

MATH 325: Honours Ordinary Differential Equations

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Abstract

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1 Introduction

Jean-Philippe Lessard (Burnside 1119). Tutorials every wednesday from 9am to 10am, ENGTR 0070, with Eunpyo Bang. Office hours thursday. No textbooks. 25% assignments (2 written assignments 15%, and 5 webworks 10%). 25% Midterm (February 16 - inclass). 50% Final. Since its honours you will deal with analysis.

2 Prerequisite knowledge

2.1 Analysis

3 Intro, Classification, Theorem of Existence & Uniqueness

3.1 Intro

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Definition 1 (Differential Equation). *A differential equation (DE) is a relation that involves an unknown function and some of its derivatives.*

To better understand what a differential equation is, consider the following example.

Imagine a ball of mass m falling, subject to gravity and air resistance (drag). Denote by $v(t)$ the velocity of the ball at time t , whereas t is the independent variable, and v the dependent variable. Let the downward direction be positive. We know the force of gravity is given by $F_g = mg$, where g is the acceleration due to gravity. The drag force is given by $F_d = -\lambda v$, where λ is the drag coefficient and is $\lambda \geq 0$. According to Newton's second law $\sum F = ma$, the net force acting on the ball is equal to its mass times its acceleration

$$m \frac{dv}{dt} = mg - \lambda v.$$

Let $y(t)$ be the position, meaning $v(t) = \frac{dy}{dt}$. Then, we can rewrite the above equation as

$$my'' + \lambda y' = mg.$$

Let's analyze another example, population growth (known as the Malthusian growth model).

Denote by $N(t)$ the size of a given population at time t . In an "unconstraint" environment, it is reasonable to assume that the rate of change of the number of individuals is proportional to the number of individuals present. This assumption leads to the following differential equation:

$$\frac{dN}{dt} = rN,$$

where r is called the growth rate (if $r > 0$), and decay rate (if $r < 0$). Assume that $N > 0$. Using the chain rule and assuming that $N(t)$ satisfies $N' = rN$

$$\frac{d}{dt} \ln(N(t)) = \frac{d\ln(N)}{dN} \cdot \frac{dN}{dt} = \frac{1}{N} \cdot N' = r,$$

integrate with respect to t

$$\ln(N(t)) = rt + C,$$

where C is the constant of integration. Exponentiating both sides, we obtain

$$N(t) = e^{\ln(N(t))} = e^C e^{rt} = k e^{rt},$$

where $\{k > 0 | k \in \mathbb{R}\}$ which could be any positive constant is the initial population size at time $t = 0$.

Assume that an initial population (condition) is given:

$$N(0) = N_0 (\text{fixed}),$$

we therefore get that $k = N_0$, and the unique solution that satisfies the initial condition is

$$N(t) = N_0 e^{rt}.$$

The problem with the answer we got in the previous example is that it is not realistic in the long run, how about we consider a carrying capacity¹. This leads us to another example: Population growth/decay with the carrying capacity of the environment.

Now assume that our growth rate depends on the population size $N(t)$ itself, therefore we get that

$$\frac{dN}{dt} = R(N)N.$$

Denote by K the number of individual that the environment can carry. K is called the carrying capacity of the environment. If $N < K$, we want growth ($R(N) > 0$) and if $N > K$, we want decay ($R(N) < 0$).

¹maximum population size that the environment can sustain indefinitely



Let's pick the simplest function $R(N)$ that satisfies $R(0) = r$, $R(K) = 0$ and is linear. We get that

$$R(N) = r\left(1 - \frac{N}{K}\right).$$

Therefore, our differential equation becomes

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N = \frac{r}{K}(K - N(t))N(t).$$

This is called the logistic equation.

Definition 2 (Ordinary Differential Equation). *An ordinary differential equation (ODE) is a differential equation whose unknown function depends on one independent variable only.*

Example of ODEs:

- $y''(t) + y'(t) + 2y(t) = \sin(t)$
- $N'(t) = rN(t)$
- $mv'(t) = mg - \lambda v(t)$
- $y'(x) + 3y(x) = e^x$

Definition 3 (Partial Differential Equation). *A partial differential equation (PDE) is a differential equation whose unknown function depends on more than one independent variable. Will not be taught in this course.*

Example of a PDE is the Heat Equation. Let $u = u(x, t)$, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. This PDE denotes the temperature of a body at time t and at position x .

