polarized-beam neutron scattering using He-3:
transport and data analysis

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1 Introduction

This is a summary and recapitulation in terms of transfer matrices of some of the information in "Polarized ^{3}He in Neutron Scattering" by T.R. Gentile, and other texts on polarized-neutron beams.

2 Setup

A general polarized beam setup for neutron-scattering spectrometers using He-3 polarization cells, P1 (polarizer) and P2 (analyzer), is represented in the following diagram.

Typically the incoming beam is unpolarized so that $N_+ = N_- = \frac{1}{2}N$, where N is the total number of neutrons incident on P1. The detector, D, does not discriminate polarization states, and so counts $n = n_+ + n_-$. The detection system, D, may include energy analysis of the scattered neutrons. Here we assume that such energy analysis would have equal efficiencies for the two neutron spin states. In the above diagram, f1 and f2 are higher order wavelength filters, m1a, m1b, m2a and m2b are low efficiency beam intensity monitors with efficiency proportional to wavelength, and F1 and F2 are spin flippers.

3 Transfer Matrices

The detected counts can be calculated using transfer matrices for each device along the beam path that affects the neutron spin, so that

$$\begin{pmatrix} n_+ \\ n_- \end{pmatrix} = T \begin{pmatrix} N_+ \\ N_- \end{pmatrix} = T \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{N}{2}. \tag{1}$$

where the transfer matrix for the total beam path is the product of the transfer matrices for each beam component.

$$T = AF_A B_A S\, B_P F_P P \tag{2}$$

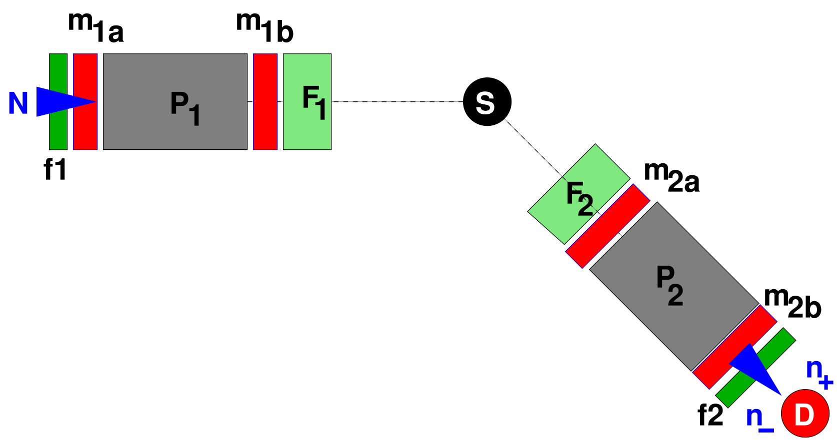

Figure 1: polarized beam triple-axis setup

and where A and P are the transfer matrices for the He-3 analyzer and polarizer, F_A and F_P are the transfer matrices for the flipper on the analyzer and polarizer sides of the sample, B_A and B_P are the transfer matrices for beam transport efficiency on the analyzer and polarizer sides of the sample, and S is the transfer matrix for the sample. It will be shown that the flipper and transport efficiency matrices commute so that the transport loss before the sample can occur anywhere between the polarizer and sample, and the analogous condition applies to the transport loss after the sample. Typically the sample transfer matrix, S, also commutes (is symmetric) with the flipper and transport loss matrices in which case the transport loss can occur anywhere between the He3 polarizer and analyzer.

3.1 He-3 polarizer/analyzer

The transmission of the He-3 polarizer is characterized by the two different absorption cross-sections: the neutron interacts with a polarized He-3 atom with its spin z-component aligned with the He-3 spin z-component, or the neutron interacts with a polarized He-3 atom with its spin z-component anti-aligned with that of the He-3. The total cross-sections for these two processes are $\sigma_+ = \sigma \uparrow\uparrow \cong 0$, and $\sigma_- = \sigma \uparrow\downarrow = 10666 barns \frac{\lambda}{1.78\text{Å}}$. Also $\sigma_0 = \frac{1}{2}(\sigma_+ + \sigma_-) \cong \frac{1}{2}\sigma_-$. The effective absorption coefficients (inverse of the absorption length) for each process are $\alpha \uparrow\uparrow = \sigma_+ n f \cong 0$, and $\alpha \uparrow\downarrow = \sigma_- n(1-f)$, where n is the number density of He-3 atoms in the polarizing cell and f measures the fraction of He-3 atoms that are polarized along the neutron spin direction. In order to symmetrize these expressions, define the He-3 polarization as

$$P_{He3} = \frac{n_{He3} \uparrow - n_{He3} \downarrow}{n_{He3} \uparrow + n_{He3} \downarrow} \quad (3)$$

where $n_{He3}\uparrow$is the number density of He-3 spins aligned with the external field quantization axis. This means

$$n_{He3}\uparrow,\downarrow=\frac{1}{2}(1\pm P_{He3})n$$

where $n=n_{He3}\uparrow+n_{He3}\downarrow$ is the He-3 number density.

Then if a neutron attempts to transit the He-3 polarizer with its spin $\uparrow$, the effective absorption coefficient is $\alpha_+=\sigma_- n_{He3}\downarrow=n\sigma_0(1-P_{He3})$, while if a neutron attempts to transit the He-3 polarizer with its spin $\downarrow$, the effective absorption coefficient is $\alpha_-=\sigma_- n_{He3}\uparrow=n\sigma_0(1+P_{He3})$

The ideal gas calculation of $n\sigma_0$ yields 0.13105 cm^{-1}x cell-pressure(bars at 293K) x neutron-wavelength/1.77 Angstroms. Then the transmission of the two neutron spin states is

$$t_\pm=t_E\exp\left(-\alpha_\pm L\right)=t_E\exp\left(-\tau_0[1\mp P_{He3}]\right) \tag{4}$$

where L is the path length through the He-3 cell, t_E is the transmission of an empty cell, and

$$\tau_0=n\sigma_0 L. \tag{5}$$

The wavelength dependence of τ is in $\sigma_0$so that

$$\tau_M=n\sigma_0(\lambda_0)L\frac{\lambda_M}{\lambda_0}=\tau_0\frac{\lambda_M}{\lambda_0},$$

and

$$\tau_{\pm M}=\tau_M\left(1\mp P_{He3}\right).$$

It will be shown at the end of this document (because the details are messy) that there are small corrections to the transmission formula due to wavelength and pathlength variations. Averaging the transmission over those variations produces the formula

$$<t_\pm>=\tilde{C}_\pm t_E\exp\left(-\tilde{\tau}_{\pm M}\right)$$

where $\tilde{C}_\pm\cong 1$ and $\tilde{\tau}_{\pm M}\cong\tau_{\pm M}$.

Once a He-3 cell is polarized and removed from the optical pumping system that produced the polarization, that polarization begins to decay exponentially with a characteristic time constant, t_C, that depends on the homogeneity of the magnetic-field on the cell (among other things). Thus

$$P_{He3}(t)=P_{He3}(t=0)\exp\left(-t/t_C\right).$$

This time dependence is an important consideration when checking transport efficiencies and performing data analysis so that it is necessary that it is measured. This is accomplished by measuring the total transmission of an unpolarized neutron beam through the He3 cell, both when it is polarized and unpolarized. It is important that these measurements are performed without higher order wavelength contamination present in the neutron beam (otherwise correcting for this is difficult).

