

# MATH 325: Honours Ordinary Differential Equations

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**Abstract**

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# 1 Introduction

Jean-Philippe Lessard (Burnside 1119). Tutorials every wednesday from 9am to 10am, ENGTR 0070, with Eunpyo Bang. Office hours thursday. No textbooks. 25% assignments (2 written assignments 15%, and 5 webworks 10%). 25% Midterm (February 16 - inclass). 50% Final. Since its honours you will deal with analysis.

## 2 Prerequisite knowledge

### 2.1 Analysis

## 3 Intro, Classification, Theorem of Existence & Uniqueness

### 3.1 Intro

January 06,  
2026.

**Definition 1** (Differential Equation). *A differential equation (DE) is a relation that involves an unknown function and some of its derivatives.*

To better understand what a differential equation is, consider the following example.

Imagine a ball of mass  $m$  falling, subject to gravity and air resistance (drag). Denote by  $v(t)$  the velocity of the ball at time  $t$ , whereas  $t$  is the independent variable, and  $v$  the dependent variable. Let the downward direction be positive. We know the force of gravity is given by  $F_g = mg$ , where  $g$  is the acceleration due to gravity. The drag force is given by  $F_d = -\lambda v$ , where  $\lambda$  is the drag coefficient and is  $\lambda \geq 0$ . According to Newton's second law  $\sum F = ma$ , the net force acting on the ball is equal to its mass times its acceleration

$$m \frac{dv}{dt} = mg - \lambda v.$$

Let  $y(t)$  be the position, meaning  $v(t) = \frac{dy}{dt}$ . Then, we can rewrite the above equation as

$$my'' + \lambda y' = mg.$$

Let's analyze another example, population growth (known as the Malthusian growth model).

Denote by  $N(t)$  the size of a given population at time  $t$ . In an "unconstraint" environment, it is reasonable to assume that the rate of change of the number of individuals is proportional to the number of individuals present. This assumption leads to the following differential equation:

$$\frac{dN}{dt} = rN,$$

where  $r$  is called the growth rate (if  $r > 0$ ), and decay rate (if  $r < 0$ ). Assume that  $N > 0$ . Using the chain rule and assuming that  $N(t)$  satisfies  $N' = rN$

$$\frac{d}{dt} \ln(N(t)) = \frac{d\ln(N)}{dN} \cdot \frac{dN}{dt} = \frac{1}{N} \cdot N' = r,$$

integrate with respect to  $t$

$$\ln(N(t)) = rt + C,$$

where  $C$  is the constant of integration. Exponentiating both sides, we obtain

$$N(t) = e^{\ln(N(t))} = e^C e^{rt} = k e^{rt},$$

where  $\{k > 0 | k \in \mathbb{R}\}$  which could be any positive constant is the initial population size at time  $t = 0$ .

Assume that an initial population (condition) is given:

$$N(0) = N_0 (\text{fixed}),$$

we therefore get that  $k = N_0$ , and the unique solution that satisfies the initial condition is

$$N(t) = N_0 e^{rt}.$$

The problem with the answer we got in the previous example is that it is not realistic in the long run, how about we consider a carrying capacity<sup>1</sup>. This leads us to another example: Population growth/decay with the carrying capacity of the environment.

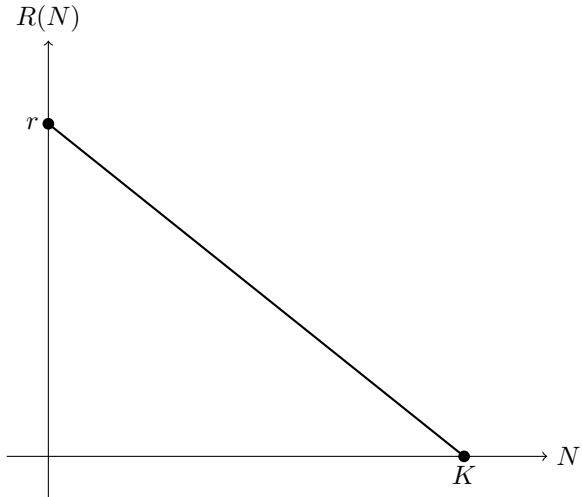
Now assume that our growth rate depends on the population size  $N(t)$  itself, therefore we get that

$$\frac{dN}{dt} = R(N)N.$$

Denote by  $K$  the number of individual that the environment can carry.  $K$  is called the carrying capacity of the environment. If  $N < K$ , we want growth ( $R(N) > 0$ ) and if  $N > K$ , we want decay ( $R(N) < 0$ ).

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<sup>1</sup>maximum population size that the environment can sustain indefinitely



Let's pick the simplest function  $R(N)$  that satisfies  $R(0) = r$ ,  $R(K) = 0$  and is linear. We get that

$$R(N) = r\left(1 - \frac{N}{K}\right).$$

Therefore, our differential equation becomes

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N = \frac{r}{K}(K - N(t))N(t).$$

This is called the logistic equation.

**Definition 2** (Ordinary Differential Equation). *An ordinary differential equation (ODE) is a differential equation whose unknown function depends on one independent variable only.*

Example of ODEs:

- $y''(t) + y'(t) + 2y(t) = \sin(t)$
- $N'(t) = rN(t)$
- $mv'(t) = mg - \lambda v(t)$
- $y'(x) + 3y(x) = e^x$

**Definition 3** (Partial Differential Equation). *A partial differential equation (PDE) is a differential equation whose unknown function depends on more than one independent variable. Will not be taught in this course.*

Example of a PDE is the Heat Equation. Let  $u = u(x, t)$ ,  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ . This PDE denotes the temperature of a body at time  $t$  and at position  $x$ .

## 3.2 Classification

### 3.2.1 The Order

**Definition 4.** *The order of an ODE is the order of the highest derivative that appears in the equation.*

**Example.**  $N' = rN$  (first order ODE)

**Example.**  $y''(t) + 2y'(t) = e^t$  (second order ODE)

### 3.2.2 Dimension of the State

**Definition 5** (Scalar ODE). *A scalar ODE has one unknown function.*

**Example** (Scalar ODE).

$$y'' + y = 0$$

**Definition 6** (System of ODEs). *A system of ODEs has several unknown functions.*

**Example** (System of ODEs).

$$\begin{cases} y'_1 = y_2 \\ y'_2 = -y_1 \end{cases}$$

**Definition 7** (Systems of first order ODEs). *A system of first order ODEs is just a collection of differential equations where all derivatives are first order. It can be written compactly as*

$$y'(t) = f(y(t), t),$$

where  $y(t)$  is a vector of unknown functions,

$$y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix},$$

and  $f$  gives the right-hand sides of the equations. Writing this out means

$$\begin{cases} y'_1(t) = f_1(y_1, \dots, y_n, t) \\ \vdots \\ y'_n(t) = f_n(y_1, \dots, y_n, t). \end{cases}$$

Any single  $n$ th-order ODE can always be turned into such a system. If

$$y^{(n)}(t) = G(t, y, y', \dots, y^{(n-1)}),$$

define new variables

$$y_1 = y, \quad y_2 = y', \quad \dots, \quad y_n = y^{(n-1)}.$$

Then each new variable has a first derivative, and the equation becomes

$$\begin{cases} y'_1 = y_2, \\ y'_2 = y_3, \\ \vdots \\ y'_{n-1} = y_n, \\ y'_n = G(t, y_1, \dots, y_n). \end{cases}$$

So one higher-order equation is the same thing as many first order equations.

