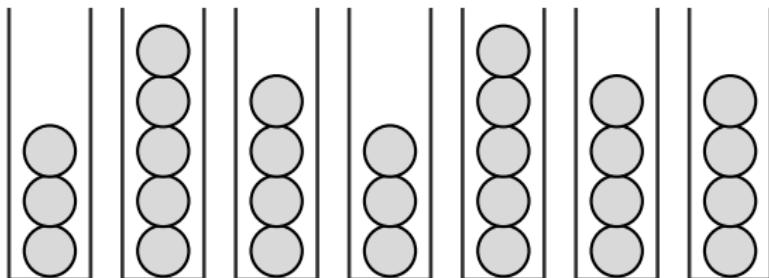


# History-Independent Load Balancing



Michael A. Bender

Stony Brook University

William Kuszmaul

CMU

Elaine Shi

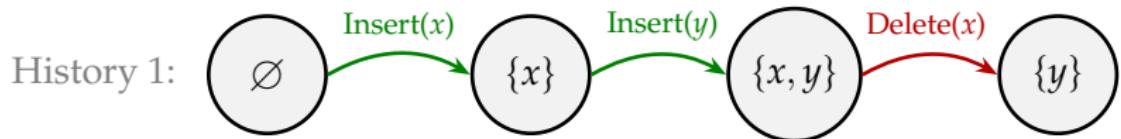
CMU

Rose Silver

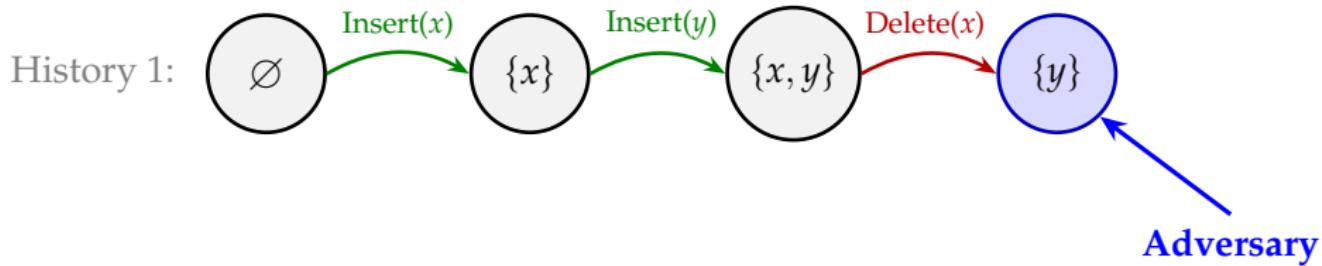
CMU

# HISTORY-INDEPENDENT DATA STRUCTURES

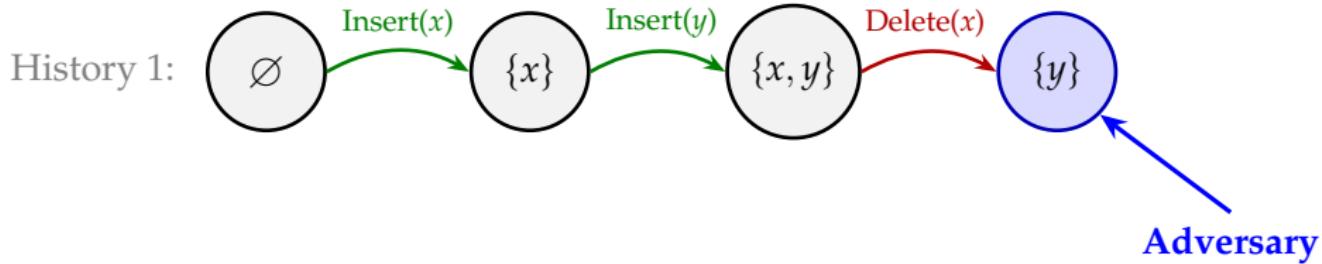
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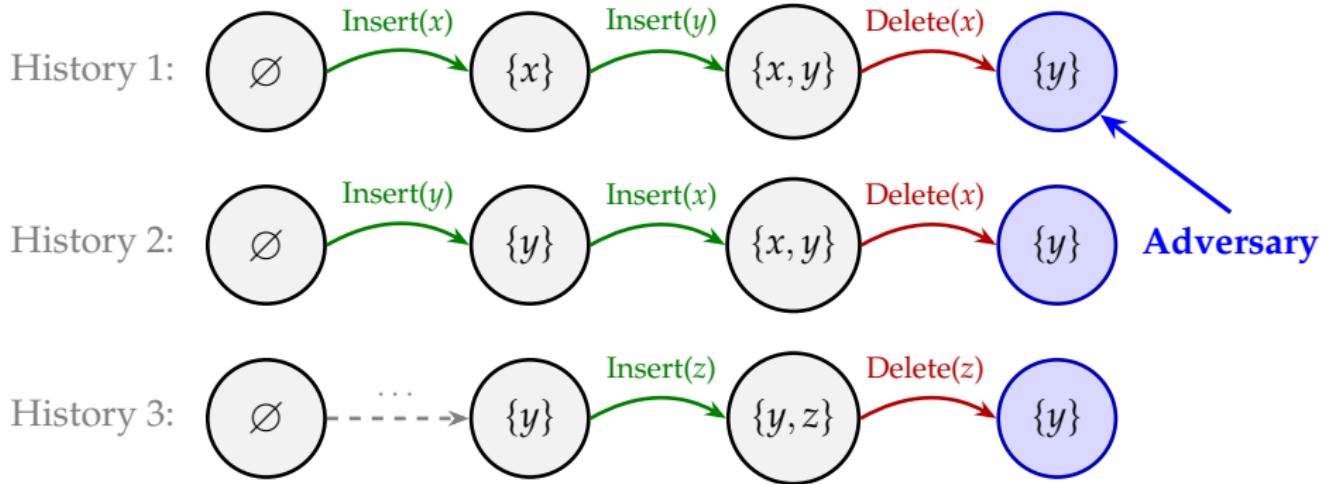
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**History Independence** (Micciancio '97, Naor & Teague '01)

- ▶ The state reveals only the current elements—**not the history of operations**.

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## A History of Applications

Hash tables, trees, memory allocation, PMAs, graph algorithms, cache-oblivious data structures, and more.

Micciancio '97, Naor & Teague '01, Buchbinder & Petrank '03, Molnar et al. '06, Blelloch & Golovin '07,  
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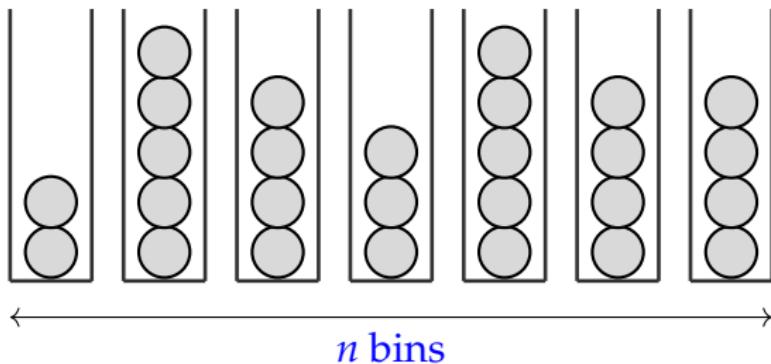
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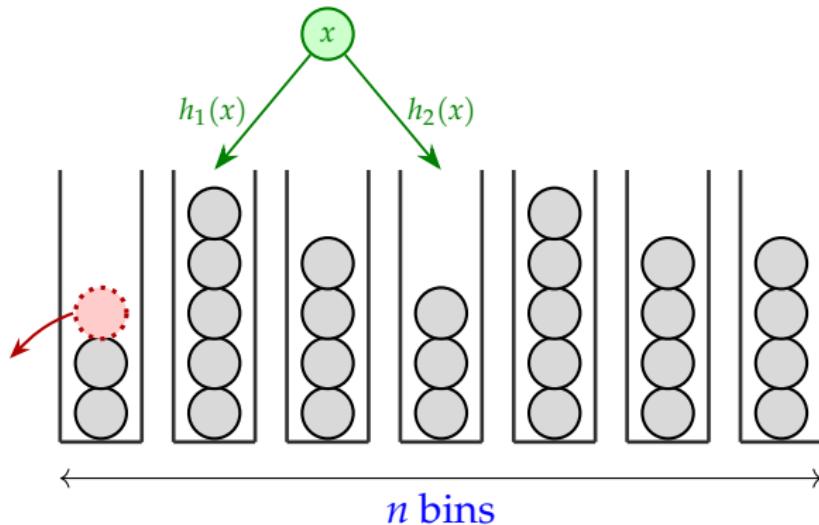
**Yet some basic questions remain open.**

**This work:** History-Independent Load Balancing

# TWO-CHOICE LOAD BALANCING

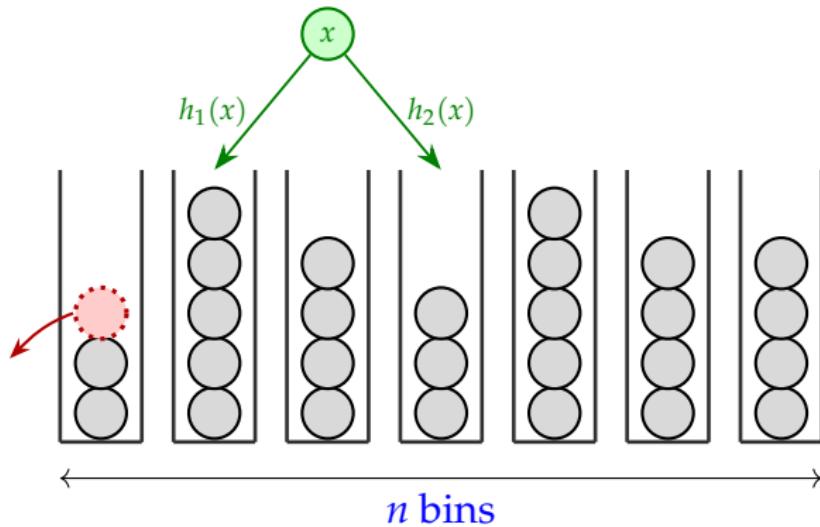


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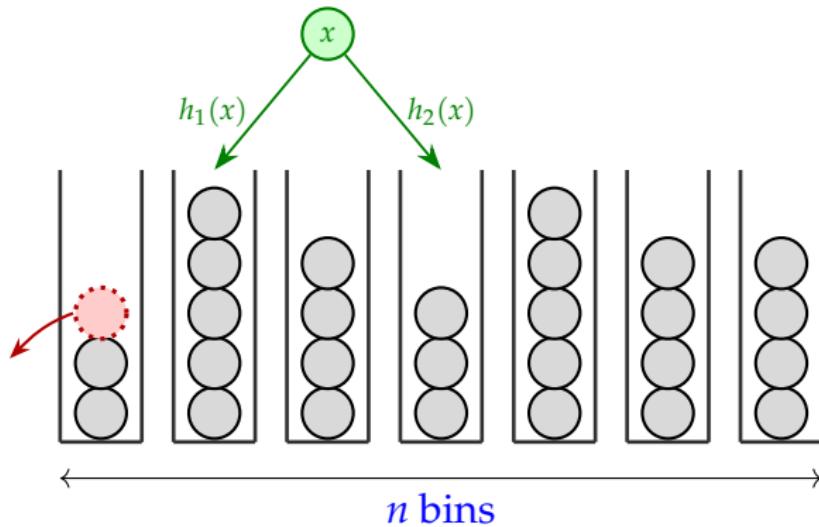
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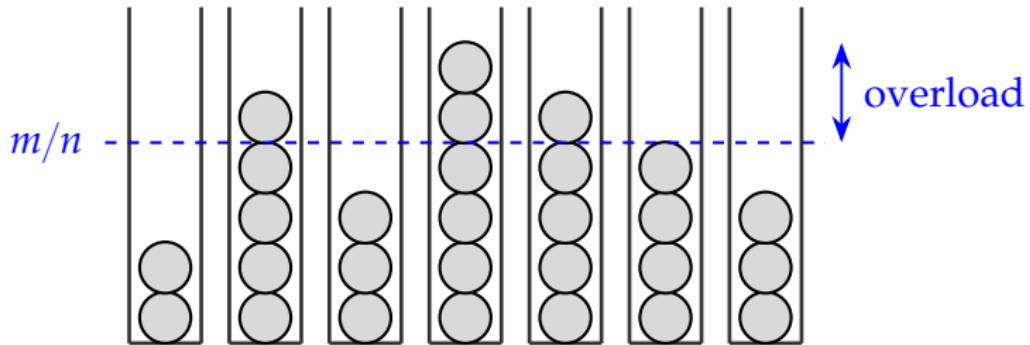
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- ▶ Each ball has two random bins where it can go.

