

ACC lower bounds (lecture notes)

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Recall that “ $\text{AC}^0[m]$ circuits” can use AND gates, OR gates, and MOD_m gates, all with unbounded fan-in. There are constants and literals at the bottom. When m is not a power of a prime, the class $\text{AC}^0[m]$ is poorly understood. As mentioned previously in this course, it is an open problem to show $\text{NP} \not\subseteq \text{AC}^0[6]$.

On the bright side, there is a line of work showing that there are “somewhat explicit” functions that cannot be computed by small $\text{AC}^0[6]$ circuits and similar models. In particular, Murray and Williams showed $\text{NQP} \not\subseteq \text{ACC}$ [MW18], where NQP is nondeterministic quasipolynomial time and $\text{ACC} = \bigcup_m \text{AC}^0[m]$.

The full proof that $\text{NQP} \not\subseteq \text{ACC}$ is beyond the scope of this course. In these lecture notes, we will sketch the proof of the weaker theorem $\text{E}^{\text{NP}} \not\subseteq \text{ACC}$, where E^{NP} denotes the class of functions that can be computed in $2^{O(n)}$ time using an NP oracle.

1 Depth reduction for ACC circuits

In this section, we will show that ACC circuits can be simulated by low-degree polynomials in a certain sense, despite the fact that we do not know how to construct low-degree probabilistic polynomials computing the MOD_6 function. Recall that $\mathbb{Z}[x_1, \dots, x_n]$ is the set of n -variate polynomials with integer coefficients.

Definition 1 (L_1 norm of a polynomial). If $h \in \mathbb{Z}[x_1, \dots, x_n]$, then we define $L_1(h)$ to be the sum of the absolute values of the coefficients.

Definition 2 (SYM^+). We define $\text{SYM}^+[k]$ to be the class of functions $C: \{0, 1\}^n \rightarrow \{0, 1\}$ of the form $C(x) = g(h(x))$, where $h \in \mathbb{Z}[x_1, \dots, x_n]$ satisfies $\deg(h) \leq k$ and $L_1(h) \leq 2^k$. Note that h is multilinear without loss of generality. The function $g: \mathbb{Z} \rightarrow \{0, 1\}$ can be arbitrary, but we emphasize that it is a function of just one integer variable.

You can double check that each function in $\text{SYM}^+[k]$ can be computed by a “SYM of AND of literals,” where the AND gates have fan-in at most k and the SYM gate has fan-in at most $2^{O(k)}$. Consequently, each function in $\text{SYM}^+[k]$ can be computed by a TC_3^0 circuit of size $2^{O(k)}$. The following theorem is a key step in the proof that $\text{E}^{\text{NP}} \not\subseteq \text{ACC}$, as well as being interesting in its own right.

Theorem 1 (Simulating $\text{AC}^0[m]$ circuits using SYM^+ circuits). *Let $m, d \in \mathbb{N}$ be constants. If $C: \{0, 1\}^n \rightarrow \{0, 1\}$ is an $\text{AC}_d^0[m]$ circuit of size $S \geq n$, then $C \in \text{SYM}^+[\text{polylog } S]$.*

When d and m are growing parameters, the best bound known is $C \in \text{SYM}^+[(\log S)^{O(d \cdot s)}]$, where s is the number of distinct prime factors of m [CP19]. In these lecture notes, we assume d and m are constant for simplicity.

1.1 Simulating MOD_m gates using MOD_p gates

Lemma 1. *Let $p, e \in \mathbb{N}$ be constants, where p is prime and $e \geq 1$. Then $\text{MOD}_{p^e} \in \text{AC}^0[p]$.*

Proof. We prove it by induction on e . The base case $e = 1$ is trivial. For the inductive step, let $e \geq 2$, let $x \in \{0, 1\}^n$, and let N be the Hamming weight of x . We claim that¹

$$\text{MOD}_{p^e}(x) = \text{MOD}_p(x) \vee \text{MOD}_{p^{e-1}} \left(\bigwedge_{i \in S_1} x_i, \dots, \bigwedge_{i \in S_{\binom{n}{p}}} x_i \right), \quad (1)$$

