This notebook is also available as a PDF file lagrange2d_tutorial.pdf

In[483]:=
```
SetDirectory[NotebookDirectory[]]
Export["resources/lagrange2d_tutorial.pdf", EvaluationNotebook[]]
```

Out[483]=
```
/Users/william/program_repos/lagrange2d
```

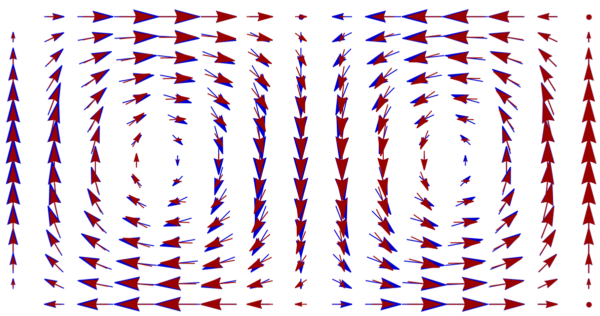
Import package and define an example flow field

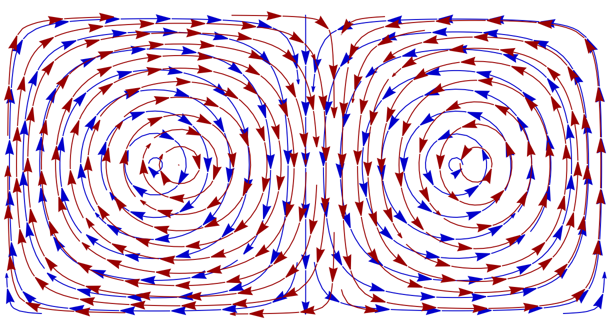
In[∘]:=
```
SetDirectory[NotebookDirectory[]]
<< lagrange2d.wl;
RandomSeed[0];
(*params = {A→0.1,ϵ→0.25,ω→π/5};*)
```

Out[∘]=
```
/Users/william/program_repos/lagrange2d
```

In[∘]:=
```
a[t_] := ϵ Sin[ω t];
b[t_] := 1 - 2 ϵ Sin[ω t];
f[t_, x_] := a[t] x^2 + b[t] x;

g[t_, x_, y_] := π A Sin[π f[t, x]] Cos[π f[t, y]](*D[f[t,x],x]*)

(* quad gyre *)
{vx[t_, x_, y_], vy[t_, x_, y_]} := {g[t, x, y], -g[t, y, x] (2 a[t] x + b[t])};

(* double gyre *)
{vx[t_, x_, y_], vy[t_, x_, y_]} :=
  {-π A Sin[π f[t, x]] Cos[π y], π A Cos[π f[t, x]] Sin[π y] (2 a[t] x + b[t])};

params = {A → 0.1, ϵ → 0.1, ω → π / 5};
```

In[•]:=
```
sys0 = {vx[t, x, y], vy[t, x, y]} /. params /. t → 0;
sys1 = {vx[t, x, y], vy[t, x, y]} /. params /. t → 2.5;
vecFig = Show[{
    VectorPlot[sys0, {x, 0, 2}, {y, 0, 1}, AspectRatio → Automatic,
     Axes → False, Frame → False, VectorStyle → Darker[Blue, .2]],
    VectorPlot[sys1, {x, 0, 2}, {y, 0, 1}, AspectRatio → Automatic,
     Axes → False, Frame → False, VectorStyle → Darker[Red, .4]]
   }]
streamFig = Show[{
    StreamPlot[sys0, {x, 0, 2}, {y, 0, 1}, AspectRatio → Automatic,
     Axes → False, Frame → False, StreamStyle → Darker[Blue, .2]],
    StreamPlot[sys1, {x, 0, 2}, {y, 0, 1}, AspectRatio → Automatic,
     Axes → False, Frame → False, StreamStyle → Darker[Red, .4]]
   }]

(*
Export["vecFig.png",vecFig,ImageResolution→300];
Export["streamFig.png",streamFig,ImageResolution→300];
*)
```

Out[•]=


Out[•]=


Demonstrations of functions

Plot pathlines

In[∘]:=
```
? pathPlot
```

Given velocity field data, generate a series of equal–time trajectories from uniform initial conditions

