MATH 223: Linear Algebra

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Abstract

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1 Introduction

2 Prerequisite knowledge

2.1 Notation

2.1.1 Sets

Sets are a grouping of objects.

Set	Meaning	Examples
$\mathbb{N}$	The set of natural numbers	(0, 1, 2, 3, ...)
$\mathbb{Z}$	The set of integers	(..., -3, -2, -1, 0, 1, 2, 3, ...)
$\mathbb{Q}$	The set of rational numbers	$\mathbb{Q} = \frac{a}{b} \mid \forall a, b \in \mathbb{Z}\ and\ b \neq 0$
$\mathbb{R}$	The set of all rational and all irrational numbers	$(..., -1, 0, \frac{1}{4}, 1, 1000, ...)$
$\mathbb{C}$	The set of all complex numbers	$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\ and\ i \subseteq \sqrt{-1}\}$.

We have the following relationships between sets:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

2.1.2 Symbols

Symbol	Meaning
$\subseteq$	is a subset of or equal to
$\subset$	is a strict subset of
$\in$	is an element of
$\forall$	for all
$\exists$	there exists
$\emptyset$	empty set
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
$\cong$	is isomorphic to

[1]

2.2 Familiarity with $\mathbb{R}^n$

Definition 1 ($\mathbb{R}^n$)**.** *Let $n \in \mathbb{N}$. The Cartesian product of n copies of $\mathbb{R}$ is called $\mathbb{R}^n$.*

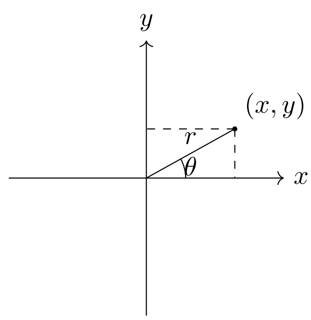
$$\mathbb{R}^n = \{ \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in \mathbb{R}\}$$

[1] is isomorphic to = structurally the same as...

2.3 Polar Coordinates

Definition 2 (Polar coordinates)**.** *Instead of describing a point by (x, y), we may describe it using polar coordinates (r, θ), where r is the distance to the origin and θ is the angle with the positive x-axis.*

Example. *Consider the point (x, y), where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. We can define (r, θ) as follows:*



2.4 Complex Algebra

2.4.1 Complex Numbers

Definition 3 (Complex Number)**.** *A complex number is of the form: $z = x + iy$ where $x, y \in \mathbb{R}$ and i is the imaginary unit $i = \sqrt{-1}$.*

Theorem 1 (Fundamental Theorem of Algebra)**.** *The FTA states that any non-constant, single-variable polynomial*[2] *with complex coefficients has at least one root in $\mathbb{C}$.*

Remark 1. *If we have a polynomial f of degree n, then it has n roots, where each root can have a multiplicity*[3]*.*

Example. *If we have a polynomial $(x-1)^2$, it has a degree of 2 but only one root, which is 1, with a multiplicity of 2.*

We can factorize a polynomial in the form of $f = a_n x^n + ... + a_1 x + a_0$ into a linear factor: $f = a(x - z_1)(x - z_2)...(x - z_n)$ where z_i are the roots of f in $\mathbb{C}$. Therefore, the FTA implies that f has a root $f(z) = a(z - z) = 0$.

2.4.2 Complex Operations

We can define operations on complex numbers as follows:

- Addition: $z + z' = (x + x') + i(y + y')$, where $x, x', y, y' \in \mathbb{R}$.

[2] Polynomial is a function such as: $f = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where $a_i \in \mathbb{R}$ or $\mathbb{C}$ and $n \in \mathbb{N}$.

[3] The multiplicity of a root represents how many times the root occurs in the polynomial.

- Multiplication: $zz' = (x+iy)(x'+iy') = (xx'-yy') + i(xy'+yx')$.
- Inverse: $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$

Multiplying by a complex number z corresponds geometrically to

$$\begin{cases} \text{a rotation by some angle } \theta, \\ \text{a rescaling by the factor } |z|. \end{cases}$$

2.4.3 Complex Conjugate

Definition 4 (Complex Conjugate)**.** *A complex conjugate is a way to "flip" the imaginary part of a complex number. For example, if we have a complex number $z = x+iy$, then the complex conjugate of z is $\overline{z} = x - iy$.*

Some basic properties of complex conjugates are:

- $\overline{z} = z$
- $\overline{z+z'} = \overline{z} + \overline{z'}$
- $\overline{z \cdot z'} = \overline{z} \cdot \overline{z'}$

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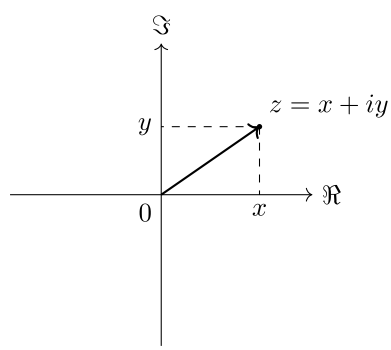
2.4.4 Geometric Interpretation of Complex Numbers

Definition 5 (Geometric interpretation)**.** *Every complex number $z = x+iy$ can be identified with a point (x, y) in the plane, called the complex plane.*

We define the complex plane as:

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.$$



We can rewrite the definition of the unit circle as follows:

$$S' = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{z \in \mathbb{C} : |z| = 1\},$$

where S' is the unit circle in the complex plane.

2.4.5 Modulus

Definition 6 (Modulus). *The modulus of a complex number z is defined by*

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}.$$

Geometrically, $|z|$ is the distance from the origin to the point (x, y). Recall r in polar coordinates, which is the same as $|z|$.

2.4.6 Polar Form of Complex Numbers

Definition 7 (Euler's formula).

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1}$$

Definition 8 (Polar and exponential form). *If $z = x + iy$ with $r = \sqrt{x^2 + y^2}$, then*

$$x = r\cos\theta, \quad y = r\sin\theta,$$

so

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Definition 9 (Multiplication in polar form).

$$z = re^{i\theta}, \quad z' = r'e^{i\theta'}, \quad zz' = rr'e^{i(\theta+\theta')}.$$

Example.

$$(1+i)^{32} = (\sqrt{2}e^{i\pi/4})^{32} = (\sqrt{2})^{32}e^{i8\pi} = 2^{16}(\cos 8\pi + i\sin 8\pi) = 2^{16}.$$

2.4.7 Roots in $\mathbb{C}$

Definition 10 (n^{th} roots in $\mathbb{C}$). *For any complex number z, an n^{th} root of z is a complex number w such that*

$$w^n = z.$$

Example. *If $z = re^{i\theta}$, then any solution of $w^n = z$ must satisfy*

$$w_k = r^{1/n}e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, 2, \dots, n-1,$$

where the n^{th} roots of z are equally spaced on a circle of radius $r^{1/n}$ centered at the origin.

Definition 11 (Roots of unity). *The n^{th} roots of unity are the solutions of a special case where $z = 1$.*

Example. *If $z = 1$, then any solution of $w^n = z$ must satisfy*

$$w_k = e^{i2\pi k/n}, \quad k = 0, 1, 2, \dots, n-1.$$

Geometrically, they lie on the unit circle and are equally spaced.

2.5 Basic Algebraic structures

2.5.1 Sets with Multiplication

Definition 12 (Set with multiplication)**.** *A set M is called a set with multiplication if you can multiply any two elements of M, and the result is still in M. In other words, for any $a, b \in M$, the product ab is also in M.*

Example. *An example of a set with multiplication is the set of all 2×2 complex matrices: $M = M_2(\mathbb{C})$. Another example is the nonzero set of all real numbers $\mathbb{R}$ with ordinary multiplication: $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.*

Example. *Let $M = \mathbb{R}$. If $a, b \in \mathbb{R}$, then $ab \in \mathbb{R}$. So the real numbers $\mathbb{R}$ form a set with multiplication.*

2.5.2 Invertibility

Definition 13 (Condition for Invertibility)**.** *Let $A \in M$ be an $n \times n$ matrix, and suppose that there exists an $n \times n$ matrix B such that $AB = I_n$ or $BA = I_n$. Where I_n is the $n \times n$ identity matrix[4] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then A is invertible, and B is called the inverse of A and is denoted by $B = A^{-1}$.*

Remark 2. *If A is invertible, then A^{-1} exists and is unique[5].*

To determine if an element A in a set with multiplication M is invertible, we can use the following examples:

Example. *Let $M = \mathbb{Z} = \{... -2, -1, 0, 1, 2, ...\}$ and $A = 2$. Is A invertible in M?*
Solution: No, because $\frac{1}{2} \notin \mathbb{Z}$.

Example. *Let $M = \mathbb{R}$ and $A = 2$, is A invertible in M?*
Solution: Yes, because $\frac{1}{2} \in \mathbb{R}$.

