

# MATH 325: Honours Ordinary Differential Equations

William Homier<sup>1</sup>

<sup>1</sup>*McGill University Physics, 3600 Rue University, Montréal, QC H3A 2T8, Canada*

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**Abstract**

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# 1 Introduction

Jean-Philippe Lessard (Burnside 1119). Tutorials every wednesday from 9am to 10am, ENGTR 0070, with Eunpyo Bang. Office hours thursday. No textbooks. 25% assignments (2 written assignments 15%, and 5 webworks 10%). 25% Midterm (February 16 - inclass). 50% Final. Since its honours you will deal with analysis.

## 2 Prerequisite knowledge

### 2.1 Analysis

## 3 Intro, Classification, Theorem of Existence & Uniqueness

### 3.1 Intro

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**Definition 1** (Differential Equation). *A differential equation (DE) is a relation that involves an unknown function and some of its derivatives.*

To better understand what a differential equation is, consider the following example.

Imagine a ball of mass  $m$  falling, subject to gravity and air resistance (drag). Denote by  $v(t)$  the velocity of the ball at time  $t$ , whereas  $t$  is the independent variable, and  $v$  the dependent variable. Let the downward direction be positive. We know the force of gravity is given by  $F_g = mg$ , where  $g$  is the acceleration due to gravity. The drag force is given by  $F_d = -\lambda v$ , where  $\lambda$  is the drag coefficient and is  $\lambda \geq 0$ . According to Newton's second law  $\sum F = ma$ , the net force acting on the ball is equal to its mass times its acceleration

$$m \frac{dv}{dt} = mg - \lambda v.$$

Let  $y(t)$  be the position, meaning  $v(t) = \frac{dy}{dt}$ . Then, we can rewrite the above equation as

$$my'' + \lambda y' = mg.$$

Let's analyze another example, population growth (known as the Malthusian growth model).

Denote by  $N(t)$  the size of a given population at time  $t$ . In an "unconstrained" environment, it is reasonable to assume that the rate of change of the number of individuals is proportional to the number of individuals present. This assumption leads to the following differential equation:

$$\frac{dN}{dt} = rN,$$

where  $r$  is called the growth rate (if  $r > 0$ ), and decay rate (if  $r < 0$ ). Assume that  $N > 0$ . Using the chain rule and assuming that  $N(t)$  satisfies  $N' = rN$

$$\frac{d}{dt} \ln(N(t)) = \frac{d \ln(N)}{dN} \cdot \frac{dN}{dt} = \frac{1}{N} \cdot N' = r,$$

integrate with respect to  $t$

$$\ln(N(t)) = rt + C,$$

where  $C$  is the constant of integration. Exponentiating both sides, we obtain

$$N(t) = e^{\ln(N(t))} = e^C e^{rt} = k e^{rt},$$

where  $\{k > 0 | k \in \mathbb{R}\}$  which could be any positive constant is the initial population size at time  $t = 0$ .

Assume that an initial population (condition) is given:

$$N(0) = N_0(\text{fixed}),$$

we therefore get that  $k = N_0$ , and the unique solution that satisfies the initial condition is

$$N(t) = N_0 e^{rt}.$$

The problem with the answer we got in the previous example is that it is not realistic in the long run, how about we consider a carrying capacity<sup>1</sup>. This leads us to another example: Population growth/decay with the carrying capacity of the environment.

Now assume that our growth rate depends on the population size  $N(t)$  itself, therefore we get that

$$\frac{dN}{dt} = R(N)N.$$

Denote by  $K$  the number of individual that the environment can carry.  $K$  is called the carrying capacity of the environment. If  $N < K$ , we want growth ( $R(N) > 0$ ) and if  $N > K$ , we want decay ( $R(N) < 0$ ).

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<sup>1</sup>maximum population size that the environment can sustain indefinitely



Let's pick the simplest function  $R(N)$  that satisfies  $R(0) = r, R(K) = 0$  and is linear. We get that

$$R(N) = r\left(1 - \frac{N}{K}\right).$$

Therefore, our differential equation becomes

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N = \frac{r}{K}(K - N(t))N(t).$$

This is called the logistic equation.

**Definition 2** (Ordinary Differential Equation). *An ordinary differential equation (ODE) is a differential equation whose unknown function depends on one independent variable only.*

Example of ODEs:

- $y''(t) + y'(t) + 2y(t) = \sin(t)$
- $N'(t) = rN(t)$
- $mv'(t) = mg - \lambda v(t)$
- $y'(x) + 3y(x) = e^x$

**Definition 3** (Partial Differential Equation). *A partial differential equation (PDE) is a differential equation whose unknown function depends on more than one independent variable. **Will not be taught in this course.***

Example of a PDE is the Heat Equation. Let  $u = u(x, t)$ ,  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ . This PDE denotes the temperature of a body at time  $t$  and at position  $x$ .

