}\right)\left(\frac{\mathrm{GeV}}{T}\right)^4 \tag{102}$$

The axion mass is strongly suppressed at temperatures that are large compared to the QCD scale, but it turns on when the temperature approaches Λ_{QCD}. In practice, the axion mass becomes important when $m_A(t) \propto t$, so it is useful to define a time t_* at which the axion mass turns on[1]:

$$m_A(t_*)t_* = 1 \tag{103}$$

Putting the potential (101) into the equation of motion (100) we obtain:

$$\boxed{\left[\partial_t^2 + 3H\partial_t - \frac{1}{a^2}\nabla_{\mathbf{x}}^2\right]\phi_A(x) + m_A^2(t)\, f_A\, \sin\left(\frac{\phi_A(x)}{f_A}\right) = 0} \tag{104}$$

On the other hand, variating the action (99) with respect to the metric tensor we can find the well known relations for the axion energy density and pressure that are:

$$\rho_A = \frac{1}{2}\dot{\phi}_A^2 + V_A \tag{105}$$

$$p_A = \frac{1}{2}\dot{\phi}_A^2 - V_A \tag{106}$$

and so:

$$\omega_A \equiv \frac{p_A}{\rho_A} = \frac{\frac{1}{2}\dot{\phi}_A^2 - V_A}{\frac{1}{2}\dot{\phi}_A^2 + V_A} \tag{107}$$

The axion field evolves according to equation (104). So once we have solved it and computed $\phi_A(x)$ we can also predict the axion energy density and pressure from the equations (105) and (107). However we need to specify the axion field initial condition that are completely random. Moreover causal disconnected region of spacetime in general have uncorrelated values of $\phi_A(x)$. Nevertheless it is well known that the size of the causal horizon grows exponentially during cosmological inflation and so it can homogenize the axion field over enormous distances. Therefore, before solving equation (104), we have to distinguish two different cases. Let us call T_{RH} the temperature of the reheating after the inflation, if:

1. $T_{\mathrm{PQ}} > T_{\mathrm{RH}}$ so the inflation occurs after the PQ symmetry breaking and the axion field is homogenized over enormous distances.

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[1] for $T \approx 1$ GeV, $t_* \simeq 2 \times 10^{-7}\mathrm{s}\left(\frac{f_a}{10^{12}\mathrm{GeV}}\right)^{1/3}$

2. $T_{\rm PQ} < T_{\rm RH}$ so the inflation occurs before the PQ symmetry breaking and the axion field has non-zero modes and carries topological defects such as strings and domain walls.

The first possibility, inflation occurring after the PQ symmetry breaking, is of course the simplest since in this way we do not have to worry about the topological defects production and the axion field is homogenized over large distances. However in what follows we will study both the situations.

5.1.1 Case 1: $T_{\rm PQ} > T_{\rm RH}$

Focus our attention on the first case $T_{\rm PQ} > T_{\rm RH}$. Inflation homogenizes the axion field over very large distances and so we can assume that the axion field does not depend on the space coordinate $\bar{\rm x}$ so that the equation of motion (104) becomes:

$$\left[\frac{d^2}{dt^2} + \frac{3}{2t}\frac{d}{dt}\right]\phi_A(t) + m_A^2(t)\, f_A \,\sin\left(\frac{\phi_A(t)}{f_A}\right) = 0 \tag{108}$$

where we have used $H = \frac{1}{2t}$. Because of our previous discussion about the time (or temperature) dependence of the axion mass $m_A(t)$, we are allowed to distinguish two different regimes $t \ll t_*$ when the axion mass can be neglected and $t \gtrsim t_*$ where the axion mass turns on.

In the regime $t \ll t_*$ we can neglect the axion mass $m_A(t \ll t_*) \approx 0$ and the equation to solve becomes:

$$\left[\frac{d^2}{dt^2} + \frac{3}{2t}\frac{d}{dt}\right]\phi_A(t) = 0 \tag{109}$$

We immediately see that the most general solution is

$$\phi_A(t) = \phi_0 + \phi_1\, t^{-\frac{1}{2}} \tag{110}$$

where ϕ_0 and ϕ_1 are constants. Therefore the expansion of the universe (in a Radiation dominated era $a(t) \propto t^{\frac{1}{2}}$) freezes the axion field to a constant value. Therefore the axion field is overdamped and frozen at its initial value by Hubble friction. The equation of state at early times is so $\omega_A = -1$ and the axion behaves as a contribution to the vacuum energy: if axions were able to dominate the energy density of the universe when they are still overdamped with equation of state $\omega < -\frac{1}{3}$, they could even drive a period of accelerated expansion.

