MATH 325: Honours Ordinary Differential Equations

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Abstract

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1 Introduction

Jean-Philippe Lessard (Burnside 1119). Tutorials every wednesday from 9am to 10am, ENGTR 0070, with Eunpyo Bang. Office hours thursday. No textbooks. 25% assignments (2 written assignments 15%, and 5 webworks 10%). 25% Midterm (February 16 - inclass). 50% Final. Since its honours you will deal with analysis.

2 Prerequisite knowledge

2.1 Analysis

3 Intro, Classification, Theorem of Existence & Uniqueness

3.1 Intro

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Definition 1 (Differential Equation)**.** *A differential equation (DE) is a relation that involves an unknown function and some of its derivatives.*

To better understand what a differential equation is, consider the following example.
Imagine a ball of mass m falling, subject to gravity and air resistance (drag). Denote by $v(t)$ the velocity of the ball at time t, whereas t is the independent variable, and v the dependent variable. Let the downward direction be positive. We know the force of gravity is given by $F_g = mg$, where g is the acceleration due to gravity. The drag force is given by $F_d = -\lambda v$, where λ is the drag coefficient and is $\lambda \geq 0$. According to Newton's second law $\sum F = ma$, the net force acting on the ball is equal to its mass times its acceleration

$$m\frac{dv}{dt} = mg - \lambda v.$$

Let $y(t)$ be the position, meaning $v(t) = \frac{dy}{dt}$. Then, we can rewrite the above equation as

$$my'' + \lambda y' = mg.$$

Let's analyze another example, population growth (known as the Malthusian growth model).
Denote by $N(t)$ the size of a given population at time t. In an "unconstraint" environment, it is reasonable to assume that the rate of change of the number of individuals is proportional to the number of individuals present. This assumption leads to the following differential equation:

$$\frac{dN}{dt} = rN,$$

where r is called the growth rate (if $r > 0$), and decay rate (if $r < 0$). Assume that $N > 0$. Using the chain rule and assuming that $N(t)$ satisfies $N' = rN$

$$\frac{d}{dt} ln(N(t)) = \frac{dln(N)}{dN} \cdot \frac{dN}{dt} = \frac{1}{N} \cdot N' = r,$$

integrate with respect to t

$$ln(N(t)) = rt + C,$$

where C is the constant of integration. Exponentiating both sides, we obtain

$$N(t) = e^{ln(N(t))} = e^C e^{rt} = ke^{rt},$$

where $\{k > 0 | k \in \mathbb{R}\}$ which could be any positive constant is the initial population size at time $t = 0$.
Assume that an initial population (condition) is given:

$$N(0) = N_0(fixed),$$

we therefore get that $k = N_0$, and the unique solution that satisfies the initial condition is

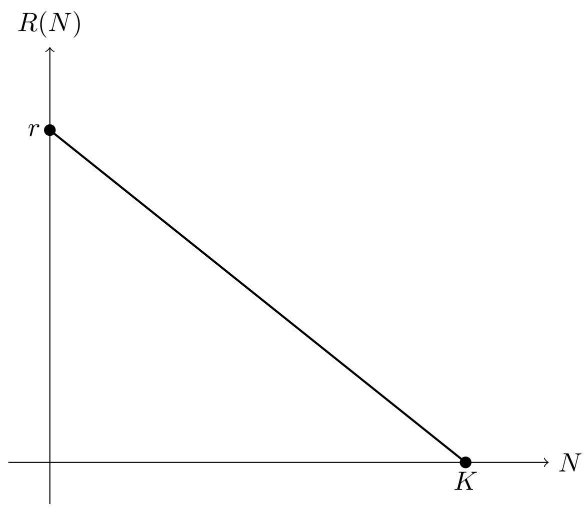
$$N(t) = N_0 e^{rt}.$$

The problem with the answer we got in the previous example is that it is not realistic in the long run, how about we consider a carrying capacity[1]. This leads us to another example: Population growth/decay with the carrying capacity of the environment.
Now assume that our growth rate depends on the population size $N(t)$ itself, therefore we get that

$$\frac{dN}{dt} = R(N)N.$$

Denote by K the number of individual that the environment can carry. K is called the carrying capacity of the environment. If $N < K$, we want growth ($R(N) > 0$) and if $N > K$, we want decay ($R(N) < 0$).