¹Recall that we defined $\text{MOD}_m(x) = 1 \iff x_1 + \dots + x_n \not\equiv 0 \pmod{m}$, which is opposite to the way many sources define it.

where $S_1, S_2, \dots, S_{\binom{n}{p}}$ is an enumeration of all size- p subsets of $[n]$. If N is not a multiple of p , this is trivial: $\text{MOD}_{p^e}(x) = \text{MOD}_p(x) = 1$. Now assume N is a multiple of p . In this case, observe that

$$\binom{N}{p} = \frac{N \cdot (N-1) \cdots (N-p+1)}{p \cdot (p-1) \cdots 1}.$$

In both the numerator and the denominator, only the first term is a multiple of p . Therefore, the exponent of p in the prime factorization of $\binom{N}{p}$ is one less than the exponent of p in the prime factorization of N . That is, $p^e \mid N$ if and only if $p^{e-1} \mid \binom{N}{p}$. Eq. (1) follows. By induction, Eq. (1) shows $\text{MOD}_{p^e} \in \text{AC}^0[p]$; note that $\text{poly}(\binom{n}{p}) = \text{poly}(n)$ since p is a constant. \square

More generally, let m be an arbitrary positive integer, with prime factorization $m = p_1^{e_1} \cdot p_2^{e_2} \cdots p_s^{e_s}$. Then

$$\text{MOD}_m(x) = \text{MOD}_{p_1^{e_1}}(x) \vee \cdots \vee \text{MOD}_{p_s^{e_s}}(x).$$

Thus, we can simulate an $\text{AC}^0[m]$ circuit using AND gates, OR gates, MOD_{p_1} gates, MOD_{p_2} gates, \dots , and MOD_{p_s} gates. The depth blows up by a constant factor and the size blows up polynomially, assuming m is a constant.

1.2 Eliminating one layer of MOD_p gates

Lemma 2 (Modulus-amplifying polynomials). *For every $k \in \mathbb{N}$, there exists a polynomial $M_k \in \mathbb{Z}[x]$ such that $\deg(M_k) = O(k)$, $L_1(M_k) = 2^{O(k)}$, and for every $x \in \mathbb{Z}$ and every $p \in \mathbb{N}$,*

$$\begin{aligned} x \equiv 0 \pmod{p} &\implies M_k(x) \equiv 0 \pmod{p^k} \\ x \equiv 1 \pmod{p} &\implies M_k(x) \equiv 1 \pmod{p^k}. \end{aligned} \tag{2}$$

Proof. Define

$$M_k(x) = \sum_{i=0}^{k-1} \binom{2k-1}{i} \cdot x^{2k-1-i} \cdot (1-x)^i.$$

The degree and L_1 bounds are straightforward. Observe that $M_k(x)$ is a multiple of x^k , which proves Eq. (2). Now suppose $x \equiv 1 \pmod{p}$. Then $1-x$ is a multiple of p , so $(1-x)^i \equiv 0 \pmod{p^k}$ whenever $i \geq k$. Consequently,

$$\begin{aligned} M_k(x) &\equiv \sum_{i=0}^{2k-1} \binom{2k-1}{i} \cdot x^{2k-1-i} \cdot (1-x)^i \pmod{p^k} \\ &= (x+1-x)^{2k-1} && \text{by the binomial theorem} \\ &= 1. \end{aligned} \quad \square$$

Lemma 3 (SYM^+ can simulate $\text{SYM}^+ \circ \text{MOD}_p$). *Let $n, k \in \mathbb{N}$, let p be prime, and let $C: \{0,1\}^n \rightarrow \{0,1\}$ be a formula consisting of variables feeding into MOD_p gates feeding into a $\text{SYM}^+[k]$ gate. Then $C \in \text{SYM}^+[O(k^3 \cdot p \cdot \log n)]$.*

