MATH 325: Honours Ordinary Differential Equations

William Homier[1]

[1] *McGill University Physics, 3600 Rue University, Montréal, QC H3A 2T8, Canada*

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Abstract

Contents

1 Introduction

Jean-Philippe Lessard (Burnside 1119). Tutorials every wednesday from 9am to 10am, ENGTR 0070, with Eunpyo Bang. Office hours thursday. No textbooks. 25% assignments (2 written assignments 15%, and 5 webworks 10%). 25% Midterm (February 16 - inclass). 50% Final. Since its honours you will deal with analysis.

2 Prerequisite knowledge

2.1 Analysis

3 Intro, Classification, Theorem of Existence & Uniqueness

3.1 Intro

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Definition 1 (Differential Equation)**.** *A differential equation (DE) is a relation that involves an unknown function and some of its derivatives.*

To better understand what a differential equation is, consider the following example.
Imagine a ball of mass m falling, subject to gravity and air resistance (drag). Denote by $v(t)$ the velocity of the ball at time t, whereas t is the independent variable, and v the dependent variable. Let the downward direction be positive. We know the force of gravity is given by $F_g = mg$, where g is the acceleration due to gravity. The drag force is given by $F_d = -\lambda v$, where λ is the drag coefficient and is $\lambda \geq 0$. According to Newton's second law $\sum F = ma$, the net force acting on the ball is equal to its mass times its acceleration

$$m\frac{dv}{dt} = mg - \lambda v.$$

Let $y(t)$ be the position, meaning $v(t) = \frac{dy}{dt}$. Then, we can rewrite the above equation as

$$my'' + \lambda y' = mg.$$

Let's analyze another example, population growth (known as the Malthusian growth model).
Denote by $N(t)$ the size of a given population at time t. In an "unconstraint" environment, it is reasonable to assume that the rate of change of the number of individuals is proportional to the number of individuals present. This assumption leads to the following differential equation:

$$\frac{dN}{dt} = rN,$$

where r is called the growth rate (if $r > 0$), and decay rate (if $r < 0$). Assume that $N > 0$. Using the chain rule and assuming that $N(t)$ satisfies $N' = rN$

$$\frac{d}{dt}ln(N(t)) = \frac{dln(N)}{dN} \cdot \frac{dN}{dt} = \frac{1}{N} \cdot N' = r,$$

integrate with respect to t

$$ln(N(t)) = rt + C,$$

where C is the constant of integration. Exponentiating both sides, we obtain

$$N(t) = e^{ln(N(t))} = e^C e^{rt} = ke^{rt},$$

where $\{k > 0 | k \in \mathbb{R}\}$ which could be any positive constant is the initial population size at time $t = 0$.
Assume that an initial population (condition) is given:

$$N(0) = N_0(fixed),$$

we therefore get that $k = N_0$, and the unique solution that satisfies the initial condition is

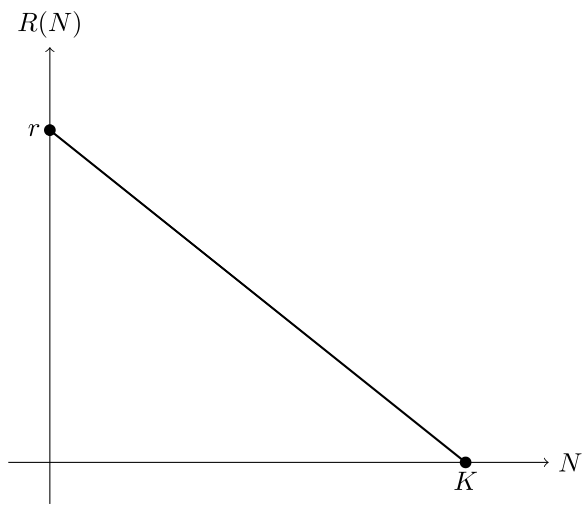
$$N(t) = N_0 e^{rt}.$$

The problem with the answer we got in the previous example is that it is not realistic in the long run, how about we consider a carrying capacity[1]. This leads us to another example: Population growth/decay with the carrying capacity of the environment.
Now assume that our growth rate depends on the population size $N(t)$ itself, therefore we get that

$$\frac{dN}{dt} = R(N)N.$$

Denote by K the number of individual that the environment can carry. K is called the carrying capacity of the environment. If $N < K$, we want growth ($R(N) > 0$) and if $N > K$, we want decay ($R(N) < 0$).

