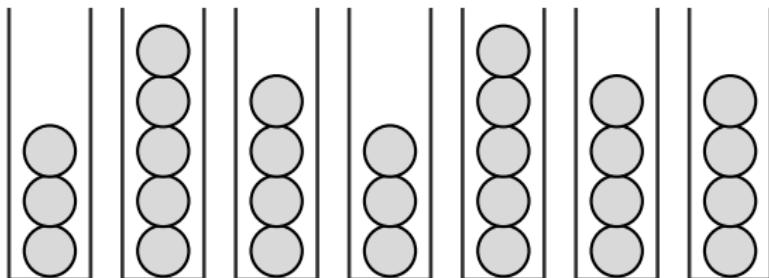


# History-Independent Load Balancing



Michael A. Bender

Stony Brook University

Bill Kuszmaul

CMU

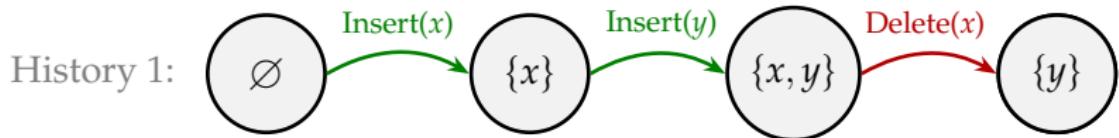
Elaine Shi

CMU

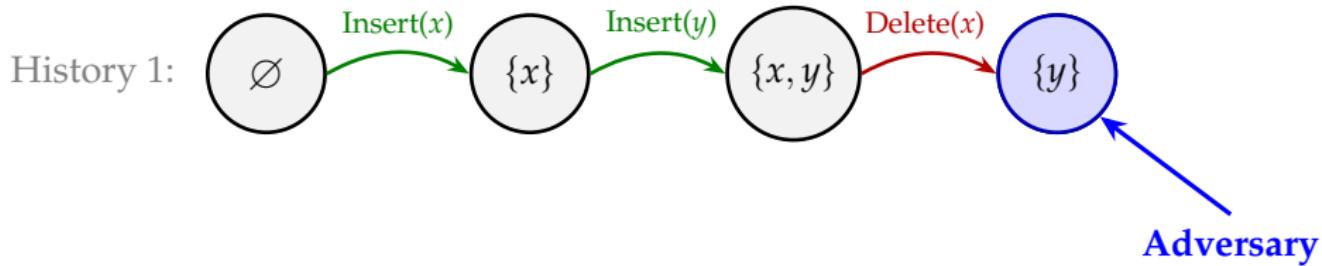
Rose Silver

CMU

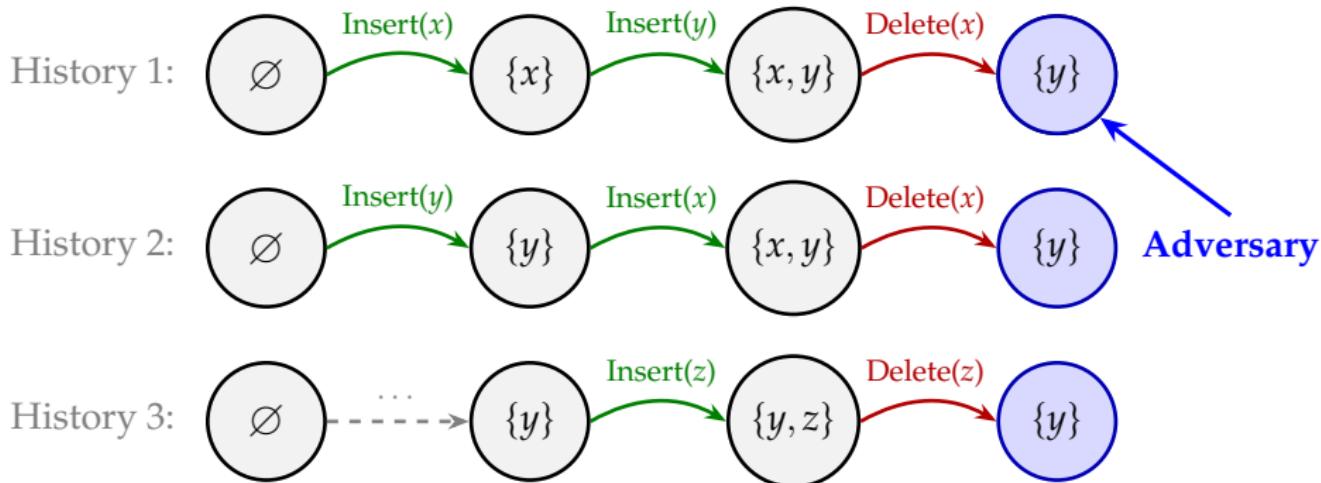
# HISTORY-INDEPENDENT DATA STRUCTURES



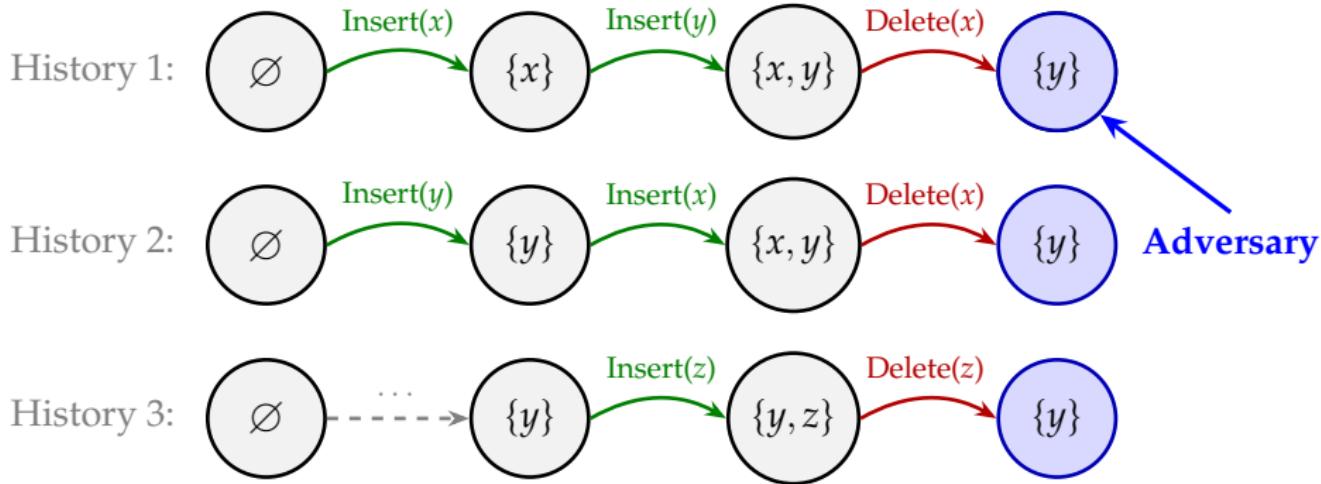
# HISTORY-INDEPENDENT DATA STRUCTURES



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**History Independence** (Micciancio '97, Naor & Teague '01)

- ▶ The state reveals only the current elements—not the history of operations.

# HISTORY INDEPENDENT DATA STRUCTURES

## A History of Applications

Hash tables, trees, memory allocation, PMAs, graph algorithms, cache-oblivious data structures, and more.

Micciancio '97, Naor & Teague '01, Buchbinder & Petrank '03, Molnar et al. '06, Blelloch & Golovin '07,  
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**Yet fundamental questions remain open.**

# HISTORY INDEPENDENT DATA STRUCTURES

## A History of Applications

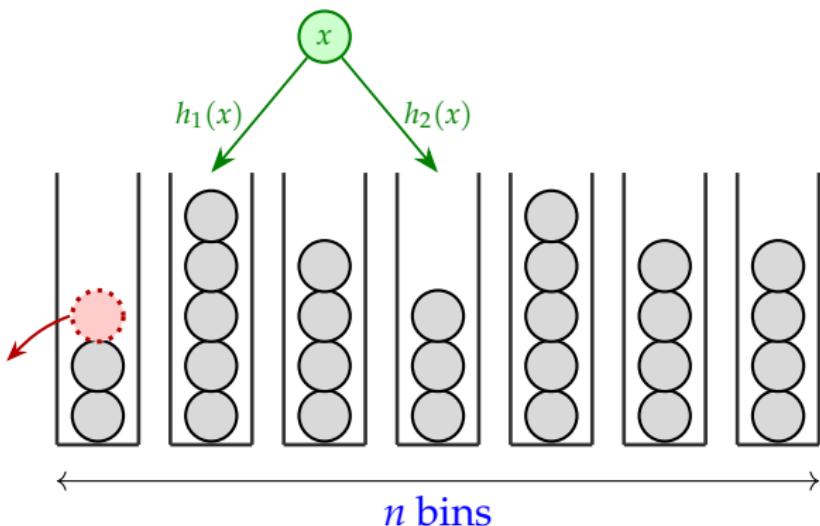
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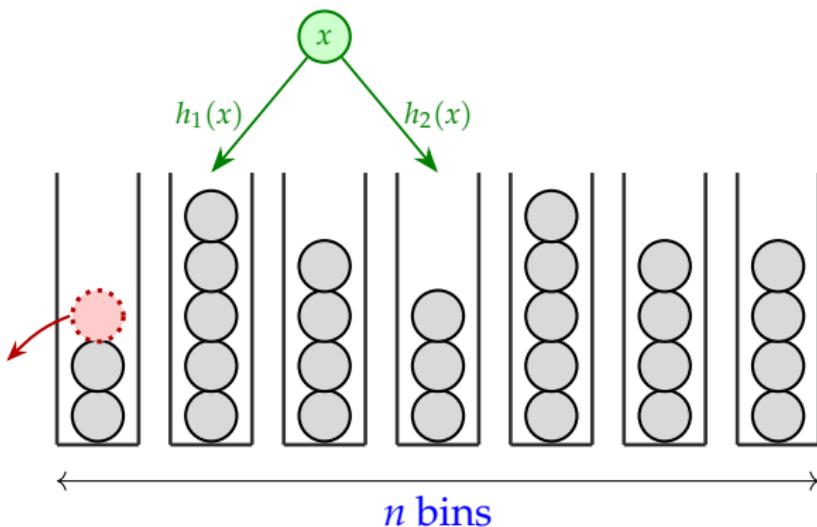
**Yet fundamental questions remain open.**

