

# MATH 325: Honours Ordinary Differential Equations

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**Abstract**

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# 1 Introduction

Jean-Philippe Lessard (Burnside 1119). Tutorials every wednesday from 9am to 10am, ENGTR 0070, with Eunpyo Bang. Office hours thursday. No textbooks. 25% assignments (2 written assignments 15%, and 5 webworks 10%). 25% Midterm (February 16 - inclass). 50% Final. Since its honours you will deal with analysis.

## 2 Prerequisite knowledge

### 2.1 Analysis

## 3 Intro, Classification, Theorem of Existence & Uniqueness

### 3.1 Intro

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**Definition 1** (Differential Equation). *A differential equation (DE) is a relation that involves an unknown function and some of its derivatives.*

To better understand what a differential equation is, consider the following example.

Imagine a ball of mass  $m$  falling, subject to gravity and air resistance (drag). Denote by  $v(t)$  the velocity of the ball at time  $t$ , whereas  $t$  is the independent variable, and  $v$  the dependent variable. Let the downward direction be positive. We know the force of gravity is given by  $F_g = mg$ , where  $g$  is the acceleration due to gravity. The drag force is given by  $F_d = -\lambda v$ , where  $\lambda$  is the drag coefficient and is  $\lambda \geq 0$ . According to Newton's second law  $\sum F = ma$ , the net force acting on the ball is equal to its mass times its acceleration

$$m \frac{dv}{dt} = mg - \lambda v.$$

Let  $y(t)$  be the position, meaning  $v(t) = \frac{dy}{dt}$ . Then, we can rewrite the above equation as

$$my'' + \lambda y' = mg.$$

Let's analyze another example, population growth (known as the Malthusian growth model).

Denote by  $N(t)$  the size of a given population at time  $t$ . In an "unconstraint" environment, it is reasonable to assume that the rate of change of the number of individuals is proportional to the number of individuals present. This assumption leads to the following differential equation:

$$\frac{dN}{dt} = rN,$$

where  $r$  is called the growth rate (if  $r > 0$ ), and decay rate (if  $r < 0$ ). Assume that  $N > 0$ . Using the chain rule and assuming that  $N(t)$  satisfies  $N' = rN$

$$\frac{d}{dt} \ln(N(t)) = \frac{d\ln(N)}{dN} \cdot \frac{dN}{dt} = \frac{1}{N} \cdot N' = r,$$

integrate with respect to  $t$

$$\ln(N(t)) = rt + C,$$

where  $C$  is the constant of integration. Exponentiating both sides, we obtain

$$N(t) = e^{\ln(N(t))} = e^C e^{rt} = k e^{rt},$$

where  $\{k > 0 | k \in \mathbb{R}\}$  which could be any positive constant is the initial population size at time  $t = 0$ .

Assume that an initial population (condition) is given:

$$N(0) = N_0 (\text{fixed}),$$

we therefore get that  $k = N_0$ , and the unique solution that satisfies the initial condition is

$$N(t) = N_0 e^{rt}.$$

The problem with the answer we got in the previous example is that it is not realistic in the long run, how about we consider a carrying capacity<sup>1</sup>. This leads us to another example: Population growth/decay with the carrying capacity of the environment.

Now assume that our growth rate depends on the population size  $N(t)$  itself, therefore we get that

$$\frac{dN}{dt} = R(N)N.$$

Denote by  $K$  the number of individual that the environment can carry.  $K$  is called the carrying capacity of the environment. If  $N < K$ , we want growth ( $R(N) > 0$ ) and if  $N > K$ , we want decay ( $R(N) < 0$ ).

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<sup>1</sup>maximum population size that the environment can sustain indefinitely



Let's pick the simplest function  $R(N)$  that satisfies  $R(0) = r, R(K) = 0$  and is linear. We get that

$$R(N) = r\left(1 - \frac{N}{K}\right).$$

Therefore, our differential equation becomes

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N = \frac{r}{K}(K - N(t))N(t).$$

This is called the logistic equation.

