MATH 223: Linear Algebra

William Homier[1]

[1] *McGill University Physics, 3600 Rue University, Montréal, QC H3A 2T8, Canada*

January 5^{th}, 2026

Abstract

Contents

1 Introduction

2 Prerequisite knowledge

2.1 Notation

2.1.1 Sets

Sets are a grouping of objects.

Set	Meaning	Examples
$\mathbb{N}$	The set of natural numbers	(0, 1, 2, 3, ...)
$\mathbb{Z}$	The set of integers	(..., -3, -2, -1, 0, 1, 2, 3, ...)
$\mathbb{Q}$	The set of rational numbers	$\mathbb{Q} = \frac{a}{b} \mid \forall a, b \in \mathbb{Z}\, and\, b \neq 0$
$\mathbb{R}$	The set of all rational and all irrational numbers	$(..., -1, 0, \frac{1}{4}, 1, 1000, ...)$
$\mathbb{C}$	The set of all complex numbers	$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\, and\, i \subseteq \sqrt{-1}\}$.

We have the following relationships between sets:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

2.1.2 Symbols

Symbol	Meaning
$\subseteq$	is a subset of or equal to
$\subset$	is a strict subset of
$\in$	is an element of
$\forall$	for all
$\exists$	there exists
$\emptyset$	empty set
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if

2.2 Complex Algebra

2.2.1 Complex Numbers

A complex number is of the form: $z = x + iy$ where $x, y \in \mathbb{R}$ and i is the imaginary unit such that $i^2 + 1 = 0$.

Definition 1 (Powers of i).

k	*0*	*1*	*2*	*3*	*4*	*5*
i^k	*1*	*i*	*-1*	*-i*	*1*	*i*

Theorem 1 (Fundamental Theorem of Algebra). *Any polynomial*[1] f *(except constant functions) has a root in* $\mathbb{C}$.

[1] Polynomial is a function such as: $f = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where $a_i \in \mathbb{R}$ or $\mathbb{C}$ and $n \in \mathbb{N}$.

Remark 1. *If we have a polynomial f of degree n, then it has n roots, where each root can have a multiplicity*[2]. *For example, if we have a polynomial $(x-1)^2$, it has a degree of 2 but only one root, which is 1, with a multiplicity of 2. This means that the root 1 appears twice in the polynomial.*

We can factorize a polynomial in the form of $f = a_n x^n + ... + a_1 x + a_0$ into a linear factor: $f = a(x - z_1)(x - z_2)...(x - z_n)$ where z_i are the roots of f in $\mathbb{C}$.

Using the FTA for a function such as $f = a_n x^n + ... + a_1 x + a_0$, we can say that the FTA implies that f has a root $f(z) = 0$.

2.2.2 Complex Operations

We can define operations on complex numbers as follows:

- Addition: $z + z' = (x + x') + i(y + y')$, where $x, x', y, y' \in \mathbb{R}$.
- Multiplication: $zz' = (x + iy)(x' + iy') = (xx' - yy') + i(xy' + yx')$.
- Inverse: $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$

From the definition of inverse, we can see that for any complex number z, its inverse $\frac{1}{z}$ is also a complex number. For example, take $z = 1 + i$, where $x = y = 1$, from the definition of inverse, we can conclude that:

$$\frac{1}{1+i} \in \mathbb{C}$$

Multiplying by a complex number z corresponds geometrically to

$$\begin{cases} \text{a rotation by some angle } \theta, \\ \text{a rescaling by the factor } |z|. \end{cases}$$

2.2.3 Complex Conjugate

A complex conjugate is a way to ”flip” the imaginary part of a complex number. For example, if we have a complex number $z = x+iy$, then the complex conjugate of z is $\overline{z} = x - iy$. Some basic properties of complex conjugates are:

- $\overline{\overline{z}} = z$
- $\overline{z + z'} = \overline{z} + \overline{z'}$
- $\overline{z \cdot z'} = \overline{z} \cdot \overline{z'}$

January 09, 2026.