[1] maximum population size that the environment can sustain indefinitely



Let's pick the simplest function $R(N)$ that satisfies $R(0) = r, R(K) = 0$ and is linear. We get that

$$R(N) = r(1 - \frac{N}{K}).$$

Therefore, our differential equation becomes

$$\frac{dN}{dt} = r(1 - \frac{N}{K})N = \frac{r}{K}(K - N(t))N(t).$$

This is called the logistic equation.

Definition 2 (Ordinary Differential Equation)**.** *An ordinary differential equation (ODE) is a differential equation whose unknown function depends on one independent variable only.*

Example of ODEs:

- $y''(t) + y'(t) + 2y(t) = sin(t)$
- $N'(t) = rN(t)$
- $mv'(t) = mg - \lambda v(t)$
- $y'(x) + 3y(x) = e^x$

Definition 3 (Partial Differential Equation)**.** *A partial differential equation (PDE) is a differential equation whose unknown function depends on more than one independent variable.* ***Will not be taught in this course.***

Example of a PDE is the Heat Equation. Let $u = u(x,t)$, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. This PDE denotes the temperature of a body at time t and at position x.

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3.2 Classification

3.2.1 The Order

Definition 4. *The order of an ODE is the order of the highest derivative that appears in the equation.*

Example. $N' = rN$ *(first order ODE)*

Example. $y''(t) + 2y'(t) = e^t$ *(second order ODE)*

Given $n \in \mathbb{N}$, an n^{th} order scalar ODE is written as

$$F(t, y(t), y'(t), y''(t), \ldots, y^{(n)}(t)) = 0,$$

where $F : \mathbb{R}^{n+2} \to \mathbb{R}$ is a map and where $y^{(k)}(t) = \frac{d^k y}{dt^k}(t), k = 1, \ldots, n.$

Systems of first order ODEs

Imagine a map $f : D \times (a, b) \to \mathbb{R}^n$, where $D \subseteq \mathbb{R}^n$ is an open set, and (a,b) is a "time" interval. A general first order system of ODEs is given by $y'(t) = f(y(t), t)$, where $y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$, $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, and $y' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$.

Remark 1. *Assume that a scalar n^{th} order ODE has the form*

$$y^{(n)}(t) = G(t, y(t), y'(t), \ldots, y^{(n-1)}(t)).$$

Let's break this down into simpler terms. Letting $y_1 = y, y_2 = y', \ldots, y_n = y^{(n-1)}$. This leads us to $y_1' = y' = y_2, y_2' = y'' = y_3, \ldots, y_{n-1}' = y^{(n-1)} = y_n, y_n' = y^{(n)} = G(t, y_1, y_2, \ldots, y_n)$. So, we can rewrite the n^{th} order ODE as a first order system of ODEs:

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ G(t, y_1, y_2, \ldots, y_n) \end{pmatrix}$$

Example (Lorenz Equation)**.**

$$y_1' = \sigma(y_2 - y_1)$$

$$y_2' = \rho y_1 - y_2 - y_1 y_3$$

$$y_3' = y_1 y_2 - \beta y_3$$

where σ, ρ, β are parameters. $n = 3$. This is a first order system of ODEs. They are nonlinear because of the products $y_1 y_3$ and $y_1 y_2$.

3.2.2 Linearity

Definition 5 (Linearity)**.** *The n^{th} order ODE $F(t, y, y', \ldots, y^{(n)}) = 0$ is linear if F is a linear polynomial in the variables $y, y', y'', \ldots, y^{(n)}$, that is, it is of the form $a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \ldots + a_{n-1}(t)y'(t) + a_n(t)y(t) = g(t)$, where $a_0, a_1, \ldots, a_n, g$ are given functions of t. Otherwise, it is nonlinear.*

In short terms, an ODE is said to be linear if it can be written as $y'(t) = A(t)y(t) + r(t)$ where, given $t \in (a, b)$, $A(t) \in M_n(\mathbb{R})$ (the set of $n \times n$ real matrices) and $r(t) \in \mathbb{R}^n$.

