Industrial Mathematics Group Project 1

Group 1

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Executive Summary

The purpose of this report is to investigate how sand piles up on one end of a conveyor belt. We explore the impact of diffusion due to vibrations, the optimal speed and length of the conveyor belt, and delivery method to best reduce the height of this pile to below one metre.

We model the height of the sand using the one-dimensional advection-diffusion equation with a source term for delivering sand and appropriate boundary conditions for sand lost at the ends of the belt. We examine this model using both analytical and numerical methods for a constant delivery of sand, for which both solutions match at a large timescale. We use our numerical solution to model the changes to the sand pile for non-constant and non-uniform delivery of sand.

We find the height of the sand pile reduces when the length of the conveyor belt is decreased, and the speed and diffusion are increased. We find that delivering sand uniformly across the length of the conveyor belt at a constant rate is the most effective delivery method for keeping the height of the sand pile as low as possible, while reducing the amount of sand lost at the start of the conveyor belt. Sanderson and Sons should use a conveyor belt five metres long which runs at 0.5m/s or faster. We cannot specify the exact delivery rate of sand as the height of the pile is dependent on the diffusion coefficient, for which future work would be needed to determine.

1 Introduction: The Model

To model the sand on the conveyor belt we use a 1-Dimensional (1D) advection-diffusion equation on a finite domain, measuring the height, $H(x, t)$, of sand. We work in space and time where $5 \leq L \leq 10$ is the length of the conveyor belt, $0 \leq x \leq L$ is the position on the conveyor belt and time $t \geq 0$. The 1D advection-diffusion equation can be written as

$$\partial_t H(x, t) + V \partial_x H(x, t) = D \partial_{xx} H(x, t) + S(x, t), \quad 0 \leq x \leq L, \tag{1}$$

subject to initial and boundary conditions. The source term $S(x, t)$ represents sand being delivered to the conveyor belt via a hopper, this is a function of space and time. D is the diffusion coefficient due to vibrations and 0.25 m/s $\leq V \leq$ 0.5m/s is the speed of the conveyor belt. To begin analysing the model we make the following assumptions:

- Sand is added to the conveyor belt instantaneously, and sand can only enter the system from the source term.
- The speed V is constant over the belt, and it is constant with respect to x.
- At initial time $T = 0$ there is no sand already on the conveyor belt.
- We can only lose sand at either end of the belt, so $H(0, t) = H(L, t) = 0$.

We have the initial condition $H(x, 0) = 0$ and Dirichlet boundary conditions $H(0, t) = H(L, t) = 0$. We are interested in how the system acts as t gets large (the steady state solution), or when $\partial_t H(x, t) = 0$. To set up a toy model we set $S(x, t) = S > 0$ to be constant (delivering a uniform layer of sand over the whole belt at a constant rate) and we will investigate how changing V, D and L affects the build up of sand analytically and numerically using

$$V \partial_x H(x, t) = D \partial_{xx} H(x, t) + S, \quad 0 \leq x \leq L. \tag{2}$$

2 Analytical solution of the steady state

To solve our toy model (2), consider a solution of the form

$$H(x, t) = Ae^{\frac{V}{D}x} + \frac{S}{V}x + \gamma.$$

where A and γ are constants to be found. Differentiating $H(x, t)$ twice we have

$$\partial_x H(x, t) = \frac{V}{D} Ae^{\frac{V}{D}x} + \frac{S}{V} \quad \text{and} \quad \partial_{xx} H(x, t) = \frac{V^2}{D^2} Ae^{\frac{V}{D}x}. \tag{3}$$

Substituting these partial derivatives into equation (2) we get

$$V(\frac{V}{D} Ae^{\frac{V}{D}x} + \frac{S}{V}) = D(\frac{V^2}{D^2} Ae^{\frac{V}{D}x}) + S \Rightarrow \frac{V^2}{D} Ae^{\frac{V}{D}x} + S = \frac{V^2}{D} Ae^{\frac{V}{D}x} + S,$$

confirming our choice of $H(x,t)$ as a solution to (2). Using the boundary conditions it can be shown that $H(0,t) = A + \gamma = 0 \Rightarrow A = -\gamma$ and

$$H(0,L) = Ae^{\frac{VL}{D}} + \frac{SL}{V} + \gamma = 0 \Rightarrow \gamma = -\frac{SL}{V(1-e^{\frac{VL}{D}})} \Rightarrow A = \frac{SL}{V(1-e^{\frac{VL}{D}})},$$

which gives us the complete solution

$$H(x,t) = \frac{SL(e^{\frac{V}{D}x}-1)}{V(1-e^{\frac{VL}{D}})} + \frac{S}{V}x = \beta(e^{\frac{V}{D}x}-1) + \frac{S}{V}x, \tag{4}$$

where $\beta = \frac{SL}{V(1-e^{\frac{VL}{D}})}$ and H is completely independent of t.