**This work:** History-Independent Load Balancing

# TWO-CHOICE LOAD BALANCING

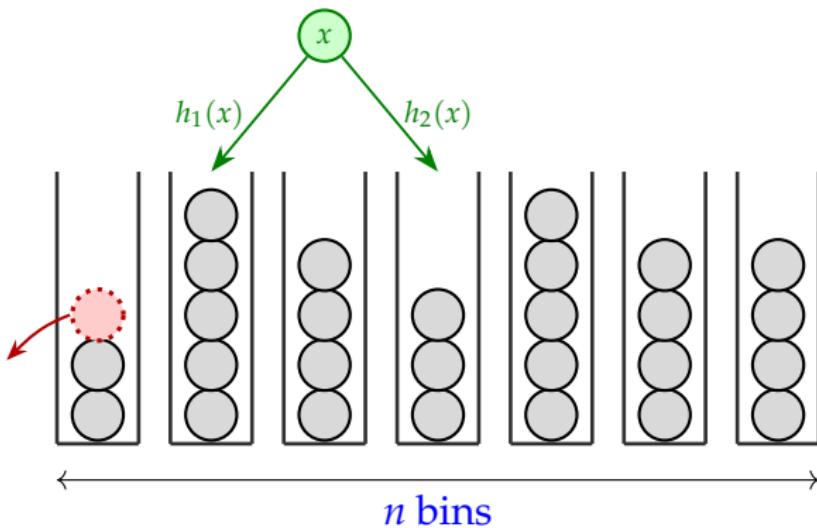


# TWO-CHOICE LOAD BALANCING



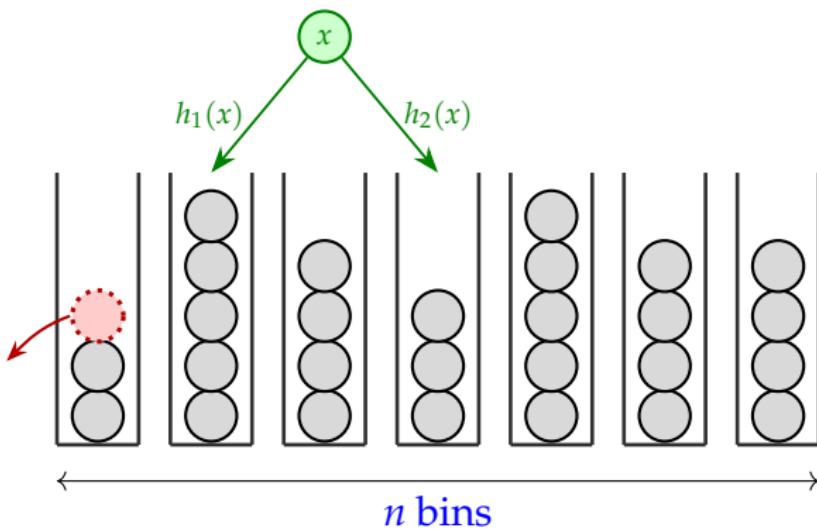
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# TWO-CHOICE LOAD BALANCING



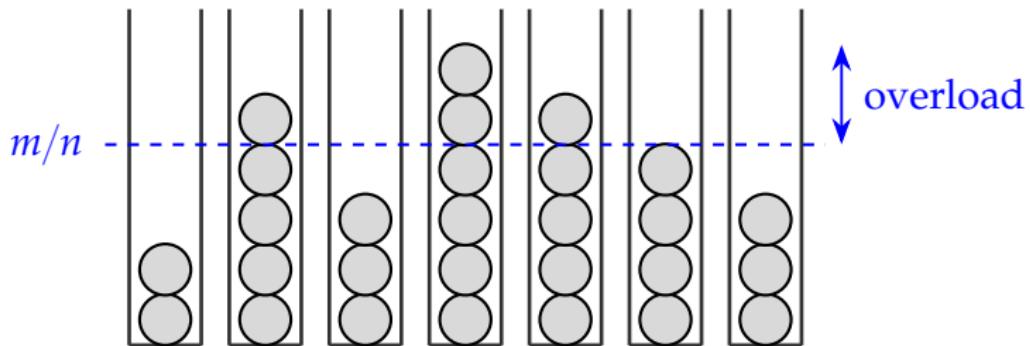
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- ▶ Each ball has two random bins where it can go.

# TWO-CHOICE LOAD BALANCING



- ▶ Balls are **inserted/deleted**, with up to  $m$  present at a time.
- ▶ Each ball has two random bins where it can go.
- ▶ We must maintain a valid assignment of balls to bins.

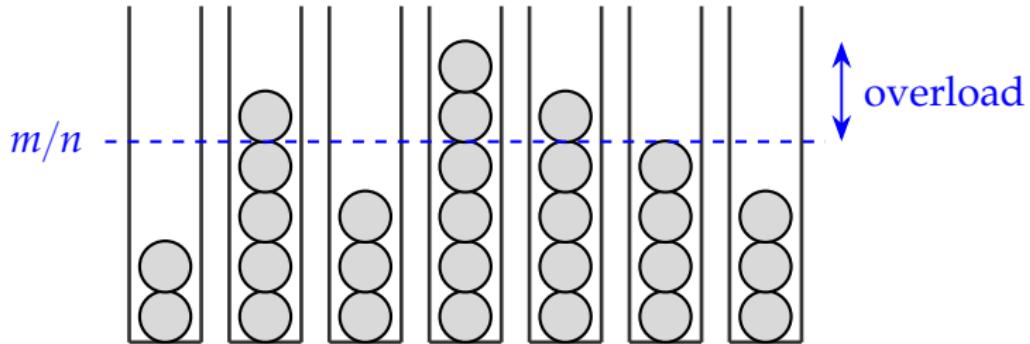
## TWO GOALS



### Minimize Overload:

- ▶ i.e., the amount by which the fullest bin exceeds  $m/n$ .

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### Minimize Overload:

- ▶ i.e., the amount by which the fullest bin exceeds  $m/n$ .

### Minimize Recourse:

- ▶ i.e., the number of balls moved around on any given insertion/deletion.

# PUTTING IT ALL TOGETHER

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# PUTTING IT ALL TOGETHER

## History-Independent Load Balancing:

- ▶ For all sets  $S$  of balls, if the current set is  $S$ , then the assignment is always  $A_S$ .

**Question:** Does there exist a history-independent solution with small recourse and small overload?

**Our Main Result:** There exists a history-independent solution with:

- ▶ High probability overload  $O(1)$
- ▶ Expected recourse  $O(\log \log(m/n))$

## PAST WORK (NOT HISTORY INDEPENDENT)

---

Overload	Recourse	Reference	Caveats
$O(\log \log n)$	0	[ABKU '94] [BCSV '00]	insertion-only
$O(1)$	$O(\log(m/n))$	[Dietzfelbinger, Weidling '07]	insertion-only
$\tilde{O}(\sqrt{m/n})$	$O(1)$	[Frieze, Petti '18]	insertion-only
$O(\log(m/n))$	0	[Bansal, Kuszmaul '22]	no reinsertions
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If we want overload  $O(1)$ , our result is a new state of the art!

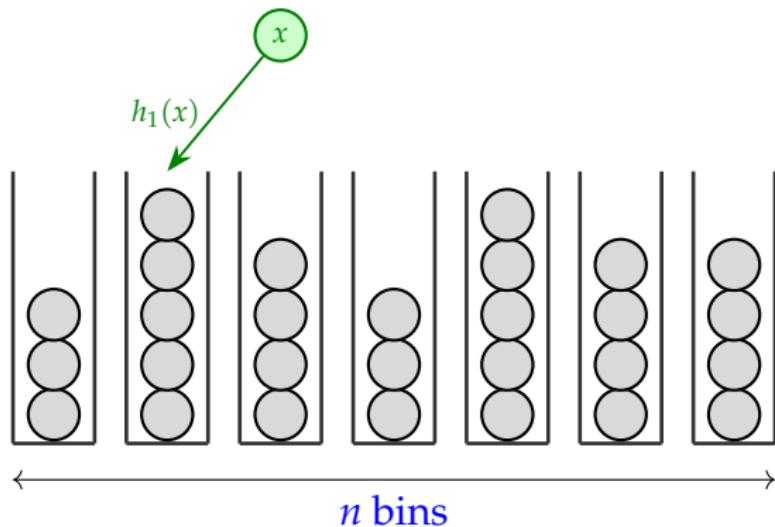
## REST OF TALK: A SIMPLE WARMUP

**Theorem:** There exists a history-independent solution with:

- ▶ High-probability overload  $\Theta(1)$   $O(\log \log n)$ .
- ▶ Expected recourse  $\Theta(\log \log(m/n))$   $O(m/n)$ .

