Ch. 1

1.1

$f: S \to S$

(S,f) pair = dynamical system

S = phase space, elements of S = states

$f: A \to B$, $g: B \to C$

$g \circ f: A \to C = g(f(x))$ = compositions

Successive compositions of f = iteration

1st iterate of f = f itself, 2nd iterate = $f \circ f$, nth iterate = $f^n = \underbrace{f \overbrace{\circ f \circ f \dots \circ}^{n-1 \text{ times}} f}_{n \text{ times}}$.

0th iterate of f = $f^0(x) = x$; f^0 = identity function.

$f^{m+n} = f^m \circ f^n$ for all m, n.

Orbit: $\underbrace{x}_{\text{initial state}}, \underbrace{f(x)}_{\text{1 time unit}}, \underbrace{f^2(x)}_{\text{2 time unit}}, f^3(x), \dots$

......

One-to-one: $(f(x) = f(y)) \Rightarrow (x = y)$. injectivity

Onto: $f: A \to B$, for all $y \in B$, $\exists\, x \in A$ s.t. $f(x) = y$ surjectivity

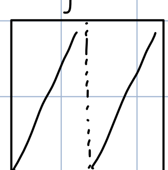
1.2

$f: A \to C$, $g: B \to C$

f is a copy, or replica, of g if there is an injective and surjective function $h: A \to B$ such that $f = g \circ h$.

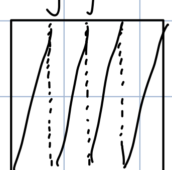
$$f(x) \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2x-1 & \frac{1}{2} \le x < 1 \end{cases}$$

f



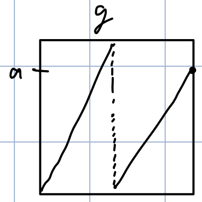
$\Rightarrow$

$f \circ f$

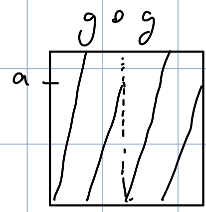


$f \circ f$ $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ are copies of f. Info is preserved, contrast with:

$$g(x) = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2ax - a & \frac{1}{2} \le x < 1 \end{cases}$$



$\Rightarrow$



$g \circ g\, [0, \frac{1}{2})$ is a copy of g, but $g \circ g\, [\frac{1}{2}, 1)$ is not a copy of g.

because if $y \in (a(2a-1), a)$ then $g \circ g\,(x)$ has a unique solution for $x \in [\frac{1}{2}, 1)$, but there are two solutions for $x \in [0, 1)$.

$\Rightarrow$ sometimes, the full range of the behavior of a "copied" function is lost, which continues with further iterations. $g \circ g$ provides both a "copy" and a "mutant" of g.

Example of a copy, g:

$f: [0,1] \to [0,1]: f(x) = x$,

$g: [0, \frac{1}{2}] \to [0,1]: g(x) = 2x$, then g is a copy of f, since

$h: [0, \frac{1}{2}] \to [0,1]: h(x) = 2x$ is injective and surjective, and $f \circ h = g$.

Proposition: $f: A \to A$ is onto, Subset $U \subseteq A$ s.t. $f|U$ of f to U is 1-1 on U and has range A. Then, $(f \circ f)|U$ is a copy of f.

In general, if $U_n = \{x : x \in U_{n-1}$ and $f(x) \in U_{n-1}\}$, then $f^{n+1}|U_n$ is a copy of f.

Self-replicating

Questions about how information is lost/gained in a dynamical system over time can arise.

- If we know the present, what can we say about the future? or past?

1.3 - knowledge & notation

Ch. 2

2.1

A particular dynamical system associated with irrational numbers:

Leopold Kronecker: if $\alpha \notin \mathbb{Q}$, then $\alpha, 2\alpha, 3\alpha, \ldots$ is dense in $[0,1)$.

"Dense" - each number in $[0,1)$ may be approximated to within any degree of accuracy by some element of the sequence.

The sequence is periodic if α is rational — DISCUSSED IN MEETING.

Also can be discussed using the Unit Circle - $t \mapsto e^{2\pi i t}$

2.2

If $\alpha \in \mathbb{R}$, def: $\operatorname{int}(\alpha)$ = largest integer $\leq \alpha$.

def: $\operatorname{frac}(\alpha) = \alpha - \operatorname{int}(\alpha)$. $\quad$ $\operatorname{frac}(\alpha)$ is in $[0,1)$

unique: $\alpha = \operatorname{int}(\alpha) + \operatorname{frac}(\alpha)$ $\quad$ $\operatorname{int}(\alpha)$ is the integer part of α.

$\operatorname{int} \in \mathbb{Z}$, $\operatorname{frac} \in [0,1)$

$\Rightarrow \operatorname{frac}(\alpha), \operatorname{frac}(2\alpha), \operatorname{frac}(3\alpha) \ldots$ lie in $[0,1)$

How is this sequence distributed in $[0,1)$? Cluster in one part? Converge?

If $\alpha \in \mathbb{Q}$ then there are finite distinct values that repeat.

If $\alpha \notin \mathbb{Q}$, all terms of the sequence are distinct, spread out throughout the interval "randomly".

Connection to Unit Circle: Observe that $e^{2\pi i \alpha} = e^{2\pi i \operatorname{frac}(\alpha)}$ on the unit circle

If $\alpha \notin \mathbb{Q}$, points never repeat and are scattered

2.3