**Problem 1** (System of First Order ODEs). Rewrite this third order ODE as a first order system:

$$y''' + 4y' - y = 0$$

**Example** (System of First Order ODEs 2). Consider the second-order ODE

$$\begin{aligned} y'' + 2y' + y &= e^t \\ \implies y'' &= -2y' - y + e^t. \end{aligned}$$

Define new variables

$$y_1 = y, \quad y_2 = y'.$$

Then the system becomes

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad r(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

**Example** (Lorenz system).

$$\begin{aligned} y'_1 &= \sigma(y_2 - y_1), \\ y'_2 &= \rho y_1 - y_2 - y_1 y_3, \\ y'_3 &= y_1 y_2 - \beta y_3, \end{aligned}$$

where  $\sigma, \rho, \beta$  are parameters. This is a nonlinear first order system in  $\mathbb{R}^3$ .

### 3.2.3 Linearity

**Definition 8** (Linearity). A linear Ordinary Differential Equation (ODE) is one where the unknown function ( $y$ ) and its derivatives ( $y', y'', \dots$ ) appear only to the first power, are not multiplied together, and are not part of special functions like  $\sin(y)$  or  $e^y$ . Essentially, they are "simple" combinations (addition/subtraction) of  $y$  and its derivatives, potentially multiplied by functions of the independent variable (like  $x$  or  $t$ ).

**Example** (Linearity). Consider the following ODEs:

- Linear:  $y' + 3y = 0$
- Linear:  $y'' - 2xy' + y = \cos(x)$
- Non-linear:  $y' + y^2 = 0$
- Non-linear:  $y'' + \sin(y) = 0$

### 3.2.4 Autonomy

**Definition 9** (Autonomy). The  $n^{th}$  order ODE  $F(t, y, y', \dots, y^{(n)}) = 0$  is autonomous if  $F$  does not depend explicitly on  $t$ , that is, if it is of the form  $F(y, y', \dots, y^{(n)}) = 0$ . Otherwise, it is non-autonomous.

**Example** (Autonomy). Consider the following ODEs:

- Non-autonomous:  $y'' + 2y' + y - e^t = 0$
- Autonomous:  $N'(t) = rN(t)$
- Non-autonomous:  $y'(t) = ty(t)$

Equivalently, a first-order system  $y' = f(y, t)$  is autonomous if it can be written as

$$y' = f(y).$$

Otherwise, it is non-autonomous.

**Example.** A classic, simple example of an autonomous first-order system is the linear growth/decay model:

$$\frac{dy}{dt} = ky$$

Here,  $f(y, t) = ky$ , which depends only on  $y$  (where  $k$  is a constant), making it autonomous. Other examples include  $\frac{dy}{dt} = 1 - y^2$  or exponential growth  $\frac{dy}{dt} = 0.5y$ .

**Remark 1.** The Lorenz system is an example of an autonomous system<sup>2</sup>.

### 3.2.5 Solutions of ODEs

**Definition 10** (Solutions of ODEs). Let  $f : D \times (a, b) \rightarrow \mathbb{R}^n$ . A solution of  $y'(t) = f(y(t), t)$  on an interval  $J \subset \mathbb{R}$  is a differentiable function  $y : J \rightarrow D \subset \mathbb{R}^n$ , such that  $y'(t) = f(y(t), t)$ ,  $\forall t \in J$ .  $t$  is the independent variable, and  $y = (y_1, \dots, y_n)$  is the dependent variable.

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<sup>2</sup>Here “system” means the unknown  $y$  is vector-valued, e.g.  $y \in \mathbb{R}^m$ , rather than scalar.

## 1. Explicit Solutions

**Example** (Explicit Solution). Consider the following ODE

$$y' + y = 1.$$

We can verify that  $y(t) = e^{-t} + 1$ , and therefore  $y'(t) = -e^{-t}$ , is a solution on  $\mathbb{R}$ . Indeed,

$$y' + y = -e^{-t} + (e^{-t} + 1) = 1.$$

In this example,  $y = y(t)$  is explicitly given as a function of  $t$  (independent variable).

## 2. Implicit solutions

**Example** (Implicit Solution). Consider the ODE

$$y \frac{dy}{dx} = x.$$

This is a nonautonomous, nonlinear, first-order scalar ODE. Separating variables gives

$$y dy = x dx.$$

Integrating,

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C,$$

or equivalently, the implicit solution

$$x^2 - y^2 = C, \quad C \in \mathbb{R}.$$

To verify, differentiate implicitly:

$$\frac{d}{dx}(x^2 - y^2) = 0 \implies 2x - 2y \frac{dy}{dx} = 0 \implies y \frac{dy}{dx} = x.$$

## 3.3 Initial Value Problems

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**Definition 11** (Initial Value Problem (IVP)). An initial value problem (IVP) is a system of ODEs with an initial condition.

**Example** (IVP). Consider the following ODE

$$y' = f(y, t), \quad f : D \times (a, b) \rightarrow \mathbb{R}^n.$$

Let  $t_0 \in (a, b)$ . An initial condition is

$$y(t_0) = y_0 \in \mathbb{R}^n.$$

An initial value problem (IVP) is

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}.$$

## 3.4 Existence and Uniqueness Theorem

### 3.4.1 Lipschitz continuity

**Definition 12** (Lipschitz continuity). Let  $D \subseteq \mathbb{R}^n$  and let  $\|\cdot\|$  be a norm on  $\mathbb{R}^n$ . A function  $f : D \rightarrow \mathbb{R}^n$  is Lipschitz continuous if there exists  $L \geq 0$  such that

$$z\|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\| \quad \forall y_1, y_2 \in D, z \in \mathbb{R}.$$

The smallest such  $L$  is called the Lipschitz constant and is denoted  $\text{Lip}(f)$ .

**Example.** Let  $f(y) = 4y - 5$ ,  $D = \mathbb{R}$ , and  $\|\cdot\| = |\cdot|$ . Then

$$|f(y_1) - f(y_2)| = |4y_1 - 4y_2| = 4|y_1 - y_2|.$$

So  $f$  is Lipschitz with  $\text{Lip}(f) = 4$ .

**Example.** Let

$$f(y) = \frac{1}{y-1}, \quad f'(y) = -\frac{1}{(y-1)^2}, \quad D = (1, +\infty).$$

This function is not Lipschitz continuous on  $D$  since as  $y \rightarrow 1^+$ , the derivative  $f'(y)$  approaches  $-\infty$ , and so the ratio  $\frac{|f(y_2) - f(y_1)|}{|y_2 - y_1|}$  approaches  $\infty$  for any  $y_1, y_2 \in D$ . Therefore, there is no Lipschitz constant for  $f$  on  $D$ .

