

# MATH 223: Linear Algebra

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**Abstract**

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# 1 Introduction

## 2 Prerequisite knowledge

### 2.1 Notation

#### 2.1.1 Sets

Sets are a grouping of objects.

Set	Meaning	Examples
$\mathbb{N}$	The set of natural numbers	$(0, 1, 2, 3, \dots)$
$\mathbb{Z}$	The set of integers	$(\dots, -3, -2, -1, 0, 1, 2, 3, \dots)$
$\mathbb{Q}$	The set of rational numbers	$\mathbb{Q} = \frac{a}{b} \mid \forall a, b \in \mathbb{Z} \text{ and } b \neq 0$
$\mathbb{R}$	The set of all rational and all irrational numbers	$(\dots, -1, 0, \frac{1}{4}, 1, 1000, \dots)$
$\mathbb{C}$	The set of all complex numbers	$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } i \subseteq \sqrt{-1}\}.$

We have the following relationships between sets:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

#### 2.1.2 Symbols

Symbol	Meaning
$\subseteq$	is a subset of or equal to
$\subset$	is a strict subset of
$\in$	is an element of
$\forall$	for all
$\exists$	there exists
$\emptyset$	empty set
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if

## 2.2 Complex Algebra

### 2.2.1 Complex Numbers

A complex number is of the form:  $z = x + iy$  where  $x, y \in \mathbb{R}$  and  $i$  is the imaginary unit such that  $i^2 + 1 = 0$ .

**Definition 1** (Powers of  $i$ ). 

$k$	$0$	$1$	$2$	$3$	$4$	$5$
$i^k$	$1$	$i$	$-1$	$-i$	$1$	$i$

**Theorem 1** (Fundamental Theorem of Algebra). *Any complex polynomial<sup>1</sup>  $f$  (except constant functions) has a root in  $\mathbb{C}$ .*

<sup>1</sup>Polynomial is a function such as:  $f = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  where  $a_i \in \mathbb{R}$  or  $\mathbb{C}$  and  $n \in \mathbb{N}$ .

**Remark 1.** If we have a polynomial  $f$  of degree  $n$ , then it has  $n$  roots, where each root can have a multiplicity<sup>2</sup>.

**Example.** If we have a polynomial  $(x - 1)^2$ , it has a degree of 2 but only one root, which is 1, with a multiplicity of 2. This means that the root 1 appears twice in the polynomial.

We can factorize a polynomial in the form of  $f = a_n x^n + \dots + a_1 x + a_0$  into a linear factor:  $f = a(x - z_1)(x - z_2)\dots(x - z_n)$  where  $z_i$  are the roots of  $f$  in  $\mathbb{C}$ .

Using the FTA for a function such as  $f = a_n x^n + \dots + a_1 x + a_0$ , we can say that the FTA implies that  $f$  has a root  $f(z) = 0$ , because  $f(z) = a(z - z) = 0$ .

### 2.2.2 Complex Operations

We can define operations on complex numbers as follows:

- Addition:  $z + z' = (x + x') + i(y + y')$ , where  $x, x', y, y' \in \mathbb{R}$ .
- Multiplication:  $zz' = (x + iy)(x' + iy') = (xx' - yy') + i(xy' + yx')$ .
- Inverse:  $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}$

From the definition of inverse, we can see that for any complex number  $z$ , its inverse  $\frac{1}{z}$  is also a complex number. For example, take  $z = 1 + i$ , where  $x = y = 1$ , from the definition of inverse, we can conclude that:

$$\frac{1}{1 + i} \in \mathbb{C}$$

Multiplying by a complex number  $z$  corresponds geometrically to

$$\begin{cases} \text{a rotation by some angle } \theta, \\ \text{a rescaling by the factor } |z|. \end{cases}$$

### 2.2.3 Complex Conjugate

A complex conjugate is a way to "flip" the imaginary part of a complex number. For example, if we have a complex number  $z = x + iy$ , then the complex conjugate of  $z$  is  $\bar{z} = x - iy$ . Some basic properties of complex conjugates are:

- $\bar{\bar{z}} = z$
- $\overline{z + z'} = \bar{z} + \bar{z'}$
- $\overline{z \cdot z'} = \bar{z} \cdot \bar{z'}$

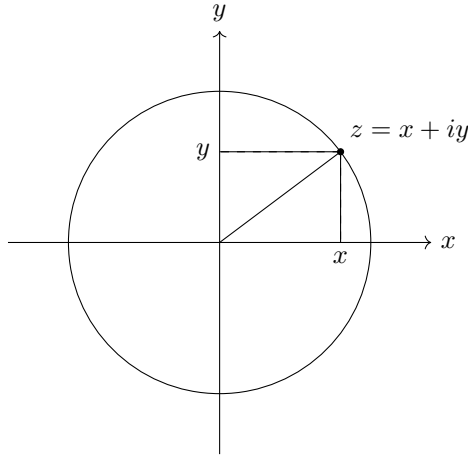
## 2.2.4 Geometric and Polar Form of Complex Numbers