Example. *Consider the second-order ODE*

$$y'' + 2y' + y = e^t$$

$$\Longrightarrow y'' = -2y' - y + e^t.$$

Define new variables

$$y_1 = y, \qquad y_2 = y'.$$

Then the system becomes

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \qquad r(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

3.2.3 Autonomy

Definition 6 (Autonomy)**.** *The n^{th} order ODE $F(t, y, y', \ldots, y^{(n)}) = 0$ is autonomous if F does not depend explicitly on t, that is, if it is of the form $F(y, y', \ldots, y^{(n)}) = 0$. Otherwise, it is non-autonomous.*

Example. $y'' + 2y' + y - e^t = 0$ *is non-autonomous.*

Example. $N'(t) = rN(t)$ *is autonomous.*

Example. $y'(t) = ty(t)$ *is non-autonomous.*

Equivalently, a first-order system $y' = f(y, t)$ is autonomous if it can be written as

$$y' = f(y).$$

Otherwise, it is non-autonomous.

Note: The Lorenz system is an example of an autonomous system[2].

[2] Here "system" means the unknown y is vector-valued, e.g. $y \in \mathbb{R}^m$, rather than scalar.

3.2.4 Solutions of ODEs

Definition 7 (Solutions of ODEs)**.** *Let $f : D \times (a,b) \to \mathbb{R}^n$. A solution of $y'(t) = f(y(t),t)$ on an interval $J \subset \mathbb{R}$ is a differentiable function $y : J \to D \subset \mathbb{R}^n$, such that $y'(t) = f(y(t),t), \forall t \in J$. t is the independent variable, and $y = (y_1, \dots, y_n)$ is the dependent variable.*

Explicit Solutions

Example. *Consider the ODE*

$$y' + y = 1.$$

We can verify that $y(t) = e^{-t} + 1$, and therefore $y'(t) = -e^{-t}$, is a solution on $\mathbb{R}$. Indeed,

$$y' + y = -e^{-t} + (e^{-t} + 1) = 1.$$

In this example, $y = y(t)$ is explicitly given as a function of t (independent variable).

Implicit solutions

Example. *Consider the ODE*

$$y\frac{dy}{dx} = x.$$

This is a nonautonomous, nonlinear, first-order scalar ODE. Separating variables gives

$$y\,dy = x\,dx.$$

Integrating,

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C,$$

or equivalently, the implicit solution

$$x^2 - y^2 = C, \qquad C \in \mathbb{R}.$$

To verify, differentiate implicitly:

$$\frac{d}{dx}(x^2 - y^2) = 0 \implies 2x - 2y\frac{dy}{dx} = 0 \implies y\frac{dy}{dx} = x.$$

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3.3 Initial Value Problems

A first-order system of ODEs is written as

$$y' = f(y,t), \quad f : D \times (a,b) \to \mathbb{R}^n.$$

Let $t_0 \in (a,b)$. An initial condition is

$$y(t_0) = y_0 \in \mathbb{R}^n.$$

An initial value problem (IVP) is

$$\begin{cases} y' = f(y,t), \\ y(t_0) = y_0. \end{cases}$$

3.4 Existence and Uniqueness Theorem

3.4.1 Lipschitz continuity

Definition 8 (Lipschitz continuity)**.** *Let $D \subseteq \mathbb{R}^n$ and let $||\cdot||$ be a norm on $\mathbb{R}^n$. A function $f : D \to \mathbb{R}^n$ is Lipschitz continuous if there exists $L \geq 0$ such that*

$$||f(y_1) - f(y_2)|| \leq L||y_1 - y_2|| \quad \forall y_1, y_2 \in D.$$

The smallest such L is called the Lipschitz constant and is denoted $Lip(f)$.

Example. *Let $f(y) = 4y - 5$, $D = \mathbb{R}$, and $||\cdot|| = |\cdot|$. Then*

$$|f(y_1) - f(y_2)| = |4y_1 - 4y_2| = 4|y_1 - y_2|.$$

So f is Lipschitz with $Lip(f) = 4$.

