Consider a sub-region Ω of a D-dimensional space with surface $\partial\Omega$. Given a probability distribution P on this space, the probability mass within and volume of Ω are respectively:

$$M = \int_\Omega P dV, \qquad V = \int_\Omega dV \tag{1}$$

In the proof below we will need the following result: If we deform the region Ω of integration of some function f, such that the surface moves by a small vector field δx, then the total integral variation can be expressed as a surface integral:

$$\delta \int_\Omega f dV = \int_{\partial\Omega} f \delta x \cdot dS \tag{2}$$

(this is a specification of Reynolds transport theorem/Leibniz integral rule).

We wish to prove that for a constant probability mass M, the minimum volume region Ω has an iso-probability surface. Introducing a Lagrange multiplier λ, we consider the standard Lagrangian and differentiate

$$\mathcal{L} = V - \lambda M \tag{3}$$

$$= \int_\Omega d\theta - \lambda \int_\Omega P(\theta) dV \tag{4}$$

$$= \int_\Omega 1 - \lambda P d\theta \tag{5}$$

$$\Rightarrow \delta\mathcal{L} = \int_{\partial\Omega} (1 - \lambda P(\theta))\, \delta x \cdot dS \tag{6}$$

We therefore find if we wish this to be zero for arbitrary surface displacements δx, we require $P = \lambda^{-1}$, so that P is constant on the surface $\partial\Omega$.