Example. *Is $1 + i$ invertible in $\mathbb{C}$?*
Solution: Yes, using our previous definition of inverse (2.4.2), we get that

$$\frac{1}{1+i} = \frac{1-i}{2} \in \mathbb{C}.$$

Problem 1 (Invertibility)**.** *Show that if an inverse of A in $\mathbb{M}$ exists, then it is unique.*

Problem 2 (Invertibility 2)**.** *Let K be a field. Prove that this matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\in M_2(K)$ is not invertible.*

[4] An identity matrix is a square matrix with 1s on its main diagonal and 0s everywhere else. It represents no change in linear transformations, and it's used in finding matrix inverses.

[5] Unique means there is exactly one such element.

2.5.3 Ring

Definition 14 (Ring)**.** *A ring is a set R where you can add and multiply elements, and the following are true:*

1. *You can add any two elements and stay in R. There is a zero, every element has a negative, and addition is commutative*[6] *and associative*[7]*.*

2. *You can multiply any two elements and stay in R. Multiplication is associative, and there is a 1.*

3. *Multiplication distributes over addition:*

$$a(b + c) = ab + ac \quad \textit{and} \quad (a + b)c = ac + bc.$$

Example. *The main example of a ring is the set of integers $\mathbb{Z}$.*

2.5.4 Field

Definition 15 (Field)**.** *A field is a ring (14) in which every nonzero element has a multiplicative inverse.*

Example. *Let F be a set with two elements $F = \{0, 1\}$ with addition and multiplication defined modulo 2. The operations are given by*

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Then 0 is the additive identity, 1 is the multiplicative identity, and the only nonzero element 1 satisfies $1^{-1} = 1$. Hence every nonzero element has a multiplicative inverse, so F is a field.

Problem 3 (Field)**.** *Construct a field with 2 elements.*

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3 Vector Spaces

3.1 Abstract Vector Spaces

A regular vector space like $\mathbb{R}^n$ has concrete vectors you can see as tuples of numbers, while an abstract vector space generalizes this idea: vectors can be anything, as long as addition and scalar multiplication satisfy the vector space axioms.

[6]Property which focuses on changing order of addition, i.e $a + b = b + a$.

[7]Property which focuses on changing grouping of addition, i.e $a + (b + c) = (a + b) + c$.

Definition 16 (Vector space over a field)**.** *Let k be a field. A vector space over k is a set V with addition and scalar multiplication such that*

$$\text{(i) } v_1 + v_2 \in V, \qquad \text{(ii) } \lambda v \in V, \qquad \text{(iii) } 0 \in V,$$

and satisfying the usual vector space axioms (see Appendix 6.2).[8]

3.2 Vectors

3.2.1 Vector operations

Vector operations are defined as follows:

- Addition: $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$.
- Scalar multiplication: $\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

Definition 17 (Linear combination)**.** *A linear combination of vectors $v_1, \ldots, v_n$ is a vector v of the form $v = \lambda_1 v_1 + \ldots + \lambda_n v_n$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.*

3.3 Span

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Definition 18 (Span)**.** *Let $A \subset \mathbb{R}^n$. The span of A, denoted* $\operatorname{span}(A)$*, is the set of all linear combinations of elements of A. In particular, if $A = \{v_1, \ldots, v_k\}$ (where k is the number of vectors in the set), then*

$$\operatorname{span}(A) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k : \lambda_i \in \mathbb{R}\}.$$

When working in $\mathbb{R}^n$, the span describes all points you can reach by scaling and adding the given vectors. Depending on the vectors, the span can be a line (if the vectors are dependent), a plane, or a higher-dimensional subspace.

Remark 3. *If $A = \{v\} \subset \mathbb{R}^2$ with $v \neq 0$, then*

$$\operatorname{span}(v) = \{\lambda v : \lambda \in \mathbb{R}\}.$$

This set consists of all scalar multiples of v, which form a line in the direction of v. In particular, taking $\lambda = 0$ gives $0 \in \operatorname{span}(v)$, so the line passes through the origin.

Example. *Let $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then* $\operatorname{span}(A) = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$*, which is a line in $\mathbb{R}^2$.*

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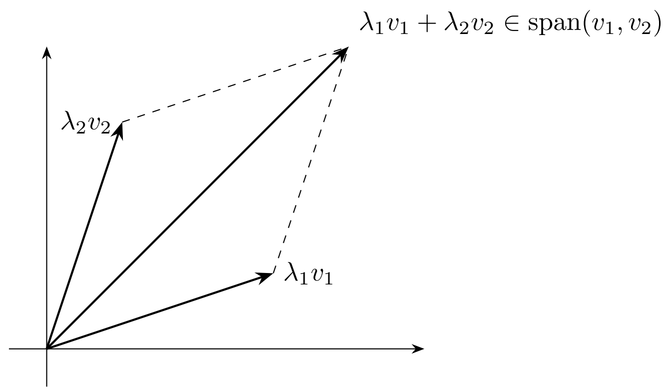
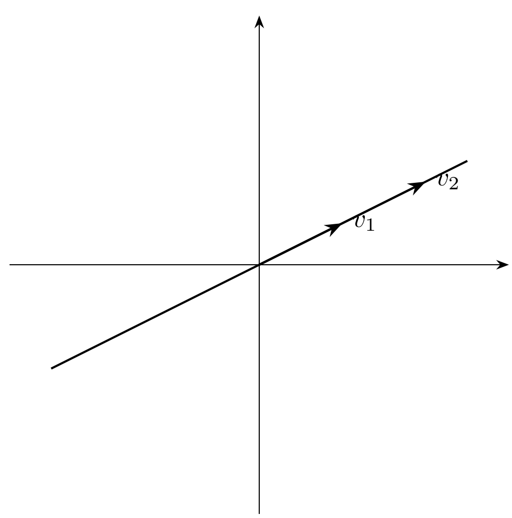
[8] For this course, the main properties we need to know are closure under addition and scalar multiplication, and the existence of the zero vector.

Problem 4 (Span)**.** *Let* $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. *Find* $\text{span}(v_1, v_2)$.

Span is a generalization of lines in $\mathbb{R}^2$. For example, let v_1 and v_2 be two linearly independent vectors in $\mathbb{R}^2$. Any vector in $\text{span}(v_1, v_2)$ has the form $\lambda_1 v_1 + \lambda_2 v_2$. Geometrically, this can be illustrated as the sum of two scaled vectors.



Furthermore, if v_1, v_2 are linearly dependent such that $v_1 = \lambda v_2$, where $\lambda \in \mathbb{R}$ and $\lambda \neq 0$, then $\text{span}(v_1, v_2)$ is a line in $\mathbb{R}^2$ and $span(v_1) = span(v_2)$. Geometrically, this is a straight line through the origin in the direction of v_1 (and v_2).



Problem 5 (Span 2)**.** *Determine whether or not the first vector is in the span of the others. If so, write it as a linear combination of the other.*

$$u = (2, 10, 7, 0) and u_1 = (3, 10, 7, 0), u_2 = (1, 3, -2, 0), u_3 = (2, 8, 1, 0), in \mathbb{R}^4.$$

3.3.1 Span in $\mathbb{C}^n$

Definition 19 (Span in $\mathbb{C}^n$)**.** *The span over $\mathbb{C}$ of vectors $v_1, \ldots, v_n \in \mathbb{C}^n$ is*

$$\operatorname{span}_{\mathbb{C}}(v_1, \ldots, v_n) = \{\lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_i \in \mathbb{C}\}.$$

Example. *Let*

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ i \end{pmatrix} \in \mathbb{C}^2.$$

Find $\operatorname{span}_{\mathbb{C}}(v_1, v_2)$*. For $\lambda_1, \lambda_2 \in \mathbb{C}$,*

$$\lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ i \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ i\lambda_2 \end{pmatrix}.$$

Since λ_1 is arbitrary and $i\lambda_2$ can represent any complex number, every vector in $\mathbb{C}^2$ can be written in this form. Hence

$$\operatorname{span}_{\mathbb{C}}(v_1, v_2) = \mathbb{C}^2.$$

3.4 Standard Basis

Definition 20 (Basis)**.** *Let V be a vector space. A set of vectors $B = \{v_1, \ldots, v_k\} \subset V$ is called a **basis** of V if*

1. *B spans V, and*
2. *B is linearly independent.*

Equivalently, every vector in V can be written uniquely as a linear combination of the vectors in B.

Definition 21 (Standard basis of $\mathbb{R}^n$)**.** *The standard basis of $\mathbb{R}^n$ is the set $\{e_1, \ldots, e_n\}$, where each vector e_i has a 1 in the i-th coordinate and 0 in all other coordinates:*

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \; \ldots, \; e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Equivalently, the vectors $e_1, \ldots, e_n$ are the columns of the $n \times n$ identity matrix I_n.