# TWO-CHOICE LOAD BALANCING



- ▶ Balls are **inserted**/**deleted**, with up to  $m$  present at a time.
- ▶ Each ball has two random bins where it can go.
- ▶ We must maintain a valid assignment of balls to bins.

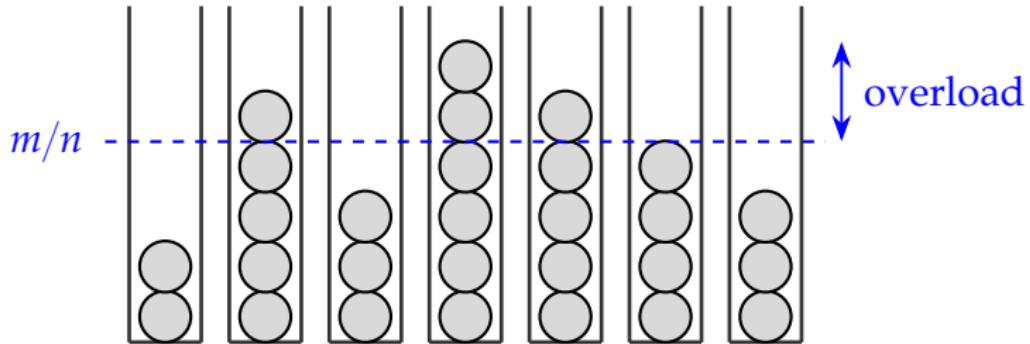
## TWO GOALS



### Minimize Overload:

- ▶ i.e., the amount by which the fullest bin exceeds  $m/n$ .

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### Minimize Recourse:

- ▶ i.e., the number of balls moved around on any given insertion/deletion.

# PUTTING IT ALL TOGETHER

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**History-Independent Load Balancing:**

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**Question:** Does there exist a **history-independent** solution with small **recourse** and small **overload**?

# PUTTING IT ALL TOGETHER

## History-Independent Load Balancing:

- ▶ For all sets  $S$  of balls: If the current set is  $S$ , then the assignment is always  $A_S$ .

**Question:** Does there exist a history-independent solution with small recourse and small overload?

**Our Main Result:** There exists a history-independent solution with:

- ▶ High probability overload  $O(1)$
- ▶ Expected recourse  $O(\log \log(m/n))$

## PAST WORK (NOT HISTORY INDEPENDENT)

---

Overload	Recourse	Reference	Caveats
$O(\log \log n)$	0	[ABKU '94] [BCSV '00]	insertion-only
$O(1)$	$O(\log(m/n))$	[Dietzfelbinger, Weidling '07]	insertion-only
$\tilde{O}(\sqrt{m/n})$	$O(1)$	[Frieze, Petti '18]	insertion-only
$O(\log(m/n))$	0	[Bansal, Kuszmaul '22]	no reinsertions
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If we want overload  $O(1)$ , our result is a new state of the art!

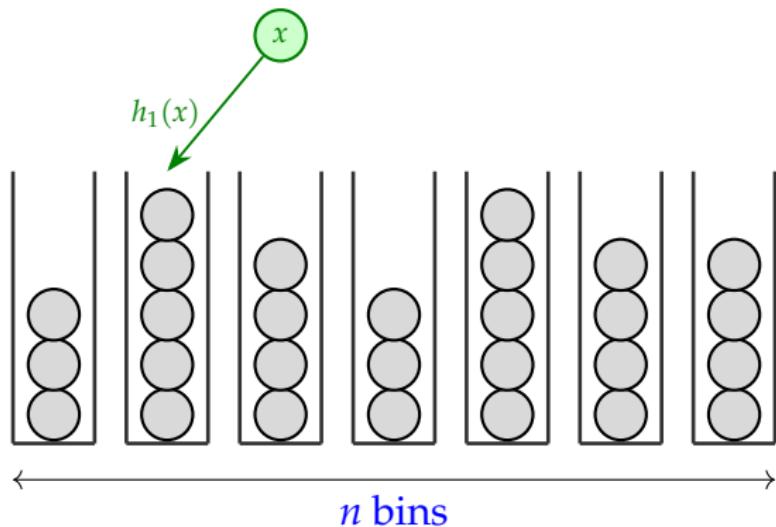
## REST OF TALK: A SIMPLE WARMUP

**Theorem:** There exists a history-independent solution with:

- ▶ High-probability overload  $\Theta(1)$   $O(\log \log n)$ .
- ▶ Expected recourse  $\Theta(\log \log(m/n))$   $O(m/n)$ .

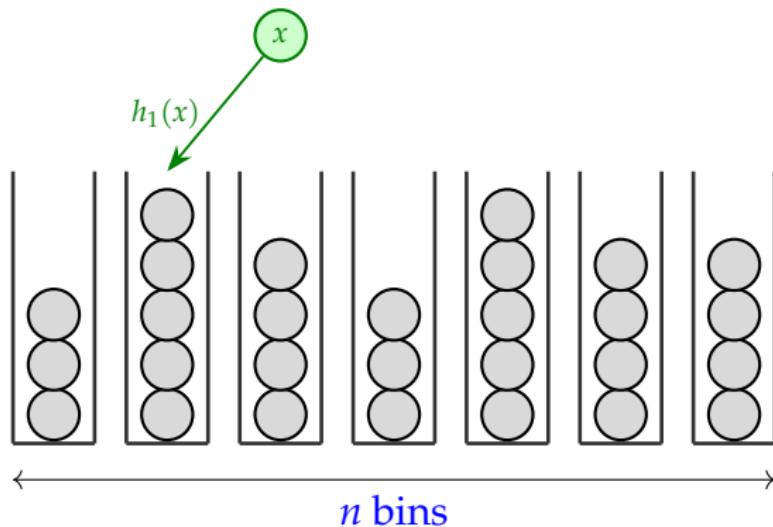
## BASELINE 1: THE SINGLE-CHOICE STRATEGY

To insert a ball  $x$ , just put it in bin  $h_1(x)$ :



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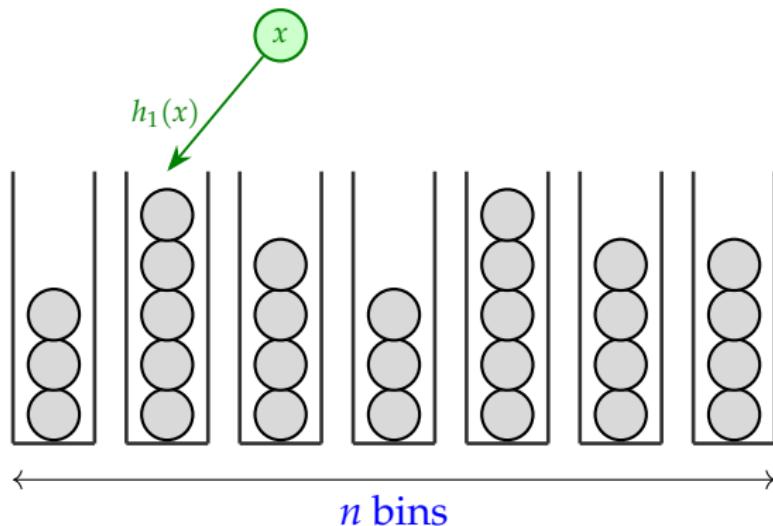
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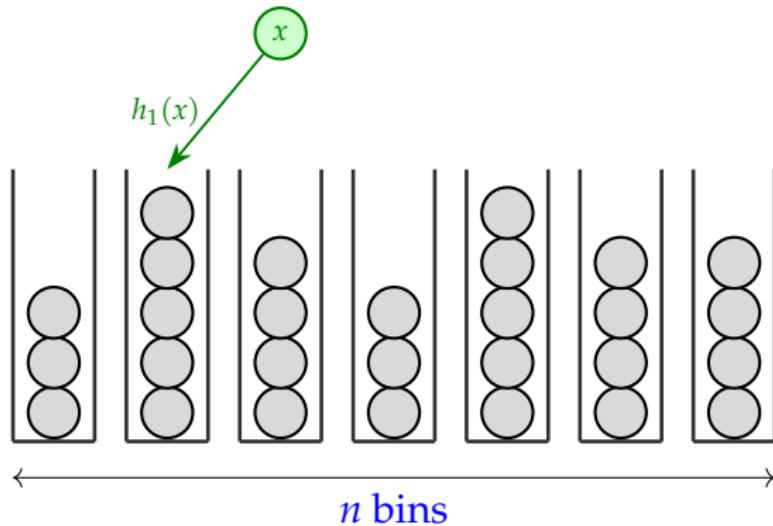
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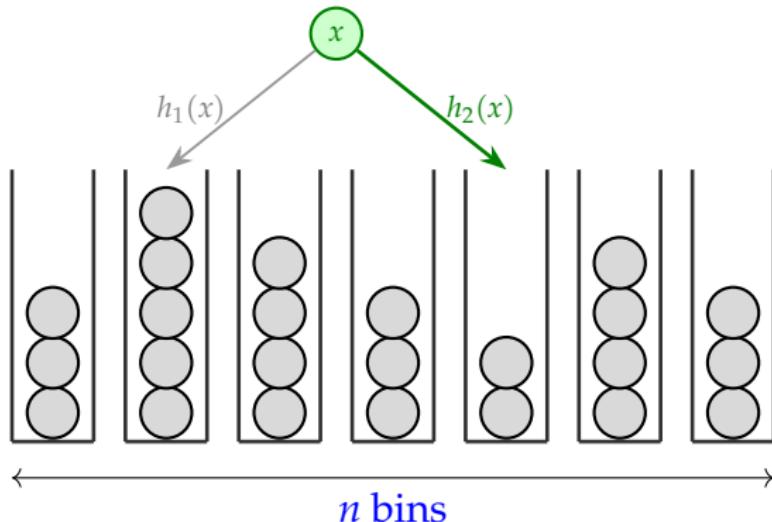
To insert a ball  $x$ , just put it in bin  $h_1(x)$ :



- ▶ This is history-independent ✓
- ▶ The recourse is 0 ✓
- ▶ But... the overload is huge, roughly  $\sqrt{m/n}$  ✗

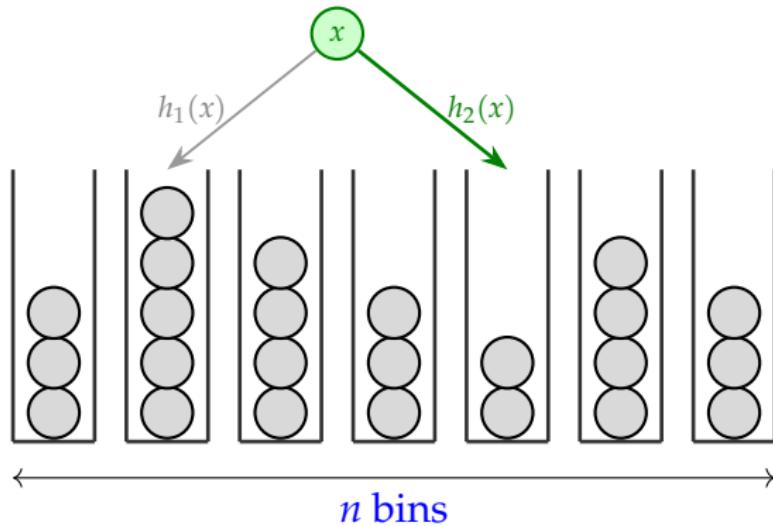
## BASELINE 2: GREEDY INSERTIONS

To insert a ball  $x$ , put it in the **emptier** of its choices:



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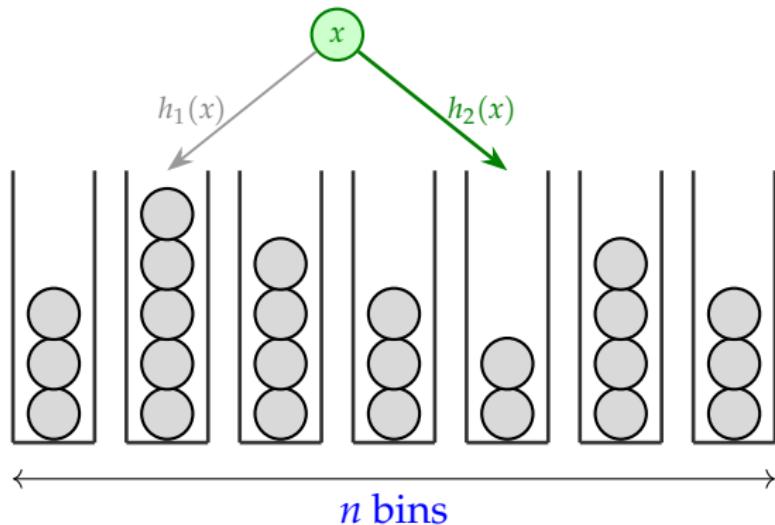
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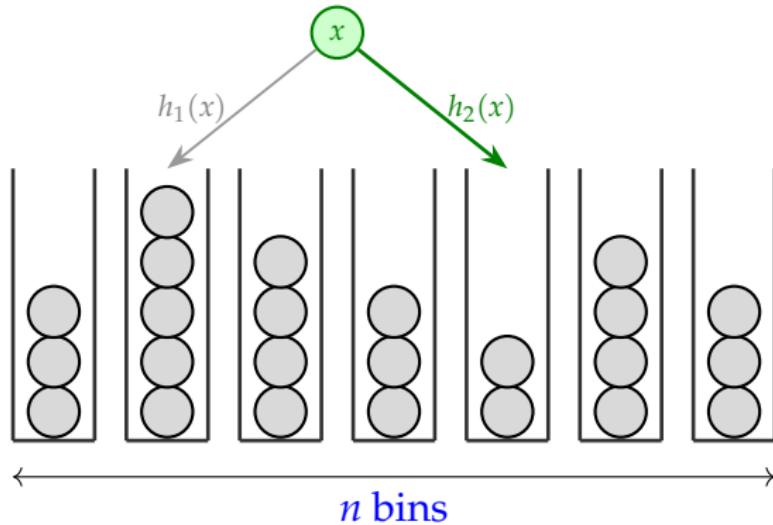
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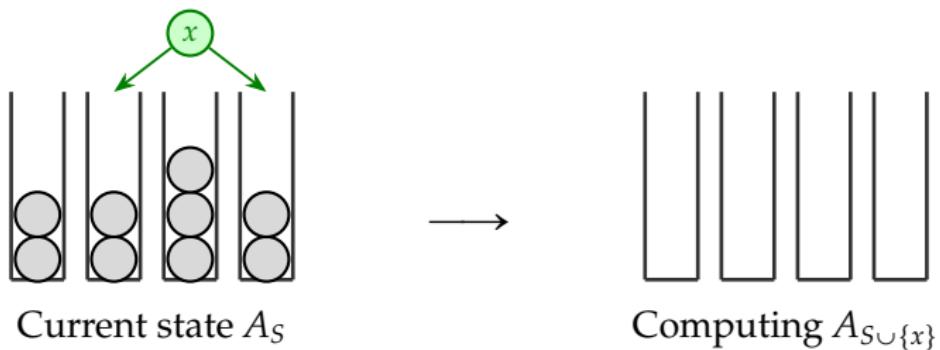
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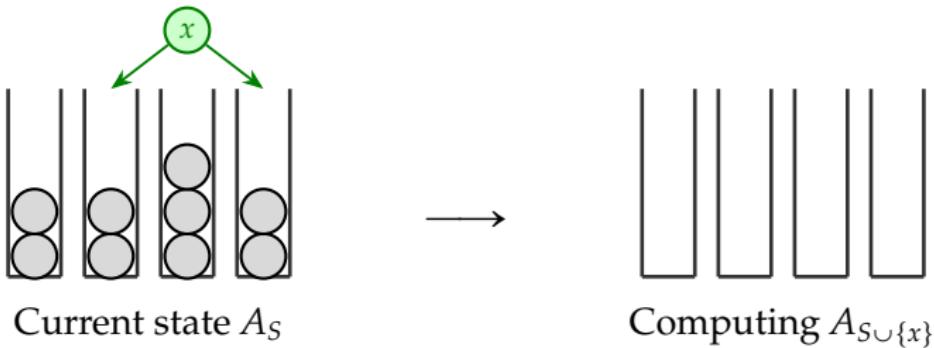
- ▶ This is **not** history-independent ✗
- ▶ The recourse is 0 ✓
- ▶ In the insertion-only case, the overload is  $O(\log \log n)$  ✓  
[Berenbrink, Czumaj, Steger, and Vöcking '00]

## WARMUP: HISTORY-INDEPENDENT GREEDY

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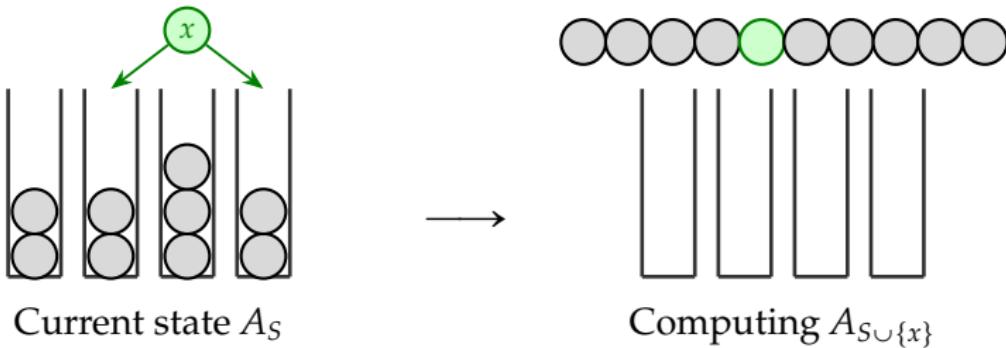
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To compute  $A_{S \cup \{x\}}$ :

1. Empty out the bins.
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3. Insert the balls in sorted order using greedy.

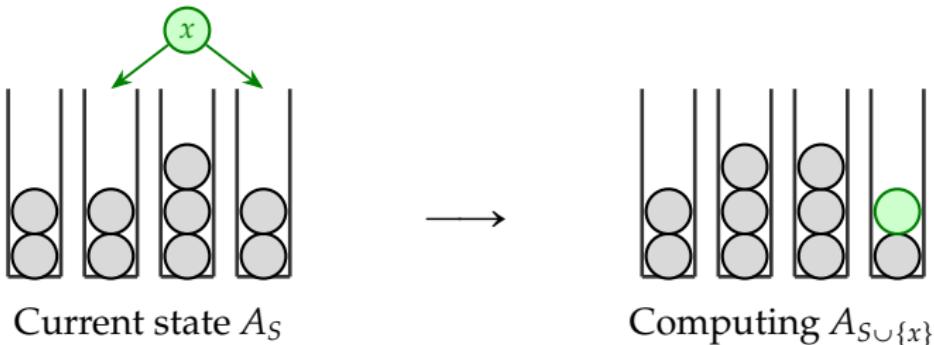
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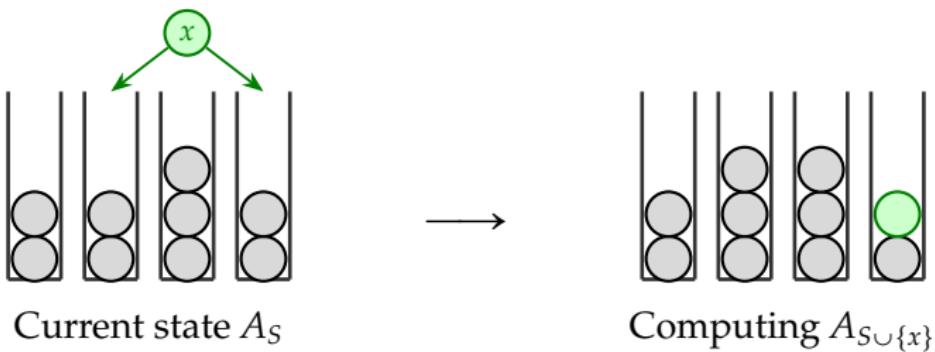
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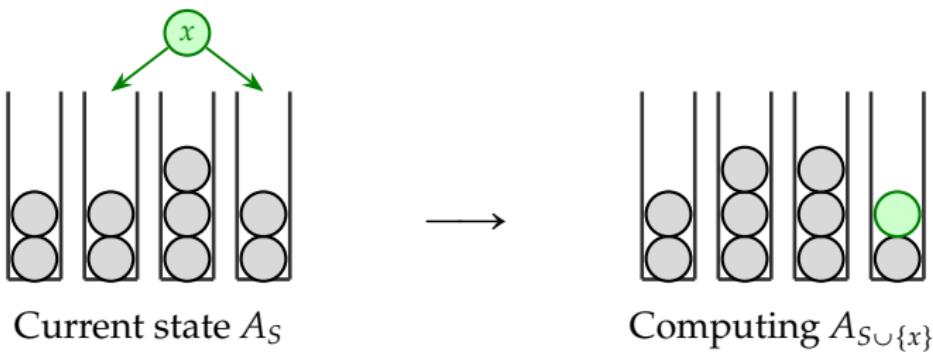
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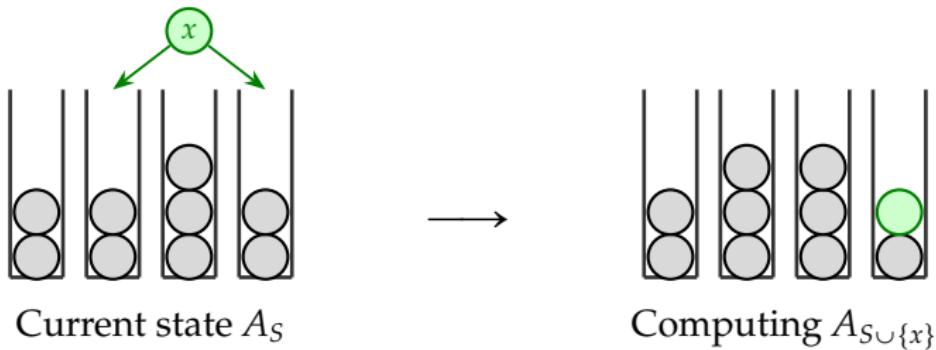


## ANALYZING HISTORY-INDEPENDENT GREEDY



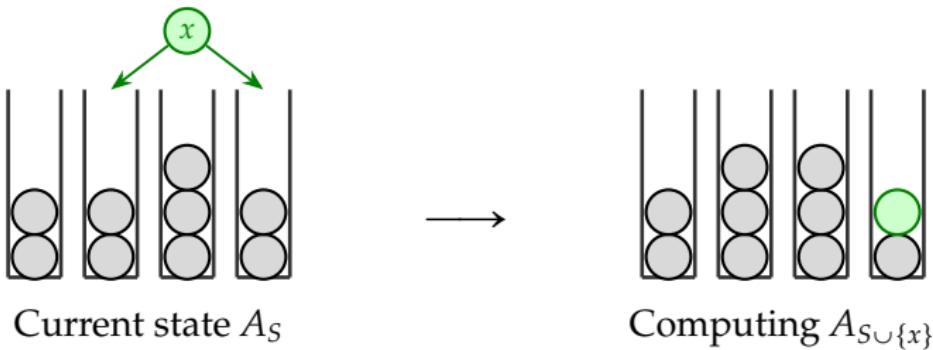
- ▶ The algorithm is history independent ✓

## ANALYZING HISTORY-INDEPENDENT GREEDY



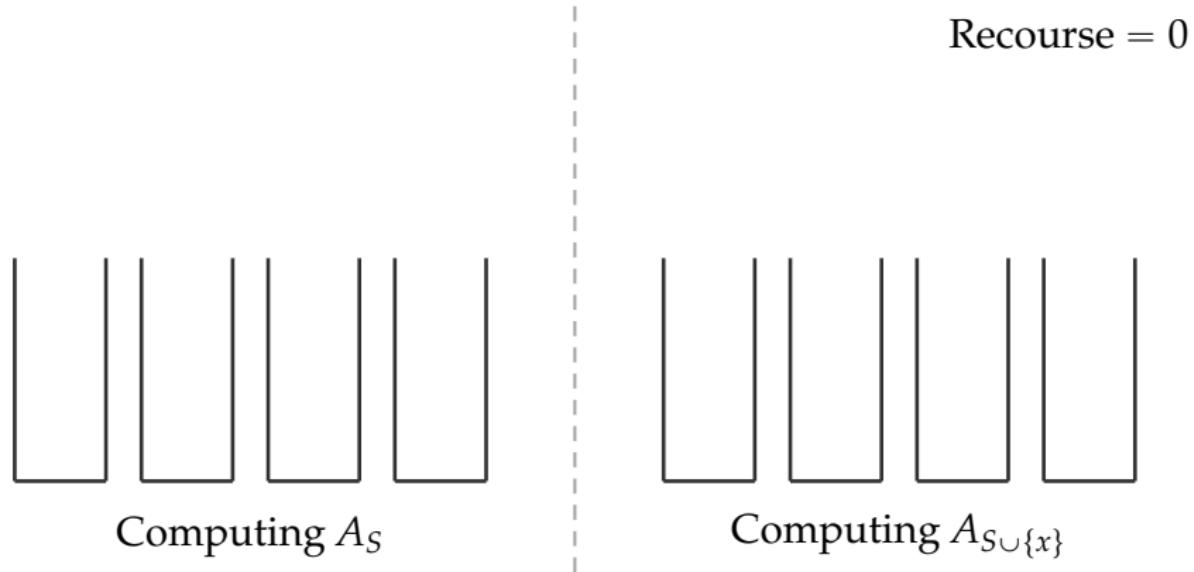
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- ▶ The overload is  $O(\log \log n)$  ✓

# ANALYZING HISTORY-INDEPENDENT GREEDY



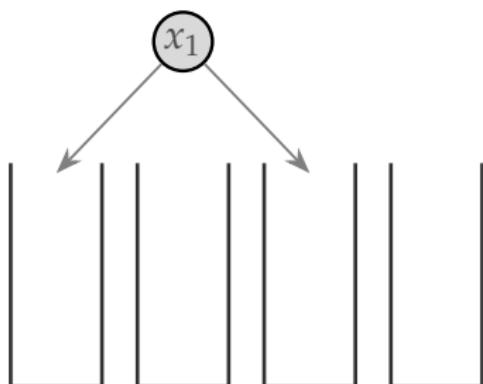
- ▶ The algorithm is history independent ✓
- ▶ The overload is  $O(\log \log n)$  ✓
- ▶ What is the recourse?

