Math 454
Fall 2020

Project 3
Dec 4, 2020

Team Leader: (William Wang) **Team Member**: (Daniel Yang)

Database:

Note: Some of the recurrence operators, critical exponents, and asymptotics were either unable to be found or not applicable, thus left empty (ex: 8, where every other number in the sequence is 0)

1. **[1,0,0],[0,1,0],[0,0,1],[2,0,0],[0,2,0],[0,0,2],[1,1,1],[2,2,2]**

Sequence:

$$[1, 7, 248, 9741, 426719, 19725956, 945573793, 46496604627, 2330130198628]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,0,0], [0,1,0], [0,0,1], [2,0,0], [0,2,0], [0,0,2], [1,1,1], and [2,2,2]

Not in the OEIS

Good Sequence:

$$[1, 2, 20, 328, 7480, 203176, 6211182, 206714074, 7336899407, 273892391945]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,0,0], [0,1,0], [0,0,1], [2,0,0], [0,2,0], [0,0,2], [1,1,1], [2,2,2] such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

Not in the OEIS

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2. **[1,0,0],[0,1,0],[0,0,1],[1,1,1],[2,2,2]]**

Sequence:

$$[1, 7, 116, 2397, 54845, 1329644, 33464881, 864627351, 22776683200, 609024723535]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,0,0], [0,1,0], [0,0,1], [1,1,1], [2,2,2].

Recurrence Operator:

$$\frac{-(3*n+17)*(n+2)^2}{((3*n+11)*(n+6)^2)} - \frac{(3*n+17)*(9*n^3+87*n^2+269*n+272)*N}{((3*n+14)*(3*n+11)*(n+6)^2} - \frac{(3*n+16)*(3*n+7)*N^2}{((3*n+14)*(3*n+11)*(n+6)^2)} + \frac{(45*n^4+780*n^3+4997*n^2+14044*n+14640)*N^3}{((3*n+14)*(3*n+11)*(n+6)^2)} + \frac{(3*n+16)*(3*n+13)*N^4}{((3*n+14)*(3*n+11)*(n+6)^2)} - \frac{(3*n+16)*(30*n^2+310*n+793)*N^5}{((3*n+14)*(n+6)^2)+N^6}$$

Growth Constant (Estimated): 29.994446134523878569082844977816299001754214184457
Critical Exponent: -1

Not published in the OEIS previously
Submitted and published in the OEIS: A339390

Good Sequence:

$$[1, 2, 11, 94, 1102, 15555, 248239, 4324125, 80451430, 1575855961, 32170583918]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,0,0], [0,1,0], [0,0,1], [1,1,1], [2,2,2] such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

Recurrence Operator:

$$\frac{-(n-3)*(n-8)*(3*n+17)}{((3*n+11)*(n+8)*(n+7))} - (3*n+17)*(9*n^3-21*n^2) - \frac{226*n+224)*N}{((3*n+11)*(n+8)*(n+7)*(3*n+14))} - \frac{28*(3*n+16)*(3*n+7)*N^2}{((3*n+11)*(n+8)*(n+7)*(3*n+14))}$$

$$+\frac{(45*n^4+510*n^3+1271*n^2-2498*n-8960)*N^3}{((3*n+11)*(n+8)*(n+7)*(3*n+14))}$$
$$+\frac{28*(3*n+16)*(3*n+13)*N^4}{((3*n+11)*(n+8)*(n+7)*(3*n+14))}$$
$$-\frac{2*(3*n+16)*(15*n^2+155*n+392)*N^5}{((n+8)*(n+7)*(3*n+14))}+N^6$$

Growth Constant (estimated): 29.994446134523878569082844977816299001754214184457
Critical Exponent: -4

Not in the OEIS

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3. **[2,1,0],[1,2,0],[0,2,1],[0,1,2],[1,0,2],[2,0,1]**

Sequence:

$[1, 0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, 8676360, 47977776, 266378112]$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the permutations of the numbers 0,1,2 as atomic steps, i.e. the atomic steps [2,1,0],[1,2,0],[0,2,1],[0,1,2],[1,0,2],[2,0,1].