Proof. By introducing dummy variables if necessary, we can write

$$C(x) = g \left(\left(\sum_{i=1}^L c_i \prod_{j=1}^k \text{MOD}_p(x_{ij1}, \dots, x_{ij\ell}) \right) \bmod p^{k+2} \right),$$

where $g: \mathbb{Z} \rightarrow \{0, 1\}$, each $c_i \in \mathbb{Z}$, we have $\sum_{i=1}^L |c_i| \leq 2^k$, and $\ell \leq n$. (Reducing mod p^{k+2} doesn't destroy any information, because the sum lies between -2^k and 2^k .) Therefore,

$$\begin{aligned}
C(x) &= g \left(\left(\sum_{i=1}^L c_i \bigwedge_{j=1}^k \left(\sum_{t=1}^{\ell} x_{ijt} \not\equiv 0 \pmod{p} \right) \right) \pmod{p^{k+2}} \right) && \text{by definition of } \text{MOD}_p \\
&= g \left(\left(\sum_{i=1}^L c_i \cdot \mathbb{1} \left[\prod_{j=1}^k \sum_{t=1}^{\ell} x_{ijt} \not\equiv 0 \pmod{p} \right] \right) \pmod{p^{k+2}} \right) && \text{because a product of nonzero elements is nonzero in any field, including } \mathbb{F}_p \\
&= g \left(\left(\sum_{i=1}^L c_i \cdot M_{k+2} \left(\left(\prod_{j=1}^k \sum_{t=1}^{\ell} x_{ijt} \right)^{p-1} \right) \right) \pmod{p^{k+2}} \right) && \text{by Fermat's little theorem and modulus amplification.}
\end{aligned}$$

The expression above has the format of SYM^+ : first we apply a multivariate polynomial, and then we apply a univariate function ("reduce mod p^{k+2} , then apply g "). The degree of the polynomial is at most $k \cdot (p-1) \cdot \deg(M_{k+2}) = O(p \cdot k^2)$. The L_1 norm of this polynomial is at most $2^k \cdot L_1(M_{k+2}) \cdot (\ell^{k \cdot (p-1)})^{\deg(M_{k+2})} = n^{O(p \cdot k^3)}$. \square

1.3 Simulating the entire circuit

Proof sketch of Theorem 1. There are several steps, but none is too difficult, given the tools that we have developed.

1. Replace each MOD_m gate with AND gates, OR gates, MOD_{p_1} gates, \dots , and MOD_{p_s} gates, as described in Section 1.1.
2. Replace each AND/OR gate with a probabilistic polynomial over the field \mathbb{F}_2 with error $0.1/S$ and degree $\ell = O(\log S)$. Note that a degree- ℓ polynomial over \mathbb{F}_2 is a $\text{MOD}_2 \circ \text{AND}_\ell$ circuit, where the MOD_2 gate has fan-in at most $S^{O(\log S)}$. Let \mathcal{D} be the resulting distribution over circuits.
3. Independently sample $t = O(n)$ circuits $C_1, \dots, C_t \sim \mathcal{D}$ and set $C(x) = \text{MAJ}_t(C_1(x), \dots, C_t(x))$. By Hoeffding's inequality and the union bound over all $x \in \{0, 1\}^n$, there is some fixing of C_1, \dots, C_t such that C computes f . Note that each C_i consists of MOD_2 gates, MOD_{p_1} gates, MOD_{p_2} gates, \dots , MOD_{p_s} gates, and AND_ℓ gates (with literals and constants at the bottom).
4. By introducing dummy gates if necessary, we can ensure that *all gates at the same level are of the same type*. In other words, we can compute f using a circuit of the following form:

$$\text{MAJ}_t \circ (\text{MOD}_2 \circ \text{MOD}_{p_1} \circ \dots \circ \text{MOD}_{p_s} \circ \text{AND}_\ell)^{O(1)}.$$

5. Note that $\text{MAJ}_t \in \text{SYM}^+[\log t]$. We eliminate the layers underneath the SYM^+ gate one by one to get a $\text{SYM}^+[k]$ circuit. To handle MOD_p layers, we use Lemma 3. To handle AND_ℓ layers, we use the trivial fact $\text{SYM}^+[k] \circ \text{AND}_\ell \subseteq \text{SYM}^+[k \cdot \ell]$. Since the number of layers is $O(1)$, we get $f \in \text{SYM}^+[\text{polylog}(S)]$. \square