# ANALYZING THE RECOURSE

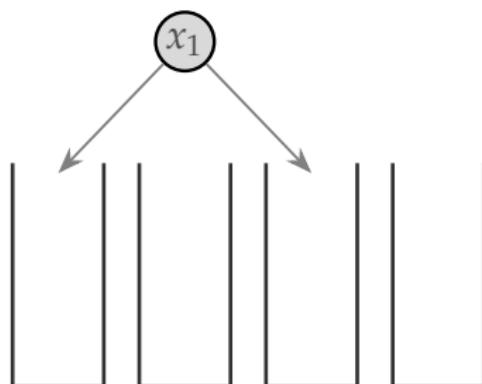


How many balls change assignments between  $A_S$  and  $A_{S \cup \{x\}}$ ?

# ANALYZING THE RECOURSE



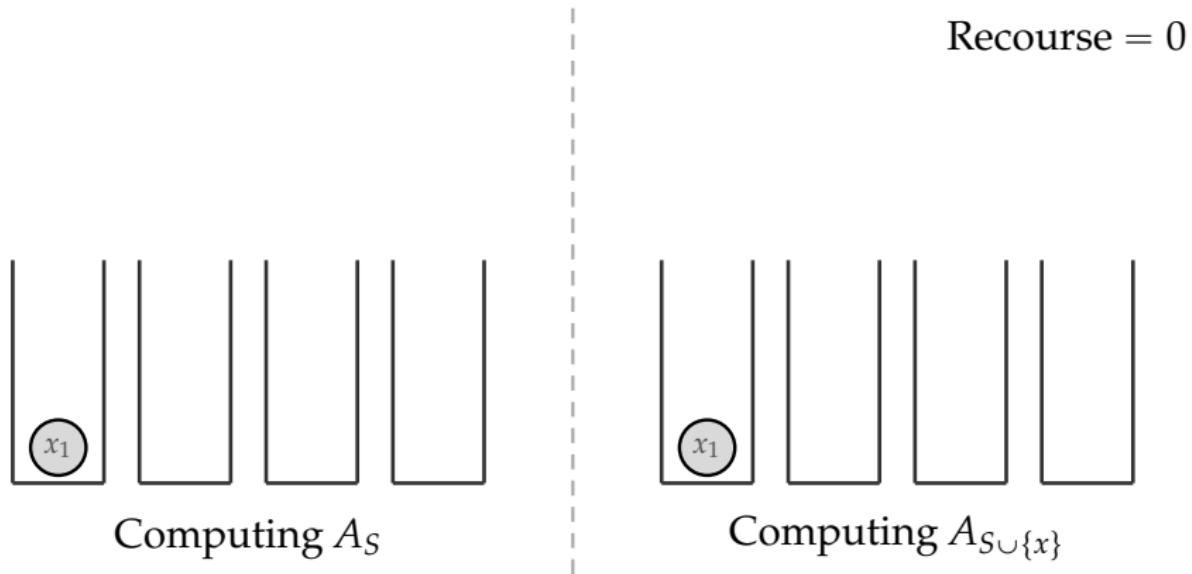
Computing  $A_S$



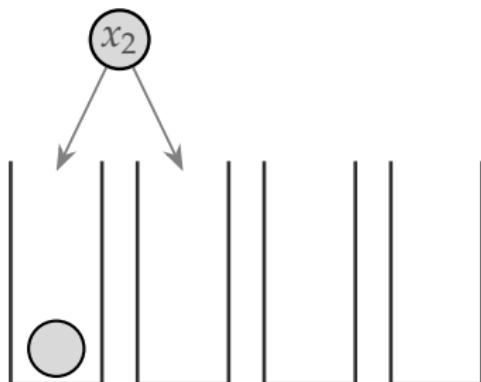
Computing  $A_{S \cup \{x\}}$

Recourse = 0

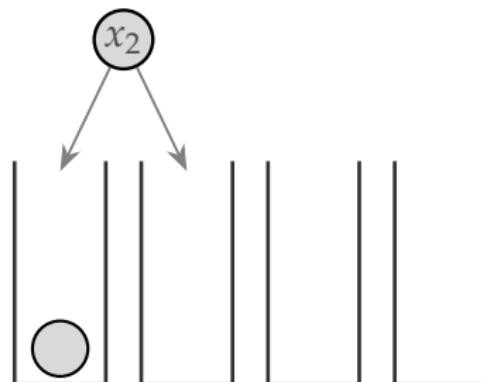
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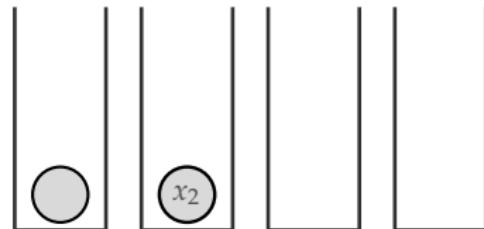
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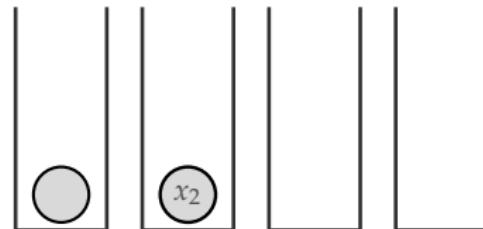
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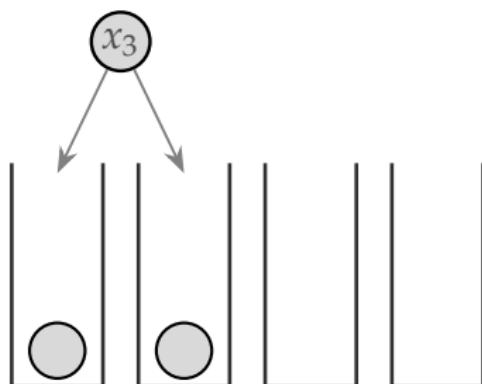
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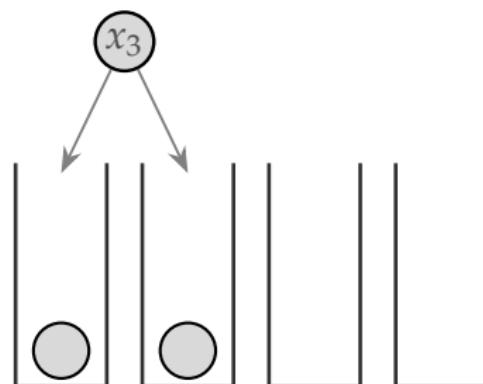
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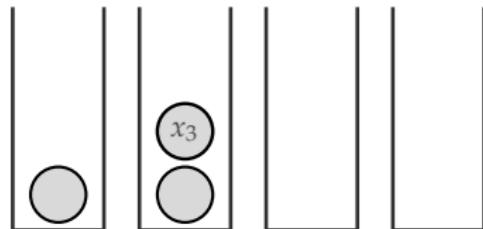
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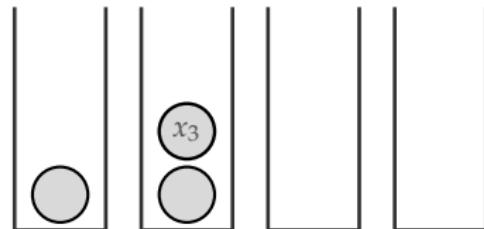
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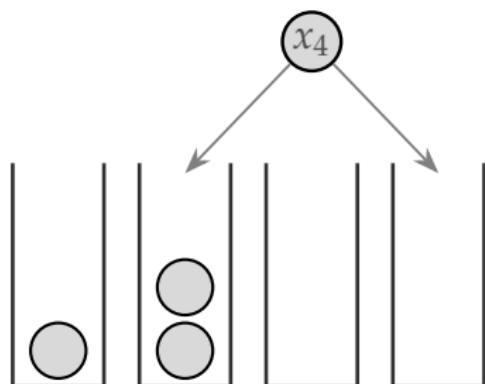
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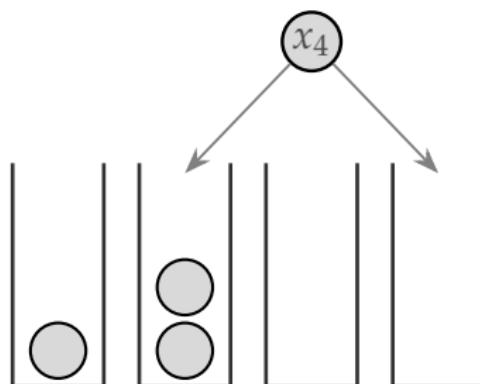
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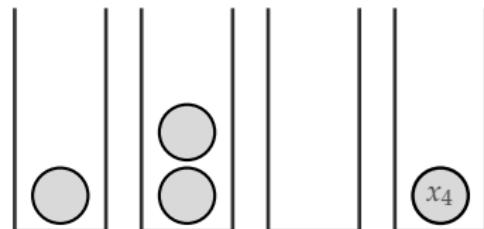
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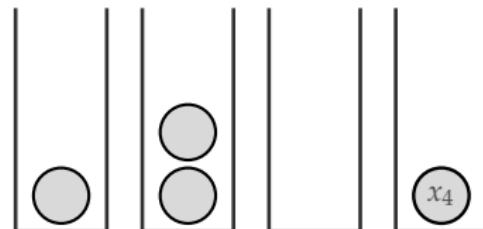
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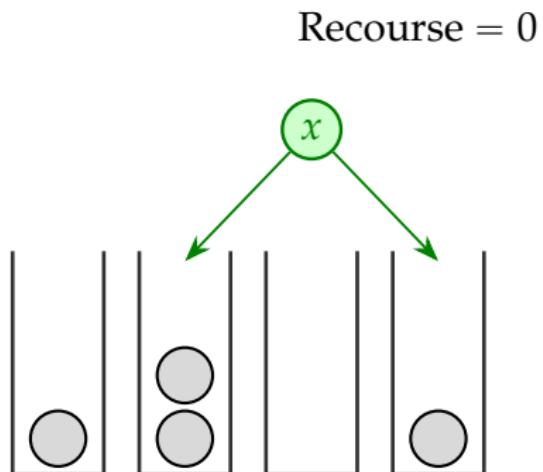
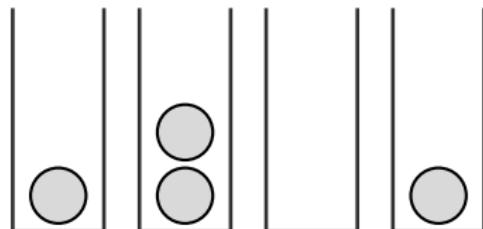
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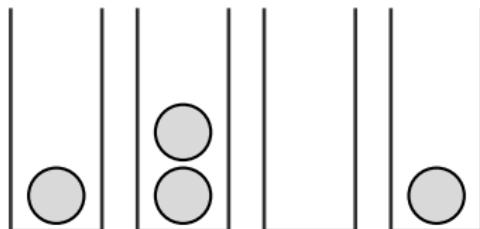
Computing  $A_{S \cup \{x\}}$