Proposition 1. *Every vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ can be written uniquely as $x = x_1 e_1 + \cdots + x_n e_n$.*

Remark 4. *To verify that the standard basis of $\mathbb{R}^n$ is indeed a basis of $\mathbb{R}^n$, we must check two properties: it spans $\mathbb{R}^n$ and it is linearly independent.*[9]

The standard basis also helps clarify the difference between real and complex vector spaces. In particular, the same vectors can generate very different spans depending on whether the scalars are real or complex.

Example (Real vs complex span of the same vectors)**.** *Let*

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2.$$

Over $\mathbb{C}$***:*** *Using complex scalars, any vector in $\mathbb{C}^2$ can be written as*

$$\lambda_1 e_1 + \lambda_2 e_2 = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{C}.$$

Thus $\operatorname{span}_{\mathbb{C}}(e_1, e_2) = \mathbb{C}^2$*, and only 2 vectors are needed:*

$$\dim_{\mathbb{C}} \mathbb{C}^2 = 2.$$

Over $\mathbb{R}$***:*** *Now only real scalars are allowed. Then*

$$a_1 e_1 + a_2 e_2 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad a_1, a_2 \in \mathbb{R},$$

which cannot produce vectors with imaginary parts. To span all of $\mathbb{C}^2$ over $\mathbb{R}$, we also need

$$ie_1 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad ie_2 = \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

Now any vector in $\mathbb{C}^2$

$$\begin{pmatrix} a + bi \\ c + di \end{pmatrix}, \quad a, b, c, d \in \mathbb{R},$$

can be expressed as a real linear combination of the 4 vectors

$$e_1, \ e_2, \ ie_1, \ ie_2.$$

Hence, over $\mathbb{R}$ the span of e_1 and e_2 requires 4 vectors, and

$$\dim_{\mathbb{R}} \mathbb{C}^2 = 4.$$

Conclusion: *The same vectors generate different spans depending on the allowed scalars. Complex scalars count as one direction, while real scalars require separate vectors for the real and imaginary parts.*

3.5 Coordinate Spaces

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Definition 22 (Coordinate Spaces)**.** *A coordinate space of dimension n over a field k ($k = \mathbb{R}$ or $k = \mathbb{C}$) is defined as*

$$k^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in k \right\}.$$

The standard basis for a coordinate space is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \dots \; , e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

3.6 Polynomial Spaces

Definition 23 (Polynomial Spaces)**.** *A polynomial space of degree at most n over a field k is*

$$P_n(k) = \{a_n x^n + \cdots + a_1 x + a_0 : a_i \in k\},$$

which forms a vector space over k. The standard basis for a polynomial space of degree n is

$$\{1, x, x^2, \dots, x^n\}.$$

Example (Polynomial Spaces)**.** *Some examples of polynomials in these spaces are*

$$f = 1 + x^2 \in P_2(\mathbb{R}), \qquad f = 1 + ix^3 \in P_3(\mathbb{C}).$$

The subscript n indicates that $\deg(f) \leq n$. All polynomials can be collected in

$$P_\infty(k) = \{a_n x^n + \cdots + a_1 x + a_0 : a_i \in k, \; n \geq 0\},$$

which is an infinite-dimensional vector space over k. We have the natural inclusions

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_\infty.$$

3.7 Matrix Spaces

Definition 24 (Matrix Spaces)**.** *The set of all $n \times n$ matrices over a field k is*

$$M_n(k) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} : a_{ij} \in k \right\},$$

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[9] To prove the standard basis is a basis, please refer to the Appendix (6.1).

which forms a vector space over k. A standard basis for $M_n(k)$ is the set of matrices $\{e_{ij} : 1 \leq i, j \leq n\}$, where e_{ij} has a 1 in the (i, j)-th entry and 0 elsewhere. The dimension of $M_n(k)$ is $\dim M_n(k) = n^2$.

Example (Matrix Spaces)**.** *For $M_2(\mathbb{R})$,*

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1e_{11} + 2e_{12} + 3e_{21} + 4e_{22}.$$

3.8 Function Spaces

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Definition 25 (Function spaces)**.** *Let D be a set and k a field. Define*

$$F(D, k) = \{f : D \to k\}.$$

With pointwise operations

$$(f + g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda f(x),$$

$F(D, k)$ is a vector space over k.

1. Finite case: standard basis Assume $D = \{1, \ldots, n\}$. For each $i \in D$, define the Kronecker delta function

$$\delta_i(j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Then $\{\delta_1, \ldots, \delta_n\}$ is a basis of $F(D, k)$. Moreover, every $f \in F(D, k)$ can be written uniquely as

$$f = \sum_{i=1}^{n} f(i)\, \delta_i.$$

Indeed, evaluating at j gives

$$\sum_{i=1}^{n} f(i) \delta_i(j) = f(j),$$

so the functions span, and linear independence is immediate.

2. Infinite case If D is infinite, $F(D, k)$ has no finite basis. The family $\{\delta_x : x \in D\}$ is linearly independent but does not span $F(D, k)$.

Example (Vector space of functions)**.** *Let $D = \{1, 2, 3\}$ and $k = \mathbb{R}$. Consider the vector space $F(D, \mathbb{R}) = \{f : D \to \mathbb{R}\}$.*
Problem: *Find a basis of $F(D, \mathbb{R})$ and express $f(1) = 2, f(2) = -1, f(3) = 3$ as a linear combination of the basis functions.*

Solution:
1. Define the standard (Kronecker delta) functions:

$$\delta_1(j) = \begin{cases} 1, & j = 1 \\ 0, & j \neq 1 \end{cases}, \quad \delta_2(j) = \begin{cases} 1, & j = 2 \\ 0, & j \neq 2 \end{cases}, \quad \delta_3(j) = \begin{cases} 1, & j = 3 \\ 0, & j \neq 3 \end{cases}.$$

Then $\{\delta_1, \delta_2, \delta_3\}$ is a basis of $F(D, \mathbb{R})$.
2. Write f as a linear combination:

$$f = f(1)\delta_1 + f(2)\delta_2 + f(3)\delta_3 = 2\delta_1 - 1\delta_2 + 3\delta_3.$$

Check: *Evaluating at each point:*

$$f(1) = 2 \cdot 1 + (-1) \cdot 0 + 3 \cdot 0 = 2, \quad f(2) = 2 \cdot 0 + (-1) \cdot 1 + 3 \cdot 0 = -1,$$
$$f(3) = 2 \cdot 0 + (-1) \cdot 0 + 3 \cdot 1 = 3.$$

Hence the decomposition is correct.

3.9 Subspaces

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Definition 26 (Subspace)**.** *A subspace is a subset of a larger vector space that is itself a vector space, meaning it contains the zero vector and is closed under vector addition and scalar multiplication. Subspaces are essential because they allow focus on smaller, self-contained structures where standard linear algebra operations (like finding spans, bases, and transformations) still hold, with examples including lines through the origin in $\mathbb{R}^2$, the null space of a matrix, or the entire space itself.*

Proposition 2 (Subspace criterion)**.** *Let V be a vector space over a field k and let $U \subseteq V$. Then U is a vector subspace of V if and only if:*

1. $0 \in U$,
2. $u, v \in U \Rightarrow u + v \in U$,
3. $u \in U,\ \lambda \in k \Rightarrow \lambda u \in U$.

Problem 6 (Subspace criterion)**.** *Let $A \in M_n(K)$ be a fixed matrix. Prove thet*

$$U = \{x \in K^n : Ax = \vec{0}\}$$

is a subspace, null space or kernel.

Problem 7 (Subspace criterion 2)**.** *Let $A \in M_n(K)$. Show that*

$$U = \{Ax : x \in K^n\}$$

is a subspace of K^n. The set U is called the image (or range) of A.

Problem 8 (Subspace criterion 3)**.** *Let $V = K$, viewed as a vector space over K. Show that the only subspaces of V are $\{0\}$ and V itself.*

Problem 9 (Subspace criterion 4)**.** *Let V be a vector space over K and let $v_1, \ldots, v_n \in V$. Show that*

$$\operatorname{span}(v_1, \ldots, v_n) = \{\lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_1, \ldots, \lambda_n \in K\}$$

is a subspace of V.

January 21, 2026

3.9.1 Membership in a span

Problem 10 (Span membership)**.** *Let*

$$v = \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix},\ v_3 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}.$$

Decide whether $v \in \operatorname{span}(v_1, v_2, v_3)$.

Problem 11 (Span membership 2)**.** *Let*

$$f = 3x^2 + 5x - 5, \quad f_1 = x^2 + 2x + 1,\ f_2 = 2x^2 + 5x + 4,\ f_3 = x^2 + 3x + 6.$$

Decide whether $f \in \operatorname{span}(f_1, f_2, f_3)$.

3.9.2 Operations on subspaces

Let W be a vector space and $U, V \subseteq W$ subspaces.

Definition 27 (Intersection)**.**

$$U \cap V = \{w \in W : w \in U \text{ and } w \in V\}.$$

The intersection of subspaces is a subspace. In particular, the smallest possible intersection is $\{0\}$.