2 A nontrivial satisfiability algorithm for ACC circuits

The next step in the proof that $\text{E}^{\text{NP}} \not\subseteq \text{ACC}$ is to design a nontrivial algorithm that *analyzes a given ACC circuit* and determines whether it is satisfiable. The algorithm is based on the following lemma, which shows how to evaluate a polynomial h on all points in $\{0, 1\}^n$ using amortized time $\text{poly}(n, \log L_1(h))$ per evaluation.

Lemma 4 (Fast multipoint evaluation of multilinear polynomials). *Let $h: \{0, 1\}^n \rightarrow \mathbb{Z}$ be a multilinear polynomial with integer coefficients. Given h , represented as a list of 2^n coefficients, it is possible to compute $h(x)$ for all $x \in \{0, 1\}^n$ in time $2^n \cdot \text{poly}(n, \log L_1(h))$.*

Proof. Let $s = L_1(f)$. If $n = 0$, then the problem is trivial. Otherwise, there are polynomials h_0, h_1 such that

$$h(x_1, \dots, x_n) = h_0(x_2, \dots, x_n) + x_1 \cdot h_1(x_2, \dots, x_n). \quad (3)$$

Considered as lists of coefficients, h_0 and h_1 are simply the first half and the second half of h , respectively. In particular, $L_1(h_0) \leq s$ and $L_1(h_1) \leq s$. We recursively compute $h_0(x)$ and $h_1(x)$ for all $x \in \{0, 1\}^{n-1}$, and then we use [Eq. \(3\)](#) to compute $h(x)$ for all $x \in \{0, 1\}^n$. The time complexity of this algorithm, $T(n, s)$, satisfies

$$T(n, s) \leq 2T(n-1, s) + 2^n \cdot \text{poly}(n, \log s),$$

because $|h_0(x)| \leq s$ and $|h_1(x)| \leq s$ for all $x \in \{0, 1\}^{n-1}$. This implies $T(n, s) \leq 2^n \cdot \text{poly}(n, \log s)$. \square

Theorem 2 (Nontrivial satisfiability algorithm for ACC). *For all constants $m, d \in \mathbb{N}$, there exists a constant $\varepsilon > 0$ such that the following holds. Given the description of a depth- d $\text{AC}_d^0[m]$ circuit $C: \{0, 1\}^n \rightarrow \{0, 1\}$ of size at most 2^{n^ε} , it is possible to determine whether C is satisfiable in time 2^{n-n^ε} .*²

Proof sketch. First, let us design a satisfiability algorithm that runs in time $2^{\text{polylog}(S)} + 2^n \cdot \text{poly}(n)$, given a circuit of size S where $n \leq S \leq 2^n$.

1. By [Theorem 1](#), it is possible to write C in the form $C(x) = g(h(x))$ where $h: \{0, 1\}^n \rightarrow \mathbb{Z}$ is a multilinear polynomial satisfying $L_1(h) \leq 2^{\text{polylog } S}$. We only showed that such a representation *exists*, but it turns out that it can be *constructed* in quasipolynomial ($2^{\text{polylog } S}$) time, if we represent h as a list of monomials with nonzero coefficients and we represent g as a list of $2^{\text{polylog}(S)}$ output values. We omit the proof that g and h can be efficiently constructed.
2. Rewrite h as a list of 2^n coefficients, many of which may be zero. This can be done in time $2^{\text{polylog } S} + 2^n \cdot \text{poly}(n)$.
3. Compute $h(x)$ for all $x \in \{0, 1\}^n$. By [Lemma 4](#), this can be done in time $2^n \cdot \text{poly}(n)$.
4. Check whether there is some $x \in \{0, 1\}^n$ such that $g(h(x)) = 1$. This can be done in time $2^n \cdot \text{poly}(n)$.