### 1. Geometric Interpretation

**Definition 2** (Geometric interpretation). *Every complex number  $z = x + iy$  can be identified with a point  $(x, y)$  in the plane, called the complex plane.*

We define the complex plane as:

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.$$



### 2. Modulus and Unit Circle

**Definition 3** (Modulus). *The modulus of a complex number  $z$  is defined by*

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

*Geometrically,  $|z|$  is the distance from the origin to the point  $(x, y)$ .*

We can rewrite the definition of the unit circle as follows:

$$S' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{z \in \mathbb{C} : |z| = 1\},$$

where  $S'$  is the unit circle in the complex plane.

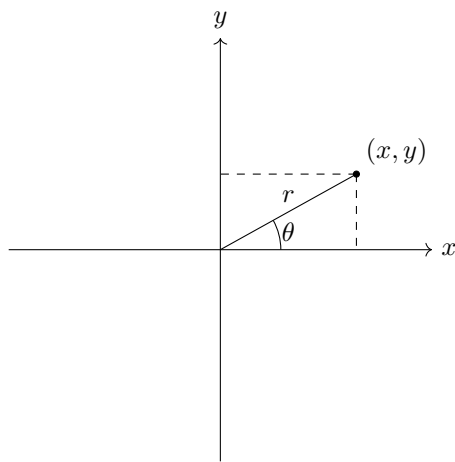
### 3. Polar Coordinates

**Definition 4** (Polar coordinates). *Instead of describing a point by  $(x, y)$ , we may describe it using polar coordinates  $(r, \theta)$ , where  $r = |z|$  is the distance to the origin and  $\theta$  is the angle with the positive  $x$ -axis.*

**Example.** *Consider the point  $(x, y)$ , where  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$ . We can define  $(r, \theta)$  as follows:*

---

<sup>2</sup>The multiplicity of a root represents how many times the root occurs in the polynomial.



Complex numbers can also be described using polar coordinates.

**Definition 5** (Polar and exponential form).

$$z = x + iy = r \cos(\theta) + ir \sin(\theta) = r(\cos(\theta) + i \sin(\theta)) = re^{i\theta}.$$

We can also define multiplication in polar form:

**Definition 6** (Multiplication in polar form).

$$z = re^{i\theta}, \quad z' = r'e^{i\theta'}, \quad zz' = rr'e^{i(\theta+\theta')}.$$

**Example.**

$$(1 + i)^{32} = (\sqrt{2}e^{i\pi/4})^{32} = (\sqrt{2})^{32}e^{i8\pi} = 2^{16}(\cos 8\pi + i \sin 8\pi) = 2^{16}.$$

Around the 1740, the mathematician Euler discovered a formula for complex numbers. The formula is known as Euler's formula.

**Definition 7** (Euler's formula).

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

This formula is quite useful when dealing with complex numbers, as it allows us to write complex numbers in a more compact form.

#### 4. Roots in $\mathbb{C}$

**Definition 8** ( $n^{\text{th}}$  roots in  $\mathbb{C}$ ). *For any complex number  $z$ , an  $n^{\text{th}}$  root of  $z$  is a complex number  $w$  such that*

$$w^n = z.$$

**Example.** If  $z = re^{i\theta}$ , then any solution of  $w^n = z$  must satisfy

$$w_k = r^{1/n} e^{i(\theta + 2\pi k)/n}, \quad k = 0, 1, 2, \dots, n-1,$$

where the  $n^{\text{th}}$  roots of  $z$  are equally spaced on a circle of radius  $r^{1/n}$  centered at the origin.

**Definition 9** (Roots of unity). The  $n^{\text{th}}$  roots of unity are the solutions of a special case where  $z = 1$ .

**Example.** If  $z = 1$ , then any solution of  $w^n = z$  must satisfy

$$w_k = e^{i2\pi k/n}, \quad k = 0, 1, 2, \dots, n-1.$$

Geometrically, they lie on the unit circle and are equally spaced.

## 3 Basic Algebraic structures

### 3.1 Sets with Multiplication

**Definition 10** (Set with multiplication). A set  $M$  is called a set with multiplication if you can multiply any two elements of  $M$ , and the result is still in  $M$ . In other words, for any  $a, b \in M$ , the product  $ab$  is also in  $M$ .

**Example.** Let  $M = \mathbb{R}$ . If  $a, b \in \mathbb{R}$ , then  $ab \in \mathbb{R}$ . So the real numbers  $\mathbb{R}$  form a set with multiplication.

**Example.** An example of a set with multiplication is the set of all  $2 \times 2$  complex matrices:  $M = M_2(\mathbb{C})$ . Another example is the nonzero set of all real numbers  $\mathbb{R}$  with ordinary multiplication:  $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ .

### 3.2 Invertibility

**Definition 11** (Condition for Invertibility). Let  $A \in M$  be an  $n \times n$  matrix, and suppose that there exists an  $n \times n$  matrix  $B$  such that  $AB = I_n$  or  $BA = I_n$ .

Where  $I_n$  is the  $n \times n$  identity matrix<sup>3</sup>  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . Then  $A$  is invertible, and  $B$  is called the inverse of  $A$  and is denoted by  $B = A^{-1}$ .

**Remark 2.** If  $A$  is invertible, then  $A^{-1}$  exists and is unique<sup>4</sup>.

To determine if an element  $A$  in a set with multiplication  $M$  is invertible, we can use the following examples:

<sup>3</sup>An identity matrix is a square matrix with 1s on its main diagonal and 0s everywhere else. It represents no change in linear transformations, and it's used in finding matrix inverses.

<sup>4</sup>Unique means there is exactly one such element.

**Example.** Let  $M = \mathbb{Z} = \{\dots - 2, -1, 0, 1, 2, \dots\}$  and  $A = 2$ . Is  $A$  invertible in  $M$ ?

*Solution:* No, because  $\frac{1}{2} \notin \mathbb{Z}$ .

**Example.** Let  $M = \mathbb{R}$  and  $A = 2$ , is  $A$  invertible in  $M$ ?

*Solution:* Yes, because  $\frac{1}{2} \in \mathbb{R}$ .

**Example.** Is  $1 + i$  invertible in  $\mathbb{C}$ ?

*Solution:* Yes, using our previous definition of inverse (2.2.2), we get that

$$\frac{1}{1+i} = \frac{1-i}{2} \in \mathbb{C}.$$

### 3.3 Ring

**Definition 12.** A **ring** is a set  $R$  where you can **add** and **multiply** elements, and the following are true:

1. You can add any two elements and stay in  $R$ . There is a zero, every element has a negative, and addition is commutative and associative.
2. You can multiply any two elements and stay in  $R$ . Multiplication is associative, and there is a 1.
3. Multiplication distributes over addition:

$$a(b + c) = ab + ac \quad \text{and} \quad (a + b)c = ac + bc.$$

The main example of a ring is the set of integers  $\mathbb{Z}$ .

### 3.4 Field

**Definition 13.** A **field** is a set of numbers which can be added, subtracted, multiplied, and divided (except for division by zero) in a way that satisfies certain rules. These rules are:

1. You can add any two elements and stay in the field. There is a zero, every element has a negative, and addition is commutative<sup>5</sup> and associative.
2. You can multiply any two elements and stay in the field. Multiplication is associative, and there is a 1.
3. Multiplication distributes over addition:

$$a(b + c) = ab + ac \quad \text{and} \quad (a + b)c = ac + bc.$$

Examples of fields include the real numbers  $\mathbb{R}$  and the complex numbers  $\mathbb{C}$ .



**Problem 1.** Construct a field with 2 elements.

**Problem 2.** Show that if an inverse of  $A$  in  $\mathbb{M}$  exists, then it is unique.

**Problem 3.** Let  $K$  be a field. Prove that this matrix  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(K)$  is not invertible.

## 4 Vector Spaces

### 4.1 Cartesian vector spaces

**Definition 14** ( $\mathbb{R}^n$ ). Let  $n \in \mathbb{N}$ . The Cartesian product of  $n$  copies of  $\mathbb{R}$  is called  $\mathbb{R}^n$ .

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}$$

### 4.2 Vectors

#### 4.2.1 Vector operations

Vector operations are defined as follows.

Addition:  $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix},$

scalar multiplication:  $\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix},$  where  $\lambda \in \mathbb{R}$ .

**Definition 15** (Linear combination). A linear combination of vectors  $v_1, \dots, v_n$  is a vector  $v$  of the form  $v = \lambda_1 v_1 + \dots + \lambda_n v_n$ , where  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ .

**Example.** A linear combination could look like this:

$$\xi((v_1 + 2v_2) + v_3) + v_4,$$

where  $\xi \in \mathbb{R}$ .

#### 4.2.2 Span

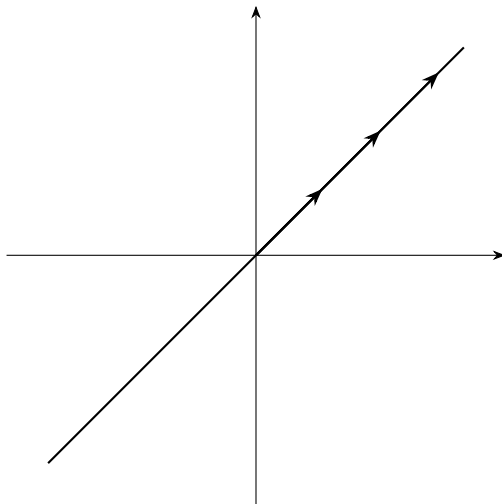
**Definition 16** (Span). Let  $A \subset \mathbb{R}^2$ . The span of  $A$ , denoted  $\text{Span}(A)$ , is the set of all linear combinations of elements of  $A$ .