Definition 28 (Union)**.**

$$U \cup V = \{w \in W : w \in U \text{ or } w \in V\}.$$

The union of subspaces is not necessarily a subspace. In general, $U \cup V$ is a subspace only if one subspace is contained in the other, i.e., $U \subseteq V$ or $V \subseteq U$.

Definition 29 (Sum of subspaces)**.**

$$U + V = \{u + v : u \in U,\, v \in V\}.$$

Proposition 3. *If U and V are subspaces of W, then $U + V$ is a subspace of W.*

Proof. We apply the subspace criterion.

1. Since $0 \in U$ and $0 \in V$, we have $0 = 0 + 0 \in U + V$.
2. Let $w_1, w_2 \in U + V$. Then
$$w_1 = u_1 + v_1, \quad w_2 = u_2 + v_2,$$
with $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Hence
$$w_1 + w_2 = (u_1 + u_2) + (v_1 + v_2) \in U + V.$$
3. Let $w \in U + V$ and $\lambda \in K$. Then $w = u + v$ and
$$\lambda w = \lambda u + \lambda v \in U + V.$$

□

Proposition 4.
$$U + V = \operatorname{span}(U \cup V).$$
Equivalently, $U + V$ is the smallest subspace of W containing both U and V.

Proof. ($\subseteq$): Let $w \in U + V$. Then $w = u + v$ with $u \in U$, $v \in V$. Since $u, v \in U \cup V$, we have $w \in \operatorname{span}(U \cup V)$.

($\supseteq$): Let $w \in \operatorname{span}(U \cup V)$. Then
$$w = a_1u_1 + \cdots + a_nu_n + b_1v_1 + \cdots + b_mv_m$$
with $u_i \in U$, $v_j \in V$. Grouping terms,
$$w = (a_1u_1 + \cdots + a_nu_n) + (b_1v_1 + \cdots + b_mv_m),$$
so $w \in U + V$. □

January 23, 2026

3.10 Direct sums

Let W be a vector space and $U, V \subseteq W$ be subspaces.

Definition 30 (Direct sum)**.** *We say that U and V are in direct sum if*
$$U \cap V = \{0\}.$$
In this case, their sum
$$U + V = \{u + v : u \in U,\ v \in V\}$$
is denoted by $U \oplus V$. We write
$$W = U \oplus V \iff \begin{cases} W = U + V, \\ U \cap V = \{0\}. \end{cases}$$

Remark 5. *To prove $W = U \oplus V$, one must show:*

- *Every $w \in W$ can be written as $w = u + v$ with $u \in U$, $v \in V$.*
- *If $w \in U$ and $w \in V$, then $w = 0$.*

3.10.1 Analogy with sets

Sets	Vector spaces
$A \cap B$	$U \cap V$
$A \cup B$	$U + V$
$A \sqcup B,\ A \cap B = \emptyset$	$U \oplus V,\ U \cap V = \{0\}$

Example (Analogy with disjoint sets). *Let*

$$D = \{1, 2, 3, 4, 5\}, \quad A = \{1, 2, 3\}, \quad B = \{4, 5\}.$$

Then

$$D = A \sqcup B.$$

3.10.2 Direct sums in function spaces

Definition 31 (Subspace of functions supported on a subset). *Let D be a set and $A \subseteq D$. Consider the function space $F(D, \mathbb{R})$. Define*

$$U = \{f \in F(D, \mathbb{R}) : f(x) = 0\, \forall x \notin A\}.$$

Then U is a subspace of $F(D, \mathbb{R})$ (it is closed under addition and scalar multiplication, and contains the zero function). Moreover, U can be naturally identified with $F(A, \mathbb{R})$, since each function in U is completely determined by its values on A

Remark 6. *Every function in U is completely determined by its values on A and vanishes outside A.*

Example (Functions supported on a subset). *Let $D = \{1, 2, 3, 4, 5\}$ and $A = \{2, 4\}$. Then*

$$U = \{f \in F(D, \mathbb{R}) : f(1) = f(3) = f(5) = 0\}.$$

Examples of functions in U include

$$f = (0, 3, 0, -1, 0), \quad g = (0, 0, 0, 7, 0).$$

Each function in U is completely determined by its values on A, i.e.,

$$f \iff (f(2), f(4)) \in F(A, \mathbb{R}) \cong \mathbb{R}^2.$$

Hence U is a 2-dimensional subspace of $F(D, \mathbb{R})$.

4 Basis and Dimension

January 26, 2026

4.1 Finite Dimensional Spaces

Let V be a vector space over a field k. Then there are two possibilities:

- The zero vector space: $V = \{0\}$
- The non-zero vector space: $V \neq \{0\}$

In the second case, there exists a non-zero vector $v_1 \in V$ such that $v_1 \neq 0$. Then $span(v_1)$ is a subspace of V, and there are two possibilities:

- $V = span(v_1)$, i.e., v_1 is a generator of V.
- $V \neq span(v_1)$, i.e., there exists a $v_2 \in V$ such that $v_2 \notin span(v_1)$.

In the second case, $span(v_1, v_2) \subseteq V$. This process can be repeated to obtain a sequence of vectors $v_1, v_2, \ldots, v_n$ such that $span(v_1, \ldots, v_n) \subseteq V$. The maximum number of linearly independent vectors in V is called the dimension of V, denoted by $\dim V$.

2 cases (2d space plane):

- $v = v_2 = span(v_1, v_2)$
- $v \neq v_2 \quad \exists v_3 \notin v_2$, where $v_3 = span(v_1, v_2, v_3) \subseteq v$

2 cases (3d space):

- $v_3 = v$
- $v_3 \neq v \quad \exists v_4 ...$

Definition 32. *V is finite dimensional if it can be spanned by finitely many vectors.*

$$\exists v_1, \ldots, v_n : V = span(v_1, \ldots, v_n).$$

$dimV$ = smallest n such that $V = span(v_1, \ldots, v_n)$.

$dim_k V = \min n$ such that $V = span_k(v_1, \ldots, v_n)$.

Example. $(\mathbb{C}^n, n = 1)$ $\mathbb{C} = span_{\mathbb{R}}(1, i) = 2$

$$x \cdot 1 + y \cdot i$$

$M_n(\mathbb{C})$

$$A = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} + y_{11}i & x_{12} + y_{12}i & \cdots & x_{1n} + y_{1n}i \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} + y_{n1}i & x_{n2} + y_{n2}i & \cdots & x_{nn} + y_{nn}i \end{pmatrix}$$

First matrix is of $dim_{\mathbb{C}} M_n(\mathbb{C}) = n^2$, and the second matrix is of $dim_{\mathbb{R}} M_n(\mathbb{C}) = 2n^2$.

Example ($\dim V = \infty$)**.**

$$P_\infty$$

$$v_1 = 1 \quad v_2 = x \quad v_3 = x^2$$

$span(v_1) =$ *constant polynomials*

$$V_1 = span(v_1, v_2) = P_1$$

$$v_{n+1} = x^n \notin P_{n-1}$$

4.2 Linear Independence

Definition 33. *$v_1, \ldots, v_n$ are linearly independent if $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0 \implies \lambda_1 = \ldots = \lambda_n = 0$.*

Example.

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Prove LI. Assume $xe_1 + ye_2 = 0$. Prove $x = y = 0$.

$$xe_1 + ye_2 = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies x = y = 0.$$

Proposition 5. *The "standard basis" are both spanning and LI. $(e_1, \ldots, e_n)$ of k^n $(1, x, \ldots, x^n)$ of P_n $(e_{11}, \ldots, e_{nn})$ of $M_n(k)$ $(\delta_1, \ldots, \delta_n)$ of $F(D)$, where $D = \{1, \ldots, n\}$*

Definition 34. *basis = set$(v_1, \ldots, v_n)$ both span and LI.*

$$dim(\forall) = \#basis$$

Problem 12. *Prove $dim P_\infty = \infty$. Assume $dim P_\infty < \infty$. Proof by contradiction. $\exists f_1, \ldots, f_n$ polynomials $P_\infty = span(f_1, \ldots, f_n)$. Let d = max degree of f_k.*

$$x^{d+1} \notin span(f_1, \ldots, f_n)$$

degree $\leq d$.

January 28, 2026

Problem 13.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}.$$

Are these vectors linearly independent? Are there $x, y, z \in \mathbb{R}$ such that

$$xv_1 + yv_2 + zv_3 = 0?$$

Solution 1. *x, y, z not all zero.*

$$x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + z\begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + 3y + 4z = 0 \\ 1 + 3y + 9z = 0 \\ 0 + 2y + 5z = 0 \end{cases}$$

Find a non-trivial solution. $x = 3$, $y = 5$, $z = -2$. Not LI: $3v_1 + 5v_2 - 2v_3 = 0$.