Now we are ready to design a satisfiability algorithm with the parameters described in the theorem statement. Given an $\text{AC}_d^0[m]$ circuit $C: \{0, 1\}^n \rightarrow \{0, 1\}$ of size 2^{n^ε} , we define $C': \{0, 1\}^{n-n^\varepsilon} \rightarrow \{0, 1\}$ by the rule

$$C'(x) = \bigvee_{y \in \{0, 1\}^{n^\varepsilon}} C(xy).$$

Then C is satisfiable if and only if C' is satisfiable. The circuit C' is an $\text{AC}_{d+1}^0[m]$ circuit of size 2^{2n^ε} , so using the algorithm described above, we can decide whether it is satisfiable in time $2^{\text{poly}(n^\varepsilon)} + 2^{n-n^\varepsilon} \cdot \text{poly}(n)$. The theorem follows by picking a small enough ε . \square

3 Nontrivial satisfiability algorithms imply circuit lower bounds

We are working toward proving $\text{E}^{\text{NP}} \not\subseteq \text{ACC}$. The last part of the proof consists of showing that nontrivial satisfiability algorithms, such as the one we constructed in the last section, imply circuit lower bounds. This last part of the proof is not specific to **ACC**; it could also be applied to a hypothetical nontrivial satisfiability algorithm for (say) TC^0 circuits. Our presentation of this last part of the proof is based on the work of Jahanjou, Miles, and Viola [[JMV18](#)].

²We assume a *random access* model of computation throughout these lecture notes.

3.1 The Nondeterministic Time Hierarchy Theorem

The **Time Hierarchy Theorem**, a classic theorem in complexity theory, basically says that given more time, it is possible to solve more problems. The proof of the theorem is a simple diagonalization trick, very similar to the proof that the halting problem is undecidable. This simple proof breaks down if we work with nondeterministic computation, because nondeterministic algorithms are “not closed under complement.” Nevertheless, it turns out that one can use more sophisticated diagonalization arguments to prove a “Nondeterministic Time Hierarchy Theorem.”

Theorem 3 (Special case of the Nondeterministic Time Hierarchy Theorem). *There exists a function $h: \{0,1\}^* \rightarrow \{0,1\}$ such that h can be decided by a nondeterministic algorithm that runs in time $O(2^n)$, but h cannot be decided by a nondeterministic algorithm that runs in time $o(2^n)$.*

We omit the proof of [Theorem 3](#).

3.2 Ultra-efficient Cook-Levin theorem

The function h from [Theorem 3](#) is somewhat “contrived” and difficult to work with. We would prefer to work with a concrete and natural problem. Thankfully, 3-SAT is NP-complete (this is the famous Cook-Levin theorem), which implies that there is an exponential-time reduction from h to 3-SAT.

Intuitively, it makes sense that the 3-SAT instance has exponential *size*. On the other hand, does the reduction really require an exponential amount of *time*? Technically, the answer is yes, simply because writing down an exponentially large 3-SAT instance requires exponential time. However, we can get around this issue by using a *succinct representation* of the 3-SAT instance.

Definition 3 (Succinct representation of a 3-CNF formula). Let $\phi: \{0,1\}^{2^b} \rightarrow \{0,1\}$ be a 3-CNF formula with 2^a clauses. A circuit $C: \{0,1\}^a \rightarrow \{0,1\}^{3 \cdot (b+1)}$ *succinctly represents* ϕ if $C(i)$ describes the i -th clause of ϕ by providing the names of the three variables in the clause, along with three additional bits to indicate the presence/absence of negations. Note that the formula ϕ does not necessarily use all 2^b available variables.

The following theorem says that there is a polynomial-time reduction that converts an instance of h into a succinct representation of an exponential-size 3-SAT instance. Furthermore, the reduction has several additional features: the 3-SAT instance has only $\tilde{O}(2^n)$ clauses, and the variable names are only $O(n)$ bits, and the succinct representation has constant depth.