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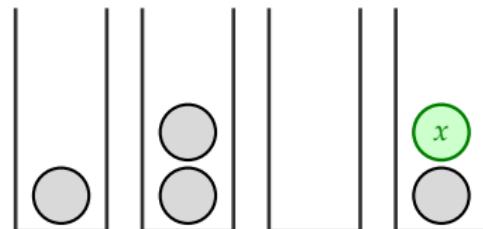
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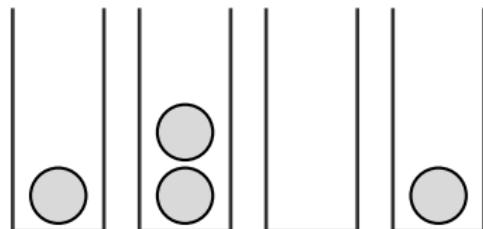
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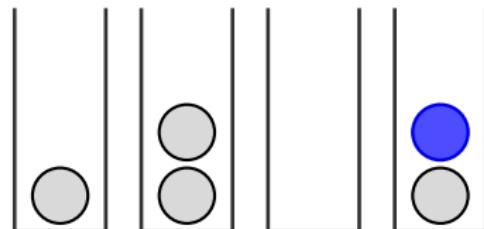
Computing  $A_{S \cup \{x\}}$

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## ANALYZING THE RECOURSE



Computing  $A_S$

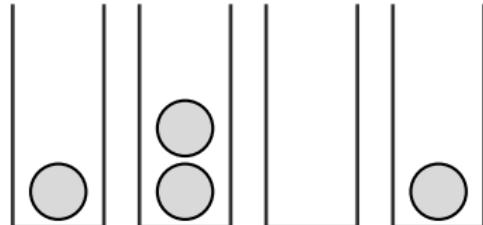


Computing  $A_{S \cup \{x\}}$

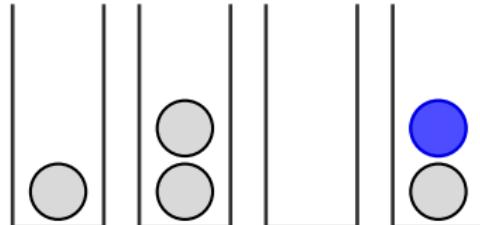
Subsequent balls will experience either:

Recourse = 0

## ANALYZING THE RECOURSE



Computing  $A_S$



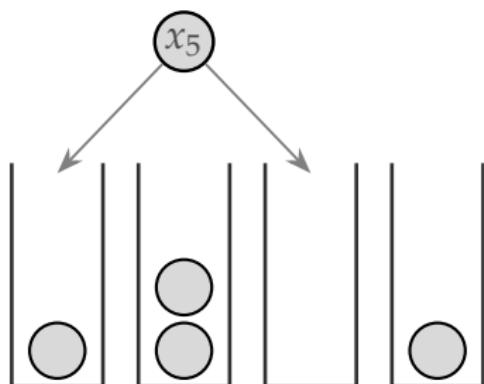
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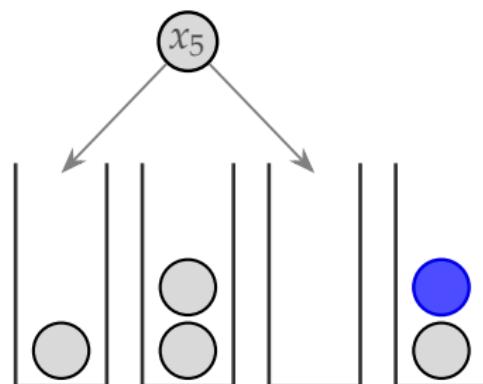
1. No recourse

$$\text{Recourse} = 0$$

## ANALYZING THE RECOURSE



Computing  $A_S$



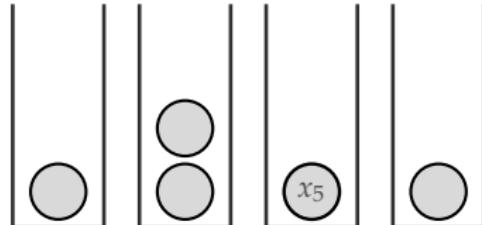
Computing  $A_{S \cup \{x\}}$

Future insertions will experience either:

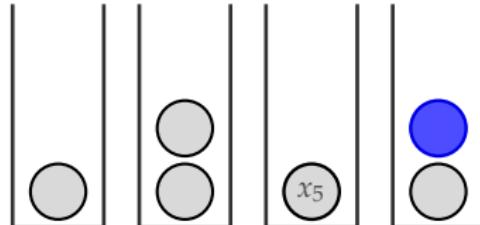
1. No recourse

Recourse = 0

# ANALYZING THE RECOURSE



Computing  $A_S$

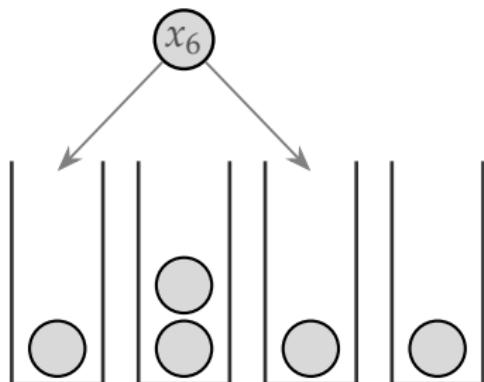


Computing  $A_{S \cup \{x\}}$

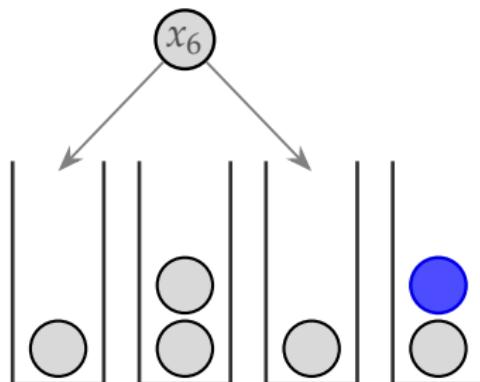
Subsequent balls will experience either:

1. No recourse

## ANALYZING THE RECOURSE



Computing  $A_S$



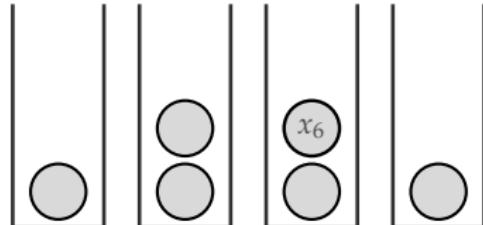
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

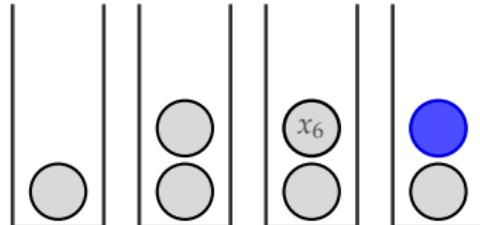
1. No recourse

Recourse = 0

# ANALYZING THE RECOURSE



Computing  $A_S$

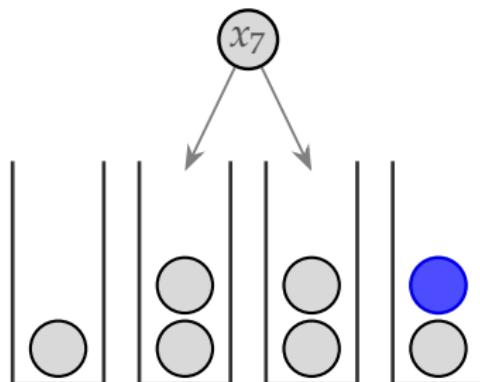
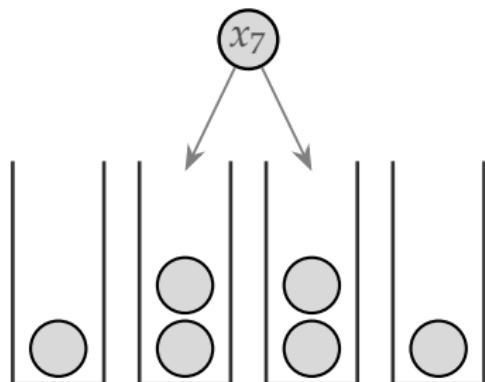


Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

1. No recourse

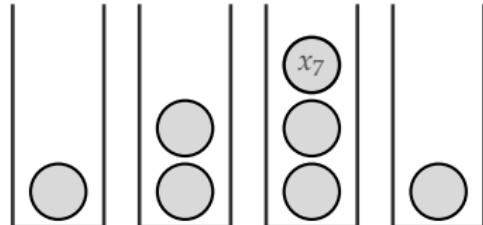
# ANALYZING THE RECOURSE



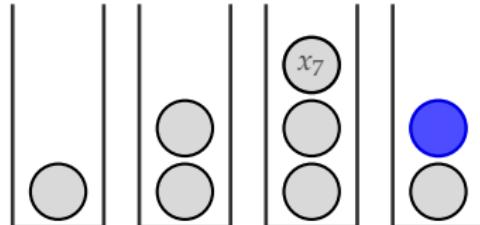
Subsequent balls will experience either:

1. No recourse

# ANALYZING THE RECOURSE



Computing  $A_S$



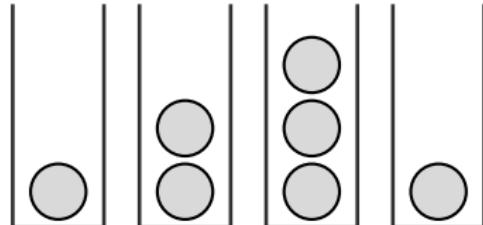
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

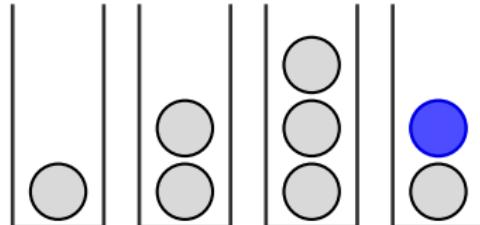
1. No recourse

$$\text{Recourse} = 0$$

# ANALYZING THE RECOURSE



Computing  $A_S$



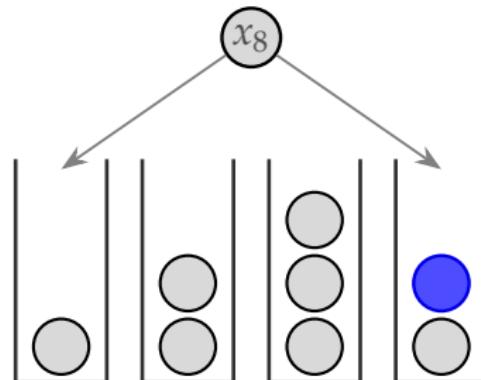
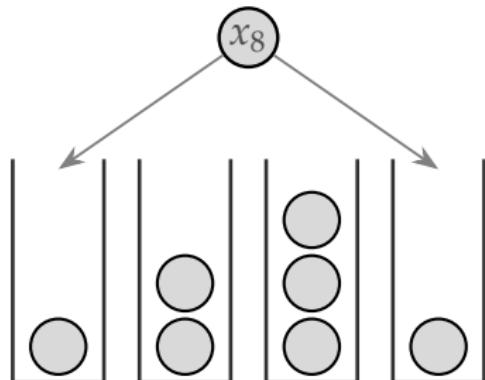
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

1. No recourse
2. Recourse

$$\text{Recourse} = 0$$

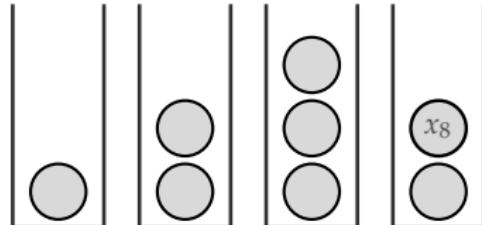
## ANALYZING THE RECOURSE



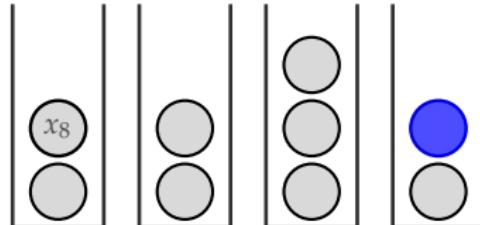
Subsequent balls will experience either:

1. No recourse
2. Recourse

# ANALYZING THE RECOURSE



Computing  $A_S$



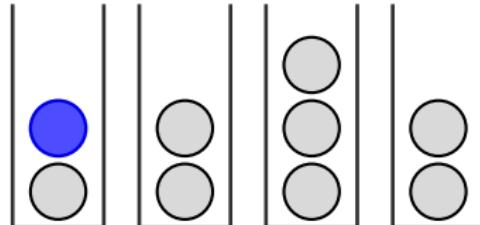
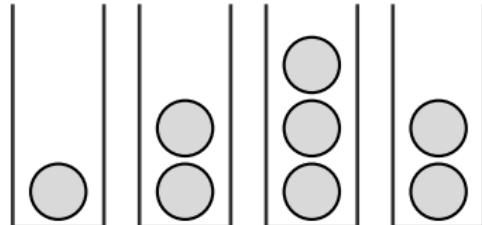
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