When t approaches t_*, the axion field starts oscillating because of the axion mass contribution. Let us suppose that $\phi_A(t) \ll f_A$ so that we can expand $f_A \sin\left(\frac{\phi_A(t)}{f_A}\right) \approx \phi_A(t)$ in the equation of motion that so becomes:

$$\left[\frac{d^2}{dt^2} + \frac{3}{2t}\frac{d}{dt}\right]\phi_A(t) + m_A^2(t)\phi_A(t) \tag{111}$$

It is useful to study this equation performing the following substitution:

$$\chi(t) = t^{\frac{3}{4}}\phi(t) \tag{112}$$

So that the equation of motion becomes:

$$\left[\frac{d^2}{dt^2} + \omega^2(t)\right]\chi(t) = 0, \tag{113}$$

where

$$\omega^2(t) = m_A^2(t) + \frac{3}{16\, t^2}. \tag{114}$$

In other words for $t \gtrsim t_*$ the axion field is oscillating with frequency $\omega \approx m_A$. The solution of equation (113) is therefore given by:

$$\chi(t) \simeq \frac{C}{\sqrt{m_A(t)}}\cos\left[\int_t {\rm d}t'\,\omega\,(t')\right] \tag{115}$$

where $C = $ const. For $\phi_A(t)$ this translates into

$$\phi_A(t) = \phi_0(t)\cos\left[\int_t {\rm d}t'\,\omega\,(t')\right] \tag{116}$$

with

$$\phi_0(t) = \frac{C\, t^{-\frac{3}{4}}}{\sqrt{m_A(t)}} \tag{117}$$

Putting the solution (116) into equation (105) one can check that

$$\rho_A \propto m_A(t)^2 \phi_0(t)^2 \propto t^{-\frac{3}{2}} \tag{118}$$

remembering that $a(t) \propto^{-\frac{1}{2}}$, we finally obtain the following very important results:

$$\boxed{\rho_A \propto a(t)^{-3}.} \tag{119}$$

This is the same behavior of ordinary matter, and this is why misalignment axions are a valid candidates for the cold dark matter. We can also estimate the late time number density of axions by saying that the axion field has a random initial value $\phi_A(t_*) = f_A\,\theta_*$ where θ_* is said initial misalignment angle and it evolves according to the equation of motion

$$\ddot{\theta}_* + 3\,H\,\dot{\theta}_* + m_A(t)^2\theta_* \tag{120}$$

Since the potential is periodic with period $2\pi\,f_A$, the relevant range of θ_* values is $[-\pi\,,\,\pi]$. The number density of axion at time t_* is given by [20, 21, 22]

$$n_A^{\text{vac}}\left(t_*\right) \sim \frac{1}{2} m_A\left(t_*\right) \phi_A^2\left(t_*\right) = \frac{f_A^2}{2t_*}\theta_*^2 \tag{121}$$

The number of axions is an adiabatic invariant after t_* and so, since they behave like matter, their number density at any given time $t > t_*$ is:

$$n_A^{\text{vac}}\left(t\right) \sim \frac{f_A^2}{2t_*}\theta_*^2 \left(\frac{a(t_*)}{a(t)}\right)^3 \tag{122}$$

and so, today ($t = t_0$):

$$n_A^{\text{vac}}\left(t_0\right) \sim \frac{f_A^2}{2t_*}\theta_*^2 \left(\frac{a(t_*)}{a(t_0)}\right)^3 \tag{123}$$

Note that from the axion number density we can obtain the Axion energy density simply multiplying for their mass:

$$\rho_A^{\text{vac}}\left(t\right) \sim \frac{m_A\,f_A^2}{2t_*}\theta_*^2 \left(\frac{a(t_*)}{a(t)}\right)^3 \tag{124}$$

This mechanism of axion production is called **vacuum realignment** or also **misalignment mechanism**.

5.1.2 Case 2: $T_{\text{PQ}} < T_{\text{RH}}$

So far we have studied in details the case when inflation homogenizes the axion field over large distances. Now we want to study the vacuum realignment in the case in which no inflation occurs and the value of the field depends on the spatial coordinates and the equation to solve is the eq. (104). As we will see the only difference with the previous case is a contribution coming from the non zero momentum modes.

Let us start considering the regime $t \ll t_*$ when the axion mass is suppressed. In this case it is convenient to expand the axion field in the Fourier space

$$\phi_A(\mathbf{x}, t) = \int \mathrm{d}^3\mathbf{k}\,\tilde{\phi}_A(\mathbf{k}, t)\,e^{i\mathbf{k}\cdot\mathbf{x}} \tag{125}$$

where the Fourier modes $\phi(\mathbf{k}, t)$ satisfy the equation of motion in the Fourier space that reads:

$$\left(\partial_t^2 + \frac{3}{2t}\partial_t + \frac{k^2}{a(t)^2}\right)\tilde{\phi}_A(\mathbf{k}, t) = 0. \tag{126}$$

As well known, in an expanding universe, the wavelength $\lambda(t) = 2\pi\frac{a(t)}{k}$ of each mode is stretched by the expansion itself and so two further different regimes arise. The evolution is in fact different depending on the larger or smaller wavelength than the causal horizon.
For the modes outside the horizon, $k/a(t) \ll H(t)$, the third term on the left hand side of Eq. (126) is dropped, and the solution is given by:

$$\tilde{\phi}_A(\mathbf{k}, t) = \phi_0(\mathbf{k}) + \phi_1(\mathbf{k})\,t^{-\frac{1}{2}} \tag{127}$$

where $\phi_0(\mathbf{k})$ and $\phi_1(\mathbf{k})$ are some k-dependent constants. Therefore, for wavelengths larger than the causal horizon, each mode goes to a constant and the axion field is frozen by causality.
On the other hand, for modes inside the horizon, $k/a(t) \gg H(t)$, we cannot neglect the third term on the left hand side of Eq. (126) that, performing the substitution

$$\chi(\mathbf{k}, t) = a(t)^{\frac{3}{2}}\,\tilde{\phi}_A(\mathbf{k}, t), \tag{128}$$

can be rewritten as follows:

$$\left[\partial_t^2 + \omega^2(t)\right] \chi(\mathbf{k}, t) = 0 \tag{129}$$

where

$$\omega^2(t) = \frac{k^2}{a^2(t)} + \frac{3}{16\, t^2} \simeq \frac{k^2}{a^2(t)}. \tag{130}$$

