3. Transformations

<Granular jamming gripper>

Recap:

Rotation matrices

$$ {}^A_B R = \begin{bmatrix} \boxed{\hat{x}_A \quad \hat{y}_B\cdot\hat{x}_A \quad \hat{z}_B\cdot\hat{x}_A} \\ \hat{x}_B\cdot \quad \hat{y}_B\cdot\hat{y}_A \quad \hat{z}_B\cdot\hat{y}_A \\ \hat{x}_B\cdot\hat{z}_A \quad \hat{y}_B\cdot\hat{z}_A \quad \hat{z}_B\cdot\hat{z}_A \end{bmatrix} \Rightarrow {}^B\hat{X}_A^T $$

$$ \Longrightarrow {}^A_B R = \left[{}^A\hat{X}_B \; {}^A\hat{Y}_B \; {}^A\hat{Z}_B\right] = \left[{}^B\hat{X}_A \; {}^B\hat{Y}_A \; {}^B\hat{Z}_A\right]^T = {}^B_A R^T $$

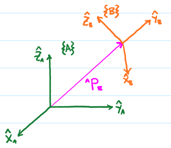
Homogenous transform

$$ T = \left[\begin{array}{c|c} {}^A_B R & {}^A P_B \\ \hline 0\;0\;0 & 1 \end{array}\right] \Longrightarrow \underset{(4\times1)}{{}^A P} = \underset{(4\times4)}{{}^A_B T}\;\underset{(4\times1)}{{}^B P} $$

* We can use the homogeneous transform for:

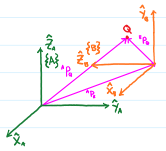
i) Description of a frame:

${}^A_B T : \{B\} = \{{}^A_B R, {}^A P_B\}$



ii) Transform mapping:

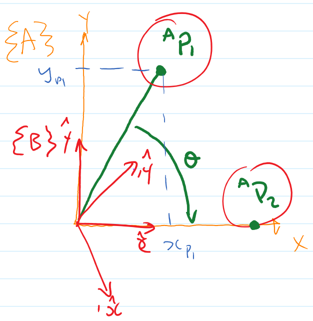
${}^A_B T : {}^B P_Q \longrightarrow {}^A P_Q$



c. Operators

* Moving points within the same frame

i Rotation



RH rule applies

A rotation matrix that rotates vectors through some rotation, R is the same as that which describes a frame rotated by R wrt the ref. frame

So the RH for direction always applies.

$= {}^B P_1$

$\longrightarrow$

So we can look at the rotation matrix as an operator in $\{A\}$

i.e. ${}^A_B R : {}^A P_1 \rightarrow {}^A P_2$

${}^A P_2 = R\,{}^A P_1$ where $R = {}^A_B R$

i.e. ${}^A_B R : {}^A P_1 \rightarrow {}^A P_2$

${}^A P_2 = R\, {}^A P_1$ where $R = {}^A_B R$

The process is the same, its just another way of viewing it.

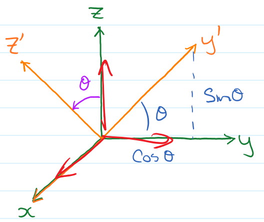
Rotational Operators — rotation around some arbitrary vector $\hat{k}$

Start with two coincident frames

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_k(\theta) : P_1 \rightarrow P_2$ Particularly when k is an axis

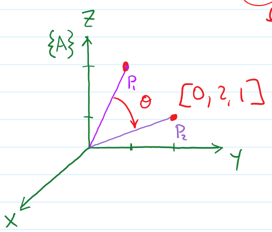
$P_2 = R_k(\theta)\, P_1$



Example : rotation around x-axis by θ

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

$C\theta = \cos\theta$
$S\theta = \sin\theta$



Recall : $R = [\, {}^A_B\hat{X} \;\; {}^A_B\hat{Y} \;\; {}^A_B\hat{Z} \,]$

$${}^A_B\hat{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad {}^A_B\hat{Y} = \begin{bmatrix} 0 \\ C\theta \\ S\theta \end{bmatrix} \quad {}^A_B\hat{Z} = \begin{bmatrix} 0 \\ -S\theta \\ C\theta \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

$${}^A P_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$\theta = \boxed{-36.87^\circ}$

$${}^A R_x(-36.87) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & -0.6 & 0.8 \end{bmatrix}$$

$$R_x(-36.87)\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

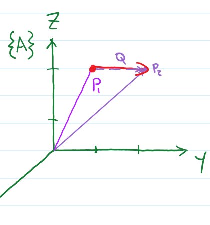
$${}^A P_2 = {}^A R_x(\theta)\, {}^A P_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

ii Translation

We can have translational operators as well



${}^A Q : {}^A P_1 \longrightarrow {}^A P_2$

${}^A P_2 = {}^A P_1 + {}^A Q$

2 points, 2 different vectors, same frame

Homogenous Transform representation — translation vector

$$D_Q = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^A P_2 = {}^A D_Q\, {}^A P_1$$

$$D_Q = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^A P_2 = {}^A D_Q \, {}^A P_1$$

