

MATH 223: Linear Algebra

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Abstract

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1 Introduction

2 Prerequisite knowledge

2.1 Notation

2.1.1 Sets

Sets are a grouping of objects.

Set	Meaning	Examples
\mathbb{N}	The set of natural numbers	$(0, 1, 2, 3, \dots)$
\mathbb{Z}	The set of integers	$(\dots, -3, -2, -1, 0, 1, 2, 3, \dots)$
\mathbb{Q}	The set of rational numbers	$\mathbb{Q} = \frac{a}{b} \mid \forall a, b \in \mathbb{Z} \text{ and } b \neq 0$
\mathbb{R}	The set of all rational and all irrational numbers	$(\dots, -1, 0, \frac{1}{4}, 1, 1000, \dots)$
\mathbb{C}	The set of all complex numbers	$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } i \subseteq \sqrt{-1}\}.$

We have the following relationships between sets:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

2.1.2 Symbols

Symbol	Meaning
\subseteq	is a subset of or equal to
\subset	is a strict subset of
\in	is an element of
\forall	for all
\exists	there exists
\emptyset	empty set
\Rightarrow	implies
\Leftrightarrow	if and only if

2.2 Complex Algebra

2.2.1 Complex Numbers

A complex number is of the form: $z = x + iy$ where $x, y \in \mathbb{R}$ and i is the imaginary unit such that $i^2 + 1 = 0$.

Definition 1 (Powers of i).

k	0	1	2	3	4	5
i^k	1	i	-1	$-i$	1	i

Theorem 1 (Fundamental Theorem of Algebra). *Any complex polynomial¹ f (except constant functions) has a root in \mathbb{C} .*

¹Polynomial is a function such as: $f = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where $a_i \in \mathbb{R}$ or \mathbb{C} and $n \in \mathbb{N}$.

Remark 1. If we have a polynomial f of degree n , then it has n roots, where each root can have a multiplicity².

Example. If we have a polynomial $(x - 1)^2$, it has a degree of 2 but only one root, which is 1, with a multiplicity of 2. This means that the root 1 appears twice in the polynomial.

We can factorize a polynomial in the form of $f = a_n x^n + \dots + a_1 x + a_0$ into a linear factor: $f = a(x - z_1)(x - z_2)\dots(x - z_n)$ where z_i are the roots of f in \mathbb{C} .

Using the FTA for a function such as $f = a_n x^n + \dots + a_1 x + a_0$, we can say that the FTA implies that f has a root $f(z) = 0$, because $f(z) = a(z - z) = 0$.

2.2.2 Complex Operations

We can define operations on complex numbers as follows:

- Addition: $z + z' = (x + x') + i(y + y')$, where $x, x', y, y' \in \mathbb{R}$.
- Multiplication: $zz' = (x + iy)(x' + iy') = (xx' - yy') + i(xy' + yx')$.
- Inverse: $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$

From the definition of inverse, we can see that for any complex number z , its inverse $\frac{1}{z}$ is also a complex number. For example, take $z = 1 + i$, where $x = y = 1$, from the definition of inverse, we can conclude that:

$$\frac{1}{1+i} \in \mathbb{C}$$

Multiplying by a complex number z corresponds geometrically to

$$\begin{cases} \text{a rotation by some angle } \theta, \\ \text{a rescaling by the factor } |z|. \end{cases}$$

2.2.3 Complex Conjugate

A complex conjugate is a way to "flip" the imaginary part of a complex number. For example, if we have a complex number $z = x + iy$, then the complex conjugate of z is $\bar{z} = x - iy$. Some basic properties of complex conjugates are:

- $\bar{\bar{z}} = z$
- $\overline{z + z'} = \bar{z} + \bar{z'}$
- $\overline{z \cdot z'} = \bar{z} \cdot \bar{z'}$

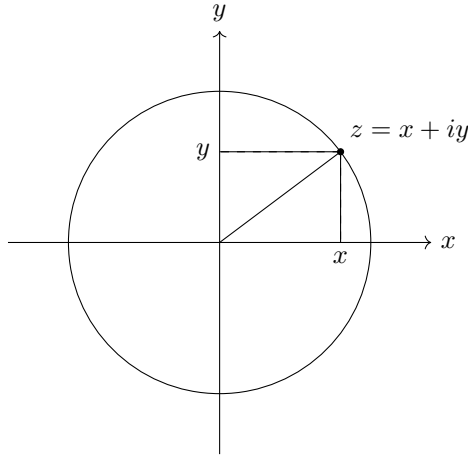
2.2.4 Geometric and Polar Form of Complex Numbers

1. Geometric Interpretation

Definition 2 (Geometric interpretation). *Every complex number $z = x + iy$ can be identified with a point (x, y) in the plane, called the complex plane.*

We define the complex plane as:

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.$$



2. Modulus and Unit Circle

Definition 3 (Modulus). *The modulus of a complex number z is defined by*

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

Geometrically, $|z|$ is the distance from the origin to the point (x, y) .

