CS271: Data Structures

Name: William Nguyen, Cheryl Nguyen

Instructor: Dr. Stacey Truex

Project 0

1. Prove inductively that a set S with cardinality $n \geq 1$ has exactly $2n$ unique subsets.

   Hypothesis: Let $P(n)$ denotes the statement "a set S with cardinality $n \geq 1$ has exactly $2n$ unique subsets."

   Base case: We have that when $n = 1$,

   The LHS of $P(1)$ is that set S has one element, and then there are 2 unique subsets which are the subset of the one element and the empty subset.

   The RHS of $P(1)$ is $2^n = 2^1 = 2$ unique subsets.

   Since the LHS and RHS of $P(n)$ are equal, the base case is true.

   Inductive hypothesis:

   Assume that a set S with cardinality $k \geq 1$ has exactly 2^k unique subsets.

   Inductive step:

   We will show that $P(k+1)$ is true, which means a set S' with cardinality $k+1$ has exactly 2^{k+1} unique subsets with $S' = S \cup \{k+1th\}$.

   We know from our hypothesis that S has exactly 2^k unique subsets that do not contain the $k+1th$ element.

   Then for each element in S combined with the $k+1th$ element, we would have another 2^k unique subsets that contain the $k+1th$ element.

   Thus, the total of unique subsets of S' is $2^k + 2^k = 2.2^k = 2^{k+1}$.

   Since the LHS and RHS of $P(k+1)$ are the same, $P(k+1)$ holds.

   Wrap-up: Since we showed that $P(1)$ is true in the base case and we showed (with $k = 2$) that $P(1) \Rightarrow P(2)$, it must be that $P(2)$ is true. Since $P(2)$ is true and we showed (with $k = 3$) that $P(2) \Rightarrow P(3)$, it must be that $P(3)$ is true, and so on. Hence, we can conclude that $P(n)$ is true for all $n \geq 1$.

2. Prove inductively that a set S with cardinality $n \geq 2$ has exactly $\frac{n(n-1)}{2}$ unique subsets of cardinality 2.

   Hypothesis: Let $P(n)$ denotes the statement "a set S with cardinality $n \geq 2$ has exactly $\frac{n(n-1)}{2}$ unique subsets of cardinality 2."

Base case: When $n = 2$,

The LHS of $P(2)$ is that a set of cardinality 2 has 1 unique subset of cardinality 2. This is true because when a set has only 2 elements, these elements can produce 1 subset containing themselves.

The RHS of $P(2)$ is $\frac{n(n-1)}{2} = 1$.

Then LHS = RHS, so the base case is true.

Inductive hypothesis:

Assume that a set S with cardinality $k \geq 2$ has exactly $\frac{n(n-1)}{2}$ unique subsets of cardinality 2.

Inductive step:

We will show that $P(k + 1)$ is true, which means a set S' with cardinality $k + 1$ has exactly $\frac{(n+1)n}{2}$ unique subsets of cardinality 2.

We know from our hypothesis that S has exactly $\frac{n(n-1)}{2}$ unique subsets of cardinality 2.

Assume that for each $n \in \mathbb{N}$, any set with cardinality n has $n(n-1)/2$ many 2-element subsets. When a new element is added to a set S of cardinality n, $|S| = n + 1$, and the number of 2-element subsets increases by n, since for each existing element in S, we can form a new subset by matching each of them with the new element, resulting in the number of 2-element subsets being $\frac{n(n-1)}{2} + n = \frac{n(n-1)+2n}{2} = \frac{(n+1)n}{2}$.

Since the LHS and RHS of $P(k + 1)$ are the same, $P(k + 1)$ holds.

Wrap-up: Since we showed that $P(2)$ is true in the base case and we showed (with $k = 3$) that $P(2) \Rightarrow P(3)$, it must be that $P(3)$ is true. Since $P(3)$ is true and we showed (with $k = 4$) that $P(3) \Rightarrow P(4)$, it must be that $P(4)$ is true, and so on. Hence, we can conclude that $P(n)$ is true for all $n \geq 2$.