## 3.2 Classification

### 3.2.1 The Order

**Definition 4.** *The order of an ODE is the order of the highest derivative that appears in the equation.*

**Example.**  $N' = rN$  (first order ODE)

**Example.**  $y''(t) + 2y'(t) = e^t$  (second order ODE)

### 3.2.2 Dimension of the State

**Definition 5** (Scalar ODE). *A scalar ODE has one unknown function.*

**Example** (Scalar ODE).

$$y'' + y = 0$$

**Definition 6** (System of ODEs). *A system of ODEs has several unknown functions.*

**Example** (System of ODEs).

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$$

**Definition 7** (Systems of first order ODEs). *A system of first order ODEs is just a collection of differential equations where all derivatives are first order. It can be written compactly as*

$$y'(t) = f(y(t), t),$$

where  $y(t)$  is a vector of unknown functions,

$$y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix},$$

and  $f$  gives the right-hand sides of the equations. Writing this out means

$$\begin{cases} y_1'(t) = f_1(y_1, \dots, y_n, t) \\ \vdots \\ y_n'(t) = f_n(y_1, \dots, y_n, t). \end{cases}$$

Any single  $n$ th-order ODE can always be turned into such a system. If

$$y^{(n)}(t) = G(t, y, y', \dots, y^{(n-1)}),$$

define new variables

$$y_1 = y, \quad y_2 = y', \quad \dots, \quad y_n = y^{(n-1)}.$$

Then each new variable has a first derivative, and the equation becomes

$$\begin{cases} y'_1 = y_2, \\ y'_2 = y_3, \\ \vdots \\ y'_{n-1} = y_n, \\ y'_n = G(t, y_1, \dots, y_n). \end{cases}$$

So one higher-order equation is the same thing as many first order equations.

**Problem 1** (System of First Order ODEs). Rewrite this third order ODE as a first order system:

$$y''' + 4y' - y = 0$$

**Example** (System of First Order ODEs 2). Consider the second-order ODE

$$\begin{aligned} y'' + 2y' + y &= e^t \\ \implies y'' &= -2y' - y + e^t. \end{aligned}$$

Define new variables

$$y_1 = y, \quad y_2 = y'.$$

Then the system becomes

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad r(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

**Example** (Lorenz system).

$$\begin{aligned} y'_1 &= \sigma(y_2 - y_1), \\ y'_2 &= \rho y_1 - y_2 - y_1 y_3, \\ y'_3 &= y_1 y_2 - \beta y_3, \end{aligned}$$

where  $\sigma, \rho, \beta$  are parameters. This is a nonlinear first order system in  $\mathbb{R}^3$ .

### 3.2.3 Linearity

**Definition 8** (Linearity). A linear Ordinary Differential Equation (ODE) is one where the unknown function ( $y$ ) and its derivatives ( $y', y'', \dots$ ) appear only to the first power, are not multiplied together, and are not part of special functions like  $\sin(y)$  or  $e^y$ . Essentially, they are "simple" combinations (addition/subtraction) of  $y$  and its derivatives, potentially multiplied by functions of the independent variable (like  $x$  or  $t$ ).

**Example (Linearity).** Consider the following ODEs:

- Linear:  $y' + 3y = 0$
- Linear:  $y'' - 2xy' + y = \cos(x)$
- Non-linear:  $y' + y^2 = 0$
- Non-linear:  $y'' + \sin(y) = 0$

### 3.2.4 Autonomy

**Definition 9** (Autonomy). The  $n^{\text{th}}$  order ODE  $F(t, y, y', \dots, y^{(n)}) = 0$  is autonomous if  $F$  does not depend explicitly on  $t$ , that is, if it is of the form  $F(y, y', \dots, y^{(n)}) = 0$ . Otherwise, it is non-autonomous.

**Example (Autonomy).** Consider the following ODEs:

- Non-autonomous:  $y'' + 2y' + y - e^t = 0$
- Autonomous:  $N'(t) = rN(t)$
- Non-autonomous:  $y'(t) = ty(t)$

Equivalently, a first-order system  $y' = f(y, t)$  is autonomous if it can be written as

$$y' = f(y).$$

Otherwise, it is non-autonomous.

**Example.** A classic, simple example of an autonomous first-order system is the linear growth/decay model:

$$\frac{dy}{dt} = ky$$

Here,  $f(y, t) = ky$ , which depends only on  $y$  (where  $k$  is a constant), making it autonomous. Other examples include  $\frac{dy}{dt} = 1 - y^2$  or exponential growth  $\frac{dy}{dt} = 0.5y$ .

**Remark 1.** The Lorenz system is an example of an autonomous system<sup>2</sup>.

### 3.2.5 Solutions of ODEs

**Definition 10** (Solutions of ODEs). Let  $f : D \times (a, b) \rightarrow \mathbb{R}^n$ . A solution of  $y'(t) = f(y(t), t)$  on an interval  $J \subset \mathbb{R}$  is a differentiable function  $y : J \rightarrow D \subset \mathbb{R}^n$ , such that  $y'(t) = f(y(t), t), \forall t \in J$ .  $t$  is the independent variable, and  $y = (y_1, \dots, y_n)$  is the dependent variable.

<sup>2</sup>Here “system” means the unknown  $y$  is vector-valued, e.g.  $y \in \mathbb{R}^m$ , rather than scalar.