To make this function Lipschitz continuous, we fix  $\delta > 1$  and define  $D_\delta = (\delta, +\infty)$ . For  $y_1, y_2 \in D_\delta$ , by the Mean Value Theorem, there exists  $z \in (y_1, y_2)$  such that

$$f(y_2) - f(y_1) = f'(z)(y_2 - y_1).$$

So

$$|f(y_2) - f(y_1)| \leq \frac{1}{(\delta-1)^2} |y_2 - y_1| \leq \frac{1}{(\delta-1)^2} |y_2 - y_1|.$$

Thus  $f$  is Lipschitz on  $D_\delta$  with

$$\text{Lip}(f) = \frac{1}{(\delta-1)^2}.$$

**Remark 2** (How to choose a Lipschitz constant). If  $f$  is differentiable on  $D \subset \mathbb{R}$  and its derivative satisfies

$$m \leq f'(y) \leq M \quad \text{for all } y \in D,$$

then

$$|f'(y)| \leq \max\{|m|, |M|\} \quad \text{for all } y \in D,$$

and  $f$  is Lipschitz on  $D$  with Lipschitz constant

$$L = \max\{|m|, |M|\}.$$

In practice, the Lipschitz constant is any uniform bound on  $|f'|$ .

**Example.** Suppose  $f$  is differentiable and satisfies

$$-4 \leq f'(y) \leq 9 \quad \text{for all } y \in \mathbb{R}.$$

Then

$$|f'(y)| \leq 9 \quad \forall y,$$

so  $f$  is Lipschitz on  $\mathbb{R}$  with Lipschitz constant  $L = 9$ .

### 3.4.2 Local Lipschitz continuity

**Definition 13** (Locally Lipschitz). Let  $D \subseteq \mathbb{R}^n$  be open. A function  $f : D \rightarrow \mathbb{R}^n$  is called locally Lipschitz if, around every point in  $D$ , there is some neighborhood where  $f$  is Lipschitz. Equivalently, for every compact set  $K \subset D$ , there exists a constant  $L > 0$  such that

$$\|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\| \quad \text{for all } y_1, y_2 \in K.$$

**Remark 3.** Lipschitz continuity and local Lipschitz continuity are not the same.

$$\text{Lipschitz} \Rightarrow \text{locally Lipschitz}, \quad \text{but not conversely.}$$

**Example** (Locally Lipschitz but not Lipschitz).

$$f(y) = y^2 \in \mathbb{R}.$$

Since  $f'(y) = 2y$  is unbounded on  $\mathbb{R}$ , no single constant works on the whole domain, so  $f$  is not Lipschitz. However, on any bounded set  $K = [-M, M]$ ,  $|f'(y)| \leq 2M$ , so  $f$  is locally Lipschitz.

**Example** (Lipschitz (hence locally Lipschitz)).

$$f(y) = \sin y.$$

Since  $|f'(y)| = |\cos y| \leq 1$ ,  $\forall y$ ,  $f$  is Lipschitz on  $\mathbb{R}$ , with  $\text{Lip}(f) = 1$ .

**Example** (Continuous but not locally Lipschitz).

$$f(y) = \sqrt{|y|}.$$

The derivative is unbounded near  $y = 0$ , so no Lipschitz constant exists even locally. Hence  $f$  is not locally Lipschitz.

### 3.4.3 Existence and Uniqueness Theorem

**Theorem 1** (Existence and Uniqueness). Let  $D \subseteq \mathbb{R}^n$  be open and let  $(a, b)$  be an open interval containing  $t_0$ . Consider the IVP

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}$$

Assume  $f : D \times (a, b) \rightarrow \mathbb{R}^n$  is continuous and locally Lipschitz in  $y$ .<sup>3</sup> If  $y_0 \in D$ , then there exists an open interval  $J$  containing  $t_0$  on which a solution exists. Moreover, this solution is unique on  $J$ .

**Problem 2** (Existence and Uniqueness Theorem). Consider the IVP

$$\begin{cases} y' = \sqrt{1 + y^2} + t^2 \\ y(1) = 0. \end{cases}$$

1. Identify  $f(y, t)$ .
2. Decide whether the hypotheses of the Existence and Uniqueness Theorem are satisfied.
3. State clearly what the theorem guarantees about solutions near  $t = 1$ .

Do not solve the ODE. Simply analyze it.

#### 3.4.4 Integral form of solutions

**Lemma 2.** A function  $y$  solves the IVP if and only if

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

*Proof.* If  $y' = f(y, t)$  and  $y(t_0) = y_0$ , then by the Fundamental Theorem of Calculus,

$$y(t) - y(t_0) = \int_{t_0}^t f(y(s), s) ds,$$

so

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

Conversely, differentiating the right-hand side gives

$$y'(t) = f(y(t), t), \quad y(t_0) = y_0.$$

□

#### 3.4.5 Picard operator

**1. Setup and notation** Let  $(y_0, t_0) \in D \times (a, b)$ . Since this set is open, there exist  $\alpha, \delta > 0$  such that

$$D_{\alpha, \delta} = \{(y, t) : \|y - y_0\| \leq \alpha, |t - t_0| \leq \delta\} \subset D \times (a, b).$$

---

<sup>3</sup> $D \times (a, b)$  denotes the set of all pairs  $(y, t)$  with  $y \in D \subset \mathbb{R}^n$  and  $t \in (a, b)$ , i.e., all allowed state-time inputs of  $f$ .

Define

$$M_{\alpha,\delta} = \sup_{(y,t) \in D_{\alpha,\delta}} \|f(y, t)\| < +\infty.$$

Let

$$\epsilon = \min \left( \delta, \frac{\alpha}{M_{\alpha,\delta}} \right), \quad J = (t_0 - \epsilon, t_0 + \epsilon).$$

## 2. Definition of the Picard Operator

**Definition 14** (Picard Operator). *For any function  $y : J \rightarrow \mathbb{R}^n$  such that  $y(t_0) = y_0$  and  $(y(t), t) \in D_{\alpha,\delta}$  for all  $t \in J$ , define*

$$T(y)(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

*The map  $T$  is called the Picard operator.*

**Example** (Picard operator). *Consider the IVP  $y' = y$ ,  $y(0) = 1$ . Define the Picard operator by*

$$T(y)(t) = 1 + \int_0^t y(s) ds.$$

*Take a trial function  $y_0(t) = 0$ . Then  $T(y_0)(t) = 1 + \int_0^t 0 ds = 1$ , so  $T(y_0) \neq y_0$ . Apply  $T$  again: let  $y_1(t) = 1$ . Then  $T(y_1)(t) = 1 + \int_0^t 1 ds = 1 + t$ , so  $T(y_1) \neq y_1$ . The true solution is  $y(t) = e^t$ . Indeed,  $T(y)(t) = 1 + \int_0^t e^s ds = e^t$ , so  $T(y) = y$ . Therefore,  $T$  is an operator that maps a function to a new one, and only the true solution is a fixed point of  $T$ .*

**3. Interpretation** The operator  $T$  takes a function  $y(t)$  and produces a new one. A function  $y$  solves the IVP if and only if  $T(y) = y$ , i.e.,  $y$  is a fixed point of  $T$ .