The total transmission for an incident unpolarized neutron beam will be

$$t_0(P_{He}) = \frac{C_H}{C_0} = \frac{1}{2} t_E \left[\widetilde{C}_+ \exp\left(-\tilde{\tau}_{M+}\right) + \widetilde{C}_- \exp\left(-\tilde{\tau}_{M-}\right) \right]. \qquad (6)$$

where $C_H$is the observed count rate with the He-3 cell in the beam, and C_0 is the count rate without the cell. Explicitly this is,

$$t_0(P_{He}) = t_E \exp\left(-\tilde{\tau}_M\right) \left[< C > \cosh(\tilde{\tau}_M P_{He3}) + \Delta \sinh(\tilde{\tau}_M P_{He3}) \right],$$

where $< C >= (\widetilde{C}_+ + \widetilde{C}_-)/2 \cong 1$ and $\Delta = (\widetilde{C}_+ - \widetilde{C}_-)/2 \ll 1$. If the He-3 cell is unpolarized

$$t_0(0) = t_{00} = \frac{C_u}{C_0} =< C > t_E \exp\left(-\tilde{\tau}_M\right)$$

where C_u is the count rate with the unpolarized He-3 cell in the beam. $\tilde{\tau}_M$ can be determined as

$$\tilde{\tau}_M = \ln\left(\frac{< C > t_E}{t_{00}} \right),$$

with the squared relative uncertainty given as

$$\sigma_{\tilde{\tau}}^2 = \tilde{\sigma}_{t_E}^2 + \tilde{\sigma}_{C_u}^2 + \tilde{\sigma}_{C_0}^2.$$

In general the squared relative uncertainty for any measured variable is

$$\tilde{\sigma}_V^2 = \frac{\sigma_V^2}{V^2}.$$

Once $\tilde{\tau}_M$ is determined, the ratio, r, of the polarized cell transmission to the unpolarized cell transmission can be used to determine the He-3 polarization, P_{He3},

$$r(P_{He3}) = \frac{t_0(P_{He3})}{t_{00}} = \frac{C_H}{C_u} = \cosh(\tilde{\tau} P_{He3}) + \Delta \sinh(\tilde{\tau} P_{He3}). \qquad (7)$$

Neglecting the correction term in Δ, the coshfunction can be inverted to give

$$\tilde{\tau} P_{He3} \cong x_0 = \ln\left(r + \sqrt{r^2 - 1} \right).$$

If the small correction coefficent, Δ, is known then

$$\tilde{\tau} P_{He3} \cong x_0 - \Delta.$$

Error analysis on the determination of the He3 polarization gives

$$\tilde{\sigma}^2_{P_{He3}} = \tilde{\sigma}^2_{\tilde{\tau}} + (\tilde{\tau} P_{He3})^{-2} \frac{r^2}{r^2-1} \tilde{\sigma}^2_r,$$

where

$$\tilde{\sigma}^2_r = \frac{\sigma^2_r}{r^2} = \tilde{\sigma}^2_{C_H} + \tilde{\sigma}^2_{C_u},$$

Once the He-3 polarization is measured, its uncertainty increases with time because the polarization decays with time, so that

$$\tilde{\sigma}^2_{P_{He3}}(t) = \tilde{\sigma}^2_{P_{He3}} + \left(\frac{t}{t_C}\right)^2 \tilde{\sigma}^2_{t_C}.$$

Care must be taken when measuring transmissions with a detector where the efficiency is wavelength dependent and higher order wavelength contamination is present. A typical beam monitor is a fission detector where the efficiency is proportional to wavlength, $\epsilon = \epsilon_0 \lambda$. Then, neglecting beam pathlength and wavelength variation corrections, the measured transmission is

$$t_0(P_{He3}) = t_E \sum_{n=1} a_n \frac{1}{n} \exp\left(-\frac{1}{n}\tilde{\tau}_M\right) \cosh\left(\frac{1}{n}\tilde{\tau}_M P_{He3}\right) / \sum_{n=1} a_n \frac{1}{n}$$

$$t_{00} = t_E \sum_{n=1} a_n \frac{1}{n} \exp\left(-\frac{1}{n}\tilde{\tau}_M\right) / \sum_{n=1} a_n \frac{1}{n},$$

where a_n are the wavelength order fractions. It is obvious that wavelength contamination is problematic for using transmission measurements to determine the He-3 cell properties, especially if using a wavelength dependent detector.

The outgoing neutron polarization, $-1 \leq P_n \leq 1$, after an incident unpolarized beam passes through a polarized He3 cell is

$$P_{neutron} = \frac{n_+ - n_-}{n_+ + n_-} = \tanh(\tilde{\tau}_M P_{He3}) + \frac{\Delta}{\cosh^2(\tilde{\tau}_M P_{He3})}. \tag{8}$$

The transfer matrices for polarizer or analyzer are

$$P, A = t_E \begin{bmatrix} \widetilde{C}_+ \exp(-\tilde{\tau_{+M}}) & 0 \\ 0 & \widetilde{C}_- \exp(-\tilde{\tau_{-M}}) \end{bmatrix} = \begin{bmatrix} t_{+P,A} & 0 \\ 0 & t_{-P,A} \end{bmatrix}. \tag{9}$$

$$\widetilde{C}_\pm = \hat{C}_\pm \left\{1 + \sum_{n=2} a_n K_{\pm n}\right\}$$

$$\langle C \rangle = \frac{1}{2}\left(\widetilde{C}_+ + \widetilde{C}_-\right)$$

$$\Delta = \frac{1}{2}\left(\widetilde{C}_+ - \widetilde{C}_-\right)$$

$$\hat{C}_\pm = 1 + \frac{1}{2}\left(\tilde{\tau}_{\pm M}\frac{\sigma_\lambda}{\lambda_M}\right)^2 - \frac{1}{2}\tilde{\tau}_{\pm M} P\left(\sigma_\gamma^2 + \sigma_\delta^2\right)$$

$$P = 1 - \frac{L}{2R}\left(1 - \frac{\langle\rho^2\rangle}{LR}\right)$$

$$K_{\pm n} = \exp\left[\left(1 - \frac{1}{n}\right)\tilde{\tau}_{\pm M}\right] - 1$$

$$\tilde{\tau}_{\pm M} = \tau_{\pm M}\left(1 - \frac{\langle\rho^2\rangle}{LR}\right)$$

$$\tau_{\pm M} = \tau_M\left(1 \mp P_{He}\right)$$

$$\tau_M = n\sigma_0(\lambda_0)L\frac{\lambda_M}{\lambda_0} = \tau_0\frac{\lambda_M}{\lambda_0}$$

$$\tilde{\tau}_M = \tau_M\left(1 - \frac{\langle\rho^2\rangle}{LR}\right)$$

where the index M can refer to the polarizer (P) or analyzer (A). See the section on wavelength and pathlength corrections for definitions of the symbols in these expressions. For most practical applications the correction factors, $\widetilde{C}_\pm$, can be approximated as unity.

3.2 spin flipper

Excluding transport losses, the transfer matrix for a spin flipper when that flipper is ON, can be written in terms of the flipping efficiency, e_F, as

$$F_{P,A} = \begin{bmatrix} 1 - e_F & e_F \\ e_F & 1 - e_F \end{bmatrix}_{P,A} \tag{10}$$

Of course when the flipper is OFF this transfer matrix is the identity matrix which can be fudged by an effective flipping efficiency of $e_F = 0$. So the flipper-state dependent transfer matrix can be written

$$F_{P,A}^{\alpha=\pm} = \begin{bmatrix} 1-e_F^\alpha & e_F^\alpha \\ e_F^\alpha & 1-e_F^\alpha \end{bmatrix}_{P,A} \tag{11}$$

With $\alpha = -1$ indicating the flipper-ON state,

$$e_F^\alpha = \delta_{\alpha,-1} e_F = \frac{1}{2}(1-\alpha)\, e_F.$$

The efficiency of a standard spin-rotation (Mezei) flipper depends on the exact angle of the neutron spin as it exits the precession coil. It is assumed that the guide field outside the precession coil is precisely in the z-direction, and the neutron spin enters the precession coil with its spin precisely along this z-direction which is parallel to the wires carrying the precession coil current. It is also assumed that that current has been set so that neutrons with wavelength, λ_M, will precess by exactly π radians as it crosses the flipper on a path that is in the x-direction perpendicular to the precession coil wire surface. Variations in λ or the path direction will result in variations in the precession angle as the neutron leaves the precession coil. In the "sudden" approximation the probability that the neutron spin has flipped is just the modulus of the overlap of the exit spinor with the z-direction state. The rotation axis of the neutron spin is the +y-axis, so that if θ is the rotation angle of the neutron spin after it crosses the precession coil, then the exit spinor in the z-up coordinate system is

$$\chi_{exit} = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{pmatrix}.$$

The probability that this spinor state will be spin-down (flipped) is then just $\sin^2(\frac{\theta}{2})$. The actual precession angle of any given neutron just depends on the time, t, it spends inside the precession coil, since the precession rate is fixed by the uniform magnetic field inside the coil. Thus, if t_M is the optimum time in the coil that produces a π flip, and L_M is the corresponding minimum pathlength, the precession angle for a neutron with actual time in the coil, t, over pathlength, L, can be written as

$$\theta = \frac{t}{t_M}\pi = \frac{L\lambda}{L_M\lambda_M}\pi = (1+\frac{1}{2}\gamma^2+\frac{1}{2}\delta^2)(1+x)\pi = \pi+\epsilon,$$

where the actual pathlength as a function of horizontal and vertical deviation angles from the optimum perpendicular to coil direction is

$$L = L_M(1+\frac{1}{2}\gamma^2+\frac{1}{2}\delta^2),$$

and $\lambda/\lambda_M = 1+x$, so that $\epsilon = \left(x+\frac{1}{2}\gamma^2+\frac{1}{2}\delta^2\right)\pi$ to second order in the deviations. Then the flipper efficiency is

$$e_F = \sin^2(\frac{\pi+\epsilon}{2}) \cong 1-\frac{1}{4}\epsilon^2 \cong 1-\frac{\pi^2}{4}\left[x^2+x(\gamma^2+\delta^2)+\frac{1}{4}(\gamma^4+2\gamma^2\delta^2+\delta^4)\right].$$

