Optimality for SoS SDPs: a Survey

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1 Introduction

. We examine constant degree sum of squares as a general tool in approximation algorithms, as it is a meta-algorithm that can be used to get very good approximations. As an algorithm, Sum of Squares is very general and adaptable to different sorts of problems, similar to how linear programming provides a framework for tackling different problems. Currently, many optimization problems require creative analyses' that are specific to the problem, but sum of squares makes use of semidefinite programming to be a general framework for solving algorithms. This gives us an approach to develop approximation algorithms for a wide class of problems.

Sum of squares is very powerful meta algorithm for minimizing a polynomial, subject to multiple polynomial constraints. It is parameterized by the degree of the polynomials, that we call d. It does this by providing a pseudistribution, $\tilde{E}$, a set of values mapping functions on variables to the real numbers, satisfying the property that $\tilde{E}[1] = 1$ and $\forall f \in \mathbb{R}[x], \tilde{E}[f^2] \geq 0$, that when applied to the polynomial we are minimizing, gives a minimum value such that there is no degree-d sum of squares proof that refutes the system. This pseudodistribution can be thought of as an actual distribution over the solutions to the polynomial equations. By doing this, we can often "round" the pseudodistribution to get an approximation to the actual solution with clever tricks, and through this process SoS gives us provably very strong approximation algorithms. When applied to polynomial systems with constant degree, these can be solved by constant degree sum of squares programs. A degree d sum of squares program can be solved in time $n^{O(d)}$, which means for many problems, this gives us a polynomial time approximation.

The organization of the rest of the note is to be as follows: Section 2 will give a general overview of approximation algorithms and what it means to have an algorithm that gives "optimal" approximation ratios. Section 3 will build on this by introducing the Unique Games Conjecture. Section 4 will discuss some results on SoS algorithms achieving such ratios. Section 5 will go in the opposite direction, discussing how sum of squares algorithms give an approach for disproving the unique games conjecture. Section 6 will then conclude by offering some final remarks, thoughts on open problems, and future possible research directions.

2 Approximation Algorithms

Approximation algorithms as a field is quite broad, so we will not go too in depth in this section. We will instead give a high level overview of some fundamental results relating to hardness of approximation.

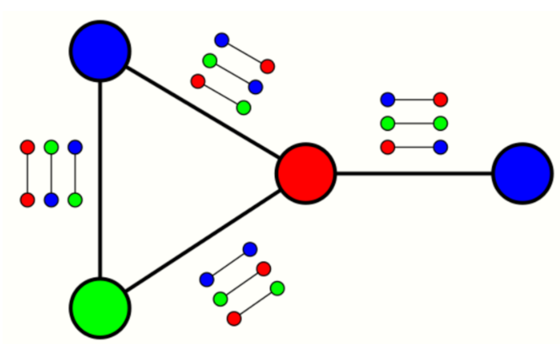
Approximation algorithms in general can be somewhat difficult to prove results for, as there are fundamental gaps in many areas. For many problems, lower bounds on approximations, or inapproximability results can be found using reductions and ideas from the PCP theorem, which gives hardness of approximation for certain constraint satisfaction problems. An example of this is the 2001 proof of Johan Hastad, proving that in the MAX-E3SAT problem (the optimization problem of 3Sat but every clause has exactly 3 literals), we have that 7/8 is the optimal approximation ratio. The randomized algorithm which assigns variables randomly in expectation satisfies $\frac{7}{8}m$ clauses out of a total m, which suggests that there is an algorithm to approximate 3-SAT within optimal (and the Karloff-Zwick algorithm shows there is). We note that this holds for the MAX-3SAT problem when it is satisfiable, but not necessarily when it is unsatisfiable.

However, for many problems there are gaps in between the known lower bounds for approximation algorithms and best known algorithm. For instance, for the metric Traveling Salesman Problem, the best bounds for the hardness approximation are 123/122 for the symmetric case, meaning assuming $P \neq NP$, it is would be optimal for an approximation algorithm to get greater than 122/123 of the distance, as shown by Karpinski, Lampis, Shmied. However, it is not known if the bound is tight, e.g. we do not know that there cannot exist polynomial time algorithms that achieve this bound. Thus there is a large gap between that and the recent approximation ratio of Karlin Klein and Gharan in 2021, $1.5 - 10^{-36}$.

An open problem within the hardness of approximation can be stated as thus: what characteristics of problems are there that differentiate problems such that it is easy vs hard to find tight inapproximability results?

3 Unique Games Conjecture

In 2002, the Unique Games Conjecture was proposed, which gave rise to many new inapproximability results, and proving optimal approximation ratios. This conjecture is somewhat unique in that there is no consensus on whether or not it is true. It can be stated somewhat similarly to the PCP Theorem, and as such can be thought of as a stronger version of it with some modifications. A commonly used statement of the unique games conjecture is as follows:



Theorem 1 (Unique Games Conjecture) *The Unique Games Conjecture as stated by Khot says For arbitrarily small constants $\zeta,\ \delta > 0$, there exists a constant $k = k(\zeta, \delta)$ such that it is*

NP-hard to determine whether a unique 2-prover game with answers from a domain of size k has value at least $1 - \zeta$ or at most δ.

To put it into terms of graphs, it can be formulated as saying it is NP-hard to approximate a unique label cover instance's value for large enough k, so at least $(1 - \zeta)$ edges being satisfied vs δ edges.