[1] maximum population size that the environment can sustain indefinitely



Let's pick the simplest function $R(N)$ that satisfies $R(0) = r, R(K) = 0$ and is linear. We get that

$$R(N) = r(1 - \frac{N}{K}).$$

Therefore, our differential equation becomes

$$\frac{dN}{dt} = r(1 - \frac{N}{K})N = \frac{r}{K}(K - N(t))N(t).$$

This is called the logistic equation.

Definition 2 (Ordinary Differential Equation)**.** *An ordinary differential equation (ODE) is a differential equation whose unknown function depends on one independent variable only.*

Example of ODEs:

- $y''(t) + y'(t) + 2y(t) = sin(t)$
- $N'(t) = rN(t)$
- $mv'(t) = mg - \lambda v(t)$
- $y'(x) + 3y(x) = e^x$

Definition 3 (Partial Differential Equation)**.** *A partial differential equation (PDE) is a differential equation whose unknown function depends on more than one independent variable.* ***Will not be taught in this course.***

Example of a PDE is the Heat Equation. Let $u = u(x,t)$, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. This PDE denotes the temperature of a body at time t and at position x.

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3.2 Classification

3.2.1 The Order

Definition 4. *The order of an ODE is the order of the highest derivative that appears in the equation.*

Example. $N' = rN$ *(first order ODE)*

Example. $y''(t) + 2y'(t) = e^t$ *(second order ODE)*

Given $n \in \mathbb{N}$, an n^{th} order scalar ODE is written as

$$F(t, y(t), y'(t), y''(t), \ldots, y^{(n)}(t)) = 0,$$

where $F : \mathbb{R}^{n+2} \to \mathbb{R}$ is a map and where $y^{(k)}(t) = \frac{d^k y}{dt^k}(t), k = 1, \ldots, n$.

Systems of first order ODEs

Consider a map $f : D \times (a, b) \to \mathbb{R}^n$, where $D \subseteq \mathbb{R}^n$ is an open set, and (a,b) is a "time" interval. A general first order system of ODEs is given by $y'(t) = f(y(t), t)$, where $y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$, $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, and $y' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$.

Remark: Assume that a scalar n^{th} order ODE has the form

$$y^{(n)}(t) = G(t, y(t), y'(t), \ldots, y^{(n-1)}(t)).$$

Letting $y_1 = y, y_2 = y', \ldots, y_n = y^{(n-1)}$. This leads us to $y_1' = y' = y_2, y_2' = y'' = y_3, \ldots, y_{n-1}' = y^{(n-1)} = y_n, y_n' = y^{(n)} = G(t, y_1, y_2, \ldots, y_n)$. Therefore, we can rewrite the n^{th} order ODE as a first order system of ODEs:

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ G(t, y_1, y_2, \ldots, y_n) \end{pmatrix}$$

Example (Lorenz Equation)**.**

$$y_1' = \sigma(y_2 - y_1)$$

$$y_2' = \rho y_1 - y_2 - y_1 y_3$$

$$y_3' = y_1 y_2 - \beta y_3$$

where σ, ρ, β are parameters. $n = 3$. This is a first order system of ODEs. They are nonlinear because of the products $y_1 y_3$ and $y_1 y_2$.

3.2.2 Linearity

Definition 5 (Linearity)**.** *The n^{th} order ODE $F(t, y, y', \ldots, y^{(n)}) = 0$ is linear if F is a linear polynomial in the variables $y, y', y'', \ldots, y^{(n)}$, that is, it is of the form $a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \ldots + a_{n-1}(t)y'(t) + a_n(t)y(t) = g(t)$, where $a_0, a_1, \ldots, a_n, g$ are given functions of t. Otherwise, it is nonlinear.*

In short terms, an ODE is said to be linear if it can be written as $y'(t) = A(t)y(t) + r(t)$ where, given $t \in (a, b)$, $A(t) \in M_n(\mathbb{R})$ (the set of $n \times n$ real matrices) and $r(t) \in \mathbb{R}^n$.

Example. *Consider the second-order ODE*

$$y'' + 2y' + y = e^t$$

$$\Longrightarrow y'' = -2y' - y + e^t.$$

Define new variables

$$y_1 = y, \qquad y_2 = y'.$$

Then the system becomes

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \qquad r(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

3.2.3 Autonomy

Definition 6 (Autonomy)**.** *The n^{th} order ODE $F(t, y, y', \ldots, y^{(n)}) = 0$ is autonomous if F does not depend explicitly on t, that is, if it is of the form $F(y, y', \ldots, y^{(n)}) = 0$. Otherwise, it is non-autonomous.*

Example. $y'' + 2y' + y - e^t = 0$ *is non-autonomous.*

Example. $N'(t) = rN(t)$ *is autonomous.*

Example. $y'(t) = ty(t)$ *is non-autonomous.*

Equivalently, a first-order system $y' = f(y, t)$ is autonomous if it can be written as

$$y' = f(y).$$

Otherwise, it is non-autonomous.