## 4. Basic Property

**Lemma 3** (Picard operator). *If  $y(t_0) = y_0$  and  $(y(t), t) \in D_{\alpha,\delta}$  for all  $t \in J$ , then  $T(y)(t_0) = y_0$  and  $(T(y)(t), t) \in D_{\alpha,\delta}$  for all  $t \in J$ .*

*Proof.* Clearly  $T(y)(t_0) = y_0$ . For  $t \in J$ ,

$$\|T(y)(t) - y_0\| \leq \int_{t_0}^t \|f(y(s), s)\| ds \leq M_{\alpha,\delta}|t - t_0| \leq M_{\alpha,\delta}\epsilon \leq \alpha.$$

Hence  $(T(y)(t), t) \in D_{\alpha,\delta}$ . □

---

<sup>4</sup>sup means the largest value in a set of numbers.

## 5. Invariant properties

**Lemma 4** (Invariant Properties). *If  $y : J \rightarrow \mathbb{R}^n$  satisfies*

1.  $y(t_0) = y_0$ ,
2.  $(y(t), t) \in D_{\alpha, \delta}$  for all  $t \in J$ ,

*then  $T(y) : J \rightarrow \mathbb{R}^n$  satisfies the same properties.*

**6. Picard iterations** Define  $y_0(t) = y_0$  (constant function), which clearly satisfies (1) and (2). For  $k \geq 1$ , define

$$y_k(t) = T(y_{k-1})(t) = y_0 + \int_{t_0}^t f(y_{k-1}(s), s) ds.$$

**4. Existence** The Picard iterations converge uniformly to a function  $y : J \rightarrow \mathbb{R}^n$  which satisfies (1) and (2), and is a solution of the IVP. To see the proof, refer to (11).

## 5. Uniqueness

*Proof of Uniqueness.* Assume  $y(t)$  and  $z(t)$  are two solutions of the IVP with the same initial condition. Then

$$\|y(t) - z(t)\| \leq \int_{t_0}^t \|f(y(s), s) - f(z(s), s)\| ds \leq L \int_{t_0}^t \|y(s) - z(s)\| ds,$$

for some Lipschitz constant  $L > 0$ . Define  $g(t) = \int_{t_0}^t \|y(s) - z(s)\| ds$ , so  $g'(t) \leq Lg(t)$  and  $g(t_0) = 0$ . Multiplying by  $e^{-L(t-t_0)}$  gives  $\frac{d}{dt}(e^{-L(t-t_0)}g(t)) \leq 0$ , which shows  $e^{-L(t-t_0)}g(t)$  is decreasing and thus  $g(t) = 0$  for all  $t$ . Hence  $\|y(t) - z(t)\| = 0$  and  $y(t) = z(t)$ , meaning two solutions with the same initial condition cannot diverge, this proves the solution is unique.  $\square$

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## 6. Examples

**Remark 4** (How  $\varepsilon$  is chosen in the theorem). *To apply the Existence and Uniqueness Theorem, we choose  $\varepsilon$  so that the solution remains inside the domain  $D \times (a, b)$ . Consider the IVP*

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}$$

*The theorem guarantees that there exists a unique solution*

$$y : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R}^n.$$

Choose  $\alpha > 0$  and  $\delta > 0$  such that

$$D_{\alpha,\delta} := \{(y, t) : \|y - y_0\| \leq \alpha, |t - t_0| \leq \delta\} \subset D \times (a, b).$$

Since  $f$  is continuous, define

$$M_{\alpha,\delta} = \sup_{(y,t) \in D_{\alpha,\delta}} \|f(y, t)\| < \infty.$$

Then one may take

$$\varepsilon = \min\left(\delta, \frac{\alpha}{M_{\alpha,\delta}}\right).$$

5

**Problem 3** (Existence and Uniqueness Theorem 2). Consider the initial value problem

$$\begin{cases} y' = y + 1, \\ y(0) = 1, \end{cases} \quad \text{with } t_0 = 0, y_0 = 1.$$

Use the Existence and Uniqueness Theorem to determine an interval on which a unique solution is guaranteed to exist. Compute  $\varepsilon$  explicitly using the theorem.

**Problem 4** (Existence and Uniqueness Theorem 3). Consider the initial value problem

$$\begin{cases} y' = y^2, \\ y(0) = 1. \end{cases}$$

Use the Existence and Uniqueness Theorem to estimate an interval of existence.

**Problem 5** (Existence and Uniqueness Theorem 4). Consider the first order differential equation

$$y' + \frac{t}{t^2 - 25}y = \frac{e^t}{t - 9}$$

For each of the initial conditions below, determine the largest interval  $a < t < b$  on which the existence and uniqueness theorem for first order linear differential equations guarantees the existence of a unique solution.

- $y(-7) = -2.1$ .
- $y(-1.5) = -3.14$ .
- $y(8.5) = 6.4$ .
- $y(13) = -0.5$ .

**Problem 6** (LLC). Consider the initial value problem

$$\begin{cases} y' = 3y^{2/3}, \\ y(0) = 0. \end{cases}$$

Investigate existence and uniqueness.

---

<sup>5</sup>In this course, you are only expected to determine  $\varepsilon$  from the theorem, and not to justify the construction further.

## 4 First-Order Scalar Equation

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A first-order scalar differential equation has the form

$$y' = f(y, t) \in \mathbb{R} \quad (n = 1)$$

### 4.1 First order linear equations

**Definition 15** (First Order Linear Equation). *A first-order linear equation is*

$$a_0(t)y' + a_1(t)y = g(t),$$

where  $a_0, a_1, g$  are functions of  $t$  and  $a_0(t) \neq 0$ . Dividing by  $a_0(t)$  gives the standard form

$$y' + p(t)y = q(t), \quad p(t) = \frac{a_1(t)}{a_0(t)}, \quad q(t) = \frac{g(t)}{a_0(t)}.$$

#### 4.1.1 Integrating Factor Method

**Definition 16** (Integrating Factor). *An integrating factor is a function  $\mu(t)$  such that*

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)q(t).$$

**Example.** Consider the first-order linear equation

$$y' + p(t)y = q(t).$$

We multiply the equation by a function  $\mu(t)$  to be determined:

$$\mu y' + \mu p(t)y = \mu q(t).$$

We require that the left-hand side be the derivative of a product:

$$\mu y' + \mu p(t)y = \frac{d}{dt}(\mu(t)y(t)). \quad (*)$$

Since

$$\frac{d}{dt}(\mu y) = \mu y' + \mu'y,$$

condition  $(*)$  holds provided that

$$\mu'(t)y = \mu(t)p(t)y,$$

which is guaranteed if

$$\mu'(t) = \mu(t)p(t).$$

Solving for  $\mu$ ,

$$\frac{\mu'(t)}{\mu(t)} = \frac{d}{dt} \ln |\mu(t)| = p(t),$$

now integrate with respect to  $t$

$$\ln|\mu(t)| = \int p(t)dt + C,$$

where  $C \in \mathbb{R}$  is a constant of integration, and now we exponentiate

$$|\mu(t)| = e^{\int p(t)dt}$$

$$\therefore \mu(t) = C_2 e^{\int p(t)dt}, \quad C_1 = \pm e^C.$$

We can therefore say choose  $\mu(t) = e^{\int p(t)dt}$  (integrating factor). This is the general solution of the first order linear equation.