Definition 35 (Linear Dependence)**.** *Linearly dependent (LD) just means not LI.*

Proposition 6. *$v_1, \ldots, v_n$ are linearly dependent if and only if one vector is in the span of the other vectors.*

Example.

$$3v_1 + 5v_2 - 2v_3 = 0$$

As we can see, v_1 is in the span of v_2 and v_3.

$$v_1 = \frac{1}{3}(-5v_2 + 2v_3)$$

Remark 7. *If one vector $v_1 = 0$ then the family is LD.*

Proof. Proof of $\longrightarrow$: Assumption $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. Not all λ_i are zero, at least one is nonzero, say $\lambda_k \neq 0$. The goal is to write v_k in the span of the other vectors.

$$v_k = \frac{-1}{\lambda_k} \sum_{i \neq k} \lambda_i v_i.$$

$$v_k \in span(v_1, \ldots, v_k, \ldots, v_n).$$

Converse: Assume one vector, say v_k is in the span of the other vectors.

$$v_k \in span(v_1, \ldots, v_k, \ldots, v_n).$$

There exists coefficients $\lambda_1, \ldots, \lambda_n$ such that

$$v_k = \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k+-1} + \lambda_{k+1} v_{k+1} + \cdots + \lambda_n v_n.$$

LD: non trivial dependency between the v_i's. $\square$

Example.

$$V = \mathbb{R}^3$$

$$v_1 = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

LI:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

LD:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.$$

Assume $xv_1 + yv_2 + zv_3 = 0$.

$$x\begin{pmatrix}a_1\\0\\0\end{pmatrix} + y\begin{pmatrix}b_1\\b_2\\0\end{pmatrix} + z\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}.$$

$$\begin{cases}a_1x + b_1y + c_1z = 0\\ b_2y + c_2z = 0\\ c_3z = 0\end{cases}$$

If $a_1 = 0$. $v_1 = \begin{pmatrix}0\\0\\0\end{pmatrix}$. *Answer: LD. Can assume* $a_1 \neq 0$. *If* $b_2 = 0$. $v_1 = \begin{pmatrix}a_1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}b_1\\0\\0\end{pmatrix}$. *Answer: LD.*

January 30, 2026

4.3 Basis

Definition 36 (Basis)**.** *A list* $(v_1, \ldots, v_n)$ *in a vector space* V *is a basis if*

1. *it spans* V*, and*
2. *it is linearly independent.*

Example (Standard basis of $\mathbb{R}^3$)**.**

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

These vectors form a basis of $\mathbb{R}^3$.

Proof. Let $v = \begin{pmatrix}x\\y\\z\end{pmatrix}$. Then

$$v = xe_1 + ye_2 + ze_3$$

so the set spans $\mathbb{R}^3$. If

$$xe_1 + ye_2 + ze_3 = 0 \implies x = y = z = 0,$$

hence the vectors are linearly independent. □

Proposition 7 (Common standard bases)**.**

- $\mathbb{R}^n$*:* $(e_1, \ldots, e_n)$
- $M_n(k)$*:* e_{ij}
- $P_n(k)$*:* $(1, x, \ldots, x^n)$
- $F(D)$ *for finite* D*:* $\{\delta_x, x \in D\}$

4.3.1 Size of spanning and independent sets

For $(v_1, \ldots, v_n) \in \mathbb{R}^3$:

- If $n < 3$ then the vectors cannot span $\mathbb{R}^3$.
- If $n > 3$ then the vectors cannot be linearly independent.

Lemma 2. *If $u_1, \ldots, u_n$ span V and $v_1, \ldots, v_m$ are linearly independent in V, then*

$$n \geq m.$$

We can conclude, that a spanning set $\geq$ LI set.

4.3.2 Characterizations of a basis

Theorem 3 (Equivalent Conditions)**.** *Let $(v_1, \ldots, v_n)$ be vectors in V. The following are equivalent:*

1. *Basis: $(v_1, \ldots, v_n)$ is a basis.*
2. *Minimal span: It spans V and n is minimal among spanning sets.*
3. *Maximal LI: It is linearly independent and n is maximal among independent sets.*
4. *Coordinates: Every $v \in V$ has a unique representation.*

$$v = x_1 v_1 + \cdots + x_n v_n.$$

The vector

$$[v]_e = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is called the coordinate vector of v.

Corollary 3.1. *Any two bases of V have the same number of vectors. This number is the dimension of V.*

Proof. Let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_m)$ be bases. Since one spans and the other is linearly independent, the lemma gives $n \geq m$ and $m \geq n$, hence $n = m$. $\square$

February 2, 2026

4.3.3 Finding a Basis

Let $(v_1, \ldots, v_n) \in k^m$ and set $U = \operatorname{span}(v_1, \ldots, v_n)$.

1. Put the vectors as rows of a matrix A.
2. Row reduce A to echelon form B.
3. The nonzero rows of B form a basis of U.

1. **Proving a set of vectors is a Basis** Let V be finite dimensional.
   Method 1 (dimension test).

$$(v_1, \ldots, v_n) \text{ is a basis } \iff \dim(\operatorname{span}(v_1, \ldots, v_n)) = n.$$

Compute the dimension using row reduction.
Method 2 (basis criterion when $\dim V = n$).

Proposition 8. *Let $\dim V = n$ and $v_1, \ldots, v_n \in V$. The following are equivalent:*

1. *$(v_1, \ldots, v_n)$ is a basis,*
2. *$(v_1, \ldots, v_n)$ spans V,*
3. *$(v_1, \ldots, v_n)$ is linearly independent.*

Idea. A spanning set of $n = \dim V$ vectors is minimal, hence linearly independent. An independent set of n vectors is maximal, hence spans. □

2. **Transfer (exchange) principle** If

$$u_1, \ldots, u_n \text{ span } V \quad \text{and} \quad v_1, \ldots, v_n \text{ are linearly independent},$$

then both lists are bases of V.

Example. *In P_n,*

$$(1, x, \ldots, x^n) \textit{ spans}, \qquad ((x-a)^0, \ldots, (x-a)^n) \textit{ is LI},$$

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so both are bases.

4.3.4 Direct Sums and Unique Decomposition

Let W be a vector space and $U, V \subseteq W$ subspaces.

Definition 37 (Direct Sum)**.** *The sum of subspaces is*

$$U + V = \{u + v : u \in U, v \in V\}.$$

We say the sum is direct, written $U \oplus V$, if

$$U \cap V = \{0\}.$$

Proposition 9. *The following are equivalent:*

1. *$U \oplus V$ (i.e. $U \cap V = \{0\}$),*
2. *every $w \in U + V$ can be written uniquely as $w = u + v$ with $u \in U$, $v \in V$.*

Proof. ($\Rightarrow$) Suppose $w = u_1 + v_1 = u_2 + v_2$. Then

$$u_1 - u_2 = v_2 - v_1 \in U \cap V = \{0\},$$

so $u_1 = u_2$ and $v_1 = v_2$.
($\Leftarrow$) If decomposition is unique and $w \in U \cap V$, then

$$0 = 0 + 0 = w + (-w),$$

so uniqueness forces $w = 0$. □

Coordinates relative to a basis If $(e_1, \dots, e_n)$ is a basis of V, every $v \in V$ can be written uniquely as

$$v = \lambda_1 e_1 + \cdots + \lambda_n e_n.$$

The vector

$$[v]_e = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is called the coordinate vector of v.

Example. *In $\mathbb{R}^3$ with the standard basis,*

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \Rightarrow \quad [v]_e = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

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example:

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, f_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$[v]_f = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

LL example: Find coordinate of $v = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$ in basis $f = (f_1, f_2, f_3)$, with $f_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Find x, y, z.

$$[v]_f = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \in \mathbb{R}^3.$$

Example: $D = \{0, 1, 2, 3, 4\}$, $V = F(D)$, $g(x) = x^2$, $g \in V$, find coordinate of y in standard basis $\delta = (\delta_0, \delta_1, \delta_2, \delta_3, \delta_4)$. $dimV = 5$. $[g]_\delta = \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_4 \end{pmatrix}$, where $g = \lambda_0\delta_0 + \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$.

Nose Basis theorem: Basis extraction theorem: $v_1, \dots, v_n$ spans V can extract a basis $\{e_1, \dots, e_n\}$ from $\{v_1, \dots, v_n\}$. Example: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 =$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ spans $\mathbb{R}^3$. Proof: Let $\{e_1, \ldots, e_n\} \subseteq \{v_1, \ldots, v_n\}$ be a subset which is spanning with m minimal. Claim: $e_1, \ldots, e_m$ is a basis. Proof by contradiction: Asumption: $(e_1, \ldots, e_n)$ not LI: Proposition: one vector is in the span of the other $(\lambda_1 e_1 + \ldots + \lambda_n e_n = 0)$. $e_k \in span(e_1, e_2, \ldots, e_k, \ldots, e_n)$.