We can rewrite the definition of the unit circle as follows:

$$S' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{z \in \mathbb{C} : |z| = 1\},$$

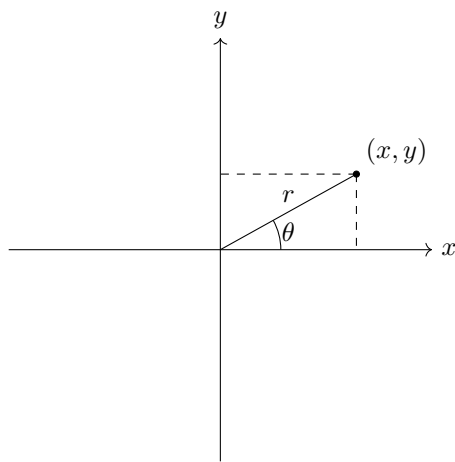
where S' is the unit circle in the complex plane.

3. Polar Coordinates

Definition 4 (Polar coordinates). *Instead of describing a point by (x, y) , we may describe it using polar coordinates (r, θ) , where $r = |z|$ is the distance to the origin and θ is the angle with the positive x -axis.*

Example. *Consider the point (x, y) , where $x = r \cos(\theta)$ and $y = r \sin(\theta)$. We can define (r, θ) as follows:*

²The multiplicity of a root represents how many times the root occurs in the polynomial.



Complex numbers can also be described using polar coordinates.

Definition 5 (Polar and exponential form).

$$z = x + iy = r \cos(\theta) + ir \sin(\theta) = r(\cos(\theta) + i \sin(\theta)) = re^{i\theta}.$$

We can also define multiplication in polar form:

Definition 6 (Multiplication in polar form).

$$z = re^{i\theta}, \quad z' = r'e^{i\theta'}, \quad zz' = rr'e^{i(\theta+\theta')}.$$

Example.

$$(1 + i)^{32} = (\sqrt{2}e^{i\pi/4})^{32} = (\sqrt{2})^{32}e^{i8\pi} = 2^{16}(\cos 8\pi + i \sin 8\pi) = 2^{16}.$$

Around the 1740, the mathematician Euler discovered a formula for complex numbers. The formula is known as Euler's formula.

Definition 7 (Euler's formula).

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

This formula is quite useful when dealing with complex numbers, as it allows us to write complex numbers in a more compact form.

4. Roots in \mathbb{C}

Definition 8 (n^{th} roots in \mathbb{C}). *For any complex number z , an n^{th} root of z is a complex number w such that*

$$w^n = z.$$

Example. If $z = re^{i\theta}$, then any solution of $w^n = z$ must satisfy

$$w_k = r^{1/n} e^{i(\theta + 2\pi k)/n}, \quad k = 0, 1, 2, \dots, n-1,$$

where the n^{th} roots of z are equally spaced on a circle of radius $r^{1/n}$ centered at the origin.

Definition 9 (Roots of unity). The n^{th} roots of unity are the solutions of a special case where $z = 1$.

Example. If $z = 1$, then any solution of $w^n = z$ must satisfy

$$w_k = e^{i2\pi k/n}, \quad k = 0, 1, 2, \dots, n-1.$$

Geometrically, they lie on the unit circle and are equally spaced.

3 Basic Algebraic structures

3.1 Sets with Multiplication

Definition 10 (Set with multiplication). A set M is called a set with multiplication if you can multiply any two elements of M , and the result is still in M . In other words, for any $a, b \in M$, the product ab is also in M .

Example. Let $M = \mathbb{R}$. If $a, b \in \mathbb{R}$, then $ab \in \mathbb{R}$. So the real numbers \mathbb{R} form a set with multiplication.

Example. An example of a set with multiplication is the set of all 2×2 complex matrices: $M = M_2(\mathbb{C})$. Another example is the nonzero set of all real numbers \mathbb{R} with ordinary multiplication: $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

3.2 Invertibility

Definition 11 (Condition for Invertibility). Let $A \in M$ be an $n \times n$ matrix, and suppose that there exists an $n \times n$ matrix B such that $AB = I_n$ or $BA = I_n$.