2.2.4 Geometric and Polar Form of Complex Numbers

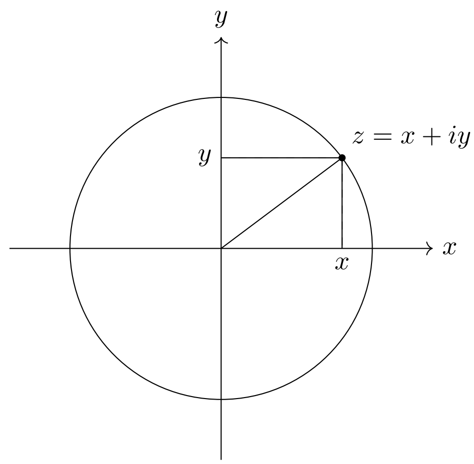
1. Geometric Interpretation

[2]The multiplicity of a root represents how many times the root occurs in the polynomial.

Definition 2 (Geometric interpretation)**.** *Every complex number $z = x + iy$ can be identified with a point (x, y) in the plane, called the complex plane.*

We define the complex plane as:

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.$$



2. Modulus and Unit Circle

Definition 3 (Modulus)**.** *The modulus of a complex number z is defined by*

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

Geometrically, $|z|$ is the distance from the origin to the point (x, y).

We can rewrite the definition of the unit circle as follows:

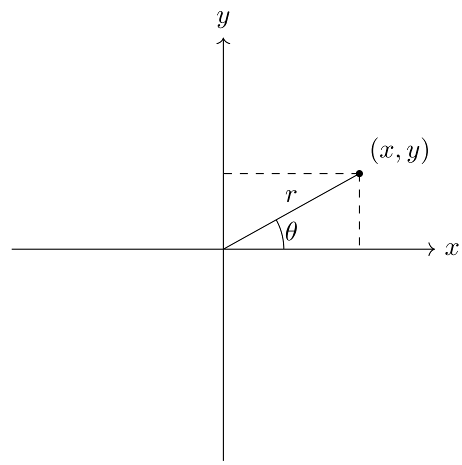
$$S' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{z \in \mathbb{C} : |z| = 1\},$$

where S' is the unit circle in the complex plane.

3. Polar Coordinates

Definition 4 (Polar coordinates)**.** *Instead of describing a point by (x, y), we may describe it using polar coordinates (r, θ), where $r = |z|$ is the distance to the origin and θ is the angle with the positive x-axis.*

Example. *Consider the point (x, y), where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. We can define (r, θ) as follows:*



Complex numbers can also be described using polar coordinates.

Definition 5 (Polar and exponential form)**.**

$$z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}.$$

We can also define multiplication in polar form:

Definition 6 (Multiplication in polar form)**.**

$$z = re^{i\theta}, \quad z' = r'e^{i\theta'}, \quad zz' = rr'e^{i(\theta+\theta')}.$$

Example.

$$(1+i)^{32} = (\sqrt{2}e^{i\pi/4})^{32} = (\sqrt{2})^{32}e^{i8\pi} = 2^{16}(\cos 8\pi + i\sin 8\pi) = 2^{16}.$$

Around the 1740, the mathematician Euler discovered a formula for complex numbers. The formula is known as Euler's formula.

Definition 7 (Euler's formula)**.**

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

This formula is quite useful when dealing with complex numbers, as it allows us to write complex numbers in a more compact form.

4. Roots

Definition 8 (n^{th} roots)**.** *An n^{th} root of z is a complex number w such that*

$$w^n = z.$$

If $z = re^{i\theta}$, then any solution of $w^n = z$ must satisfy

$$w^n = re^{i\theta}.$$

Writing $w = \rho e^{i\varphi}$, we obtain

$$\rho^n = r, \quad n\varphi = \theta + 2\pi k, \quad k \in \mathbb{Z}.$$

Hence, all n^{th} roots of z are

$$w_k = r^{1/n} e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, 2, \ldots, n-1.$$

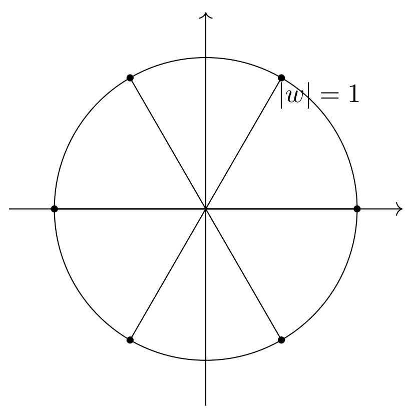
Definition 9 (Roots of unity)**.** *The n^{th} roots of unity are the solutions of*

$$w^n = (e^{i2\pi/n})^n = e^{i2\pi} = 1, \; w \in \mathbb{C}.$$

Since $1 = e^{i2\pi k}$, they are given by

$$w_k = e^{i2\pi k/n}, \; k = 0, 1, 2, \ldots, n-1 \in \mathbb{Z}.$$

Geometrically, the n^{th} roots of unity lie on the unit circle and are equally spaced.