Theorem 4 (Ultra-efficient Cook-Levin theorem for exponential time [\[JMV18\]](#)). *Let $h: \{0,1\}^* \rightarrow \{0,1\}$, and assume h can be computed by a nondeterministic algorithm that runs in time $O(2^n)$. Then for every $x \in \{0,1\}^n$, there exists a 3-CNF formula $\phi_x: \{0,1\}^{2^b} \rightarrow \{0,1\}$ with the following properties.*

1. ϕ_x is satisfiable if and only if $h(x) = 1$.
2. There is a circuit $C_x: \{0,1\}^a \rightarrow \{0,1\}^{3 \cdot (b+1)}$ such that C_x succinctly represents ϕ_x and $C_x \in \text{NC}^0$.³
3. We have $2^a = 2^n \cdot \text{poly}(n)$ and $2^b = 2^{O(n)}$.
4. Given x , the description of C_x can be computed in $\text{poly}(n)$ time.

We omit the proof of [Theorem 4](#).

³To be clear, when we say $C_x \in \text{NC}^0$, we are saying that each output bit of C_x depends on only c input bits, where c is a constant that does not depend on n or x or anything else.

3.3 The “guess-and-SAT” method

Theorem 5. $E^{NP} \not\subseteq ACC$.

Proof sketch. Let h be a function that can be computed in nondeterministic time $O(2^n)$, but not $o(2^n)$ (Theorem 3). Let A denote the function computed by the following E^{NP} algorithm. Given $x \in \{0, 1\}^n$ and $i \in [2^n \cdot \text{poly}(n)]$:

1. Compute the full (non-succinct) description of the 3-CNF $\phi_x: \{0, 1\}^{2^b} \rightarrow \{0, 1\}$ from Theorem 4. We can do this in time $2^{O(n)}$ by first constructing the succinct representation C_x , and then computing $C_x(j)$ for each $j \in [2^a]$.
2. Construct a satisfying assignment $z \in \{0, 1\}^{2^b}$ for ϕ_x , if one exists. We can do this in time $2^{O(n)}$ by repeatedly using the NP oracle to extend z by a single bit.
3. Output $A(x, i) = z_i$.

Assume for the sake of contradiction that A can be computed by an $AC_d^0[m]$ circuit of size $S = \text{poly}(n)$, where d and m are constants. In this case, given $x \in \{0, 1\}^n$, we can compute $h(x)$ by the following “guess-and-SAT” algorithm:

1. Let $\phi_x: \{0, 1\}^{2^b} \rightarrow \{0, 1\}$ be the exponential-size 3-CNF formula from Theorem 4, and let C_x be its succinct representation.
2. Nondeterministically guess an $AC_d^0[m]$ circuit $\hat{A}_x: \{0, 1\}^b \rightarrow \{0, 1\}$ of size S . We think of \hat{A}_x as a succinct representation of an *exponentially large assignment* $\sigma \in \{0, 1\}^{2^b}$.
3. For each $i \in [2^a]$, let $D_x(i)$ indicate whether σ violates clause i of ϕ_x . Construct a circuit that computes D_x by starting with C_x , feeding its output into three copies of \hat{A}_x , then applying three XOR gates to account for negations, and finally applying an AND_3 gate. Altogether, D_x is an $AC^0[m]$ circuit of depth $d + O(1) = O(1)$ and size $\text{poly}(n)$.
4. Use the ACC satisfiability algorithm (Theorem 2) to check whether D_x is satisfiable. If so, reject, otherwise, accept.

If $h(x) = 1$, then we can nondeterministically guess a circuit computing the function $\hat{A}_x(i) = A(x, i)$, in which case σ satisfies ϕ_x , hence D_x is unsatisfiable. On the other hand, if $h(x) = 0$, then ϕ_x is unsatisfiable, so σ violates at least one clause of ϕ_x , hence D_x is satisfiable. Constructing D_x takes $\text{poly}(n)$ time, and testing whether it is satisfiable takes $2^{a - a^{\Omega(1)}} = 2^{n - n^{\Omega(1)}}$ time, contradicting the Nondeterministic Time Hierarchy Theorem. \square

References

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