

MATH 325: Honours Ordinary Differential Equations

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Abstract

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1 Introduction

Jean-Philippe Lessard (Burnside 1119). Tutorials every wednesday from 9am to 10am, ENGTR 0070, with Eunpyo Bang. Office hours thursday. No textbooks. 25% assignments (2 written assignments 15%, and 5 webworks 10%). 25% Midterm (February 16 - inclass). 50% Final. Since its honours you will deal with analysis.

2 Prerequisite knowledge

2.1 Analysis

3 Intro, Classification, Theorem of Existence & Uniqueness

3.1 Intro

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Definition 1 (Differential Equation). *A differential equation (DE) is a relation that involves an unknown function and some of its derivatives.*

To better understand what a differential equation is, consider the following example.

Imagine a ball of mass m falling, subject to gravity and air resistance (drag). Denote by $v(t)$ the velocity of the ball at time t , whereas t is the independent variable, and v the dependent variable. Let the downward direction be positive. We know the force of gravity is given by $F_g = mg$, where g is the acceleration due to gravity. The drag force is given by $F_d = -\lambda v$, where λ is the drag coefficient and is $\lambda \geq 0$. According to Newton's second law $\sum F = ma$, the net force acting on the ball is equal to its mass times its acceleration

$$m \frac{dv}{dt} = mg - \lambda v.$$

Let $y(t)$ be the position, meaning $v(t) = \frac{dy}{dt}$. Then, we can rewrite the above equation as

$$my'' + \lambda y' = mg.$$

Let's analyze another example, population growth (known as the Malthusian growth model).

Denote by $N(t)$ the size of a given population at time t . In an "unconstrained" environment, it is reasonable to assume that the rate of change of the number of individuals is proportional to the number of individuals present. This assumption leads to the following differential equation:

$$\frac{dN}{dt} = rN,$$

where r is called the growth rate (if $r > 0$), and decay rate (if $r < 0$). Assume that $N > 0$. Using the chain rule and assuming that $N(t)$ satisfies $N' = rN$

$$\frac{d}{dt} \ln(N(t)) = \frac{d \ln(N)}{dN} \cdot \frac{dN}{dt} = \frac{1}{N} \cdot N' = r,$$

integrate with respect to t

$$\ln(N(t)) = rt + C,$$

where C is the constant of integration. Exponentiating both sides, we obtain

$$N(t) = e^{\ln(N(t))} = e^C e^{rt} = k e^{rt},$$

where $\{k > 0 | k \in \mathbb{R}\}$ which could be any positive constant is the initial population size at time $t = 0$.

Assume that an initial population (condition) is given:

$$N(0) = N_0(\text{fixed}),$$

we therefore get that $k = N_0$, and the unique solution that satisfies the initial condition is

$$N(t) = N_0 e^{rt}.$$

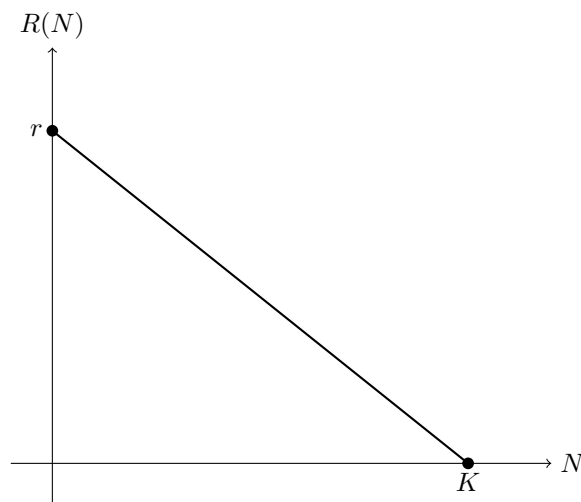
The problem with the answer we got in the previous example is that it is not realistic in the long run, how about we consider a carrying capacity¹. This leads us to another example: Population growth/decay with the carrying capacity of the environment.

Now assume that our growth rate depends on the population size $N(t)$ itself, therefore we get that

$$\frac{dN}{dt} = R(N)N.$$

Denote by K the number of individual that the environment can carry. K is called the carrying capacity of the environment. If $N < K$, we want growth ($R(N) > 0$) and if $N > K$, we want decay ($R(N) < 0$).

¹maximum population size that the environment can sustain indefinitely



Let's pick the simplest function $R(N)$ that satisfies $R(0) = r, R(K) = 0$ and is linear. We get that

$$R(N) = r\left(1 - \frac{N}{K}\right).$$

Therefore, our differential equation becomes

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N = \frac{r}{K}(K - N(t))N(t).$$

This is called the logistic equation.

