Discharging Method on the Planar Graph

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Introduction

- The discharging method is a technique used to prove lemmas in structural graph theory
- Discharging is most well known for its central role in the proof of the Four Color Theorem
- The discharging method is used to prove that every graph in a certain class contains some subgraph from a specified list.
- The presence of the desired subgraph is then often used to prove a coloring result.

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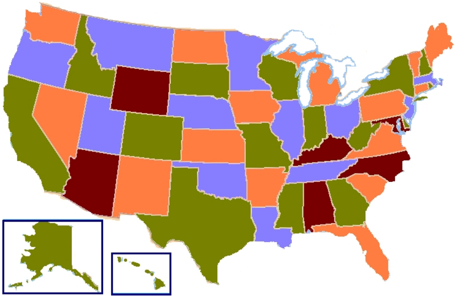
Four color theorem



- In 1904, Wernicke introduced the discharging method to prove a theorem which was part of an attempt to prove the four color theorem.
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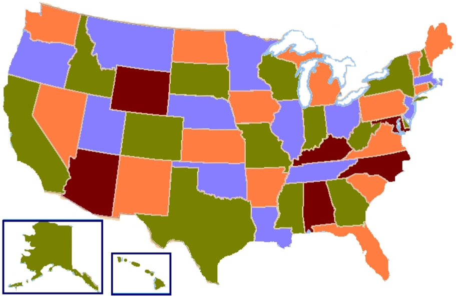

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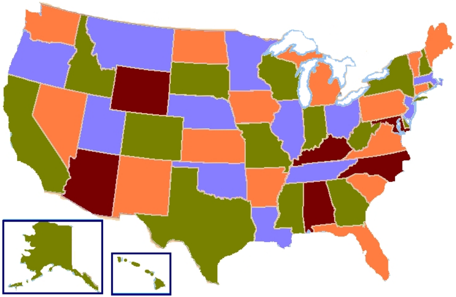

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Method

- Euler's formula : For a connected planar graph :

$$V(G) - E(G) + F(G) = 2$$

- Multiply Euler's Formula by -6 and split the term for edges to obtain

$$-6V(G) + 2E(G) + 4E(G) - 6F(G) = -12$$

- Since $E(G) = \frac{1}{2}\sum_{v \in V(G)} d(v)$ and $E(G) = \frac{1}{2}\sum_{f \in F(G)} l(f)$ substitute to the equation we get

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2l(f) - 6) = -12$$

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If we multiply Euler's formula by -6 and split E(G) as similar

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or multiply by -4 split the edge as similar

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- **Proposition :** Let G be a connected plane graph then the following hold for G

$$\sum_{v \in V(G)} (d(v)-6) + \sum_{f \in F(G)} (2l(f)-6) = -12 \text{ (vertex charging)}$$

$$\sum_{v \in V(G)} (2d(v)-6) + \sum_{f \in F(G)} (l(f)-6) = -12 \text{ (face charging)}$$

$$\sum_{v \in V(G)} (d(v)-4) + \sum_{f \in F(G)} (l(f)-4) = -8 \text{ (balance charging)}$$

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Example

Definition : Assign each edge a list of color. A **k-edge-choosable** means that the list on each edge has length k, and that from any such set of lists, the graph G can be properly colored.

Vizing's theorem : Every graph is $\Delta + 1$ edge colorable
Vizing's conjecture : Every graph is $\Delta + 1$ edge choosable

We are going to prove a smaller case of Vizing's conjecture

Theorem : If G is a plane graph and no two 3-faces sharing an edge, then G is $\max\{\Delta(G) + 1, 8\}$ -edge choosable

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Lemma 1 : Let G be a connected plane graph with no two 3-faces sharing an edge. If $\Delta \leq 5$ then G has an edge with weight at most 8. If $\Delta = 6$, then G has an edge with weight at most 9.

Proof : Let e be any edge in G. Let $w(e)$ denotes the weight of e. So $w(e) \leq 2\Delta \leq 8$ for all $\Delta(G) \leq 4$

Consider the case when $\Delta = 5$ or 6. We will show that G has an edge e such that $w(e) \leq \Delta + 3$.

Suppose every edge $e \in E(G)$ has $w(e) \geq \Delta + 4$

Use balance charging for initial charge : $d(v) - 4$, $l(f) - 4$

Need charge : 3-faces with initial charge -1.

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Rule : Each 3-face take $\frac{1}{2}$ from its k-vertices where $k \geq 5$.

Every 3-face has at least two k-vertices so 3-faces is happy.

