# Comparison of Numerical Integration Techniques

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## Abstract

Numerical integration, more-or-less synonymous with *quadrature*, is the process of determining the area of a particular figure or region[1]. While many numerical integration techniques exist, in this project I will discuss the application of a few numerical integration techniques: Gaussian quadrature, Romberg's method, and Tanh-Sinh quadrature.

Starting with the oldest technique, An n-point Gauss-Legendre quadrature rule involves an approximation of the definite integral of a function over an interval $[-1, 1]$ with suitable choices of nodes $x_i$ and weights $w_i$ for $i = 1, ..., n$. The rule is stated as

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i) \tag{1}$$

and is accurate for polynomials of degree $2n - 1$ or less[2]. Gaussian-Legendre quadrature rules are a subset of Gaussian quadrature, and while other Gaussian quadrature rules may perform better, they may be subject to other/additional conditions.

Next, Romberg's method is a Newton-Cotes formula, which is a well-known class of formulas for numerical integration. Romberg's method essentially involves approximating a definite integral on an interval $[a, b]$ at equally spaced points by applying Richardson extrapolation on the Trapezoid or Midpoint rule[3].

Lastly, Tanh-Sinh quadrature, or the Double-Exponential (DE) formula, first involves a change of variables $x = \tanh(\frac{1}{2}\sinh(t))$, where $f(x)$ is the integrand, transforming an integral on the interval $x \in [-1, 1]$ to the interval $t \in (-\infty, \infty)$[4]. The quadrature rule is stated as

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=-\infty}^{\infty} w_i f(x_i) \tag{2}$$

where $h$ is the step size, the weight $w_i$ is

$$w_i = \frac{\frac{1}{2}h\pi\cosh(ih)}{\cosh^2(\frac{1}{2}\pi\sinh(ih))} \tag{3}$$

and the node $x_i$ is

$$x_i = \tanh(\frac{1}{2}\pi\sinh(ih)) \tag{4}$$

By performing this change of variables, it conveniently makes the approximation resistant to endpoint behavior and in many cases causes the the approximation to quickly converge[5].

The main goals of this project will be to 1) investigate and compare the performance of each of the techniques, and 2) test each technique for various functions $f(x)$.

## References

[1] *Numerical integration* (2021). Retrieved March 28, 2021 from https://en.wikipedia.org/wiki/Numerical_integration

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[3] *Romberg's method* (2021). Retrieved March 28, 2021 from https://en.wikipedia.org/wiki/Romberg%27s_method

[4] D. Bailey. *Tanh-Sinh High Precision Quadrature* (2006). Retrieved March 27, 2021 from https://www.davidhbailey.com/dhbpapers/dhb-tanh-sinh.pdf

[5] *Tanh-sinh quadrature* (2021). Retrieved March 28, 2021 from https://en.wikipedia.org/wiki/Tanh-sinh_quadrature

## Other Sources

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