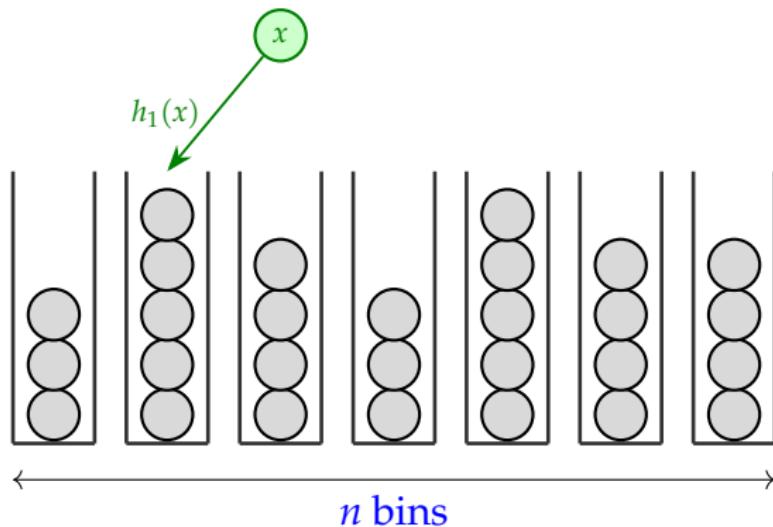
## BASELINE 1: THE SINGLE-CHOICE STRATEGY

To insert a ball  $x$ , just put it in bin  $h_1(x)$ :



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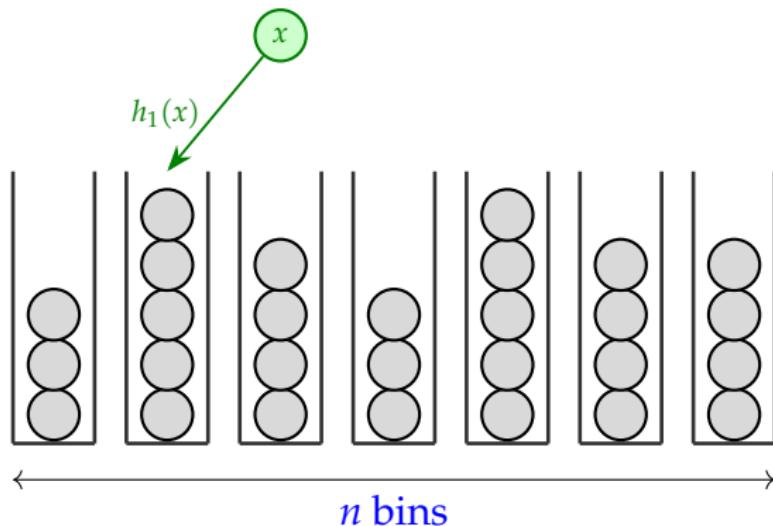
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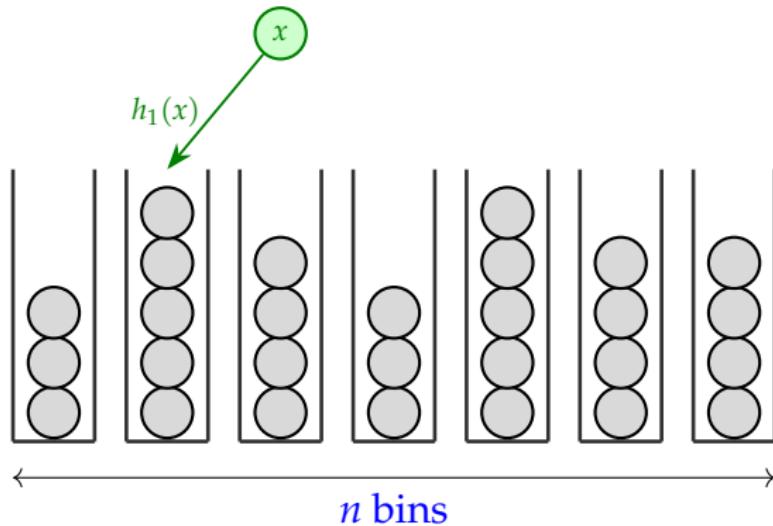
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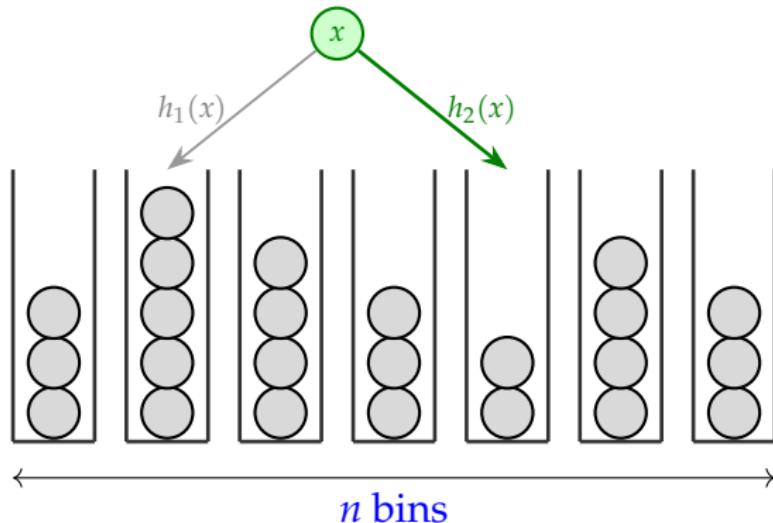
To insert a ball  $x$ , just put it in bin  $h_1(x)$ :



- ▶ This is history-independent ✓
- ▶ The recourse is 0 ✓
- ▶ But... the overload is huge, roughly  $\sqrt{m/n}$  ✗

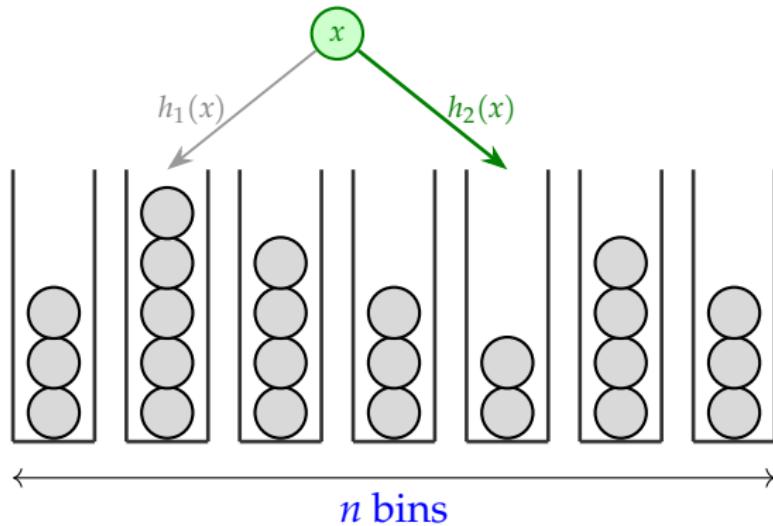
## BASELINE 2: GREEDY INSERTIONS

To insert a ball  $x$ , put it in the **emptier** of its choices:



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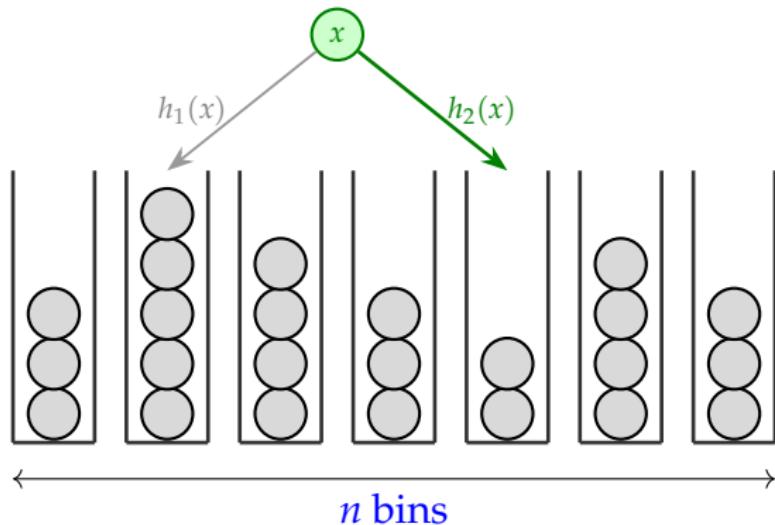
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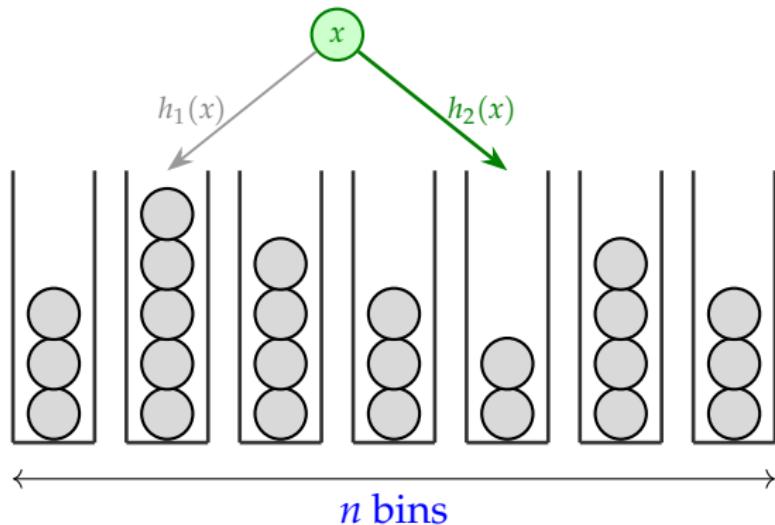
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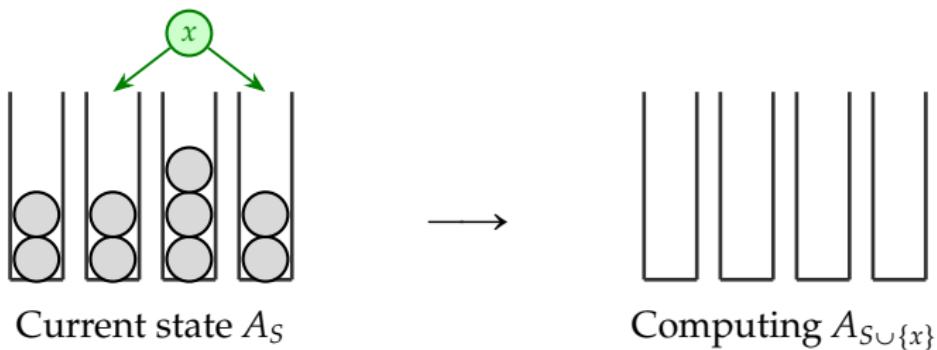
To insert a ball  $x$ , put it in the **emptier** of its choices:



- ▶ This is **not** history-independent ✗
- ▶ The recourse is 0 ✓
- ▶ In the insertion-only case, the overload is  $O(\log \log n)$  ✓

[Azar, Broder, Karlin and Upfal '94]

# A SIMPLE HISTORY-INDEPENDENT ALGORITHM



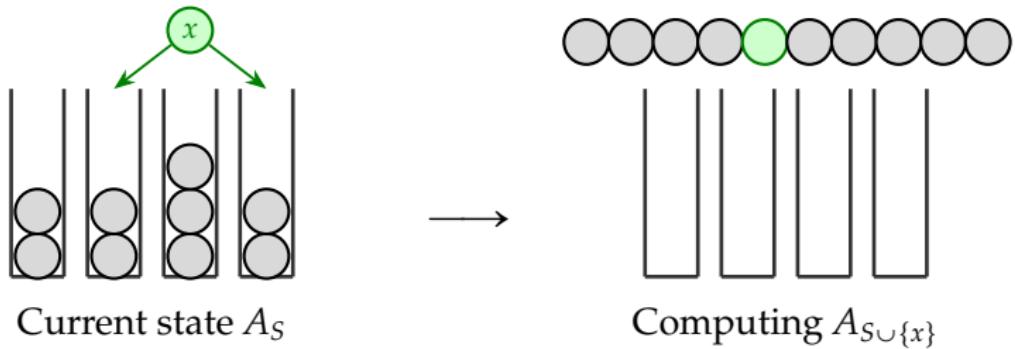
# A SIMPLE HISTORY-INDEPENDENT ALGORITHM



To insert a new ball  $x$ , we compute  $A_{S \cup \{x\}}$ :

1. Empty out the bins.
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3. Insert the balls in sorted order using greedy.

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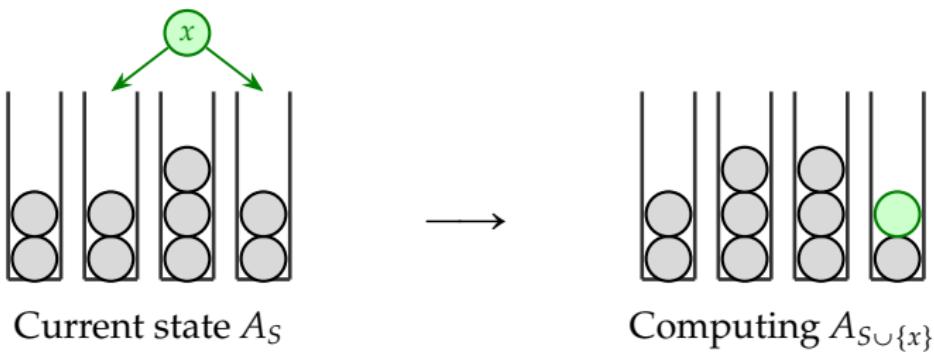
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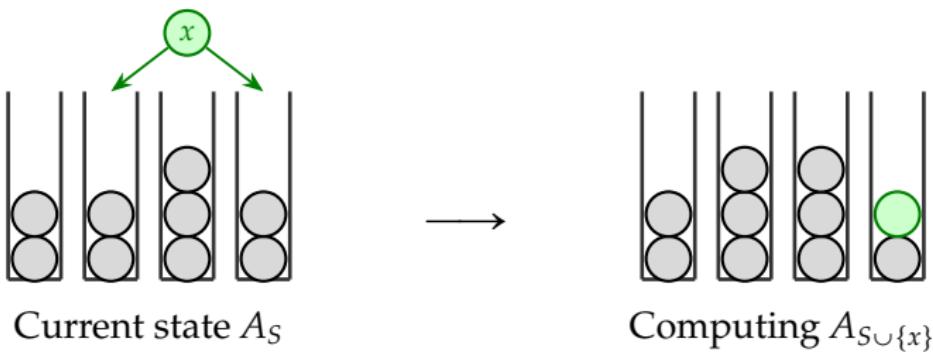
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## ANALYZING HISTORY-INDEPENDENT GREEDY

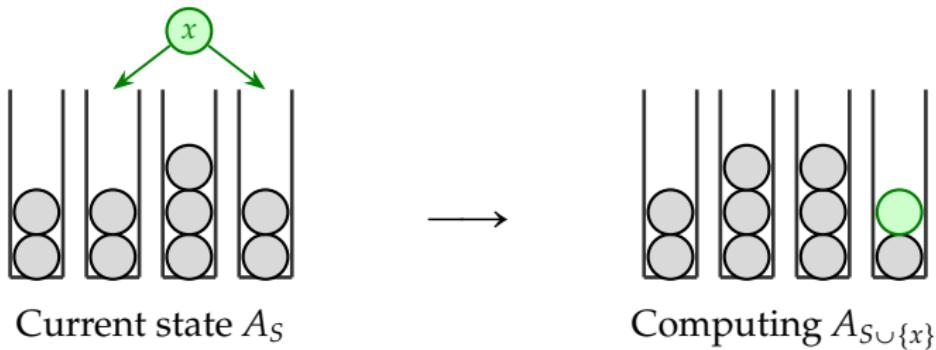


## ANALYZING HISTORY-INDEPENDENT GREEDY



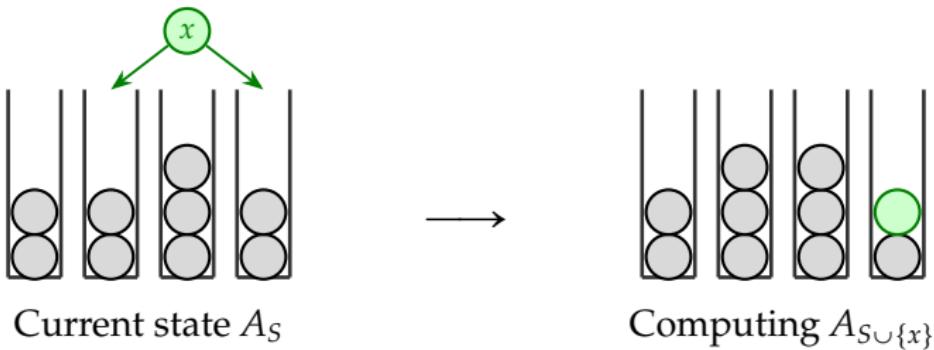
- ▶ The algorithm is history independent ✓

## ANALYZING HISTORY-INDEPENDENT GREEDY



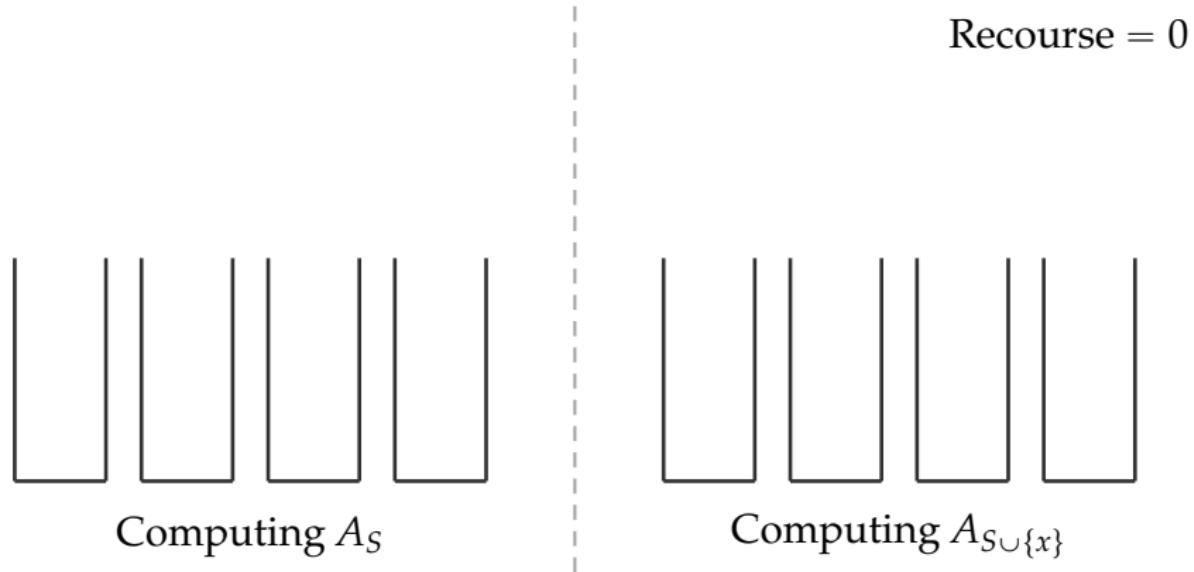
- ▶ The algorithm is history independent ✓
- ▶ The overload is  $O(\log \log n)$  ✓

# ANALYZING HISTORY-INDEPENDENT GREEDY



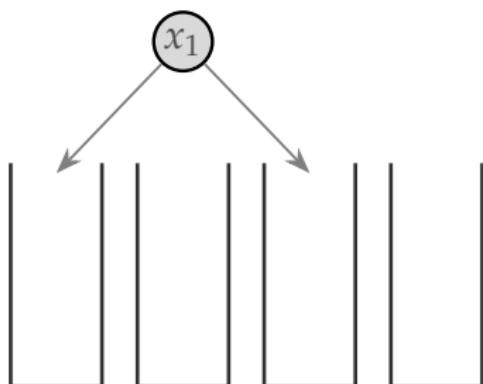
- ▶ The algorithm is history independent ✓
- ▶ The overload is  $O(\log \log n)$  ✓
- ▶ What is the recourse?