**Remark 3.** If  $A = \{v\}$  contains one nonzero vector, then  $\text{Span}(v)$  is a line through the origin.

**Example.** Let  $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . Then

$$\text{Span}(A) = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

which is a line in  $\mathbb{R}^2$ .



### 1. Span in $\mathbb{R}^n$

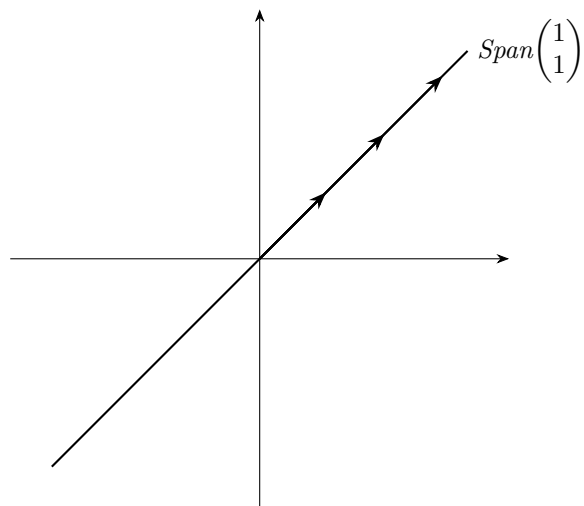
**Example.** The span of  $n$  vectors  $v_1, \dots, v_n$  is the set of all linear combinations of  $v_1, \dots, v_n$ . Then any vector in  $\text{Span}(v_1, \dots, v_n)$  has the form

$$\text{Span}(v_1, \dots, v_n) = \{ \lambda_1 v_1 + \dots + \lambda_n v_n : \lambda_i \in \mathbb{R} \}$$

When working in  $\mathbb{R}^n$ , the span describes all points you can reach by scaling and adding the given vectors. Depending on the vectors, the span can be a line (if the vectors are dependent), a plane, or a higher-dimensional subspace. The following examples show what spans look like in  $\mathbb{R}^2$ .

**Example.** Let  $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ,  $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ . Then  $\text{Span}(v_1, v_2) = \{ \lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in \mathbb{R} \}$

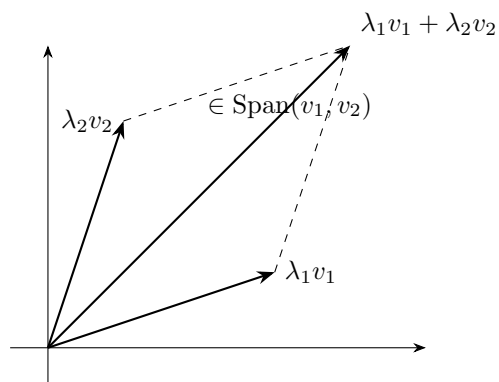
**Example.** Let  $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . Then  $\text{Span}(A) = \{ \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \}$



**Example.** Let

$$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Any vector in  $\text{Span}(v_1, v_2)$  has the form  $\lambda_1 v_1 + \lambda_2 v_2$ . Geometrically, this is illustrated below.

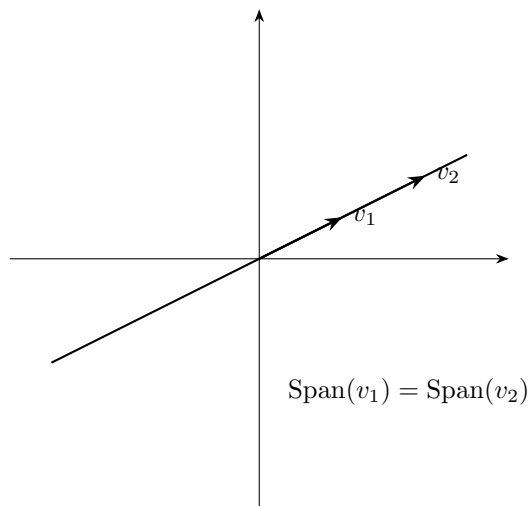


**Example.** Let

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Then  $v_2 = 2v_1$ , so

$$\text{Span}(v_1) = \text{Span}(v_2).$$



### Span in $\mathbb{C}^n$

**Definition 17.** The span over  $\mathbb{C}$  of vectors  $v_1, \dots, v_n$  is

$$\text{Span}_{\mathbb{C}}(v_1, \dots, v_n) = \{\lambda_1 v_1 + \dots + \lambda_n v_n : \lambda_i \in \mathbb{C}\}.$$

**Example.** For example,

$$\begin{pmatrix} 2i \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 1+i \end{pmatrix} \in \mathbb{C}^2.$$

So we distinguish between real and complex vector spaces:

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}, \quad \mathbb{C}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{C} \right\}.$$

### 4.2.3 Standard Basis

**Definition 18.** The standard basis for  $\mathbb{R}^n$  is the set  $\{e_1, \dots, e_n\}$ , where  $e_i$  is the  $i$ th standard basis vector.