Arguments:
xlo: Real
    The lower bound on x
xhi: Real
    The upper bound on x
ylo: Real
    The lower bound on y
yhi: Real
    The upper bound on y
tlo: Real
    The starting value for t
thi: Real
    The ending value for t

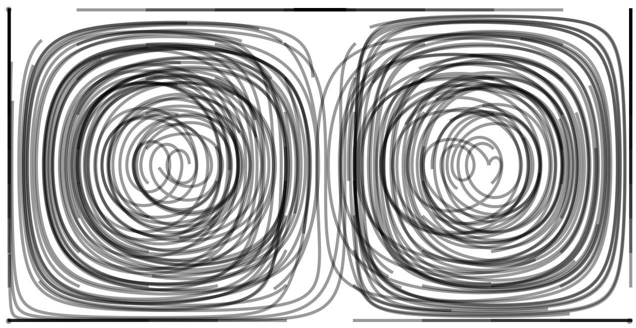
Arguments (Optional):
n : Integer
    The number of points to use to discretize the domain
seeds1 : List of {Real, Real}
    An explicit list of starting points to use for the integration
Options[ParametricPlot] : Any plotting options that would normally be passed to ParametricPlot

Plot pathlines of a flow and modify the plot properties

In[∘]:=
```
pathFig = pathPlot[{vx[t, x, y], vy[t, x, y]} /. params, {x, 0, 2},
   {y, 0, 1}, {t, 0, 15}, PlotStyle → {{Black, Opacity[.4]}}, Axes → False]
(*Export["pathFig.png",pathFig,ImageResolution→300]*)
```

... NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

Out[∘]=

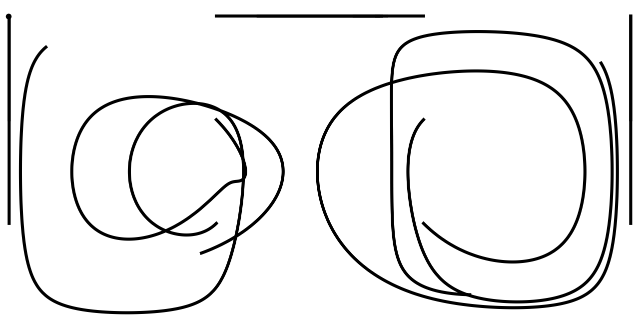


Specify the resolution of the streamline plotting

```
pathPlot[
 {vx[t, x, y], vy[t, x, y]} /. params,
 {x, 0, 2},
 {y, 0, 1},
 {t, 0, 15},
 {n → 20},
 PlotStyle → Black,
 Axes → False]
```

... NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

Out[∘]=

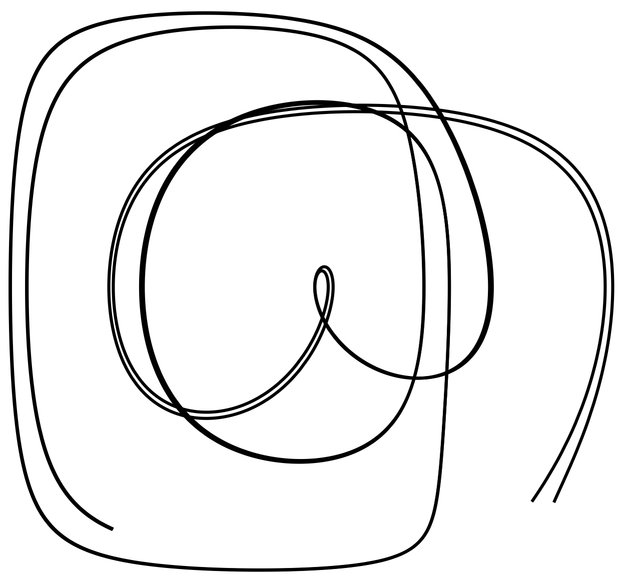


Pass an explicit list of starting points

```
pathPlot[
 {vx[t, x, y], vy[t, x, y]} /. params,
 {x, 0, 2}, {y, 0, 1}, {t, 0, 45},
 {seeds1 → {{.2, .1}, {.201, .101}}},
 PlotStyle → Black, Axes → False
]
```

... NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

Out[∘]=



Visualize advection of a blob

In[∘]:=

```
? advectPoints
```

Given a collection of initial conditions, advect them forward in time

Arguments:

seedPoints : N–ist of 2–Lists;

A list of starting points at which to initialize trajectories

vfield : Pair of functions in x,y;

Cartesian expression of a 2D vector field

timeLimit : Real;

The amount of time to integrate

In[∘]:=
```
blobPoints = makeMesh[{.2, .6}, {.1, .5}, 50, 50];
ptLocs = advectPoints[
    blobPoints,
    {vx[t, x, y], vy[t, x, y]} /. params,
    {t, 0, 20}, x, y];
im = Show[{
     ListPlot[Flatten[{x[t], y[t]} /. ptLocs /. t → 1, 1],
      PlotStyle → Lighter[Blue, .9], PlotMarkers → {Automatic, 3}],
     ListPlot[Flatten[{x[t], y[t]} /. ptLocs /. t → 5, 1],
      PlotStyle → Lighter[Blue, .6], PlotMarkers → {Automatic, 3}],
     ListPlot[Flatten[{x[t], y[t]} /. ptLocs /. t → 20, 1],
      PlotStyle → Blue, PlotMarkers → {Automatic, 3}]
    },
    PlotRange → {{0, 2}, {0, 1}}, AspectRatio → Automatic, Axes → False];
transportFig = ImageReflect[ImageReflect[Image[im], Top], Left]

(*Export["transportFig.png",im,ImageResolution→300]*)
```

... NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

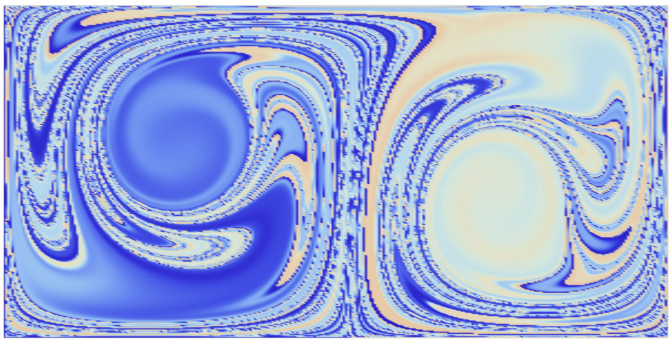
Out[∘]=

Visualize the final locations of particles

```
blobPoints = makeMesh[{0, 2}, {0, 1}, 300, 300];
tLim = 50;
(* 3 GB *)
ptLocs = advectPointsFinal[
    blobPoints,
    {vx[t, x, y], vy[t, x, y]} /. params,
    {t, 0, tLim}, x, y];
finalLocs = Transpose[Join[Transpose[blobPoints], {Transpose[ptLocs][[1]] - 1}]];
```

... NDSolveValue: Initial condition x0 is not a number or a rectangular array of numbers.

In[ ]:=
```
ListDensityPlot[
 finalLocs,
 InterpolationOrder → 3,
 ColorFunction → "ThermometerColors",
 AspectRatio → Automatic,
 Axes → False,
 Frame → False
]
```

Out[ ]=


Visualize the field of maximal Lyapunov exponents

In[ ]:=
```
? findMaxFTLEField
```

Given a velocity field and a set of coordinate points, compute the max FTLE field at a given timepoint

Arguments:
vfield : Pair of functions in x,y
Cartesian expression of a 2D vector field
seeds : N–ist of 2–Lists
A list of starting points at which to initialize trajectories
tlo : Real
The time to start integration
thi : Real
The time to stop integration