Where I_n is the $n \times n$ identity matrix³ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then A is invertible, and B is called the inverse of A and is denoted by $B = A^{-1}$.

Remark 2. If A is invertible, then A^{-1} exists and is unique⁴.

To determine if an element A in a set with multiplication M is invertible, we can use the following examples:

³An identity matrix is a square matrix with 1s on its main diagonal and 0s everywhere else. It represents no change in linear transformations, and it's used in finding matrix inverses.

⁴Unique means there is exactly one such element.

Example. Let $M = \mathbb{Z} = \{\dots - 2, -1, 0, 1, 2, \dots\}$ and $A = 2$. Is A invertible in M ?

Solution: No, because $\frac{1}{2} \notin \mathbb{Z}$.

Example. Let $M = \mathbb{R}$ and $A = 2$, is A invertible in M ?

Solution: Yes, because $\frac{1}{2} \in \mathbb{R}$.

Example. Is $1 + i$ invertible in \mathbb{C} ?

Solution: Yes, using our previous definition of inverse (2.2.2), we get that

$$\frac{1}{1+i} = \frac{1-i}{2} \in \mathbb{C}.$$

Problem 1 (Invertibility). Show that if an inverse of A in \mathbb{M} exists, then it is unique.

Problem 2 (Invertibility 2). Let K be a field. Prove that this matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(K)$ is not invertible.

3.3 Ring

Definition 12. A **ring** is a set R where you can **add** and **multiply** elements, and the following are true:

1. You can add any two elements and stay in R . There is a zero, every element has a negative, and addition is commutative and associative.
2. You can multiply any two elements and stay in R . Multiplication is associative, and there is a 1.
3. Multiplication distributes over addition:

$$a(b+c) = ab+ac \quad \text{and} \quad (a+b)c = ac+bc.$$

The main example of a ring is the set of integers \mathbb{Z} .

3.4 Field

Definition 13. A **field** is a set of numbers which can be added, subtracted, multiplied, and divided (except for division by zero) in a way that satisfies certain rules. These rules are:

1. You can add any two elements and stay in the field. There is a zero, every element has a negative, and addition is commutative⁵ and associative⁶.

⁵Property which focuses on changing order of addition, i.e $a+b=b+a$.

⁶Property which focuses on changing grouping of addition, i.e $a+(b+c)=(a+b)+c$.

2. You can multiply any two elements and stay in the field. Multiplication is associative, and there is a ⁷1.
3. Multiplication distributes over addition:

$$a(b + c) = ab + ac \quad \text{and} \quad (a + b)c = ac + bc.$$

Examples of fields include the set of real numbers \mathbb{R} and the set of complex numbers \mathbb{C} .

Problem 3 (Field). Construct a field with 2 elements.

4 Vector Spaces

4.1 Cartesian vector spaces

Definition 14 (\mathbb{R}^n). Let $n \in \mathbb{N}$. The Cartesian product of n copies of \mathbb{R} is called \mathbb{R}^n .

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}$$

4.2 Vectors

4.2.1 Vector operations

Vector operations are defined as follows:

Addition:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}.$$

Scalar multiplication:

$$\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix},$$

where $\lambda \in \mathbb{R}$.

Definition 15 (Linear combination). A linear combination of vectors v_1, \dots, v_n is a vector v of the form $v = \lambda_1 v_1 + \dots + \lambda_n v_n$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

⁷Like literally the number 1.

Example. A linear combination could look like this:

$$\xi((v_1 + 2v_2) + v_3) + v_4,$$

where $\xi \in \mathbb{R}$.

4.2.2 Span

Definition 16 (Span). Let $A \subset \mathbb{R}^n$. The span of A , denoted $\text{Span}(A)$, is the set of all linear combinations of elements of A . In particular, if $A = \{v_1, \dots, v_n\}$, then

$$\text{Span}(v_1, \dots, v_n) = \{\lambda_1 v_1 + \dots + \lambda_n v_n : \lambda_i \in \mathbb{R}\}.$$

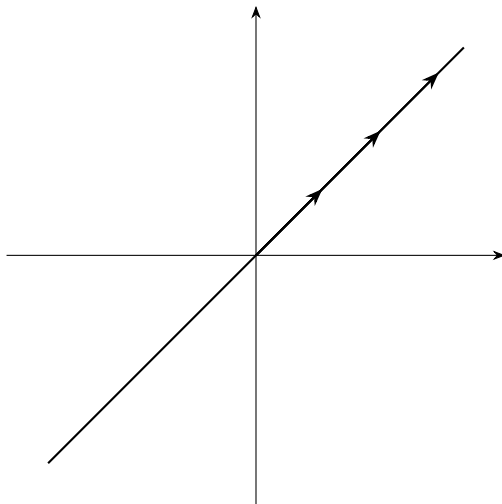
When working in \mathbb{R}^n , the span describes all points you can reach by scaling and adding the given vectors. Depending on the vectors, the span can be a line (if the vectors are dependent), a plane, or a higher-dimensional subspace. The following examples show what spans look like in \mathbb{R}^2 .