**Definition 2** (Ordinary Differential Equation). *An ordinary differential equation (ODE) is a differential equation whose unknown function depends on one independent variable only.*

Example of ODEs:

- $y''(t) + y'(t) + 2y(t) = \sin(t)$
- $N'(t) = rN(t)$
- $mv'(t) = mg - \lambda v(t)$
- $y'(x) + 3y(x) = e^x$

**Definition 3** (Partial Differential Equation). *A partial differential equation (PDE) is a differential equation whose unknown function depends on more than one independent variable. Will not be taught in this course.*

Example of a PDE is the Heat Equation. Let  $u = u(x, t)$ ,  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ . This PDE denotes the temperature of a body at time  $t$  and at position  $x$ .

## 3.2 Classification

### 3.2.1 The Order

**Definition 4.** *The order of an ODE is the order of the highest derivative that appears in the equation.*

**Example.**  $N' = rN$  (first order ODE)

**Example.**  $y''(t) + 2y'(t) = e^t$  (second order ODE)

Given  $n \in \mathbb{N}$ , an  $n^{\text{th}}$  order scalar ODE is written as

$$F(t, y(t), y'(t), y''(t), \dots, y^{(n)}(t)) = 0,$$

where  $F : \mathbb{R}^{n+2} \rightarrow \mathbb{R}$  is a map and where  $y^{(k)}(t) = \frac{d^k y}{dt^k}(t)$ ,  $k = 1, \dots, n$ .

### Systems of first order ODEs

Consider a map  $f : D \times (a, b) \rightarrow \mathbb{R}^n$ , where  $D \subseteq \mathbb{R}^n$  is an open set, and  $(a, b)$  is a "time" interval. A general first order system of ODEs is given by

$$y'(t) = f(y(t), t), \text{ where } y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \text{ and } y' = \begin{pmatrix} y'_1 \\ \vdots \\ y'_n \end{pmatrix}.$$

**Remark:** Assume that a scalar  $n^{\text{th}}$  order ODE has the form

$$y^{(n)}(t) = G(t, y(t), y'(t), \dots, y^{(n-1)}(t)).$$

Letting  $y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$ . This leads us to  $y'_1 = y' = y_2, y'_2 = y'' = y_3, \dots, y'_{n-1} = y^{(n-1)} = y_n, y'_n = y^{(n)} = G(t, y_1, y_2, \dots, y_n)$ . Therefore, we can rewrite the  $n^{\text{th}}$  order ODE as a first order system of ODEs:

$$\begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_{n-1} \\ y'_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ G(t, y_1, y_2, \dots, y_n) \end{pmatrix}$$

**Example** (Lorenz Equation).

$$y'_1 = \sigma(y_2 - y_1)$$

$$y'_2 = \rho y_1 - y_2 - y_1 y_3$$

$$y'_3 = y_1 y_2 - \beta y_3$$

where  $\sigma, \rho, \beta$  are parameters.  $n = 3$ . This is a first order system of ODEs. They are nonlinear because of the products  $y_1 y_3$  and  $y_1 y_2$ .

### 3.2.2 Linearity

**Definition 5** (Linearity). *The  $n^{\text{th}}$  order ODE  $F(t, y, y', \dots, y^{(n)}) = 0$  is linear if  $F$  is a linear polynomial in the variables  $y, y', y'', \dots, y^{(n)}$ , that is, it is of the form  $a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \dots + a_{n-1}(t)y'(t) + a_n(t)y(t) = g(t)$ , where  $a_0, a_1, \dots, a_n, g$  are given functions of  $t$ . Otherwise, it is nonlinear.*

In short terms, an ODE is said to be linear if it can be written as  $y'(t) = A(t)y(t) + r(t)$  where, given  $t \in (a, b)$ ,  $A(t) \in M_n(\mathbb{R})$  (the set of  $n \times n$  real matrices) and  $r(t) \in \mathbb{R}^n$ .