**Example.** Solve the first-order linear equation

$$y' - 2y = 3e^t.$$

First, identify the functions  $p(t)$  and  $q(t)$ :

$$p(t) = -2, \quad q(t) = 3e^t.$$

Next, compute the integrating factor:

$$\mu(t) = e^{\int p(t)dt} = e^{-2t}.$$

Multiply through by  $\mu(t)$  to rewrite the left-hand side as a derivative:

$$e^{-2t}(y' - 2y) = e^{-2t}3e^t.$$

$$\implies \frac{d}{dt}(e^{-2t}y) = 3e^{-t}.$$

Integrate both sides with respect to  $t$ :

$$e^{-2t}y = \int 3e^{-t}dt + C = -3e^{-t} + C.$$

Solve for  $y(t)$ :

$$y(t) = e^{2t}(-3e^{-t} + C) = -3e^t + Ce^{2t}.$$

Using the initial condition  $y(0) = 1$ , we find  $C = 4$ , giving the final solution:

$$y(t) = -3e^t + 4e^{2t}.$$

## 4.2 Existence and Uniqueness for Linear Equations

**Theorem 5** (Existence & Uniqueness). For

$$y' + p(t)y = q(t), \quad y(t_0) = y_0$$

if  $p$  and  $q$  are continuous on an interval containing  $t_0$ , there exists a unique solution on that interval.

This theorem guarantees that any first-order linear equation we solve, including real-world applications like falling objects or mixing problems, has a unique solution on the interval where  $p$  and  $q$  are continuous.

**Example.**

$$\begin{cases} y' + \frac{1}{t-1}y = \frac{1}{\cos t} \\ y(0) = 1 \end{cases}$$

Maximal interval:  $J_{max} = (-\frac{\pi}{2}, 1)$

### 4.3 Applications

**Example** (Falling Object with Air Resistance). Consider an object falling under gravity while experiencing air resistance proportional to its velocity. The differential equation is

$$v' + \frac{\gamma}{m}v = g.$$

Using the integrating factor  $\mu(t) = e^{\frac{\gamma}{m}t}$ , we rewrite the equation as

$$\frac{d}{dt}(\mu v) = g\mu \implies v = \frac{gm}{\gamma} + Ce^{-\frac{\gamma}{m}t}.$$

Applying the initial condition  $v(0) = \frac{gm}{\gamma}$  gives  $C = 0$ , so the velocity reaches a constant terminal value:

$$v(t) = \frac{gm}{\gamma}.$$

This example shows how first-order linear equations model real-world motion with resistive forces.

**Example** (Mixing Problem). A tank initially contains 100 L of brine with  $y_0$  g of dissolved salt. Pure salt solution enters at 50 g/L at rate  $R$  L/s, and the well-mixed solution leaves at the same rate. Let  $y(t)$  be the amount of salt at time  $t$ . The equation is

$$y' = 50R - \frac{R}{100}y \implies y' + \frac{R}{100}y = 50R, \quad y(0) = y_0.$$

Using the integrating factor  $\mu(t) = e^{\frac{R}{100}t}$ , we solve:

$$y(t) = 5000 + (y_0 - 5000)e^{-\frac{R}{100}t}.$$

The solution shows that over time the salt concentration approaches a stable threshold of 5000 g, illustrating exponential approach in mixing problems.

### 4.4 Separable Equations

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**Definition 17.** Let  $f(x, y)$  be a function of two variables. Then  $f$  is separable if there exist two functions  $f_1(x)$  and  $f_2(y)$  such that  $f(x, y) = f_1(x)f_2(y)$ .

**Example.** This following ODE is separable

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$f_1(x) = x, \quad f_2(y) = -\frac{1}{y}$$

**Example.**

$$\frac{dy}{dx} = x^2 + y^2$$

This ODE is not separable because it is not in the form  $f_1(x)f_2(y)$ .

#### 4.4.1 General Method for Separable Equations

Defining the general form of a separable ODE to be

$$\frac{dy}{dx} = f_1(x)f_2(y).$$

We can rewrite as

$$M(x) + N(y)\frac{dy}{dx} = 0,$$

where  $M(x) = -f_1(x)$  and  $N(y) = \frac{1}{f_2(y)}$ .

Consider  $H_1(x)$  and  $H_2(y)$  as antiderivatives of  $M(x)$  and  $N(y)$ , respectively. Using the Chain Rule,

$$\frac{d}{dx} H_2(y(x)) = H'_2(y) \frac{dy}{dx}.$$

The ODE can be written as

$$H'_1(x) + H'_2(y) \frac{dy}{dx} = 0 \implies \frac{d}{dx}(H_1(x) + H_2(y(x))) = 0.$$

Therefore,

$$H_1(x) + H_2(y(x)) = C,$$

which is the implicit general solution.

In integral form, the solution is

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + C, \quad C \in \mathbb{R},$$

which provides a systematic way to solve any separable equation by integrating the  $x$ -part and  $y$ -part separately.

**Example (Logistic Equation).** Consider the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

This ODE is separable. Using partial fractions:

$$\frac{K}{N(N-K)} = \frac{-1}{N} + \frac{1}{N-K},$$

so

$$\int \frac{dN}{N} - \int \frac{dN}{N-K} = \int rdt \implies \ln \left| \frac{N-K}{N} \right| = -rt + C.$$

Solving for  $N(t)$  gives

$$N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}},$$

where  $N_0 = N(0)$ .

This shows that  $N(t)$  approaches the carrying capacity  $K$  as  $t \rightarrow \infty$ , with special cases  $N_0 = 0 \implies N(t) = 0$  and  $N_0 = K \implies N(t) = K$  for all  $t$ .

#### 4.4.2 Potential Functions and Integral Curves

**Definition 18** (Potential Function). A potential function is a function  $\Psi(x, y)$  whose level curves define the solutions of a differential equation. For separable equations, we can take

$$\Psi(x, y) = H_1(x) + H_2(y),$$

where  $H_1$  and  $H_2$  are antiderivatives of  $M(x)$  and  $N(y)$  in the rewritten ODE  $M(x) + N(y)y' = 0$ .

**Definition 19** (Level Curve). A level curve of a potential function is a curve along which the potential function is constant:

$$\Psi(x, y) = C.$$

These curves are tangent to the vector field defined by the ODE and represent the general solution in implicit form.

**Definition 20** (Integral Curve). An integral curve is a level curve of the potential function that passes through a specific initial condition  $(x_0, y_0)$ . This gives the unique solution curve corresponding to that initial condition.

An integral curve is a geometric object that contains (possibly many) solutions. The unique integral curve that contains the initial condition  $y(x_0) = y_0$  is defined by

$$\Gamma_{(x_0, y_0)} = \{(x, y) | \Psi(x, y) = \Psi(x_0, y_0) = C(x_0, y_0)\},$$

and in the separable case,

$$\frac{d}{dx} \left[ \int f_1(x) dx + \int \frac{dy}{f_2(y)} \right] = 0.$$

**Remark 5.** For a separable equation, the general solution  $\Psi(x, y) = C$  describes all possible solution curves, and picking a particular  $C$  gives the integral curve passing through a given point.

**Example.** Solve the separable equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}.$$

Separate variables and integrate both sides:

$$\int (1 - y^2) dy = \int x^2 dx + C.$$

This gives the implicit general solution

$$\Psi(x, y) = y - \frac{y^3}{3} - \frac{x^3}{3} = C,$$

which defines the family of solution curves for this ODE.