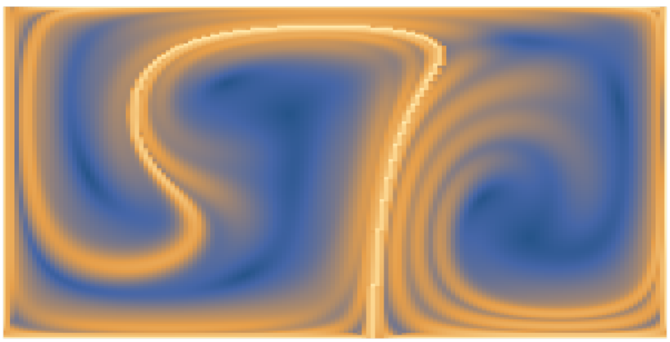
Returns:
ftleField : List of 3–Lists
A list of {x,y,lambda} points denoting the max finite time Lyapunov exponents at each location

```
In[•]:= params =  {A → 0.1, ϵ → 0.25, ω → π / 5};
domainPoints = makeMesh[{0, 2}, {0, 1}, 150, 150];
ftleField = findMaxFTLEField[
   {vx[t, x, y], vy[t, x, y]} /. params, domainPoints, {t, 0, 10}, x, y];
ftlePlot = ListDensityPlot[ftleField, InterpolationOrder → 0,
  AspectRatio → Automatic, Axes → False, Frame → False]
(*Export["ftlePlot.png",ftlePlot,ImageResolution→300]*)
```

NDSolveValue: Initial condition x0 is not a number or a rectangular array of numbers.

ReplaceAll: {allTraj$340444} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

Out[•]=



Visualize the Kaplan-Yorke exponents

```
In[•]:= ? findFTLEField
? findKYDim
```

Given a velocity field and a set of coordinate points, compute the FTLE field
at a given timepoint

Arguments:
vfield : Pair of functions in x,y
Cartesian expression of a 2D vector field
seeds : N–ist of 2–Lists
A list of starting points at which to initialize trajectories
tlo : Real
The time to start integration
thi : Real
The time to stop integration

Returns:
ftleField : List of 3–Lists
A list of {x,y,{lamba1,lambda2} points denoting the two finite time
Lyapunov exponents at each location

Given a list of Lyapunov exponents, compute the Kaplan–Yorke fractal dimension

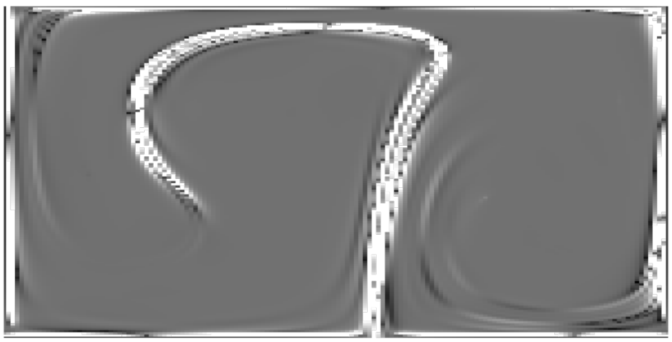
Arguments:
ptList : A List of {x,y,{lamba1,lambda2}} points denoting the two Lyapunov
exponents at each location

Returns:
kyField : List of 3–Lists
A list of {x,y,K} points denoting the Kaplan–Yorke dimension at
each location

```
domainPoints = makeMesh[{0, 2}, {0, 1}, 150, 150];
ftleVals =
  findFTLEField[{vx[t, x, y], vy[t, x, y]} /. params, domainPoints, {t, 0, 10}, x, y];
kyVals = findKYDim[ftleVals];

signedLog = Sign[#] Log[1 + Abs[#]] &; (* for visualization *)
kyfig = ListDensityPlot[{#[[1]], #[[2]], signedLog[#[[3]]]} & /@ kyVals,
  InterpolationOrder → 0, AspectRatio → Automatic, PlotRange → {Automatic, 1.5},
  ColorFunction → GrayLevel, Axes → False, Frame → False]
(*Export["kyfig.png",kyfig,ImageResolution→300]*)
```

NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

Out[ ]=



Out[ ]= kyfig.png

Draw lines of maximal stretching

In[ ]:=
```
? findStretchlines
```

Given a velocity field and a set of coordinate points, compute the vector field
corresponding to the maximum or minimum stretching direction

Arguments:
 vfield : Pair of functions in x,y
  Cartesian expression of a 2D vector field
seeds : N–ist of 2–Lists
  A list of starting points at which to initialize trajectories
tlo : Real
  The time to start integration
 thi : Real
  The time to stop integration
ordering : String
  Whether to compute the field for maximum or minimum streching