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

(Basis = span + linear independence)

**Claim** These vectors span  $\mathbb{R}^n$ .

*Proof.* We show that any vector in  $\mathbb{R}^n$  can be written as a linear combination of  $e_1, \dots, e_n$ :

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

□

**Example** (Complex vs Real Span).

$$\mathbb{C}^2 = \text{Span}_{\mathbb{C}}(e_1, e_2) = \{z_1 e_1 + z_2 e_2 : z_1, z_2 \in \mathbb{C}\},$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here  $\dim_{\mathbb{C}} \mathbb{C}^2 = 2$  but  $\dim_{\mathbb{R}} \mathbb{C}^2 = 4$ . Every complex scalar can be written as  $z_k = x_k + iy_k$  with  $x_k, y_k \in \mathbb{R}$ . So

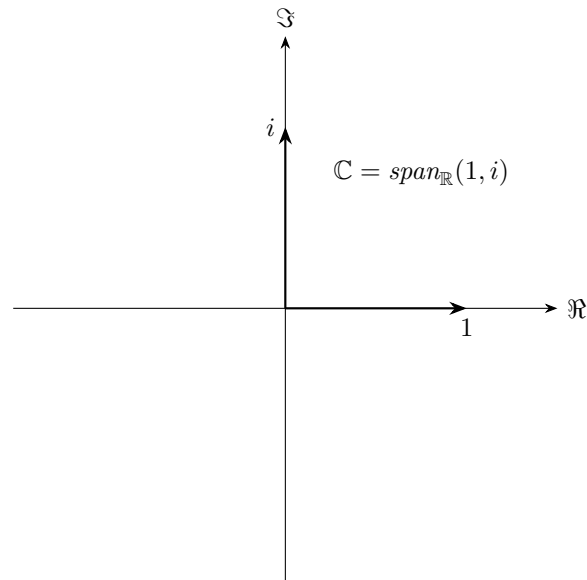
$$\text{Span}_{\mathbb{C}}(e_1, e_2) = \{(x_1 + iy_1)e_1 + (x_2 + iy_2)e_2 : x_1, x_2, y_1, y_2 \in \mathbb{R}\}.$$

Separating real and imaginary parts,

$$\text{Span}_{\mathbb{C}}(e_1, e_2) = x_1 e_1 + x_2 e_2 + y_1 (ie_1) + y_2 (ie_2).$$

Therefore,

$$\text{Span}_{\mathbb{C}}(e_1, e_2) = \text{Span}_{\mathbb{R}} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix} \right).$$



#### 4.2.4 Abstract Vector Spaces

An abstract vector space is just a set with two operations: vector addition  $(v_1 + v_2)$  and scalar multiplication  $(\lambda v)$ , where  $\lambda$  comes from a field  $k$ , satisfying certain axioms.

**Definition 19.** *Let  $k$  be a field. A **vector space over  $k$**  is a set  $V$  together with two operations*

$$\begin{cases} v_1 + v_2 \in V, & v_1, v_2 \in V, \\ \lambda v \in V, & v \in V, \lambda \in k, \end{cases}$$

*satisfying 8 axioms.*

**Axioms** (1) Rules for addition

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

There exists a zero vector  $0 \in V$  such that  $0 + v = v$ . For every  $v \in V$ , there exists  $-v \in V$ .

$$u + v = v + u$$

(2) Rules for scalar multiplication

$$1 \cdot v = v$$

$$\lambda_1(\lambda_2 v) = (\lambda_1 \lambda_2)v$$

(3) Compatibility (distributive laws)

$$(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$$

$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$$

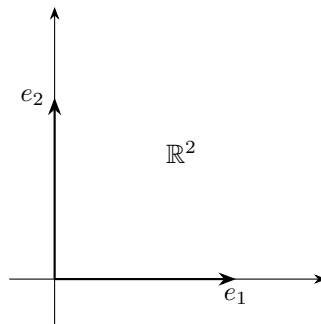
**Example.** *Let  $K$  be a field. Then*

$$K^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in K \right\}$$

*is a vector space over  $K$ .*

**Remark 4.**

$$\text{Span}_{\mathbb{Z}}(e_1, e_2) = \{\lambda_1 e_1 + \lambda_2 e_2 : \lambda_i \in \mathbb{Z}\}.$$



### 4.2.5 Examples of Vector Spaces

We now list important examples of vector spaces.