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V vector space (finite dimensional) $v_1, \ldots, v_n$ LI set. Can extend $v_1, \ldots, v_n$ to a basis of V. Proof: $span(v_1, \ldots, v_n) = U \subseteq V$. Step 1: Case 1: If $U = V \rightarrow$ basis (LI + span) Case 2: $U \neq V \implies \exists v_{n+1} \in V\ U$ Claim: $(v_1, \ldots, v_{n+1})$ LI V minus U.

$$U_{n+1} = span(v_1, \ldots, v_{n+1})$$

Step 2: Case 1: $U_{n+1} = V \implies (v_1, \ldots, v_{n+1})$ is a basis. Case 2: $U_{n+1} \neq V \implies$ a m add v_{m+2}

Step 3: …This stops at v_n for some $n \geq m$, $n = dimV \implies n$ LI vectors in V.

Coronary: $U \subseteq V$, $dimU \leq dimV$ or $U \nsubseteq V$ and $dimU < dimV$.[10]. If for $U \subseteq V$, $U = V \iff dimU = dimV$.

Coronary: $U \subseteq V$. U admits a supplement, meaning there exists a subspace $U' \subseteq V$ such that $V = U \oplus U'$. $dimV = dimU + dimU'$.

Basis concatenation: Theorem: Suppose $U \oplus U'$ are in direct sum in V. $(e_1, \ldots, e_n)$ basis of U. $(f_1, \ldots, f_m)$ basis of U'. $\implies (e_1, \ldots, e_n, f_1, \ldots, f_m)$ is a basis of $U \oplus U'$, which means $dimU \oplus U' = m + n = dimU + dimU'$. Theorem: $U, U' \subset V$, $dimU + U' = dimU + dimU' - dim(U \cap U')$.

proof of : Suppose $U \oplus U'$ are in direct sum in V Assume: $\begin{cases} e_1, \ldots, e_n basis of U \\ f_1, \ldots, f_m basis of U' \end{cases}$

Prove $(e_1, \ldots, e_n, f_1, \ldots, f_m)$ is a basis LI: $a_1e_1 + \ldots + a_ne_n + b_1f_1 + \ldots + b_mf_m = 0$ $w = a_1e_1 + \ldots + a_ne_n = -b_1f_1 - \ldots - b_mf_m$. $w \in U$, $w \in U' \implies w \neq 0 \implies a_1e_1 + \ldots a_ne_n = 0 \implies a_i = 0. \implies b_1f_1 + \ldots b_nf_n = 0 \implies b_i = 0.$

February 11, 2026

$Span : v \in U + U' \implies v = u + u'$

5 Linear Transformation

Today general trans/function/map U,V are sets where U is the domain and V is the codomain. $u \in U$ and $T(u) \in V$. Image of $T = Im(T) = \{T(u) : u \in U\} \subseteq V$. Example: $sin : \mathbb{R} \rightarrow \mathbb{R}$, $x \rightarrow sin(x)$. $Im(sin) = (-1, 1)$. $F(D)$ is a vector space with D set, codomain $K = \mathbb{R} or \mathbb{C}$. Definition: $T : U \rightarrow V$, $u \rightarrow T(u)$ is surjective, if $\Im(T) = V$. Definition: $T : U \rightarrow V$, $u \rightarrow T(u)$ is injective, if you have u and v where $u \neq v$, but $T(u) = T(v)$. Injective: if $u, v \in U$ satisfy $T(u) = T(v)$, then $u = v$. Example: x^3 is injective. Definition: T is a bijection if it is both injective and surjective. Example: x^3 is bijective whilst x^2 is not. More precisely: $f : \mathbb{R} \rightarrow \mathbb{R}$ $x \rightarrow x^2$ is not injective nor surjective. Example: $y : \mathbb{C} \rightarrow \mathbb{C}$ $z \rightarrow z^2$ injective? $(-1)^2 = (1)^2$. surjective? $z^2 = w$ where $w \in \mathbb{C}$ is any complex number. Fix w, solve for z, $g(z) = w$.

[10] $\nsubseteq$ means it is a subset but not equal to.

Composition of transformations:

$$R :\longrightarrow T \longrightarrow V \longrightarrow S \longrightarrow W$$

$$u \longrightarrow T(u) \longrightarrow S(T(u))$$

Composition:

$$R : U \longrightarrow W$$

$$u \longrightarrow S(T(u))$$

$$g : \mathbb{R} \xrightarrow[A]{x^2} \mathbb{R} \xrightarrow[B]{sin} \mathbb{R}$$

$$g = sin(x^2)$$

$$R : U \xrightarrow{T} V \xrightarrow{S} W$$

$$R(u) = S(T(u))$$

$$R = S \circ T$$

($\circ$) is called composition. A inverse = a matrix B such that $AB = Id$ and $BA = Id$. Proposition: $T : U \to V$ is bjective $\iff$ T admits an inverse, $\exists S$ such that $\begin{cases} ST = id \\ TS = id \end{cases}$. Let U be a vector space:

$$T : U \longrightarrow U \text{Id transformation}$$

$$u \longrightarrow u$$

$$U = \mathbb{R}^n$$

$$T \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Example: $U = \mathbb{R}^n$, $V = \mathbb{R}^n$, $A \in M_n(\mathbb{R})$, $T_A : u \to v$, T_A = trans with A.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$A \begin{pmatrix} x \\ y \end{pmatrix}$$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$$

Definition: U,V vector spaces over K. $T : U \to V$ transformation. Say T is linear if $T(u+v) = T(u) + T(v)$ and $T(\lambda u) = \lambda T(u)$. Proposition: S,T are linear, here $S \circ T$ is linear.

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Linear transformations:

$$T : U \longrightarrow V$$

$$u \longrightarrow T(u)$$

Imagine two big circles one is U the othre V, in each a small dot $u \in U$ and $v \in V$ and an arrow from u to v, and this arrow is T. T linear:

1. Additive: $T(u+u') = T(u) + T(u')$
2. Scalars: $T(\lambda u) = \lambda T(u)$

Main example: $T : \mathbb{R}^n \to \mathbb{R}^n$ (a matrix).

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Example 1:

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$u \to 2u$$

$$v \to v$$

Fix a basis (u, v) of $\mathbb{R}^2$. draw the answer Example 2:

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$u \to u$$

$$v \to 2v$$

draw something Definition: A linear transformation $T : U \to U$ is called scaling if there exists a basis $(u_1, \ldots, u_n)$ of U, $\lambda_1, \ldots, \lambda_n \in k$ such that $T(u_i) = \lambda_i u_i$. How to define T: Proposition: Let $(u_1, \ldots, u_n)$ be a basis of U. $v_1, \ldots, v_n$ arbitrary vectors in V. There exists a unique linear transformation $T : U \to V$ such that $T(u_i) = v_i$. Example:

$$T : U \to V$$

$$u_i \to 0$$

$u_i =$ basis of U. T extends uniquely to U. $T(u) = 0$.

$$[u]_n = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

Proof: $u = \lambda_1 u_1 + \cdots + \lambda_n u_n$. $T(u) = \lambda_1 T(u_1) + \cdots + \lambda_n T(u_n) = 0$. Proof: 2 proofs:

1. T exists.
2. T is unique.

Assumptions: $\begin{cases} (u_1, \ldots, u_n) basis \\ T(u_i) = v_i \end{cases}$ Proof of existence: Find T linear such that $T(u_i) = v_i$. Let $u \in U$. Need to define $T(u)$, $u = \lambda_1 u_1 + \ldots + \lambda_n u_n$. Define $T(u) = \lambda_1 T(u_1) + \cdots + \lambda_n T(u_n) \in V$.

1. Prove $T(u_i) = v_i$.
2. T linear.

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Proof of $T(u_i) = v_i$: To define T on u_i: coordinates of u in basis $u_i = 0u_1 + \ldots + 1u_i + \ldots + 0u_n$. $T(u_i) = 0v_1 + \cdots + 1v_i + \cdots + 0v_n$.

Examples: Example 1: The derivative

$$D : P_n \to P_n, \quad f \to f'$$

where $D(x^n) = nx^{n-1}$, Linear map:

- $(f + g)' = f' + g'$
- $(\lambda f)' = \lambda f'$, where $\lambda \in \mathbb{R}$.

Is D injective? No, $D(1) = 0$. $1, 0 \in P_n$, constant polynomials : $r + 0x + 0x^2 + \cdots$. $D(r) = 0$, where $r \in \mathbb{R}$. Is D surjective? No, there is no f such that $f' = 1$. $x^n \notin \Im(D)$. $f \in P_n$ such that $D(f) = x^n$. No sinsce every polynomials n in $\Im(D)$ has degree $\leq n - 1$.

$$D(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + na_n x^{n-1}.$$

$D : P_n \to P_n$ is not surjective.

