Exercises 15-16

Circuit Complexity, Autumn 2024, University of Chicago
Instructor: William Hoza (williamhoza@uchicago.edu)

Submission. Solutions are due **Wednesday, December 4** at 5pm Central time. Submit your solutions through Gradescope. You are encouraged, but not required, to typeset your solutions using a LaTeX editor such as Overleaf.

The policies below can also be found on the course webpage.

Collaboration. You are encouraged to collaborate with your classmates on homework, but you must adhere to the following rules.

- Work on each exercise on your own for at least fifteen minutes before discussing it with your classmates.
- Feel free to explain your ideas to your classmates in person, and feel free to use whiteboards/chalkboards/etc. However, do not share any written/typeset solutions with your classmates for them to study on their own time. This includes partial solutions.
- Write your solutions on your own. While you are writing your solutions, do not consult any notes that you might have taken during discussions with classmates.
- In your write-up, list any classmates who helped you figure out the solution. The fact that student A contributed to student B's solution does not necessarily mean that student B contributed to student A's solution.

Permitted Resources for Full Credit. In addition to discussions with me and discussions with classmates as discussed above, you may also use any slides or notes posted in the "Course Timeline" section of the course webpage, and you may also use Wikipedia. If you wish to receive full credit on an exercise, you may not use any other resources.

Outside Resources for Partial Credit. If you wish, you may use outside resources (ChatGPT, Stack Exchange, etc.) to solve an exercise for partial credit. If you decide to go this route, you must make a note of which outside resources you used when you were working on each exercise. You must disclose using a resource even if it was ultimately unhelpful for solving the exercise. Furthermore, if you consult an outside resource while working on an exercise, then you must not discuss that exercise with your classmates.

For a function $C\colon \{0,1\}^n \to \{0,1\}$ and a string $x \in \{0,1\}^n$, we define the *sensitivity of C at x* to be the number of coordinates i such that the output value $C(x)$ changes if we flip the i-th bit of x:

$$\mathsf{sens}(C,x) = |\{i \in [n] : C(x) \neq C(x^{\oplus i})\}|$$

where $x^{\oplus i} := (x_1, x_2, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n)$.

Exercise 15 (7 points)**.** Let $C\colon \{0,1\}^n \to \{0,1\}$ be an AC_d^0 circuit of size S. Use the AC^0 Criticality Theorem to prove that

$$\mathop{\mathbb{E}}_{x \in \{0,1\}^n}[\mathsf{sens}(C,x)] \leq O(\log S)^{d-1}.$$

Hint: Sample $\rho \sim R_p$ and condition on the event $|\rho^{-1}(\star)| = 1$.

Exercise 16 (12 points)**.**

(a) Let $T\colon \{\pm 1\}^n \to \{\pm 1\}$ be a depth-D decision tree. Prove that for every $S \subseteq [n]$, the Fourier coefficient $\widehat{T}(S)$ is an integer multiple of 2^{-D}.

Hint: Use D random bits to sample an integer-valued random variable with expectation equal to $\widehat{T}(S)$.

(b) Let $T\colon \{\pm 1\}^n \to \{\pm 1\}$ be a function. Let $\varepsilon \in (0,1)$, and suppose every Fourier coefficient $\widehat{T}(S)$ is an integer multiple of ε. Prove that $\sum_{S\subseteq[n]} |\widehat{T}(S)| \leq 1/\varepsilon$.

(c) Let $C\colon \{\pm 1\}^n \to \{\pm 1\}$ be an AC^0_d circuit of size s. Use the AC^0 Criticality Theorem and the previous parts of this exercise to prove that for every $k \in \mathbb{N}$, we have

$$\sum_{\substack{S\subseteq[n]\\|S|=k}} \left|\widehat{C}(S)\right| \leq O(\log s)^{(d-1)\cdot k}.$$

Hint: Begin by deriving a formula for $\mathbb{E}_{\rho\sim R_p}\left[\widehat{C|_\rho}(S)\right]$ for an arbitrary function C.

(d) Let $n \in \mathbb{N}$. For $\mu \in (0,1)$, let X^μ denote the distribution over $\{\pm 1\}^n$ in which the bits are i.i.d. and each has expectation equal to μ. Use part (d) to prove that for every $s, d \in \mathbb{N}$, there is a value $\mu = 1/O(\log s)^{d-1}$ such that X^μ fools size-s AC^0_d circuits with error 0.1.

(You may find it interesting to compare this exercise to Exercise 4a.)