3.2 Classification

3.2.1 The Order

Definition 4. *The order of an ODE is the order of the highest derivative that appears in the equation.*

Example. $N' = rN$ (first order ODE)

Example. $y''(t) + 2y'(t) = e^t$ (second order ODE)

Given $n \in \mathbb{N}$, an n^{th} order scalar ODE is written as

$$F(t, y(t), y'(t), y''(t), \dots, y^{(n)}(t)) = 0,$$

where $F : \mathbb{R}^{n+2} \rightarrow \mathbb{R}$ is a map and where $y^{(k)}(t) = \frac{d^k y}{dt^k}(t)$, $k = 1, \dots, n$.

Systems of first order ODEs

Consider a map $f : D \times (a, b) \rightarrow \mathbb{R}^n$, where $D \subseteq \mathbb{R}^n$ is an open set, and (a, b) is a "time" interval. A general first order system of ODEs is given by

$$y'(t) = f(y(t), t), \text{ where } y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \text{ and } y' = \begin{pmatrix} y'_1 \\ \vdots \\ y'_n \end{pmatrix}.$$

Remark: Assume that a scalar n^{th} order ODE has the form

$$y^{(n)}(t) = G(t, y(t), y'(t), \dots, y^{(n-1)}(t)).$$

Letting $y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$. This leads us to $y'_1 = y' = y_2, y'_2 = y'' = y_3, \dots, y'_{n-1} = y^{(n-1)} = y_n, y'_n = y^{(n)} = G(t, y_1, y_2, \dots, y_n)$. Therefore, we can rewrite the n^{th} order ODE as a first order system of ODEs:

$$\begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_{n-1} \\ y'_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ G(t, y_1, y_2, \dots, y_n) \end{pmatrix}$$

Example (Lorenz Equation).

$$y'_1 = \sigma(y_2 - y_1)$$

$$y'_2 = \rho y_1 - y_2 - y_1 y_3$$

$$y'_3 = y_1 y_2 - \beta y_3$$

where σ, ρ, β are parameters. $n = 3$. This is a first order system of ODEs. They are nonlinear because of the products $y_1 y_3$ and $y_1 y_2$.

3.2.2 Linearity

Definition 5 (Linearity). *The n^{th} order ODE $F(t, y, y', \dots, y^{(n)}) = 0$ is linear if F is a linear polynomial in the variables $y, y', y'', \dots, y^{(n)}$, that is, it is of the form $a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \dots + a_{n-1}(t)y'(t) + a_n(t)y(t) = g(t)$, where a_0, a_1, \dots, a_n, g are given functions of t . Otherwise, it is nonlinear.*

The system $y' = f(y, t)$ is said to be linear if it can be written as $y'(t) = A(t)y(t) + r(t)$ where, given $t \in (a, b)$, $A(t) \in M_n(\mathbb{R})$ (the set of $n \times n$ real matrices) and $r(t) \in \mathbb{R}^n$. Otherwise, it is nonlinear.

Example. Consider the second-order ODE

$$\begin{aligned} y'' + 2y' + y &= e^t \\ \implies y'' &= -2y' - y + e^t. \end{aligned}$$

Define new variables

$$y_1 = y, \quad y_2 = y'.$$

Then the system becomes

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad r(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

3.2.3 Autonomy

Definition 6 (Autonomy). *The n^{th} order ODE $F(t, y, y', \dots, y^{(n)}) = 0$ is autonomous if F does not depend explicitly on t , that is, if it is of the form $F(y, y', \dots, y^{(n)}) = 0$. Otherwise, it is non-autonomous.*

Example. $y'' + 2y' + y - e^t = 0$ is non-autonomous.

Example. $N'(t) = rN(t)$ is autonomous.

Example. $y'(t) = ty(t)$ is non-autonomous.

Equivalently, a first-order system $y' = f(y, t)$ is autonomous if it can be written as

$$y' = f(y).$$

Otherwise, it is non-autonomous.

Note: The Lorenz system is an example of an autonomous system².

²Here “system” means the unknown y is vector-valued, e.g. $y \in \mathbb{R}^m$, rather than scalar.