**Example.** Consider the second-order ODE

$$\begin{aligned} y'' + 2y' + y &= e^t \\ \implies y'' &= -2y' - y + e^t. \end{aligned}$$

Define new variables

$$y_1 = y, \quad y_2 = y'.$$

Then the system becomes

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad r(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

### 3.2.3 Autonomy

**Definition 6** (Autonomy). *The  $n^{\text{th}}$  order ODE  $F(t, y, y', \dots, y^{(n)}) = 0$  is autonomous if  $F$  does not depend explicitly on  $t$ , that is, if it is of the form  $F(y, y', \dots, y^{(n)}) = 0$ . Otherwise, it is non-autonomous.*

**Example.**  $y'' + 2y' + y - e^t = 0$  is non-autonomous.

**Example.**  $N'(t) = rN(t)$  is autonomous.

**Example.**  $y'(t) = ty(t)$  is non-autonomous.

Equivalently, a first-order system  $y' = f(y, t)$  is autonomous if it can be written as

$$y' = f(y).$$

Otherwise, it is non-autonomous.

**Note:** The Lorenz system is an example of an autonomous system<sup>2</sup>.

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<sup>2</sup>Here “system” means the unknown  $y$  is vector-valued, e.g.  $y \in \mathbb{R}^m$ , rather than scalar.

### 3.2.4 Solutions of ODEs

**Definition 7** (Solutions of ODEs). Let  $f : D \times (a, b) \rightarrow \mathbb{R}^n$ . A solution of  $y'(t) = f(y(t), t)$  on an interval  $J \subset \mathbb{R}$  is a differentiable function  $y : J \rightarrow D \subset \mathbb{R}^n$ , such that  $y'(t) = f(y(t), t)$ ,  $\forall t \in J$ .  $t$  is the independent variable, and  $y = (y_1, \dots, y_n)$  is the dependent variable.

#### Explicit Solutions

**Example.** Consider the ODE

$$y' + y = 1.$$

We can verify that  $y(t) = e^{-t} + 1$ , and therefore  $y'(t) = -e^{-t}$ , is a solution on  $\mathbb{R}$ . Indeed,

$$y' + y = -e^{-t} + (e^{-t} + 1) = 1.$$

In this example,  $y = y(t)$  is explicitly given as a function of  $t$  (independent variable).

#### Implicit solutions

**Example.** Consider the ODE

$$y \frac{dy}{dx} = x.$$

This is a nonautonomous, nonlinear, first-order scalar ODE. Separating variables gives

$$y dy = x dx.$$

Integrating,

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C,$$

or equivalently, the implicit solution

$$x^2 - y^2 = C, \quad C \in \mathbb{R}.$$

To verify, differentiate implicitly:

$$\frac{d}{dx}(x^2 - y^2) = 0 \implies 2x - 2y \frac{dy}{dx} = 0 \implies y \frac{dy}{dx} = x.$$

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### 3.3 Initial Value Problems

A first-order system of ODEs is written as

$$y' = f(y, t), \quad f : D \times (a, b) \rightarrow \mathbb{R}^n.$$

Let  $t_0 \in (a, b)$ . An initial condition is

$$y(t_0) = y_0 \in \mathbb{R}^n.$$

An initial value problem (IVP) is

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}$$

## 3.4 Existence and Uniqueness Theorem

### 3.4.1 Lipschitz continuity

**Definition 8** (Lipschitz continuity). Let  $D \subseteq \mathbb{R}^n$  and let  $\|\cdot\|$  be a norm on  $\mathbb{R}^n$ . A function  $f : D \rightarrow \mathbb{R}^n$  is Lipschitz continuous if there exists  $L \geq 0$  such that

$$\|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\| \quad \forall y_1, y_2 \in D.$$

The smallest such  $L$  is called the Lipschitz constant and is denoted  $\text{Lip}(f)$ .

**Example.** Let  $f(y) = 4y - 5$ ,  $D = \mathbb{R}$ , and  $\|\cdot\| = |\cdot|$ . Then

$$|f(y_1) - f(y_2)| = |4y_1 - 4y_2| = 4|y_1 - y_2|.$$

So  $f$  is Lipschitz with  $\text{Lip}(f) = 4$ .

**Example.** Let

$$f(y) = \frac{1}{y-1}, \quad D = (1, +\infty).$$

Then  $f$  is not Lipschitz on  $D$ .

