Discharging Method on the Planar Graph

Huy Bui

Texas A&M University

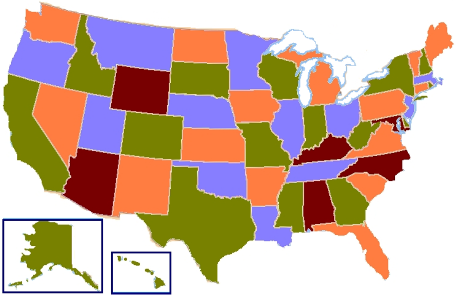
July 16th 2019

Table on Contents

Introduction

- The discharging method is a technique used to prove lemmas in structural graph theory
- Discharging is most well known for its central role in the proof of the Four Color Theorem
- The discharging method is used to prove that every graph in a certain class contains some subgraph from a specified list.
- The presence of the desired subgraph is then often used to prove a coloring result.

Four color theorem



- In 1904, Wernicke introduced the discharging method to prove a theorem which was part of an attempt to prove the four color theorem.
- In 1976, the four color theorem was proved by Appel and Haken using discharging method. The proof is very complex, over 400 pages, and it heavily relies on computer.

Method

Euler's formula : For a connected planar graph,

$$V(G) - E(G) + F(G) = 2$$

Multiply Euler's Formula by -6 and split the term for edges to obtain

$$-6V(G) + 2E(G) + 4E(G) - 6F(G) = -12$$

Since $E(G) = \frac{1}{2}\sum_{v \in V(G)} d(v)$ and $E(G) = \frac{1}{2}\sum_{f \in F(G)} l(f)$ substitute to the equation we get

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2l(f) - 6) = -12$$

If we multiply Euler's formula by -6 and split E(G) as similar

$$-6V(G) + 4E(G) + 2E(G) - 6F(G) = -12$$

or multiply by -4 split the edge as similar

$$-4V(G) + 2E(G) + 2E(G) - 4F(G) = -8$$

and complete the substitution, we obtain the following proposition.

Proposition : Let G be a connected plane graph then the following hold for G

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2l(f) - 6) = -12 \text{ (vertex charging)}$$

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (l(f) - 6) = -12 \text{ (face charging)}$$

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (l(f) - 4) = -8 \text{ (balance charging)}$$

General structure of later proofs :

- Assume by contradiction that G has none of the configuration.
- Initialize charge
- Establish discharging rule
- Calculate final charge (which would contradict with the proposition)

Theorem 1

Lemma 1.1 : Every plane graph G with $\delta(G) \geq 3$ has two 3-faces with a common edge, or a j-face with $4 \leq j \leq 9$, or a 10-face whose vertices all have degree 3.

Proof : Suppose the graph has none of these configuration which means :

- **C1** : No 3-faces with a common edge
- **C2** : Every j-face satisfies $j = 3$ or $j \geq 10$
- **C3** : Every 10-face have at least one 4^+-vertex

Use face charging : $2d(v) - 6$ for each vertex, $l(f) - 6$ for each face. The only thing that need charge are 3-faces and they begin with charge -3.

- **R1** : Each triangle takes 1 charge from each neighboring face.
- **R2** : Each face f takes 1 charge from each incident 4^+-vertex lying on the triangle sharing an edge with f

Every 3-face is happy because of R1. 3-vertices also happy.
Let v be a j-vertex where $j \geq 4$ and we investigate how much charge v can lose. A triangle incident to v can have at most 2 neighboring-faces that each can take 1 charge from v. This also require at least 2 extra edges. These extra edges can also be shared by 2 triangles. Thus, the worst case scenario is when we have triangles and edges alternating.
Note : if $3|j$ then v would lose at most $\frac{2j}{3}$

Therefore, for every $j \geq 4$, v would lose at most $\left\lfloor \frac{2j}{3} \right\rfloor$ and so the final charge of v is at least :

$$(j-6) - \frac{2j}{3} = \left\lceil \frac{4j}{3} \right\rceil - 6 \geq 0$$

Consider a j-face f for $j \geq 10$. Let p be a path along its boundary such that neighboring faces are triangles. For each p, if both end points of p have degree 3 then then f only lose 1 charge.