3 Basic Algebraic structures

3.1 Sets with Multiplication

Definition 10 (Set with multiplication)**.** *A set with multiplication is a set M where you can multiply any two elements of M, and the result is still an element of M. Formally, this means there is a rule $\cdot$ that takes any pair (a, a) from M and produces an element ab that is also in M:*

$$\cdot : M \times M \to M$$

3.2 Invertibility

Definition 11 (Condition for Invertibility)**.** *Let $A \in M$ be an $n \times n$ matrix, and suppose that there exists an $n \times n$ matrix B such that $AB = I_n$ or $BA = I_n$. Where I_n is the $n \times n$ identity matrix*[3] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *Then A is invertible, and B is called the inverse of A and is denoted by $B = A^{-1}$.*

Remark 2. *If A is invertible, then A^{-1} exists and is unique*[4].

To determine if an element A in a set with multiplication M is invertible, we can use the following examples:

Example. *Let $M = \mathbb{Z} = \{... - 2, -1, 0, 1, 2, ...\}$ and $A = 2$. Is A invertible in M?*
Solution: No, because $\frac{1}{2} \notin \mathbb{Z}$.

Example. *Let $M = \mathbb{R}$ and $A = 2$, is A invertible in M?*
Solution: Yes, because $\frac{1}{2} \in \mathbb{R}$.

Example. *Is $1 + i$ invertible in $\mathbb{C}$?*
Solution: Yes, using our previous definition of inverse (2.2.2), we get that

$$\frac{1}{1+i} = \frac{1-i}{2} \in \mathbb{C}.$$

3.3 Ring

Definition 12. *A **ring** is a set $\mathbb{R}$ where you can **add** and **multiply** elements, and the following are true:*

1. *You can add any two elements and stay in $\mathbb{R}$. There is a zero, negatives exist, and the order of addition does not matter.*
2. *You can multiply any two elements and stay in $\mathbb{R}$. There is a 1, and multiplication is associative.*
3. *Multiplication distributes over addition:*

$$a(b+c) = ab + ac \quad \text{and} \quad (a+b)c = ac + bc.$$

The main example of a ring is the set of integers $\mathbb{Z}$.

[3] An identity matrix is a square matrix with 1s on its main diagonal and 0s everywhere else. It represents no change in linear transformations, and it's used in finding matrix inverses.
[4] Unique means there is exactly one such element.

3.4 Field

Definition 13. *A field is a commutative ring in which every element is invertible.*

The main example of a field is the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Mathematically, $K = \mathbb{R}$ or $\mathbb{C}$, where K is the field.
In the following explanation, we will denote M to be a set with multiplication. An example of a set with multiplication is the set of all 2×2 complex matrices: $M = M_2(\mathbb{C})$. Another example is the nonzero set of all real numbers $\mathbb{R}$ with ordinary multiplication: $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Problem 1. *Construct a field with 2 elements.*

Problem 2. *Show that if an inverse of A in $\mathbb{M}$ exists, then it is unique.*

Problem 3. *Let K be a field. Prove that this matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(K)$ is not invertible.*

4 Vector Spaces

January 09, 2026.

4.1 Cartesian vector spaces

Definition 14 (R^n)**.** *Let $n \in \mathbb{N}$. The Cartesian product of n copies of $\mathbb{R}$ is called $\mathbb{R}^n$.*

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in \mathbb{R} \right.$$

4.2 Vectors

4.2.1 Vector operations

Vector operations are defined as follows.
Addition: $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$,
scalar multiplication: $\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

Definition 15 (Linear combination)**.** *A linear combination of vectors $v_1, \dots, v_n$ is a vector v of the form $v = \lambda_1 v_1 + \ldots + \lambda_n v_n$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.*

Example. *A linear combination could look like this:*

$$\xi((v_1 + 2v_2) + v_3) + v_4,$$

where $\xi \in \mathbb{R}$.

4.2.2 Span

January 12, 2026.