**1. Coordinate spaces** Let  $k$  be a field ( $k = \mathbb{R}$  or  $k = \mathbb{C}$ ).

$$k^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in k \right\}$$

is a vector space over  $k$ .

The standard basis is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

**2. Polynomial spaces** A polynomial is an expression

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad a_i \in k.$$

$$P_n(k) = \{a_n x^n + \dots + a_1 x + a_0 : a_i \in k\}$$

is a vector space over  $k$ .

Examples:

$$1 + x^2 \in P_2(\mathbb{R}), \quad 1 + ix^3 \in P_3(\mathbb{C}).$$

The subscript must satisfy  $\deg(f) \leq n$ .

All polynomials:

$$P_\infty = \{a_n x^n + \dots + a_1 x + a_0 : a_i \in k, n \geq 0\}.$$

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \dots \subseteq P_\infty.$$

A standard basis is  $\{1, x, x^2, \dots, x^n\}$ . Every  $f \in P_n$  can be written

$$f = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \dots + \lambda_n x^n.$$

**3. Matrix spaces**

$$M_n(k) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} : a_{ij} \in k \right\}$$

is a vector space over  $k$ .

Standard basis matrices:

$$e_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \dots$$

$$\dim M_n(k) = n^2.$$

Example in  $M_2(\mathbb{R})$ :

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1e_{11} + 2e_{12} + 3e_{21} + 4e_{22}.$$

**4. Function spaces** Let  $D$  be a set and  $k$  a field.

$$F(D, k) = \{f : D \rightarrow k\}$$

is a vector space over  $k$ .

When  $D = \mathbb{R}$  and  $k = \mathbb{R}$ , functions are drawn with horizontal axis  $D$  and vertical axis  $k$ .

$F(D)$  is a function from  $D$  to  $K$ , where  $K = \mathbb{R}$  or  $\mathbb{C}$ . No "standard basis" in general.

Standard basis if  $D$  is a finite set eg  $D = \{1, \dots, n\}$ .

$$(f \in F(D)). f(1) = 2, f(2) = 4, \dots, f(n) = 2^n$$

Kronecker function: Let  $x \in D$ ,  $\delta_x \in F(D)$ , defined by  $\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

Then  $\{\delta_x : x \in D\}$  is a standard basis of  $F(D)$ .

Standard basis :  $(\delta_1, \dots, \delta_n)$ , for  $F(\{1, \dots, n\})$ .

Remark:  $\{\delta_x : x \in \mathbb{R}\}$  do not span  $F(\mathbb{R})$ .

Proof that  $(\delta_1, \dots, \delta_n)$  spans  $F\{1, \dots, n\}$ . Take any

$$f \in F(D)$$

Find  $a_1, \dots, a_n \in K$  such that

$$f = a_1\delta_1 + \dots + a_n\delta_n$$

$$f(x), \quad x \in D = \{1, \dots, n\}$$

$$g(1) = a_1\delta_1(1) + a_2\delta_2(1) + \dots + a_n\delta_n(1) = a_1$$

Claim:  $a_k = f(k) \in K$  Claim:  $f = \sum f(k)\delta_k$ , where  $f(k) \in K$  and  $\delta_k$  is the basis vector.

Proof:

$$g(l) = f(1)\delta_1(l) + \dots + f(l)\delta_l(l) + \dots + f(n)\delta_n(l) = f(l)$$



Subspace criterion. Let  $V$  be a vector space. To check a subset  $u \subseteq V$  is a vector space. In principle need to check: 1.  $U$  is stable under addition  $u, v \in U$  Hence  $u + v \in U$ . 2.  $U$  is stable under scalar multiplication.  $\lambda \in K, u \in U$  Hence  $\lambda u \in U$ . 3. 8 axioms hold in  $U$ .

Proposition. A subset  $U \subseteq V$  is a vector space (8 axioms are true) if f: (0)  $U$  is not empty ( $0 \in U$ ) (1)  $U$  is stable under addition:  $u, v \in U \rightarrow u + v \in U$ . (2)  $U$  is stable under scalar multiplication:  $u \in U, \lambda \in K \rightarrow \lambda u \in U$ .

Subspaces

**Problem 4** (Subspace criterion). Let  $A \in M_n(K)$  be a fixed matrix. Prove that

$$U = \{x \in K^n : Ax = \vec{0}\}$$

is a subspace, null space or kernel.

**Problem 5** (Subspace criterion 2). The set  $U = \{Ax : x \in K^n\}$  is a subspace, image of  $A$ .

**Problem 6** (Subspace criterion 3). Let  $V = K$  (vector space of dim 1) show that the only 2 subspaces of  $V$  are  $\{0\}$  and  $V$  itself.

**Problem 7** (Subspace criterion 4).  $\text{Span}(v_1, \dots, v_n) = \{\lambda_1 v_1 + \dots + \lambda_n v_n : \lambda_1, \dots, \lambda_n \in K\}$  is a subspace.