## 1. Explicit Solutions

**Example** (Explicit Solution). *Consider the following ODE*

$$y' + y = 1.$$

*We can verify that  $y(t) = e^{-t} + 1$ , and therefore  $y'(t) = -e^{-t}$ , is a solution on  $\mathbb{R}$ . Indeed,*

$$y' + y = -e^{-t} + (e^{-t} + 1) = 1.$$

*In this example,  $y = y(t)$  is explicitly given as a function of  $t$  (independent variable).*

## 2. Implicit solutions

**Example** (Implicit Solution). *Consider the ODE*

$$y \frac{dy}{dx} = x.$$

*This is a nonautonomous, nonlinear, first-order scalar ODE. Separating variables gives*

$$y \, dy = x \, dx.$$

*Integrating,*

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C,$$

*or equivalently, the implicit solution*

$$x^2 - y^2 = C, \quad C \in \mathbb{R}.$$

*To verify, differentiate implicitly:*

$$\frac{d}{dx}(x^2 - y^2) = 0 \implies 2x - 2y \frac{dy}{dx} = 0 \implies y \frac{dy}{dx} = x.$$

## 3.3 Initial Value Problems

**Definition 11** (Initial Value Problem (IVP)). *An initial value problem (IVP) is a system of ODEs with an initial condition.*

**Example** (IVP). *Consider the following ODE*

$$y' = f(y, t), \quad f : D \times (a, b) \rightarrow \mathbb{R}^n.$$

*Let  $t_0 \in (a, b)$ . An initial condition is*

$$y(t_0) = y_0 \in \mathbb{R}^n.$$

*An initial value problem (IVP) is*

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}.$$

### 3.4 Existence and Uniqueness Theorem

#### 3.4.1 Lipschitz continuity

**Definition 12** (Lipschitz continuity). Let  $D \subseteq \mathbb{R}^n$  and let  $\|\cdot\|$  be a norm on  $\mathbb{R}^n$ . A function  $f : D \rightarrow \mathbb{R}^n$  is Lipschitz continuous if there exists  $L \geq 0$  such that

$$z\|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\| \quad \forall y_1, y_2 \in D, z \in \mathbb{R}.$$

The smallest such  $L$  is called the Lipschitz constant and is denoted  $Lip(f)$ .

**Example.** Let  $f(y) = 4y - 5$ ,  $D = \mathbb{R}$ , and  $\|\cdot\| = |\cdot|$ . Then

$$|f(y_1) - f(y_2)| = |4y_1 - 4y_2| = 4|y_1 - y_2|.$$

So  $f$  is Lipschitz with  $Lip(f) = 4$ .

**Example.** Let

$$f(y) = \frac{1}{y-1}, \quad f'(y) = -\frac{1}{(y-1)^2}, \quad D = (1, +\infty).$$

This function is not Lipschitz continuous on  $D$  since as  $y \rightarrow 1^+$ , the derivative  $f'(y)$  approaches  $-\infty$ , and so the ratio  $\frac{|f(y_2) - f(y_1)|}{|y_2 - y_1|}$  approaches  $\infty$  for any  $y_1, y_2 \in D$ . Therefore, there is no Lipschitz constant for  $f$  on  $D$ .

To make this function Lipschitz continuous, we fix  $\delta > 1$  and define  $D_\delta = (\delta, +\infty)$ . For  $y_1, y_2 \in D_\delta$ , by the Mean Value Theorem, there exists  $z \in (y_1, y_2)$  such that

$$f(y_2) - f(y_1) = f'(z)(y_2 - y_1).$$

So

$$|f(y_2) - f(y_1)| \leq \frac{1}{(z-1)^2} |y_2 - y_1| \leq \frac{1}{(\delta-1)^2} |y_2 - y_1|.$$

Thus  $f$  is Lipschitz on  $D_\delta$  with

$$Lip(f) = \frac{1}{(\delta-1)^2}.$$

#### 3.4.2 Local Lipschitz continuity

**Definition 13** (Locally Lipschitz). Let  $D \subseteq \mathbb{R}^n$  be open. A function  $f : D \rightarrow \mathbb{R}^n$  is called locally Lipschitz if, around every point in  $D$ , there is some neighborhood where  $f$  is Lipschitz. Equivalently, for every compact set  $K \subset D$ , there exists a constant  $L > 0$  such that

$$\|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\| \quad \text{for all } y_1, y_2 \in K.$$

**Remark 2.** Lipschitz continuity and local Lipschitz continuity are not the same.

Lipschitz  $\Rightarrow$  locally Lipschitz, but not conversely.

**Example** (Locally Lipschitz but not Lipschitz).

$$f(y) = y^2 \in \mathbb{R}.$$

Since  $f'(y) = 2y$  is unbounded on  $\mathbb{R}$ , no single constant works on the whole domain, so  $f$  is not Lipschitz. However, on any bounded set  $K = [-M, M]$ ,  $|f'(y)| \leq 2M$ , so  $f$  is locally Lipschitz.

**Example** (Lipschitz (hence locally Lipschitz)).

$$f(y) = \sin y.$$

Since  $|f'(y)| = |\cos y| \leq 1$ ,  $\forall y$ ,  $f$  is Lipschitz on  $\mathbb{R}$ .

**Example** (Continuous but not locally Lipschitz).

$$f(y) = \sqrt{|y|}.$$

The derivative is unbounded near  $y = 0$ , so no Lipschitz constant exists even locally. Hence  $f$  is not locally Lipschitz.

### 3.4.3 Existence and Uniqueness Theorem

**Theorem 1** (Existence and Uniqueness). Let  $D \subseteq \mathbb{R}^n$  be open and let  $(a, b)$  be an open interval containing  $t_0$ . Consider the IVP

$$\begin{cases} y' = f(y, t), \\ y(t_0) = y_0. \end{cases}$$

Assume  $f : D \times (a, b) \rightarrow \mathbb{R}^n$  is continuous and locally Lipschitz in  $y$ .<sup>3</sup> If  $y_0 \in D$ , then there exists an open interval  $J$  containing  $t_0$  on which a solution exists. Moreover, this solution is unique on  $J$ .