Remark 3. If $A = \{v\}$ contains one nonzero vector, then $\text{Span}(v)$ is a line through the origin.

Example. Let $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then

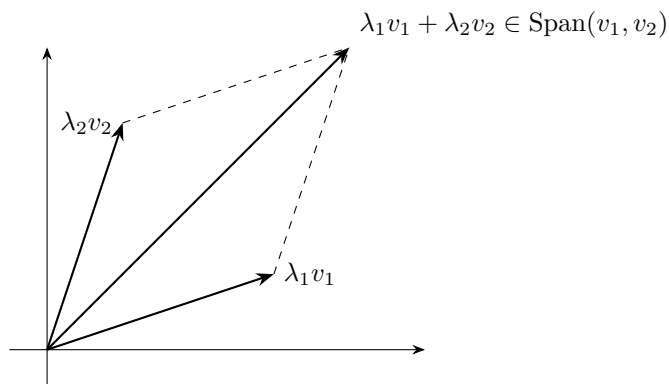
$$\text{Span}(A) = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

which is a line in \mathbb{R}^2 .



Problem 4 (Span). Let $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Find $\text{Span}(v_1, v_2)$.

Span is a generalization of lines in \mathbb{R}^2 . For example, let $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Any vector in $\text{Span}(v_1, v_2)$ has the form $\lambda_1 v_1 + \lambda_2 v_2$. Geometrically, this can be illustrated as the sum of two scaled vectors.

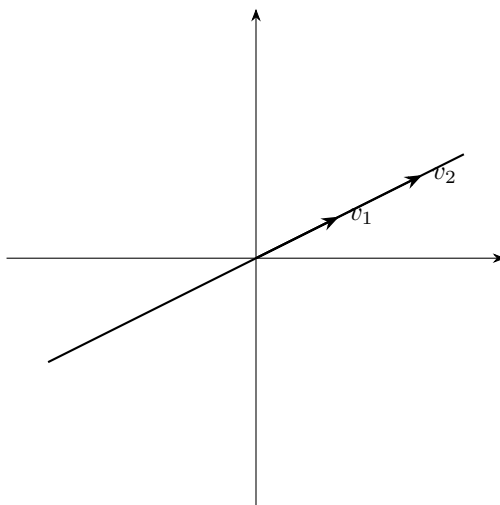


Furthermore, if v_1, v_2 are linearly dependent, then $\text{Span}(v_1, v_2)$ is a line in \mathbb{R}^2 . For example, let $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. Then $v_2 = 2v_1$, so $\text{Span}(v_1) = \text{Span}(v_2)$. Geometrically, this is a straight line through the origin in the direction of v_1 (and v_2). Let

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Then $v_2 = 2v_1$, so

$$\text{Span}(v_1) = \text{Span}(v_2).$$



Span in \mathbb{C}^n The definition of span is the same in complex vector spaces, except the scalars are complex numbers.

Definition 17. *The span over \mathbb{C} of vectors $v_1, \dots, v_n \in \mathbb{C}^n$ is*

$$\text{Span}_{\mathbb{C}}(v_1, \dots, v_n) = \{\lambda_1 v_1 + \dots + \lambda_n v_n : \lambda_i \in \mathbb{C}\}.$$

Thus, $\text{Span}_{\mathbb{C}}(v_1, \dots, v_n)$ consists of all vectors obtained by complex linear combinations of v_1, \dots, v_n .

4.2.3 Standard Basis

Definition 18 (Standard basis of \mathbb{R}^n). *The standard basis of \mathbb{R}^n is the set $\{e_1, \dots, e_n\}$, where*

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Every vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ can be written uniquely as

$$x = x_1 e_1 + \dots + x_n e_n.$$

We now verify that the standard basis really is a basis, by checking that it spans \mathbb{R}^n and is linearly independent.

Claim 1. The vectors e_1, \dots, e_n span \mathbb{R}^n .