Averaging over independent Gaussian probability distributions for the angle and wavelength deviations yields

$$e_F \cong 1 - \frac{\pi^2}{4}\left\{\left(\frac{\sigma_\lambda}{\lambda_M}\right)^2 + \frac{3}{4}\left(\sigma_\gamma^4 + \sigma_\delta^4\right) + \frac{1}{2}\sigma_\gamma^2\sigma_\delta^2\right\}.$$

Note that the angle deviations contibute to fourth order while the wavelength deviation contributes to second order. If $\sigma_\lambda/\lambda_M = 0.02$ the flipper efficiency is about 0.999.

3.3 transport losses

In order to account for transport losses in terms of a transport efficiency, e_t, use the matrix

$$B_{P,A} = \frac{1}{2}\begin{bmatrix} 1 + e_{tP,A} & 1 - e_{tP,A} \\ 1 - e_{tP,A} & 1 + e_{tP,A} \end{bmatrix} \tag{12}$$

Transport losses produce neutrons that have equal probability of being spin-up or spin-down (depolarized). Note that multiplying two transport loss matrices results in a transport loss matrix where the transport efficiency is just the product of the two separate efficiencies. Also the product of a spin-flip matrix and a beam transport loss matrix is

$$F^\alpha B = \frac{1}{2}\begin{bmatrix} 1 - e_t(2e_F^\alpha - 1) & 1 + e_t(2e_F^\alpha - 1) \\ 1 + e_t(2e_F^\alpha - 1) & 1 - e_t(2e_F^\alpha - 1) \end{bmatrix} \tag{13}$$

The symmetric matrices for the spin-flipper and transport losses commute.

How does transport loss occur? Consider a neutron travelling in a guide magnetic field along the z-direction that encounters a magnetic field perturbation. Take the magnetic field to vary in the reference frame of the neutron as

$$\overrightarrow{B}(t) = B_z\hat{z} + B_x G(t)\hat{x}$$

so that the time dependence is in the magnetic field component along the $\hat{x}$ direction. Take, for example, a Gaussian time perturbation of B_x,

$$G(t) = \exp\left[-\frac{1}{2}\left(\frac{t - t_0}{\tau}\right)^2\right].$$

If $B_x$is sufficiently small, a time-dependent perturbation solution based on an expansion in terms of the eigenstates when B_x is zero can be used. Such an expansion is

$$\overrightarrow{\chi}(t) = c_+(t)\,\exp(i\omega_z t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_-(t)\,\exp(-i\omega_z t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $\omega_z = \tilde{\gamma_n} B_z$ ($\tilde{\gamma_n} = 0.916x10^8 s^{-1} T^{-1}$is half the neutron gyromagnetic ratio). Substituting this solution into the spinor Schroedinger equation yields

$$\dot{c}_+ = i\omega_x G(t) c_- \exp(-2i\omega_z t)$$

$$\dot{c}_- = i\omega_x G(t) c_+ \exp(+2i\omega_z t)$$

where $\omega_x = \tilde{\gamma_n} B_x$. Satisfying the initial condition that $\overrightarrow{\chi}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, means that $c_+(0) = 1$ and $c_-(0) = 0$. Then the approximation is that c_+remains near 1 and c_-remains near zero during the perturbation, so solve only

$$\dot{c}_- = i\omega_x G(t) \exp(+2i\omega_z t).$$

Thus

$$c_-(T) = i\omega_x \int_0^T \exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\tau}\right)^2\right] \exp(+2i\omega_z t)\, dt.$$

or

$$c_-(\infty) = i\sqrt{2\pi}\omega_x \tau \ \exp\left[-2(\omega_z \tau)^2 + 2i\omega_z t_0\right].$$

Then the probability that the neutron ends up in the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state is

$$|c_-(\infty)|^2 = 2\pi \left(\omega_x \tau\right)^2 \ \exp\left[-4(\omega_z \tau)^2\right].$$

This result shows that spin transport loss increases as the square of the magnetic field perturbation, and decreases exponentially with the square of the number of Larmor precessions the neutron makes during the time of the perturbation (i.e. large guide field magnitude is better for this term). This indicates why spin transport may be problematic, since the magnitude of field perturbations may be proportional to the magnitude of the guide field. The conclusion is that to keep the depolarization minimal we have the competing conditions, $\omega_z \tau \gg 1$ and $|B_x/B_z| \ll 1$.

3.4 sample transfer matrix

The transfer matrix for the sample is

$$S = \begin{bmatrix} S^{++} & S^{+-} \\ S^{-+} & S^{--} \end{bmatrix} \tag{14}$$

where S^{++}refers to the cross-section for scattering a neutron from a spin-up state to a spin-up state, and S^{+-}refers to the cross-section for scattering a neutron from a spin-up state to a spin-down state (spin-flip scattering). It is important to note that in general $S^{++} \neq S^{--}$and $S^{+-} \neq S^{-+}$, so that the sample transfer matrix does not commute with the spin-flipper and transport loss matrices (which do commute with one another).

3.5 total transfer matrix

Combining all of the above transfer matrices, the detected counts for the up and down spin channels, which depend on the flipper states, α and β can be written,

$$\begin{bmatrix} n_+ \\ n_- \end{bmatrix} = AF_A^\beta B_A S B_P F_P^\alpha P \begin{bmatrix} N/2 \\ N/2 \end{bmatrix} = T \begin{bmatrix} N/2 \\ N/2 \end{bmatrix} \tag{15}$$

Recall that the product of flip and transport efficiency matrices can be written

$$\begin{aligned} (F^\alpha B_,)_{P,A} &= \frac{1}{2}\begin{bmatrix} 1-e_t(2e_F^\alpha-1) & 1+e_t(2e_F^\alpha-1) \\ 1+e_t(2e_F^\alpha-1) & 1-e_t(2e_F^\alpha-1) \end{bmatrix}_{P,A} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{2}e_{P,A}^\alpha \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \end{aligned}$$

where $e_{P,A}^\alpha = \{e_t(2e_F^\alpha - 1)\}_{P,A}$. The following matrix products are then required

$$\begin{bmatrix} t_{+A} & 0 \\ 0 & t_{-A} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} S \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t_{+P} & 0 \\ 0 & t_{-P} \end{bmatrix} = \sigma_{++++}\begin{bmatrix} t_{+A}t_{+P} & t_{+A}t_{-P} \\ t_{-A}t_{+P} & t_{-A}t_{-P} \end{bmatrix}$$

$$\begin{bmatrix} t_{+A} & 0 \\ 0 & t_{-A} \end{bmatrix}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} S \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}\begin{bmatrix} t_{+P} & 0 \\ 0 & t_{-P} \end{bmatrix} = \sigma_{+--+}\begin{bmatrix} t_{+A}t_{+P} & -t_{+A}t_{-P} \\ -t_{-A}t_{+P} & t_{-A}t_{-P} \end{bmatrix}$$

$$\begin{bmatrix} t_{+A} & 0 \\ 0 & t_{-A} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} S \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}\begin{bmatrix} t_{+P} & 0 \\ 0 & t_{-P} \end{bmatrix} = \sigma_{+-+-}\begin{bmatrix} t_{+A}t_{+P} & -t_{+A}t_{-P} \\ t_{-A}t_{+P} & -t_{-A}t_{-P} \end{bmatrix}$$

$$\begin{bmatrix} t_{+A} & 0 \\ 0 & t_{-A} \end{bmatrix}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} S \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t_{+P} & 0 \\ 0 & t_{-P} \end{bmatrix} = \sigma_{++--}\begin{bmatrix} t_{+A}t_{+P} & t_{+A}t_{-P} \\ -t_{-A}t_{+P} & -t_{-A}t_{-P} \end{bmatrix}.$$