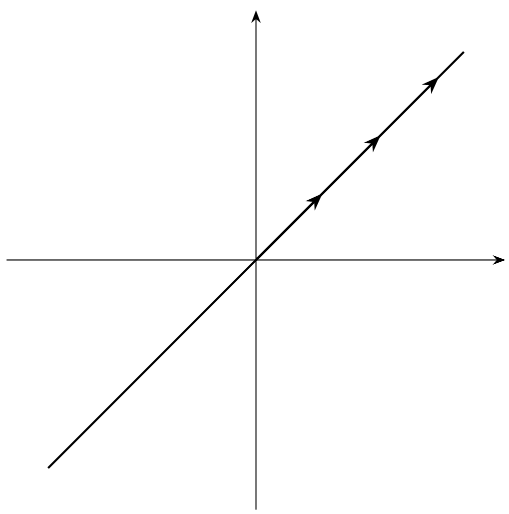
Definition 16 (Span)**.** *Let $A \subset \mathbb{R}^2$. The span of A, denoted $Span(A)$, is the set of all linear combinations of elements of A.*

Remark 3. *If $A = \{v\}$ contains one nonzero vector, then $\mathrm{Span}(v)$ is a line through the origin.*

Example. *Let $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then*

$$\mathrm{Span}(A) = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

which is a line in $\mathbb{R}^2$.



1. Span in $\mathbb{R}^n$

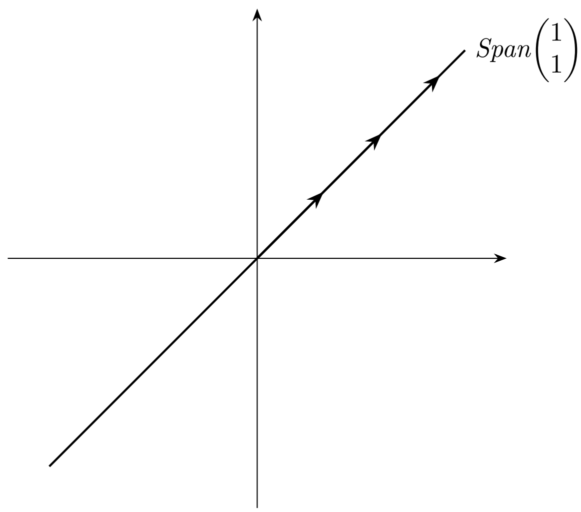
Example. *The span of n vectors $v_1, \ldots, v_n$ is the set of all linear combinations of $v_1, \ldots, v_n$. Then any vector in $\mathrm{Span}(v_1, \ldots, v_n)$ has the form*

$$Span(v_1, \ldots, v_n) = \{\lambda_2 v_1 + \ldots + \lambda_n v_n : \lambda_i \in \mathbb{R}\}$$

When working in $\mathbb{R}^n$, the span describes all points you can reach by scaling and adding the given vectors. Depending on the vectors, the span can be a line (if the vectors are dependent), a plane, or a higher-dimensional subspace. The following examples show what spans look like in $\mathbb{R}^2$.

Example. *Let* $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. *Then* $\mathrm{Span}(v_1, v_2) = \{\lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}$

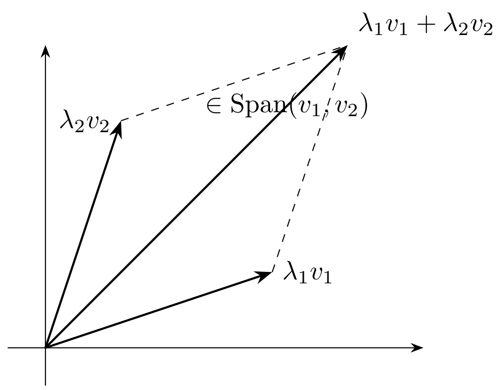
Example. *Let* $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. *Then* $\mathrm{Span}(A) = \{\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R}\}$



Example. *Let*

$$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Any vector in $\mathrm{Span}(v_1, v_2)$ *has the form* $\lambda_1 v_1 + \lambda_2 v_2$. *Geometrically, this is illustrated below.*