Example. *Let*

$$f(y) = \frac{1}{y-1}, \quad D = (1, +\infty).$$

Then f is not Lipschitz on D.

Now fix $\delta > 1$ and define $D_\delta = (\delta, +\infty)$. For $y_1, y_2 \in D_\delta$, by the Mean Value Theorem, there exists $z \in (y_1, y_2)$ such that

$$f(y_2) - f(y_1) = f'(z)(y_2 - y_1), \quad f'(y) = -\frac{1}{(y-1)^2}.$$

So

$$|f(y_2) - f(y_1)| \leq \frac{1}{(z-1)^2}|y_2 - y_1| \leq \frac{1}{(\delta-1)^2}|y_2 - y_1|.$$

Thus f is Lipschitz on D_δ with

$$Lip(f) = \frac{1}{(\delta-1)^2}.$$

3.4.2 Local Lipschitz continuity

Definition 9 (Locally Lipschitz)**.** *Let $D \subseteq \mathbb{R}^n$ be open. A function $f : D \to \mathbb{R}^n$ is locally Lipschitz if for every compact set $K \subset D$, there exists $L(K)$ such that*

$$||f(y_1) - f(y_2)|| \leq L(K)||y_1 - y_2|| \quad \forall y_1, y_2 \in K.$$

Problem 1. *(See tutorial 1) If $f \in C^1(D)$, then f is locally Lipschitz.*

3.4.3 Existence and Uniqueness Theorem

Theorem 1 (Existence and Uniqueness)**.** *Let $D \subseteq \mathbb{R}^n$ be open and let (a, b) be an open interval containing t_0. Consider the IVP*

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}$$

Assume $f : D \times (a, b) \to \mathbb{R}^n$ is continuous and locally Lipschitz in y.

If $y_0 \in D$, then there exists an open interval J containing t_0 on which a solution exists. Moreover, this solution is unique on J.

3.4.4 Integral form of solutions

Lemma 2. *A function y solves the IVP if and only if*

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\, ds.$$

Proof. If $y' = f(y, t)$ and $y(t_0) = y_0$, then by the Fundamental Theorem of Calculus,

$$y(t) - y(t_0) = \int_{t_0}^{t} f(y(s), s)\, ds,$$

so

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\, ds.$$

Conversely, differentiating the right-hand side gives

$$y'(t) = f(y(t), t), \quad y(t_0) = y_0.$$

□

3.4.5 Picard operator

Setup and notation Let $(y_0, t_0) \in D \times (a, b)$. Since this set is open, there exist $\alpha, \delta > 0$ such that

$$D_{\alpha,\delta} = \{(y, t) : ||y - y_0|| \leq \alpha, |t - t_0| \leq \delta\} \subset D \times (a, b).$$

Define

$$M_{\alpha,\delta} = \sup_{(y,t) \in D_{\alpha,\delta}} {}^{3}||f(y, t)|| < +\infty.$$

Let

$$\epsilon = \min\left(\delta, \frac{\alpha}{M_{\alpha,\delta}}\right), \quad J = (t_0 - \epsilon, t_0 + \epsilon).$$

Definition of the Picard operator

Lemma 3 (Picard operator)**.** *For any function y such that $y(t_0) = y_0$ and $(y(t), t) \in D_{\alpha,\delta}$ for all $t \in J$, define*

$$T(y)(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\, ds.$$

Then $T(y)(t_0) = y_0$ and $(T(y)(t), t) \in D_{\alpha,\delta}$ for all $t \in J$.

[3] *sup* means the largest value in a set of numbers.

Proof.

$$T(y)(t_0) = y_0.$$

For $t \in J$,

$$||T(y)(t) - y_0|| \leq \int_{t_0}^{t} ||f(y(s), s)||ds \leq M_{\alpha,\delta}|t - t_0| \leq M_{\alpha,\delta}\epsilon \leq \alpha.$$

So $(T(y)(t), t) \in D_{\alpha,\delta}$. □

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Invariant properties

Lemma 4. *If $y : J \to \mathbb{R}^n$ satisfies*

1. $y(t_0) = y_0$,
2. $(y(t), t) \in D_{\alpha,\delta}$ *for all* $t \in J$,

then $T(y) : J \to \mathbb{R}^n$ satisfies the same properties.