The most general solution is

$$\tilde{\phi}(\mathbf{k}, t) = \frac{C}{a(t)} \cos\left[\int^t \mathrm{d}t' \omega\,(t')\right] \tag{131}$$

where $C =$ const. This is an oscillating solution with a frequency $\omega \simeq k/a(t)$ and whose amplitude decreases with time as $\frac{1}{a(t)}$.

When $t \gtrsim t_*$ the axion mass term becomes non-negligible. The modes outside the causal Horizon, $k/a(t) \ll H(t)$, start oscillating with frequency $\omega \simeq m_A(t)$. In fact the equation of motion becomes exactly the same than that studied in the previous scenario with $T_{\rm PQ} > T_{\rm RH}$: we call these modes zero modes. As concerns the modes inside the Horizon, $k/a(t) \gg H(t)$, they contribute to the higher momentum modes.
The axion number density in this case is therefore given by the sum of the contribution of the zero momentum modes $n_A^{\rm vac,0}(t)$ and the contribution of the higher momentum modes $n_A^{\rm vac,1}(t)$:

$$n_A^{\rm vac}(t) = n_A^{\rm vac,0}(t) + n_A^{\rm vac,1}(t). \tag{132}$$

The zero momentum modes contribution is given by equation (122), but since in this case the initial misalignment angle is different from one QCD horizon to another and since the average of θ_*^2 is of order one we have

$$n_A^{\rm vac,0}\,(t) \sim \frac{f_A^2}{2t_*}\left(\frac{a(t_*)}{a(t)}\right)^3. \tag{133}$$

On the other hand, one can show [14, 23] that the higher momentum modes contribution at the time $t = t_*$ is:

$$n_A^{\rm vac,1}\,(t_*) \sim \frac{N^2 f_A^2}{2t_*} \tag{134}$$

where the factor N is defined saying that the typically variation of the Axion field from one horizon to the next is $\sim N\, f_A$. Since after t_* almost all these axions are non-relativistic they behave like ordinary matter and so:

$$n_A^{\rm vac,1}\,(t) \sim \frac{N^2 f_A^2}{2t_*}\left(\frac{a(t_*)}{a(t)}\right)^3 \tag{135}$$

5.2 Topological Defects

Since the physics of the early universe is described by guage theories which undergo spontaneous symmetry breaking, the universe is expected to have gone through various phase transitions as it cooled after the big bang. These phase transitions can give rise to topological defects. The Peccei Quinn phase transition is one of the primordial phase transition that can produce such topological defects. If inflation occurs after the PQ phase transition ($T_{\rm PQ} > T_{\rm RH}$), so the topological defects are exponentially diluted in such a way that today it extremely improbable to observe them in our universe. Otherwise, if inflation occurs before the PQ phase transition, topological defects are not diluted and so their evolution can have left a trace in our observable universe. So in this section we will focus on this scenario studying the different topological defects and the Axion production from topological defects. We do not pretend to be mathematically precise. For a more formal description the interested reader can see refs [24, 25, 26, 27].
Before studying the Axion production from topological defects, we need to be accurate to classify the different type of defects. The type of topological defects depends on the topology of the gauge theory and in particular on the topology of the vacuum manifold $\mathcal{M}$. In what follows let us assume that the universe is correctly described by a gauge theory which undergoes spontaneous symmetry breaking of its symmetry group G at some critical temperature. The generic field ϕ (not necessary the axion field ϕ_A) undergoing spontaneous symmetry breaking can be taken to have a minimum at $\phi = 0$ in the high temperature phase while it takes a vacuum expectation value $\langle 0|\phi|0\rangle = \langle\phi\rangle \neq 0$. Let us suppose that $\langle\phi\rangle$ is invariant under sub-group transformations $H \in G$. The vacuum manifold $\mathcal{M}$ therefore is given by $\mathcal{M} = \frac{G}{H}$ the space of degenerate states which breaks the original symmetry. During the phase transition, $\phi(x)$ will take a vacuum expectation value in $\mathcal{M}$ that will be uncorrelated in causal disconnected regions of spaces. To say the truth, for energetic reasons (because in general a spatial derivative term appears in the Hamiltonian) a constant or slowly varying vacuum expectation value is preferred but, depending on the topology of $\mathcal{M}$, some boundaries may survive between domains where

instead $\phi(x) = 0$. These are nothing else that the topological defects: topologically stable configurations of the higher-energy state.
We are basically interested in two different types of topological defects: **Domain Walls** and **Cosmic strings**.