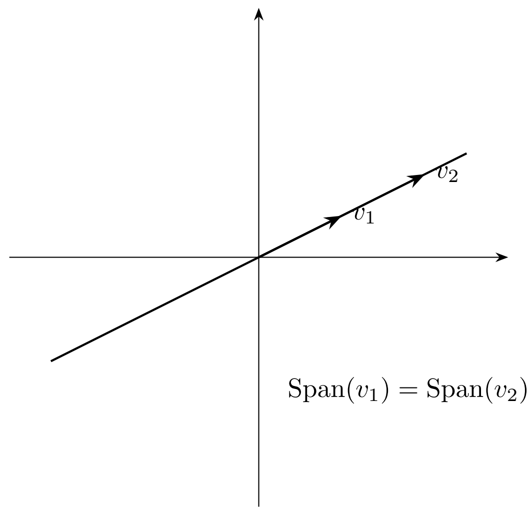


Example. *Let*

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Then $v_2 = 2v_1$, *so*

$$\mathrm{Span}(v_1) = \mathrm{Span}(v_2).$$



Span in $\mathbb{C}^n$

Definition 17. *The span over $\mathbb{C}$ of vectors $v_1, \ldots, v_n$ is*

$$Span_{\mathbb{C}}(v_1, \ldots, v_n) = \{\lambda_1 v_1 + \ldots + \lambda_n v_n : \lambda_i \in \mathbb{C}\}.$$

Example. *For example,*

$$\begin{pmatrix} 2i \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 1+i \end{pmatrix} \in \mathbb{C}^2.$$

So we distinguish between real and complex vector spaces:

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}, \qquad \mathbb{C}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{C} \right\}.$$

4.2.3 Standard Basis

Definition 18. *The standard basis for $\mathbb{R}^n$ is the set $\{e_1, \ldots, e_n\}$, where e_i is the ith standard basis vector.*

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

(Basis = span + linear independence)

Claim These vectors span $\mathbb{R}^n$.

Proof. We show that any vector in $\mathbb{R}^n$ can be written as a linear combination of $e_1, \ldots, e_n$:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

□

Example (Complex vs Real Span)**.**

$$\mathbb{C}^2 = \mathrm{Span}_{\mathbb{C}}(e_1, e_2) = \{z_1 e_1 + z_2 e_2 : z_1, z_2 \in \mathbb{C}\},$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here $\dim_{\mathbb{C}} \mathbb{C}^2 = 2$ *but* $\dim_{\mathbb{R}} \mathbb{C}^2 = 4$*. Every complex scalar can be written as* $z_k = x_k + iy_k$ *with* $x_k, y_k \in \mathbb{R}$*. So*

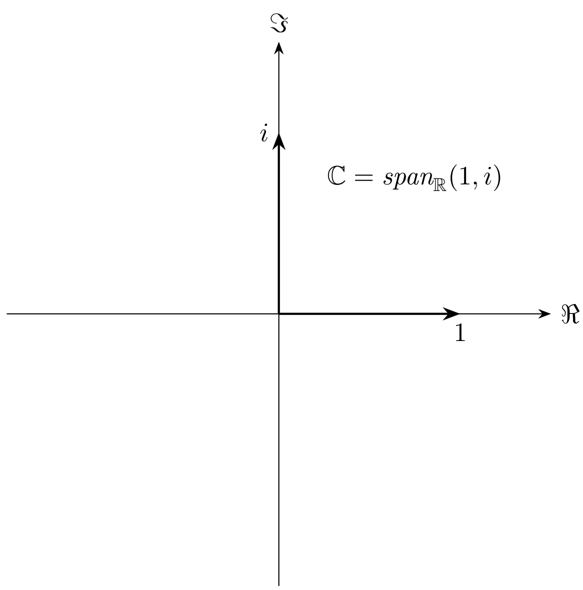
$$\mathrm{Span}_{\mathbb{C}}(e_1, e_2) = \{(x_1 + iy_1)e_1 + (x_2 + iy_2)e_2 : x_1, x_2, y_1, y_2 \in \mathbb{R}\}.$$

Separating real and imaginary parts,

$$\mathrm{Span}_{\mathbb{C}}(e_1, e_2) = x_1 e_1 + x_2 e_2 + y_1(ie_1) + y_2(ie_2).$$

Therefore,

$$\mathrm{Span}_{\mathbb{C}}(e_1, e_2) = \mathrm{Span}_{\mathbb{R}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix}\right).$$



4.2.4 Abstract Vector Spaces

An abstract vector space is just a set with two operations: vector addition (v_1+v_2) and scalar multiplication (λv), where λ comes from a field k, satisfying certain axioms.

