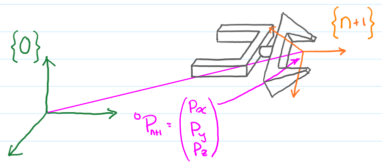
2. Transformations

<Passive walking robot>

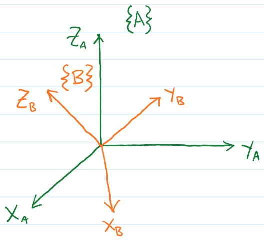
Recap: We can describe the position and orientation of a rigid body by attaching a frame to it and describing this frame wrt a reference frame



$$^0P_{n+1} = \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}$$

A Description

☆ How can we describe one frame relative to anoth



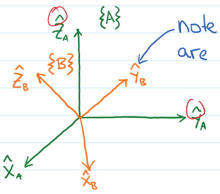
i Rotation matrix

parent / ref frame

$$^A_BR = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

child frame

3x3 for 3D



note the 'hats' - these are now unit vectors

ref frame

$$^A\hat{X}_B = {}^A_BR \; \boxed{^B\hat{X}_B}$$

thing desc.

Description of $\hat{X}_B$ in frame {B} $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Description of $\hat{X}_B$ in frame {A}

Q What is $^B\hat{X}_B$? $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$^A\hat{X}_B = {}^A_BR \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad ^A\hat{Y}_B = {}^A_BR \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad ^A\hat{Z}_B = {}^A_BR \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

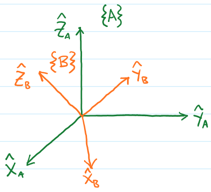
$$\Longrightarrow \; ^A\hat{X}_B = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow {}^A\hat{X}_B = \begin{array}{c} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\ \begin{array}{ccc} X & Y & Z \end{array} \end{array} \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ {}^B\hat{X}_B \end{array}$$

$$= \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

$$\Longrightarrow {}^A_BR = \left[{}^A\hat{X}_B \ \ {}^A\hat{Y}_B \ \ {}^A\hat{Z}_B \right]$$

★ The rotation matrix is simply the components of $\hat{x}_B$, $\hat{y}_B$, and $\hat{z}_B$ in frame A



- This definition can often help you intuitively find/understand the rotation matrix

$${}^A_BR = \left[{}^A\hat{X}_B \ \ {}^A\hat{Y}_B \ \ {}^A\hat{Z}_B \right]$$

Q How do we find ${}^A\hat{X}_B$ (or ${}^A\hat{Y}_B$, ${}^A\hat{Z}_B$)

Vector dot product

because we're dealing with unit vectors $\|a\| = \|b\| = 1$

$\Rightarrow \hat{a} \cdot \hat{b} = \cos\theta$

The dot product: $(\hat{a} \cdot \hat{b} = \|a\|\|b\| \cos\theta)$

$${}^A\hat{X}_B = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A \end{bmatrix} \begin{array}{l} x\text{-comp} \\ y\text{-comp} \\ z\text{-comp.} \end{array}$$

$$\Longrightarrow {}^A_BR = \begin{array}{c} \begin{array}{ccc} \phantom{\hat{X}_B \cdot \hat{X}_A} & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{array} \\ \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} \end{array}$$

These 9 values are called "direction cosines" as they come from the dot product

Q What do you notice about the rows of ${}^A_BR$?

$${}^A_BR = \begin{array}{c} \begin{array}{ccc} {}^A\hat{X}_B & \phantom{\hat{Y}_B \cdot \hat{X}_A} & \phantom{\hat{Z}_B \cdot \hat{X}_A} \end{array} \\ \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} \end{array} \begin{array}{l} \Rightarrow {}^B\hat{X}_A^T \\ \\ \\ \end{array}$$

$$\Longrightarrow {}^A_BR = \left[{}^A\hat{X}_B \ {}^A\hat{Y}_B \ {}^A\hat{Z}_B \right] = \begin{bmatrix} {}^B\hat{X}_A^T \\ {}^B\hat{Y}_A^T \\ {}^B\hat{Z}_A^T \end{bmatrix} = \left[{}^B\hat{X}_A \ {}^B\hat{Y}_A \ {}^B\hat{Z}_A \right]^T = {}^B_AR^T$$