Picard iterations Define $y_0(t) = y_0$ (constant function), which clearly satisfies (1) and (2).

For $k \geq 1$, define

$$y_k(t) = T(y_{k-1})(t) = y_0 + \int_{t_0}^{t} f(y_{k-1}(s), s)ds.$$

Existence The Picard iterations converge uniformly to a function $y : J \to \mathbb{R}^n$ which satisfies (1) and (2), and is a solution of the IVP.

Proof. Pick $t \in [t_0, t_0 + \epsilon]$ (the proof is similar for $t \in [t_0 - \epsilon, t_0]$).

The goal is to show that $\{y_k(t)\}_{k=0}^{\infty}$ is a Cauchy sequence[4] in $\mathbb{R}^n$.

We prove by induction that

$$(**) \quad ||y_m(t) - y_{m-1}(t)|| \leq L^{m-1} M_{\alpha,\delta} \frac{(t - t_0)^m}{m!}, \ \forall m \geq 1.$$

Base case $m = 1$:

$$||y_1(t) - y_0(t)|| = \left\| \int_{t_0}^{t} f(y_0, s)ds \right\| \leq \int_{t_0}^{t} ||f(y_0, s)||ds \leq M_{\alpha,\delta}|t - t_0| \leq \alpha.$$

Induction step:

$$||y_{m+1}(t) - y_m(t)|| \leq \int_{t_0}^{t} ||f(y_m(s), s) - f(y_{m-1}(s), s)||ds.$$

[4] Cauchy sequence is a sequence that has a limit in a metric space $\mathbb{R}^n$.

Since $D_{\alpha,\delta}$ is compact and f is Lipschitz on $D_{\alpha,\delta}$, there exists L such that

$$||f(x,t) - f(y,t)|| \leq L||x - y||.$$

Thus,

$$\leq L \int_{t_0}^{t} ||y_m(s) - y_{m-1}(s)|| ds.$$

Using (**),

$$\leq L^m M_{\alpha,\delta} \frac{1}{(m-1)!} \int_{t_0}^{t} (s - t_0)^{m-1} ds = L^m M_{\alpha,\delta} \frac{(t - t_0)^m}{m!}.$$

Hence (**) holds.
In particular, for all $\rho \geq 1$,

$$||y_\rho(t) - y_{\rho-1}(t)|| \leq M_{\alpha,\delta} \frac{(L(t - t_0))^\rho}{(\rho)!} < \frac{M_{\alpha,\delta}}{L} \frac{(L\epsilon)^\rho}{\rho!}.$$

Let $m, p \geq 1$:

$$||y_{m+p}(t) - y_{m+1}(t)|| \leq \sum_{k=1}^{p-1} ||y_{m+k+1}(t) - y_{m+k}(t)||.$$

So,

$$< \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{m+p} \frac{(L\epsilon)^j}{j!}.$$

Since $e^{L\epsilon} = \sum_{j=0}^{\infty} \frac{(L\epsilon)^j}{j!}$ converges,

$$\to_{m,p \to +\infty} 0.$$

Thus $\{y_k(t)\}$ is Cauchy and converges to $y(t)$.
Taking limits in the iteration,

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s) ds.$$

$\square$

[5]

[5] Evaluated on compacted cylinder, understanding epsilon, the L lipschitz constant coming from somehwere, not on the analysis background such as...