Here $\sigma_{\pm\pm\pm\pm}$ refers to a sum of the four cross-sections, $S^{\pm\pm}$, with the sign of each term given by the corresponding $\pm$ index of σ. For example, $\sigma_{++++} =$

$S^{++} + S^{+-} + S^{-+} + S^{--}$. The four elements of the flipper-state-dependent total transfer matrix can then be written as

$$T_{\mu\nu}^{\alpha\beta} = E_{\mu\nu}^{\alpha\beta} t_{\mu A} t_{\nu P}.$$

Then the observed counts for spin-up and spin-down are

$$\begin{bmatrix} n_+ \\ n_- \end{bmatrix} = T_{\mu\nu}^{\alpha\beta} \begin{bmatrix} N/2 \\ N/2 \end{bmatrix}$$

and the total detected counts for each combination of polarizer and analyzer flipper states, $\alpha = 1$ for analyzer flipper OFF, $\alpha = -1$ for analyzer flipper ON, $\beta = 1$ for polarizer flipper OFF and $\beta = -1$ for polarizer flipper ON, are

$$Counts^{\alpha\beta} = C^{\alpha\beta} = \frac{N}{2} \Sigma_{\mu\nu} T_{\mu\nu}^{\alpha\beta} = \frac{N}{2} \Sigma_{\mu\nu} E_{\mu\nu}^{\alpha\beta} t_{\mu A} t_{\nu P}$$

where

$$E_{\mu\nu}^{\alpha\beta} = \frac{1}{4} \Sigma_{\mu'\nu'} S^{\mu'\nu'} \left(1 - \mu'\mu e_A^{\alpha}\right) \left(1 - \nu'\nu e_P^{\beta}\right).$$

Recall that $t_{\pm A,P} = \widetilde{C}_{\pm A,P} t_E \exp\left(-\tilde{\tau}_{\pm A,P}\right)$ are the transmission factors from the He-3 analyzer and polarizer. Now the expected count rates can be written as a linear function of the four polarized beam cross-sections

$$C^{\alpha\beta} = \frac{N}{2} \Sigma_{\mu\nu} c_{\mu\nu}^{\alpha\beta} S^{\mu\nu}$$

where the matrix of coefficients is

$$c_{\mu\nu}^{\alpha\beta} = \frac{1}{4} \sum_{\mu'} \left(1 - \mu'\mu e_A^{\alpha}\right) t_{\mu' A} \sum_{\nu'} \left(1 - \nu'\nu e_P^{\beta}\right) t_{\nu' P}. \tag{16}$$

so that each matrix element is the product of factors from before and after the sample. Recall that the efficiency coefficients, $e_{A,P}^{\alpha}$, are given by,

$$e_{A,P}^{\alpha} = e_{tA,P}(2e_{FA,P}^{\alpha} - 1)$$

which is a product involving the transport and spin-flip efficiencies. P refers to before the sample and A refers to after the sample.

In order to understand the matrix of coefficients, examine the simplest case. Assume that the transport and flipping efficiencies are unity so that $e_{A,P}^{\alpha} = -\alpha$ (since $e_F^{\alpha} = \frac{1}{2}\left(1 - \alpha\right) e_F$). Then

$$c_{\mu\nu}^{\alpha\beta} = \frac{1}{4} \sum_{\mu'} \left(1 + \mu'\mu\alpha\right) t_{\mu' A} \sum_{\nu'} \left(1 + \nu'\nu\beta\right) t_{\nu' P}.$$

Now

$$\sum_{\mu'} (1 + \mu'\mu\alpha)\, t_{\mu' A} = t_{+A} + t_{-A} + \mu\alpha\,(t_{+A} - t_{-A}) = 2t_{(\mu\alpha)A}.$$

so that the matrix elements have simplified to

$$c^{\alpha\beta}_{\mu\nu} = t_{(\mu\alpha)A} t_{(\nu\beta)P}.$$

For example, if both flippers are off, $c^{++}_{\mu\nu} = t_{\mu A} t_{\nu P}$ and the total detected counts are

$$BothFlippersOFFCounts = C^{++} = \left(S^{++} t_{+A} t_{+P} + S^{+-} t_{+A} t_{-P} + S^{-+} t_{-A} t_{+P} + S^{--} t_{-A} t_{-P}\right) \frac{N}{2}.$$

The He-3 transmission factors for the perferred spin-states, $t_{+A}$and $t_{+P}$are typically much larger than the transmission factors for the non-preferred states, so one approximately measures S^{++}.

$$C^{++} \cong \left(S^{++} t_{+A} t_{+P}\right) \frac{N}{2}.$$

Now, turning on the polarizer flipper gives $c^{+-}_{\mu\nu} = t_{\mu A} t_{-\nu P}$, and

$$PolarizerFlipperONCounts = C^{+-} = \left(S^{++} t_{+A} t_{-P} + S^{+-} t_{+A} t_{+P} + S^{-+} t_{-A} t_{-P} + S^{--} t_{-A} t_{+P}\right) \frac{N}{2}.$$

$S^{+-}$is multiplied by the largest transmission factors so that

$$C^{+-} \cong \left(S^{+-} t_{+A} t_{+P}\right) \frac{N}{2}.$$

Similarly

$$AnalyzerFlipperONCounts = C^{-+} \cong \left(S^{-+} t_{+A} t_{+P}\right) \frac{N}{2},$$

and

$$BothFlippersONCounts = C^{--} \cong \left(S^{--} t_{+A} t_{+P}\right) \frac{N}{2}.$$

These expressions have to be corrected since the transmission factors for the non-preferred states, $t_{-A}$and t_{-P} are likely not zero.

4 correcting polarized beam data

Since neutron polarizing, flipping and transport devices may not be perfectly efficient, it is necessary to examine the corrections that need to be made to raw polarized beam data in order to extract the the cross-sections that produce observed count rates. In this section it is assumed that the efficiencies of the polarized beam transport have already been determined (that determination is discussed in section **??**). Recall from the section that derived the transfer matrix, that the expected count rates can be written as a linear function of the cross-sections

$$C^{\alpha\beta} = \frac{N}{2}\Sigma_{\mu\nu}c^{\alpha\beta}_{\mu\nu}S^{\mu\nu}$$

where the elements of the matrix of coefficients are

$$c^{\alpha\beta}_{\mu\nu} = \frac{1}{4}\sum_{\mu'}\left(1-\mu'\mu e^{\alpha}_{A}\right)t_{\mu' A}\sum_{\nu'}\left(1-\nu'\nu e^{\beta}_{P}\right)t_{\nu' P}.$$

Do the sums on μ' and ν' by defining

$$\sum_{\mu'} t_{\mu' X} = t_{+X} + t_{-X} = t_{sX}$$

$$\sum_{\mu'} \mu' t_{\mu' X} = t_{+X} - t_{-X} = t_{aX}.$$

From the expressions for the transmission coefficients, t_{sX} and t_{aX} can be expanded to

$$t_{sX} = 2t_{EX}C_{sX}\exp\left(-\tilde{\tau}_X\right)\cosh\left(\tilde{\tau}_X P_{He3X}\right)$$

$$t_{aX} = 2t_{EX}C_{aX}\exp\left(-\tilde{\tau}_X\right)\sinh\left(\tilde{\tau}_X P_{He3X}\right)$$

where

$$C_{sX} = \langle C_X\rangle\left\{1+\frac{\Delta_X}{\langle C_X\rangle}\tanh\left(\tilde{\tau}_X P_{He3X}\right)\right\}$$

$$C_{aX} = \langle C_X\rangle\left\{1+\frac{\Delta_X}{\langle C_X\rangle}\coth\left(\tilde{\tau}_X P_{He3X}\right)\right\}.$$

See the section on correcting the transmission for wavelength and pathlength deviations, **??**, for the definitions of $\langle C_X\rangle \cong 1$ and $\Delta_X \ll 1$. Then the matrix elements are

$$c^{\alpha\beta}_{\mu\nu} = \frac{1}{4}\left(t_{sA}-\mu e^{\alpha}_{A}t_{aA}\right)\left(t_{sP}-\nu e^{\beta}_{P}t_{aP}\right).$$

In general the matrix of these coefficients will require numerical inversion to solve for the cross-sections corresponding to observed count rates. There are special cases, however that can be handled algebraically.