☆ $^A_BR = {^B_AR}^T$

$\Rightarrow$ Inverse of rotation matrices $\{B\} \rightleftarrows \{A\}$

$$\underline{^A_BR^{-1}} = \underline{^B_AR} = {^A_BR}^T$$

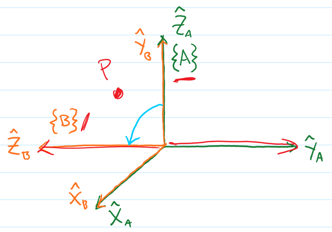
describes frm A in B (Parent)
A: child

This is an important property of the rotation matrix

It derives from the fact that R is formed from orthogonal unit vectors

$$^A_BR^{-1} = {^A_BR}^T$$ Orthonormal matrix

Example:



$^A_BR_x(90) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

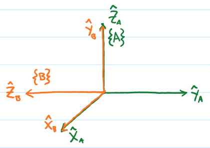
Apply 90° rot to A $\rightarrow$ get B

then $^A_BR = R_x$

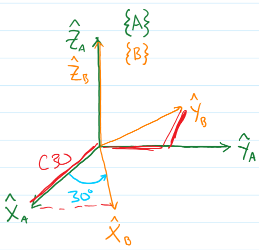
Q $\quad ^A_BR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \leftarrow ?\ ^B Y_A$ $\qquad {^AP} = {^A_BR} \begin{bmatrix} ^BP_x \\ P_y \\ P_z \end{bmatrix}$

$$^A_BR = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \leftarrow {^B\hat{X}_A^T} \\ \leftarrow {^B\hat{Y}_A^T} \\ \leftarrow {^B\hat{Z}_A^T} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ ^A\hat{X}_B & ^A\hat{Y}_B & ^A\hat{Z}_B \end{array} \end{array}$$

$$^B_AR = {^A_BR^{-1}} = {^A_BR}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$



Another example:



$^A\hat{X}_B = \hat{X}_A \cdot \hat{X}_B =$ $\qquad$ Cos 30 = 0.866

Q $\quad ^AP = \left(\begin{bmatrix} C30 & -S30 \end{bmatrix} \; 0 \right]$

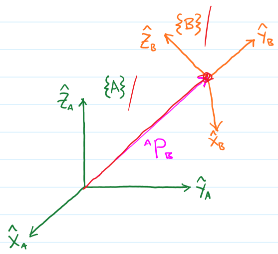
Q $\quad {}^A_B R = \begin{bmatrix} C30 & -S30 & 0 \\ S30 & C30 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$^A\hat{X}_B = \hat{X}_A \cdot \hat{X}_B =$
$\hat{Y}_A \cdot \hat{X}_B$
$\hat{Z}_A \cdot \hat{X}_B$

Cos 30 = 0.866
Sin 30 = 0.5
0 = 0

* The rotation matrix allows us to describe the orientation of one frame relative to another

We also need to describe the position



Frame B is fully described by: $\boxed{^A\hat{X}_B, {}^A\hat{Y}_B, {}^A\hat{Z}_B, {}^AP_B}$

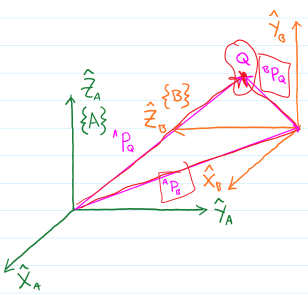
$\Longrightarrow \{B\} = \{ \underbrace{{}^A_B R}_{\text{orientation}}, \underbrace{{}^AP_B}_{\text{position}} \}$

B. Mapping:

* Changing descriptions of a point or vector from one frame to another

Example:

We have a point, Q in frame B, defined by the vector ^BP_Q



$${}^A_B R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$${}^AP_B = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$${}^BP_Q = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

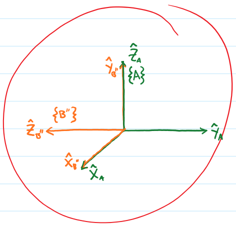
What are the coordinates of Q in $\{A\}$?