If one end point of p have degree more than 3 then f does not lose charge. Thus, f lose the most charge when it is adjacent to triangles that share two 3-vertices. There are at most $\left\lfloor \frac{j}{2} \right\rfloor$ triangle. So when $j \geq 11$, the final charge of f is :

$$f - 6 - \left\lfloor \frac{j}{2} \right\rfloor = \left\lceil \frac{j}{2} \right\rceil - 6 \geq 0$$

When $j = 10$, we can only have at most 4 triangles adjacent to f because of $\overline{C}$. Thus final charge of 10-faces is at least $(10 - 6) - 4 = 0$. Now, everyone is happy $\square$

Theorem 1 : Every plane graph having no 4-cycle and no j-face with $5 \leq j \leq 9$ is 3-colorable.

Proof :
Let G having a j-vertex v where $j \in \{1, 2\}$. If neighbor of v are 3-colorable then there is always a choice for v. Thus we can assume $\delta(G) \geq 3$.

Since there is no 4-cycle, no 3-faces share an edge. We can apply lemma 1.1 which implies G has at least one 10-face, namely C, with all incident vertices having degree 3.

Let f be a proper 3-coloring of $G - V(C)$. Since each vertex on C has exactly one neighbor outside C, two color remain available at each vertex of C.
Since even cycle are 2-choosable, the coloring can be completed. $\square$.

Theorem 2

Definition : Assign each edge a list of color. A k**-edge-choosable** means that the list on each edge has length k, and that from any such set of lists, the graph G can be properly colored.

Vizing's theorem : Every graph is $\Delta + 1$ edge colorable

Vizing's conjecture : Every graph is $\Delta + 1$ edge choosable

We are going to prove a smaller case of Vizing's conjecture

Theorem 2 : If G is a plane graph and no two 3-faces sharing an edge, then G is $\max\{\Delta(G) + 1, 8\}$ -edge choosable

Lemma 2.1

Lemma 2.1 : Let G be a connected plane graph with no two 3-faces sharing an edge. If $\Delta \leq 5$ then G has an edge with weight at most 8. If $\Delta = 6$, then G has an edge with weight at most 9.

Proof :
Let e be any edge in G. Let $w(e)$ denotes the weight of e. So $w(e) \leq 2\Delta \leq 8$ for all $\Delta(G) \leq 4$
Consider the case when $\Delta = 5$ or 6. We will show that G has an edge e such that $w(e) \leq \Delta + 3$.
Suppose every edge $e \in E(G)$ has $w(e) \geq \Delta + 4$
Use balance charging with initial charge : $d(v) - 4$, $l(f) - 4$.
Thus, the configuration that need charge is 3-faces since its initial charge is -1.

Rule : Each 3-face take $\frac{1}{2}$ from its k-vertices where $k \geq 5$.

Every 3-face has at least two k-vertices so 3-faces is happy.
Each k-vertex has at most $\lfloor k/2 \rfloor$ incident triangle so its final charge is at least

$$(k-4) - \frac{1}{2} \lfloor k/2 \rfloor = \frac{1}{2} \left\lceil \frac{3k}{2} \right\rceil - 4 \geq 0$$

for any $k \in \{5, 6\}$ $\Rightarrow\!\Leftarrow$

Lemma 2.2

Lemma 2.2 : Let G be a connected plane graph with $\Delta(G) \geq 7$. Prove that G has either two 3-faces with a common edge or an edge with weight at most $\Delta(G) + 2$.

Proof : Assume the graph has none of these configurations :

- **C1 :** there is no two 3-faces with a common edge.
- **C2 :** every edge has weight at least $\Delta + 3$.

