

## Exercises 7 & 8

Analysis of Boolean Functions, Autumn 2025, University of Chicago

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**Submission.** Solutions are due **Friday, October 31** at 11:59pm Central time. Submit your solutions through Gradescope. You are encouraged, but not required, to typeset your solutions using a L<sup>A</sup>T<sub>E</sub>X editor such as **Overleaf**.

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The policies below can also be found on the [course webpage](#).

**Collaboration.** You are encouraged to collaborate with your classmates on exercises, but you must adhere to the following rules.

- Work on each exercise on your own for at least five minutes before discussing it with classmates.
- Feel free to explain your ideas to your classmates in person, and feel free to use whiteboards/chalkboards/etc. However, do not share any written/typeset solutions with your classmates for them to study on their own time. This includes partial solutions.
- Write your solutions on your own. While you are writing your solutions, do not consult any notes that you might have taken during discussions with classmates.
- In your write-up, list any classmates who helped you figure out the solution. The fact that student A contributed to student B's solution does not necessarily mean that student B contributed to student A's solution.

**Permitted Resources for Full Credit.** In addition to discussions with me and discussions with classmates as discussed above, you may also use the course textbook, any slides or notes posted in the “Course Timeline” section of the course webpage, and Wikipedia. If you wish to receive full credit on an exercise, you may not use any other resources.

**Outside Resources for Partial Credit.** If you wish, you may use outside resources (ChatGPT, Stack Exchange, etc.) to solve an exercise for partial credit. If you decide to go this route, you must make a note of which outside resources you used when you were working on each exercise. You must disclose using a resource even if it was ultimately unhelpful for solving the exercise. Furthermore, if you consult an outside resource while working on an exercise, then you must not discuss that exercise with your classmates.

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**Exercise 7** (10 points).

- (a) Suppose we have access to random examples from an unknown monotone function  $f: \{\pm 1\}^n \rightarrow \{\pm 1\}$ . Prove that we can estimate  $\text{Inf}_i[f]$  to within error  $\pm \varepsilon$  given  $i$ , except with failure probability  $\delta$ , in time  $\text{poly}(n, 1/\varepsilon, \log(1/\delta))$ .

*Hint:* Use **Hoeffding's inequality**.

- (b) Let  $\varepsilon \in (0, 1)$  be any constant. Prove that size- $s$  monotone decision trees are learnable from random examples with error  $\varepsilon$  in time  $\text{poly}(n, s)$ .

*Hint:* Use Theorems 3.29 and 9.28 in the textbook. You don't need to submit proofs of those theorems, but make sure for your own sake that you understand the proofs. (We essentially proved both theorems in class, but maybe not in the specific forms stated in the textbook.)

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**Exercise 8** (10 points).

- (a) Let  $\alpha_0, \alpha_1 \in [0, 1]$ . Let  $\alpha = \frac{\alpha_0 + \alpha_1}{2}$ . Prove that

$$\alpha_0 \cdot \log_2(1/\alpha_0) + \alpha_1 \cdot \log_2(1/\alpha_1) + |\alpha_0 - \alpha_1| \geq 2\alpha \cdot \log_2(1/\alpha),$$

using the convention  $0 \cdot \log(1/0) = 0$ . You may take for granted the inequality  $H(p) \geq 4p \cdot (1-p)$ , where  $H(p)$  is the binary entropy function, i.e.,  $H(p) = p \cdot \log_2(\frac{1}{p}) + (1-p) \cdot \log_2(\frac{1}{1-p})$ .

- (b) (Theorem 2.39 in the textbook) Let  $f: \{0, 1\}^n \rightarrow \{0, 1\}$ . Prove that

$$I \left[ (-1)^f \right] \geq 2 \mathbb{E}[f] \cdot \log_2(1/\mathbb{E}[f]). \quad (1)$$

*Hint:* Use induction on  $n$ . For  $b \in \{0, 1\}$  and  $x \in \{0, 1\}^{n-1}$ , let  $f_b(x) = f(xb)$ .

*Context:* [Eq. \(1\)](#) is an example of an “isoperimetric inequality.” You might be more familiar with the isoperimetric inequality in the plane, which says that if a curve encloses a region with area  $A$ , then the length of the curve must be at least  $\sqrt{4\pi \cdot A}$ , with equality when the curve is a circle. In general, an isoperimetric inequality says that every set with large “volume” must have a large “boundary.” In our case,  $\mathbb{E}[f]$  measures the “volume” of  $f^{-1}(1)$ , and  $I[(-1)^f]$  measures the “boundary” of  $f^{-1}(1)$ .

- (c) Let  $f: \{0, 1\}^n \rightarrow \{0, 1\}$  be a conjunction of literals. Prove that

$$I \left[ (-1)^f \right] = 2 \mathbb{E}[f] \cdot \log_2(1/\mathbb{E}[f]).$$

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