Uniqueness

Proof. Assume $y(t)$ and $z(t)$ solve the IVP. Then

$$||y(t) - z(t)|| \leq \int_{t_0}^{t} ||f(y(s), s) - f(z(s), s)||ds \leq L \int_{t_0}^{t} ||y(s) - z(s)||ds.$$

Define $g(t) = \int_{t_0}^{t} ||y(s) - z(s)||ds$. Then

$$g'(t) \leq Lg(t).$$

So,

$$\frac{d}{dt}(e^{-L(t-t_0)}g(t)) \leq 0.$$

Thus $e^{-L(t-t_0)}g(t)$ is decreasing and

$$0 \leq g(t) \leq g(t_0) = 0.$$

Hence $g(t) = 0$ and $y(t) = z(t)$. □

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Examples on how to use the theorem.

$$\begin{cases} y' = f(y, t) \\ y(t_0) = y_0 \end{cases}$$

Th. ∃&! says there is a solution for this IVP, and it is unique.

$$y : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^n$$

solves the IVP

$$\epsilon = min(\delta, \frac{\alpha}{M_{\alpha,\delta}})$$

$$f : D \times (a, b) \to \mathbb{R}^n$$

Choose α, δ such that $D_{\alpha,\delta} \subset D \times (a, b)$.

$$\{(y, t) : ||y - y_0|| \leq \alpha, |t - t_0| \leq \delta\} \subset D \times (a, b)$$

$$M_{\alpha,\delta} = sup_{(y,t)\in D_{\alpha,\delta}} ||f(y, t)|| < +\infty$$

Demanded to obtain epsilon from the theorem in this ODE course, nothing else, since it is not an analysis course.

Example.

$$\begin{cases} y' = y + 1 \\ y(0) = 1 = y_0 \\ t_0 = 0 \end{cases}$$

$$f(y) = y + 1$$

If f is a C^1 function, then it is locally Lipschitz in y and the theorem applies.

$$f : D \to \mathbb{R}, D = \mathbb{R} = (-\infty, +\infty)$$

Since f is defined everywhere, then there are no constraints on α, δ.

$$D_{\alpha,\delta} = \{(y, t) : ||y - 1|| \leq \alpha, |t - 0| \leq \delta\} \subset \mathbb{R} \times \mathbb{R} | \forall \alpha, \delta > 0$$

$$D_{\alpha,\delta} = [1 - \alpha, 1 + \alpha] \times [-\delta, \delta] \subset \mathbb{R} \times \mathbb{R}$$

$$M_{\alpha,\delta} = sup_{(y,t)\in D_{\alpha,\delta}} ||f(y, t)y + 1|| = sup_{y\in[1-\alpha,1+\alpha]} ||y + 1|| = 2 + \alpha$$

$$\epsilon = min(\delta, \frac{\alpha}{2 + \alpha})$$

Pick $\alpha = 1, \delta = 1$

$$\epsilon = min(1, \frac{1}{3}) = \frac{1}{3}$$

Therefore, there exists a unique solution on the interval $(-\frac{1}{3}, \frac{1}{3})$. Pick $\alpha = 3, \delta = 2$

$$\epsilon = min(2, \frac{3}{5}) = \frac{3}{5}$$

Therefore, there exists a unique solution on the interval $(-\frac{3}{5}, \frac{3}{5})$. Since there are no constraints on α, δ, we can make ϵ as large as we want, therefore the solution exists and is unique on $\mathbb{R}$. The "maximal" times interval guaranteed by the theorem of $\exists$&! is $J = (-1, 1)$. In chapter 2 we will see that the solution is $y(t) = 2e^t - 1$. In fact the maximal time interval on which the solution is defined is $J_max = \mathbb{R}$

Example.

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

Let $y \in C^1 \to LLC$, therefore there exists a solution.

$$M_{\alpha,\delta} = sup_{y\in[1-\alpha,1+\alpha]} ||y^2|| = (1 + \alpha)^2$$

$$\epsilon = min(\delta, \frac{\alpha}{(1 + \alpha)^2})$$

$$h(\alpha) = \frac{\alpha}{(1 + \alpha)^2}$$

$$h'(\alpha) = \frac{1 - \alpha}{(1 + \alpha)^3}$$

$$h'(\alpha) = \frac{1 - \alpha^2}{blablabla} = 0$$

Pick $\alpha = 1$, $\delta = 104073$

$$\epsilon = \frac{1}{4}$$

∃! solution on $y : (-\frac{1}{4}, \frac{1}{4}) \to \mathbb{R}$ to the IVP. Chapter 2 will show that the answer is in the form

$$y(t) = \frac{1}{1-t}$$

draw a cartesian plane and an exponential function and the asymptote is named the finite time blowup.