Proof. We show that any vector in \mathbb{R}^n can be written as a linear combination of e_1, \dots, e_n :

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n.$$

□

Claim 2. The vectors e_1, \dots, e_n are linearly independent.

Proof. Suppose

$$\lambda_1 e_1 + \dots + \lambda_n e_n = 0.$$

Then

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

so $\lambda_1 = \cdots = \lambda_n = 0$. Therefore e_1, \dots, e_n are linearly independent. \square

The standard basis also helps clarify the difference between real and complex vector spaces. In particular, the same vectors can generate very different spans depending on whether the scalars are real or complex.

Example (Standard basis and real vs complex span). *Consider the standard basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in \mathbb{C}^2 .*

- *Over \mathbb{C} , any vector in \mathbb{C}^2 can be written as*

$$z_1 e_1 + z_2 e_2, \quad z_1, z_2 \in \mathbb{C}.$$

So the complex span is

$$\text{Span}_{\mathbb{C}}(e_1, e_2) = \mathbb{C}^2, \quad \dim_{\mathbb{C}} \mathbb{C}^2 = 2.$$

- *Each complex scalar can be written as $z_k = x_k + iy_k$, $x_k, y_k \in \mathbb{R}$. Then*

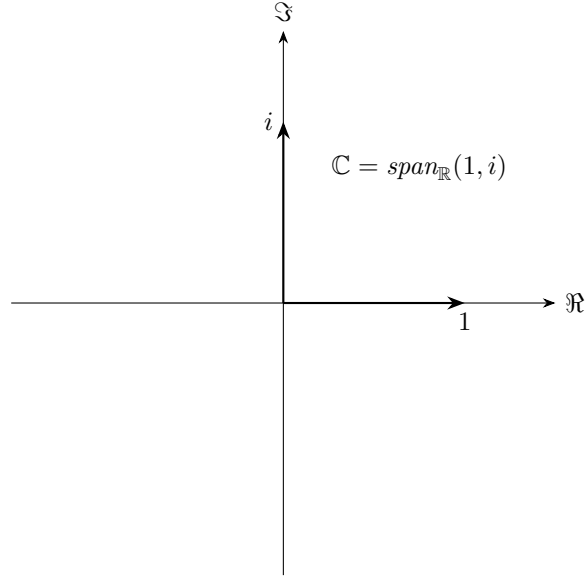
$$z_1 e_1 + z_2 e_2 = x_1 e_1 + y_1 (ie_1) + x_2 e_2 + y_2 (ie_2),$$

showing that every complex linear combination is also a real linear combination of the four vectors

$$e_1, e_2, ie_1, ie_2.$$

Hence, as a real vector space, \mathbb{C}^2 has dimension 4:

$$\dim_{\mathbb{R}} \mathbb{C}^2 = 4.$$



4.2.4 Abstract Vector Spaces

An abstract vector space is just a set with two operations: vector addition ($v_1 + v_2$) and scalar multiplication (λv), where λ comes from a field k , satisfying certain axioms.

Definition 19. *Let k be a field. A **vector space over k** is a set V together with two operations*

$$\begin{cases} v_1 + v_2 \in V, & v_1, v_2 \in V, \\ \lambda v \in V, & v \in V, \lambda \in k, \end{cases}$$

satisfying 8 axioms.

Axioms (1) Rules for addition

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

There exists a zero vector $0 \in V$ such that $0 + v = v$. For every $v \in V$, there exists $-v \in V$.

$$u + v = v + u$$

(2) Rules for scalar multiplication

$$1 \cdot v = v$$

$$\lambda_1(\lambda_2 v) = (\lambda_1 \lambda_2) v$$

(3) Compatibility (distributive laws)

$$(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$$

$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$$

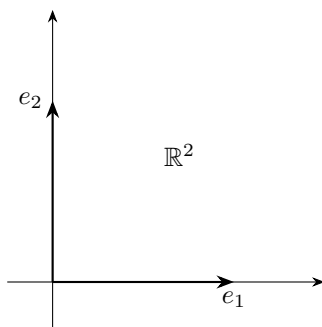
Example. Let K be a field. Then

$$K^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in K \right\}$$

is a vector space over K .

Remark 4.

$$\text{Span}_{\mathbb{Z}}(e_1, e_2) = \{\lambda_1 e_1 + \lambda_2 e_2 : \lambda_i \in \mathbb{Z}\}.$$



4.2.5 Examples of Vector Spaces

We now list important examples of vector spaces.