Returns:
stretchField : List of 3–Lists
  A list of {x,y,{v1, v2}} points denoting the stretching field
  at each location

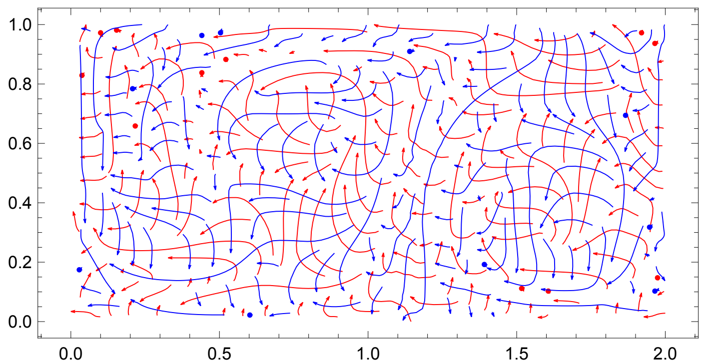
Compute the vector fields associated with maximum and minimal stretching at each location, and overlay them in different colors

In[∘]:=
```
blobPoints = makeMesh[{0, 2}, {0, 1}, 100, 100];
stretchVecsPositive = findStretchlines[
   {vx[t, x, y], vy[t, x, y]} /. params, blobPoints, {t, 0, 10}, x, y, "Positive"];
stretchVecsNegative = findStretchlines[{vx[t, x, y], vy[t, x, y]} /. params,
   blobPoints, {t, 0, 10}, x, y, "Negative"];

Show[{
  ListStreamPlot[{{#[[1]], #[[2]]}, #[[3]]} & /@ stretchVecsPositive, AspectRatio →
    Automatic, PlotRange → All, StreamStyle → Red, StreamScale → {10, 500, .005}],
  ListStreamPlot[{{#[[1]], #[[2]]}, #[[3]]} & /@ stretchVecsNegative,
   AspectRatio → Automatic, PlotRange → All,
   StreamStyle → Blue, StreamScale → {10, 500, .005}]
 }]
```

... NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

... NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

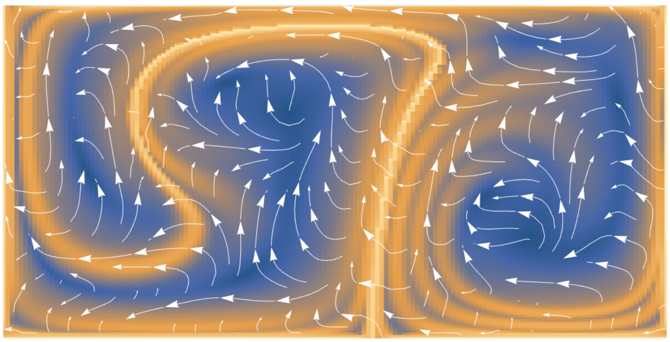
Out[∘]=



Overlay stretching lines on the FTLE field

```
stretchFig = Show[
  {
   ListDensityPlot[ftleField, InterpolationOrder → 0, AspectRatio → Automatic],
   (*ListVectorPlot[{{#[[1]],#[[2]]},#[[3]]}&/@stretchVecs,
    AspectRatio→Automatic,PlotRange→All,VectorStyle→White]*)
   ListStreamPlot[{{#[[1]], #[[2]]}, #[[3]]} & /@ stretchVecsPositive,
    AspectRatio → Automatic, PlotRange → All, StreamStyle → White]
  }, Frame → False]
(*Export["stretchFig.png",stretchFig,ImageResolution→300]*)
```

Out[ ]=



Out[ ]= stretchFig.png

Find flushing times

In[*]:=
```
? flushingTimes
```

Given a velocity field and a domain, calculate the flushing time field

Arguments:
vfield : Pair of functions in x,y,t
Cartesian expression of a 2D time–dependent vector field
xlo: Real
The lower bound on x
xhi: Real
The upper bound on x
ylo: Real
The lower bound on y
yhi: Real
The upper bound on y
tlo : Real
The time to start integration
thi : Real
The time to stop integration (the maximum flushing time)

