3.6 - independent sets & events

independence - "unaffected by previous events" - $\mu(A \cap B) = \mu(A)\,\mu(B)$

Proposition - A ind. $B \Rightarrow A$ ind. B^c

Ex - sequence of $A_1, A_2, \dots A_n$

if $B_k = A_k$ or A_k^c, then $B_1, B_2, \dots B_n$ are all independent

— changing one set to complement doesn't affect independence of multiple sets.

3.7

Random results still exhibit order sometimes - coinflip example

"higher degrees of order" events take longer to observe

☆ Just because temporary order is observed does not mean there is order in the underlying process

$\Rightarrow$ Apply to binary expansions

Lemma 3.15 - pg. 127:

The set of all binary digits of a sequence $d(x)$ is a subinterval of $[0,1)$ of length 2^{-n}.

Proposition 3.17:

Let $\theta_1, \theta_2, \dots \theta_s$ be a finite sequence of 0s and 1s.

Let D be the set of all $x \in [0,1)$ s.t. the sequence above does not occur in the binary expansion of x,

$\Rightarrow D$ has measure zero.

The Recurrence Theorem - there is a subset Z of $[0,1)$ w/ measure zero and the following property:

if $x \in Z$, every finite sequence of 0s and 1s occurs in the binary digit sequence of x.

Avg. # of coinflips until heads $= \sum_{n=1}^{\infty} n \cdot \frac{1}{2^n} = 2$; mean =

3.8 Rademacher Functions

$$\text{Rademacher - } r_n(x) \begin{cases} -1 & \text{if } x \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \text{ and } k \text{ is odd} \\ 1 & x \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \text{ and } k \text{ is even.} \end{cases}$$

i.e. only 1 or -1 depending on the binary digit $(2d_0 - 1)$

Like a binary switch - 0 or 1. Behaves like coinflip when you sample randomly from x.

$\int r = 0$. integral of a rademacher is zero

Alternate between -1 and 1 along an interval of length 2^{-n}

☆ almost every number between $[0,1)$, excluding a small measure zero set, has the same number of 0s and 1s in its binary expansion.

Property - integral of the products of distinct Rademacher functions is zero. — very strong

3.9 randomness, law of averages

Borel's Theorem - $\lim_{n \to \infty} \frac{S_n(x)}{2} = \frac{1}{2}$ for all $x \in [0,1)$ except for some small measure zero set.

- i.e. almost all numbers in $[0,1)$ have an equal number of 0 and 1 in binary expansion.

"Normal Numbers"

Proof was way too long.

3.10 - Dynamical Systems Approach

$$f(x) \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2x-1 & \frac{1}{2} \le x < 1 \end{cases}$$

Shifts binary representation of x leftward, and discards integer part.

$\Rightarrow$ if we apply f repeatedly to x, half of the time its orbit will be in $[0, \frac{1}{2})$ and other half in $[\frac{1}{2}, 1)$.

Behaves similarly to the left shift operator

Structural equivalence between the dynamical system $([0,1), f)$ and the system of binary sequences $(\mathbb{Z}_2, \sigma)$.

i.e. studying the action of f on real numbers = studying how binary sequences are shifted leftward.

Proof of countability:

Why is $\mathbb{Q}$ measure zero?

infemum - "greatest lower bound", like a minimum.

defn. of measure (μ): $\inf \left\{ \sum_{j=1}^{\infty} \ell(I_j) \,\middle|\, A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$

ex. (a, b) - no minimum value, since you can get as close as you want to a

Add up the lengths of all intervals:

Consider a, b, c on a number line. $I_{1,2,3}$ of length $\frac{\varepsilon}{3}$, $\varepsilon > 0$

$\Rightarrow$ the length of all $I_{1 \to 3}$ is $\le \varepsilon$.

$\Rightarrow$ the measure of length $a, b, c \le \varepsilon$. But ε can be any number, so measure $(a, b, c) = 0$.

$(a_j)_{j \in \mathbb{N}} = \{a_1, a_2, a_3, \dots a_j\}$ - ordered set, can sum since countably infinite

$\mu(a(j)\ j \in \mathbb{N})$

fix $\varepsilon > 0$,

$I_1 = \{a_1 - \frac{\varepsilon}{4}, a_1 + \frac{\varepsilon}{4}\}$ $\ell(I_1) = \frac{\varepsilon}{2}$

$I_2 = \{a_2 - \frac{\varepsilon}{8}, a_2 + \frac{\varepsilon}{8}\}$ $\ell(I_2) = \frac{\varepsilon}{2^2}$

$\vdots$ $\vdots$

$I_k = \{a_k - \frac{\varepsilon}{2^{k+1}}, a_k + \frac{\varepsilon}{2^{k+1}}\}$ $\ell(I_k) = \frac{\varepsilon}{2^k}$.

$\mu(a) \leq \mu\left(\bigcup_{k=1}^{\infty} I_k\right) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k}$ — converges to ε

$\mu(a) \leq \varepsilon$