By C2, $\delta(G) = 3$.
Use balance charging :$d(v) - 4$, $l(f) - 4$.
Need charges : 3-vertices and 3-faces both have -1 charge

- **R1** **:**Each 3-vertex takes $\frac{1}{3}$ charge from its neighbors.
- **R2** **:** Each 3-face takes $\frac{1}{2}$ charge from its k-vertices with $k \geq \frac{\Delta+3}{2}$

Every 3-vertices are happy. Each triangle must have at least two k-vertices. Thus 3-faces are happy. Let v be a k-vertex.

Case 1 : $\frac{\Delta+3}{2} \leq k < \Delta$
v only lose charge to incident 3-faces and there are at most $\left\lfloor \frac{k}{2} \right\rfloor$ triangles incident to v. Thus, for $k \geq 5$ final charge of v is

$$(k-4) - \frac{1}{2}\left\lfloor \frac{k}{2} \right\rfloor = \frac{1}{2}\left\lceil \frac{3k}{2} \right\rceil - 4 \geq 0$$

Case 2 : $k = \Delta$

v loses charge to incident 3-faces and 3-vertices. Let t be the number of triangles and n be the number of 3-vertices. We have the inequality

$$k - 2t + t \geq n$$

and

$$t \leq \lfloor k/2 \rfloor$$

Substitute these two inequalities to the final charge equation to obtain :

$$k - 4 - \frac{t}{2} - \frac{n}{3} \geq \frac{1}{6}\left\lfloor \frac{7k}{2} \right\rfloor - 4 \geq 0$$

for $k = \Delta = 7$ Thus, everything is happy $\Rightarrow\!\Leftarrow$

Theorem 2

Theorem 2 If G is a plane graph and no two 3-faces sharing an edge, then G is $\max\{\Delta(G)+1, 8\}$ -edge choosable

Proof :
Let $M_G = \max\{\Delta(G)+1, 8\}$
If $E(G) \leq 8$, then each edge would have at most 7 edges incident to it. Thus it is 8 choosable, which implies M_G -edge choosable

Inductive hypothesis : Let's assume that every plane graph G with $E(G) = k$ and no two 3-faces sharing an edge is M_G -edge choosable. Consider a plane graph H with no two 3-faces sharing an edge and $E(H) = k+1$.

If H is disconnected, each component H_i of H would have $E(H_i) \leq k$. Thus, H_i is M_{H_i} -edge choosable for every i. Therefore, the edge choosibility of H is :

$$\max_{i \in I}\{M_{H_i}\} \leq M_H$$

since $\Delta(H_i) \leq \Delta(H)$.
Suppose H is connected. Let e be the edge such that :

$$w(e) = \begin{cases} 8 & \text{if } \Delta(H) = 5 \\ 9 & \text{if } \Delta(H) = 6 \\ \Delta + 2 & \text{if } \Delta(H) \geq 7 \end{cases}$$

Let $I(e)$ be number of incident edges of e. So $I(e) = w(e) - 2$.

Consider the graph $H - e$. WLOG assume by inductive hypothesis that $H - e$ is connected and M_H choosable.

Note : M_{H-e}-choosable implies M_H choosable)

Showing that $I(e) \leq M_H - 1$ implies that we have enough color for e such that H is still M_H-choosable.

In fact :

- If $\Delta(H) = 5$, $I(e) = 6 \leq 7 = M_H - 1$
- If $\Delta(H) = 6$, $I(e) = 7 \leq 7 = M_H - 1$
- If $\Delta(H) \geq 7$, $I(e) = \Delta(H) = M_H - 1$
- If $\Delta(H) \leq 4$, we can pick any edge and still obtain the same inequality :

$$I(e) \leq 2\Delta(H) - 2 < 7 = M_H - 1$$

Thus, H is M_H edge choosable $\square$