**Problem 2** (Existence and Uniqueness Theorem). Consider the IVP

$$\begin{cases} y' = \sqrt{1 + y^2} + t^2 \\ y(1) = 0. \end{cases}$$

1. Identify  $f(y, t)$ .
2. Decide whether the hypotheses of the Existence and Uniqueness Theorem are satisfied.
3. State clearly what the theorem guarantees about solutions near  $t = 1$ .

Do not solve the ODE. Simply analyze it.

---

<sup>3</sup> $D \times (a, b)$  denotes the set of all pairs  $(y, t)$  with  $y \in D \subset \mathbb{R}^n$  and  $t \in (a, b)$ , i.e., all allowed state-time inputs of  $f$ .

### 3.4.4 Integral form of solutions

**Lemma 2.** *A function  $y$  solves the IVP if and only if*

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

*Proof.* If  $y' = f(y, t)$  and  $y(t_0) = y_0$ , then by the Fundamental Theorem of Calculus,

$$y(t) - y(t_0) = \int_{t_0}^t f(y(s), s) ds,$$

so

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

Conversely, differentiating the right-hand side gives

$$y'(t) = f(y(t), t), \quad y(t_0) = y_0.$$

□

### 3.4.5 Picard operator

**Setup and notation** Let  $(y_0, t_0) \in D \times (a, b)$ . Since this set is open, there exist  $\alpha, \delta > 0$  such that

$$D_{\alpha, \delta} = \{(y, t) : \|y - y_0\| \leq \alpha, |t - t_0| \leq \delta\} \subset D \times (a, b).$$

Define

$$M_{\alpha, \delta} = \sup_{(y, t) \in D_{\alpha, \delta}} {}^4\|f(y, t)\| < +\infty.$$

Let

$$\epsilon = \min \left( \delta, \frac{\alpha}{M_{\alpha, \delta}} \right), \quad J = (t_0 - \epsilon, t_0 + \epsilon).$$

### Definition of the Picard operator

**Lemma 3** (Picard operator). *For any function  $y$  such that  $y(t_0) = y_0$  and  $(y(t), t) \in D_{\alpha, \delta}$  for all  $t \in J$ , define*

$$T(y)(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

*Then  $T(y)(t_0) = y_0$  and  $(T(y)(t), t) \in D_{\alpha, \delta}$  for all  $t \in J$ .*

---

<sup>4</sup>*sup* means the largest value in a set of numbers.

*Proof.*

$$T(y)(t_0) = y_0.$$

For  $t \in J$ ,

$$\|T(y)(t) - y_0\| \leq \int_{t_0}^t \|f(y(s), s)\| ds \leq M_{\alpha, \delta} |t - t_0| \leq M_{\alpha, \delta} \epsilon \leq \alpha.$$

So  $(T(y)(t), t) \in D_{\alpha, \delta}$ .

□

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### Invariant properties

**Lemma 4.** *If  $y : J \rightarrow \mathbb{R}^n$  satisfies*

1.  $y(t_0) = y_0$ ,
2.  $(y(t), t) \in D_{\alpha, \delta}$  for all  $t \in J$ ,

*then  $T(y) : J \rightarrow \mathbb{R}^n$  satisfies the same properties.*

**Picard iterations** Define  $y_0(t) = y_0$  (constant function), which clearly satisfies (1) and (2).

For  $k \geq 1$ , define

$$y_k(t) = T(y_{k-1})(t) = y_0 + \int_{t_0}^t f(y_{k-1}(s), s) ds.$$

**Existence** The Picard iterations converge uniformly to a function  $y : J \rightarrow \mathbb{R}^n$  which satisfies (1) and (2), and is a solution of the IVP.

*Proof.* Pick  $t \in [t_0, t_0 + \epsilon]$  (the proof is similar for  $t \in [t_0 - \epsilon, t_0]$ ).

The goal is to show that  $\{y_k(t)\}_{k=0}^\infty$  is a Cauchy sequence<sup>5</sup> in  $\mathbb{R}^n$ .

We prove by induction that

$$(**) \quad \|y_m(t) - y_{m-1}(t)\| \leq L^{m-1} M_{\alpha, \delta} \frac{(t - t_0)^m}{m!}, \quad \forall m \geq 1.$$

**Base case  $m = 1$ :**

$$\|y_1(t) - y_0(t)\| = \left\| \int_{t_0}^t f(y_0(s), s) ds \right\| \leq \int_{t_0}^t \|f(y_0(s), s)\| ds \leq M_{\alpha, \delta} |t - t_0| \leq \alpha.$$

**Induction step:**

$$\|y_{m+1}(t) - y_m(t)\| \leq \int_{t_0}^t \|f(y_m(s), s) - f(y_{m-1}(s), s)\| ds.$$

---

<sup>5</sup>Cauchy sequence is a sequence that has a limit in a metric space  $\mathbb{R}^n$ .

Since  $D_{\alpha,\delta}$  is compact and  $f$  is Lipschitz on  $D_{\alpha,\delta}$ , there exists  $L$  such that

$$\|f(x, t) - f(y, t)\| \leq L\|x - y\|.$$

Thus,

$$\leq L \int_{t_0}^t \|y_m(s) - y_{m-1}(s)\| ds.$$

Using (\*\*),

$$\leq L^m M_{\alpha,\delta} \frac{1}{(m-1)!} \int_{t_0}^t (s - t_0)^{m-1} ds = L^m M_{\alpha,\delta} \frac{(t - t_0)^m}{m!}.$$

Hence (\*\*) holds.