Definition 19. *Let k be a field. A* ***vector space over*** *k is a set V together with two operations*

$$\begin{cases} v_1 + v_2 \in V, & v_1, v_2 \in V, \\ \lambda v \in V, & v \in V,\ \lambda \in k, \end{cases}$$

satisfying 8 axioms.

Axioms (1) Rules for addition

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

There exists a zero vector $0 \in V$ such that $0 + v = v$. For every $v \in V$, there exists $-v \in V$.

$$u + v = v + u$$

(2) Rules for scalar multiplication

$$1 \cdot v = v$$

$$\lambda_1(\lambda_2 v) = (\lambda_1\lambda_2)v$$

(3) Compatibility (distributive laws)

$$(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$$

$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$$

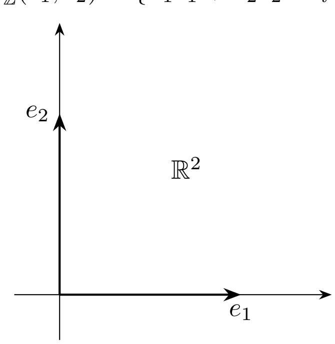
Example. *Let K be a field. Then*

$$K^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in K \right\}$$

is a vector space over K.

Remark 4.

$$\mathrm{Span}_{\mathbb{Z}}(e_1, e_2) = \{\lambda_1 e_1 + \lambda_2 e_2 : \lambda_i \in \mathbb{Z}\}.$$



14 January, 2026

4.2.5 Examples of Vector Spaces

We now list important examples of vector spaces.

1. Coordinate spaces Let k be a field ($k = \mathbb{R}$ or $k = \mathbb{C}$).

$$k^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in k \right\}$$

is a vector space over k.

The standard basis is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \dots \ , e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

2. Polynomial spaces A polynomial is an expression

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in k.$$

$$P_n(k) = \{a_n x^n + \cdots + a_1 x + a_0 : a_i \in k\}$$

is a vector space over k.

Examples:

$$1 + x^2 \in P_2(\mathbb{R}), \qquad 1 + ix^3 \in P_3(\mathbb{C}).$$

The subscript must satisfy $\deg(f) \leq n$.

All polynomials:

$$P_\infty = \{a_n x^n + \cdots + a_1 x + a_0 : a_i \in k,\ n \geq 0\}.$$

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_\infty.$$

A standard basis is $\{1, x, x^2, \dots, x^n\}$. Every $f \in P_n$ can be written

$$f = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_n x^n.$$

3. Matrix spaces

$$M_n(k) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} : a_{ij} \in k \right\}$$

is a vector space over k.

Standard basis matrices:

$$e_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \ \ldots$$

$$\dim M_n(k) = n^2.$$

Example in $M_2(\mathbb{R})$:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1e_{11} + 2e_{12} + 3e_{21} + 4e_{22}.$$

4. Function spaces Let D be a set and k a field.

$$F(D,k) = \{f : D \to k\}$$

is a vector space over k.

When $D = \mathbb{R}$ and $k = \mathbb{R}$, functions are drawn with horizontal axis D and vertical axis k.

16 January, 2026

$F(D)$ is a function from D to K, where $K = \mathbb{R}$ *or* $\mathbb{C}$. No "standard basis" in general.

Standard basis if D is a finite set *eg* $D = \{1, \ldots, n\}$.

$$(f \in F(D).f(1) = 2, f(2) = 4, \ldots, f(n) = 2^n$$

Kronecker function: Let $x \in D$, $\delta_x \in F(D)$, defined by $\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

Then $\{\delta_x : x \in D\}$ is a standard basis of $F(D)$.

Standard basis :$(\delta_1, \ldots, \delta_n)$, for $F(\{1, \ldots, n\})$.

Remark: $\{\delta_x : x \in \mathbb{R}\}$ do not span $F(\mathbb{R})$.

Proof that $(\delta_1, \ldots, \delta_n)$ spans $F\{1, \ldots, n\}$. Take any

$$f \in F(D)$$

Find $a_1, \ldots, a_n \in K$ such that

$$f = a_1\delta_1 + \ldots + a_n\delta_n$$

$$f(x), \quad x \in D = \{1, \ldots, n\}$$

$$g(1) = a_1\delta_1(1) + a_2\delta_2(1) + \ldots + a_n\delta_n(1) = a_1$$

Claim: $a_k = f(K) \in K$ Claim: $f = \sum f(k)\delta_k$, where $f(k) \in K$ and δ_k is the basis vector.