Definition 2 (Ordinary Differential Equation). *An ordinary differential equation (ODE) is a differential equation whose unknown function depends on one independent variable only.*

Example of ODEs:

- $y''(t) + y'(t) + 2y(t) = \sin(t)$
- $N'(t) = rN(t)$
- $mv'(t) = mg - \lambda v(t)$
- $y'(x) + 3y(x) = e^x$

Definition 3 (Partial Differential Equation). *A partial differential equation (PDE) is a differential equation whose unknown function depends on more than one independent variable. **Will not be taught in this course.***

Example of a PDE is the Heat Equation. Let $u = u(x, t)$, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. This PDE denotes the temperature of a body at time t and at position x .

3.2 Classification

3.2.1 The Order

Definition 4. *The order of an ODE is the order of the highest derivative that appears in the equation.*

Example. $N' = rN$ (first order ODE)

Example. $y''(t) + 2y'(t) = e^t$ (second order ODE)

Given $n \in \mathbb{N}$, an n^{th} order scalar ODE is written as

$$F(t, y(t), y'(t), y''(t), \dots, y^{(n)}(t)) = 0,$$

where $F : \mathbb{R}^{n+2} \rightarrow \mathbb{R}$ is a map and where $y^{(k)}(t) = \frac{d^k y}{dt^k}(t), k = 1, \dots, n$.

Systems of first order ODEs

Consider a map $f : D \times (a, b) \rightarrow \mathbb{R}^n$, where $D \subseteq \mathbb{R}^n$ is an open set, and (a, b) is a "time" interval. A general first order system of ODEs is given by

$$y'(t) = f(y(t), t), \text{ where } y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \text{ and } y' = \begin{pmatrix} y'_1 \\ \vdots \\ y'_n \end{pmatrix}.$$

Remark: Assume that a scalar n^{th} order ODE has the form

$$y^{(n)}(t) = G(t, y(t), y'(t), \dots, y^{(n-1)}(t)).$$

Letting $y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$. This leads us to $y'_1 = y' = y_2, y'_2 = y'' = y_3, \dots, y'_{n-1} = y^{(n-1)} = y_n, y'_n = y^{(n)} = G(t, y_1, y_2, \dots, y_n)$. Therefore, we can rewrite the n^{th} order ODE as a first order system of ODEs:

$$\begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_{n-1} \\ y'_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ G(t, y_1, y_2, \dots, y_n) \end{pmatrix}$$

Example (Lorenz Equation).

$$y'_1 = \sigma(y_2 - y_1)$$

$$y'_2 = \rho y_1 - y_2 - y_1 y_3$$

$$y'_3 = y_1 y_2 - \beta y_3$$

where σ, ρ, β are parameters. $n = 3$. This is a first order system of ODEs. They are nonlinear because of the products $y_1 y_3$ and $y_1 y_2$.

3.2.2 Linearity

Definition 5 (Linearity). *The n^{th} order ODE $F(t, y, y', \dots, y^{(n)}) = 0$ is linear if F is a linear polynomial in the variables $y, y', y'', \dots, y^{(n)}$, that is, it is of the form $a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \dots + a_{n-1}(t)y'(t) + a_n(t)y(t) = g(t)$, where a_0, a_1, \dots, a_n, g are given functions of t . Otherwise, it is nonlinear.*

In short terms, an ODE is said to be linear if it can be written as $y'(t) = A(t)y(t) + r(t)$ where, given $t \in (a, b)$, $A(t) \in M_n(\mathbb{R})$ (the set of $n \times n$ real matrices) and $r(t) \in \mathbb{R}^n$.

Example. Consider the second-order ODE

$$\begin{aligned} y'' + 2y' + y &= e^t \\ \implies y'' &= -2y' - y + e^t. \end{aligned}$$

Define new variables

$$y_1 = y, \quad y_2 = y'.$$

Then the system becomes

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad r(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

3.2.3 Autonomy

Definition 6 (Autonomy). *The n^{th} order ODE $F(t, y, y', \dots, y^{(n)}) = 0$ is autonomous if F does not depend explicitly on t , that is, if it is of the form $F(y, y', \dots, y^{(n)}) = 0$. Otherwise, it is non-autonomous.*

Example. $y'' + 2y' + y - e^t = 0$ is non-autonomous.

Example. $N'(t) = rN(t)$ is autonomous.

Example. $y'(t) = ty(t)$ is non-autonomous.

Equivalently, a first-order system $y' = f(y, t)$ is autonomous if it can be written as

$$y' = f(y).$$

Otherwise, it is non-autonomous.