In particular, for all  $\rho \geq 1$ ,

$$\|y_\rho(t) - y_{\rho-1}(t)\| \leq M_{\alpha,\delta} \frac{(L(t - t_0))^\rho}{(\rho)!} < \frac{M_{\alpha,\delta}}{L} \frac{(L\epsilon)^\rho}{\rho!}.$$

Let  $m, p \geq 1$ :

$$\|y_{m+p}(t) - y_{m+1}(t)\| \leq \sum_{k=1}^{p-1} \|y_{m+k+1}(t) - y_{m+k}(t)\|.$$

So,

$$< \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{m+p} \frac{(L\epsilon)^j}{j!}.$$

Since  $e^{L\epsilon} = \sum_{j=0}^{\infty} \frac{(L\epsilon)^j}{j!}$  converges,

$$\rightarrow_{m,p \rightarrow +\infty} 0.$$

Thus  $\{y_k(t)\}$  is Cauchy and converges to  $y(t)$ .

Taking limits in the iteration,

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds.$$

□

---

<sup>6</sup>Evaluated on compacted cylinder, understanding epsilon, the L lipschitz constant coming from somewhere, not on the analysis background such as...

### Uniqueness

*Proof.* Assume  $y(t)$  and  $z(t)$  solve the IVP. Then

$$\|y(t) - z(t)\| \leq \int_{t_0}^t \|f(y(s), s) - f(z(s), s)\| ds \leq L \int_{t_0}^t \|y(s) - z(s)\| ds.$$

Define  $g(t) = \int_{t_0}^t \|y(s) - z(s)\| ds$ . Then

$$g'(t) \leq Lg(t).$$

So,

$$\frac{d}{dt}(e^{-L(t-t_0)}g(t)) \leq 0.$$

Thus  $e^{-L(t-t_0)}g(t)$  is decreasing and

$$0 \leq g(t) \leq g(t_0) = 0.$$

Hence  $g(t) = 0$  and  $y(t) = z(t)$ . □

Examples on how to use the theorem.

$$\begin{cases} y' = f(y, t) \\ y(t_0) = y_0 \end{cases}$$

Th.  $\exists \epsilon!$  says there is a solution for this IVP, and it is unique.

$$y : (t_0 - \epsilon, t_0 + \epsilon) \rightarrow \mathbb{R}^n$$

solves the IVP

$$\epsilon = \min(\delta, \frac{\alpha}{M_{\alpha, \delta}})$$

$$f : D \times (a, b) \rightarrow \mathbb{R}^n$$

Choose  $\alpha, \delta$  such that  $D_{\alpha, \delta} \subset D \times (a, b)$ .

$$\{(y, t) : \|y - y_0\| \leq \alpha, |t - t_0| \leq \delta\} \subset D \times (a, b)$$

$$M_{\alpha, \delta} = \sup_{(y, t) \in D_{\alpha, \delta}} \|f(y, t)\| < +\infty$$

Demand to obtain epsilon from the theorem in this ODE course, nothing else, since it is not an analysis course.

**Example.**

$$\begin{cases} y' = y + 1 \\ y(0) = 1 = y_0 \\ t_0 = 0 \end{cases}$$

$$f(y) = y + 1$$

If  $f$  is a  $C^1$  function, then it is locally Lipschitz in  $y$  and the theorem applies.

$$f : D \rightarrow \mathbb{R}, D = \mathbb{R} = (-\infty, +\infty)$$

Since  $f$  is defined everywhere, then there are no constraints on  $\alpha, \delta$ .

$$D_{\alpha, \delta} = \{(y, t) : \|y - 1\| \leq \alpha, |t - 0| \leq \delta\} \subset \mathbb{R} \times \mathbb{R} \forall \alpha, \delta > 0$$

$$D_{\alpha, \delta} = [1 - \alpha, 1 + \alpha] \times [-\delta, \delta] \subset \mathbb{R} \times \mathbb{R}$$

$$M_{\alpha, \delta} = \sup_{(y, t) \in D_{\alpha, \delta}} \|f(y, t)y + 1\| = \sup_{y \in [1 - \alpha, 1 + \alpha]} \|y + 1\| = 2 + \alpha$$

$$\epsilon = \min(\delta, \frac{\alpha}{2 + \alpha})$$

Pick  $\alpha = 1, \delta = 1$

$$\epsilon = \min(1, \frac{1}{3}) = \frac{1}{3}$$

Therefore, there exists a unique solution on the interval  $(-\frac{1}{3}, \frac{1}{3})$ . Pick  $\alpha = 3, \delta = 2$

$$\epsilon = \min(2, \frac{3}{5}) = \frac{3}{5}$$

Therefore, there exists a unique solution on the interval  $(-\frac{3}{5}, \frac{3}{5})$ . Since there are no constraints on  $\alpha, \delta$ , we can make  $\epsilon$  as large as we want, therefore the solution exists and is unique on  $\mathbb{R}$ . The "maximal" times interval guaranteed by the theorem of  $\exists \&! is  $J = (-1, 1)$ . In chapter 2 we will see that the solution is  $y(t) = 2e^t - 1$ . In fact the maximal time interval on which the solution is defined is  $J_{max} = \mathbb{R}$$

**Example.**

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

Let  $y \in C^1 \rightarrow LLC$ , therefore there exists a solution.