I

no rotation

iii General Operator

Obviously, we can combine the rotation and translation operators

rotation

translation

$$P_2 = \left[\begin{array}{c|c} R_k(\theta) & Q \\ \hline 0\ 0\ 0 & 1 \end{array}\right] P_1$$

$${}^A P_2 = T \, {}^A P_1$$

ALWAYS Pre-multiply (for column vectors)

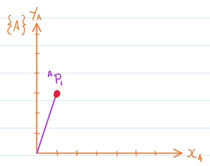
This is the same homogenous transform as we've used to describe and map - Just a different interpretation

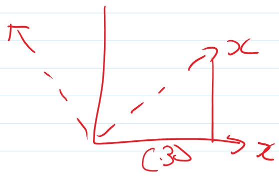
Example:

We have a point ${}^A P_1$ that we wish to rotate around the Z-axis by 30° and translate by 3 units in the x-direction and 1 unit in the y-direction

Find ${}^A P_2$

$${}^A P_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$





$$\begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^A R_Z(30) = \begin{bmatrix} C30 & -S30 & 0 \\ S30 & C30 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^A Q = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

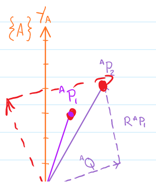
$$T = \begin{bmatrix} 0.866 & -0.5 & 0 & 3 \\ 0.5 & 0.866 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

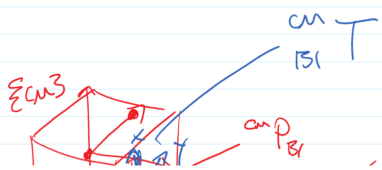
$$\Rightarrow T = \begin{bmatrix} 0.866 & -0.5 & 0 & 3 \\ 0.5 & 0.866 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

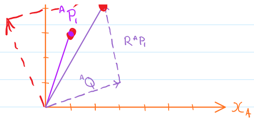
$${}^A P_2 = T \, {}^A P_1$$

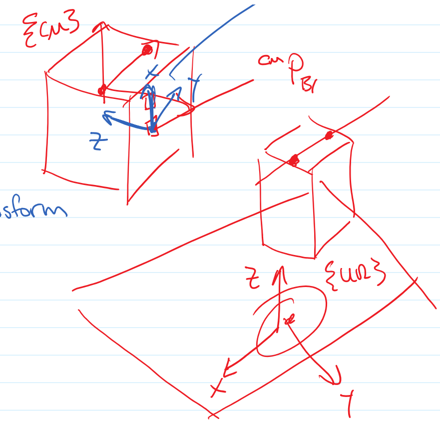
$$= \begin{bmatrix} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \ \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.37 \\ 4.10 \\ 0 \\ \cancel{1} \end{bmatrix}$$





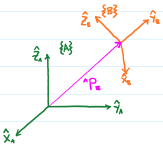




* Interpretations of the homogeneous transform

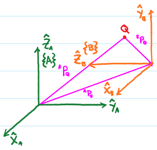
i) Description of a frame:

$^A_BT : \{B\} = \{{}^A_BR, {}^A_BP\}$

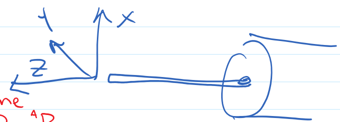


ii) Transform mapping:

$^A_BT : {}^BP_Q \longrightarrow {}^AP_Q$

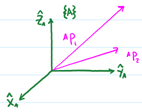


$^{UR}P_{B1} = {}^{UR}_{CM}T\;{}^{CM}P_{B1}$



iii) Transform operator:

$T : {}^AP_1 \longrightarrow {}^AP_2$



Can operate on a frame

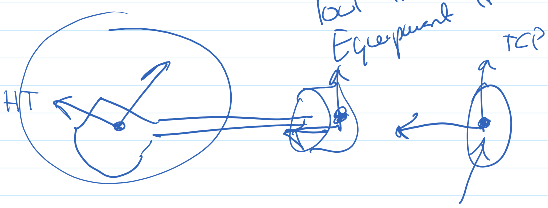
$R : \{A\} \longrightarrow \{B\} \Rightarrow R = {}^A_BR$

$\boxed{^{UR}_{B1}T} = \boxed{^{UR}_{CM}T}\,\boxed{^{CM}_{B1}T}$

where we send the robot

Tool frames

Equipment frames



D. Inversion

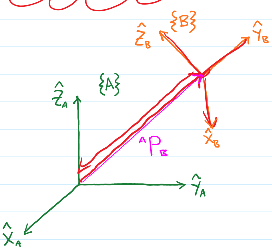
Recall the inverse of a rotation matrix:

$^A_BR^{-1} = {}^B_AR = {}^A_BR^T$

It's not quite so simple for an HT

$$^A_BT = \left[\begin{array}{c|c} ^A_BR & ^AP_B \\ \hline 0\;0\;0 & 1 \end{array}\right]$$

$$T^{-1} \neq T^T$$



^A_BT

$^A_BR^T = {}^B_AR$

$$^A_BT^{-1} = {}^B_AT = \left[\begin{array}{c|c} \boxed{^A_BR^T} & -{}^A_BR^T\,{}^AP_B \\ \hline 0\;0\;0 & 1 \end{array}\right] \; ? \quad -{}^B_AR\;{}^AP_B$$

Example:

Invert the Transform operator from before:

$$R_Z(30^\circ) = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$T = \left[\begin{array}{c|c} R_Z(30^\circ) & Q \end{array}\right]$$

$$T = \begin{bmatrix} R_z(30^\circ) & Q \\ 0\,0\,0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} R_z(30)^T & \boxed{-R_z(30)^T Q} \\ 0\,0\,0 & 1 \end{bmatrix}$$

$$R_z(30^\circ)^T = \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-R_z(30^\circ)^T Q = -\begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3.1 \\ 0.63 \\ 0 \end{bmatrix}$$

$$\Rightarrow T^{-1} = \begin{bmatrix} 0.866 & 0.5 & 0 & -3.1 \\ -0.5 & 0.866 & 0 & 0.63 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is exactly what you get if you use >> inv(T) in Matlab

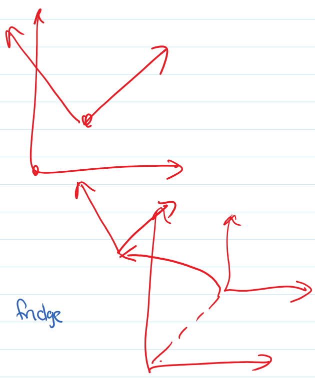
To test it out:

$${}^AP_1 = T^{-1}\,{}^AP_2$$

$$= \begin{bmatrix} 0.866 & 0.5 & 0 & -3.1 \\ -0.5 & 0.866 & 0 & 0.63 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2.37 \\ 4.10 \\ 0 \\ 1 \end{bmatrix}$$

answer from previous

$$= \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \checkmark$$

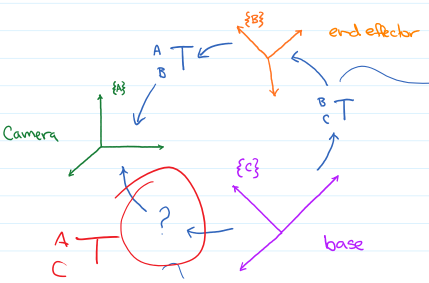


E. Transform Equation (Compound transformations)

Lets say that you have a camera pointed at your fridge and you intend to "improve" your turtlebot

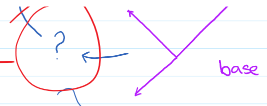
Your camera gives you the position and orientation of the end effector in its frame

You know the relationship between the base and end effector



Arrows indicate can transform points from – to

^A_BT transforms points from $\{B\}$ to $\{A\}$

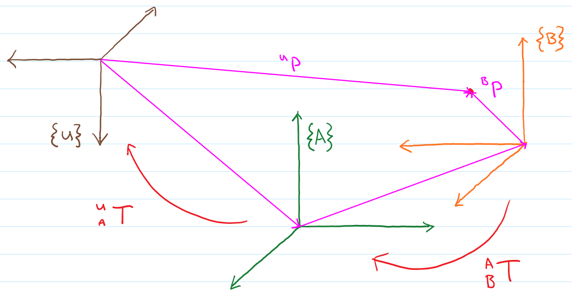
${}^A_CT$ 

is $\{B\}$ to $\{A\}$

want to know location of base in camera frame

The transform equation allows you to "close the loop" and calculate ${}^A_CT$, giving the position and orientation of the base in the camera frame