1. No recourse
2. Recourse

Recourse = 1

## ANALYZING THE RECOURSE

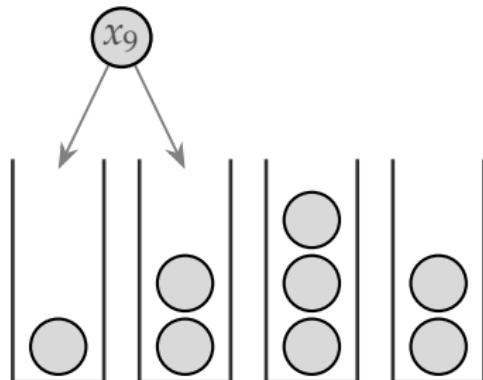


Recourse = 1

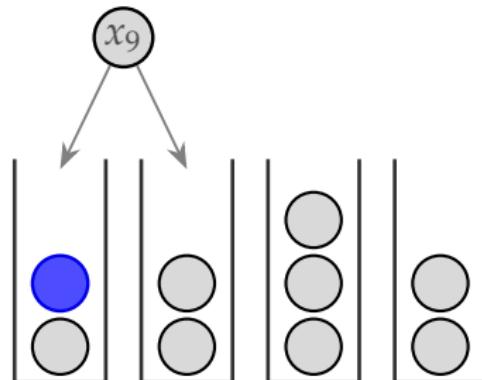
Subsequent balls will experience either:

1. No recourse
2. Recourse

# ANALYZING THE RECOURSE



Computing  $A_S$



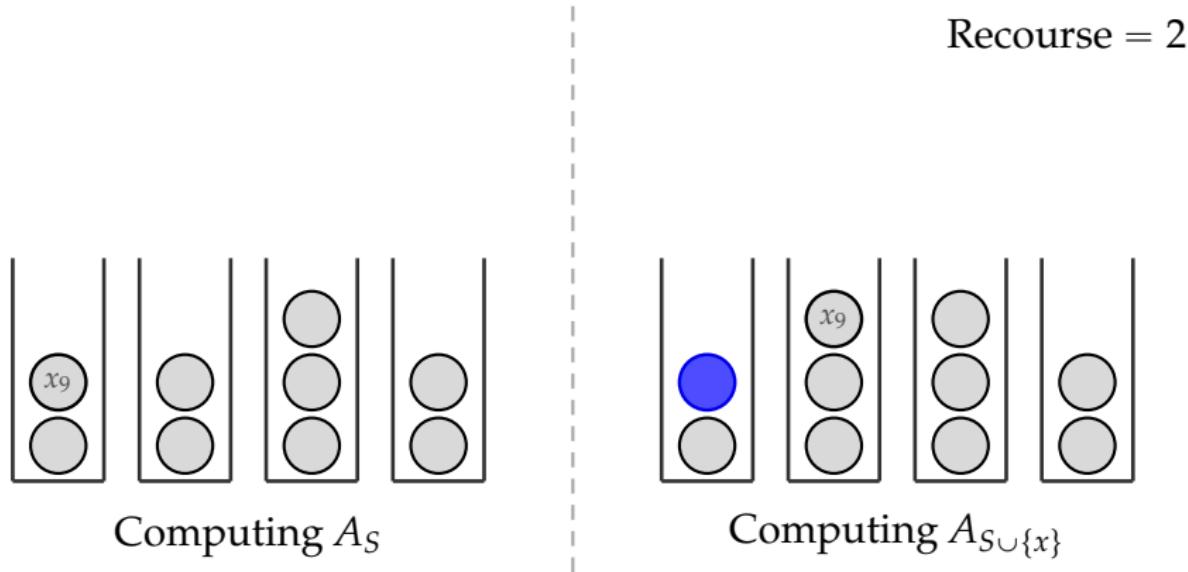
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

1. No recourse
2. Recourse

Recourse = 1

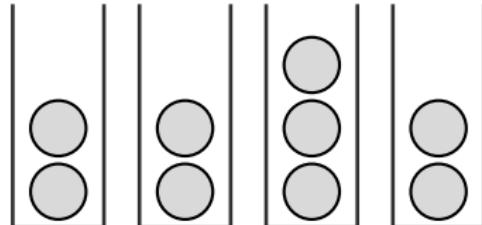
# ANALYZING THE RECOURSE



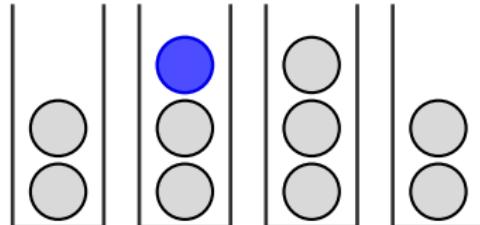
Subsequent balls will experience either:

1. No recourse
2. Recourse

# ANALYZING THE RECOURSE



Computing  $A_S$



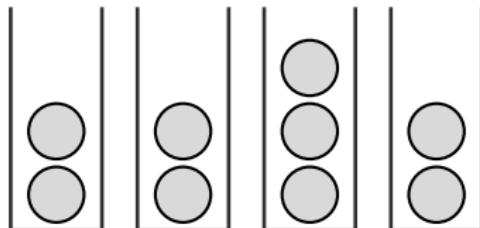
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

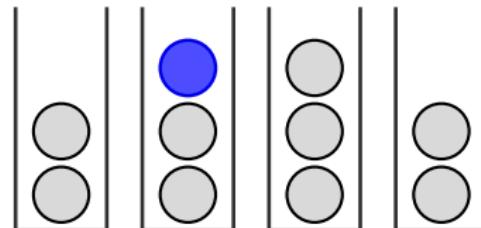
1. No recourse
2. Recourse

Recourse = 2

# ANALYZING THE RECOURSE



Computing  $A_S$

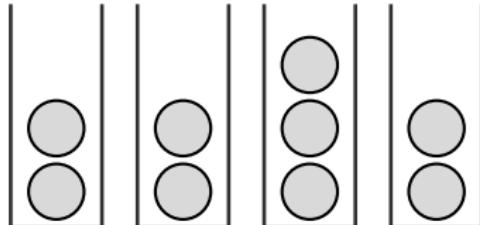


Computing  $A_{S \cup \{x\}}$

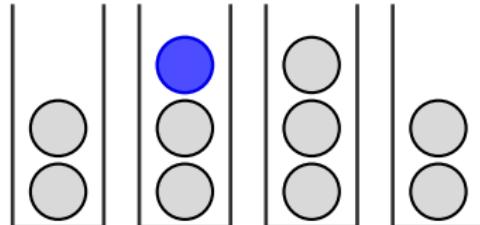
Two key observations:

Recourse = 2

# ANALYZING THE RECOURSE



Computing  $A_S$



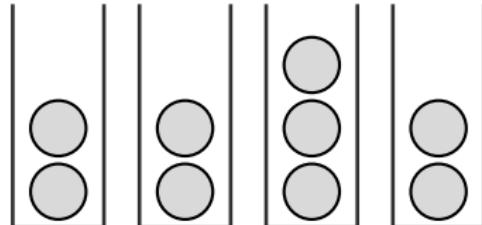
Computing  $A_{S \cup \{x\}}$

Two key observations:

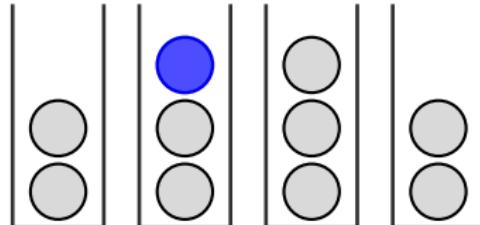
1. There's always one special bin with an extra ball

Recourse = 2

# ANALYZING THE RECOURSE



Computing  $A_S$



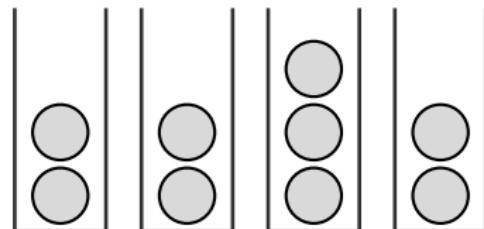
Computing  $A_{S \cup \{x\}}$

Two key observations:

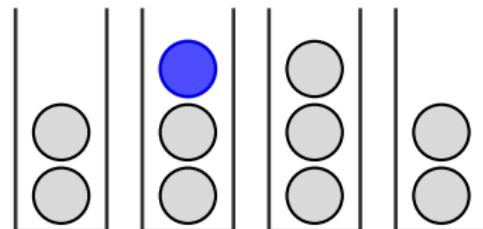
1. There's always one special bin with an extra ball
2. If a ball incurs recourse, one of its choices is the special bin

Recourse = 2

# ANALYZING THE RECOURSE



Computing  $A_S$

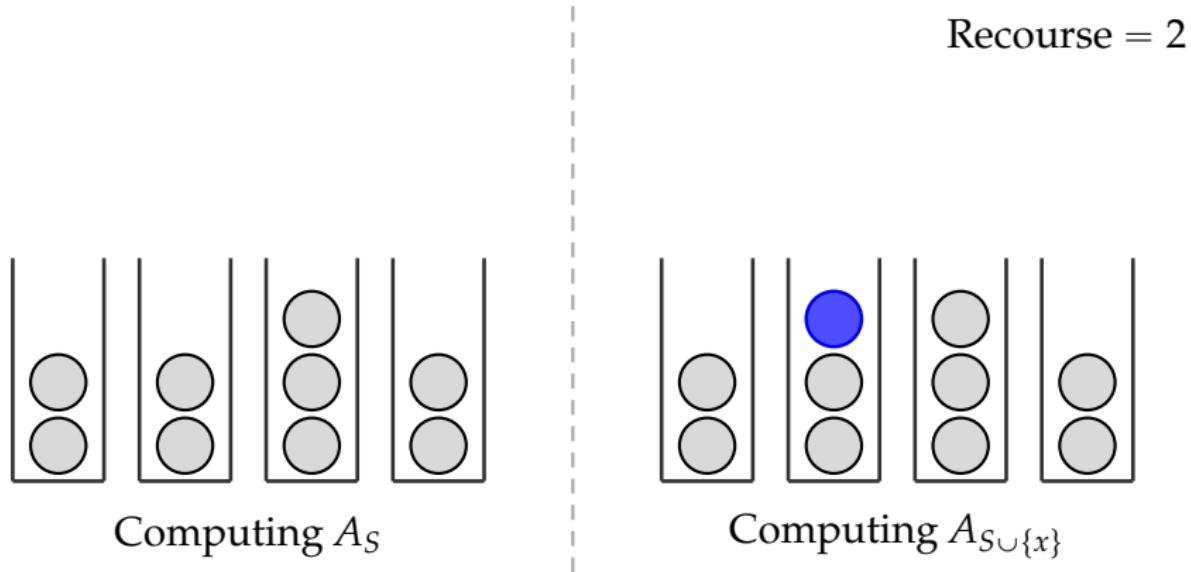


Computing  $A_{S \cup \{x\}}$

$$\Pr[\text{ball } x_i \text{ incurs recourse}] = O(1/n)$$

Recourse = 2

# ANALYZING THE RECOURSE



$$\Pr[\text{ball } x_i \text{ incurs recourse}] = O(1/n)$$

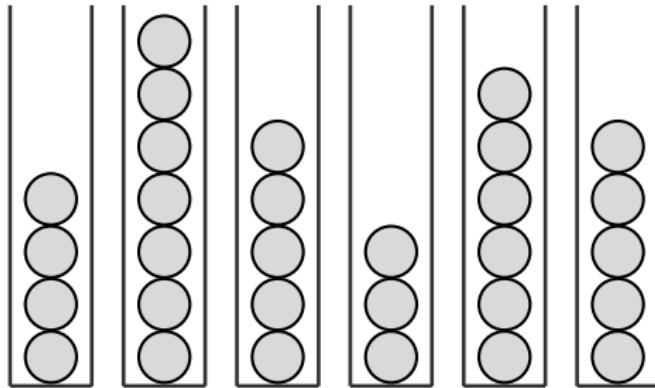
$$\implies \mathbb{E}[\text{total recourse}] = \sum_i \Pr[\text{ball } x_i \text{ incurs recourse}] = O(m/n)$$

## A SIMPLE WARMUP

**Theorem:** There exists a history-independent solution with:

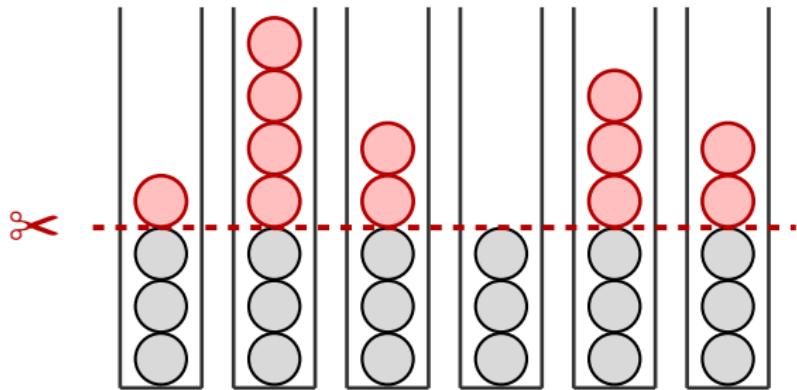
- ▶ High-probability overload  $\Theta(1)$   $O(\log \log n)$ .
- ▶ Expected recourse  $\Theta(\log \log(m/n))$   $O(m/n)$ .