5.2.1 Domain Walls

Domain walls are two-dimensional topological defects, which are formed when the vacuum manifold $\mathcal{M}$ is disconnected. This happens for example when M is made up by only two points corresponding to two different values of the vacuum expectation value $\langle\phi\rangle = \pm\eta$. In each causal disconnected region $\langle\phi\rangle$ can be $+\eta$ or $-\eta$ randomly and then causal disconnected neighboring regions will tend to fall randomly into the different states (see figure 2). The common boundary surface between these regions are what we call domain walls. Since one cannot pass from $\langle\phi\rangle = +\eta$ to $\langle\phi\rangle = -\eta$ without passing through a region where $\langle\phi\rangle = 0$, so the domain walls are such that $\langle\phi\rangle = 0$. As one can see from fig. 2 regions with $\langle\phi\rangle = 0$ are in a higher energy state and so high energy walls are formed. The cosmological evolution of a domain wall is determined by its surface tension. The

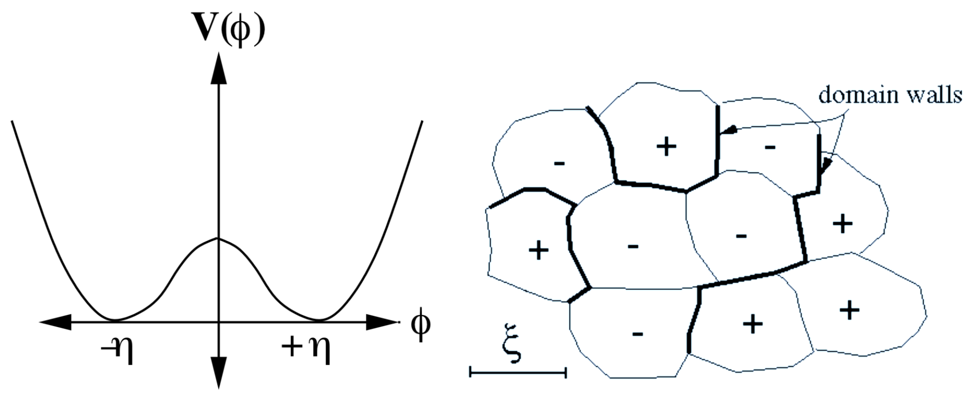

Figure 2: On the left a typical potential that has a disconnected vacuum manifold $\mathcal{M}$. On the right a schematic representation of the domain walls: regions denoted with "+" are those where $\langle\phi\rangle = +\eta$ while "-" indicates regions with $\langle\phi\rangle = -\eta$. The black common boundary surface represents the domain walls where $\langle\phi\rangle = 0$.

structures will grow with time until they are comparable to the Hubble scale, leading to large inhomogeneities in the cosmic background radiation that are not observed [26, 28].

5.2.2 Cosmic Strings

Cosmic strings are one dimensional topological defects which are formed when the vacuum manifold $\mathcal{M}$ is not simply connected. Cosmic strings require a more complicated theory than Domain walls. Consider a complex scalar field ϕ. Its vacuum state will have a U(1) symmetry: $\langle\phi\rangle = \eta\, e^{i\theta}$: at each point in space the field can assume a phase $\theta \in [0\,,\,2\pi]$. Since $\langle\phi\rangle$ is single valued, the total change of θ around any closed loop must be $\Delta\theta = 2\pi n$. If $n \neq 0$ the loop cannot be shrunk to a point and there will be at least one point inside the loop where θ is undefined. If θ is undefined so it must be $\langle\phi\rangle = 0$. The loop can be deformed in order to find another point of false vacuum. All these false vacuum points connect together to form a tube of false vacuum that is what we call cosmic string.

5.2.3 Topological defects for Axion theory

Let us try to understand how topological defects may arise from the invisible axion model. This requires a brief description of the model itself. Let us call T_{PQ} the temperature at which the PQ symmetry is spontaneously broken, we have that

$$T_{\mathrm{PQ}} \simeq v_A \tag{136}$$

where v_A is the vacuum expectation value of a complex field $\sigma(x)$, named Peccei Quinn filed, whose Lagrangian is:

$$\mathcal{L}_\sigma = \frac{1}{2}\partial_\mu\sigma^\dagger\partial^\mu\sigma - \frac{\lambda}{4}\left(\sigma^\dagger\sigma - v_A^2\right)^2 + \mathcal{L}_{\mathrm{int}}\left[\sigma(x)\,,\,\psi_i\right] \tag{137}$$

where $\mathcal{L}_{\mathrm{int}}\left[\sigma(x)\,,\,\psi_i\right]$ is the Lagrangian of interactions between $\sigma(x)$ and the other fields in the theory. When $T > T_{\mathrm{PQ}}$ the minimum of energy is at $\sigma(x) = 0$ and the PQ symmetry is unbroken. However as the universe cools down $T < T_{\mathrm{PQ}}$ and the temperature of the universe becomes comparable to the QCD scale $\Lambda_{\mathrm{QCD}} \sim 100\mathrm{MeV}$, so the non perturbative nature of QCD becomes relevant, the axions acquire mass and the effective potential arises [2].

$$V_A(\phi_A) = m_A^2 f_a^2 \left[1 - \cos\left(\frac{N_{\mathrm{DW}}\phi_A}{f_A}\right)\right]. \tag{138}$$