# ANALYZING THE RECOURSE

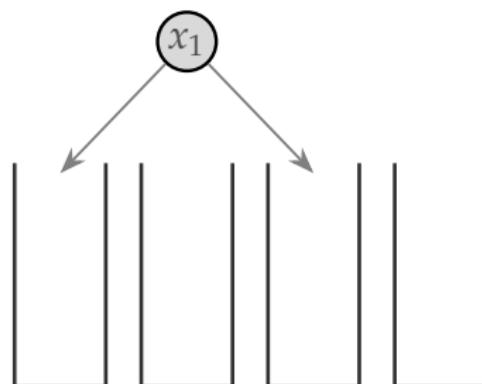


How many balls change assignments between  $A_S$  and  $A_{S \cup \{x\}}$ ?

# ANALYZING THE RECOURSE



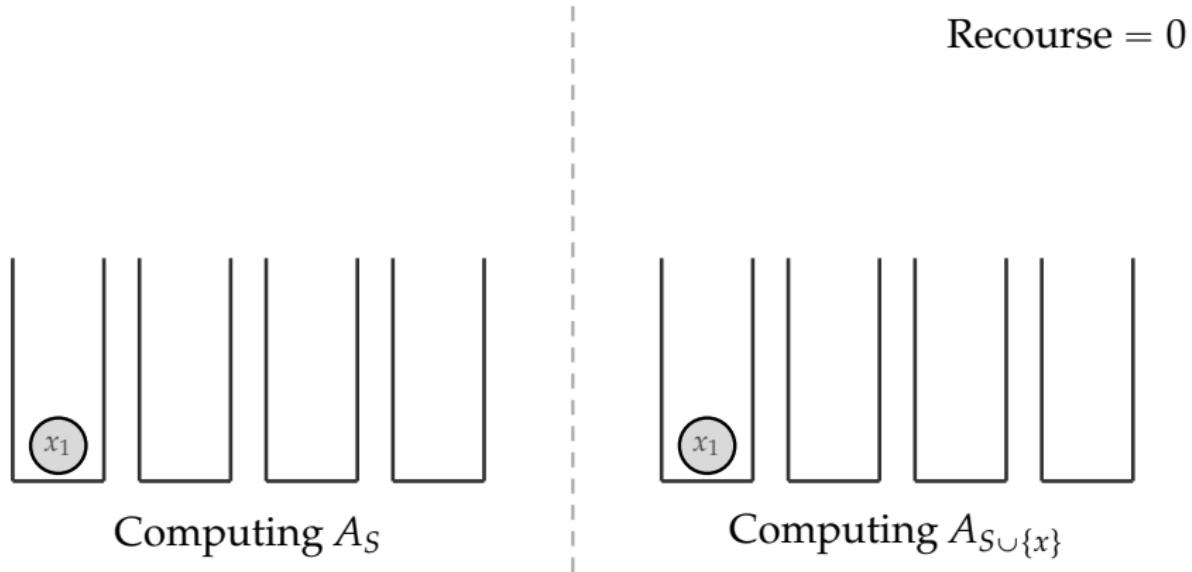
Computing  $A_S$



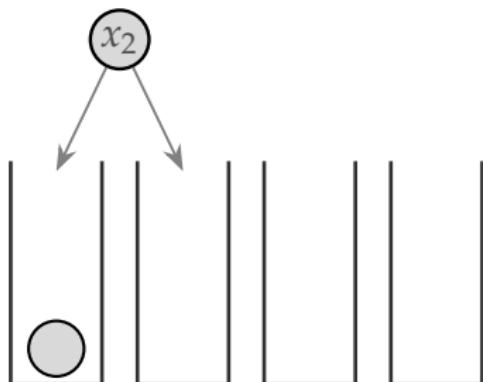
Computing  $A_{S \cup \{x\}}$

Recourse = 0

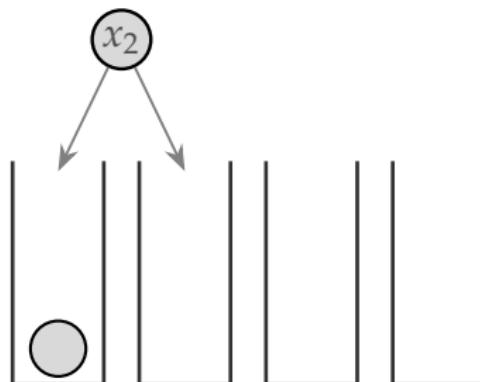
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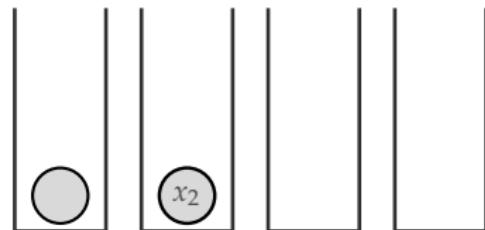
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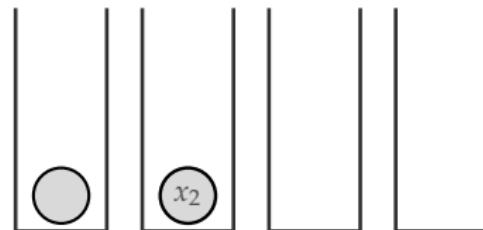
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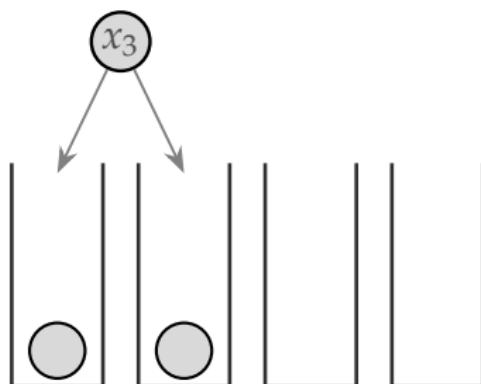
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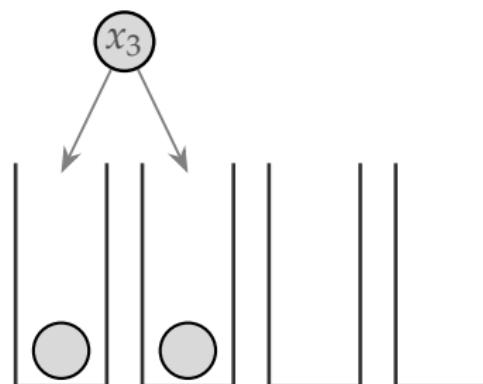
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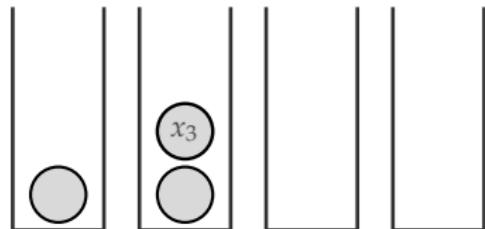
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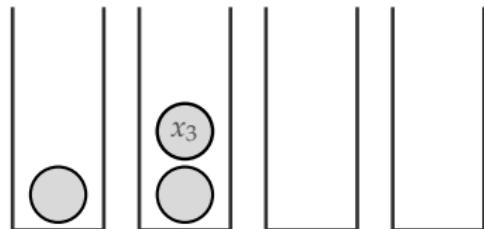
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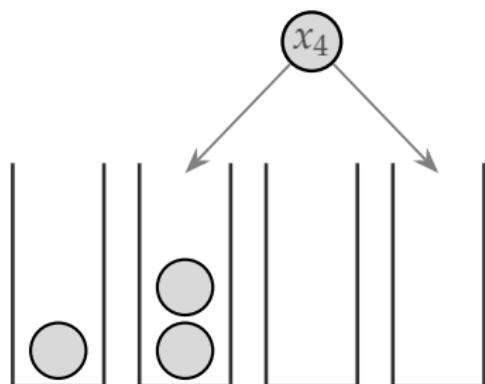
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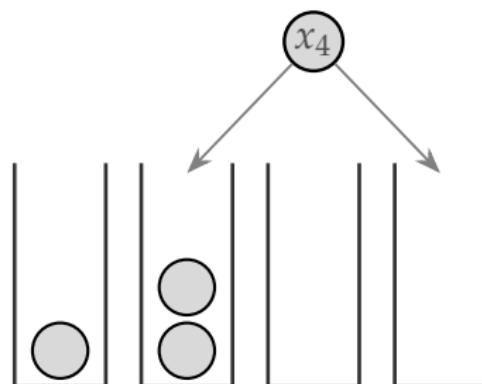
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## ANALYZING THE RECOURSE