# SLICE AND SPREAD



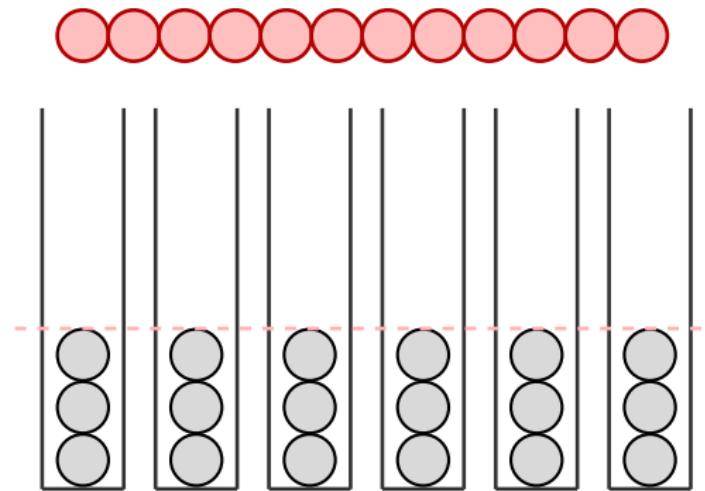
1. **Slice** off the jagged surface

## SLICE AND SPREAD



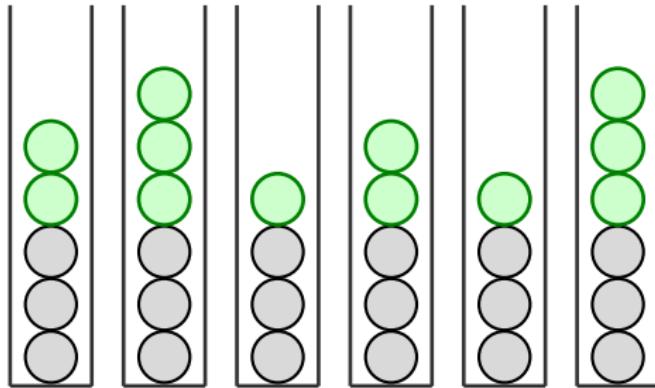
1. **Slice** off the jagged surface

## SLICE AND SPREAD



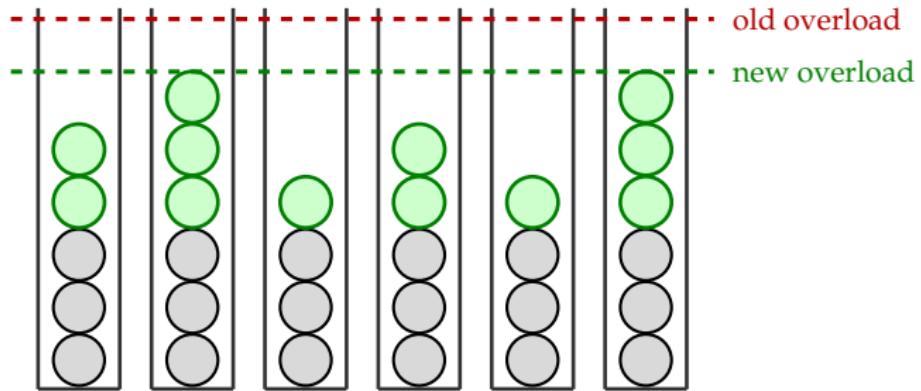
1. **Slice** off the jagged surface
2. **Spread** balls to their second-choice bins

# SLICE AND SPREAD



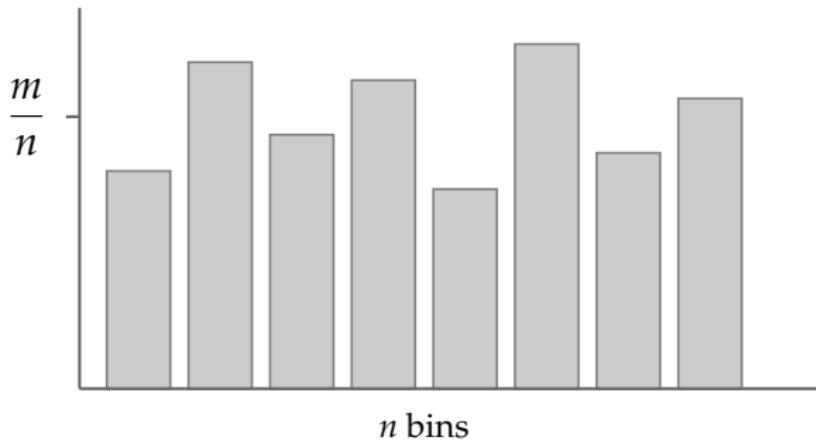
1. **Slice** off the jagged surface
2. **Spread** balls to their second-choice bins

# SLICE AND SPREAD

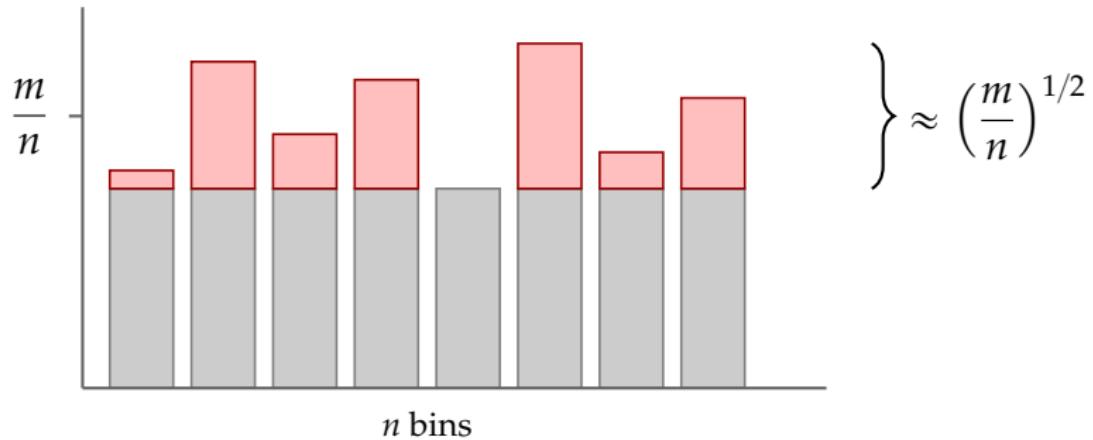


1. **Slice** off the jagged surface
2. **Spread** balls to their second-choice bins

## SLICE AND SPREAD REDUCES OVERLOAD



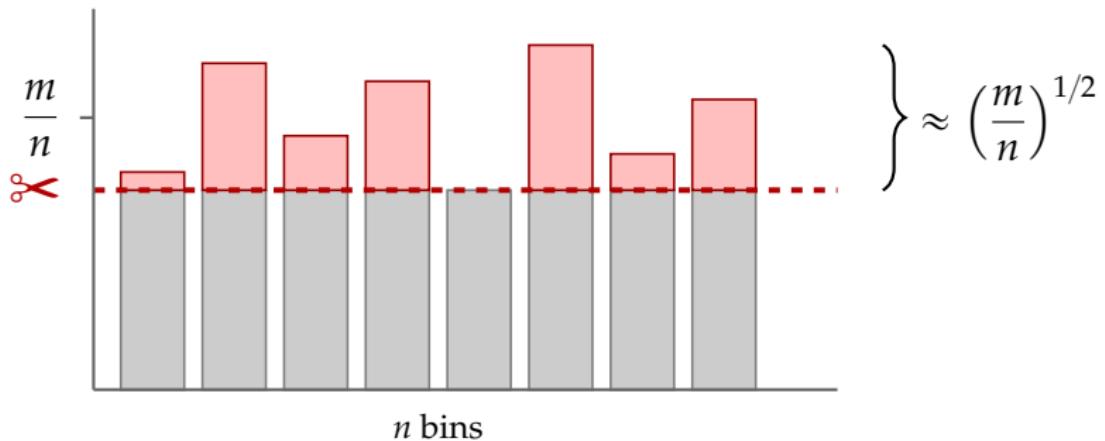
## SLICE AND SPREAD REDUCES OVERLOAD



### Key Fact

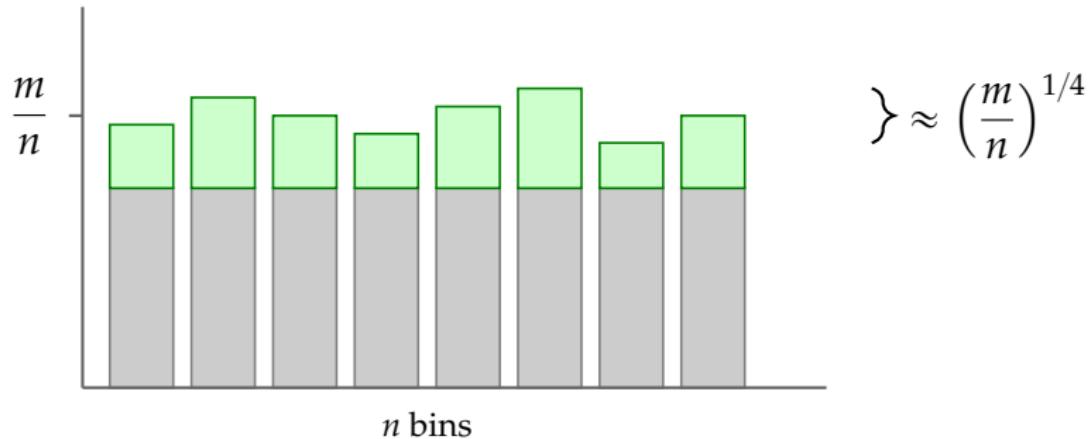
After throwing  $m \gg n$  balls uniformly at random into  $n$  bins, the bin loads are within roughly  $\approx \sqrt{m/n}$  with high probability in  $n$ .

## SLICE AND SPREAD REDUCES OVERLOAD



- ▶ Balls above dotted line  $\approx (mn)^{1/2}$

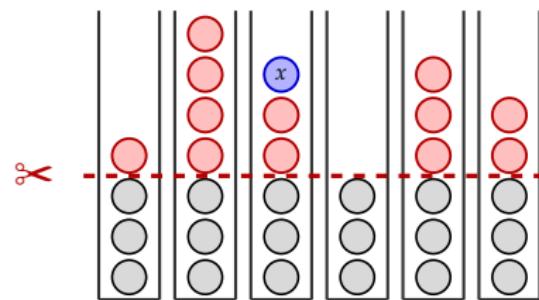
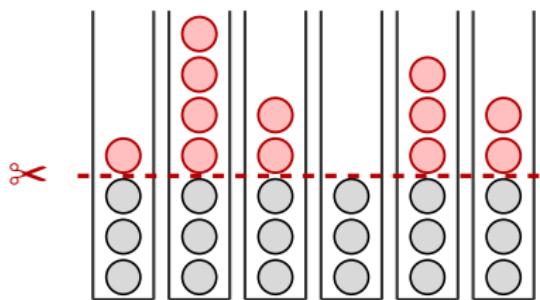
## SLICE AND SPREAD REDUCES OVERLOAD



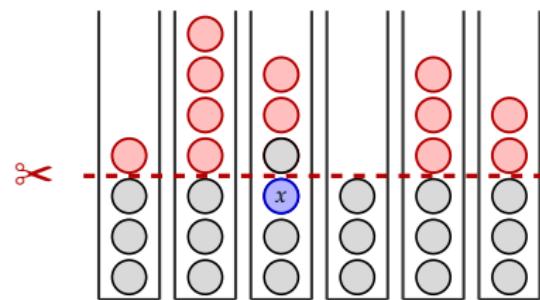
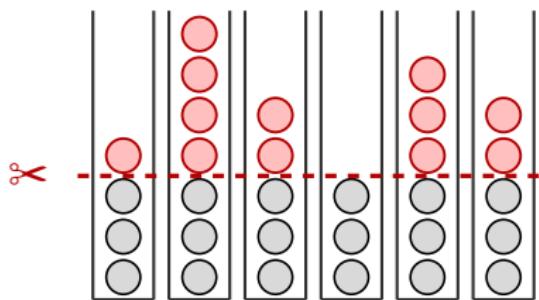
- ▶ Balls above dotted line  $\approx (mn)^{1/2}$
- ▶ By Key Fact, bin loads within  $\approx (m/n)^{1/4}$

WHAT'S THE RE COURSE?