Proof:

$$g(l) = f(1)\delta_1(l) + \ldots + f(l)\delta_l(l) + \ldots + f(n)\delta_n(l) = f(l)$$

Subspace criterion. Let V be a vector space. To check a subset $u \subseteq V$ is a vector space. In principle need to checck: 1. U is stable under addition $u, v \in U$ Hence $u + v \in U$. 2. U is stable under scalar multiplication. $\lambda \in K, u \in U$ Hence $\lambda u \in U$. 3. 8 axioms hold in U.

Proposition. A subset $U \subseteq V$ is a vector space (8 axioms are true) if f: (0) U is not empty ($0 \in U$) (1) U is stable under addition: $u, v \in U \to u + v \in U$.

January 19, 2026

(2) U is stable under scalar multiplication: $u \in U, \lambda \in K \to \lambda u \in U$.

Subspaces

Problem 4 (Subspace criterion)**.** *Let $A \in M_n(K)$ be a fixed matrix. Prove thet*

$$U = \{x \in K^n : Ax = \vec{0}\}$$

is a subspace, null space or kernel.

Problem 5 (Subspace criterion 2)**.** *The set $U = \{Ax : x \in K^n\}$ is a subspace, image of A.*

Problem 6 (Subspace criterion 3)**.** *Let $V = K$ (vector space of dim 1) show that the only 2 subspaces of V are $\{0\}$ and V itself.*

Problem 7 (Subspace criterion 4)**.** *$Span(v_1, \ldots, v_n) = \{\lambda_1 v_1 + \ldots + \lambda_n v_n : \lambda_1, \ldots, \lambda_n \in K\}$ is a subspace.*

Problem 8. *$Span(v_1, \ldots, v_n)$ is the smallest subspace c*

January 21, 2026

Problem 9 ($K = \mathbb{R}$)**.** *Is $v_i \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix}$ in the space of $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ In otherwords: are there $x, y, z \in \mathbb{R}$ such that $v = xv_1 + yv_2 + zv_3$?*

Problem 10. *Is $f : 3x^2 + 5x - 5$ in the span of $f_1 = x^2 + 2x + 1, f_2 = 2x^2 + 5x + 4, f_3 = x^2 + 3x + 6$?*

$$f = xf_1 + yf_2 + zf_3$$

Intersection, sums, direct sums. W vector space. $U, V \subseteq W$ two subspaces. Definition: $U \cap V = \{v \in W : v \in U \text{ and } v \in V\}$ smallest intersection $\{0\}$.

The union of 2 subspaces in NOT a subspace (in general).

$W = \mathbb{R}^2$: draw a cartesian plane with vertical being V and horizontal being U and draw the cross $(u + v)$. $U \cup V =$ cross, no quadrant, not a subspace

Sum of 2 subspaces: $U + V = \{u + v : u \in U, v \in V\}$ line U + line V = plane

Propoisition: U, V are subspaces of W. $U + V$ is a subspace. U cannot rescale a vector space, therefore $U + U = U$. Proof: $\subseteq$: Let $u + u' \in U + U$ prove $u + u' \in U$. Yes subce U is a subspace stable under addition. $\supseteq$: Take

$u \in U$, prove $u \in U + U$. $u = u + 0 \in U + U$. Proof: subspace criterion. (0) $0 \in U + V$ since $0 \in U, 0 \in V$. (1) Stable under addition: let $w_1, w_2 \in U + V$. Prove $w_1 + w_2 \in U + V$.

$$w_1 = u_1 + v_1, u_1 \in U, v_1 \in V$$

$$w_2 = u_2 + v_2, u_2 \in U, v_2 \in V$$

$$w_1 + w_2 = (u_1 + u_2) + (v_1 + v_2)$$

$$u_1 + u_2 \in U, v_1 + v_2 \in V$$

$$w_1 + w_2 \in U + V$$

(2) Stable under scalar multiplication: let $w \in U + V, \lambda \in K$. Prove $\lambda w = \lambda u + \lambda v \in U + V$.