1. Coordinate spaces Let k be a field ($k = \mathbb{R}$ or $k = \mathbb{C}$).

$$k^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in k \right\}$$

is a vector space over k .

The standard basis is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

2. Polynomial spaces A polynomial is an expression

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad a_i \in k.$$

$$P_n(k) = \{a_n x^n + \dots + a_1 x + a_0 : a_i \in k\}$$

is a vector space over k .

Examples:

$$1 + x^2 \in P_2(\mathbb{R}), \quad 1 + ix^3 \in P_3(\mathbb{C}).$$

The subscript must satisfy $\deg(f) \leq n$.

All polynomials:

$$P_\infty = \{a_n x^n + \cdots + a_1 x + a_0 : a_i \in k, n \geq 0\}.$$

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_\infty.$$

A standard basis is $\{1, x, x^2, \dots, x^n\}$. Every $f \in P_n$ can be written

$$f = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_n x^n.$$

3. Matrix spaces

$$M_n(k) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} : a_{ij} \in k \right\}$$

is a vector space over k .

Standard basis matrices:

$$e_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \dots$$

$$\dim M_n(k) = n^2.$$

Example in $M_2(\mathbb{R})$:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1e_{11} + 2e_{12} + 3e_{21} + 4e_{22}.$$

4. Function spaces

Let D be a set and k a field.

$$F(D, k) = \{f : D \rightarrow k\}$$

is a vector space over k .

When $D = \mathbb{R}$ and $k = \mathbb{R}$, functions are drawn with horizontal axis D and vertical axis k .

$F(D)$ is a function from D to K , where $K = \mathbb{R}$ or \mathbb{C} . No "standard basis" in general.

Standard basis if D is a finite set eg $D = \{1, \dots, n\}$.

$$(f \in F(D)). f(1) = 2, f(2) = 4, \dots, f(n) = 2^n$$

Kronecker function: Let $x \in D$, $\delta_x \in F(D)$, defined by $\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

Then $\{\delta_x : x \in D\}$ is a standard basis of $F(D)$.

Standard basis : $(\delta_1, \dots, \delta_n)$, for $F(\{1, \dots, n\})$.

Remark: $\{\delta_x : x \in \mathbb{R}\}$ do not span $F(\mathbb{R})$.

Proof that $(\delta_1, \dots, \delta_n)$ spans $F\{1, \dots, n\}$. Take any

$$f \in F(D)$$

Find $a_1, \dots, a_n \in K$ such that

$$f = a_1\delta_1 + \dots + a_n\delta_n$$

$$f(x), \quad x \in D = \{1, \dots, n\}$$

$$g(1) = a_1\delta_1(1) + a_2\delta_2(1) + \dots + a_n\delta_n(1) = a_1$$

Claim: $a_k = f(k) \in K$ Claim: $f = \sum f(k)\delta_k$, where $f(k) \in K$ and δ_k is the basis vector.

Proof:

$$g(l) = f(1)\delta_1(l) + \dots + f(l)\delta_l(l) + \dots + f(n)\delta_n(l) = f(l)$$

Subspace criterion. Let V be a vector space. To check a subset $u \subseteq V$ is a vector space. In principle need to check: 1. U is stable under addition $u, v \in U$ Hence $u + v \in U$. 2. U is stable under scalar multiplication. $\lambda \in K, u \in U$ Hence $\lambda u \in U$. 3. 8 axioms hold in U .

Proposition. A subset $U \subseteq V$ is a vector space (8 axioms are true) if f: (0) U is not empty ($0 \in U$) (1) U is stable under addition: $u, v \in U \rightarrow u + v \in U$. (2) U is stable under scalar multiplication: $u \in U, \lambda \in K \rightarrow \lambda u \in U$.

Subspaces

Problem 5 (Subspace criterion). Let $A \in M_n(K)$ be a fixed matrix. Prove that

$$U = \{x \in K^n : Ax = \vec{0}\}$$

is a subspace, null space or kernel.

Problem 6 (Subspace criterion 2). The set $U = \{Ax : x \in K^n\}$ is a subspace, image of A .

Problem 7 (Subspace criterion 3). Let $V = K$ (vector space of dim 1) show that the only 2 subspaces of V are $\{0\}$ and V itself.

Problem 8 (Subspace criterion 4). $\text{Span}(v_1, \dots, v_n) = \{\lambda_1 v_1 + \dots + \lambda_n v_n : \lambda_1, \dots, \lambda_n \in K\}$ is a subspace.