3.2.4 Solutions of ODEs

Definition 7 (Solutions of ODEs). Let $f : D \times (a, b) \rightarrow \mathbb{R}^n$. A solution of $y'(t) = f(y(t), t)$ on an interval $J \subset \mathbb{R}$ is a differentiable function $y : J \rightarrow D \subset \mathbb{R}^n$, such that $y'(t) = f(y(t), t)$, $\forall t \in J$. t is the independent variable, and $y = (y_1, \dots, y_n)$ is the dependent variable.

Explicit Solutions

Example. Consider the ODE

$$y' + y = 1.$$

We can verify that $y(t) = e^{-t} + 1$, and therefore $y'(t) = -e^{-t}$, is a solution on \mathbb{R} . Indeed,

$$y' + y = -e^{-t} + (e^{-t} + 1) = 1.$$

In this example, $y = y(t)$ is explicitly given as a function of t (independent variable).

Implicit solutions

Example. Consider the ODE

$$y \frac{dy}{dx} = x.$$

This is a nonautonomous, nonlinear, first-order scalar ODE. Separating variables gives

$$y dy = x dx.$$

Integrating,

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C,$$

or equivalently, the implicit solution

$$x^2 - y^2 = C, \quad C \in \mathbb{R}.$$

To verify, differentiate implicitly:

$$\frac{d}{dx}(x^2 - y^2) = 0 \implies 2x - 2y \frac{dy}{dx} = 0 \implies y \frac{dy}{dx} = x.$$

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3.3 Initial Value Problems

$$y' = f(y, t), \quad f : D \times (a, b) \rightarrow \mathbb{R}^n.$$

Consider $t_0 \in (a, b)$ (initial time) An initial condition is given by $y(t_0) = y_0 \in \mathbb{R}^n$, where y_0 is given. An initial value problem is of the form

$$y' = f(y, t), \quad y(t_0) = y_0.$$

3.4 Theorem of Existence & Uniqueness

Definition 8. Consider a set $D \subseteq \mathbb{R}^n$ and choose a norm $\|\cdot\|$ on \mathbb{R}^n (e.g. $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ the Euclidean norm). A function $f : D \rightarrow \mathbb{R}^n$ is Lipschitz continuous if there exists a constant $L \geq 0$ such that $\forall y_1, y_2 \in D, \|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\|$.

The smallest L which satisfies the inequality is called the Lipschitz constant and is denoted $\text{Lip}(f)$.

Example. Consider $f(y) = 4y - 5$ ($n = 1, D = \mathbb{R}$) $f : \mathbb{R} \rightarrow \mathbb{R}$. $\|\cdot\| = |\cdot|$ (absolute value). Let $y_1, y_2 \in \mathbb{R}$,

$$|f(y_1) - f(y_2)| = |4y_1 - 4y_2| = 4|y_1 - y_2| \quad (\text{LC})\text{Lip}(f) = 4.$$

Example. Let

$$f(y) = \frac{1}{y-1}$$

and $D = (1, \infty)$.

Draw the function, with the asymptote at $x = 1$.

$\therefore f : D \rightarrow \mathbb{R}$ is not LC.

Now, fix $\delta > 1$, and let $D_\delta = (\delta, +\infty)$.

Draw the function, with the asymptote at $x = 1$, but now add a line on the x axis starting at a later point than $x = 1$ and this new point is delta with a line continuing to infinity called D sub delta, and pick two points on D sub delta named y_1 and y_2 .

Then, $f : D_\delta \rightarrow \mathbb{R}$ is LC. Indeed, let $y_1 < y_2 \in D_\delta$. By the Mean Value Theorem, there exists $z \in (y_1, y_2)$ such that

$$f(y_2) - f(y_1) = f'(z)(y_2 - y_1),$$

where $f'(y) = -\frac{1}{(y-1)^2}$

$$f(y_2) - f(y_1) = \left| -\frac{1}{(z-1)^2} \right| |y_2 - y_1|$$

$$f(y_2) - f(y_1) \leq \left(\sup_{z \in D_\delta} \frac{1}{(z-1)^2} \right) |y_2 - y_1| = \frac{1}{(\delta-1)^2} |y_2 - y_1|,$$

where $\frac{1}{(\delta-1)^2}$ is $L = \text{Lip}(f)$.