Arguments (Optional):
n : Integer
The number of points to use to discretize the domain
seeds1 : List of {Real, Real}
An explicit list of starting points to use for the integration
Options[ParametricPlot] : Any plotting options that would normally be
passed to ParametricPlot

Returns:
flushField : List of 3–Lists
A list of {x,y,t} points denoting the flushing time at
each spatial location

In[•]:=
```
flushField = flushingTimes[
    {vx[t, x, y], vy[t, x, y]} /. params,
    {x, -1, 1},
    {y, 0, 1},
    {t, 0, 50},
    {n → 20 000}];
```

NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

NDSolve: Event location failed to converge to the requested accuracy or precision within 100 iterations between t = 37.49311322693752` and t = 37.5576377456767`.

NDSolve: Event location failed to converge to the requested accuracy or precision within 100 iterations between t = 39.83633961797111` and t = 39.90742071252945`.

NDSolve: Event location failed to converge to the requested accuracy or precision within 100 iterations between t = 37.46618896718305` and t = 37.529873341428306`.

General: Further output of NDSolve::evcvmit will be suppressed during this calculation.

```
flushFig = ListDensityPlot[flushField,
  InterpolationOrder → 0, AspectRatio → Automatic, Frame → False]
(*Export["flushFig.png",flushFig,ImageResolution→300]*)
```

Out[•]=

Out[•]= flushFig.png

Make video of a time-varying flow

In[ ]:=
```
? animateFlow
```

Given a velocity field and a domain, create a video of the flow as it evolves in time

Arguments:

vfield : Pair of functions in x,y,t

Cartesian expression of a 2D time–dependent vector field

seeds : Length N List of 2–Lists

A list of starting points at which to initialize trajectories

xlo: Real

The lower bound on x

xhi: Real

The upper bound on x

ylo: Real

The lower bound on y

yhi: Real

The upper bound on y

tlo : Real

The time to start integration

thi : Real

The time to stop integration (the maximum flushing time)

frameRate : Integer

The number of frames per second of the video

playbackSpeed : Integer

The playback rate, 1x, 2x, etc.

filename : String

The name of the video. The extension .avi may also be used

Returns: None

```
blobPoints = makeMesh[{1.2, 1.6}, {.1, .5}, 50, 50];
animateFlow[
  {vx[t, x, y], vy[t, x, y]} /. params,
  blobPoints,
  {t, 0, 20},
  {x, 0, 2},
  {y, 0, 1},
  20, 1/10];

(*makeMesh[{1.2,1.6},{.1,.5},50,50],
  {vx[t,x,y],vy[t,x,y]}/.params,
  {t,0,20},x,y];*)
```

··· NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

··· NDSolve: Initial condition x0 is not a number or a rectangular array of numbers.

Import an experimental velocity field

This section shows how to import an experimental velocity field, and then use it to calculate a maximal FTLE field. The process of interpolating a velocity field can be computationally expensive for large datasets.

Import dataset

```
rawData = Import["resources/bco_dmo_lakedata_729461.mat"];
(* Source: https://www.bco-dmo.org/dataset-deployment/730831 *)

(* Data must have the format (t, x, y, vx, vy) *)
data = rawData[[2 ;; 6]] ;

(* Get bounds of dataset *)
{xMin, xMax} = {Min[#], Max[#]} &@data[[3]];
{yMin, yMax} = {Min[#], Max[#]} &@data[[4]];

(* Interpolate velocity field functions *)
{vx, vy} = fitVField[data];
```

Compute maximal FTLE field

```
domainPoints = makeMesh[{xMin, .3 * xMax}, {yMin, .3 * yMax}, 30, 30];
ftleField = Quiet@findMaxFTLEField[{vx[t, x, y], vy[t, x, y]},
    domainPoints, {t, .001, 10}, x, y]; (* Compute expensive *)
```

Out[•]= $Aborted

```
In[•]:= ListDensityPlot[{#[[1]], #[[2]], Log[#[[3]]]} & /@ ftleField,
  InterpolationOrder → 0,
  ScalingFunctions → {"Linear", "Linear", "Log"},
  AspectRatio → Automatic,
  Axes → False,
  Frame → False]
```

Out[•]=