The existence of the QCD potential explicitly breaks the $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry and the vacuum expectation value of the field σ becomes a circle whose radius quickly approaches v_A:

$$\langle 0|\sigma(x)|0\rangle = v_A e^{i\frac{\phi_A(x)}{v_A}} \tag{139}$$

where $\phi_A(x)$ is the axion field. As we can see, now we have a non trivial vacuum manifold $\mathcal{M}$ that is non simply connected and we are in the situation described in section 5.2.2 and so cosmic strings are formed. Let us consider the discrete subgroup $Z_{N_{\mathrm{DW}}}$ of the shift symmetry

$$\phi_A \to \phi_A + 2\,k\,\pi\,\frac{f_A}{N_{\mathrm{DW}}} \tag{140}$$

with $k = \{0, 1 ..., N_{\mathrm{DW}} - 1\}$ that leaves the potential unchanged. Also this $Z_{N_{\mathrm{DW}}}$ symmetry is spontaneously broken because of the vacuum expectation value (139) of the axion field. Therefore the vacuum manifold $\mathcal{M}$ becomes more complicated since we have N_{DW} degeneracy vacuums that are equidistant in the curve of the minima. In other words $\mathcal{M}$ not only is not simply connected but it is also disconnected and so Domain walls can be produced. Cosmic strings occur much earlier than the formation of domain walls: the domain walls are formed when the Hubble parameter becomes comparable to the axion mass: $H \sim m_A$.

5.2.4 Axions from string decay

Let us consider the axion cosmic strings. Because they are strongly coupled to the axion field, the strings decay very efficiently into axions. We want to estimate this process. For more details one can read [23, 25, 27].
The energy per unit length of an axion string is

$$\mu = \pi\,v_A^2\,\ln(v_A L) \tag{141}$$

where L is an infra-red cutoff approximately equal to the distance of the nearest neighbor string. As we have seen, since without inflation[3] the size of the causal horizon is of order t, so $\phi_A(x)$ is completely uncorrelated over distances larger than t. At a given time t, there is at least the order of one string per horizon. At the beginning these strings are in the primordial plasma and they are stretched by the Hubble expansion $a(t) \propto t^{-\frac{1}{2}}$. During this time the density of strings grows and they become much more than one per horizon. However with the spacetime expansion, when the temperature of the universe becomes lower than approximatively [29] $T_{\mathrm{free}} \sim 2 \times 10^7\mathrm{GeV}\left(\frac{f_A}{10^{12}\mathrm{GeV}}\right)^2$, they decouple from the primordial plasma and they make up a network of axion strings moving freely at relativistic speeds. Axions are expected to be largely and efficiently produced by the string loops collapse. Moreover long strings (that by definition are stretched across horizons) reconnect to form loops that then collapse freely in axions. The number density of axions radiated by strings n_A^{str} is given by [14]

$$\frac{\mathrm{d}n_A^{\mathrm{str}}}{\mathrm{d}t} = -3Hn_A^{\mathrm{str}} + \frac{1}{\omega(t)}\frac{\mathrm{d}\rho_{\mathrm{str}\to A}}{\mathrm{d}t} \tag{142}$$

where $\omega(t)$ is the average energy of axions radiated in string-decay processes at a given time[4]

$$\frac{1}{\omega(t)} = \left(\frac{\mathrm{d}\rho_{\mathrm{str}\to A}}{\mathrm{d}t}\right)^{-1}\int\frac{\mathrm{d}k}{k}\frac{\mathrm{d}^2\rho_{\mathrm{str}\to A}}{\mathrm{d}t\mathrm{d}k} \tag{143}$$

while $\frac{\mathrm{d}\rho_{\mathrm{str}\to A}}{\mathrm{d}t}$ is the rate at which energy density is converted from strings to axions

$$\frac{\mathrm{d}\rho_{\mathrm{str}\to A}}{\mathrm{d}t} = -\frac{\mathrm{d}\rho_{\mathrm{str}}}{\mathrm{d}t} - 2H\rho_{\mathrm{str}} \tag{144}$$

where ρ_{str} is the long sting energy density [14] $\rho_{\mathrm{str}}(t) = \xi\frac{\mu}{t^2}$. The parameter ξ determines the density of the string network, $\xi = 1$ corresponds to a density of one long string per horizon. If we want that global strings can decay efficiently into axions we have to require that $\xi \approx 1$. Numerical simulation of global string networks

[2] So far we have set $N_{\mathrm{DW}} = 1$, but now let us relax this assumption

[3] Remember that in this scenario inflation occurs before the PQ symmetry breaking, otherwise it will dilute the abundance of topological defects in such a way that their contribution to the universe energy density will be negligible

[4] Note that $\frac{\mathrm{d}^2\rho_{\mathrm{str}\to A}}{\mathrm{d}t\mathrm{d}k}$ figuring in the equation (143) is nothing else that the spectrum of the axions produced.

in an expanding universe found that effectively $\xi \simeq 1$ [14, 30].
Combining all these equations, maintaining only the terms of order $\ln(v_A\, t)$, one obtains

$$n_A^{\rm str}(t) \simeq \frac{\xi \pi f_a^2 N^2}{t^{3/2}} \int_{t_{\rm PQ}}^{t} {\rm d}t' \frac{\ln\left(v_a t'\right)}{t'^{3/2} \omega\left(t'\right)} \tag{145}$$