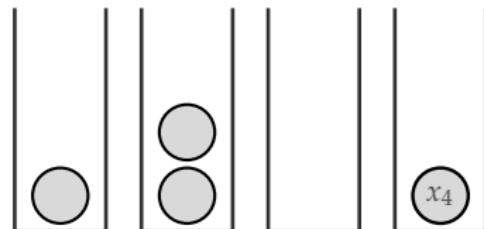
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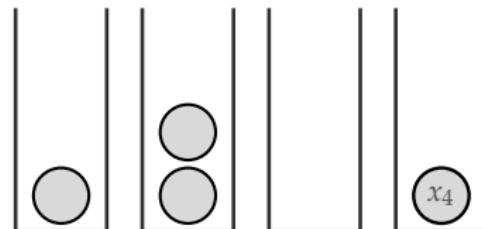
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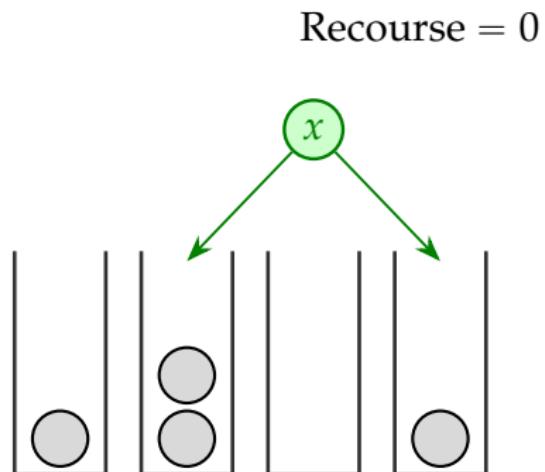
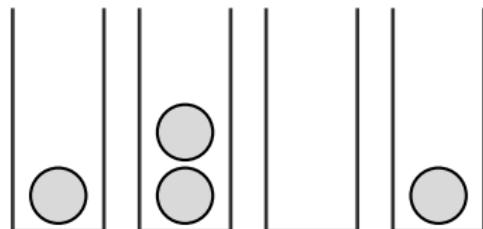
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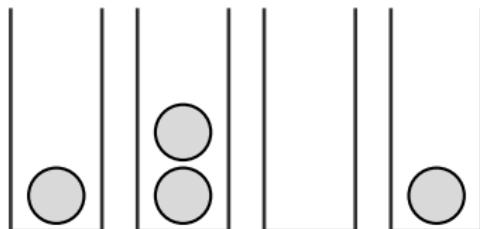
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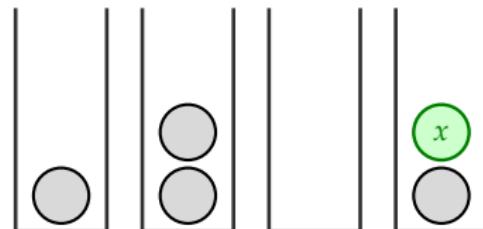
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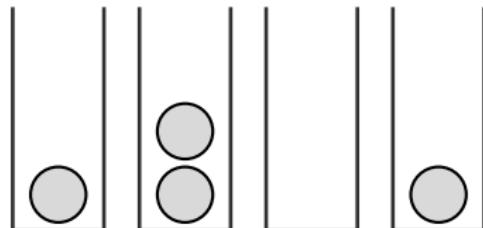


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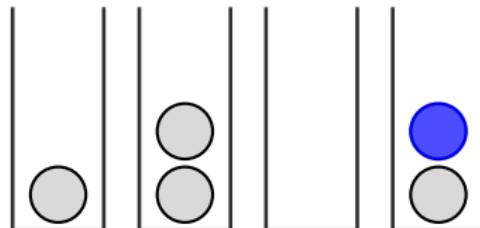


Recourse = 0

## ANALYZING THE RECOURSE



Computing  $A_S$

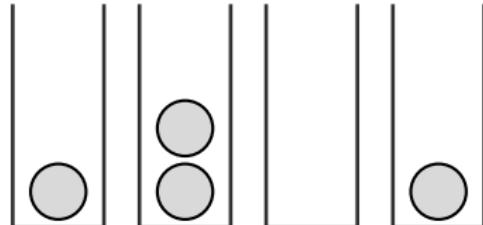


Computing  $A_{S \cup \{x\}}$

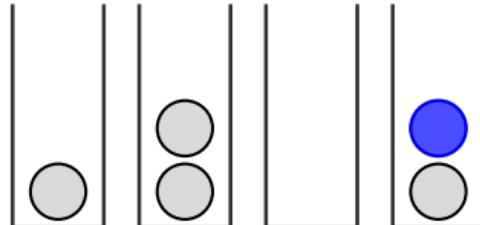
Subsequent balls will experience either:

Recourse = 0

## ANALYZING THE RECOURSE



Computing  $A_S$



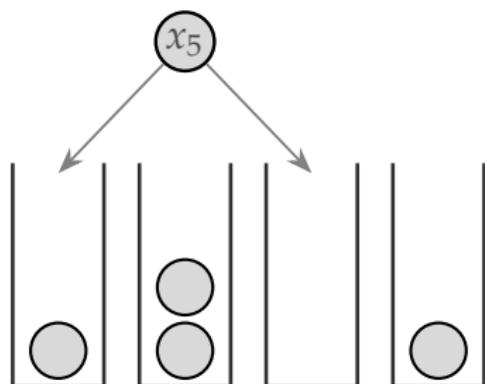
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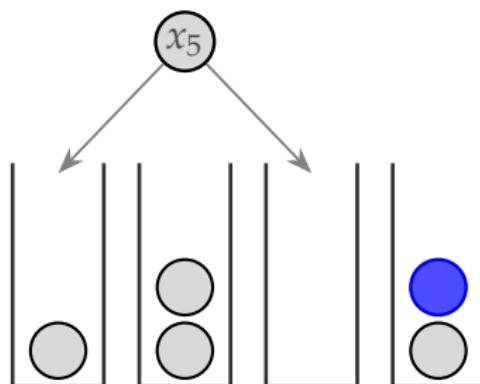
1. No recourse

$$\text{Recourse} = 0$$

## ANALYZING THE RECOURSE



Computing  $A_S$



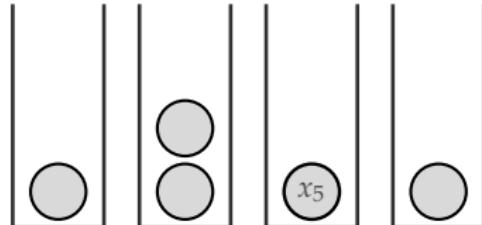
Computing  $A_{S \cup \{x\}}$

Future insertions will experience either:

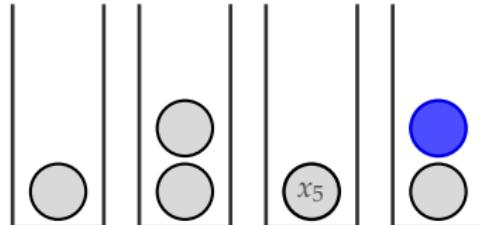
1. No recourse

Recourse = 0

# ANALYZING THE RECOURSE



Computing  $A_S$

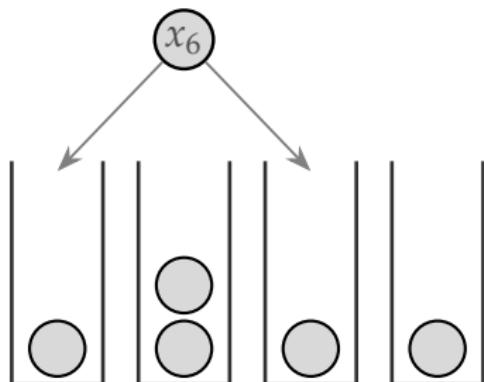


Computing  $A_{S \cup \{x\}}$

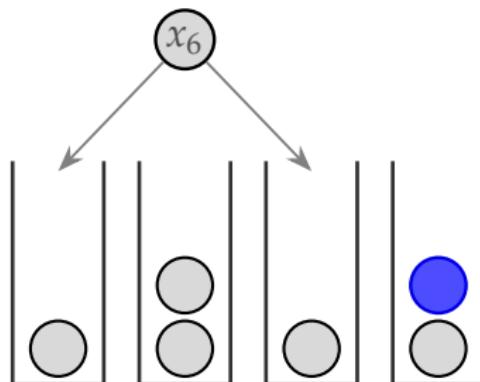
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1. No recourse

## ANALYZING THE RECOURSE



Computing  $A_S$



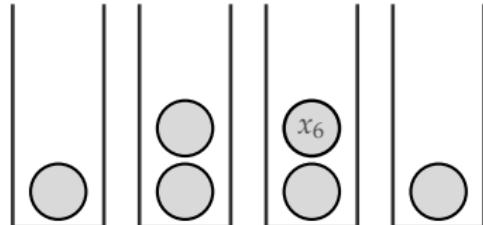
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

