2.4

Kronecker's Theorem:

For the sequence frac(α), frac(2α), ...

if α is irrational, the sequence is spread throughout the interval

such that every non-empty open subinterval of $[0,1)$ contains terms of the sequence.

$$\boxed{\alpha \notin \mathbb{Q},\ x \in [0,1],\ \varepsilon > 0,\ \text{there is } m \in \mathbb{N} \text{ s.t. } |x - \text{frac}(m\alpha)| < \varepsilon.}$$

2.5

Kronecker's theorem can be regarded as saying that every open and non-void subinterval of $[0,1)$ will contain points in the orbit of 0, under τ_α.

A dynamical system of the form $([0,1), \tau_\alpha)$ is called a Kronecker system.

2.6 - music discs: $\frac{2\pi r k}{v_1}$, $\frac{2\pi r k}{v_2}$ — if $\frac{v_1}{v_2} \notin \mathbb{Q}$, then the music will never start exactly where it began, but can be estimated to a finite degree of accuracy

2.7 - Weyl's theorem on irrational numbers

The proportion of terms in the sequence frac(α), frac(2α), ... that lie in a subinterval J, is just equal to the length of J over the length of the entire thing.

* no "preference" to a specific interval.

2.8 Proof of Weyl's Thm.

2.9 Chaos in Kronecker Systems

In a chaotic system, even if the evolution of the system follows an exact rule, future behavior of the system cannot be predicted.

"Transitive" - if we let the system run long enough, we can "approximately" get from x to near the state y for some $\varepsilon > 0$

A Kronecker System $([0,1), \tau_\alpha)$ is transitive iff $\alpha \notin \mathbb{Q}$. (α is irrational)

"Sensitive to initial conditions" - an arbitrarily small deviation in the initial conditions of the system will lead to a comparatively large deviation in some future states of the system. "Lost information"

Exercises for Ch. 2

1) prove $\frac{\text{frac}(2\sqrt{2})}{\text{frac}(\sqrt{2})}$ is rational

$$\frac{\text{frac } 2\sqrt{2} = 2\sqrt{2}-2}{\text{frac } \sqrt{2} = \sqrt{2}-1} = \frac{2\sqrt{2}-2}{\sqrt{2}-1} = \frac{2(\sqrt{2}-1)}{\sqrt{2}-1} = 2 = \text{rational} \checkmark$$

3.1

Basis of probability, infinite coin tosses / infinite sample space (binary expansion)

Probability of a frequency can be length (Weyl's Thm.)

Normal Numbers Theorem - every binary sequence digit expansion appears in the expected proportion.

Benford's Law

3.2 Binary expansions of digits

$x = \sum_{n=1}^{\infty} \frac{d_n(x)}{b^n}$ - the expansion of x to the base b

$d_n(x)$ is called the n^{th} digit of x relative to the base b.

when $b = 2$, the expansion of x is called the binary expansion of x.

$d_n(x)$ may be called the n^{th} binary digit

3.3 expansions of rational numbers

Periodic expansion - if $d_1, d_2, \ldots$ is a sequence, if $\exists\, k \in \mathbb{N}$ s.t. $d_{n+k} = d_n$ for all $n \in \mathbb{N}$

Eventually periodic - for all $n \geq i$ instead of $n \in \mathbb{N}$

For every rational number, its sequence of digits in all bases is eventually periodic. (p. 124)

x is rational iff $d_1, d_2, \ldots$ (the sequence of its digits) is eventually periodic - if $\exists\, i, k \in \mathbb{N}$ s.t. $d_n = d_{n+k}$ for all $n \geq i$.

ex. $\left(\frac{2}{5}\right)_{\text{base } 7} =$

$$\begin{array}{r} .25\ldots \\ 5\overline{)2.0_7 0} \\ \underline{1.3_7} \\ 0.40_7 \\ \underline{.34_7} \\ .03_7 \end{array}$$

$0.a_1 a_2 a_3 \ldots$ in base b $= \sum_{n=1}^{\infty} \frac{a_n}{b^n}$ = converging geometric series by comparison

$\frac{a_n}{b^n} < \frac{b}{b^n} = \frac{1}{b^{n-1}}$

There exists a function F such that $F: \left\{ \begin{matrix} \text{base } 6 \\ \text{expansion} \end{matrix} \right\} \to \mathbb{R}$ (a map) – not injective, but is surjective

For which inputs is F not injective? i.e. more than 1 input, same output (decimal expansions aren't unique)

A: for every finite decimal expansion number. ($\frac{1}{3}$ has a unique representation in base 10, but not base 3)

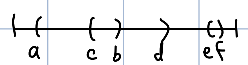
↑ measure zero set; very small (in comparison) and countable.

3.4 sets, length, probability

A subset A of $\mathbb{R}$ is basic if it is the union of a finite number of intervals.

Let S be an interval, $\mathcal{B}$ denote the family of all basic subsets of S. Then,

1) if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$
2) if $A, B \in \mathcal{B}$, then $A \cap B \in \mathcal{B}$
3) every finite union of sets in $\mathcal{B}$ is also in $\mathcal{B}$,
   and every finite intersection of sets in $\mathcal{B}$ is also in $\mathcal{B}$. (obvious)
4) If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$.
5) A subset of S is in $\mathcal{B}$ iff it is a finite, disjoint union of intervals. i.e. any basic set can be written as a disjoint union of intervals.



$S = (a,b) \cup (c,d) \cup (e,f) = (a,d) \cup (e,f)$

3.5 Sets of measure zero – null sets

A subset B of the real numbers is called a set of <u>measure zero</u> if, for each $\varepsilon > 0$, there is a corresponding sequence (J_n) of intervals such that:

$$B \subseteq \bigcup_{n=1}^{\infty} J_n \quad \text{and} \quad \sum_{n=1}^{\infty} \underbrace{\mu}_{\text{measure}}(J_n) < \varepsilon.$$

i.e. a set of measure zero is one which can be covered by an infinite sequence of intervals, the sum of whose lengths is as small as any $\varepsilon > 0$.

Examples of measure zero sets:

1) A single point is measure zero.
2) The union of a sequence of sets, each of which is measure zero, is measure zero.
3) subset of a measure zero set is measure zero.
4) translate of a "

more specific:

1) Every countable subset of $\mathbb{R}$ has measure zero.
2) $\Rightarrow$ The set $\mathbb{Q}$, all rational numbers, has measure zero. ☆