4.2 Length-preserving transformations

Dynamical system (S, f). If f is length-preserving:

$\mu(f^{-n}(B)) = \mu(B)$ assuming B is a basic set

"stationary"

Binary transformation or Borel System:

$$f(x) \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ 2x-1 & \frac{1}{2} \leq x < 1 \end{cases}$$

$\Rightarrow$ is disjoint

$$\mu(f^{-1}([a,b))) = \mu\left(\left[\frac{a}{2}, \frac{b}{2}\right)\right) + \mu\left(\left[\frac{a+1}{2}, \frac{b+1}{2}\right)\right)$$

$$= \frac{b}{2} - \frac{a}{2} + \left(\frac{b+1}{2} - \frac{a+1}{2}\right)$$

$$= \frac{b}{2} + \frac{b}{2} - \frac{a}{2} - \frac{a}{2} = b - a \Rightarrow \mu([a,b)).$$

i.e. $\mu(f^{-1}(B)) = \mu(B)$ for any basic set/interval B.

Lemma: Let $J \subseteq S$, J is a subinterval of S.

$\Rightarrow f^{-1}(J)$ basic subset of S

Then, if $\mu(f^{-1}(J)) = \mu(J)$, f is length-preserving.

One class of length-preserving functions: <u>piecewise-linear</u> functions

4.3 Poincaré recurrence

Proposition 4.5 - Let S be a bounded interval, U is basic subset of S w/ positive length, and f is length-preserving transformation

Then, $\exists\, x \in U$, $n \in \mathbb{N}$ s.t. $f^n(x) \in U$.

Poincaré's recurrence theorem: Not just $\exists\, x \in U$, but for almost all $x \in U$ excluding some measure zero subset.

4.4 Recurrent points

Poincaré's recurrence theorem: in a length-preserving dynamical system on some bounded interval, almost any point will return to some set of positive measure, but not necessarily original location.

Recurrence definition: a point x is recurrent if, for every open neighborhood V of x, $\exists\, n \in \mathbb{N}$ s.t. $f^n(x) \in V$.

- All periodic points are recurrent, but not necessarily the converse.
- Orbit gets "arbitrarily close" to the original point, infinitely often (for a recurrent point).

Poincaré's recurrence theorem (Strengthened): For a length-preserving transformation f on a bounded interval S, almost every point in S is recurrent.

4.5 Kac's Theorem, Average Return Times

Kac's Theorem - for a length-preserving transformation f on bounded interval S, and subset U of S,

$$\text{Avg. recurrence time} = \frac{M(S)}{M(U)} ; \; M = \text{Lebesgue measure}$$

INDEPENDENT of f!

Examples of Kac's - coin tossing until a specific pattern occurs.
- irrational rotations, binary expansions

4.6 – Application of Kac's to Kronecker and Borel

4.7 Standard Deviation of Recurrence Time

Q: How variable are recurrence times compared to average?

Explicit formula using $M(S)$ and $M(U)$.

Std. dev is 0 if recurrence time is constant

Std. dev peaks when $M(U) = \frac{2M(S)}{3}$, and becomes 0 if $M(U) = M(S)$ or $M(U) = \frac{M(S)}{2}$ under some cases.

Poincaré and Kac's results – quantitative & qualitative properties of recurrence:
- Quantitative – recurrence time is inversely proportional to measure of the target set.
- Qualitative – Almost all points come back arbitrarily close to their start.

5.1 – Averaging in Time and Space – Dynamical Systems

Birkhoff's Individual Ergodic Theorem – time averages converge to space averages for almost all starting points.
- Borel's Theorem: almost all real #'s in $[0,1)$, frequency of 1's approaches $\frac{1}{2}$.
- Weyl's Theorem: equidistribution of sequences mod 1.
- Benford's Law: distribution of digits in various base/number systems.

All cases of time averages being consistent w/ space averages.

Let (S, f) be a dynamical system, and $A \subseteq S$. Orbit $= x, f(x), f^2(x), \dots f^n(x)$

Define χ_A : 1 if $x \in A$, 0 otherwise.

Time average: $\frac{1}{n}\sum_{j=0}^{n-1} \chi_A(f^j(x))$ – counts the proportion of time the orbit x spends in A.

– should converge to $\frac{\mu(A)}{\mu(S)}$, proportion of the space occupied by A.

5.2 – Outer Measure

"Measure" – a function that takes in sets and returns the "size" – some $\mathbb{R} \geq 0$.
(length)

Properties: $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

$A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$

$A_i \cap A_j = \emptyset$, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$
$\forall i, j \in \mathbb{N}$

in $\mathbb{R}$: $\mu(A) = \mu(x + A)$ translation

$\mu([a, b]) = b - a$ length

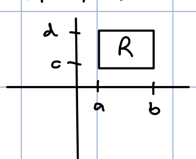
"Outer Measure" vs. "Measure" –

$\nu_{out}(A) = \text{infemum}\left\{ \sum_{n=1}^{\infty} l(I_n) \mid I_k\text{'s disjoint intervals}, A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$ – Cover set with intervals, add up lengths, and take the smallest possible value.

$\mathcal{B} = \{A$ st. $A =$ countable int. or union of intervals (or the complements) $\}$ works in $\mathbb{R}$

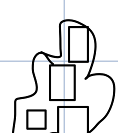
Borel – sigma algebra..

How to extend to $\mathbb{R}^2$? e.g. how do we measure area?



$\mu(R) = (b-a)(d-c)$

Measure of an arbitrary shape – bunch of rectangles



So if $A \subseteq \mathbb{R}^2$, $\mu(A) = \text{infemum}\left\{ \sum_{n=1}^{\infty} \mu(R_n) \mid R_k\text{'s are disjoint rectangles} \right\}$

$A \subseteq \bigcup_{k \in \mathbb{N}} R_k$

Extend to $\mathbb{R}^3$ – use cubes, not rectangles