$$M_{\alpha, \delta} = \sup_{y \in [1 - \alpha, 1 + \alpha]} \|y^2\| = (1 + \alpha)^2$$

$$\epsilon = \min(\delta, \frac{\alpha}{(1 + \alpha)^2})$$

$$h(\alpha) = \frac{\alpha}{(1 + \alpha)^2}$$

$$h'(\alpha) = \frac{1 - \alpha}{(1 + \alpha)^3}$$

$$h'(\alpha) = \frac{1 - \alpha^2}{blablabla} = 0$$

Pick  $\alpha = 1, \delta = 104073$

$$\epsilon = \frac{1}{4}$$



$\exists!$  solution on  $y : (-\frac{1}{4}, \frac{1}{4}) \rightarrow \mathbb{R}$  to the IVP. Chapter 2 will show that the answer is in the form

$$y(t) = \frac{1}{1-t}$$

draw a cartesian plane and an exponential function and the asymptote is named the finite time blowup.

**Example.**

$$\begin{cases} y' = t^2 + y^2 = f(y, t) \\ y(0) = 0 \end{cases}$$

$$D_{\alpha, \delta} = [-\alpha, \alpha] \times [-\delta, \delta]$$

$$M_{\alpha, \delta} = \sup_{(y, t) \in D_{\alpha, \delta}} |t^2 + y^2| = \delta^2 + \alpha^2$$

$$\epsilon = \min(\delta, \frac{\alpha}{\delta^2 + \alpha^2})$$

Pick  $\alpha = 10^{12}, \delta = 1$

$$\epsilon = \min(1, \frac{10^{12}}{1 + 10^{24}}) = \frac{10^{12}}{1 + 10^{24}} \approx 0$$

**Example** (LLC matters).

$$\begin{cases} y' = 3y^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$$

$$y(t) = 0, \forall y \text{ is a solution}$$

$$f(y) = 3y^{\frac{2}{3}}$$

$$f'(y) = 2y^{-\frac{1}{3}}$$

This function is not locally Lipschitz continuous at  $y = 0$ . In chapter 2, we will learn how to solve this separable ODE (assuming  $y \neq 0$ ) we will get  $y_2(t) =$

$$\begin{cases} t^3, t \geq 0 \\ 0, t < 0 \end{cases}.$$

$$y'_2 = \frac{d}{dt}(t^3) = 3t^2 = 3(y_2(t))^{\frac{2}{3}}$$

The solution is not unique.

## 4 First-Order Scalar Equation

January 22,  
2026

$$y' = f(y, t) \in \mathbb{R} \quad (n = 1) \text{ One equation}$$

## 4.1 First order linear equations

$$a_0(t)y' + a_1(t)y = g(t)$$

where  $a_0, a_1, g$  are functions of  $t$ . Dividing by  $a_0$  leads to

$$y' + p(t)y = q(t)$$

where  $p(t) = \frac{a_1(t)}{a_0(t)}$  and  $q(t) = \frac{g(t)}{a_0(t)}$ . Multiply the ODE by an integrating factor  $\mu(t)$  (to be determined).

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)q(t)$$

Requirement in  $\mu$  is that  $\mu(t)y' + \mu(t)p(t)y = \frac{d}{dt}(\mu(t)y(t))$  (\*) holds. In this case, integrate with respect to  $t$ .

$$\mu(t)y(t) = \int \mu(t)q(t)dt + C,$$

where  $C \in \mathbb{R}$  is a constant of integration. We therefore get

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)q(t)dt + C.$$

Let us find  $\mu$  so that (\*) holds.

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)y' + \mu(t)p(t)y = \mu'(t)y(t) + \mu(t)y',$$

this implies that

$$\mu'(t)y(t) = \mu(t)p(t)y(t).$$

It is sufficient to have that

$$\mu'(t) = \mu(t)p(t).$$

Now we can solve for  $p(t)$

$$\frac{d}{dt} \ln|\mu(t)| = \frac{\mu'(t)}{\mu(t)} = p(t),$$

now integrate with respect to  $t$

$$\ln|\mu(t)| = \int p(t)dt + C,$$

where  $C \in \mathbb{R}$  is a constant of integration, and now we exponentiate

$$|\mu(t)| = e^{\int p(t)dt + C_1} = e^{C_1} e^{\int p(t)dt}$$

$$\mu(t) = (\pm e^{C_1}) e^{\int p(t)dt}.$$

We can therefore say choose  $\mu(t) = e^{\int p(t)dt}$  (integrating factor). This is the general solution of the first order linear equation.

**Example.**

$$\begin{aligned}
 y' - 2y &= 3e^t \\
 p(t) &= -2, q(t) = 3e^t \\
 \mu(t) &= e^{\int -2dt} = e^{-2t} \\
 e^{-2t}(y' - 2y) &= e^{-2t}3e^t = 3e^{-t} \\
 e^{-2t}(y' - 2y) &= \frac{d}{dt}(e^{-2t}y(t))
 \end{aligned}$$

integrate with respect to  $t$

$$e^{-2t}y(t) = \int e^{-2t}3e^t dt + C = -3e^{-t} + C$$

Using this explicit solution

$$y(t) = \frac{1}{\mu(t)} \left[ \int \mu(t)q(t)dt + C \right]$$

we can find the general solution

$$y(t) = e^{2t}(-3e^{-t} + C) = -3e^t + Ce^{2t}$$

If  $y(0) = 1$ , then  $y(0) = -3 + C = 1$ , so  $C = 4$ .

$$y(t) = -3e^t + 4e^{2t}$$

**Theorem 5** (Existence & Uniqueness for Linear Equations).

$$\begin{cases} y' + p(t)y = q(t) \\ y(t_0) = y_0 \end{cases}$$

Assume that the functions  $p$  and  $q$  are continuous on some interval  $(a, b)$ , where  $(-\infty \leq a < b \leq \infty)$ , and assume that  $t_0 \in (a, b)$ . Then there is a unique solution  $y(t)$  to the IVP and  $J_{max} = (a, b)$ .