4.1 spin-flip and non-spin-flip cross-sections only

One important special case is when the cross-sections have the often occuring symmetry that $S^{++} = S^{--} = S^{nsf}$, and $S^{+-} = S^{-+} = S^{sf}$. In this case the master equation for the expected count rates reduces to

$$C^{\alpha\beta}/\frac{N}{2} = \left(c^{\alpha\beta}_{++} + c^{\alpha\beta}_{--}\right) S^{nsf} + \left(c^{\alpha\beta}_{+-} + c^{\alpha\beta}_{-+}\right) S^{sf}.$$

Cross terms cancel when the coefficients are added in this way, so that

$$C^{\alpha\beta}/\frac{N}{2} = \frac{1}{2}\left(t_+ + e^{\alpha}_{A} e^{\beta}_{P} t_-\right) S^{nsf} + \frac{1}{2}\left(t_+ - e^{\alpha}_{A} e^{\beta}_{P} t_-\right) S^{sf},$$

where

$$t_+ = t_{sA} t_{sP} = t_{+A} t_{+P} + t_{-A} t_{-P} + t_{+A} t_{-P} + t_{-A} t_{+P}$$

$$t_- = t_{aA} t_{aP} = t_{+A} t_{+P} + t_{-A} t_{-P} - t_{+A} t_{-P} - t_{-A} t_{+P}.$$

The expansions of t_+ and t_- follow simply from the expansions of t_{sX} and t_{aX}. These equations can easily be inverted to obtain the cross-sections as a function of the count-rates. This is typically accomplished by measuring the non-spin-flip counts as $C^{++} = C^{nsf}$, and using one flipper to measure either C^{+-} or C^{-+} as the spin-flip count rate, C^{sfX}, where X indicates which flipper is used. This system of equations is

$$\begin{pmatrix} C^{++} \\ C^{sfX} \end{pmatrix} = \frac{N}{2}\frac{1}{2}\begin{pmatrix} t_+ + e_t A t_- & t_+ - e_t A t_- \\ t_+ - e_t B t_- & t_+ + e_t B t_- \end{pmatrix}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix}$$

where $e_t = e_{tA} e_{tP}$ is the aggregate beam transport efficiency, $A = 1$, $B = 2e_{FX} - 1$, and $X = P, A$ depending on which flipper is used. The determinant of the matrix is $4 e_t e_{FX} t_+ t_-$ so that matrix inversion gives the result

$$\frac{N}{2}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix} = \frac{1}{2 e_t e_{FX}}\begin{pmatrix} a_{+B} & -a_{-A} \\ -a_{-B} & a_{+A} \end{pmatrix}\begin{pmatrix} C^{++} \\ C^{sfX} \end{pmatrix}$$

where the elements of the inverted matrix are found from

$$a_{\pm A} = \frac{t_+ \pm e_t A t_-}{t_+ t_-}$$

$$a_{\pm B} = \frac{t_+ \pm e_t B t_-}{t_+ t_-}.$$

In order to do the error analysis on this solution, write

$$S^{nsf} e_t e_{FX} N = \left(\frac{e_t B}{t_+} + \frac{1}{t_-}\right) C^{++} + \left(\frac{e_t A}{t_+} - \frac{1}{t_-}\right) C^{sfX}$$

$$S^{sf} e_t e_{FX} N = \left(\frac{e_t B}{t_+} - \frac{1}{t_-}\right) C^{++} + \left(\frac{e_t A}{t_+} + \frac{1}{t_-}\right) C^{sfX}$$

which is

$$NS^{nsf} = K^{++} C^{++} + K^{+-} C^{sfX}$$

$$NS^{sf} = K^{-+} C^{++} + K^{--} C^{sfX}$$

where

$$K^{\alpha\beta} = A^{\alpha\beta} + B^{\alpha\beta} + C^{\alpha\beta}$$

and

$$A^{\alpha\beta} = \frac{\alpha\beta}{e_t e_{FX}} \frac{1}{t_-}$$

$$B^{\alpha\beta} = \frac{2\delta_{\beta +}}{t_+}$$

$$C^{\alpha\beta} = \frac{-\beta}{e_{FX}} \frac{1}{t_+}.$$

The partial derivatives of t_+ and t_- are

$$\frac{\partial t_\pm}{\partial \tilde{\tau}_{P,A}} = t_\pm \left[P_{He3P,A} \tanh^{\pm 1} \left(\tilde{\tau} P_{He3}\right)_{P,A} - 1 \right]$$

$$\frac{\partial t_\pm}{\partial P_{He3P,A}} = t_\pm \left[\tilde{\tau}_{P,A} \tanh^{\pm 1} \left(\tilde{\tau} P_{He3}\right)_{P,A} \right].$$

Then the error propagation for the coefficients is

$$\sigma^2_{K^{\alpha\beta}} = \left(A^{\alpha\beta}\right)^2 \tilde{\sigma}^2_{e_t} + \left(A^{\alpha\beta} + C^{\alpha\beta}\right)^2 \tilde{\sigma}^2_{e_{FX}} + \sum_{X = \tilde{\tau}_A \tilde{\tau}_P P_A P_P} \left(W^{\alpha\beta}_X\right)^2 \sigma^2_X$$

where

$$W^{\alpha\beta}_X = A^{\alpha\beta} \left[\bar{x} \coth\left(x\overline{x}\right) - \delta_{x\tau}\right] + \left(B^{\alpha\beta} + C^{\alpha\beta}\right) \left[\bar{x} \tanh\left(x\overline{x}\right) - \delta_{x\tau}\right]$$

and $\bar{x}$ is the partner variable for x in the pairs $\tilde{\tau}_A P_{He3A}$ and $\tilde{\tau}_P P_{He3P}$. The error propagation for the X variables $\tilde{\tau}$ and P is in section **??**. Also, in general

$$\tilde{\sigma}^2_X = \frac{\sigma^2_X}{X^2}.$$

The final error propagation to the cross-sections is

$$\sigma^2_{NS^{nsf}} = \left(C^{++}\sigma_{K^{++}}\right)^2 + \left(C^{sfX}\sigma_{K^{+-}}\right)^2 + \left(K^{++}\sigma_{C^{++}}\right)^2 + \left(K^{+-}\sigma_{C^{sfX}}\right)^2$$

$$\sigma^2_{NS^{sf}} = \left(C^{++}\sigma_{K^{-+}}\right)^2 + \left(C^{sfX}\sigma_{K^{--}}\right)^2 + \left(K^{-+}\sigma_{C^{++}}\right)^2 + \left(K^{--}\sigma_{C^{sfX}}\right)^2$$

If count rates are measured for all four cross-sections then the counts in the spin-flip and non-spin flip channels can be added so that

$$\frac{1}{2}\begin{pmatrix} C^{++}+C^{--} \\ C^{+-}+C^{-+} \end{pmatrix} = \frac{N}{2}\frac{1}{2}\begin{pmatrix} t_+ + e_t A t_- & t_+ - e_t A t_- \\ t_+ - e_t B t_- & t_+ + e_t B t_- \end{pmatrix}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix}$$

where

$$A = \frac{1}{2}\left[1 + \left(2e_{FA}-1\right)\left(2e_{FP}-1\right)\right]$$

$$B = \frac{1}{2}\left[\left(2e_{FA}-1\right) + \left(2e_{FP}-1\right)\right]$$

The determinant of this matrix is $4e_t e_{FA} e_{FP} t_+ t_-$ so that the inversion becomes

$$\frac{N}{2}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix} = \frac{1}{2e_t e_{FA} e_{FP}}\begin{pmatrix} a_{+B} & -a_{-A} \\ -a_{-B} & a_{+A} \end{pmatrix}\begin{pmatrix} \langle C^{nsf}\rangle \\ \langle C^{sf}\rangle \end{pmatrix}$$

where

$$a_{\pm A} = \frac{t_+ \pm e_t A t_-}{t_+ t_-}$$

$$a_{\pm B} = \frac{t_+ \pm e_t B t_-}{t_+ t_-}.$$

The error analysis procedes as before by separating the solution coefficients into terms that depend on transport and flipper efficiencies,

$$S^{nsf} e_t e_{FA} e_{FP} N = \left(\frac{e_t B}{t_+} + \frac{1}{t_-}\right)\langle C^{nsf}\rangle + \left(\frac{e_t A}{t_+} - \frac{1}{t_-}\right)\langle C^{sf}\rangle$$