3 Numerical solution of the steady state

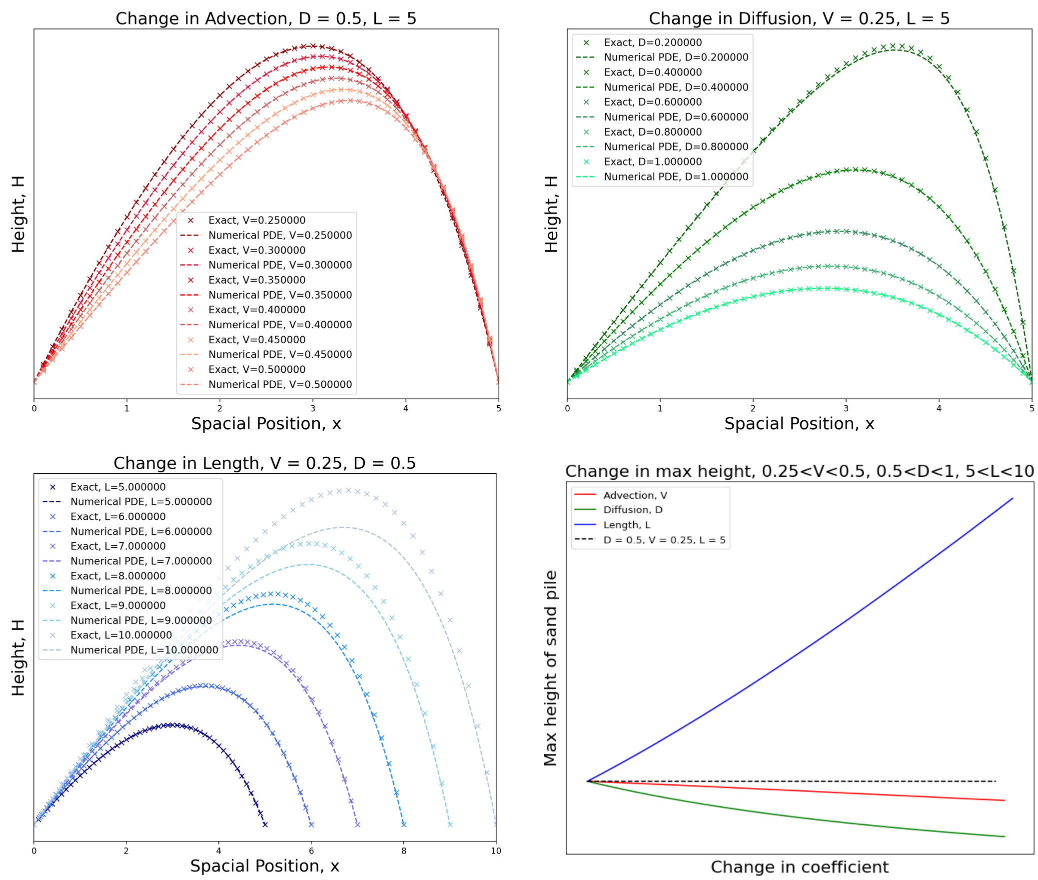

Figure 1: Visualisation of the sand pile at its steady sate for changing parameters

Using forward Euler we find the finite difference equation to be

$$H_m^{n+1} = (\frac{\alpha}{2} + \sigma)H_{m-1}^n + (1-2\sigma)H_m^n + (\sigma - \frac{\alpha}{2})H_{m+1}^n + S_m^n \Delta t, \tag{5}$$

where $\alpha = V\frac{\Delta t}{\Delta x}$ and $\sigma = D\frac{\Delta t}{(\Delta x)^2}$ [1]. For stability we require $\alpha \leq 1$ and $\sigma \leq \frac{1}{2}$. By implementing this solution in python and plotting it with the exact solution (Figure 5.1) we see that:

- The numerical and analytical solutions match for a large timescale, validating our solutions.
- The sand does indeed pile up toward the end of the conveyor belt, validating our model.
- Increasing Length increases the maximum height of the sand pile and decreases accuracy of the numerical solution.
- Increasing speed and diffusion decreases the maximum height of the sand pile.