Indefinite Integral:

$$\int : P_\infty \to P_\infty$$

[Linear] Injective? $\int 1 = x$, $deg(f) = n$, $deg(\int f) = n + 1$. $\int f = 0$ implies $f = 0$. So injective. $D(\int f) = f$ not surjective. No $f \in P_\infty$ such that $\int f = 1$. Definition: Assume $V = U \oplus U'$, direct sum decomposition of V. $U \cap U' = \{0\}$, $V \in U + U'$. Definition of $P(v)$: Since $V = U \oplus U'$, every $v \in V$ admits a unique sum decomposition $v = u + u'$, by definition $P(v) := u$. $P(v)$ is called the projection of v onto U along U' parallel to U'. Proposition: P is a linear map Proof: $\begin{cases} P(v_1 + v_2) = P(v_1) + P(v_2) \\ P(\lambda v) = \lambda P(v) \end{cases}$

1. Prove : $P(v_1 + v_2) = P(v_1) + P(v_2)$.
2. $v_1 = u_1 + u_1' \in U \oplus U', P(v_1) = u_1$, $v_2 = u_2 + u_2' \in U \oplus U', P(v_2) = u_2$.
3. $P(v_1 + v_2) =?$, $v_1 + v_2 = u_1 + u_2 + u_1' + u_2'$, $P(v_1 + v_2) = u_1 + u_2 = P(v_1 + v_2)$.

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Kernel and Image: Kernel: $0 \in \ker(T) = \{u \in U : T(u) = 0\}$. Image: $0 \in \Im(T) = \{T(u) : u \in U\} \subseteq V$ Proposition: $\ker(T) \subseteq U$ and $\Im(T) \subseteq V$ are subspaces. Proof: Subspace Criterion Proposition: T surjective $\iff \Im(T) = V$, obvious by definition of surjective. Proposition: T linear $T : U \to V$. Statement: T injective $\iff \ker(T) = \{0\}$, linear system. Kernel is only useful for T linear. $T = \sin$ is non linear, $\ker(sin) = \{x \in \mathbb{R} : sin(x) = 0\}$, not a subspace. Proof: $\ker(T) \neq \{0\}$ then T non-injective $\to$ T injective $\to \ker(T) = \{0\}$. Basically since $[A \implies B] \iff [Not(B) \implies Not(A)]$. Assumption: $\ker(T) \neq \{0\}$ which means $\exists u \in \ker(T), u \neq 0$. Second proof (converse): If T not injective then $\ker(T) \neq \{0\}$. $T(u) = T(u')$. Definition: $\exists u, u' \in U$ s.t $u \neq u'$ and $T(u) = T(u')$. Prove that $T(u) = 0$ for some $v \neq 0$.

$$T(u) - T(u') = 0$$

$$T(u - u') = 0$$

by linearity of T $v \neq 0$. We will show $v := u - u'$ v nonzero and verify $T(v) = 0$. Example: $\Im(P) = U$, $\ker(P) = U'$. $\{v \in V : P(v) = 0\}$. $V = U \oplus U'$, $v = u + u'$, and $P(v) = u \in U$: projection. If $u \in U$ then $P(u) = u$. $\Im(P) = U$: $\Im(P) \subseteq U$: take $U \in \Im(P)$, $u = P(v) \in U$. Then $u = P(v)$ for some V (det of Im). To compute $P(u)$, write $v = U + U'$, $P(v) = u$. $U \subseteq \Im(P)$: $\forall u \in U$, $\exists v : P(v) = u$. $P(v) = u$, take $v = u$, $P(v) = P(u) = u$. $P(v) = u$, $T(u) = 0$.

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$T_A : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}$$

Find $\ker(T_A)$.

$$\ker(T_A) = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : T_A \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

$$\implies \ker(T_A) = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \begin{cases} x + 2y = 0 \\ 3x + 4y = 0 \end{cases}\}$$

$\begin{pmatrix} x \\ y \end{pmatrix} \in \ker(T) \iff \begin{pmatrix} x \\ y \end{pmatrix}$ solution to $\begin{cases} x + 2y = 0 \\ 3x + 4y = 0 \end{cases}$. Solve, find $x = y = 0$:

$$\implies \ker(T) = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

which implies T is injective.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad T_A : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\ker(T_A) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases} \}$$

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Basis solution: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. And $\ker(T_A) = \text{span}(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix})$.

Image of a linear transformation $T : U \to V$ is linear, $\Im(T) = \{T(u) : u \in U\}$. Reminder, for a transformation to be linear $T(\lambda u) = \lambda T(u)$, where $\lambda \in k$. Example: $T = T_A$, $T : \mathbb{R}^2 \to \mathbb{R}^2$, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. $T(e_1) = Ae_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $T(e_2) = Ae_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. $\Im(T) = \text{span}(T(e_1), T(e_2)) = \text{span}(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}) = \mathbb{R}^2$. Proposition: $(e_1, \ldots, e_n)$ basis of U, $\Im(T) = \text{span}(T(e_1), \ldots, T(e_n))$. Main Idea: $\Im(T) = \{T(u) : \forall u \in U\}$, take $u \in U$, meaning $u = \lambda_1 e_1 + \ldots + \lambda_n e_n$, then $T(u) = T(\lambda_1 e_1 + \ldots + \lambda_n e_n) = \lambda_1 T(e_1) + \cdots + \lambda_n T(e_n) \in \text{span}(T(e_i))$. Proof that $\Im(T) \subseteq \text{span}(T(e_1), \ldots, T(e_n))$: Step 1: Let $v \in \Im(T)$, prove $v \in$ span. Step 2: $\exists u$ such that $v \in T(u)$. Show there exist $\lambda_1, \ldots, \lambda_n$ such that $v = \lambda_1 T(e_1) + \cdots + \lambda_n T(e_n)$. Step 3: Find $\lambda_1, \ldots, \lambda_n$, e basis. $\exists \lambda_1, \ldots, \lambda_n$ such that $u = \lambda_1 e_1 + \cdots + \lambda_n e_n$. Apply T: $T(u) = \lambda_1 T(e_1) + \cdots + \lambda_n T(e_n)$.

Matrix of a transformation: dim 1: Proposition: $T : \mathbb{R} \to \mathbb{R}$ linear is determined by a single real (1x1 matrix) number Propostion: $T : \mathbb{R} \to \mathbb{R}$ linear map $T(x) = T(x - 1) = xT(1) - ax$. Matrix of T in basis e: denoted $A = [T]_e$, $[T]_e = ([T(e_1)]_e, \cdots, [T(e_n)]_e) = A$. To compute matrix of $T : U \to U$ in $e =$ basis of U:

1. Compute coord of $T(e_1)$ in basis e.
2. Write these coord as columns in A.

Example: $U = \mathbb{R}^2$, $e =$ std basis,$T : \mathbb{R}^2 \to \mathbb{R}^2$, $[T]_e = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Derivative: Matrix of $D(f) = f'$, $D : P_n \to P_n$, $e = (x^n, x^{n-1}, \ldots, x^1, 1)$.

6 Appendix

6.1 Proof: Standard Basis is a Basis

Claim 1. The vectors $e_1, \ldots, e_n$ span $\mathbb{R}^n$.

Proof. We show that any vector in $\mathbb{R}^n$ can be written as a linear combination of $e_1, \ldots, e_n$:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

□

Claim 2. The vectors $e_1, \ldots, e_n$ are linearly independent.

Proof. Suppose

$$\lambda_1 e_1 + \cdots + \lambda_n e_n = 0.$$

Then

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

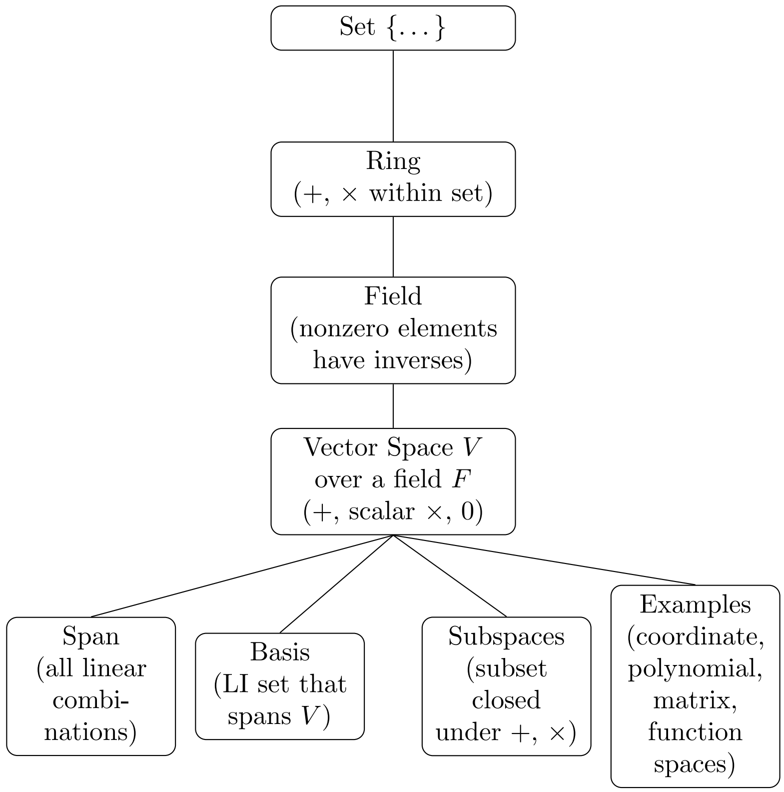
so $\lambda_1 = \cdots = \lambda_n = 0$. Therefore $e_1, \ldots, e_n$ are linearly independent. □