Note: The Lorenz system is an example of an autonomous system[2].

[2] Here "system" means the unknown y is vector-valued, e.g. $y \in \mathbb{R}^m$, rather than scalar.

3.2.4 Solutions of ODEs

Definition 7 (Solutions of ODEs)**.** *Let $f : D \times (a,b) \to \mathbb{R}^n$. A solution of $y'(t) = f(y(t),t)$ on an interval $J \subset \mathbb{R}$ is a differentiable function $y : J \to D \subset \mathbb{R}^n$, such that $y'(t) = f(y(t),t), \forall t \in J$. t is the independent variable, and $y = (y_1,\dots,y_n)$ is the dependent variable.*

Explicit Solutions

Example. *Consider the ODE*

$$y' + y = 1.$$

We can verify that $y(t) = e^{-t} + 1$, and therefore $y'(t) = -e^{-t}$, is a solution on $\mathbb{R}$. Indeed,

$$y' + y = -e^{-t} + (e^{-t} + 1) = 1.$$

In this example, $y = y(t)$ is explicitly given as a function of t (independent variable).

Implicit solutions

Example. *Consider the ODE*

$$y\frac{dy}{dx} = x.$$

This is a nonautonomous, nonlinear, first-order scalar ODE. Separating variables gives

$$y\,dy = x\,dx.$$

Integrating,

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C,$$

or equivalently, the implicit solution

$$x^2 - y^2 = C, \qquad C \in \mathbb{R}.$$

To verify, differentiate implicitly:

$$\frac{d}{dx}(x^2 - y^2) = 0 \implies 2x - 2y\frac{dy}{dx} = 0 \implies y\frac{dy}{dx} = x.$$

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3.3 Initial Value Problems

$$y' = f(y,t), \quad f : D \times (a,b) \to \mathbb{R}^n.$$

Consider $t_0 \in (a,b)$ (initial time) An initial condition is given by $y(t_0) = y_0 \in \mathbb{R}^n$, where y_0 is given. An initial value problem is of the form

$$y' = f(y,t), \quad y(t_0) = y_0.$$

3.4 Theorem of Existence & Uniqueness

Definition 8. *Consider a set $D \subseteq \mathbb{R}^n$ and choose a norm $||\cdot||$ on $\mathbb{R}^n$ (e.g $||x||_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ the Euclidean norm). A function $f \cdot D \to \mathbb{R}^n$ is Lipschitz continuous if there exists a constant $L \geq 0$ such that $\forall y_1, y_2 \in D, ||f(y_1) - f(y_2)|| \leq l||y_1 - y_2||$.*

The smallest L which satisfies the inequality is called the Lipschitz constant and is denoted $Lip(f)$.

Example. *Consider $f(y) = 4y - 5 \quad (n = 1, D = \mathbb{R}) \quad f : \mathbb{R} \to \mathbb{R}$. $||\cdot|| = |\cdot|$ (absolute value). Let $y_1, y_2 \in \mathbb{R}$,*

$$|f(y_1) - f(y_2)| = |4y_1 - 4y_2| = 4|y_1 - y_2| \quad (LC)Lip(f) = 4.$$

Example. *Let*

$$f(y) = \frac{1}{y-1}$$

and $D = (1, \inf)$.
Draw the function, with the asymptote at x = 1.

$$\therefore f \cdot D \to \mathbb{R} \textit{ is not LC.}$$

Now, fix $\delta > 1$, and let $D_\delta = (\delta, +\inf)$.

Draw the function, with the asymptote at x = 1, but now add a line on the x axis starting at a later point than x = 1 and this new point is delta wih a line continuing to infinity called D sub delta, and pick two points on D sub delta named y1 and y2.

Then, $f \cdot D_\delta \to \mathbb{R}$ is LC. Indeed, let $y_1 < y_2 \in D_\delta$. By the Mean Value Theorem, there exists $z \in (y_1, y_2)$ such that

$$f(y_2) - f(y_1) = f'(z)(y_2 - y_1),$$

where $f'(y) = -\frac{1}{(y-1)^2}$

$$f(y_2) - f(y_1) = |-\frac{1}{(z-1)^2}||y_2 - y_1|$$

$$f(y_2) - f(y_1) \leq (sup_{z \in D_\delta} \frac{1}{(z-1)^2})|y_2 - y_1| = \frac{1}{(\delta - 1)^2}|y_2 - y_1|,$$

where $\frac{1}{(\delta-1)^2}$ is $L = Lip(f)$.