Therefore, we will fix the length to the minimum of 5m and speed to the maximum of 0.5m/s. We would suggest increasing the speed further if you want to increase the rate of delivery. Note: In implementation we treat $S(x,t)$ as constant and deliver an arbitrary amount of sand uniformly at each timestep, scaling if we change L - this gives us an arbitrary height of the pile so more work is needed to determine a suitable diffusion coefficient to allow choice of S which reduces the height to $H \leq 1$m.

4 Industry Research

Unfortunately there is not much data similar to this problem so our solutions cannot be validated. However, we found a typical 6m conveyor belt runs at speed 0.75m/s [3]. We also found a diffusion coefficient for sand to be 4.29 x 10^{-6} [4]. This gives us an indication that the the diffusion coefficient of sand due to vibrations may be very small relative to the advection coefficient. Although this is useful when calculating a realistic pile height we will instead use $D = 0.5$ so that experiments are easily distinguishable.

5 Exploring the optimal solution with $S(x,t)$

We will now explore how changing our source term with respect to space and time impacts the height of the sand pile.

5.1 Changing $S(x,t)$ as a function of space

We write our $S(x,t)$ functions in Python to be delivering the same total volume of sand at each time step so that the results are comparable, even though in reality there may be a limit on the volume deliverable. We investigate dropping sand at increasing intervals from length 1m to 5m which always start at $x = 0$; dropping sand at moving intervals of length 1m; and dropping sand in a few other patterns (sine, cosine, and every other x).

From the graphs in Figure 2 we see a few delivery methods which are feasible to implement and minimise height: For sand delivery at the last metre ($x \in [4,5]$) we have a very small maximum height, but this model makes little practical sense. For sand delivery in the first

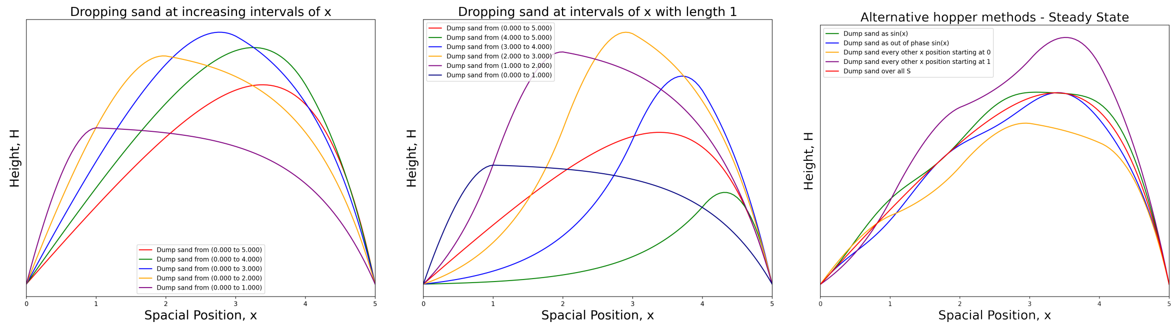

Figure 2: Visualisation of the sand pile at its steady state for alternative hopper methods

metre ($x \in [0, 1]$) we see a flat curve and a low maximum height, however the steep gradient between 0 and 1 suggests we are losing lots of sand at the $x = 0$ end. For sand delivery uniform in x and sand delivery every other x starting at $x = 0$ we find similar heights, but would like to distinguish between them more rigorously. This gives us motivation to study the sand lost (flux) at the $x = 0$ end.

5.2 Changing $S(x, t)$ as a function of time

For this section we will deliver sand uniformly over space which allows us to implement some time-delivery methods as seen in Figure 3.