$$S^{sf} e_t e_{FA} e_{FP} N = \left(\frac{e_t B}{t_+} - \frac{1}{t_-}\right)\langle C^{nsf}\rangle + \left(\frac{e_t A}{t_+} + \frac{1}{t_-}\right)\langle C^{sf}\rangle,$$

which as before is

$$NS^{nsf} = K^{++}C^{++} + K^{+-}C^{sfX}$$

$$NS^{sf} = K^{-+}C^{++} + K^{--}C^{sfX}$$

where

$$K^{\alpha\beta} = A^{\alpha\beta} + B^{\alpha\beta} + C^{\alpha\beta} + D^{\alpha\beta} + E^{\alpha\beta}$$

and

$$A^{\alpha\beta} = \frac{\alpha\beta}{e_t e_{FA} e_{FP}} \frac{1}{t_-}$$

$$B^{\alpha\beta} = \frac{2\delta_{\beta -}}{t_+}$$

$$C^{\alpha\beta} = \frac{\beta}{e_{FA}} \frac{1}{t_+}$$

$$D^{\alpha\beta} = \frac{\beta}{e_{FP}} \frac{1}{t_+}$$

$$E^{\alpha\beta} = -\frac{\beta}{e_{FA} e_{FP}} \frac{1}{t_+}.$$

Then the error propagation for the coefficients is

$$\sigma^2_{K^{\alpha\beta}} = \left(A^{\alpha\beta}\right)^2 \tilde{\sigma}^2_{e_t} + \left(W^{\alpha\beta}_{e_{FA}}\right)^2 \tilde{\sigma}^2_{e_{FA}} + \left(W^{\alpha\beta}_{e_{FP}}\right)^2 \tilde{\sigma}^2_{e_{FP}} + \sum_{X=\tilde{\tau}_A \tilde{\tau}_P P_A P_P} \left(W^{\alpha\beta}_X\right)^2 \sigma^2_X$$

where

$$W^{\alpha\beta}_{e_{FA}} = A^{\alpha\beta} + C^{\alpha\beta} + E^{\alpha\beta}$$

$$W^{\alpha\beta}_{e_{FP}} = A^{\alpha\beta} + D^{\alpha\beta} + E^{\alpha\beta}$$

$$W^{\alpha\beta}_X = A^{\alpha\beta}\left[\bar{x}\coth\left(x\overline{x}\right) - \delta_{x\tau}\right] + \left(B^{\alpha\beta} + C^{\alpha\beta} + D^{\alpha\beta} + E^{\alpha\beta}\right)\left[\bar{x}\tanh\left(x\overline{x}\right) - \delta_{x\tau}\right]$$

The final error propagation to the cross-sections is

$$\sigma^2_{NS^{nsf}} = \left(\left\langle C^{nsf}\right\rangle \sigma_{K^{++}}\right)^2 + \left(\left\langle C^{sf}\right\rangle \sigma_{K^{+-}}\right)^2 + \left(K^{++}\sigma_{\langle C^{nsf}\rangle}\right)^2 + \left(K^{+-}\sigma_{\langle C^{sf}\rangle}\right)^2$$

$$\sigma^2_{NS^{sf}} = \left(\left\langle C^{nsf}\right\rangle \sigma_{K^{-+}}\right)^2 + \left(\left\langle C^{sf}\right\rangle \sigma_{K^{--}}\right)^2 + \left(K^{-+}\sigma_{\langle C^{nsf}\rangle}\right)^2 + \left(K^{--}\sigma_{\langle C^{sf}\rangle}\right)^2$$

4.1.1 simplification using a flipping ratio measurement

Up to this point it has been assumed that the full time dependence of the He-3 cell transmissions must be taken into account, especially in the error propagation. If a flipping ratio measurement is available at the approximate time of data collection, then much of the uncertainty produced by the time dependences can be replaced by just the uncertainty in measuring the flipping ratio. This procedure will work for elastic-scattering data, since the flipping ratio is measured under elastic scattering conditions (although, as shown later, it is possible to apply this procedure to inelastic data by making a wavelength dependence correction to the flipping ratio). To this end, rewrite the correction formulae in terms of the flipping ratio measured at the same time as the data. Here it is assumed that $C^{nsf} = C^{++}$and the spin-flip counts can be collected with either flipper. The expression for the non-spin-flip flipping ratio, that is $R = C^{++}/C^{+-}$, is

$$R = \frac{t_+ + e_t t_-}{t_+ - e_t(2e_F - 1)t_-} = \frac{1 + e_t t_-/t_+}{1 - e_t(2e_F - 1)t_-/t_+}.$$

Solving for t_-/t_+ in terms of R

$$\frac{t_-}{t_+} = \frac{1}{e_t}\frac{R-1}{R(2e_F - 1) + 1}.$$

Now the cross-section asymmetry solutions can be written in terms of R,

$$S^{nsf}NK_e = \left(\frac{1 + e_{tF}t_-/t_+}{1 + e_t t_-/t_+}\right)C^{nsf} - \frac{1}{R}\left(\frac{1 - e_t t_-/t_+}{1 - e_{tF}t_-/t_+}\right)C^{sf} \tag{17}$$

or

$$S^{nsf}NK_e = \left[\left(2 - \frac{1}{e_F}\right) + \frac{1}{R}\left(\frac{1}{e_F} - 1\right)\right]C^{nsf} - \left[\frac{1}{R}\frac{1}{e_F} - \left(\frac{1}{e_F} - 1\right)\right]C^{sf}.$$

Separate out the flipper efficiency dependence by using $1/e_F = 1 + (1 - e_F)/e_F$, to find

$$S^{nsf}NK_e = C^{nsf} - \frac{1}{R}C^{sf} - \epsilon\left(C^{nsf} - C^{sf}\right),$$

where

$$\epsilon = \left(\frac{1 - e_F}{e_F}\right)\left(\frac{R-1}{R}\right).$$

The formula for the spin-flip cross-section in terms of the flipping ratio is

$$S^{sf}NK_e = C^{sf} - \frac{1}{R}C^{nsf}, \tag{18}$$

and the formula for S^{nsf} takes this same simple form when the flipping efficiency is unity.

This is a good approach if the ratio S^{sf}/S^{nsf}(or its inverse) is of interest, since this ratio depends only on the measured counts, the measured flipping ratio and a small correction for flipping efficiency. However, in the following it will be shown that the formula and error analysis is even simpler in terms of the cross-section asymmetry and count-rate asymmetry. Using these results, the solution for the ratio S^{nsf}/S^{sf} is

$$s_{nsf} = \frac{S^{nsf}}{S^{sf}} = \frac{C^{nsf} - \frac{1}{R}C^{sf} - \epsilon\left(C^{nsf} - C^{sf}\right)}{C^{sf} - \frac{1}{R}C^{nsf}}$$

with squared relative error

$$\tilde{\sigma}^2_{s_{nsf}} = W_R^2\tilde{\sigma}_R^2 + W_{e_F}^2\tilde{\sigma}_{e_F}^2 + W_C^2\left(\tilde{\sigma}^2_{C^{nsf}} + \tilde{\sigma}^2_{C^{sf}}\right)$$

where

$$W_R = R\frac{\left(C^{nsf}\right)^2 - \left(C^{sf}\right)^2}{\left(RC^{nsf} - C^{sf}\right)\left(RC^{sf} - C^{nsf}\right)}$$

$$W_{e_F} = \frac{C^{nsf} - C^{sf}}{RC^{nsf} - C^{sf}}\frac{R-1}{e_F}$$

$$W_C = \frac{C^{nsf}C^{sf}\left(R^2-1\right)}{\left(RC^{nsf} - C^{sf}\right)\left(RC^{sf} - C^{nsf}\right)}.$$

If the ratio S^{sf}/S^{nsf} is of interest, simply invert the formula above and interchange C^{nsf} and C^{sf} in the error analysis. $\tilde{\sigma}_R^2$ is defined in the section on flipping ratios **??**.