Note: The Lorenz system is an example of an autonomous system².

²Here “system” means the unknown y is vector-valued, e.g. $y \in \mathbb{R}^m$, rather than scalar.

3.2.4 Solutions of ODEs

Definition 7 (Solutions of ODEs). *Let $f : D \times (a, b) \rightarrow \mathbb{R}^n$. A solution of $y'(t) = f(y(t), t)$ on an interval $J \subset \mathbb{R}$ is a differentiable function $y : J \rightarrow D \subset \mathbb{R}^n$, such that $y'(t) = f(y(t), t), \forall t \in J$. t is the independent variable, and $y = (y_1, \dots, y_n)$ is the dependent variable.*

Explicit Solutions

Example. *Consider the ODE*

$$y' + y = 1.$$

We can verify that $y(t) = e^{-t} + 1$, and therefore $y'(t) = -e^{-t}$, is a solution on \mathbb{R} . Indeed,

$$y' + y = -e^{-t} + (e^{-t} + 1) = 1.$$

In this example, $y = y(t)$ is explicitly given as a function of t (independent variable).

Implicit solutions

Example. *Consider the ODE*

$$y \frac{dy}{dx} = x.$$

This is a nonautonomous, nonlinear, first-order scalar ODE. Separating variables gives

$$y \, dy = x \, dx.$$

Integrating,

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C,$$

or equivalently, the implicit solution

$$x^2 - y^2 = C, \quad C \in \mathbb{R}.$$

To verify, differentiate implicitly:

$$\frac{d}{dx}(x^2 - y^2) = 0 \implies 2x - 2y \frac{dy}{dx} = 0 \implies y \frac{dy}{dx} = x.$$

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3.3 Initial Value Problems

A first-order system of ODEs is written as

$$y' = f(y, t), \quad f : D \times (a, b) \rightarrow \mathbb{R}^n.$$

Let $t_0 \in (a, b)$. An initial condition is

$$y(t_0) = y_0 \in \mathbb{R}^n.$$

An initial value problem (IVP) is

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}$$

3.4 Existence and Uniqueness Theorem

3.4.1 Lipschitz continuity

Definition 8 (Lipschitz continuity). Let $D \subseteq \mathbb{R}^n$ and let $\|\cdot\|$ be a norm on \mathbb{R}^n . A function $f : D \rightarrow \mathbb{R}^n$ is Lipschitz continuous if there exists $L \geq 0$ such that

$$\|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\| \quad \forall y_1, y_2 \in D.$$

The smallest such L is called the Lipschitz constant and is denoted $Lip(f)$.

Example. Let $f(y) = 4y - 5$, $D = \mathbb{R}$, and $\|\cdot\| = |\cdot|$. Then

$$|f(y_1) - f(y_2)| = |4y_1 - 4y_2| = 4|y_1 - y_2|.$$

So f is Lipschitz with $Lip(f) = 4$.

Example. Let

$$f(y) = \frac{1}{y-1}, \quad D = (1, +\infty).$$

Then f is not Lipschitz on D .

Now fix $\delta > 1$ and define $D_\delta = (\delta, +\infty)$. For $y_1, y_2 \in D_\delta$, by the Mean Value Theorem, there exists $z \in (y_1, y_2)$ such that

$$f(y_2) - f(y_1) = f'(z)(y_2 - y_1), \quad f'(y) = -\frac{1}{(y-1)^2}.$$

So

$$|f(y_2) - f(y_1)| \leq \frac{1}{(z-1)^2}|y_2 - y_1| \leq \frac{1}{(\delta-1)^2}|y_2 - y_1|.$$

Thus f is Lipschitz on D_δ with

$$Lip(f) = \frac{1}{(\delta-1)^2}.$$

3.4.2 Local Lipschitz continuity

Definition 9 (Locally Lipschitz). Let $D \subseteq \mathbb{R}^n$ be open. A function $f : D \rightarrow \mathbb{R}^n$ is locally Lipschitz if for every compact set $K \subset D$, there exists $L(K)$ such that

$$\|f(y_1) - f(y_2)\| \leq L(K)\|y_1 - y_2\| \quad \forall y_1, y_2 \in K.$$

Problem 1. (See tutorial 1) If $f \in C^1(D)$, then f is locally Lipschitz.

3.4.3 Existence and Uniqueness Theorem

Theorem 1 (Existence and Uniqueness). Let $D \subseteq \mathbb{R}^n$ be open and let (a, b) be an open interval containing t_0 . Consider the IVP

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}$$

Assume $f : D \times (a, b) \rightarrow \mathbb{R}^n$ is continuous and locally Lipschitz in y .