**Problem 8.**  $\text{Span}(v_1, \dots, v_n)$  is the smallest subspace  $c$

**Problem 9** ( $K = \mathbb{R}$ ). Is  $v_i = \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix}$  in the space of  $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$  In otherwords: are there  $x, y, z \in \mathbb{R}$  such that  $v = xv_1 + yv_2 + zv_3$ ?

**Problem 10.** Is  $f : 3x^2 + 5x - 5$  in the span of  $f_1 = x^2 + 2x + 1, f_2 = 2x^2 + 5x + 4, f_3 = x^2 + 3x + 6$ ?

$$f = xf_1 + yf_2 + zf_3$$

Intersection, sums, direct sums.  $W$  vector space.  $U, V \subseteq W$  two subspaces. Definition:  $U \cap V = \{v \in W : v \in U \text{ and } v \in V\}$  smallest intersection  $\{0\}$ .

The union of 2 subspaces is NOT a subspace (in general).

$W = \mathbb{R}^2$  : draw a cartesian plane with vertical being  $V$  and horizontal being  $U$  and draw the cross  $(u + v)$ .  $U \cup V = \text{cross}$ , no quadrant, not a subspace

Sum of 2 subspaces:  $U + V = \{u + v : u \in U, v \in V\}$  line  $U$  + line  $V$  = plane

Proposition:  $U, V$  are subspaces of  $W$ .  $U + V$  is a subspace.  $U$  cannot rescale a vector space, therefore  $U + U = U$ . Proof:  $\subseteq$ : Let  $u + u' \in U + U$  prove  $u + u' \in U$ . Yes subce  $U$  is a subspace stable under addition.  $\supseteq$ : Take

$u \in U$ , prove  $u \in U + U$ .  $u = u + 0 \in U + U$ . Proof: subspace criterion. (0)  $0 \in U + V$  since  $0 \in U, 0 \in V$ . (1) Stable under addition: let  $w_1, w_2 \in U + V$ . Prove  $w_1 + w_2 \in U + V$ .

$$w_1 = u_1 + v_1, u_1 \in U, v_1 \in V$$

$$w_2 = u_2 + v_2, u_2 \in U, v_2 \in V$$

$$w_1 + w_2 = (u_1 + u_2) + (v_1 + v_2)$$

$$u_1 + u_2 \in U, v_1 + v_2 \in V$$

$$w_1 + w_2 \in U + V$$

(2) Stable under scalar multiplication: let  $w \in U + V, \lambda \in K$ . Prove  $\lambda w = \lambda u + \lambda v \in U + V$ .

Prop:  $U + V = \text{span}(U \cup V)$ .  $U + V =$  smallest subspace containing  $U \cup V$ .  
Proof:  $\subseteq$ : Take  $w \in U + V$  prove  $w \in \text{span}(U \cup V)$ .

$$w = u + v, u \in U, v \in V$$

Prove  $w \in \text{span}(U \cup V)$ . (You could also write it like  $\text{span}(U, V)$  since line can be of both in  $U$  or  $V$ ).  $\supseteq$ : Take  $w \in \text{span}(U \cup V)$  prove  $w \in U + V$ .

$$w = a_1 u_1 + \dots + a_n u_n + b_1 v_1 + \dots + b_m v_m$$

where  $u_i \in U, v_j \in V, a_i, b_j \in K$ .

$$w = (a_1 u_1 + \dots + a_n u_n) + (b_1 v_1 + \dots + b_m v_m)$$

$$w = u + v, u \in U + V$$

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2026

### 4.3 Direct sums

$U, V$  subspaces in  $W$  (ambient space).

**Definition 20.**  $U$  and  $V$  are in direct sum if  $U \cap V = \{0\}$ . Its a sum where the intersection is zero. Sum:

$$U + V = \{u + v : u \in U, v \in V\}$$

Now if  $U \cap V = \{0\}$  we write  $U \oplus V$ , and  $U, V \subseteq W$ . Sol:  $W = U \oplus V$  means

$$\begin{cases} W = U + V \\ U \cap V = \{0\} \end{cases} \quad \text{To prove } W = U \oplus V,$$

- $W = U + V, \subseteq$  Let  $w \in W$ . Prove: there exist  $u \in U, v \in V$  such that  $w = u + v$ .
- $U \cap V = \{0\}$ .  $\supseteq$  automatic. Pick  $w \in \{0\}$ . Prove  $w \in U \cap V$ . By assumption  $w = 0$ . Prove  $w \in U$  and  $w \in V$  hold by definition of subspace. Only need to show  $U \cap V \subseteq \{0\}$ . Let  $w \in U$  and  $w \in V$ . Prove  $w = 0$ .