Now fix  $\delta > 1$  and define  $D_\delta = (\delta, +\infty)$ . For  $y_1, y_2 \in D_\delta$ , by the Mean Value Theorem, there exists  $z \in (y_1, y_2)$  such that

$$f(y_2) - f(y_1) = f'(z)(y_2 - y_1), \quad f'(y) = -\frac{1}{(y-1)^2}.$$

So

$$|f(y_2) - f(y_1)| \leq \frac{1}{(\delta-1)^2}|y_2 - y_1| \leq \frac{1}{(\delta-1)^2}|y_2 - y_1|.$$

Thus  $f$  is Lipschitz on  $D_\delta$  with

$$\text{Lip}(f) = \frac{1}{(\delta-1)^2}.$$

### 3.4.2 Local Lipschitz continuity

**Definition 9** (Locally Lipschitz). Let  $D \subseteq \mathbb{R}^n$  be open. A function  $f : D \rightarrow \mathbb{R}^n$  is locally Lipschitz if for every compact set  $K \subset D$ , there exists  $L(K)$  such that

$$\|f(y_1) - f(y_2)\| \leq L(K)\|y_1 - y_2\| \quad \forall y_1, y_2 \in K.$$

**Problem 1.** (See tutorial 1) If  $f \in C^1(D)$ , then  $f$  is locally Lipschitz.

### 3.4.3 Existence and Uniqueness Theorem

**Theorem 1** (Existence and Uniqueness). Let  $D \subseteq \mathbb{R}^n$  be open and let  $(a, b)$  be an open interval containing  $t_0$ . Consider the IVP

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}$$

Assume  $f : D \times (a, b) \rightarrow \mathbb{R}^n$  is continuous and locally Lipschitz in  $y$ .

If  $y_0 \in D$ , then there exists an open interval  $J$  containing  $t_0$  on which a solution exists. Moreover, this solution is unique on  $J$ .

### 3.4.4 Integral form of solutions

**Lemma 2.** *A function  $y$  solves the IVP if and only if*

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

*Proof.* If  $y' = f(y, t)$  and  $y(t_0) = y_0$ , then by the Fundamental Theorem of Calculus,

$$y(t) - y(t_0) = \int_{t_0}^t f(y(s), s) ds,$$

so

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

Conversely, differentiating the right-hand side gives

$$y'(t) = f(y(t), t), \quad y(t_0) = y_0.$$

□

### 3.4.5 Picard operator

**Setup and notation** Let  $(y_0, t_0) \in D \times (a, b)$ . Since this set is open, there exist  $\alpha, \delta > 0$  such that

$$D_{\alpha, \delta} = \{(y, t) : \|y - y_0\| \leq \alpha, |t - t_0| \leq \delta\} \subset D \times (a, b).$$

Define

$$M_{\alpha, \delta} = \sup_{(y, t) \in D_{\alpha, \delta}} {}^3||f(y, t)|| < +\infty.$$

Let

$$\epsilon = \min \left( \delta, \frac{\alpha}{M_{\alpha, \delta}} \right), \quad J = (t_0 - \epsilon, t_0 + \epsilon).$$

#### Definition of the Picard operator

**Lemma 3** (Picard operator). *For any function  $y$  such that  $y(t_0) = y_0$  and  $(y(t), t) \in D_{\alpha, \delta}$  for all  $t \in J$ , define*

$$T(y)(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

*Then  $T(y)(t_0) = y_0$  and  $(T(y)(t), t) \in D_{\alpha, \delta}$  for all  $t \in J$ .*

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<sup>3</sup>*sup* means the largest value in a set of numbers.

*Proof.*

$$T(y)(t_0) = y_0.$$

For  $t \in J$ ,

$$\|T(y)(t) - y_0\| \leq \int_{t_0}^t \|f(y(s), s)\| ds \leq M_{\alpha, \delta} |t - t_0| \leq M_{\alpha, \delta} \epsilon \leq \alpha.$$

So  $(T(y)(t), t) \in D_{\alpha, \delta}$ . □

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### Invariant properties

**Lemma 4.** *If  $y : J \rightarrow \mathbb{R}^n$  satisfies*

1.  $y(t_0) = y_0$ ,
2.  $(y(t), t) \in D_{\alpha, \delta}$  for all  $t \in J$ ,

*then  $T(y) : J \rightarrow \mathbb{R}^n$  satisfies the same properties.*

**Picard iterations** Define  $y_0(t) = y_0$  (constant function), which clearly satisfies (1) and (2).

For  $k \geq 1$ , define

$$y_k(t) = T(y_{k-1})(t) = y_0 + \int_{t_0}^t f(y_{k-1}(s), s) ds.$$

**Existence** The Picard iterations converge uniformly to a function  $y : J \rightarrow \mathbb{R}^n$  which satisfies (1) and (2), and is a solution of the IVP.