To go further we need to know $\omega(t)$, the average energy of axions radiated at time t. Many analytics approximations or computational techniques can be used in order to estimate $\omega(t)$, see for example [14] and the reference within. What really interests us is that in the range $t^{-1} \lesssim k \lesssim (t_{\rm PQ} t)^{-1/2}$ the axions coming from string decay have a spectrum $\propto k^{-2}$ and so, at the time t_* they have a momentum $\propto \frac{1}{t_*}$. This means that when the axions acquire a mass they become non-relativistic soon after. Axions produced by cosmic strings decay contribute to cold dark matter and we have that, after t_*

$$\rho_A^{\rm str}(t) = m_A\, n_A^{\rm str}(t_*) \left(\frac{a(t_*)}{a(t)}\right)^3 \tag{146}$$

and so today ($t = t_0$)

$$\rho_A^{\rm str}(t_0) = m_A\, n_A^{\rm str}(t_*) \left(\frac{a(t_*)}{a(t_0)}\right)^3 \tag{147}$$

5.2.5 Axion and domain walls

When the axion mass turns on, at time t_*, each axion string becomes the edge of $N_{\rm DW}$ domain walls. The domain walls produce a cosmological disaster unless there is inflation after the PQ phase-transition (case 1) or unless $N_{\rm DW} = 1$. Basically what happens is that, when the axion mass turns on, each axion string becomes the edge of $N_{\rm DW}$ domain walls. Let us consider that $N_{\rm DW} \geq 2$, since there are two or more exactly degenerate vacuum states so there is at least the order of one domain wall per causal horizon. The energy density in domain walls is then

$$\rho_{\rm DW}(t) \gtrsim \frac{\sigma}{t} \tag{148}$$

where σ is the wall energy per unit surface [14]

$$\sigma \simeq 9 f_A^2 m_A \simeq 5.5 \times 10^{10} {\rm GeV}^3 \left(\frac{f_A}{10^{12}{\rm GeV}}\right) \tag{149}$$

Therefore we estimate the domain wall energy density today $t = t_0 \simeq 14$ Gyr obtaining:

$$\rho_{\rm DW}\left(t_0\right) \gtrsim \frac{\sigma}{t_0} \simeq 2 \times 10^{-14} {\rm g\,cm}^{-3} \left(\frac{f_A}{10^{12}{\rm GeV}}\right) \tag{150}$$

The critical density today (i.e. the energy density required in order to have a flat universe) is

$$\rho_c(t_0) \equiv \frac{3H_0^2}{8\pi G} \simeq 10^{-29} {\rm g\,cm}^{-3}. \tag{151}$$

We immediately see that the domain wall contribution to the energy density exceeds alone by many orders the magnitude of the critical energy density for closing the universe. Domain Walls would over-close the universe. There are at least two options to avoid the axion domain-wall problem[5]: the first trivial option is to have inflation with $t_{\rm RH} < T_{\rm PQ}$ so that the axion field is then homogenized over large distances, and there are no strings or domain walls. The second option is to postulate $N_{\rm DW} = 1$. In this way, when the axion mass turns on, each string becomes the boundary of a single domain wall. In this case what typically happens is that each string accelerates to relativistic speeds, in the direction of the wall to which it is attached, in less than a Hubble time unzipping the wall and releasing the stored energy in the form of barely relativistic axions [14]. The part of the domain walls that does not decay in axions can decay in gravitational waves. For example this is the case when one considers small symmetry breaking. Domain walls are gravitationally repulsive, a detailed description of their behavior can be found in [31, 32, 33]. Here we just summarize the following results. Domain walls accelerate away from each other with an acceleration $2\,\pi\, G\,\sigma$ and, after a time of order $(2\,\pi\, G\,\sigma)^{-1}$, they recede at the speed of light. By averaging over volumes containing many cells separated by walls, the equation of state of a wall dominated universe is $p_{\rm DW} = -\frac{2}{3}\rho_{\rm DW}$. This implies that the energy density of Domain Walls scales as

$$\rho_{\rm DW} \propto \frac{1}{a(t)}. \tag{152}$$

[5]The domain problem $N_{\rm DW} > 1$ can also be avoided if one assumes a small symmetry breaking in such a way that the true vacuum takes over before the walls dominate the energy density. On the other hand, it must be small enough so that the PQ mechanism still works and so there is a very small portion in the parameter space in which such option works and some fine tuning problems arise.

This, combined with the Freedman equations, implies that a domain-wall-dominated universe expands as $a(t) \propto t^2$. A domain-wall-dominated universe has an accelerated expansion and so one could be tempted to say that the present-day accelerated expansion of the universe is due to the domain walls. However a domain wall dominated universe would have some proprieties that are far away from what we observe today [14] and we are forced to reject this option.