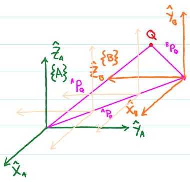
Q $\quad$ Can we simply add them, $^AP_Q = {}^AP_B + {}^BP_Q$? No

because frame B and frame A are not aligned – in terms of their axes.

We cannot simply add, as the axes are rotated.

→ We treat this problem in 2 parts: Rotation, Translation





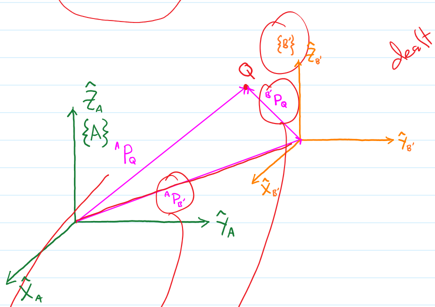
i Rotation

$$^{B'}P_Q = {}^A_BR\,{}^BP_Q$$

gives us ^AP_Q … ^BP_Q

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

dealt with the rotation



Now that the axes are aligned, we can add vectors:

ii Translation

$$^AP_Q = {}^AP_{B'} + {}^{B'}P_Q$$

$$= \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$^AP_Q = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

We have mapped the coordinates of point Q from frame B to frame A:

$$\star\quad {}^AP_Q = {}^A_BR\,{}^BP_Q + {}^AP_B$$

✶ Mapping: Changing descriptions from frame to frame

- The point or vector remains the same - just its description changes

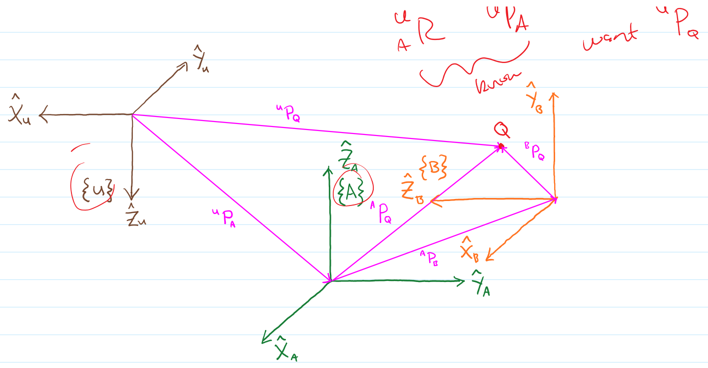
Now, what if we wanted to do an additional mapping to a 3rd frame, U ??

^U_AR $\quad$ ^UP_A $\quad$ want ^UP_Q

known

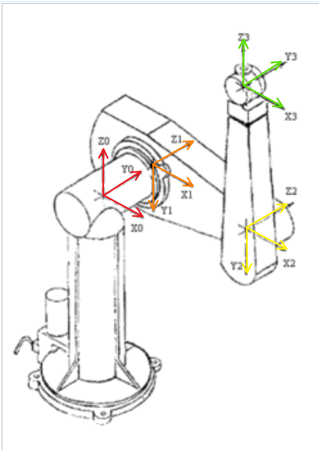
$\hat{Y}_U$

$\hat{Y}$



$${}^{u}P_Q = {}^{u}_{A}R\,(\underbrace{{}^{A}_{B}R\,{}^{B}P_Q + {}^{A}P_B}_{{}^{A}P_Q}) + {}^{u}P_A$$

This will quickly get very messy! Especially if we're thinking about a manipulator with many links:



We really want to have our transform in the form of:

$${}^{A}P = {}^{A}_{B}T\,{}^{B}P$$

$\Rightarrow$ If we move to 4 dimensions, we can have this:

iii Homogeneous transform

$${}^{A}P_Q = {}^{A}_{B}R\,{}^{B}P_Q + {}^{A}P_B$$

$$\Downarrow$$

$$\begin{bmatrix} {}^{A}P_Q \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_B \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P_Q \\ 1 \end{bmatrix}$$