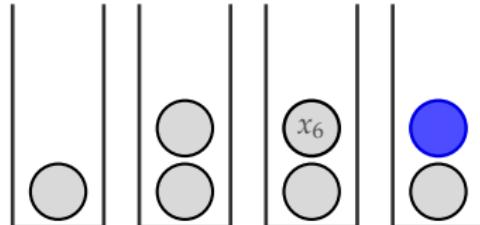
1. No recourse

Recourse = 0

# ANALYZING THE RECOURSE



Computing  $A_S$

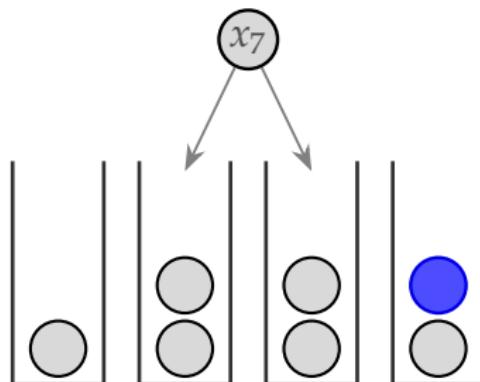
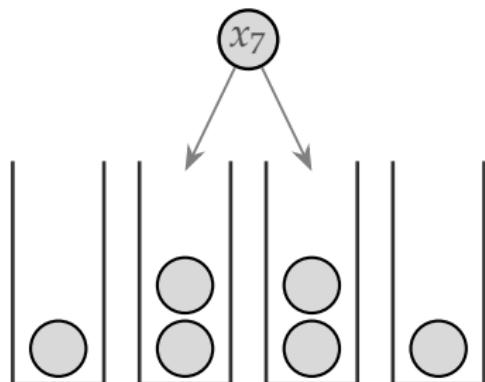


Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

1. No recourse

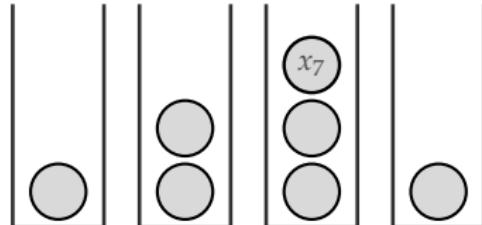
# ANALYZING THE RECOURSE



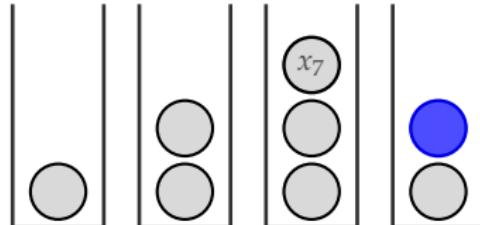
Subsequent balls will experience either:

1. No recourse

# ANALYZING THE RECOURSE



Computing  $A_S$



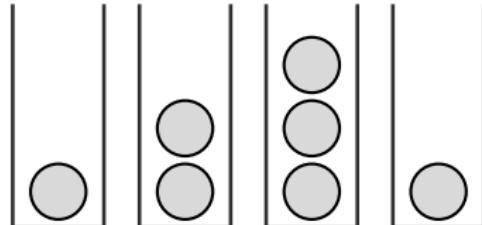
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

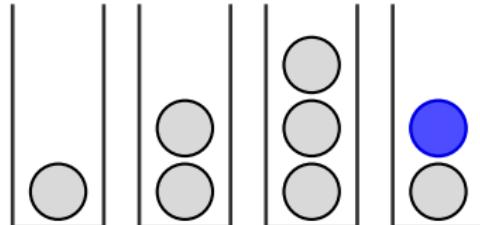
1. No recourse

$$\text{Recourse} = 0$$

# ANALYZING THE RECOURSE



Computing  $A_S$



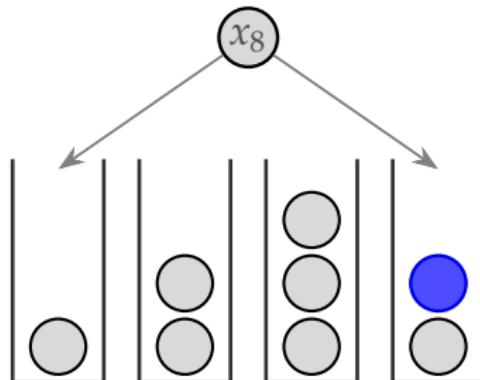
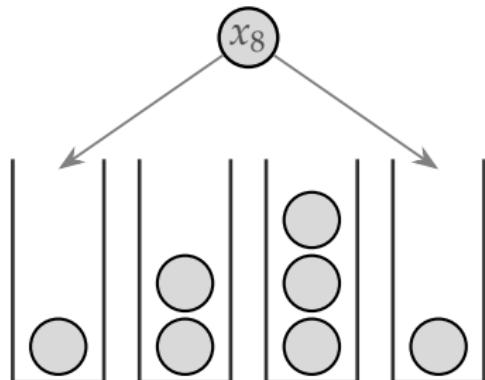
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

1. No recourse
2. Recourse

$$\text{Recourse} = 0$$

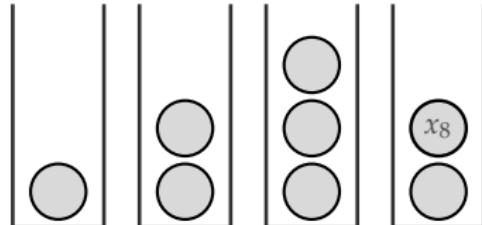
## ANALYZING THE RECOURSE



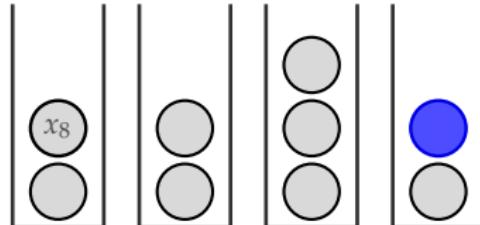
Subsequent balls will experience either:

1. No recourse
2. Recourse

# ANALYZING THE RECOURSE



Computing  $A_S$



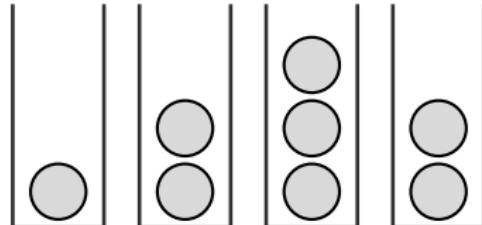
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

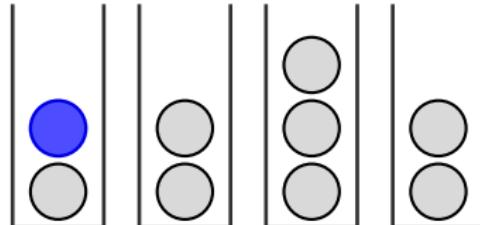
1. No recourse
2. Recourse

Recourse = 1

## ANALYZING THE RECOURSE



Computing  $A_S$



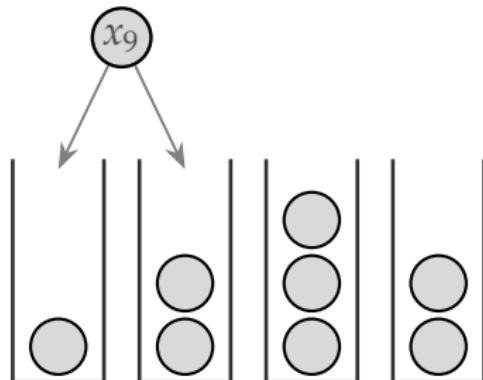
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

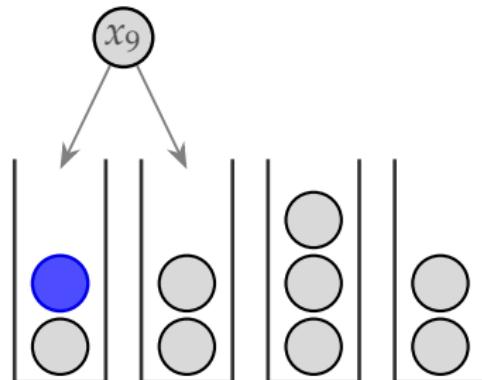
1. No recourse
2. Recourse

Recourse = 1

# ANALYZING THE RECOURSE



Computing  $A_S$



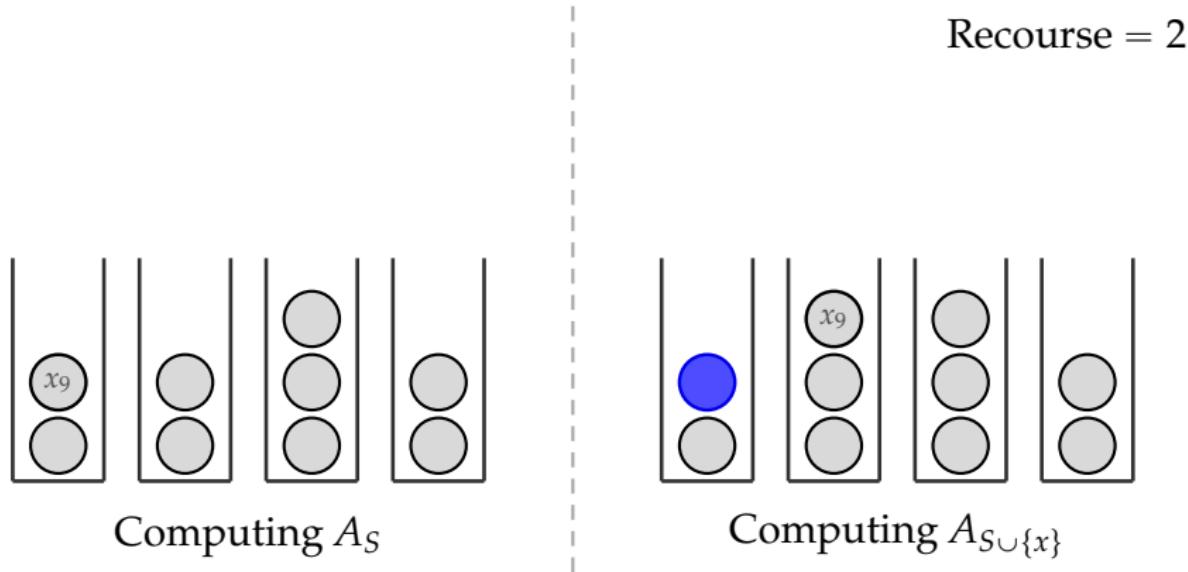
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