3. Prove inductively that the complement of the union of any n sets S_1, S_2, ..., S_n is equivalent to the intersection of each of their individual complements (i.e., that $\overline{S_1 \cup S_2 \cup ... \cup S_n} = \overline{S_1} \cap \overline{S_2} \cap ... \cap \overline{S_n}$) for all $n \geq 1$. Hint: it may be helpful to remember De Morgan's Law:
$$\overline{S \cup T} = \overline{S} \cap \overline{T}$$

Hypothesis: Let $P(n)$ denotes the statement " the complement of the union of any n sets S_1, S_2, ..., S_n is equivalent to the intersection of each of their individual complements" for all $n \geq 1$.

Base case: When $n = 1$,

The LHS of $P(1)$ is $\overline{S_1}$.

The RHS of $P(1)$ is $\overline{S_1}$.

Then LHS = RHS, so the base case is true.

When $n = 2$,

The LHS of $P(2)$ is $\overline{S_1 \cup S_2}$.

The RHS of $P(2)$ is $\overline{S_1} \cap \overline{S_2}$.

By De Morgan's Law, which states that $\overline{S \cup T} = \overline{S} \cap \overline{T}$, we see that the LHS and RHS of $P(2)$ are equal, so $P(2)$ holds.

Inductive hypothesis:

Assume that the complement of the union of any k sets S_1, S_2, ..., S_k is equivalent to the intersection of each of their individual complements. It follows that $\overline{S_1 \cup S_2 \cup ... \cup S_k} = \overline{S_1} \cap \overline{S_2} \cap ... \cap \overline{S_k}$.

Inductive step:

We will show that the complement of the union of any $k+1$ sets S_1, S_2, ..., S_k, S_{k+1} is equivalent to the intersection of each of their individual complements. This means that $\overline{S_1 \cup S_2 \cup ... \cup S_k \cup S_{k+1}} = \overline{S_1} \cap \overline{S_2} \cap ... \cap \overline{S_k} \cap \overline{S_{k+1}}$.

In the LHS, we have $\overline{S_1 \cup S_2 \cup ... \cup S_k \cup S_{k+1}}$.

Let $S_1 \cup S_2 \cup ... \cup S_k = T$, we have:

LHS $= \overline{T \cup S_{k+1}} = \overline{T} \cap \overline{S_{k+1}} = \overline{S_1 \cup S_2 \cup ... \cup S_k} \cap \overline{S_{k+1}}$ according to De Morgan's Law.

From the inductive hypothesis, we have $\overline{S_1 \cup S_2 \cup ... \cup S_k} = \overline{S_1} \cap \overline{S_2} \cap ... \cap \overline{S_k}$.

Thus, LHS is $\overline{S_1} \cap \overline{S_2} \cap ... \cap \overline{S_k} \cap \overline{S_{k+1}}$

RHS: $\overline{S_1} \cap \overline{S_2} \cap ... \cap \overline{S_k} \cap \overline{S_{k+1}}$

Thus LHS = RHS, so the induction hypothesis holds for $k + 1$.

Wrap-up: We showed that $P(1)$ and $P(2)$ are true in the base case. Since we know that $P(1)$ and $P(2)$ are true, by the inductive step this means that $P(3)$ is true. Since we know that $P(2)$ and $P(3)$ are true, by the inductive step this means that $P(4)$ is true. Continuing in this manner, we can show that $P(n)$ is true for all $n \geq 1$.

4. Prove by contradiction that the intersection of any set S_1 with the difference of any set S_2 and S_1 is the empty set (i.e., $S_1 \cap (S_2 \setminus S_1) = \emptyset$).

   Assume for a contradiction that the intersection of any set S_1 with the difference of any set S_2 and S_1 is not the empty set, which means there is at least 1 element in this intersection (i.e, $S_1 \cap (S2 \setminus S1) \neq \emptyset$).

   Then, the elements in this set must be present in both S_1 and the difference of S_2 and S_1. To be in the difference of S_2 and S_1, they have to be elements of S_2 and not

S_1. This is a contradiction because we assume that the elements must be present in the set S_1.

Thus, the intersection of any set S_1 with the difference of any set S_2 and S_1 is the empty set (i.e., $S1 \cap (S2 \setminus S1) = \emptyset$).