Now the remaining time dependence is in K_e which is

$$K_e = \frac{e_t e_F t_-}{1 + e_t t_-/t_+} \tag{19}$$

Making the same replacement for t_-/t_+ in terms of R, $K_e$can be rewritten as

$$K_e = \frac{1}{2}e_t\left(2e_F - 1 + \frac{1}{R}\right)t_-$$

where recall that $t_-$is

$$t_- = 4t_{EA}t_{EP}C_{\Delta-}\exp\left(-\tilde{\tau}_A\right)\exp\left(-\tilde{\tau}_P\right)\sinh\left(\tilde{\tau}_A P_{He3A}\right)\sinh\left(\tilde{\tau}_P P_{He3P}\right).$$

Be aware that the transport efficiency may be angle dependent. To be precise, the transport and flipping efficiencies should be measured at the same spectrometer settings and guide field settings used to measure C^{nsf} and C^{sf} and those efficiencies should then be used to make the corrections to obtain S^{nsf} and S^{sf}. A classic case is the use of the neutron polarization direction to vary the amount of magnetic scattering that contributes to the spin-flip and non-spin-flip channels. This dependence arises from the fact that spin-flip magnetic scattering is due to the neutron scattering from sample magnetic moment components that are perpendicular to the neutron polarization direction, and conversely the non-spin-flip magnetic scattering is due to the neutron scattering from sample magnetic moment components that are parallel to the neutron polarization direction. This dependence is utillised experimentally by controlling the neutron polarization direction at the sample with either a vertical (to scattering plane) or horizontal guide field. If the horizontal sample guide field is aligned along the scattering vector, Q, then all magnetic scattering must be in the spin-flip channel, since the neutron spin scatters only from sample magnetic moment components that are perpendicular to Q, and these same sample magnetic moments are also perpendicular to the neutron spin. Since there are other possible contributions to the scattering in the spin-flip channel, the usual procedure is to subtract off the spin-flip scattering observed when the sample guide field is vertical. This vertical field spin-flip scattering will have a different amount of magnetic scattering but all the other scattering processes will be the same as in the horizontal field case. Since the transport efficiencies may be different between the vertical and horizontal field cases it is important to correct the observed counts (using the efficiencies) to obtain the true cross-sections before making such a subtraction.

4.1.2 cross-section asymmetry solution

When the ratio of cross-sections is the quantity of interest, the ideal analysis is in terms of the cross-section asymmetry, $-1 \leq s \leq 1$, defined as

$$s = \frac{S^{nsf} - S^{sf}}{S^{nsf} + S^{sf}}$$

and the count-rate asymmetry, $-1 \leq c \leq 1$, defined as

$$c = \frac{C^{nsf} - C^{sf}}{C^{nsf} + C^{sf}},$$

$$(1-c)\, C^{nsf} = (1+c)\, C^{sf}.$$

The result for s in terms of c is

$$s = \frac{t_+}{t_-} \frac{c/e_t}{e_F - (1-e_F)c} = \frac{R\,(2e_F - 1) + 1}{R-1} \frac{c}{e_F - (1-e_F)c} \cong \frac{R+1}{R-1} \frac{c}{e_F},$$

where the approximation is for e_F near to unity. All of the time dependence and beam transport efficiency is now contained in the measured flipping ratio, R. This result becomes quite simple when the flipper efficiency can be assumed to be unity. This can be combined with

$$S^{nsf} + S^{sf} = \frac{1}{t_+} \frac{(2e_F - 1)C^{nsf} + C^{sf}}{e_F},$$

if it is necessary to extract the individual values S^{nsf} and S^{sf}.

The error analysis on s produces

$$\sigma_s^2 = W_R^2 \tilde{\sigma}_R^2 + W_{e_F}^2 \tilde{\sigma}_{e_F}^2 + W_C^2 \left(\tilde{\sigma}_{C^{nsf}}^2 + \tilde{\sigma}_{C^{sf}}^2 \right)$$

where

$$W_R = s \frac{2e_F R}{[R-1]\left[R\left(2e_F - 1\right) + 1\right]} \cong s \frac{2e_F R}{R^2 - 1}$$

$$W_{e_F} = s \frac{2e_F R}{\left[R\left(2e_F - 1\right) + 1\right]} - s \frac{e_F\left(1 + c\right)}{e_F - (1 - e_F)c} \cong s \frac{2e_F R}{R + 1} - s\left(1 + c\right)$$

$$W_C = 2s \frac{e_F}{e_F - (1 - e_F)c} \frac{C^{nsf} C^{sf}}{\left(C^{nsf}\right)^2 - \left(C^{sf}\right)^2} \cong 2 \frac{R+1}{R-1} \frac{C^{nsf} C^{sf}}{\left(C^{nsf} + C^{sf}\right)^2}.$$

Note that σ_R is just the error for the flipping ratio measurement, which simply depends on the count rates measured in obtaining R as shown in a following section.

4.2 all four cross-sections with perfect flippers

In the case that the spin flipper efficiencies are unity, the full transfer matrix can be solved for all four cross-sections. Recall that the formula for the transfer matrix coefficients in

$$C^{\alpha\beta} = \frac{N}{2} \Sigma_{\mu\nu} c_{\mu\nu}^{\alpha\beta} S^{\mu\nu}$$

was

$$c_{\mu\nu}^{\alpha\beta} = \frac{1}{4} \left(t_{sA} - \mu e_A^{\alpha} t_{aA}\right) \left(t_{sP} - \nu e_P^{\beta} t_{aP}\right).$$

Also, $e_{P,A}^{\alpha} = \{e_t(2e_F^{\alpha} - 1)\}_{P,A}$ can be rewritten as

$$e_{P,A}^{\alpha} = -\alpha \left(1 - \epsilon_{P,A}^{\alpha}\right)$$

where the small transport inefficiency parameter, $\epsilon_{P,A}^{\alpha}$, is

$$\epsilon^{+} = 1 - e_t$$

and

$$\epsilon^{-} = 1 - e_{tF} = 1 - e_t\,(2e_F - 1)$$

If the spin-flipper efficiency is unity then $\epsilon^{+}_{P,A} = \epsilon^{-}_{P,A} = \epsilon_{P,A} = 1 - e_{tA,P}$ is independent of α so that $e^{\alpha}_{P,A} = -\alpha\,(1 - \epsilon_{P,A}) = -\alpha e_{tA,P}$. Then

$$4c^{\alpha\beta}_{\mu\nu} = [t_{sA} + \mu\alpha e_{tA} t_{aA}]\,[t_{sP} + \nu\beta e_{tP} t_{aP}]\,,$$

so that μ and α appear only as their product, and the same for ν and β. This means

$$c^{\alpha\beta}_{\mu\nu} = c^{\bar{\alpha}\beta}_{\bar{\mu}\nu} = c^{\alpha\bar{\beta}}_{\mu\bar{\nu}} = c^{\bar{\alpha}\bar{\beta}}_{\bar{\mu}\bar{\nu}}$$

where $\bar{\alpha} = -\alpha$, so that there are only four distinct elements in the matrix. These four distinct elements can be generated by fixing $\mu = \nu = +1$, and they are

$$4c^{++} = [t_{sA} + e_{tA} t_{aA}]\,[t_{sP} + e_{tP} t_{aP}] = t_{+} + e_{tA} e_{tP} t_{-} + e_{tA} t_{A} + e_{tP} t_{P}$$

$$4c^{--} = [t_{sA} - e_{tA} t_{aA}]\,[t_{sP} - e_{tP} t_{aP}] = t_{+} + e_{tA} e_{tP} t_{-} - e_{tA} t_{A} - e_{tP} t_{P}$$

$$4c^{+-} = [t_{sA} + e_{tA} t_{aA}]\,[t_{sP} - e_{tP} t_{aP}] = t_{+} - e_{tA} e_{tP} t_{-} + e_{tA} t_{A} - e_{tP} t_{P}$$

$$4c^{-+} = [t_{sA} - e_{tA} t_{aA}]\,[t_{sP} + e_{tP} t_{aP}] = t_{+} - e_{tA} e_{tP} t_{-} - e_{tA} t_{A} + e_{tP} t_{P},$$

where

$$t_A = t_{aA} t_{sP} = (t_{+A} - t_{-A})\,(t_{+P} + t_{-P})$$

$$t_P = t_{sA} t_{aP} = (t_{+A} + t_{-A})\,(t_{+P} - t_{-P})\,.$$

Now form the symmetric two by two matrices

$$\mathbf{c}_n = \begin{pmatrix} c^{++} & c^{--} \\ c^{--} & c^{++} \end{pmatrix}$$

and

$$\mathbf{c}_f = \begin{pmatrix} c^{+-} & c^{-+} \\ c^{-+} & c^{+-} \end{pmatrix},$$

with the corresponding two component vectors

$$\mathbf{C}_n = \begin{pmatrix} C^{++} \\ C^{--} \end{pmatrix}$$

$$\mathbf{C}_f = \begin{pmatrix} C^{+-} \\ C^{-+} \end{pmatrix}$$

$$\mathbf{S}_n = \begin{pmatrix} S^{++} \\ S^{--} \end{pmatrix}$$

$$\mathbf{S}_f = \begin{pmatrix} S^{+-} \\ S^{-+} \end{pmatrix}$$

and the system of equations can then be written

$$\begin{pmatrix} \mathbf{C}_n \\ \mathbf{C}_f \end{pmatrix} = \frac{N}{2} \begin{pmatrix} \mathbf{c}_n & \mathbf{c}_f \\ \mathbf{c}_f & \mathbf{c}_n \end{pmatrix} \begin{pmatrix} \mathbf{S}_n \\ \mathbf{S}_f \end{pmatrix}.$$