5.3 Total cold Axion energy density

The total amount of the cold axion energy density Ω_A can be computed taking into account the different cold dark matter contribution described before. Ignoring the contribution of domain walls and assuming that the contribution of cosmic strings is

$$\rho_A^{\rm str}(t) \sim 2\frac{f_A^2}{t_*}\left(\frac{a(t_*)}{a(t)}\right)^3 m_A \tag{153}$$

for the two different cases described above one obtains

$$\Omega_A \sim \left(\frac{f_A}{10^{12}\mathrm{GeV}}\right)^{7/6}\left(\frac{0.7}{h}\right)^2 \times \begin{cases} 0.15\,\theta_*^2 & \text{for } T_{\rm PQ} > T_{\rm RH} \\ 0.7 & \text{for } T_{\rm PQ} < T_{\rm RH} \end{cases} \tag{154}$$

where h is defined by $H_0 = h\,100\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$ and $\Omega_A = \frac{\rho_A}{\rho_c}$. Note that if we want axions to be (part of) Cold Dark Matter of course the total amount of cold axions cannot exceed the total amount of cold dark matter estimated in the universe $\Omega_{\rm CDM} \approx 0.22$:

$$\Omega_A \leq \Omega_{\rm CDM} \approx 0.22. \tag{155}$$

If we are in the situation in which inflation occurs before the PQ symmetry breaking (so that the axion energy density does not depend on the initial misalignment angle θ_*) this translates into a direct constraint on the PQ symmetry breaking scale

$$f_A < 0.37 \quad 10^{12}\mathrm{GeV} \qquad \text{for } T_{\rm PQ} < T_{\rm RH} \tag{156}$$

while if we are in the situation of a broken PQ symmetry during inflation this translates into a relation between the initial misalignment angle and the PQ symmetry breaking scale

$$\left|\frac{\theta_*}{\pi}\right| < 0.4\left(\frac{10^{12}\mathrm{GeV}}{f_A}\right)^{7/12} \qquad \text{for } T_{\rm PQ} > T_{\rm RH}. \tag{157}$$

Let us consider figure 1, again. This time let us consider the black lines in the plot. They represent the cold axion energy density for a fixed value of the initial misalignment angle as a function of the PQ symmetry breaking scale or equivalently the axion mass. The horizontal dashed line represents the value at which all the Cold Dark Matter can be explained with axions, above this line the cold axions exceed the total amount of the cold dark matter observed in the universe and so we are forced to stay down that limit. As you can see from the plot, an initial misalignment angle of order one corresponds to $f_A \sim 10^{12}$ GeV while, in order to allow regions with $f_A \gg 10^{12}$ (such as $f_A \sim 10^{13} - 10^{14}$) we need to require a very fined tuned value of $\theta_* \approx 0$. Clearly being θ_* an angle it can assume all the value between $-\pi$ and π with the same probability, but to avoid a fine tuning problem (again), we prefer regions of the parameter space with $f_A \lesssim 10^{12}$ GeV corresponding to $m_A \gtrsim 10^{-5}$. Another interesting characteristic that is clearly showed from the figure 1 is that as the axion mass is smaller as the axion energy density is bigger. This because axions are bosons and so, if the decoupling is non thermal, their momentum is very slow and they behave as a condensate. This is an important difference between axions and other cold dark matter candidates such as WIMPs. Let us conclude this section saying that the axion mass window $m_A \in [1\,,\,100]$ μeV is the most promising in order to have axion cold dark matter and most of the present experimental efforts on axions are focused on searching axions in this range.

6 Axion Isocurvature Perturbations

We now turn to isocurvature perturbations of the axion field produced by quantum fluctuation during the inflation. As pointed out in the previous section, we basically have two possibilities: inflation can occur before or later the Peccei Quinn phase transition. If the reheating temperature after inflation is less than the Peccei Quinn temperature $T_{\rm PQ}$, the axion field is present during inflation and it is subjected to quantum-mechanical fluctuations, just like the inflaton field. Since in this case the axion field is massless and weakly coupled as well as the the inflaton field, so the two fields have the same spectrum of fluctuation [6]. So the axion field perturbations $\delta\phi_A(\mathbf{x}, t)$ are given by [14]

$$\delta\phi_A(\mathbf{x}, t) \tag{158}$$

while the spectrum of the axion field perturbation is given by[6]:

$$P_A(k) = \int \frac{d^3\mathbf{x}}{(2\pi)^3} \left\langle \delta\phi_A(\mathbf{x},t)\delta\phi_A\left(\mathbf{x}',t\right)\right\rangle e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \left(\frac{H_\mathrm{I}}{2\pi}\right)^2 \frac{2\pi^2}{k^3} \tag{159}$$

At the start of the QCD phase-transition, the local value of the axion field $\phi_A(x)$ determines the local number density of cold axions produced by the vacuum realignment mechanism

$$n_A\left(\mathbf{x},t_*\right) = \frac{f_A^2}{2t_*}\theta^2\left(\mathbf{x},t_*\right) \tag{160}$$

where $\theta\left(\mathbf{x},t_*\right) \equiv \frac{\phi_A(\mathbf{x},t_*)}{f_A}$ is the initial misalignment angle. Therefore it is clear that the axion field perturbation generates perturbations in the number density of cold axions, in such a way that:

$$\frac{\delta n_A^\mathrm{iso}}{n_A} = \frac{2\delta\phi_A}{\phi_A(\mathbf{x},t_*)} = \frac{H_\mathrm{I}}{\pi f_A\theta_*} \tag{161}$$