For any $k \subset D$ compact (for instance $k = [\delta, \beta]$), $L = \frac{1}{(\delta-1)^2}$.

Definition 9. Consider $D \subseteq \mathbb{R}^n$ open. $f : D \rightarrow \mathbb{R}^n$ is locally LC, if any compact set $k \subset D$ (closed & bounded), there exists a constant $L = L(k)$ such that, for all $y_1, y_2 \in k$, $\|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\|$.

Problem 1. (See tutorial 1)

$$f \in C^1(D) \rightarrow f \text{ is LLC.}$$

Theorem 1 (Existence & Uniqueness). Consider $D \subseteq \mathbb{R}^n$ open and an open interval (a, b) which contains t_0 . Such as

$$\begin{cases} y' = f(y, t) \\ y(t_0) = y_0 \end{cases}$$

$f : D \times (a, b) \rightarrow \mathbb{R}^n$ is continuous and that, for all compact set $K \subset D \times (a, b) \subset \mathbb{R}^{n+1}$, there exists $L = L(K)$ such that (IVP):

$$||f(x, t) - f(y, t)|| \leq L ||x - y||$$

for all $(x, t), (y, t) \in K$. If $y_0 \in D$, then there exists an open interval J , containing t_0 , over which a solution to the (IVP) is defined. Moreover, the IVP has only one solution defined on J .

Lemma 2. y solves the IVP

$$\leftrightarrow y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds$$

Proof.

$$y'(t) = f(y(t), t), y(t_0) = y_0$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} y(t) - y(t_0) &= \int_{t_0}^t f(y(s), s) ds \\ y(t_0) &= y_0 + \int_{t_0}^{t_0} f(y(s), s) ds = y_0 \end{aligned}$$

Differentiating the integral equation:

$$y'(t) = \frac{d}{dt}(y_0 + \int_{t_0}^t f(y(s), s) ds) = 0 + f(y(t), t)$$

Since $(y_0, t_0) \in D \times (a, b)$ (open in \mathbb{R}^n), there exist $\alpha, \delta > 0$ such that the compact cylinder

$$D_{\alpha, \delta} = \{(y, t) \in \mathbb{R}^{n+1} \mid ||y - y_0|| \leq \alpha, |t - t_0| \leq \delta\} \subset D \times (a, b)$$

3D graph check phone (there is alpha and y0) Let $M_{\alpha, \delta} = \sup_{(y, t) \in D_{\alpha, \delta}} ||f(y, t)|| < +\infty$ \square

Lemma 3 (Picard Operator). Let $\epsilon > 0$ be defined by $\epsilon = \min(\delta, \frac{\alpha}{M_{\alpha, \delta}})$. Let $J = (t_0 - \epsilon, t_0 + \epsilon)$. Then for any function $y(t)$ which satisfies $y(t_0) = y_0$ and $(y(t), t) \in D_{\alpha, \delta}$ for all $t \in J$, then $T(y)$ defined by

$$T(y)(t) = y_0 + \int_{t_0}^t f(y(s), s) ds$$

also satisfies $(y(t_0) = y_0)$ and $(y(t), t) \in D_{\alpha, \delta}$ for all $t \in J$.

Proof.

$$T(y)(t_0) = y_0 + 0 = y_0$$

$$(T(y)(t), t) \in D_{\alpha,\epsilon} \forall t \in J?$$

Show:

$$\|T(y)(t) - y_0\| \leq \alpha$$

$$\begin{aligned}\|T(y)(t) - y_0\| &= \|y_0 + \int_{t_0}^t f(y(s), s) ds - y_0\| \\ &= \left\| \int_{t_0}^t f(y(s), s) ds \right\| \leq \int_{t_0}^t \|f(y(s), s)\| ds \leq M_{\alpha,\delta} \int_{t_0}^t ds \\ &= M_{\alpha,\delta} |t - t_0| (\text{where } t \in J) \leq M_{\alpha,\delta} \cdot \epsilon \leq \frac{\alpha}{M_{\alpha,\delta}} \cdot M_{\alpha,\delta}\end{aligned}$$

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10 Solutions

11 Appendix

12 Useful Links