Consistent comparison of BGK and ES-BGK equations

Neglecting body forces the ES-BGK equation is

$$\frac{\partial \varphi}{\partial t} + \eta_i \frac{\partial \varphi}{\partial x_i} = n^2 K_{ES}[\Psi - f] = n K_{ES}[\psi - \varphi] \tag{1}$$

where I have absorbed one factor of n into the bracketed term to write an equation in terms of φ. The standard BGK equation in a similar form would be

$$\frac{\partial \varphi}{\partial t} + \eta_i \frac{\partial \varphi}{\partial x_i} = n K_{BGK}[\varphi_M - \varphi] \tag{2}$$

Introducing our standard scaling on the each side of the ES-BGK equation,

$$\eta_{ref} \equiv \sqrt{\frac{kT_{ref}}{m}}, \lambda_{ref} = \frac{1}{n_{ref}\sigma_{ref}}; \hat{f} = \eta_{ref}^3 f, \hat{n} = \frac{n}{n_{ref}}, \hat{\varphi} \equiv \hat{n}\hat{f} \Rightarrow \varphi = \frac{n_{ref}}{\eta_{ref}^3}\hat{\varphi}; \frac{\partial}{\partial t} = \frac{\eta_{ref}}{L_{ref}}\frac{\partial}{\partial \hat{t}} \tag{3}$$

we get

$$\frac{\partial \hat{\varphi}}{\partial \hat{t}} + \hat{\eta}_i \frac{\partial \hat{\varphi}}{\partial \hat{x}_i} = \frac{L_{ref} n_{ref}}{\eta_{ref}} \hat{n} K_{ES}[\hat{\psi} - \hat{\varphi}]. \tag{4}$$

Defining the Knudsen number $\mathrm{Kn} = \lambda_{ref}/L_{ref}$ the scaled ES-BGK equation becomes

$$\frac{\partial \hat{\varphi}}{\partial \hat{t}} + \hat{\eta}_i \frac{\partial \hat{\varphi}}{\partial \hat{x}_i} = \frac{\hat{K}_{ES}}{\mathrm{Kn}} \hat{n}[\hat{\psi} - \hat{\varphi}], \qquad \hat{K}_{ES} \equiv \frac{K_{ES}}{\sigma_{ref}\eta_{ref}} \tag{5}$$

Here

$$\hat{\psi} = \frac{\hat{n}}{(2\pi)^{3/2}(\det \hat{\Lambda}_{ij})^{1/2}} \exp\left[-\frac{1}{2}\hat{C}_i \hat{\Lambda}_{ij}^{-1} \hat{C}_j\right] \tag{6}$$

$$\hat{\Lambda}_{ij} = (1 - \lambda^*)\hat{T}\delta_{ij} + \lambda^* \hat{M}_{ij} \tag{7}$$

and to get the correct value of the Prandtl number, Pr=2/3 we require $\lambda^* = -1/2$.

Similarly the scaled BGK equation in this form is

$$\frac{\partial \hat{\varphi}}{\partial \hat{t}} + \hat{\eta}_i \frac{\partial \hat{\varphi}}{\partial \hat{x}_i} = \frac{\hat{K}_{BGK}}{\mathrm{Kn}} \hat{n}[\hat{\varphi}_M - \hat{\varphi}], \qquad \hat{K}_{BGK} \equiv \frac{K_{BGK}}{\sigma_{ref}\eta_{ref}} \tag{8}$$

Note that $\mathrm{Kn} = 1$ for the shock wave problem.

To make the most consistent comparisons of shock profiles computed using different model forms of the collision integral (BGK vs ES-BGK) and molecular models (hard sphere vs pseudo-Maxwell VHS) let us assume that we choose our models based on matching the coefficient of viscosity μ. Then for a monatomic gas we have

$$K_{ES} = Pr\frac{kT}{\mu} = \frac{2}{3}\frac{kT}{\mu} \qquad K_{BGK} = \frac{kT}{\mu} \tag{9}$$

Now in the Chapman-Enskog limit for a single component gas we have

$$\mu = \frac{5}{8}\frac{kT}{\Omega_{pp}^{(2,2)}} \Rightarrow K_{ES} = \frac{16}{15}\Omega_{pp}^{(2,2)}, K_{BGK} = \frac{8}{5}\Omega_{pp}^{(2,2)} \tag{10}$$

From Omeiri and Djafri (2010) and using $\sigma_{ref} = \pi d_{ref}^2$ for VHS molecules we have

$$\Omega_{pp}^{(2,2)} = \frac{1}{12}\left(\frac{1}{2\pi}\right)^{1/2} \sigma_{ref}(5-2\omega_p)(7-2\omega_p)\left(\frac{2kT}{m}\right)^{1-\omega_p}\left(\frac{2kT_{ref}}{m}\right)^{\omega_p-\frac{1}{2}} \tag{11}$$

For VHS pseudo-Maxwell (pM) molecules $\omega_p = 1$ and we get

$$\Omega_{pp,pM}^{(2,2)} = \frac{15}{12}\left(\frac{1}{2\pi}\right)^{1/2} \sigma_{ref}\left(\frac{2kT_{ref}}{m}\right)^{\frac{1}{2}} \Rightarrow K_{ES,pM} = \frac{4}{3\sqrt{\pi}}\sigma_{ref}\eta_{ref} \tag{12}$$

For hard sphere (HS) molecules $\omega_p = 1/2$ and we get

$$\Omega_{pp,HS}^{(2,2)} = \left(\frac{2}{\pi}\right)^{1/2} \sigma_{ref}\eta_{ref}\left(\frac{T}{T_{ref}}\right)^{\frac{1}{2}} \Rightarrow K_{ES,HS} = \frac{16}{15}\left(\frac{2}{\pi}\right)^{1/2} \sigma_{ref}\eta_{ref}\left(\frac{T}{T_{ref}}\right)^{\frac{1}{2}} \tag{13}$$

It follows that

$$\hat{K}_{ES,pM} = \frac{4}{3\sqrt{\pi}}, \qquad \hat{K}_{ES,HS} = \frac{16}{15}\sqrt{\frac{2}{\pi}}\hat{T}^{\frac{1}{2}} \tag{14}$$

while

$$\hat{K}_{BGK,pM} = \frac{2}{\sqrt{\pi}}, \qquad \hat{K}_{BGK,HS} = \frac{8}{5}\sqrt{\frac{2}{\pi}}\hat{T}^{\frac{1}{2}} \tag{15}$$