Recurrence Operator:

$$\frac{-36*(n+2)*(n+1)}{(n+3)^2} - \frac{24*(n+2)^2*N}{(n+3)^2} - \frac{(n+2)*N^2}{(n+3)} + N^3$$

Growth Constant (Estimated): 6
Critical Exponent: -1

In the OEIS: A002898, but our description of the sequence was not mentioned previously
Submitted and published a new comment for A002898, which was our description

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Good Sequence:

$[1, 0, 1, 0, 4, 4, 31, 76, 376, 1332, 5994, 24828, 112016, 500044, 2313815, 10787288, 51270984, ...]$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [2,1,0], [1,2,0], [0,2,1], [0,1,2], [1,0,2], [2,0,1] such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

Not in the OEIS

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4. **[1,2,3],[1,3,2],[2,1,3],[2,3,1],[3,1,2],[3,2,1]**

Sequence:

$$[1, 0, 0, 0, 6, 0, 12, 0, 90, 0, 360, 0, 2040, 0, 10080, 0, 54810, 0, 290640, 0, 1588356]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using permutations of the numbers 1,2,3 as atomic steps, i.e. the atomic steps [1,2,3], [1,3,2], [2,1,3], [2,3,1], [3,1,2], [3,2,1]

Not in the OEIS

Good Sequence:

$$[1, 0, 0, 0, 1, 0, 0, 0, 4, 0, 4, 0, 31, 0, 76, 0, 376, 0, 1332, 0, 5994]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,2,3], [1,3,2], [2,1,3], [2,3,1], [3,1,2], [3,2,1] such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

Not in the OEIS

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5. **[1,0,0],[0,1,0],[0,0,1],[2,0,0],[0,2,0],[0,0,2],[3,0,0],[0,3,0],[0,0,3],[1,1,1],[2,2,2],[3,3,3]**

Sequence:

$$[1, 7, 248, 11380, 560089, 29125351, 1569958128, 86788339340, 4888825879881]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,0,0], [0,1,0], [0,0,1], [2,0,0], [0,2,0], [0,0,2], [3,0,0], [0,3,0], [0,0,3], [1,1,1], [2,2,2], [3,3,3].

Not in the OEIS

Good Sequence:

$$[1, 2, 20, 392, 10076, 308794, 10635713, 398441947, 15910609458, 668028916680]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,0,0], [0,1,0], [0,0,1], [2,0,0], [0,2,0], [0,0,2], [3,0,0], [0,3,0], [0,0,3], [1,1,1], [2,2,2], [3,3,3] such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

Not in the OEIS

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6. **[1,1,0],[1,0,1],[0,1,1]**

Sequence:

$$[1, 0, 6, 0, 90, 0, 1680, 0, 34650, 0, 756756, 0, 17153136, 0,$$

$$399072960, 0, 9465511770, 0, 227873431500, 0, 5550996791340]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,1,0],[1,0,1],[0,1,1].

Not in the OEIS

Good Sequence:

$$[1, 0, 1, 0, 5, 0, 42, 0, 462, 0, 6006, 0, 87516, 0, 1385670, 0, 23371634, 0, 414315330, 0]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,1,0],[1,0,1],[0,1,1] such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

Not in the OEIS

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7. **[1,1,0],[1,0,1],[0,1,1],[1,1,1]**

Sequence:

$$[1, 1, 7, 25, 151, 751, 4411, 24697, 146455, 862351, 5195257, 31392967, 191815339]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,1,0],[1,0,1],[0,1,1],[1,1,1].

Recurrence Operator:

$$\frac{-(3*n+7)*(n+1)^2}{((3*n+4)*(n+3)^2)} - \frac{(3*n+5)*(24*n^2+88*n+75)*N}{((3*n+4)*(n+3)^2)} - \frac{(9*n^3+57*n^2+116*n+74)*N^2}{((3*n+4)*(n+3)^2)} + N^3$$

Critical Exponent: -1

In the OEIS: A208425, but our description of the sequence was not mentioned previously
Submitted and published a new comment for A208425, which was our description

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Good Sequence:

$$[1, 1, 2, 5, 16, 56, 218, 897, 3907, 17677, 82864, 399191, 1970684, 9928426, 50931050]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,1,0], [1,0,1], [0,1,1], [1,1,1] such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

Recurrence Operator:

$$\frac{-n*(3*n+7)*(n-1)}{((n+5)*(3*n+4)*(n+7))} - \frac{4*(3*n+5)*(6*n^2+22*n+21)*N}{((n+5)*(3*n+4)*(n+7))} - \frac{(n+1)*(9*n^2+75*n+140)*N^2}{((n+5)*(3*n+4)*(n+7))} + N^3$$

Critical Exponent: -4

Not in the OEIS

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8. **[1,1,0],[1,0,1],[0,1,1],[2,2,0],[2,0,2],[0,2,2]**

Sequence:

$$[1, 0, 6, 0, 222, 0, 8280, 0, 347850, 0, 15381828, 0, 705379416, 0, 33176670912, 0, 1590179139450, \\ 0, 77338582832940, 0, 3805317108650772]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,1,0], [1,0,1], [0,1,1], [2,2,0], [2,0,2], [0,2,2].