If $y_0 \in D$, then there exists an open interval J containing t_0 on which a solution exists. Moreover, this solution is unique on J .

3.4.4 Integral form of solutions

Lemma 2. *A function y solves the IVP if and only if*

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

Proof. If $y' = f(y, t)$ and $y(t_0) = y_0$, then by the Fundamental Theorem of Calculus,

$$y(t) - y(t_0) = \int_{t_0}^t f(y(s), s) ds,$$

so

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

Conversely, differentiating the right-hand side gives

$$y'(t) = f(y(t), t), \quad y(t_0) = y_0.$$

□

3.4.5 Picard operator

Let $(y_0, t_0) \in D \times (a, b)$. Since this set is open, there exist $\alpha, \delta > 0$ such that

$$D_{\alpha, \delta} = \{(y, t) : \|y - y_0\| \leq \alpha, |t - t_0| \leq \delta\} \subset D \times (a, b).$$

Define

$$M_{\alpha, \delta} = \sup_{(y, t) \in D_{\alpha, \delta}} \|f(y, t)\| < +\infty.$$

Let

$$\epsilon = \min \left(\delta, \frac{\alpha}{M_{\alpha, \delta}} \right), \quad J = (t_0 - \epsilon, t_0 + \epsilon).$$

Lemma 3 (Picard operator). *For any function y such that $y(t_0) = y_0$ and $(y(t), t) \in D_{\alpha, \delta}$ for all $t \in J$, define*

$$T(y)(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

Then $T(y)(t_0) = y_0$ and $(T(y)(t), t) \in D_{\alpha, \delta}$ for all $t \in J$.

Proof.

$$T(y)(t_0) = y_0.$$

For $t \in J$,

$$\|T(y)(t) - y_0\| \leq \int_{t_0}^t \|f(y(s), s)\| ds \leq M_{\alpha, \delta} |t - t_0| \leq M_{\alpha, \delta} \epsilon \leq \alpha.$$

So $(T(y)(t), t) \in D_{\alpha, \delta}$.

□

Lemma 4. *If $y : J \rightarrow \mathbb{R}^n$ satisfies*

1. $y(t_0) = y_0$,
2. $(y(t), t) \in D_{\alpha, \delta}$ for all $t \in J$,

then $T(y) : J \rightarrow \mathbb{R}^n$ satisfies the same properties.

Picard iterations Define $y_0(t) = y_0$ (constant function) clearly satisfies (1) and (2). For $k \geq 1$, define the Picard iterations by

$$y_k(t) = T(y_{k-1}(t)) = y_0 + \int_{t_0}^t f(y_{k-1}(s), s) ds.$$

Existance The Picard iterations converge uniformly to a function $y : J \rightarrow \mathbb{R}^n$ which satisfies (1) and (2), and is a solution of the IVP.

Proof. Pick $t \in [t_0, t_0 + \epsilon]$ (the proof is similar for $t \in [t_0 - \epsilon, t_0]$). The goal is to show first that $\{y_k(t)\}_{k=0}^\infty$ is a Cauchy sequence⁴ in \mathbb{R}^n . We prove by induction that

$$(**) : \|y_m(t) - y_{m-1}(t)\| \leq L^{m-1} M_{\alpha, \delta} \frac{(t - t_0)^m}{m!}, \quad \forall m \geq 1.$$

For $m = 1$,

$$\|y_1(t) - y_0(t)\| = \|y_0 + \int_{t_0}^t f(y_0, s) ds - y_0\| \leq \int_{t_0}^t \|f(y_0, s)\| ds \leq M_{\alpha, \delta} |t - t_0| \leq \alpha.$$

Assume $(**)$ holds for m and show that it holds for $m + 1$

$$\|y_{m+1}(t) - y_m(t)\| = \|y_0 + \int_{t_0}^t f(y_m(s), s) ds - y_0 - \int_{t_0}^t f(y_{m-1}(s), s) ds\| \leq \int_{t_0}^t \|f(y_m(s), s) - f(y_{m-1}(s), s)\| ds.$$

Since $D_{\alpha, \delta}$ is compact and f is (LLC) Lipschitz on $D_{\alpha, \delta}$ with Lipschitz constant L . Therefore, there exists $L = L(D_{\alpha, \delta})$ such that

$$\|f(x, t) - f(y, t)\| \leq L \|x - y\|, \quad \forall (x, t), (y, t) \in D_{\alpha, \delta}.$$

Which implies

$$\begin{aligned} &\leq \int_{t_0}^t \|y_m(s) - y_{m-1}(s)\| ds. \leq {}^5 L \int_{t_0}^t L^{m-1} M_{\alpha, \delta} \frac{(t - t_0)^{m-1}}{(m-1)!} ds \leq L^{m-1} M_{\alpha, \delta} \frac{(s - t_0)^m}{m!} ds \\ &= L^m \frac{M_{\alpha, \delta}}{m!} \int_{t_0}^t (s - t_0)^m ds = L^m M_{\alpha, \delta} \frac{(t - t_0)^{m+1}}{(m+1)!}. \end{aligned}$$

³sup means the largest value in a set of numbers.