**Example.** Solve the separable equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Separate variables and integrate:

$$\int y dy = \int -x dx + C.$$

This gives the implicit general solution

$$\Psi(x, y) = \frac{y^2}{2} + \frac{x^2}{2} = C, \quad (C \geq 0),$$

which defines the family of solution curves for this ODE.

**Example.** Find the particular solution passing through  $(x_0, y_0) = (0, 2)$  for

$$x^2 + y^2 = (\sqrt{2C})^2.$$

Using the initial condition:

$$\frac{x_0^2}{2} + \frac{y_0^2}{2} = 2 = C,$$

so the unique integral curve is

$$\frac{x^2}{2} + \frac{y^2}{2} = 2 \implies y^2 = 4 - x^2 \implies y(x) = \pm\sqrt{4 - x^2}.$$

Taking the positive branch to satisfy the initial condition, we get

$$y(x) = \sqrt{4 - x^2}, \quad J_{max} = (-2, 2).$$

## 4.5 Exact Equations

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (1)$$

Define  $\frac{dy}{dx} = f(x, y)$ , where  $f(x, y) = -\frac{M(x, y)}{N(x, y)}$ .

**Definition 21** (Exact Equation). *The ODE is exact in a domain  $D$  if there exists a potential function  $\Psi(x, y)$  such that  $M(x, y) = \frac{\partial \Psi}{\partial x}(x, y), \forall (x, y) \in D$  and  $N(x, y) = \frac{\partial \Psi}{\partial y}(x, y)$ . Of the ODE is exact, then it can be re-written as*

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{dy}{dx} = 0 \iff \frac{d}{dx}(\Psi(x, y(x))) = 0.$$

General solution:  $\Psi(x, y) = C$ .

**Theorem 6.** Let  $D \subset \mathbb{R}^2$  be an open simply connected (no holes) domain in the plane. Assume that  $M, N, M_y = \frac{\partial M}{\partial y}, N_x = \frac{\partial N}{\partial x}$  are continuous on  $D$ . The ODE 1 is exact in  $D \iff M_y(x, y) = N_x(x, y), \forall (x, y) \in D$ .

*Proof.* ( $\implies$ ) The ODE is exact  $\implies \exists$  potential function  $\Psi$  such that  $M = \Psi_x, N = \Psi_y$ .

$$\begin{aligned} \implies M_y &= \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\Psi_x) = \frac{\partial^2 \Psi}{\partial y \partial x} \\ N_x &= \frac{\partial N}{\partial x} = \frac{\partial^2 \Psi}{\partial x \partial y} \end{aligned}$$

. ( $\iff$ ) See the notes when  $D$  is an open rectangle. The proof for general simply connected domains is more involved.  $\square$

**Example.**

$$\begin{aligned} y\cos(x) + 2xe^y + (\sin(x) + x^2e^y - 1) \frac{dy}{dx} &= 0 \\ M(x, y) &= y\cos(x) + 2xe^y \\ N(x, y) &= \sin(x) + x^2e^y - 1 \end{aligned}$$

Here,  $M, N, M_y, N_x$  are continuous on  $D = \mathbb{R}^2$  (open simply connected).

$$M_y = \cos(x) + 2xe^y = N_x = \cos(x) + 2xe^y \implies \text{ODE is exact}$$

By the theorem,  $\exists \Psi$  such that  $M = \Psi_x$  (E1) and  $N = \Psi_y$  (E2). Using (E1) first  $\implies \Psi(x, y) = \int M(x, y) dx + c(y)$ , where  $c$  is the constant of integration that possibly depends on  $y$ .

$$\begin{aligned} &= \int y\cos(x) + 2xe^y dx + c(y) \\ &= y\sin x + 2xe^y + c(y) \end{aligned}$$

Using (E2):

$$\begin{aligned}\frac{\partial \Psi}{\partial y} &= \frac{\partial}{\partial y}(ysinx + 2xe^y + c(y)) = N(x, y) = sin(x) + x^2e^y - 1 \\ &\implies c'(y) = -1\end{aligned}$$

$\implies c(y) = -y \implies \Psi(x, y) = ysinx + 2xe^y - y$  [Potential Function, where  $\Psi(x, y) = C$  is a general solution].

Starting with E2 instead:

$$\Psi(x, y) = \int N(x, y) dy + c(x) = ysinx + x^2e^y - y + c(x)$$

Now using E1:

$$\begin{aligned}\Psi_x &= ycos(x) + 2xe^y + c'(x) = M = ycos(x) + 2xe^y \implies c'(x) = 0, \quad c(x) = 0 \\ &\implies \Psi(x, y) = ysinx + x^2e^y - y.\end{aligned}$$

**Remark 6.** Separable equations are exact over simply connected domains.

$$\begin{aligned}M(x) + N(y) \frac{dy}{dx} &= 0 \\ M_y &= \frac{\partial}{\partial y}(M(x)) = 0 = \frac{\partial}{\partial x}(N(y)) = N_x\end{aligned}$$

## 4.6 Integrating Factors for Exact Equations

Assume that  $M(x, y) + N(x, y) \frac{dy}{dx} = 0$  is not exact. Let us multiply that ODE by a function  $\mu(x, y)$  (the integrating factor):

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y) \frac{dy}{dx} = 0 \quad (*2)$$

Equation (\*2) is exact if  $(\mu M)_y = (\mu N)_x$ ,  $\mu_y M + \mu M_y = \mu_x N + \mu N_x$  (\*3). Goal: Find  $\mu(x, y)$  such that (\*3) holds, but this is a PDE ;-. If  $\mu = \mu(x)$ , then  $\mu_y = 0$ . (\*3) becomes

$$\begin{aligned}\mu M_y &= \frac{d\mu}{dx} N + \mu N_x \\ \iff \frac{d\mu}{dx} &= \mu \frac{(N_x - M_y)}{N} \quad (*4)\end{aligned}$$

If  $\frac{N_x - M_y}{N}$  only depends on x, we can solve the separable equation (\*4), find  $\mu(x)$ , then solve the exact ODE (\*2).

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$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

Let  $\mu = \mu(x, y)$ . Then

$$(\mu M)_y = (\mu N)_x$$

$$\iff \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

If  $\mu = \mu(x)$ , then  $\mu_y = 0 \implies \mu \frac{(M_y - N_x)}{N} = \mu_x = \frac{d\mu}{dx}$ . Hence, IF the function  $\frac{(M_y - N_x)}{N}$  only depends on x, we can solve the separable ODE  $\mu \frac{(M_y - N_x)}{N} = \mu_x = \frac{d\mu}{dx}$  and we find the integrating factor  $\mu(x)$ , we use it to transform the original ODE into an exact ODE, which we then solve.