Not in the OEIS

Good Sequence:

$$[1, 0, 1, 0, 14, 0, 227, 0, 5095, 0, 133766, 0, 3939013, 0, 125968801, 0, 4290568003, \\ 0, 153574639342, 0, 5721989787415]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [1,1,0], [1,0,1], [0,1,1], [2,2,0], [2,0,2], [0,2,2] such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

Not in the OEIS

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9. **[3,1,0],[1,3,0],[0,3,1],[0,1,3],[1,0,3],[3,0,1]**

Sequence:

$$[1, 0, 0, 0, 12, 0, 0, 0, 900, 0, 0, 0, 124320, 0, 0, 0, 20404692, 0, 0, 0, 3565834272]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by moving through the 3D lattice like a long chess knight in 3D (3 steps in one direction, 1 in perpendicular direction), i.e. using the atomic steps [3,1,0], [1,3,0], [0,3,1], [0,1,3], [1,0,3], [3,0,1]

Not in the OEIS

Good Sequence:

$$[1, 0, 0, 0, 0, 0, 0, 0, 6, 0, 0, 0, 368, 0, 0, 0, 30305, 0, 0, 0, 2914078]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [3,1,0], [1,3,0], [0,3,1], [0,1,3], [1,0,3], [3,0,1] such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

Not in the OEIS

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10. **[i,0,0],[0,i,0],[0,0,i], i=1..n**

Sequence:

$$[1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, 4223303759148]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by moving through the 3D lattice like a chess rook in 3D, i.e. the atomic steps [n,0,0], [0,n,0], [0,0,n],n=1..i

In the OEIS: A144045

Good Sequence:

$$[1, 1, 14, 290, 7680, 238636, 8285506, 312077474, 12509563082, 526701471002]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [n,0,0],[0,n,0],[0,0,n],n=1..i such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$

In the OEIS: A227580

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11. **[i,0,0],[0,i,0],[0,0,i],[i,i,i],i=1..n**

Sequence:

$$[1, 7, 248, 11380, 577124, 30970588, 1724240804, 98508192580, 5736813639188]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by moving through the 3D lattice like a chess queen in 3D, i.e. the atomic steps [n,0,0],[0,n,0],[0,0,n],[n,n,n],n=1..i

In the OEIS: A229482

Good Sequence:

$$[1, 2, 20, 392, 10488, 333672, 11915064, 462573560, 19135907480, 832159886696]$$

Description:
Number of paths from (0,0,0) to (i,i,i) by only using the atomic steps [n,0,0], [0,n,0], [0,0,n], [n,n,n], n=1..i such that the first coordinate is always greater than or equal to the second coordinate, and the second coordinate is always greater than or equal to the third coordinate, i.e. $x \geq y \geq z$
Not in the OEIS

Sequence List Dictionary (lexicographic)

(4) [1, 0, 0, 0, 6, 0, 12, 0, 90, 0, 360, 0, 2040, 0, 10080, 0, 54810, 0, 290640, 0, 1588356]

(9) [1, 0, 0, 0, 12, 0, 0, 0, 900, 0, 0, 0, 124320, 0, 0, 0, 20404692, 0, 0, 0, 3565834272]

(6) [1, 0, 6, 0, 90, 0, 1680, 0, 34650, 0, 756756, 0, 17153136, 0, 399072960, 0, 9465511770, 0, 227873431500, 0, 5550996791340]

(8) [1, 0, 6, 0, 222, 0, 8280, 0, 347850, 0, 15381828, 0, 705379416, 0, 33176670912, 0, 1590179139450, 0, 77338582832940, 0, 3805317108650772]

(3) [1, 0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, 8676360, 47977776, 266378112]

(7) [1, 1, 7, 25, 151, 751, 4411, 24697, 146455, 862351, 5195257, 31392967, 191815339]

(10) [1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, 4223303759148]

(2) [1, 7, 116, 2397, 54845, 1329644, 33464881, 864627351, 22776683200, 609024723535]

(1) [1, 7, 248, 9741, 426719, 19725956, 945573793, 46496604627, 2330130198628]

(5) [1, 7, 248, 11380, 560089, 29125351, 1569958128, 86788339340, 4888825879881]

(11) [1, 7, 248, 11380, 577124, 30970588, 1724240804, 98508192580, 5736813639188]

Good Sequence List Dictionary (lexicographic)

(9) [1, 0, 0, 0, 0, 0, 0, 0, 6, 0, 0, 0, 368, 0, 0, 0, 30305, 0, 0, 0, 2914078]

(4) [1, 0, 0, 0, 1, 0, 0, 0, 4, 0, 4, 0, 31, 0, 76, 0, 376, 0, 1332, 0, 5994]

(3) [1, 0, 1, 0, 4, 4, 31, 76, 376, 1332, 5994, 24828, 112016, 500044, 2313815, 10787288, 51270984]

(6) [1, 0, 1, 0, 5, 0, 42, 0, 462, 0, 6006, 0, 87516, 0, 1385670, 0, 23371634, 0, 414315330, 0]