**Example.**

$$\begin{cases} y' + \frac{1}{t-1}y = \frac{1}{\cos(t)} \\ y(0) = 1 \end{cases}$$

$$J_{max} = \left(-\frac{\pi}{2}, 1\right)$$

**Example** (Falling Object). Imagine a falling object, subject to gravity and air drag.

$$v'(t) + \frac{\gamma}{m}v(t) = g$$

This is a first order linear equation. The integrating factor is

$$\mu(t) = e^{\int \frac{\gamma}{m} dt} = e^{\frac{\gamma}{m}t}$$

$$\frac{d}{dt}(e^{\frac{\gamma}{m}t}v(t)) = e^{\frac{\gamma}{m}t}(v' + \frac{\gamma}{m}v) = e^{\frac{\gamma}{m}t}g$$

We get

$$e^{\frac{\gamma}{m}t}v(t) = g \int e^{\frac{\gamma}{m}t} dt + C = \frac{gm}{\gamma} e^{\frac{\gamma}{m}t} + C$$

$$v(t) = \frac{gm}{\gamma} + Ce^{-\frac{\gamma}{m}t}$$

where  $C \in \mathbb{R}$ .

$$v(0) = \frac{gm}{\gamma} = \frac{gm}{\gamma} + C$$

Which implies

$$C = 0$$

$$y(t) = \frac{gm}{\gamma}$$

**Example** (Mixing Problems). *Imagine a tank, at  $t = 0$ , the tank contains 100 litres of brine water, in which  $y_0$  grams of salt is dissolved. You pour in 50g of salt per litre in the tank, and there is a flow in of water at a rate of  $R$  litres per second. Water is flowing out of the tank with a rate of  $R$  litres per seconds. Denote by  $y(t)$  the quantity of salt in grams in the tank at time  $t$  in seconds.  $\frac{dy}{dt}$  is the rate in - the rate out.*

$$\frac{dy}{dt} = 50[\frac{\text{grams}}{\text{litres}}] \times R[\frac{\text{litres}}{\text{seconds}}] - \frac{y(t)}{100}[\frac{g \text{ of salt}}{\text{litres}}]$$

We get now that

$$y' = 50R - \frac{R}{100}y$$

Rewrite it as

$$y' + \frac{R}{100}y = 50R$$

and

$$y(0) = y_0$$

Now we can solve this IVP.

$$\mu(t) = e^{\frac{Rt}{100}}$$

$$\frac{d}{dt}(e^{\frac{Rt}{100}}y(t)) = e^{\frac{Rt}{100}} \cdot 50R$$

$$e^{\frac{Rt}{100}}y(t) = 50R \frac{100}{R} e^{\frac{Rt}{100}} + C = 5000e^{\frac{Rt}{100}} + C$$

$$y(t) = 5000 + Ce^{-\frac{Rt}{100}}$$

$$y(0) = 5000 + C = y_0$$

$$C = y_0 - 5000$$

$$y(t) = 5000 + (y_0 - 5000)e^{-\frac{Rt}{100}}$$

Draw graph with threshold 5000, and two exponentials approaching the threshold from the top and below.

## 4.2 Separable Equations

Notes:

$$\frac{dy}{dx} = f(x, y) = f_1(x)f_2(y) \rightarrow$$

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + C$$

We call the ODE separable if  $f(x, y) = f_1(x)f_2(y)$

**Example.**

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$f_1(x) = x, \quad f_2(y) = -\frac{1}{y}$$

**Example.** This following ODE is not separable

$$\frac{dy}{dx} = x^2 + y^2$$

$$\frac{dy}{dx} = f_1(x)f_2(y) \leftrightarrow -f_1(x) + \frac{1}{f_2(y)} \frac{dy}{dx} = 0 \leftrightarrow M(x) + N(y) \frac{dy}{dx} = 0$$

Consider  $H_1(x)$  and  $H_2(y)$  antiderivatives of  $M(x)$  and  $N(y)$ , respectively. From the Chain Rule  $\frac{d}{dx} H_2(y(x)) = H_2'(y) \frac{dy}{dx}$

$$H_1'(x) + H_2'(y) \frac{dy}{dx} = 0 \leftrightarrow \frac{d}{dx} (H_1(x) + H_2(y(x))) = 0$$

$$\rightarrow H_1(x) + H_2(y(x)) = C$$

is an implicit general solution.

$$\leftrightarrow \int M(x)dx + \int N(y)dy = C$$

$$\leftrightarrow \int -f_1(x)dx + \int \frac{dy}{f_2(y)} = C$$

$$\leftrightarrow \int \frac{dy}{f_2(y)} = \int f_1(x)dx + C, \quad C \in \mathbb{R}$$

**Example.**

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

$$\int (1-y^2)dy = \int x^2dx + C$$

$$\Psi(x, y) = y - \frac{y^3}{3} - \frac{x^3}{3} = C$$

**Example** (Logistic Equation).

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

This ODE is non-linear but is separable.