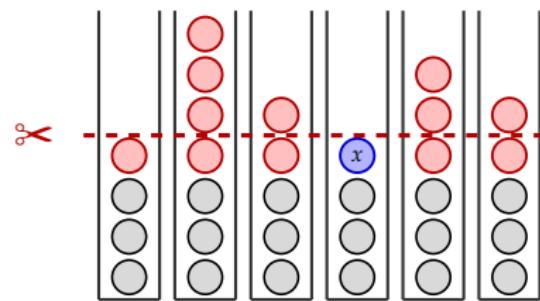
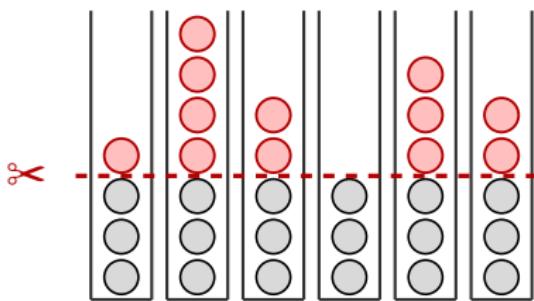
# WHAT'S THE RECOURSE?



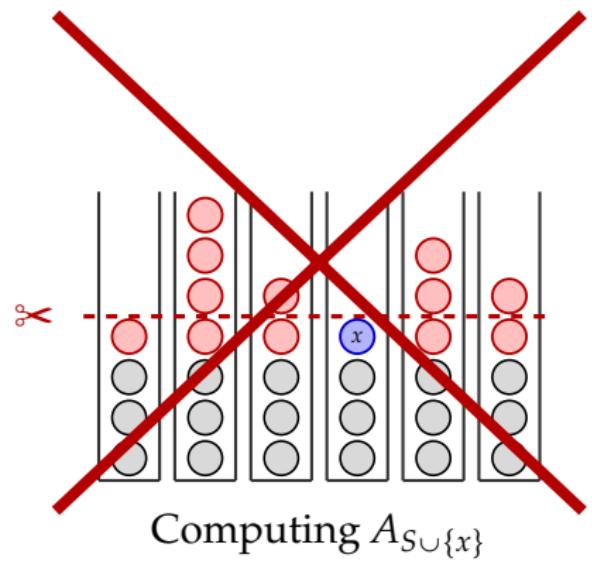
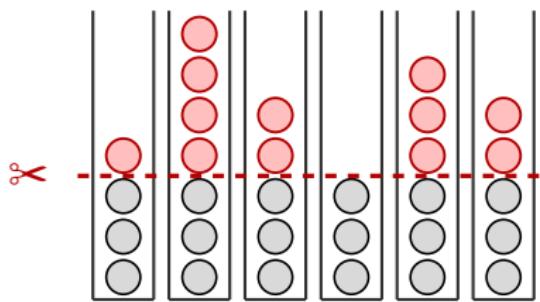
# WHAT'S THE RECOURSE?



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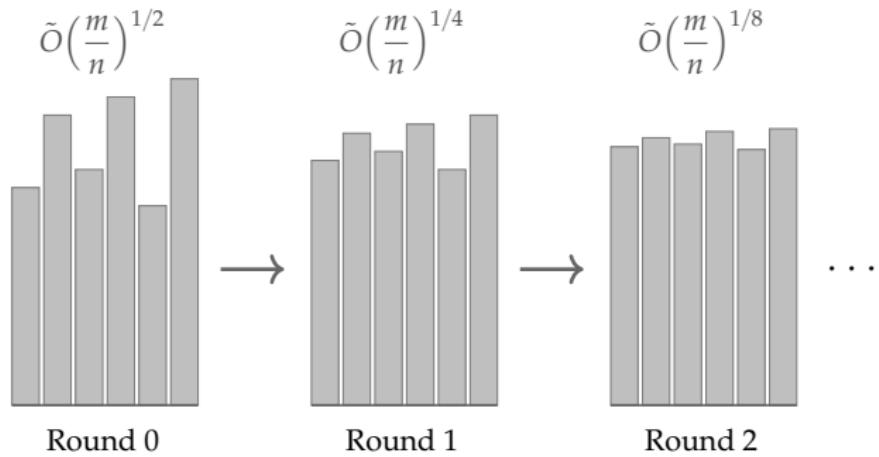


# WHAT'S THE RECOURSE?

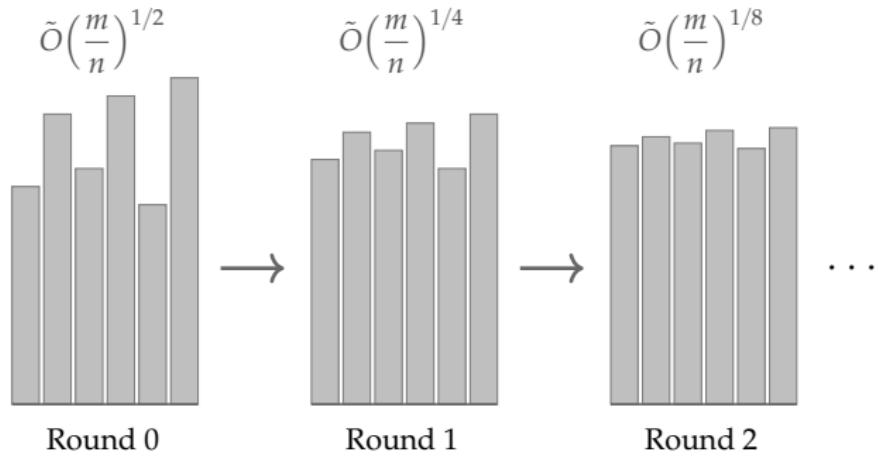


To keep recourse low, fix the slicing line (as a function of the maximum number of balls in the system).

# REPEATEDLY SLICING AND SPREADING



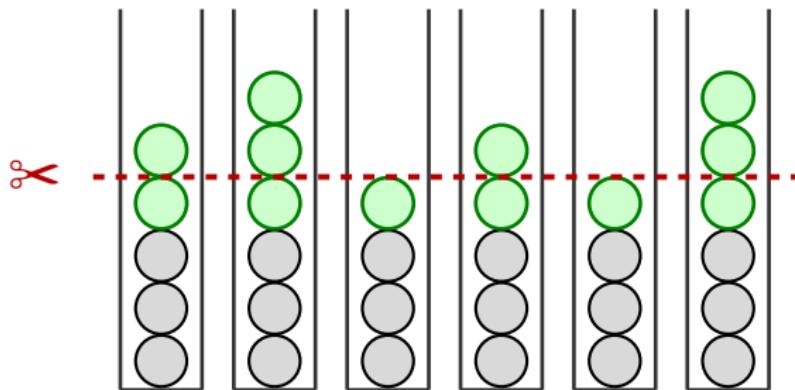
# REPEATEDLY SLICING AND SPREADING



## Proposition

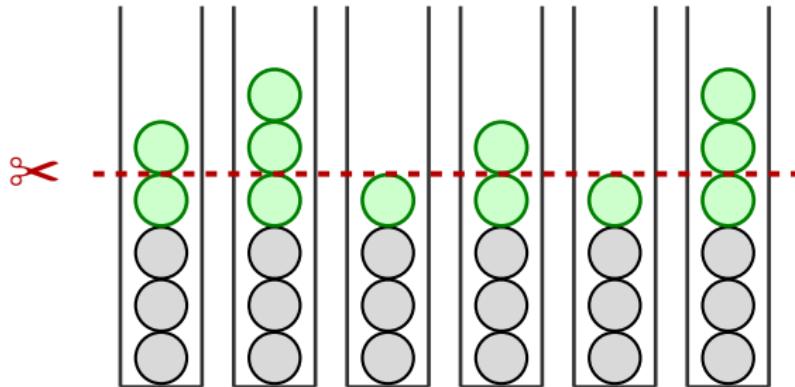
After  $O(\log \log(m/n))$  rounds of slice-and-spread, the cumulative overload is  $O(n)$  with high probability in  $n$ . The expected recourse is  $O(\log \log(m/n))$ .

# ALGORITHMIC QUESTION



**Question:** Which balls do we slice in each round?

# ALGORITHMIC QUESTION

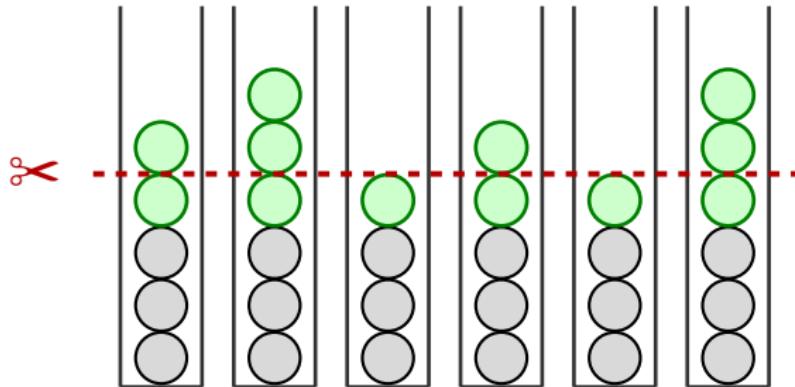


**Question:** Which balls do we slice in each round?

- ▶ **Option 1:** Scrape off the top

✗ Reuses stale randomness

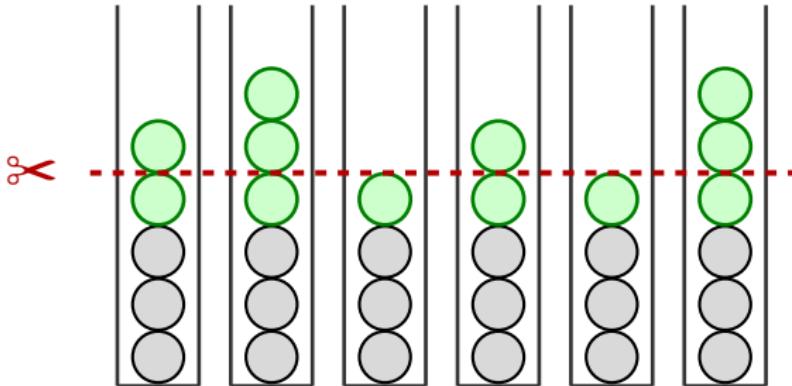
# ALGORITHMIC QUESTION



**Question:** Which balls do we slice in each round?

- ▶ **Option 1:** Scrape off the top ✗ Reuses stale randomness
- ▶ **Option 2:** Priority queue per bin ✗ Exploding recourse

# ALGORITHMIC QUESTION



**Question:** Which balls do we slice in each round?

- ▶ **Option 1:** Scrape off the top ✗ Reuses stale randomness
- ▶ **Option 2:** Priority queue per bin ✗ Exploding recourse
- ▶ **Our approach:** Option 2 + Assign every ball a **round number**

## CHALLENGE 1: SLICING FAILURES

**Challenge:** There may not be enough fresh randomness in each bin. Not enough balls assigned to that round in that bin.

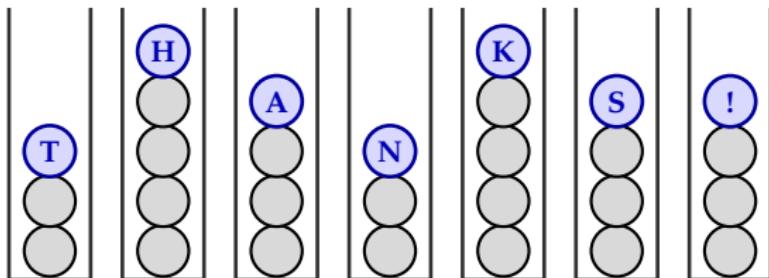
**Result:** We won't be able to slice off enough balls, and so the jaggedness will remain.

## CHALLENGE 2: SPREADING FAILURES

**Challenge:** Spreading step got unlucky and didn't spread very evenly. There's more jaggedness than we want

**Result:** Either we have to slice more balls than we want and may not get the overload down, or we have to slice less balls than we want and may not get the jaggedness down.

# History-Independent Load Balancing



Michael A. Bender  
Stony Brook University

William Kuszmaul  
CMU

Elaine Shi  
CMU

**Rose Silver**  
CMU