Expand out the matrix equation:

$${}^{A}P_Q = {}^{A}_{B}R\,{}^{B}P_Q + {}^{A}P_B$$

$$1 = 1$$

* The 4×4 matrix, $T = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_B \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$ is called an

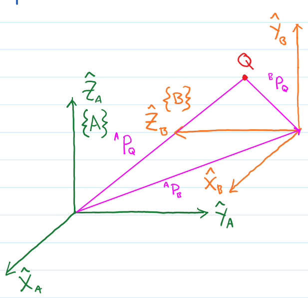
Homogeneous transform

$$\Rightarrow \quad \underset{(4\times1)}{{}^{A}P} = \underset{(4\times4)}{{}^{A}_{B}T}\ \underset{(4\times1)}{{}^{B}P}$$

(4×1) (4×4) (4×1)

This is simply a construction to allow us to treat frame transformations in a convenient way.

Example:



$$^A_BR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$^AP_B = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$^BP_Q = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Q What is the homogenous transform, ^A_BT for mapping frame B to frame A ?

$$^A_BT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^A_BT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad ^AP_Q = {}^A_BT\, {}^BP_Q$$

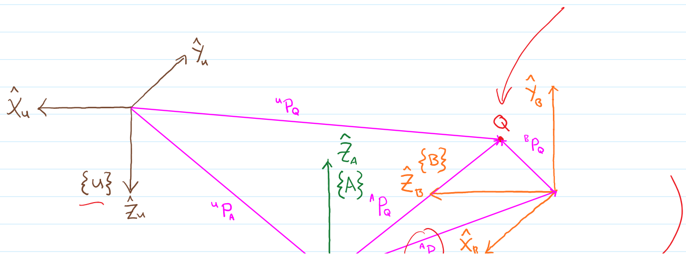
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

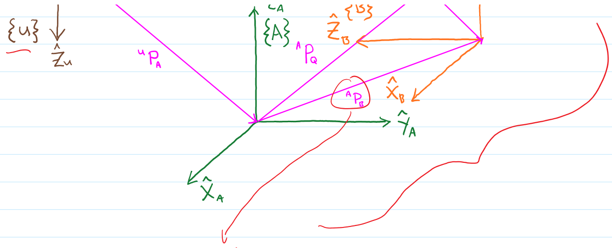
$$= \begin{bmatrix} 1 \\ 1 \\ 4 \\ \cancel{1} \end{bmatrix}$$

✓ Same as before

The homogeneous transform makes multiple transformations fun and easy!

Recall this:





$$^{u}P_Q = {}^{u}_{A}R\,\underbrace{\left({}^{A}_{B}R\,{}^{B}P_Q + {}^{A}P_B\right)}_{{}^{A}_{B}T\,{}^{B}P_Q} + {}^{u}P_A$$

$$\underbrace{\qquad\qquad}_{{}^{u}_{A}T\left({}^{A}_{B}T\,{}^{B}P_Q\right)}$$

$$\Rightarrow {}^{u}P_Q = {}^{u}_{A}T\,{}^{A}_{B}T\,{}^{B}P_Q$$

Summary:

* Rotation matrix

$$^{A}_{B}R = \left[{}^{A}\hat{X}_B \; {}^{A}\hat{Y}_B \; {}^{A}\hat{Z}_B\right] = \left[{}^{B}\hat{X}_A \; {}^{B}\hat{Y}_A \; {}^{B}\hat{Z}_A\right]^T = {}^{B}_{A}R^T$$

* Homogeneous Transform

$$T = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_B \\ \hline 0\;0\;0 & 1 \end{array}\right] \Longrightarrow \underset{(4\times1)}{{}^{A}P} = \underset{(4\times4)}{{}^{A}_{B}T}\;\underset{(4\times1)}{{}^{B}P}$$

* We can use the homogeneous transform to:
  - Describe one frame relative to another
  - Map vectors from one frame to another