**Example.**

$$(3xy + y^2) + (x^2 + xy) \frac{dy}{dx} = 0$$

$$M_y = 3x + 2y \neq N_x = 2x + y \quad \text{Not exact}$$

$$\text{Now } \frac{M_y - N_x}{N} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}.$$

$$\implies \mu(x) \frac{1}{x} = \frac{d\mu}{dx}$$

$$\implies \int \frac{dx}{x} = \int \frac{d\mu}{\mu} + C$$

$$\ln|x| = \ln|\mu| + C \implies |x| = e^C |\mu| \implies x = \mu$$

$$\implies x(3xy + y^2) + x(x^2 + xy) \frac{dy}{dx} = 0$$

$$\implies M = 3x^2y + xy^2, \quad N = x^3 + x^2y$$

$$\implies M_y = 3x^2 + 2xy, \quad N_x = 3x^2 + 2xy$$

$\exists \Psi$  such that  $M = \Psi_x$  and  $N = \Psi_y$ . Using  $M = \Psi_x$ :

$$\Psi(x, y) = \int M(x, y) dx = \int 3x^2y + yx^2 dx = x^3y + \frac{x^2y^2}{2} + c(y)$$

Using  $N = \Psi_y$ :

$$\Psi_y = x^3 + x^2y + c'(y) = N = x^3 + x^2y \implies c'(y) = 0 \implies c(y) = 0 \implies \Psi(x, y) = x^3y + \frac{x^2y^2}{2}$$

$$\psi(x, y) = x^3y + \frac{x^2y^2}{2} = C(\text{general implicit solution}).$$

## 5 Systems of Linear Equations

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We study first order linear systems of ODEs

$$y'(t) = A(t)y(t) + r(t), \quad t \in I. \tag{3}$$

**Notation.**

$$y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} \in \mathbb{R}^n, \quad A(t) \in M_n(\mathbb{R}), \quad r(t) = \begin{pmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{pmatrix}.$$

$$y, r : I \rightarrow \mathbb{R}^n, \quad A : I \rightarrow M_n(\mathbb{R}).$$

## 5.1 From higher order scalar ODEs to linear systems

**Example.** Convert the third order equation

$$y''' - 2y'' + 4y' - 5y = \sin t + t^3$$

into a first order system.

Let

$$y_1 = y, \quad y_2 = y', \quad y_3 = y''.$$

Then

$$\begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -4 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sin t + t^3 \end{pmatrix}.$$

Thus the equation is written in the form  $y' = Ay + r$ , with  $J_{\max} = \mathbb{R}$ .

## 5.2 Theory of first order linear systems

**Theorem 7** (Existence and uniqueness). If every entry of  $A(t)$  and  $r(t)$  is continuous on an interval  $I$ , then for any

$$t_0 \in I, \quad y_0 \in \mathbb{R}^n,$$

the IVP

$$\begin{cases} y'(t) = A(t)y(t) + r(t), \\ y(t_0) = y_0 \end{cases}$$

has a unique solution on all of  $I$ . Hence  $J_{\max} = I$ .

**Definition 22.** If  $r(t) = 0$ , the system is **homogeneous**. Otherwise it is **non-homogeneous**.

We first study the homogeneous system

$$y'(t) = A(t)y(t). \tag{4}$$

Let

$$S = \{ y : I \rightarrow \mathbb{R}^n \mid y \text{ solves (4)} \} \subset C^1(I).$$

## 5.3 Structure of the solution space

**Definition 23.** Functions  $y_1, \dots, y_n : I \rightarrow \mathbb{R}^n$  are **linearly independent** if

$$c_1 y_1(t) + \dots + c_n y_n(t) = 0 \text{ for all } t \Rightarrow c_1 = \dots = c_n = 0.$$

**Theorem 8.** The solution set  $S$  is an  $n$ -dimensional vector subspace of  $C^1(I)$ .

*Proof.* **Step 1: Vector space.**

If  $y_1, y_2 \in S$  and  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,

$$(\alpha_1 y_1 + \alpha_2 y_2)' = \alpha_1 y_1' + \alpha_2 y_2' = A(t)(\alpha_1 y_1 + \alpha_2 y_2),$$

so  $\alpha_1 y_1 + \alpha_2 y_2 \in S$ . Also  $0 \in S$ .

**Step 2: Construct a basis.**

Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbb{R}^n$ .

For each  $i$ , let  $y_i$  be the unique solution of

$$\begin{cases} y' = A(t)y, \\ y(t_0) = e_i. \end{cases}$$

Define

$$B = \{y_1, \dots, y_n\}.$$

Every linear combination of the  $y_i$  solves the system, so  $\text{span}(B) \subset S$ .

If  $z \in S$  and

$$z(t_0) = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix},$$

then uniqueness gives

$$z(t) = \sum_{i=1}^n z_i y_i(t),$$

so  $S \subset \text{span}(B)$ .

**Step 3: Independence.**

If

$$c_1 y_1(t) + \cdots + c_n y_n(t) = 0,$$

evaluating at  $t_0$  gives

$$c_1 e_1 + \cdots + c_n e_n = 0,$$

hence  $c_1 = \cdots = c_n = 0$ .

Thus  $B$  is a basis and  $\dim S = n$ .  $\square$

## 6 Second and Higher-Order Scalar Linear Equations

## 7 Stability, Phase Portraits and Orbits

## 8 Laplace Transform

## 9 Power Series Solutions and Numerical Methods

## 10 Solutions

**Solution 1** (System of First Order ODEs). Define new variables:

$$y_1 = y, \quad y_2 = y', \quad y_3 = y''.$$

Now we can rewrite the third order ODE into a first order system:

$$\begin{cases} y'_1 = y_2 \\ y'_2 = y_3 \\ y'_3 = -4y_2 + y_1 \end{cases}$$

**Solution 2** (Existence and Uniqueness Theorem). We have the IVP

$$\begin{cases} y' = \sqrt{1+y^2} + t^2 \\ y(1) = 0 \end{cases}$$

1. The function is simply  $f(y, t) = \sqrt{1+y^2} + t^2$ .

2. Check the hypotheses:

- $f$  is continuous  $\forall y \in \mathbb{R}, t \in \mathbb{R}$  because square root of  $1+y^2$  and  $t^2$  are continuous everywhere.
- Check local Lipschitz in  $y$ :  $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{1+y^2}}$ , which is continuous  $\forall y$ . Therefore  $f$  is locally Lipschitz in  $y$ . So the hypotheses of E & U th. are satisfied in  $\mathbb{R} \times \mathbb{R}$ .

3. Since  $f$  is continuous and locally Lipschitz in  $y$ , and since  $y(1) = 0$  with  $0 \in \mathbb{R}$ , the Existence and Uniqueness Theorem guarantees that there exists an open interval  $J$  containing  $t = 1$  on which a solution exists, and this solution is unique on  $J$ .