*Proof.* Pick  $t \in [t_0, t_0 + \epsilon]$  (the proof is similar for  $t \in [t_0 - \epsilon, t_0]$ ).

The goal is to show that  $\{y_k(t)\}_{k=0}^{\infty}$  is a Cauchy sequence<sup>4</sup> in  $\mathbb{R}^n$ .

We prove by induction that

$$(**) \quad \|y_m(t) - y_{m-1}(t)\| \leq L^{m-1} M_{\alpha, \delta} \frac{(t - t_0)^m}{m!}, \quad \forall m \geq 1.$$

**Base case**  $m = 1$ :

$$\|y_1(t) - y_0(t)\| = \left\| \int_{t_0}^t f(y_0, s) ds \right\| \leq \int_{t_0}^t \|f(y_0, s)\| ds \leq M_{\alpha, \delta} |t - t_0| \leq \alpha.$$

**Induction step:**

$$\|y_{m+1}(t) - y_m(t)\| \leq \int_{t_0}^t \|f(y_m(s), s) - f(y_{m-1}(s), s)\| ds.$$

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<sup>4</sup>Cauchy sequence is a sequence that has a limit in a metric space  $\mathbb{R}^n$ .

Since  $D_{\alpha,\delta}$  is compact and  $f$  is Lipschitz on  $D_{\alpha,\delta}$ , there exists  $L$  such that

$$||f(x,t) - f(y,t)|| \leq L||x - y||.$$

Thus,

$$\leq L \int_{t_0}^t ||y_m(s) - y_{m-1}(s)|| ds.$$

Using (\*\*),

$$\leq L^m M_{\alpha,\delta} \frac{1}{(m-1)!} \int_{t_0}^t (s-t_0)^{m-1} ds = L^m M_{\alpha,\delta} \frac{(t-t_0)^m}{m!}.$$

Hence (\*\*) holds.

In particular, for all  $\rho \geq 1$ ,

$$||y_\rho(t) - y_{\rho-1}(t)|| \leq M_{\alpha,\delta} \frac{(L(t-t_0))^\rho}{(\rho)!} < \frac{M_{\alpha,\delta}}{L} \frac{(L\epsilon)^\rho}{\rho!}.$$

Let  $m, p \geq 1$ :

$$||y_{m+p}(t) - y_{m+1}(t)|| \leq \sum_{k=1}^{p-1} ||y_{m+k+1}(t) - y_{m+k}(t)||.$$

So,

$$< \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{m+p} \frac{(L\epsilon)^j}{j!}.$$

Since  $e^{L\epsilon} = \sum_{j=0}^{\infty} \frac{(L\epsilon)^j}{j!}$  converges,

$$\rightarrow_{m,p \rightarrow +\infty} 0.$$

Thus  $\{y_k(t)\}$  is Cauchy and converges to  $y(t)$ .

Taking limits in the iteration,

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

□

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<sup>5</sup>Evaluated on compacted cylinder, understanding epsilon, the L lipschitz constant coming from somewhere, not on the analysis background such as...

## Uniqueness

*Proof.* Assume  $y(t)$  and  $z(t)$  solve the IVP. Then

$$\|y(t) - z(t)\| \leq \int_{t_0}^t \|f(y(s), s) - f(z(s), s)\| ds \leq L \int_{t_0}^t \|y(s) - z(s)\| ds.$$

Define  $g(t) = \int_{t_0}^t \|y(s) - z(s)\| ds$ . Then

$$g'(t) \leq Lg(t).$$

So,

$$\frac{d}{dt}(e^{-L(t-t_0)} g(t)) \leq 0.$$

Thus  $e^{-L(t-t_0)} g(t)$  is decreasing and

$$0 \leq g(t) \leq g(t_0) = 0.$$

Hence  $g(t) = 0$  and  $y(t) = z(t)$ .  $\square$

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**9 Solutions**

**10 Appendix**

**11 Useful Links**