Prop: $U + V = span(U \cup V)$. $U + V =$ smalles subspace containing $U \cup V$. Proof: $\subseteq$: Take $w \in U + V$ prove $w \in span(U \cup V)$.

$$w = u + v, u \in U, v \in V$$

Prove $w \in span(U \cup V)$. (You could also write it like $span(U, V)$ since line can be of both in U or V). $\supseteq$: Take $w \in span(U \cup V)$ prove $w \in U + V$.

$$w = a_1 u_1 + \ldots + a_n u_n + b_1 v_1 + \ldots + b_m v_m$$

where $u_i \in U, v_j \in V, a_i, b_j \in K$.

$$w = (a_1 u_1 + \ldots + a_n u_n) + (b_1 v_1 + \ldots + b_m v_m)$$

$$w = u + v, u \in U + V$$

5 Appendix

6 Solutions

Solution 1. *A field with 2 elements can be constructed as follows: Let* $F = \{0, 1\}$ *be a set with two elements. We define addition and multiplication operations on* F *as follows:*

- $0 + 0 = 0$
- $0 + 1 = 1$
- $1 + 0 = 1$
- $1 + 1 = 0$
- $0 \times 0 = 0$
- $0 \times 1 = 0$

- $1 \times 0 = 0$
- $1 \times 1 = 1$

Solution 2. *Suppose B and B' are both inverses of A. Then*

$$B = BI = B(AB') = (BA)B' = IB' = B'.$$

Therefore, $B = B'$, so the inverse is unique.

Solution 3. *We can answer this problem with proof by contradiction. Let's suppose this matrix is invertible. By definition there exists $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We can rewrite this equation into: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{-1}$. The inverse of our matrix can be rewritten as $\frac{1}{0*0-1*0} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$[5]. But this is undefined since division by 0 is undefined. Therefore, our initial assumption that the matrix is invertible is false, and thus the matrix is not invertible.*

Solution 4 (Subspace criterion)**.** *(0) is $0 \in U$? Yes, because $A0 = 0$.*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(1) Addition: Let $x, y \in U$. Prove $x + y \in U$.

$$A(x + y) = Ax + Ay = 0 + 0 = 0$$

(2) Scalar multiplication: Let $x \in U, \lambda \in K$. Prove $\lambda x \in U$.

$$A(\lambda x) = \lambda Ax = \lambda 0 = 0$$

Solution 5 (Subspace criterion 2)**.** *(0) is $0 \in U$? Find $x \in K^n$ such that $Ax = 0$. $x = 0$.*

(1) Stability under addition: $y_1, y_2 \in U \to y_1 + y_2 \in U$. There exist x, such that $y_1 = Ax_1, y_2 = Ax_2$.

$$y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2) \in U$$

(2) Stability under scalar multiplication: If $f = Ax$, for some x, $\lambda y = A(\lambda x)$

Solution 6 (Subspace criterion 3)**.** *Let $U \subseteq \mathbb{R}$ be a subspace. Prove: $U = \{0\}$ or $U = \mathbb{R}$.*

Case 1: $U = \{0\}$.

Case 2: $U \neq \{0\} \to$ there exists $v \in \mathbb{R}$, $v \neq 0$. Prove: $U = \mathbb{R}$. Let $x \in \mathbb{R}$ be any real number. Prove: $x \in U$. Since U is a subspace, it is stable under scalar multiplication.

$$x \in U \to Ax \in U$$

Therefore, $x \in U$.

[5] Recall that an inverse of a 2×2 matrix is equal to its determinant multiplied with its conjugate

Solution 7 (Subspace criterion 4)**.** *(0)* $0 \in span(v_1, \dots, v_n)$*, because*

$$\lambda_1 = \dots \lambda_n = 0 \qquad \lambda_1 v_1 + \dots + \lambda_n v_n = 0.$$

(1) addition: $u, v \in span$ *Prove* $u + v \in span.$

$$u = a_1 v_1 + \dots + a_n v_n$$

$$v = b_1 v_1 + \dots + b_n v_n$$

$$u + v = (a_1 + b_1) v_1 + \dots + (a_n + b_n) v_n \in span(v_1, \dots, v_n)$$

(2) scalar multiplication:

Solution 8.

$$\begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} + z \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{cases} x + 2y + z = 3 \\ 2x + 5y + 3z = 5 \\ x + 4y + 6z = -5 \end{cases} \qquad \to x = 3, y = 1, z = -2$$

A: yes since

$$v = 3v_1 + v_2 + 2v_3$$

7 Useful Links