Example.

$$\begin{cases} y' = t^2 + y^2 = f(y,t) \\ y(0) = 0 \end{cases}$$

$$D_{\alpha,\delta} = [-\alpha, \alpha] \times [-\delta, \delta]$$

$$M_{\alpha,\delta} = sup_{(y,t)\in D_{\alpha,\delta}} |t^2 + y^2| = \delta^2 + \alpha^2$$

$$\epsilon = min(\delta, \frac{\alpha}{\delta^2 + \alpha^2})$$

Pick $\alpha = 10^{12}, \delta = 1$

$$\epsilon = min(1, \frac{10^{12}}{1 + 10^{24}}) = \frac{10^{12}}{1 + 10^{24}} \approx 0$$

Example (LLC matters)**.**

$$\begin{cases} y' = 3y^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$$

$$y(t) = 0, \forall y is\ a\ solution$$

$$f(y) = 3y^{\frac{2}{3}}$$

$$f'(y) = 2y^{-\frac{1}{3}}$$

This function is not locally Lipschitz continuous at $y = 0$. In chapter 2, we will learn how to solve this separable ODE (assuming $y \neq 0$) we will get $y_2(t) = \begin{cases} t^3, t \geq 0 \\ 0, t < 0 \end{cases}$.

$$y_2' = \frac{d}{dt}(t^3) = 3t^2 = 3(y_2(t))^{\frac{2}{3}}$$

The solution is not unique.

4 First-Order Scalar Equation

January 22, 2026

$$y' = f(y,t) \in \mathbb{R} \quad (n = 1)\ One\ equation$$

4.1 First order linear equations

$$a_0(t)y' + a_1(t)y = g(t)$$

where a_0, a_1, g are functions of t. Dividing by a_0 leads to

$$y' + p(t)y = q(t)$$

where $p(t) = \frac{a_1(t)}{a_0(t)}$ and $q(t) = \frac{g(t)}{a_0(t)}$ Multiply the ODE by an integrating factor $\mu(t)$ (to be determined).

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)q(t)$$

Requirement in μ is that $\mu(t)y' + \mu(t)p(t)y = \frac{d}{dt}(\mu(t)y(t))$ (*) holds. In this case, integrate with respect to t.

$$\mu(t)y(t) = \int \mu(t)q(t)dt + C,$$

where $C \in \mathbb{R}$ is a constant of integration. We therefore get

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)q(t)dt + C.$$

Let us find μ so that (*) holds.

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)y' + \mu(t)p(t)y = \mu'(t)y(t) + \mu(t)y',$$

this implies that

$$\mu'(t)y(t) = \mu(t)p(t)y(t).$$

It is sufficient to have that

$$\mu'(t) = \mu(t)p(t).$$

Now we can solve for $p(t)$

$$\frac{d}{dt}ln|\mu(t)| = \frac{\mu'(t)}{\mu(t)} = p(t),$$

now integrate with respect to t

$$ln|\mu(t)| = \int p(t)dt + C,$$

where $C \in \mathbb{R}$ is a constant of integration, and now we exponentiate

$$|\mu(t)| = e^{\int p(t)dt + C_1} = e^{C_1}e^{\int p(t)dt}$$

$$\mu(t) = (\pm e^{C_1})e^{\int p(t)dt}.$$

We can therefore say choose $\mu(t) = e^{\int p(t)dt}$ (integrating factor). This is the general solution of the first order linear equation.