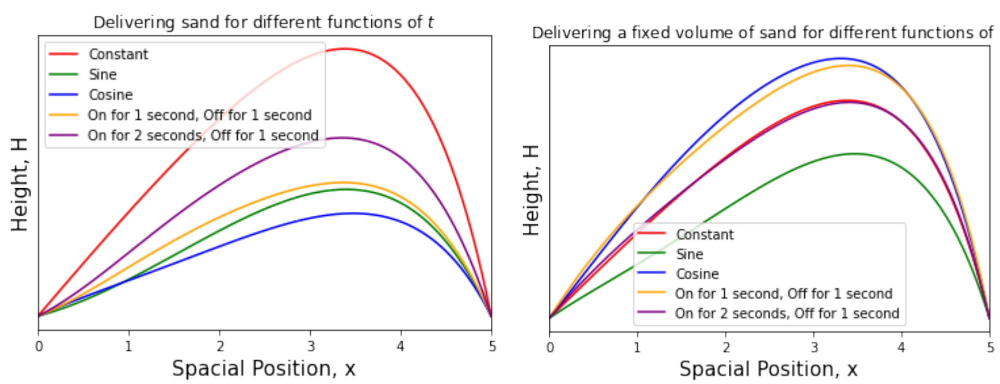

Figure 3: Dropping sand non-uniformly in time

We see that time-delivery methods with more interruption give rise to a smaller height. For example, pausing delivery every other second ("pulse") decreases the height by approximately a half. This is expected, but not optimal.
With this in mind we fix the total volume of sand delivered over a 10π second time period to be equal for all delivery methods. We find that for constant delivery over time, the sand pile gradually increases and reaches a maximum. All other methods oscillate above and below the maximum height of the constant delivery. The sine and cosine methods are adjusted to be out of phase, showing the maximum variation around the constant delivery method. If the sand pile getting too large impacts other machinery or interferes with Sanderson and Sons' business then we suggest using the constant delivery method.

5.3 Flux of sand at the belt ends

By sections 5.1 and 5.2 we would like to pick the best of a few delivery methods with similar heights. This section focuses on understanding how much sand is falling off either end of the belt, to reduce the volume of sand falling off the wrong ($x = 0$) end of the belt.
Let $\Phi(x)$ denote the total flux of sand at a given point x on the belt. In the steady state, movement of sand is driven solely by advection and diffusion, and $\Phi(x)$ is obtained by adding up the advective and diffusive components of the flux [2]:

$$\Phi(x) = \Phi_{advec} + \Phi_{diff} = VH(x) - D\partial_x H(x).$$

By substituting this into (3) and (4) we get the result

$$\Phi(x) = V\left(\frac{SL(e^{\frac{V}{D}x} - 1)}{V(1 - e^{\frac{VL}{D}})} + \frac{S}{V}x\right) - D\left(\frac{\frac{SLV}{D}e^{Vx/D}}{V(1 - e^{\frac{VL}{D}})} + \frac{S}{V}\right) = ... = Sx - \frac{1}{1 - e^{\frac{VL}{D}}} - \frac{DS}{V}. \quad (6)$$

This is a well behaved linear result, which we can sanity check. At the steady state, the volume of sand being delivered per unit time and the volume falling off the ends per unit time must be equal, or

$$|\Phi(0)| + |\Phi(L)| = \left| -\frac{1}{1 - e^{\frac{VL}{D}}} - \frac{DS}{V}\right| + \left|SL - \frac{1}{1 - e^{\frac{VL}{D}}} - \frac{DS}{V}\right| = ... = SL,$$

which is expected. By (1) and using the boundary conditions in (2) we can compute the total flux at the left end of the belt as $\Phi(0) = VH(0) - D\partial_x H(0) = -D\partial_x H(0)$, where $\partial_x H(0)$ is approximated by $\frac{H_2^N - H_1^N}{\Delta x}$. Using this we calculate some important fluxes when $V = 0.25, D = 0.5, L = 5$: From this we gather that even though delivering sand in the first

Total Sand lost at $x = 0$ from $T = 0$ to $T = 10\pi$ for different delivery methods						
Method in x or t	$x \in [0, 5]$	$x \in [0, 1]$	$x \in [0, 1], [2, 3], [4, 5]$	Uniform t	"Pulse"	$sin(t)$
Sand lost (ms^{-1})	27.2	88.4	33.3	27.2	27.1	43.5

metre or every other metre reduces height it is at the expense of losing more sand at the wrong end, so a uniform approach may be beneficial and easier to implement. We can also see that the uniform in t and a "pulse" method are almost identical, but a uniform method will be easier to implement, so we would suggest delivering sand constant in time.

6 Conclusion and Recommendations

The main conclusion from our investigation is that to minimise the height of the sand pile speed (V) must be maximised to $0.5ms^{-1}$, or more, and Length (L) must be minimised to $5m$. The realistically optimal method of delivering sand is to distribute it evenly across the length of the conveyor belt, uniformly in time. To reduce sand lost at the beginning of the conveyor belt, sand should be dropped at $x \geq 1$ uniformly in time. As the diffusion coefficient (D) is caused by vibrations in the conveyor belt, future work should be done to investigate an accurate order of magnitude for this value as it greatly affects the height of the sand pile. Furthermore, we cannot accurately give a rate of delivery which results in w$H \leq 1$ as this is dependent on D also.

References

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