Problem 9. $\text{Span}(v_1, \dots, v_n)$ is the smallest subspace c

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Problem 10 ($K = \mathbb{R}$). Is $v_i \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix}$ in the space of $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ In otherwords: are there $x, y, z \in \mathbb{R}$ such that $v = xv_1 + yv_2 + zv_3$?

Problem 11. Is $f : 3x^2 + 5x - 5$ in the span of $f_1 = x^2 + 2x + 1, f_2 = 2x^2 + 5x + 4, f_3 = x^2 + 3x + 6$?

$$f = xf_1 + yf_2 + zf_3$$

Intersection, sums, direct sums. W vector space. $U, V \subseteq W$ two subspaces.
Definition: $U \cap V = \{v \in W : v \in U \text{ and } v \in V\}$ smallest intersection $\{0\}$.

The union of 2 subspaces in NOT a subspace (in general).

$W = \mathbb{R}^2$: draw a cartesian plane with vertical being V and horizontal being U and draw the cross $(u + v)$. $U \cup V =$ cross, no quadrant, not a subspace

Sum of 2 subspaces: $U + V = \{u + v : u \in U, v \in V\}$ line $U +$ line $V =$ plane

Proposition: U, V are subspaces of W . $U + V$ is a subspace. U cannot rescale a vector space, therefore $U + U = U$. Proof: \subseteq : Let $u + u' \in U + U$ prove $u + u' \in U$. Yes subce U is a subspace stable under addition. \supseteq : Take $u \in U$, prove $u \in U + U$. $u = u + 0 \in U + U$. Proof: subspace criterion. (0) $0 \in U + V$ since $0 \in U, 0 \in V$. (1) Stable under addition: let $w_1, w_2 \in U + V$. Prove $w_1 + w_2 \in U + V$.

$$w_1 = u_1 + v_1, u_1 \in U, v_1 \in V$$

$$w_2 = u_2 + v_2, u_2 \in U, v_2 \in V$$

$$w_1 + w_2 = (u_1 + u_2) + (v_1 + v_2)$$

$$u_1 + u_2 \in U, v_1 + v_2 \in V$$

$$w_1 + w_2 \in U + V$$

(2) Stable under scalar multiplication: let $w \in U + V, \lambda \in K$. Prove $\lambda w = \lambda u + \lambda v \in U + V$.

Prop: $U + V = \text{span}(U \cup V)$. $U + V =$ smallest subspace containing $U \cup V$.
Proof: \subseteq : Take $w \in U + V$ prove $w \in \text{span}(U \cup V)$.

$$w = u + v, u \in U, v \in V$$

Prove $w \in \text{span}(U \cup V)$. (You could also write it like $\text{span}(U, V)$ since line can be of both in U or V). \supseteq : Take $w \in \text{span}(U \cup V)$ prove $w \in U + V$.

$$w = a_1u_1 + \dots + a_nu_n + b_1v_1 + \dots + b_mv_m$$

where $u_i \in U, v_j \in V, a_i, b_j \in K$.

$$w = (a_1u_1 + \dots + a_nu_n) + (b_1v_1 + \dots + b_mv_m)$$

$$w = u + v, u \in U + V$$

4.3 Direct sums

U, V subspaces in W (ambient space).

Definition 20. U and V are in direct sum if $U \cap V = \{0\}$. Its a sum where the intersection is zero. Sum:

$$U + V = \{u + v : u \in U, v \in V\}$$

Now if $U \cap V = \{0\}$ we write $U \oplus V$, and $U, V \subseteq W$. Sol: $W = U \oplus V$ means

$$\begin{cases} W = U + V \\ U \cap V = \{0\} \end{cases} \quad \text{To prove } W = U \oplus V,$$

- $W = U + V, \subseteq$ Let $w \in W$. Prove: there exist $u \in U, v \in V$ such that $w = u + v$.
- $U \cap V = \{0\}$. \supseteq automatic. Pick $w \in \{0\}$. Prove $w \in U \cap V$. By assumption $w = 0$. Prove $w \in U$ and $w \in V$ hold by definition of subspace. Only need to show $U \cap V \subseteq \{0\}$. Let $w \in U$ and $w \in V$. Prove $w = 0$.