Example.

$$y' - 2y = 3e^t$$
$$p(t) = -2, q(t) = 3e^t$$
$$\mu(t) = e^{\int -2dt} = e^{-2t}$$
$$e^{-2t}(y' - 2y) = e^{-2t}3e^t = 3e^{-t}$$
$$e^{-2t}(y' - 2y) = \frac{d}{dt}(e^{-2t}y(t))$$

integrate with respect to t

$$e^{-2t}y(t) = \int e^{-2t}3e^{-t}dt + C = -3e^{-t} + C$$

Using this explicit solution

$$y(t) = \frac{1}{\mu(t)}[\int \mu(t)q(t)dt + C]$$

we can find the general solution

$$y(t) = e^{2t}(-3e^{-t} + C) = -3e^t + Ce^{2t}$$

If $y(0) = 1$*, then* $y(0) = -3 + C = 1$*, so* $C = 4$*.*

$$y(t) = -3e^t + 4e^{2t}$$

Theorem 5 (Existence & Uniqueness for Linear Equations)**.**

$$\begin{cases} y' + p(t)y = q(t) \\ y(t_0) = y_0 \end{cases}$$

Assume that the functions p *and* q *are continuous on some interval* (a, b)*, where* $(-\infty \leq a < b \leq \infty)$*, and assume that* $t_0 \in (a, b)$*. Then there is a unique solution* $y(t)$ *to the IVP and* $J_{max} = (a, b)$*.*

Example.

$$\begin{cases} y' + \frac{1}{t-1}y = \frac{1}{cos(t)} \\ y(0) = 1 \end{cases}$$

$$J_{max} = (-\frac{\pi}{2}, 1)$$

Example (Falling Object)**.** *Imagine a falling object, subject to gravity and air drag.*

$$v'(t) + \frac{\gamma}{m}v(t) = g$$

This is a first order linear equation. The integrating factor is

$$\mu(t) = e^{\int \frac{\gamma}{m}dt} = e^{\frac{\gamma}{m}t}$$

$$\frac{d}{dt}(e^{\frac{\gamma}{m}t}v(t)) = e^{\frac{\gamma t}{m}}(v' + \frac{\gamma}{m}v) = e^{\frac{\gamma t}{m}g}$$

We get

$$e^{\frac{\gamma t}{m}}v(t) = g\int e^{\frac{\gamma t}{m}}dt + C = \frac{gm}{\gamma}e^{\frac{\gamma t}{m}} + C$$

$$v(t) = \frac{gm}{\gamma} + Ce^{-\frac{\gamma t}{m}}$$

where $C \in \mathbb{R}$.

$$v(0) = \frac{gm}{\gamma} = \frac{gm}{\gamma} + C$$

Which implies

$$C = 0$$

$$y(t) = \frac{gm}{\gamma}$$

Example (Mixing Problems)**.** *Imagine a tank, at $t = 0$, the tank contains 100 litres of brine water, in which y_0 grams of salt is dissolved. You pour in 50g of salt per litre in the tank, and there is a flow in of water at a rate of R litres per second. Water is flowing out of the tank with a rate of R litres per seconds. Denote by $y(t)$ the quantity of salt in grams in the tank at time t in seconds. $\frac{dy}{dt}$ is the rate in - the rate out.*

$$\frac{dy}{dt} = 50[\frac{grams}{litres}] \times R[\frac{litres}{seconds}] - \frac{y(t)}{100}[\frac{g\ of\ salt}{litres}]$$

We get now that

$$y' = 50R - \frac{R}{100}y$$

Rewrite it as

$$y' + \frac{R}{100}y = 50R$$

and

$$y(0) = y_0$$

Now we can solve this IVP.

$$\mu(t) = e^{\frac{Rt}{100}}$$

$$\frac{d}{dt}(e^{\frac{Rt}{100}}y(t)) = e^{\frac{Rt}{100}} \cdot 50R$$

$$e^{\frac{Rt}{100}}y(t) = 50R\frac{100}{R}e^{\frac{Rt}{100}} + C = 5000e^{\frac{Rt}{100}} + C$$

$$y(t) = 5000 + Ce^{-\frac{Rt}{100}}$$

$$y(0) = 5000 + C = y_0$$

$$C = y_0 - 5000$$

$$y(t) = 5000 + (y_0 - 5000)e^{-\frac{Rt}{100}}$$

Draw graph with threshold 5000, and two exponentials approaching the threshold from the top and below.

5 Systems of Linear Equations

6 Second and Higher-Order Scalar Linear Equations

7 Stability, Phase Portraits and Orbits

8 Laplace Transform

9 Power Series Solutions and Numerical Methods

10 Solutions

11 Appendix

12 Useful Links