Sets	Vector Spaces
intersection $A \cap B$	intersection $U \cap V$
union $A \cup B$	union $U + V$
Disjoint union $A \sqcup B$ and $A \cap B = \emptyset$	Direct sum $U \oplus V = \{0\}$

**Example.**

$$D = \{1, 2, 3, 4, 5\}$$

$$A = \{1, 2, 3\}, B = \{4, 5, 6\}$$

Then

$$D = A \sqcup B$$

→ function spaces  $F(D)$  = functions on D.

$$f \in F(D)$$

Example of a function:  $f(1) = 2, f(2) = \sqrt{2}, f(3) = \pi \dots$  View  $F(A)$  as a subspace of  $F(D)$ . Claim:  $F(A)$  can be "identified" with a subspace  $U$  of  $F(D)$ .

$$U = \{f : D \rightarrow \mathbb{R} : f(x) = 0, x \notin A\}$$

## 5 Appendix

## 6 Solutions

**Solution 1.** A field with 2 elements can be constructed as follows: Let  $F = \{0, 1\}$  be a set with two elements. We define addition and multiplication operations on  $F$  as follows:

- $0 + 0 = 0$
- $0 + 1 = 1$
- $1 + 0 = 1$
- $1 + 1 = 0$
- $0 \times 0 = 0$
- $0 \times 1 = 0$
- $1 \times 0 = 0$
- $1 \times 1 = 1$

**Solution 2.** Suppose  $B$  and  $B'$  are both inverses of  $A$ . Then

$$B = BI = B(AB') = (BA)B' = IB' = B'.$$

Therefore,  $B = B'$ , so the inverse is unique.

**Solution 3.** We can answer this problem with proof by contradiction. Let's suppose this matrix is invertible. By definition there exists  $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  such that  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . We can rewrite this equation into:  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{-1}$ . The inverse of our matrix can be rewritten as  $\frac{1}{0*0-1*0} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ <sup>6</sup>. But this is undefined since division by 0 is undefined. Therefore, our initial assumption that the matrix is invertible is false, and thus the matrix is not invertible.

**Solution 4** (Subspace criterion). (0) is  $0 \in U$ ? Yes, because  $A0 = 0$ .

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(1) Addition: Let  $x, y \in U$ . Prove  $x + y \in U$ .

$$A(x + y) = Ax + Ay = 0 + 0 = 0$$

(2) Scalar multiplication: Let  $x \in U, \lambda \in K$ . Prove  $\lambda x \in U$ .

$$A(\lambda x) = \lambda Ax = \lambda 0 = 0$$

**Solution 5** (Subspace criterion 2). (0) is  $0 \in U$ ? Find  $x \in K^n$  such that  $Ax = 0$ .  $x = 0$ .

(1) Stability under addition:  $y_1, y_2 \in U \rightarrow y_1 + y_2 \in U$ . There exist  $x$ , such that  $y_1 = Ax_1, y_2 = Ax_2$ .

$$y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2) \in U$$

(2) Stability under scalar multiplication: If  $f = Ax$ , for some  $x$ ,  $\lambda y = A(\lambda x)$

**Solution 6** (Subspace criterion 3). Let  $U \subseteq \mathbb{R}$  be a subspace. Prove:  $U = \{0\}$  or  $U = \mathbb{R}$ .

Case 1:  $U = \{0\}$ .

Case 2:  $U \neq \{0\} \rightarrow$  there exists  $v \in \mathbb{R}, v \neq 0$ . Prove:  $U = \mathbb{R}$ . Let  $x \in \mathbb{R}$  be any real number. Prove:  $x \in U$ . Since  $U$  is a subspace, it is stable under scalar multiplication.

$$x \in U \rightarrow Ax \in U$$

Therefore,  $x \in U$ .

**Solution 7** (Subspace criterion 4). (0)  $0 \in \text{span}(v_1, \dots, v_n)$ , because

$$\lambda_1 = \dots \lambda_n = 0 \quad \lambda_1 v_1 + \dots + \lambda_n v_n = 0.$$

---

<sup>6</sup>Recall that an inverse of a  $2 \times 2$  matrix is equal to its determinant multiplied with its conjugate

(1) *addition:*  $u, v \in \text{span}$  Prove  $u + v \in \text{span}$ .

$$u = a_1v_1 + \dots + a_nv_n$$

$$v = b_1v_1 + \dots + b_nv_n$$

$$u + v = (a_1 + b_1)v_1 + \dots + (a_n + b_n)v_n \in \text{span}(v_1, \dots, v_n)$$

(2) *scalar multiplication:*

**Solution 8.**

$$\begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} + z \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{cases} x + 2y + z = 3 \\ 2x + 5y + 3z = 5 \\ x + 4y + 6z = -5 \end{cases} \rightarrow x = 3, y = 1, z = -2$$

*A: yes since*

$$v = 3v_1 + v_2 + 2v_3$$

## 7 Useful Links