(8) [1, 0, 1, 0, 14, 0, 227, 0, 5095, 0, 133766, 0, 3939013, 0, 125968801, 0, 42905680030, 0, 153574639342, 0, 5721989787415]

(7) [1, 1, 2, 5, 16, 56, 218, 897, 3907, 17677, 82864, 399191, 1970684, 9928426, 50931050]

(10) [1, 1, 14, 290, 7680, 238636, 8285506, 312077474, 12509563082, 526701471002]

(2) [1, 2, 11, 94, 1102, 15555, 248239, 4324125, 80451430, 1575855961, 32170583918]

(1) [1, 2, 20, 328, 7480, 203176, 6211182, 206714074, 7336899407, 273892391945]

(5) [1, 2, 20, 392, 10076, 308794, 10635713, 398441947, 15910609458, 668028916680]

(11) [1, 2, 20, 392, 10488, 333672, 11915064, 462573560, 19135907480, 832159886696]

Interesting Patterns:

1. An interesting pattern that can be observed is that the atomic steps in 3 are nested within the atomic steps seen in 4

   For example:

   Same as atomic steps in 3

   (a) SeqW([0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0], 20);

   [0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, 8676360, 47977776, 266378112, 1488801600, 8355739392, 47104393050, 266482019232, 1512589408044, 8610448069080, 49144928795820]

   Same as atomic steps in 4

   (b) SeqW([1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1], 20);

   $$[0, 0, 0, 6, 0, 12, 0, 90, 0, 360, 0, 2040, 0, 10080, 0, 54810, 0, 290640, 0, 1588356]$$

   (c) SeqW([2, 3, 4], [2, 4, 3], [3, 2, 4], [3, 4, 2], [4, 2, 3], [4, 3, 2], 20);

   $$[0, 0, 0, 0, 0, 6, 0, 0, 12, 0, 0, 90, 0, 0, 360, 0, 0, 2040, 0,]$$

   (d) SeqW([3, 4, 5], [3, 5, 4], [4, 3, 5], [4, 5, 3], [5, 3, 4], [5, 4, 3], 20);

   $$[0, 0, 0, 0, 0, 0, 0, 6, 0, 0, 0, 12, 0, 0, 0, 90, 0, 0, 0, 360]$$

2. A Similar pattern can be seen with the same list in Good Walks

   For example:

   Same as atomic steps in

   (a) SeqGW([0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0], 20);

   [0, 1, 0, 4, 4, 31, 76, 376, 1332, 5994, 24828, 112016, 500044, 2313815, 10787288, 51270984, 246265136, 1198208064, 5887369312, 29212675530]

   Same as atomic steps in

   (b) SeqGW([1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1], 20);

   $$[0, 0, 0, 1, 0, 0, 0, 4, 0, 4, 0, 31, 0, 76, 0, 376, 0, 1332, 0, 5994]$$

   (c) SeqGW([2, 3, 4], [2, 4, 3], [3, 2, 4], [3, 4, 2], [4, 2, 3], [4, 3, 2], 20);

   $$[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 4, 0, 0, 4, 0, 0, 31, 0, 0]$$

   (d) SeqGW([3, 4, 5], [3, 5, 4], [4, 3, 5], [4, 5, 3], [5, 3, 4], [5, 4, 3], 20);

   $$[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 4]$$

Conjecture: By increasing the values of each atomic step by 1 in every list of atomic steps, a 0 is added before the original integer sequence and between every value of the original integer sequence. i.e. If $x_1, x_2, x_3, x_4, x_5, x_6, ...$ is some integer sequence corresponding to some list of atomic steps $\{[i], [j], [k]\}$, then the integer sequence corresponding to the list of atomic steps $\{[i\ ], [j\ ], [k\ ]\}$, where i is equal to i with 1 added to every value in i is $0, x_1, 0, x_2, 0, x_3, 0, x_4, 0, x_5, 0, x_6, ...$

Application Does this work for all lists of atomic steps? Let's test:

ListNo. 9

(a) SeqW([0, 1, 3], [0, 3, 1], [1, 0, 3], [1, 3, 0], [3, 0, 1], [3, 1, 0], 20);

[0, 0, 0, 12, 0, 0, 0, 900, 0, 0, 0, 124320, 0, 0, 0, 20404692, 0, 0, 0, 3565834272]

(b) SeqW([1, 2, 4], [1, 4, 2], [2, 1, 4], [2, 4, 1], [4, 1, 2], [4, 2, 1], 20);

$$[0, 0, 0, 0, 0, 0, 12, 0, 0, 0, 0, 0, 0, 900, 0, 0, 0, 0, 0, 0, 0]$$

Conjecture seems to work!