Where $\phi_A(\mathbf{x},t_*)$ is the value of the axion field at the start of the QCD phase-transition that, because of inflation, is common to our entire visible universe. Since the vacuum-realignment mechanism converts the quark - gluon plasma energy into axion rest mass energy, the energy density of these perturbations obeys to:

$$\delta\rho_A^\mathrm{iso}\left(t_*\right) = -\delta\rho_\mathrm{rad}^\mathrm{iso}\left(t_*\right) \tag{162}$$

while, as known, the density perturbations produced by the fluctuations of the inflaton field satisfy the adiabatic condition:

$$\frac{\delta\rho_{\text{ matter}}}{\rho_{\text{ matter}}} = \frac{3}{4}\frac{\delta\rho_{\text{ rad}}}{\rho_{\text{ rad}}}. \tag{163}$$

Therefore the perturbations produced by the axion field fluctuations are not adiabatic (as those produced by the inflaton field fluctuations) but they are called *isocurvature perturbations*. However note that in this case the density perturbations in the cold axion fluid have both adiabatic and isocurvature components: the adiabatic component is given by the quantum mechanical fluctuations of the inflaton field during inflation, while the isocurvature perturbations are produced by the quantum mechanical fluctuations of the axion field during that same epoch. The adiabatic and axion isocurvature components are of course uncorrelated. The isocurvature perturbations leave trace on the cosmic microwave background (CMB) that is different from those coming from the adiabatic perturbations. In order to constrain the kind of perturbation observed in the CMB, in general one uses the primordial isocurvature fraction β_iso defined as the ratio between the isocurvature perturbation spectrum over the sum of the isocurvature and adiabatic perturbation spectrum. The present CMB data are consistent with models allowing pure adiabatic perturbations

$$\beta_\mathrm{iso} = \frac{\mathcal{P}_\mathrm{iso}}{\mathcal{P}_\mathrm{iso} + \mathcal{P}_\mathrm{ad}} \lesssim 0.04. \tag{164}$$

This limit can be used to exclude regions in the parameter space of the PQ scale and the scale of inflation H_I, since they are related via [34]

$$H_I \simeq 0.96\times 10^7\mathrm{GeV}\left(\frac{\beta_\mathrm{iso}}{0.04}\right)^{1/2}\left(\frac{\Omega_\mathrm{A}}{0.120}\right)^{1/2}\left(\frac{f_\mathrm{A}}{10^{11}\mathrm{GeV}}\right)^{0.408} \tag{165}$$

If we consider the limit (164) and if we assume all the cold dark matter to be made of axions, we have:

$$H_I \leq 0.87\times 10^7\mathrm{GeV}\left(\frac{f_\mathrm{A}}{10^{11}\mathrm{GeV}}\right)^{0.408} \tag{166}$$

Note that another independent estimator of the energy scale of the inflation is the amplitude of the primordial gravitational waves or, equivalently, the tensor to scalar ratio

$$r = \frac{\mathcal{P}_T}{\mathcal{P}_S} \tag{167}$$

The tensor to scalar ration give us information on the energy scale of inflation since $\left(\frac{H_I}{2\pi}\right)^2 = M_{pl}^2\, A_s\, \frac{r}{8}$. It is so clear that a future measurement of the primordial gravitational waves will constrain the PQ scale.

[6] As well known, outside the causal horizon, fluctuations in each axion mode are frozen.

Part III
Outline of the Work Plan

A Thermal Axions

We plan to modify the standard ΛCDM model analyzing a scenario with both axions and neutrinos as extra hot thermal relics. We will focus on the following points:

- The standard extra radiation density will change since the presence of thermal axions will increase the value of the effective number of relativistic degrees of freedom $N_{\rm eff}$. So we want to investigate the constrain on $N_{\rm eff}$ from the latest Planck data release in order to understand if deviations from the standard value are allowed and consequently to constrain the thermal axion mass.

- We plan to include massive neutrinos in the scenarios in order to study the degeneracy among the axion mass and the masses of the neutrinos.

- We plan to study the impact of the additional parameters on the standard ΛCDM parameters focusing on the possible degeneracy in the parameter space.

B Cold Axions

We plan to modify the standard ΛCDM model introducing cold axions and investigating the possibility that they are (part of) cold dark matter. We will focus on the following points:

- We plan to constrain the axion mass and the PQ scale in the scenario in which inflation occurs after the PQ symmetry breaking assuming the cold dark matter to be made of cold axions produced by the vacuum realignment mechanism. In this scenario we also want to analyze the dependence of the results from the initial misalignment angle.

- We plan to constrain the axion mass and the PQ scale in the scenario in which inflation occurs before the PQ symmetry breaking assuming the cold dark matter to be made of cold axions produced both by the vacuum realignment mechanism and by the decay of topological defects (parameterizing this further contribution opportunely).

- We plan to constrain the Isocurvature component in the primordial fluctuations and so to analyze the consequent constraints on the PQ scale.

- We plan to better and further constrain the PQ scale using the constraints on the energy scale of the Inflation and on the primordial gravitational waves.

We would like to clearly state that this is only a first draft of the work plan and that we reserve the possibility to make changes in future. Of course many other aspects will be analyzed and studied step by step following the hints of the results of our work.

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