This could also be written as for any $\delta > 0$ there is a value k such that given equations $x_i - x_j = c_{ij} \mod p$, it is NP hard to distinguish between there being a solution that satisfies $1 - \delta$ fraction vs saying no solution can satisfy more than δ.

Similar to the Unique Games Conjecture is the small set expansion hypothesis, which we will add for completeness. It is a stronger form of the unique games conjecture, but it as well is unknown if it is true or false. The small set expansion hypothesis implies the unique games conjecture, but not the other way around.

Theorem 2 (Small Set Expansion Hypothesis) *The Small Set Expansion Hypothesis as stated by Barak and Steurer states that for any undirected d-regular graph $G = (V, E)$ and $\delta > 0$, we let the "expansion profile" be $\min_{S \subset V, |S| < \delta V} |E(S, \bar{S})|/(d|S|)$. The conjecture is that for every $\epsilon > 0$, there is some δ such that it is NP-hard to distinguish between the case that the expansion profile is less than ϵ vs greater than 1 - ϵ.*

Many approaches tackling the unique games conjecture instead look to tackle the small set expansion hypothesis as well, under the assumption that an approach for one will give closure on the other.

4 Sum of Squares and optimal ratios

The unique games conjecture is inherently related to sum of squares semidefinite programs. Under the unique games conjecture, reductions can be made from the unique games problems to instances of constraint satisfaction problems that are known to be optimally solvable by certain semidefinite programs.

These reductions give us more powerful approximation ratios compared to previous ones that relied on just assuming $P \neq NP$, but also imply algorithms. For instance, the Goemans-Williamson algorithm in 1995 gives an approximation for the Max Cut problem that solves to an approximation ratio of 0.878, which is provably optimal under the unique games conjecture. It does this by using a semidefinite program to assign each vertex in the graph to a point in R^n, and take a random hyperplane through the origin. Then using the expectation of the number of edges cut by the hyperplane, this gives us an approximation for the Max cut, which is optimal under the unique games conjecture.

For general constraint satisfaction problems as well, in 2008 Raghavendra proved that every general constraint satisfaction problem, the unique games conjecture gives an optimal approximation ratio. Furthermore this can be reached by a simple semidefinite program. This is useful because we have that sum of squares programs rely on semidefinite programming as well, so constant degree sum of squares algorithms will also be optimal for general CSPs.

5 Sum of Squares and disproving Unique Games

While we do have that SoS algorithms can attain "optimal" approximation results under the unique games conjecture and small set expansion hypothesis, SoS arguments seems to have some promise of disproving the small set expansion hypothesis. Because of the relationship between the small set expansion hypothesis, this will likely provide some progress towards disproving the unique games conjecture itself.

The small set expansion hypothesis can be thought of as finding "sparse" vectors in a linear subspace. Barak and Steurer formalize this relation, with the following.

For notation, let $G = (V, E)$ be a $d-$regular graph with Laplacian L. Also for $p > 1$ and $\delta \in (0, 1)$, we say a vector in $x \in R^n$ is (δ, p)-sparse if $E_i x_i^{2p} \geq \delta^{1-p}(E_i x_i^2)^p$.

Theorem 3 (Equivalence of SSEH and finding sparse vectors) *For every $p \geq 2$ and $\varphi \in (0, 1)$,*

1. *Non expanding small sets imply sparse vectors. If there exists $S \in V$ with $|S| = o(|V|)$ and $\phi_G(S) \leq \varphi$ then there exists an $(o(1), p)-$sparse vector $x \in W_{\leq \varphi + o(1)}$ where for every $\lambda, W_{\leq \lambda}$ denotes the span of the eigenvectors of L with eigenvalue smaller than λ.*

2. *Sparse vectors imply non-expanding small sets. If there exists a $(o(1), p)-$sparse vector $x \in W_\varphi$, then there exists $S \in V$ with $|S| = o(|V|)$, $\phi_G(S) \leq \rho$ for some constant $\rho < 1$ depending on φ*

So the first statement claims that if the graph is determined to not expand significantly, there is a sparse vector in the subspace given by the span of small eigenvectors of the Laplacian. The proof of this equivalency is nontrivial, so it will not be included. The importance of this equivalency is that the determination of the minimum of $\phi_G(S)$ is close to 1 or close to 0 is the small set expansion hypothesis, and related to the idea of the unique games conjecture. But this is reduced into bounding the maximum of $E_i x_i^{2p}$ over unit vectors in a subspace.

This is a polynomial optimization problem, so the current approach to tackling this relies on bounding the degree of the sum of squares proof needed to verify this. It is known for a random subspace it can be done with a constant degree proof, but because of the current reliance on the degree of the subspace, which is related to the Laplacian, it is currently unknown whether or not this approach will bear fruit.

Lots of "evidence" as well for UGC requires showing that for some "hard" instances of problems makes certain algorithms we know fail, but SoS SDP generally can solve them with constant degree.

6 Conclusion and Ending Remarks

While we understand that this is not as focused on high dimensional statistics as other notes, the techniques of sum of squares programming and approximation algorithms can be applied to statistics as well, specifically with applications lower bounding statistical problems. This can be done by giving a sample pseudodistribution that appears feasible while being incorrect. This note can also be thought of as more of a general survey of the field of approximation algorithms and progress on sum of squares applications, instead of a specific technique.

Some open questions remaining are whether or not the unique games conjecture and small set expansion hypothesis are equivalent. Another one is that approximation algorithms guarantee us a best approximation, but for what problems can sum of squares algorithms give us a good average-case guarantee.

7 Citations

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