Sets	Vector Spaces
intersection $A \cap B$	intersection $U \cap V$
union $A \cup B$	union $U + V$
Disjoint union $A \sqcup B$ and $A \cap B = \emptyset$	Direct sum $U \cap V = \{0\}$

Example.

$$D = \{1, 2, 3, 4, 5\}$$

$$A = \{1, 2, 3\}, B = \{4, 5, 6\}$$

Then

$$D = A \sqcup B$$

\rightarrow function spaces $F(D) =$ functions on D .

$$f \in F(D)$$

Example of a function: $f(1) = 2, f(2) = \sqrt{2}, f(3) = \pi \dots$ View $F(A)$ as a subspace of $F(D)$. Claim: $F(A)$ can be "identified" with a subspace U of $F(D)$.

$$U = \{f : D \rightarrow \mathbb{R} : f(x) = 0, x \notin A\}$$

5 Appendix

6 Solutions

Solution 1 (Invertibility). Suppose B and B' are both inverses of A . Then

$$B = BI = B(AB') = (BA)B' = IB' = B'.$$

Therefore, $B = B'$, so the inverse is unique.

Solution 2 (Invertibility 2). We can answer this problem with proof by contradiction. Let's suppose this matrix is invertible. By definition there exists $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We can rewrite this equation into: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{-1}$. The inverse of our matrix can be rewritten as $\frac{1}{0 \cdot 0 - 1 \cdot 0} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ ⁸. But this is undefined since division by 0 is undefined. Therefore, our initial assumption that the matrix is invertible is false, and thus the matrix is not invertible.

Solution 3 (Field). A field with 2 elements can be constructed as follows: Let $F = \{0, 1\}$ be a set with two elements. We define addition and multiplication operations on F as follows:

- $0 + 0 = 0$
- $0 + 1 = 1$
- $1 + 0 = 1$
- $1 + 1 = 0$
- $0 \times 0 = 0$
- $0 \times 1 = 0$
- $1 \times 0 = 0$
- $1 \times 1 = 1$

Solution 4 (Span).

$$\text{Span}(v_1, v_2) = \{xv_1 + yv_2 : x, y \in \mathbb{R}\} = \left\{ \begin{pmatrix} 3x + y \\ x + 3y \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

Solution 5 (Subspace criterion). (0) is $0 \in U$? Yes, because $A0 = 0$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(1) Addition: Let $x, y \in U$. Prove $x + y \in U$.

$$A(x + y) = Ax + Ay = 0 + 0 = 0$$

(2) Scalar multiplication: Let $x \in U, \lambda \in K$. Prove $\lambda x \in U$.

$$A(\lambda x) = \lambda Ax = \lambda 0 = 0$$

⁸Recall that an inverse of a 2×2 matrix is equal to its determinant multiplied with its conjugate

Solution 6 (Subspace criterion 2). (0) is $0 \in U$? Find $x \in K^n$ such that $Ax = 0$. $x = 0$.

(1) Stability under addition: $y_1, y_2 \in U \rightarrow y_1 + y_2 \in U$. There exist x , such that $y_1 = Ax_1, y_2 = Ax_2$.

$$y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2) \in U$$

(2) Stability under scalar multiplication: If $f = Ax$, for some x , $\lambda y = A(\lambda x)$

Solution 7 (Subspace criterion 3). Let $U \subseteq \mathbb{R}$ be a subspace. Prove: $U = \{0\}$ or $U = \mathbb{R}$.

Case 1: $U = \{0\}$.

Case 2: $U \neq \{0\} \rightarrow$ there exists $v \in \mathbb{R}, v \neq 0$. Prove: $U = \mathbb{R}$. Let $x \in \mathbb{R}$ be any real number. Prove: $x \in U$. Since U is a subspace, it is stable under scalar multiplication.

$$x \in U \rightarrow Ax \in U$$

Therefore, $x \in U$.

Solution 8 (Subspace criterion 4). (0) $0 \in \text{span}(v_1, \dots, v_n)$, because

$$\lambda_1 = \dots \lambda_n = 0 \quad \lambda_1 v_1 + \dots + \lambda_n v_n = 0.$$

(1) addition: $u, v \in \text{span}$ Prove $u + v \in \text{span}$.

$$u = a_1 v_1 + \dots + a_n v_n$$

$$v = b_1 v_1 + \dots + b_n v_n$$

$$u + v = (a_1 + b_1)v_1 + \dots + (a_n + b_n)v_n \in \text{span}(v_1, \dots, v_n)$$

(2) scalar multiplication:

Solution 9.

$$\begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} + z \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{cases} x + 2y + z = 3 \\ 2x + 5y + 3z = 5 \\ x + 4y + 6z = -5 \end{cases} \rightarrow x = 3, y = 1, z = -2$$

A: yes since

$$v = 3v_1 + v_2 + 2v_3$$

7 Useful Links