1. No recourse
2. Recourse

Recourse = 1

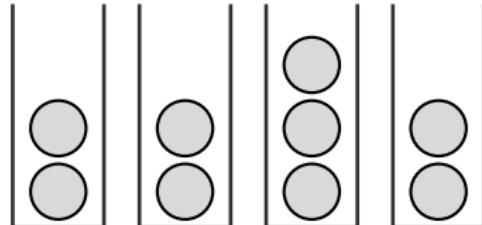
# ANALYZING THE RECOURSE



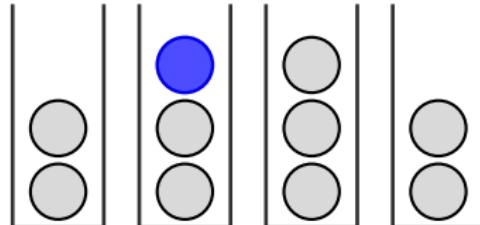
Subsequent balls will experience either:

1. No recourse
2. Recourse

# ANALYZING THE RECOURSE



Computing  $A_S$



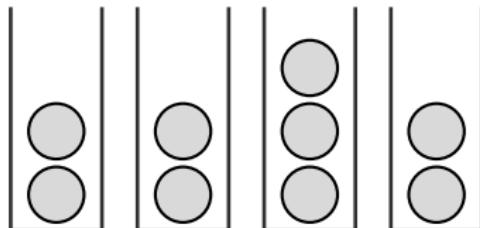
Computing  $A_{S \cup \{x\}}$

Subsequent balls will experience either:

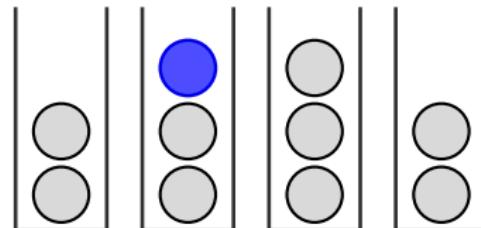
1. No recourse
2. Recourse

Recourse = 2

# ANALYZING THE RECOURSE



Computing  $A_S$

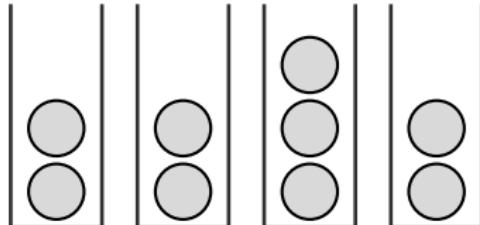


Computing  $A_{S \cup \{x\}}$

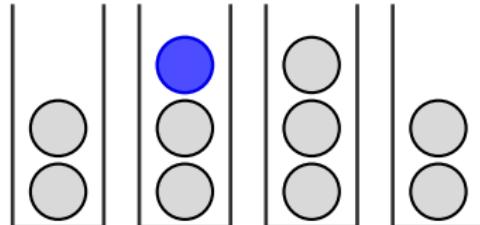
Two key observations:

Recourse = 2

# ANALYZING THE RECOURSE



Computing  $A_S$



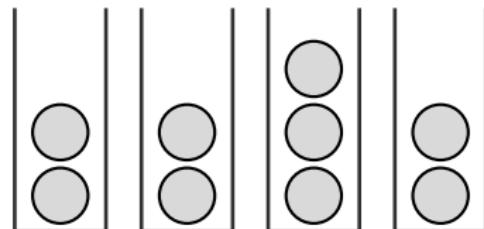
Computing  $A_{S \cup \{x\}}$

Two key observations:

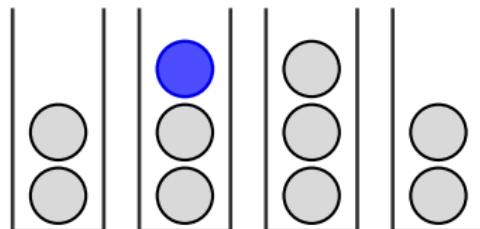
1. There's always one special bin with an extra ball

Recourse = 2

# ANALYZING THE RECOURSE



Computing  $A_S$



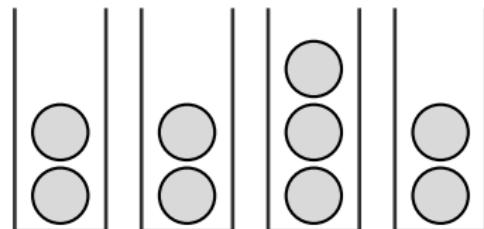
Computing  $A_{S \cup \{x\}}$

Two key observations:

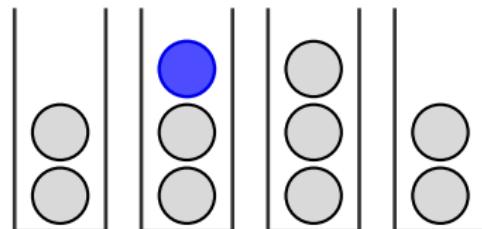
1. There's always one special bin with an extra ball
2. If a ball incurs recourse, one of its choices is the special bin

Recourse = 2

# ANALYZING THE RECOURSE



Computing  $A_S$

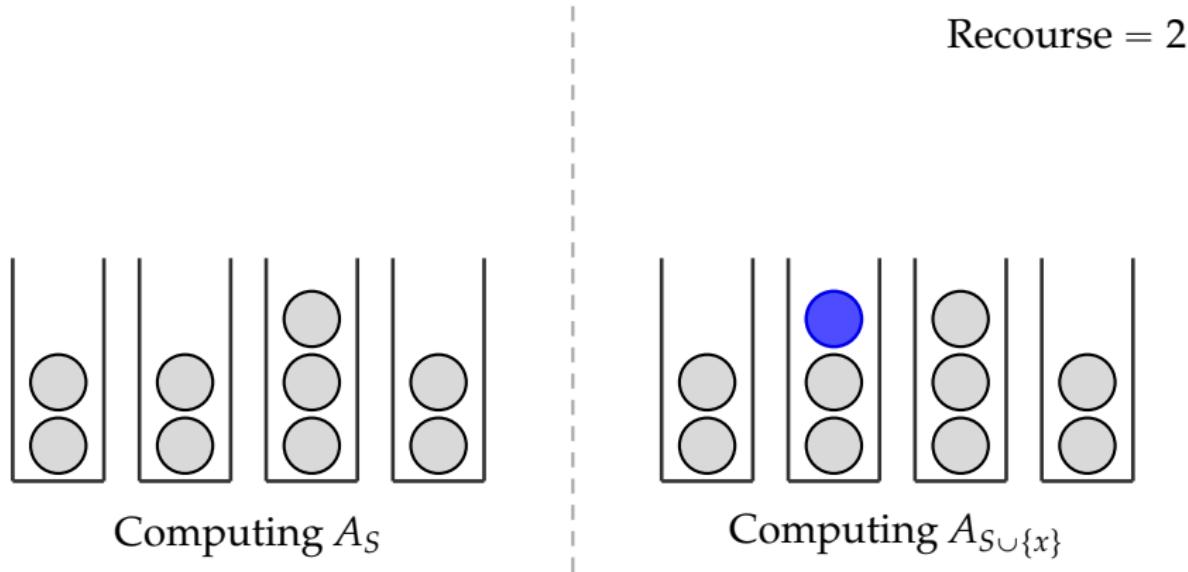


Computing  $A_{S \cup \{x\}}$

$$\Pr[\text{ball } x_i \text{ incurs recourse}] = O(1/n)$$

Recourse = 2

# ANALYZING THE RECOURSE



$$\Pr[\text{ball } x_i \text{ incurs recourse}] = O(1/n)$$

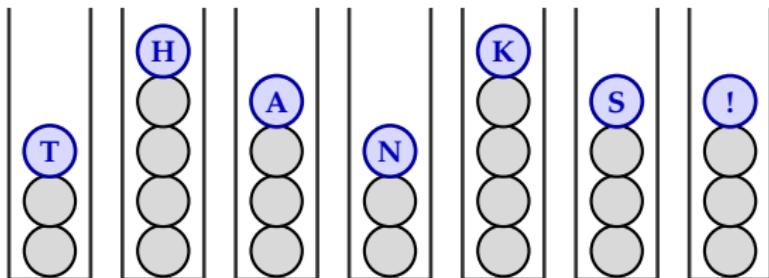
$$\implies \mathbb{E}[\text{total recourse}] = \sum_i \Pr[\text{ball } x_i \text{ incurs recourse}] = O(m/n)$$

## A SIMPLE WARMUP

**Theorem:** There exists a history-independent solution with:

- ▶ High-probability overload  $\Theta(1)$   $O(\log \log n)$ .
- ▶ Expected recourse  $\Theta(\log \log(m/n))$   $O(m/n)$ .

# History-Independent Load Balancing



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