Each k-vertex has at most $\lfloor k/2 \rfloor$ incident triangle so its final charge is at least

$$(k-4) - \frac{1}{2}\lfloor k/2 \rfloor = \frac{1}{2}\left\lceil \frac{3k}{2} \right\rceil - 4 \geq 0$$

for any $k \in \{5, 6\}$ $\Rightarrow\!\Leftarrow$

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Lemma 2 : Let G be a connected plane graph with $\Delta(G) \geq 7$. Prove that G has either two 3-faces with a common edge or an edge with weight at most $\Delta(G) + 2$.

Proof : Assume the graph has none of these configurations :

- **C1 :** there is no two 3-faces with a common edge.
- **C2 :** every edge has weight at least $\Delta + 3$.

By C2, $\delta(G) = 3$.

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- **R1 :** Each 3-vertex takes $\frac{1}{3}$ charge from its neighbors.
- **R2 :** Each 3-face takes $\frac{1}{2}$ charge from its k-vertices with $k \geq \frac{\Delta+3}{2}$

Every 3-vertices are happy.

Each triangle must have at least two k-vertices. Thus 3-faces are happy.

Let v be a k-vertex

Case 1 : $\frac{\Delta+3}{2} \leq k < \Delta$

v only lose charge to incident 3-faces and there are at most $\left\lfloor \frac{k}{2} \right\rfloor$ triangles incident to v. Thus, for $k \geq 5$ final charge of v is

$$(k-4) - \frac{1}{2}\left\lfloor \frac{k}{2} \right\rfloor = \frac{1}{2}\left\lceil \frac{3k}{2} \right\rceil - 4 \geq 0$$

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- **R2 :** Each 3-face takes $\frac{1}{2}$ charge from its k-vertices with $k \geq \frac{\Delta+3}{2}$

  Every 3-vertices are happy.

  Each triangle must have at least two k-vertices. Thus 3-faces are happy.

  Let v be a k-vertex

  Case 1 : $\frac{\Delta+3}{2} \leq k < \Delta$

  v only lose charge to incident 3-faces and there are at most $\left\lfloor \frac{k}{2} \right\rfloor$ triangles incident to v. Thus, for $k \geq 5$ final charge of v is

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Case 2 : $k = \Delta$

v loses charge to incident 3-faces and 3-vertices. Let t be the number of triangles and n be the number of 3-vertices. We have the inequality

$$k - 2t + t \geq n$$

and

$$t \leq \lfloor k/2 \rfloor$$

Substitute these two inequalities to the final charge equation to obtain

$$k - 4 - \frac{t}{2} - \frac{n}{3} \geq \frac{1}{6}\left\lfloor \frac{7k}{2} \right\rfloor - 4 \geq 0$$

for $k = \Delta = 7$

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Theorem If G is a plane graph and no two 3-faces sharing an edge, then G is $\max\{\Delta(G)+1, 8\}$ -edge choosable

Proof : Let $M_G = \max\{\Delta(G)+1, 8\}$

If $E(G) \leq 8$, then each edge would have at most 7 edges incident to it. Thus it is 8 choosable, which implies M_G -edge choosable

Inductive hypothesis : let's assume that every plane graph G with $E(G) = k$ and no two 3-faces sharing an edge is M_G -edge choosable.

Consider a plane graph H with no two 3-faces sharing an edge and $E(H) = k+1$.

If H is disconnected, each component H_i of H would have $E(H_i) \leq k$. Thus, H_i is M_{H_i} -edge choosable for every i. Therefore, the edge choosibility of H is :

$$\max_{i \in I}\{M_{H_i}\} \leq M_H$$

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Suppose H is connected. Let e be the edge such that :

$$w(e) = \begin{cases} 8 & \text{if } \Delta(H) = 5 \\ 9 & \text{if } \Delta(H) = 6 \\ \Delta + 2 & \text{if } \Delta(H) \geq 7 \end{cases}$$

Let $I(e)$ be number of incident edges of e. So $I(e) = w(e) - 2$.

Consider the graph $H - e$. WLOG assume by inductive hypothesis that $H - e$ is connected and M_H choosable. (Note that M_{H-e}-choosable implies M_H choosable)

Showing that $I(e) \leq M_H - 1$ implies that we have enough color for e such that H is still M_H-choosable.

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If $\Delta(H) = 5$, $I(e) = 6 \leq 7 = M_H - 1$

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If $\Delta(H) \geq 7$, $I(e) = \Delta(H) = M_H - 1$

For the case when $\Delta(H) \leq 4$, we can pick any edge and still obtain the same inequality :

$$I(e) \leq 2\Delta(H) - 2 < 7 = M_H - 1$$

Thus, H is M_H edge choosable $\square$

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Thank you!