**Solution 3** (Existence and Uniqueness Theorem 2). We write

$$f(y, t) = y + 1.$$

Since  $f$  is a  $C^1$  function, it is locally Lipschitz in  $y$ , and the Existence and Uniqueness Theorem applies. The domain is

$$f : D \rightarrow \mathbb{R}, \quad D = \mathbb{R} = (-\infty, +\infty).$$

Since  $f$  is defined everywhere, there are no constraints on  $\alpha$  and  $\delta$ . Define

$$D_{\alpha,\delta} = \{(y, t) : \|y - 1\| \leq \alpha, |t - 0| \leq \delta\} = [1 - \alpha, 1 + \alpha] \times [-\delta, \delta] \subset \mathbb{R} \times \mathbb{R}.$$

Then

$$M_{\alpha,\delta} = \sup_{(y,t) \in D_{\alpha,\delta}} \|f(y, t)\| = \sup_{y \in [1-\alpha, 1+\alpha]} |y + 1| = 2 + \alpha.$$

The theorem allows us to take

$$\varepsilon = \min\left(\delta, \frac{\alpha}{2 + \alpha}\right).$$

Pick  $\alpha = 1, \delta = 1$ :

$$\varepsilon = \min\left(1, \frac{1}{3}\right) = \frac{1}{3}.$$

Therefore, there exists a unique solution on  $(-\frac{1}{3}, \frac{1}{3})$ . Pick  $\alpha = 3, \delta = 2$ :

$$\varepsilon = \min\left(2, \frac{3}{5}\right) = \frac{3}{5}.$$

Therefore, there exists a unique solution on  $(-\frac{3}{5}, \frac{3}{5})$ . Since there are no constraints on  $\alpha$  and  $\delta$ , we can make  $\varepsilon$  as large as we want. Hence, the solution exists and is unique on  $\mathbb{R}$ . The “maximal” time interval guaranteed by the Existence and Uniqueness Theorem is

$$J = (-1, 1).$$

In Chapter 2 we will see that the explicit solution is

$$y(t) = 2e^t - 1.$$

In fact, the maximal interval on which the solution is defined is

$$J_{\max} = \mathbb{R}.$$

**Solution 4** (Existence and Uniqueness Theorem 3). Since  $f(y) = y^2$  is  $C^1$ , it is locally Lipschitz, so a solution exists and is unique locally. On  $D_{\alpha,\delta} = [1 - \alpha, 1 + \alpha] \times [-\delta, \delta]$ ,

$$M_{\alpha,\delta} = \sup_{y \in [1-\alpha, 1+\alpha]} |y^2| = (1 + \alpha)^2.$$

Thus,

$$\varepsilon = \min\left(\delta, \frac{\alpha}{(1 + \alpha)^2}\right).$$

Define

$$h(\alpha) = \frac{\alpha}{(1+\alpha)^2}, \quad h'(\alpha) = \frac{1-\alpha}{(1+\alpha)^3}.$$

Setting  $h'(\alpha) = 0$  gives  $\alpha = 1$ . Pick  $\alpha = 1$ ,  $\delta = 104073$ :

$$\varepsilon = \frac{1}{4}.$$

Therefore, there exists a unique solution

$$y : \left(-\frac{1}{4}, \frac{1}{4}\right) \rightarrow \mathbb{R}.$$

Chapter 2 will show the solution is

$$y(t) = \frac{1}{1-t}.$$

The vertical asymptote illustrates finite-time blow-up.

**Solution 5** (Existence and Uniqueness Theorem 4). The differential equation is

$$y' + \frac{t}{t^2 - 25}y = \frac{e^t}{t - 9}.$$

The existence and uniqueness theorem requires the coefficients

$$p(t) = \frac{t}{t^2 - 25}, \quad g(t) = \frac{e^t}{t - 9}$$

to be continuous. Discontinuities occur when denominators are zero:

$$t^2 - 25 = 0 \Rightarrow t = \pm 5, \quad t - 9 = 0 \Rightarrow t = 9.$$

So the critical points are  $t = -5, 5, 9$ .

- For  $y(-7) = -2.1$ : the nearest discontinuity is  $t = -5$ . Largest interval:

$$(-\infty, -5).$$

- For  $y(-1.5) = -3.14$ : the nearest discontinuities are  $t = -5$  and  $t = 5$ . Largest interval:

$$(-5, 5).$$

- For  $y(8.5) = 6.4$ : the nearest discontinuities are  $t = 5$  and  $t = 9$ . Largest interval:

$$(5, 9).$$

- For  $y(13) = -0.5$ : the nearest discontinuity is  $t = 9$ . Largest interval:

$$(9, \infty).$$

**Solution 6** (LLC). We have

$$f(y) = 3y^{2/3}, \quad f'(y) = 2y^{-1/3}.$$

This derivative is not bounded near  $y = 0$ , so  $f$  is not locally Lipschitz at  $y = 0$ . The function

$$y_1(t) = 0$$

is a solution. In Chapter 2, solving the separable equation (assuming  $y \neq 0$ ) gives another solution

$$y_2(t) = \begin{cases} t^3, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Indeed,

$$y'_2(t) = 3t^2 = 3(y_2(t))^{2/3}.$$

Thus, the solution is not unique.

## 11 Appendix

*Proof of Picard Iterations Convergence.* Pick  $t \in [t_0, t_0 + \epsilon]$  (the proof is similar for  $t \in [t_0 - \epsilon, t_0]$ ). The goal is to show that  $\{y_k(t)\}_{k=0}^\infty$  is a Cauchy sequence<sup>6</sup> in  $\mathbb{R}^n$ . We prove by induction that

$$(**) \quad \|y_m(t) - y_{m-1}(t)\| \leq L^{m-1} M_{\alpha, \delta} \frac{(t - t_0)^m}{m!}, \quad \forall m \geq 1.$$

**Base case**  $m = 1$ :

$$\|y_1(t) - y_0(t)\| = \left\| \int_{t_0}^t f(y_0, s) ds \right\| \leq \int_{t_0}^t \|f(y_0, s)\| ds \leq M_{\alpha, \delta} |t - t_0| \leq \alpha.$$

**Induction step:**

$$\|y_{m+1}(t) - y_m(t)\| \leq \int_{t_0}^t \|f(y_m(s), s) - f(y_{m-1}(s), s)\| ds.$$

Since  $D_{\alpha, \delta}$  is compact and  $f$  is Lipschitz on  $D_{\alpha, \delta}$ , there exists  $L$  such that

$$\|f(x, t) - f(y, t)\| \leq L \|x - y\|.$$

Thus,

$$\leq L \int_{t_0}^t \|y_m(s) - y_{m-1}(s)\| ds.$$

Using (\*\*),

$$\leq L^m M_{\alpha, \delta} \frac{1}{(m-1)!} \int_{t_0}^t (s - t_0)^{m-1} ds = L^m M_{\alpha, \delta} \frac{(t - t_0)^m}{m!}.$$

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<sup>6</sup>Cauchy sequence is a sequence that has a limit in a metric space  $\mathbb{R}^n$ .

Hence (\*\*\*) holds. In particular, for all  $\rho \geq 1$ ,

$$\|y_\rho(t) - y_{\rho-1}(t)\| \leq M_{\alpha,\delta} \frac{(L(t-t_0))^\rho}{(\rho)!} < \frac{M_{\alpha,\delta}}{L} \frac{(L\epsilon)^\rho}{\rho!}.$$

Let  $m, p \geq 1$ :

$$\|y_{m+p}(t) - y_{m+1}(t)\| \leq \sum_{k=1}^{p-1} \|y_{m+k+1}(t) - y_{m+k}(t)\|.$$

So,

$$< \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{m+p} \frac{(L\epsilon)^j}{j!}.$$

Since  $e^{L\epsilon} = \sum_{j=0}^{\infty} \frac{(L\epsilon)^j}{j!}$  converges,

$$\rightarrow_{m,p \rightarrow +\infty} 0.$$

Thus  $\{y_k(t)\}$  is Cauchy and converges to  $y(t)$ . Taking limits in the iteration,

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

□

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## 12 Useful Links

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<sup>7</sup>Evaluated on compacted cylinder, understanding epsilon, the L Lipschitz constant coming from somewhere, not on the analysis background such as the Banach fixed point theorem.