6.2 Vector Space Axioms

1. Commutativity of addition: $v_1 + v_2 = v_2 + v_1$
2. Associativity of addition: $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$
3. Existence of additive identity: $\exists 0 \in V$ such that $v + 0 = v$
4. Existence of additive inverses: $\forall v \in V, \exists - v \in V$ with $v + (-v) = 0$
5. Compatibility of scalar multiplication with field multiplication: $\lambda(\mu v) = (\lambda\mu)v$
6. Identity element of scalar multiplication: $1v = v$
7. Distributivity of scalar multiplication over vector addition: $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$
8. Distributivity of scalar multiplication over field addition: $(\lambda + \mu)v = \lambda v + \mu v$

6.3 Algebraic Structures: Rings, Fields, and Vector Spaces

Genealogy of Algebraic Structures. Conceptually, we can visualize the "family tree" of algebraic structures as follows:



This tree shows how vector spaces arise from fields (which arise from rings), and highlights the main concepts inside a vector space: spans, bases, subspaces, and common examples.

Rings vs Vector Spaces. Although rings and vector spaces both have addition and a zero element, they differ fundamentally in how multiplication works:

- **Ring:** A set R with addition and multiplication between elements of R. Addition forms an abelian group, multiplication is associative, and distributes over addition. Scalars are elements *inside the set.*

- **Vector Space:** A set V with addition and multiplication by scalars from a *field* F. Addition forms an abelian group, scalar multiplication distributes appropriately. There is no multiplication between vectors themselves; only scalar multiplication is defined.

7 Solutions

Solution 2 (Invertibility)**.** *Suppose B and B' are both inverses of A. Then*

$$B = BI = B(AB') = (BA)B' = IB' = B'.$$

Therefore, $B = B'$, so the inverse is unique.

Solution 3 (Invertibility 2)**.** *We can answer this problem with proof by contradiction. Let's suppose this matrix is invertible. By definition there exists $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We can rewrite this equation into: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{-1}$. The inverse of our matrix can be rewritten as $\frac{1}{0*0-1*0} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$[11]. But this is undefined since division by 0 is undefined. Therefore, our initial assumption that the matrix is invertible is false, and thus the matrix is not invertible.*

Solution 4 (Field)**.** *A field with 2 elements can be constructed as follows: Let $F = \{0, 1\}$ be a set with two elements. We define addition and multiplication operations on F as follows:*

- $0 + 0 = 0$
- $0 + 1 = 1$
- $1 + 0 = 1$
- $1 + 1 = 0$
- $0 \times 0 = 0$
- $0 \times 1 = 0$
- $1 \times 0 = 0$
- $1 \times 1 = 1$

Solution 5 (Span)**.**

$$\operatorname{span}(v_1, v_2) = \{xv_1 + yv_2 : x, y \in \mathbb{R}\} = \left\{ \begin{pmatrix} 3x + y \\ x + 3y \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

Solution 6 (Span 2)**.** *Assume $u = au_1 + bu_2 + cu_3$, with $a, b, c \in \mathbb{R}$. This gives the system of equations*

$$\left(\begin{array}{ccc|c} 3 & 1 & 2 & 2 \\ 10 & 3 & 8 & 10 \\ 7 & -2 & 1 & 7 \end{array}\right).$$

Solving via Gaussian elimination, we find $a = \frac{2}{21}$, $b = -\frac{46}{21}$, and $c = \frac{41}{21}$. Hence $u \in \operatorname{span}(u_1, u_2, u_3)$.

[11] Recall that an inverse of a 2×2 matrix is equal to its determinant multiplied with its conjugate

Solution 7 (Subspace Criterion)**.** *Let $A \in M_n(K)$ be fixed and define*

$$U = \{x \in K^n : Ax = \vec{0}\}.$$

We verify the subspace criterion.

(0) Non-empty: Since $A0 = 0$, we have $0 \in U$.

(1) Closed under addition: Let $x, y \in U$. Then $Ax = 0$ and $Ay = 0$. Hence

$$A(x + y) = Ax + Ay = 0 + 0 = 0,$$

so $x + y \in U$.

(2) Closed under scalar multiplication: Let $x \in U$ and $\lambda \in K$. Then

$$A(\lambda x) = \lambda Ax = \lambda 0 = 0,$$

so $\lambda x \in U$.

Therefore, by the subspace criterion, U is a subspace of K^n. It is called the null space (kernel) of A.

Solution 8 (Subspace Criterion 2)**.** *We verify the subspace criterion.*

(0) Non-empty: Since $A0 = 0$, we have $0 \in U$.

(1) Closed under addition: Let $y_1, y_2 \in U$. Then there exist $x_1, x_2 \in K^n$ such that

$$y_1 = Ax_1, \quad y_2 = Ax_2.$$

Hence,

$$y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2) \in U.$$

(2) Closed under scalar multiplication: Let $y \in U$ and $\lambda \in K$. Then $y = Ax$ for some $x \in K^n$, and

$$\lambda y = \lambda Ax = A(\lambda x) \in U.$$

Therefore, by the subspace criterion, U is a subspace of K^n. It is called the image of A.

Solution 9 (Subspace Criterion 3)**.** *Let $U \subseteq K$ be a subspace. We show that either $U = \{0\}$ or $U = K$. If $U = \{0\}$, we are done. Otherwise, $U \neq \{0\}$. Then there exists $v \in U$ with $v \neq 0$. We prove that $U = K$. Let $x \in K$ be arbitrary. Since $v \neq 0$, there exists $\lambda \in K$ such that*

$$x = \lambda v.$$

Because U is closed under scalar multiplication, $\lambda v \in U$, hence $x \in U$. Therefore every $x \in K$ belongs to U, so $U = K$. Conclusion: the only subspaces of K are $\{0\}$ and K.

Solution 10 (Subspace Criterion 4)**.** *We verify the subspace criterion. (0) Non-empty: Taking $\lambda_1 = \cdots = \lambda_n = 0$ gives*

$$0 = 0v_1 + \cdots + 0v_n \in \operatorname{span}(v_1, \ldots, v_n).$$

(1) Closed under addition: Let $u, v \in \operatorname{span}(v_1, \ldots, v_n)$*. Then there exist scalars* $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$ *such that*

$$u = a_1 v_1 + \cdots + a_n v_n, \quad v = b_1 v_1 + \cdots + b_n v_n.$$

Hence,

$$u + v = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n \in \operatorname{span}(v_1, \ldots, v_n).$$

(2) Closed under scalar multiplication: Let $u \in \operatorname{span}(v_1, \ldots, v_n)$ *and* $\lambda \in K$*. Then*

$$u = a_1 v_1 + \cdots + a_n v_n$$

for some scalars a_i*, and*

$$\lambda u = (\lambda a_1)v_1 + \cdots + (\lambda a_n)v_n \in \operatorname{span}(v_1, \ldots, v_n).$$

Therefore, $\operatorname{span}(v_1, \ldots, v_n)$ *is a subspace of* V*.*

Solution 11 (Span Membership)**.** *We look for scalars* $x, y, z \in \mathbb{R}$ *such that*

$$xv_1 + yv_2 + zv_3 = v.$$

That is,

$$x \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} + z \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix}.$$

This gives the system

$$\begin{cases} x + 2y + z = 3, \\ 2x + 5y + 3z = 5, \\ x + 4y + 6z = -5. \end{cases}$$

Solving, we obtain

$$x = 3, \quad y = 1, \quad z = -2.$$

Therefore,

$$v = 3v_1 + v_2 - 2v_3,$$

so $v \in \operatorname{span}(v_1, v_2, v_3)$*.*

Solution 12 (Span Membership 2)**.** *We seek scalars* $x, y, z \in \mathbb{R}$ *such that*

$$f = xf_1 + yf_2 + zf_3.$$

Comparing coefficients,

$$x(x^2 + 2x + 1) + y(2x^2 + 5x + 4) + z(x^2 + 3x + 6) = 3x^2 + 5x - 5,$$

which gives

$$\begin{cases} x + 2y + z = 3, \\ 2x + 5y + 3z = 5, \\ x + 4y + 6z = -5. \end{cases}$$

Solving,

$$x = 3, \quad y = 1, \quad z = -2.$$

Hence,

$$f = 3f_1 + f_2 - 2f_3,$$

and therefore $f \in \mathrm{span}(f_1, f_2, f_3)$.

8 Useful Links