For any $k \subset D$ compact (for instance $k = [\delta, \beta]$), $L = \frac{1}{(\delta-1)^2}$.

Definition 9. *Consider $D \subseteq \mathbb{R}^n$ open. $f : D \to \mathbb{R}^n$ is locally LC, if any compact set $k \subset D$(closed & bounded), there exists a constant $L = L(k)$ such that, for all $y_1, y_2 \in k$, $||f(y_1) - f(y_2)|| \leq L||y_1 - y_2||$.*

Problem 1. *(See tutorial 1)*

$$f \in C^1(D) \to f \textit{ is LLC.}$$

Theorem 1 (Existance & Uniqueness)**.** *Consider $D \subseteq \mathbb{R}^n$ open and an open interval (a,b) which contains t_0. Such as*

$$\begin{cases} y' = f(y,t) \\ y(t_0) = y_0 \end{cases}$$

$f : D \times (a,b) \to \mathbb{R}^n$ is continuous and that, for all compact set $k \subset D \times (a,b) \subset \mathbb{R}^{n+1}$, there exists $L = L(k)$ such that (IVP):

$$||f(x,t) - f(y,t)|| \leq L||x-y||$$

for all $(x,t),(y,t) \in K$. If $y_0 \in D$, then there exists an open interval J, containing t_0, over which a solution to the (IVP) is defined. Moreover, the IVP has only one solution defined on J.

Lemma 2. *y solves the IVP*

$$\leftrightarrow y(t) = y_0 + \int_{t_0}^{t} f(y(s),s)ds$$

Proof.

$$y'(t) = f(y(t),t), y(t_0) = y_0$$

By the Fundamental Theorem of Calculus,

$$y(t) - y(t_0) = \int_{t_0}^{t} f(y(s),s)ds$$

$$y(t_0) = y_0 + \int_{t_0}^{t_0} f(y(s),s)ds = y_0$$

Differentiating the integral equation:

$$y'(t) = \frac{d}{dt}(y_0) + \frac{d}{dt}(\int_{t_0}^{t} f(y(s),s)ds) = 0 + f(y(t),t)$$

Since $(y_0,t_0) \in D \times (a,b)$ (open in $\mathbb{R}^n$), there exist $\alpha, \delta > 0$ such that the compact cylinder

$$D_{\alpha,\delta} = \{(y,t) \in \mathbb{R}^{n+1} | ||y-y_0|| \leq \alpha, |t-t_0| \leq \delta\} \subset D \times (a,b)$$

3D graph check phone (there is alpha and y0) Let $M_{\alpha,\delta} = sup_{(y,t)\in D_{\alpha,\delta}} ||f(y,t)|| < +\inf$ □

Lemma 3 (Picard Operator)**.** *Let $\epsilon > 0$ be defined by $\epsilon = min(\delta, \frac{\alpha}{M_{\alpha,\delta}})$. Let $J = (t_0 - \epsilon, t_0 + \epsilon)$. Then for any function $y(t)$ which satisfies $y(t_0) = y_0$ and $(y(t),t) \in D_{\alpha,\delta}$ for all $t \in J$, then $T(y)$ defined by*

$$T(y)(t) = y_0 + \int_{t_0}^{t} f(y(s),s)ds$$

also satisfies ($y(t_0) = y_0$ and $(y(t),t) \in D_{\alpha,\delta}$ for all $t \in J$).

Proof.

$$T(y)(t_0) = y_0 + 0 = y_0$$
$$(T(y)(t), t) \in D_{\alpha,\epsilon} \forall t \in J?$$

Show:

$$||T(y)(t) - y_0|| \leq \alpha$$

.

$$||T(y)(t) - y_0|| = ||y_0 + \int_{t_0}^{t} f(y(s), s)ds - y_0||$$
$$= || \int_{t_0}^{t} f(y(s), s)ds|| \leq \int_{t_0}^{t} ||f(y(s), s)||ds \leq M_{\alpha,\delta} \int_{t_0}^{t} ds$$
$$= M_{\alpha,\delta}||t - t_0|(where\, t \in J) \leq M_{\alpha,\delta} \cdot \epsilon \leq \frac{\alpha}{M_{\alpha,\delta}} \cdot M_{\alpha,\delta}$$

□

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10 Solutions

11 Appendix

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