We touched on this earlier when introducing the HT



$${}^uP = {}^u_AT\,{}^AP \qquad {}^AP = {}^A_BT\,{}^BP$$

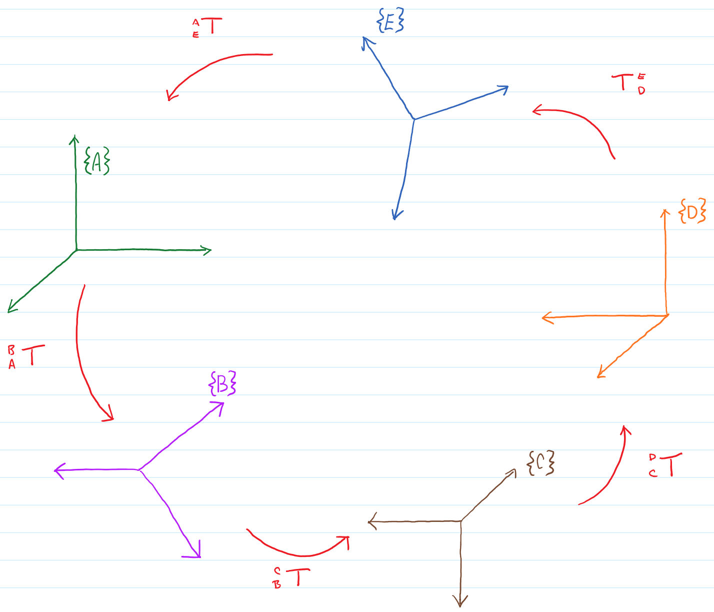
$$= {}^u_AT\left({}^A_BT\,{}^BP\right)$$

$$\Rightarrow {}^u_BT = {}^u_AT\,{}^A_BT$$ * Cancel the super/sub-scripts

$${}^u_BT = \begin{bmatrix} {}^u_AR\,{}^A_BR & {}^u_AR\,{}^AP_B + {}^uP_A \\ 0\ 0\ 0 & 1 \end{bmatrix} \;?$$

If we use the HT, all this messy-ness is automatically taken care of

With a few more frames...



If we walk all the way around this loop, from A to A we should obtain an HT that is?

We can see:

Q $${}^A_ET\,{}^E_DT\,{}^D_CT\,{}^C_BT\,{}^B_AT = {}^A_AT\;? = I$$

This is called the transform equation

Say we know four of these transforms and want to find the fifth, ${}^A_ET$

multiply both sides by inverse

$$\left({}^A_ET\right)^{-1}\,{}^A_ET\,{}^E_DT\,{}^D_CT\,{}^C_BT\,{}^B_AT = \left({}^A_ET\right)^{-1}$$

The [illegible], ${}^{E}T$

$$({}^{A}_{E}T)^{-1}\,{}^{A}_{E}T\,{}^{E}_{D}T\,{}^{D}_{C}T\,{}^{C}_{B}T\,{}^{B}_{A}T = ({}^{A}_{E}T)^{-1}$$

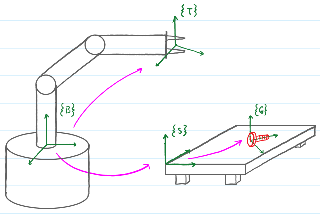
multiply both sides by inverse

$$ {}^{E}_{D}T\,{}^{D}_{C}T\,{}^{C}_{B}T\,{}^{B}_{A}T = {}^{E}_{A}T$$

now we can solve for this

Example:

We know ${}^{B}_{T}T$ from the kinematics, ${}^{B}_{S}T$ from the design of the workspace, and ${}^{S}_{G}T$



know

${}^{B}_{T}T$

${}^{B}_{S}T$

${}^{S}_{G}T$

want ${}^{T}_{G}T$

What is the position and orientation of the screw relative to the end effector, ${}^{T}_{G}T$?

$${}^{T}_{G}T = \cancel{{}^{T}_{B}T\;{}^{B}_{S}T\;{}^{S}_{G}T}$$

$$({}^{B}_{T}T)^{-1}\;{}^{B}_{S}T\;{}^{S}_{G}T = {}^{T}_{G}T$$

Summary:

* Interpretations of the homogeneous transform

i) Description of a frame: ${}^{A}_{B}T$: $\{B\} = \{{}^{A}_{B}R, {}^{A}_{B}P\}$

ii) Transform mapping: ${}^{A}_{B}T : {}^{B}P_Q \rightarrow {}^{A}P_Q$

iii) Transform operator: $T : {}^{A}P_1 \rightarrow {}^{A}P_2$

operator to rotate a frame is the same as the transform to describe the rotated frame in the original frame

* Inversion of the HT

$$T^{-1} \neq T^{T}$$

$${}^{A}_{B}T = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_B \\ \hline 0\;0\;0 & 1 \end{array}\right]$$

$${}^{A}_{B}T^{-1} = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}P_B \\ \hline 0\;0\;0 & 1 \end{array}\right] = {}^{B}_{A}T$$

* Transform equation:

$${}^{A}_{E}T\;{}^{E}_{D}T\;{}^{D}_{C}T\;{}^{C}_{B}T\;{}^{B}_{A}T = I$$

Can be used to find an unknown T