⁴Cauchy sequence is a sequence that has a limit in a metric space \mathbb{R}^n .

⁵By the $(**)$

In particular, for all $\rho \geq 1$,

$$\|y_\rho(t) - y_{\rho-1}(t)\| \leq M_{\alpha,\delta} \frac{(L(t-t_0))^\rho}{(\rho)!} < \frac{M_{\alpha,\delta}}{L} \frac{(L\epsilon)^\rho}{\rho!}.$$

Pick $m, p \geq 1$ be two fixed integers

$$\|y_{m+p}(t) - y_{m+1}(t)\| = \|y_{m+p} - y_{m+p-1} + y_{m+p-1} + \dots - y_{m+2} + y_{m+2} - y_{m+1}\|.$$

Triangle inequality gives

$$\begin{aligned} &\leq \sum_{k_1}^{p-1} \|y_{m+k+1}(t) - y_{m+k}(t)\| \\ &< \sum_{k=1}^{p-1} \frac{(L\epsilon)^{m+k+1}}{(m+k+1)!} = \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{m+p} \frac{(L\epsilon)^j}{j!}. \end{aligned}$$

Where $j = m + k + 1$. Now

$$\rightarrow_{m,p \rightarrow +\infty} 0$$

since $e^{L\epsilon} = \sum_{j=0}^{+\infty} \frac{(L\epsilon)^j}{j!}$ converges. $\rightarrow \{y_k(t)\}_{k=0}^\infty$ is a Cauchy sequence in \mathbb{R}^n . Since \mathbb{R}^n is complete (all cauchy sequences converge) $y_k(t)$ converges to a limit, that we denote $y(t)$. Take the limit when $p \rightarrow +\infty$ in (***) :

$$y(t) - y_{m+1}(t) \leq \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{+\infty} \frac{(L\epsilon)^j}{j!} \rightarrow_{m \rightarrow +\infty} 0.$$

This implies that $y_k(t)$ converges uniformly to $y(t)$. By construction, $y(t_0) = y_0$ is continuous, y_1, y_2, y_3, \dots are also continuous. By the uniform convergence theorem, $y(t)$ is continuous. By Picard's iterations,

$$y_k(t) = y_0 + \int_{t_0}^t f(y_{k-1}(s), s) ds.$$

Taking the limit when $k \rightarrow +\infty$

$$y(t) = y_0 + \lim_{k \rightarrow +\infty} \int_{t_0}^t f(y_{k-1}(s), s) ds = y_0 + \int_{t_0}^t \lim_{k \rightarrow +\infty} f(y_{k-1}(s), s) ds = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

□

⁶ (***)

⁷ Uniform convergence

⁸ Evaluated on compacted cylinder, understanding epsilon, the L lipschitz constant coming from somewhere, not on the analysis background such as...

Uniqueness. Assume that $y(t)$ and $z(t)$ satisfy the IVP

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}$$

Which implies that

$$\begin{cases} y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds, \\ z(t) = y_0 + \int_{t_0}^t f(z(s), s) ds. \end{cases}$$

$$\|y(t) - z(t)\| \leq \int_{t_0}^t \|f(y(s), s) - f(z(s), s)\| ds \leq \int_{t_0}^t \|y(s) - z(s)\| ds.$$

Define $g(t) = \int_{t_0}^t \|y(s) - z(s)\| ds$. Therefore,

$$g'(t) = \|y(t) - z(t)\| \leq Lg(t).$$

Which implies that

$$g'(t) - Lg(t) \leq 0.$$

Multiply both sides by $e^{-L(t-t_0)}$

$$\frac{d}{dt}(e^{-L(t-t_0)}g(t)) = (g'(t) - Lg(t))e^{-L(t-t_0)} \leq 0.$$

Which implies that $e^{-L(t-t_0)}g(t)$ is decreasing. Therefore, for all $t \geq t_0$, $e^{-L(t-t_0)}g(t) \leq e^{-L(t-t_0)}g(t_0)$ □

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9 Solutions

10 Appendix

11 Useful Links