$$\begin{aligned} \longrightarrow \int \frac{KdN}{N(N-K)} &= - \int rdt + C \\ \int \frac{K}{N(N-K)} &= -rt + C \end{aligned}$$

Using the method of partial fractions.

$$\frac{K}{N(N-K)} = \frac{A}{N} + \frac{B}{N-K} = \frac{A(N-K) + BN}{N(N-K)}$$

$$K = (A+B)N - AK \longrightarrow A+B=0, \quad -AK=K, \quad \therefore A=-1, B=1$$

$$\longrightarrow -\ln|N| + \ln|N-K| = -rt + C$$

$$\longrightarrow \ln\left|\frac{N-K}{N}\right| = -rt + C$$

$$\longrightarrow \left|\frac{N-K}{N}\right| = e^{-rt+C} = C_1 e^{-rt}$$

$$\longrightarrow \frac{N-K}{N} = (\pm C_1) e^{-rt} = C_2 e^{-rt}$$

At  $t=0$ ,  $N(0) = N_0$

$$N-K = C_2 N e^{-rt}$$

$$\longrightarrow N(t) = \frac{K}{1 - C_2 e^{-rt}} = \frac{K}{\frac{N_0}{N_0} - \left(\frac{N_0-K}{N_0}\right) e^{-rt}}$$

$$\longrightarrow N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}$$

If  $N_0 = 0$ , then  $N(t) = 0, \forall t$ . If  $N_0 = K$ , then  $N(t) = K, \forall t$ . If  $N_0 \in (0, K)$ ,  $N_0 \leq N_0 + (K - N_0)e^{-rt}$  as  $N_0$  decreases  $\longrightarrow N(t) \rightarrow \frac{KN_0}{N_0} = K$ . If  $N_0 > K$ , verify that  $N(t) \rightarrow K$  as  $t \rightarrow +\infty$ .

Denote  $\Psi(x, y) = H_1(x) + H_2(y)$  is the potential function. Note that the general solution  $\Psi(x, y) = C$  is given by the level curves of the potential function.

**Definition 14** (Integral Curve). An integral curve is a level curve of the potential function.

**Example.**

$$\frac{dy}{dx} = -\frac{x}{y} \longrightarrow \int ydy = \int -xdx + C$$

$$\longrightarrow \Psi(x, y) = \frac{y^2}{2} + \frac{x^2}{2} = C, \quad (C \geq 0)$$

this is a general implicit solution.

**Example.**

$$x^2 + y^2 = (\sqrt{2C})^2$$

Assume that  $y(0) = 2$  and  $x_0 = 0$

$$\frac{x_0^2}{2} + \frac{y_0^2}{2} = 2 = C$$

The unique integral curve that contains the initial condition  $(x_0, y_0) = (0, 2)$  is

$$\frac{x^2}{2} + \frac{y^2}{2} = 2$$

$$\longrightarrow y^2 = 4 - x^2$$

$$\longrightarrow y(x) = \pm \sqrt{4 - x^2}$$

$$\longrightarrow y(x) = \sqrt{4 - x^2}, \quad J_{max} = (-2, 2)$$

An integral curve is a geometric object that contains (possibly many) solutions. The unique integral curve that contains the initial condition  $y(x_0) = y_0$  is defined by

$$\Gamma_{(x_0, y_0)} = \{(x, y) | \Psi(x, y) = \Psi(x_0, y_0) = C(x_0, y_0)\}$$

$$\frac{d}{dx} \left[ \int f_1(x) dx + \int \frac{dy}{f_2(y)} = C \right]$$

**Definition 15** (Separable Equations). *An equation of the form*

$$\begin{cases} y' + p(t)y = q(t) \\ y(0) = y_0 \end{cases}$$

where  $p(t)$  and  $q(t)$  are functions of  $t$  is called a separable equation.

## 5 Systems of Linear Equations

## 6 Second and Higher-Order Scalar Linear Equations

## 7 Stability, Phase Portraits and Orbits

## 8 Laplace Transform

## 9 Power Series Solutions and Numerical Methods

## 10 Solutions

**Solution 1** (System of First Order ODEs). *Define new variables:*

$$y_1 = y, \quad y_2 = y', \quad y_3 = y''.$$

*Now we can rewrite the third order ODE into a first order system:*

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -4y_2 + y_1 \end{cases}$$

**Solution 2** (Existence and Uniqueness Theorem). 1. *For this IVP,  $y' = \sqrt{1+y^2} + t^2$ , the function is simply  $f(y, t) = \sqrt{1+y^2} + t^2$ .*

2. *Check the hypotheses:*

- *$f$  is continuous  $\forall y \in \mathbb{R}, t \in \mathbb{R}$  because square root of  $1+y^2$  and  $t^2$  are continuous everywhere.*
- *Check local Lipschitz in  $y$ :  $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{1+y^2}}$ , which is continuous  $\forall y$ . Therefore  $f$  is locally Lipschitz in  $y$ . So the hypotheses of  $E \& U$  th. are satisfied in  $\mathbb{R} \times \mathbb{R}$ .*
- *Since  $f$  is continuous and locally Lipschitz in  $y$ , and since  $y(1) = 0$  with  $0 \in \mathbb{R}$ , the Existence and Uniqueness Theorem guarantees that there exists an open interval  $J$  containing  $t = 1$  on which a solution exists, and this solution is unique on  $J$ .*

## 11 Appendix

## 12 Useful Links