Because the matrices $\mathbf{c}_n$ and $\mathbf{c}_f$ are symmetric and thus commute with each other, the inversion of this matrix problem is

$$\begin{pmatrix} \mathbf{c}_n & -\mathbf{c}_f \\ -\mathbf{c}_f & \mathbf{c}_n \end{pmatrix} \begin{pmatrix} \mathbf{C}_n \\ \mathbf{C}_f \end{pmatrix} = \frac{N}{2} \begin{pmatrix} \mathbf{c}_n^2 - \mathbf{c}_f^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_n^2 - \mathbf{c}_f^2 \end{pmatrix} \begin{pmatrix} \mathbf{S}_n \\ \mathbf{S}_f \end{pmatrix}$$

which can be checked by substituting the solution for $(\mathbf{C}_n, \mathbf{C}_f)$ from the previous equation. Letting $\mathbf{c}_n^2 - \mathbf{c}_f^2 = \mathbf{c}_d$, which is another symmetric matrix, the solution becomes

$$\begin{pmatrix} \mathbf{s}_n & \mathbf{s}_f \\ \mathbf{s}_f & \mathbf{s}_n \end{pmatrix} \begin{pmatrix} \mathbf{C}_n \\ \mathbf{C}_f \end{pmatrix} = \begin{pmatrix} \mathbf{c}_d^{-1}\mathbf{c}_n & -\mathbf{c}_d^{-1}\mathbf{c}_f \\ -\mathbf{c}_d^{-1}\mathbf{c}_f & \mathbf{c}_d^{-1}\mathbf{c}_n \end{pmatrix} \begin{pmatrix} \mathbf{C}_n \\ \mathbf{C}_f \end{pmatrix} = \frac{N}{2} \begin{pmatrix} \mathbf{S}_n \\ \mathbf{S}_f \end{pmatrix}.$$

With

$$\mathbf{c}_d = \begin{pmatrix} e & f \\ f & e \end{pmatrix},$$

$$e = \left(c^{++}\right)^2 + \left(c^{--}\right)^2 - \left(c^{+-}\right)^2 - \left(c^{-+}\right)^2$$

$$f = 2c^{++}c^{--} - 2c^{+-}c^{-+}$$

$$\mathbf{c}_d^{-1} = \left(e^2 - f^2\right)^{-1} \begin{pmatrix} e & -f \\ -f & e \end{pmatrix}$$

then

$$\mathbf{c}_d^{-1}\mathbf{c}_n = \mathbf{s}_n = \left(e^2 - f^2\right)^{-1} \begin{pmatrix} u & v \\ v & u \end{pmatrix}$$

where

$$u = c^{++}\left[\left(c^{++}\right)^2 - \left(c^{--}\right)^2 - \left(c^{+-}\right)^2 - \left(c^{-+}\right)^2\right] + 2c^{--}c^{+-}c^{-+}$$

$$v = c^{--}\left[\left(c^{--}\right)^2 - \left(c^{++}\right)^2 - \left(c^{+-}\right)^2 - \left(c^{-+}\right)^2\right] + 2c^{++}c^{+-}c^{-+}.$$

Similarly,

$$-\mathbf{c}_d^{-1}\mathbf{c}_f = \mathbf{s}_f = \left(e^2 - f^2\right)^{-1}\begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

where

$$-x = c^{+-}\left[\left(c^{++}\right)^2 + \left(c^{--}\right)^2 - \left(c^{+-}\right)^2 + \left(c^{-+}\right)^2\right] - 2c^{-+}c^{++}c^{--}$$

$$-y = c^{-+}\left[\left(c^{++}\right)^2 + \left(c^{--}\right)^2 + \left(c^{+-}\right)^2 - \left(c^{-+}\right)^2\right] - 2c^{+-}c^{++}c^{--}$$

Calculation shows that

$$e^2 - f^2 = e_{tA}^2 e_{tP}^2 t_+ t_- t_A t_P$$

$$\frac{4}{e_t}u = (t_+ + e_t t_-)\, t_A t_P + t_+ t_- \left(e_{tA} t_A + e_{tP} t_P\right)$$

$$\frac{4}{e_t}v = (t_+ + e_t t_-)\, t_A t_P - t_+ t_- \left(e_{tA} t_A + e_{tP} t_P\right)$$

$$-\frac{4}{e_t}x = (t_+ - e_t t_-)\, t_A t_P + t_+ t_- \left(e_{tA} t_A - e_{tP} t_P\right)$$

$$-\frac{4}{e_t}y = (t_+ - e_t t_-)\, t_A t_P - t_+ t_- \left(e_{tA} t_A - e_{tP} t_P\right)$$

and as usual $e_t = e_{tA}e_{tP}$ is the aggregate beam transport efficiency. Thus the submatrix solutions are

$$\mathbf{s}_n = \frac{+1}{4e_t}\begin{pmatrix} a_+ + b_+ & a_+ - b_+ \\ a_+ - b_+ & a_+ + b_+ \end{pmatrix}$$

$$\mathbf{s}_f = \frac{-1}{4e_t}\begin{pmatrix} a_- + b_- & a_- - b_- \\ a_- - b_- & a_- + b_- \end{pmatrix}$$

where

$$a_\pm = \frac{t_+ \pm e_t t_-}{t_+ t_-} = \frac{1}{t_-} \pm \frac{e_t}{t_+}$$

$$b_\pm = \frac{e_{tA} t_A \pm e_{tP} t_P}{t_A t_P} = \frac{e_{tA}}{t_P} \pm \frac{e_{tP}}{t_A}.$$

If at this point it is found that $S^{++} = S^{--} = S^{nsf}$ and $S^{+-} = S^{-+} = S^{sf}$ then the pairs of equations in the solution can be added to reproduce the previous result in terms of average count-rates, but now with the flipper efficiency set to unity,

$$\frac{N}{2} \begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix} = \frac{1}{2e_t} \begin{pmatrix} a_+ & -a_- \\ -a_- & a_+ \end{pmatrix} \begin{pmatrix} \langle C^{nsf} \rangle \\ \langle C^{sf} \rangle \end{pmatrix}.$$

The system of equations for the general solution with flipper efficiency unity can now be written

$$N S^{\alpha\beta} = K^{\alpha\beta}_{\mu\nu} C^{\mu\nu},$$

where

$$K^{\alpha\beta}_{\mu\nu} = \frac{1}{2e_t} \nu\beta \left(\mu\alpha a_{(\mu\alpha\nu\beta)} + b_{(\mu\alpha\nu\beta)} \right).$$

The partial derivatives of t_+ and t_- are

$$\frac{\partial t_\pm}{\partial \tilde{\tau}_{P,A}} = t_\pm \left[P_{He3P,A} \tanh^{\pm 1} \left(\tilde{\tau} P_{He3} \right)_{P,A} - 1 \right]$$

$$\frac{\partial t_\pm}{\partial P_{He3P,A}} = t_\pm \left[\tilde{\tau}_{P,A} \tanh^{\pm 1} \left(\tilde{\tau} P_{He3} \right)_{P,A} \right].$$

The partial derivatives of t_A and t_P are

$$\frac{\partial t_X}{\partial \tilde{\tau}_Y} = t_X \left[P_{He3Y} \tanh^{-XY} \left(\tilde{\tau} P_{He3} \right)_Y - 1 \right]$$

$$\frac{\partial t_X}{\partial P_{He3Y}} = t_X \left[\tilde{\tau}_Y \tanh^{-XY} \left(\tilde{\tau} P_{He3} \right)_Y \right],$$

where $X = P, A$ and $Y = P, A$ and A is equivalent to -1, while P is equivalent to $+1$ for the purpose of calculating the tanh exponent. In order to do the error propagation divide $K^{\alpha\beta}_{\mu\nu}$ into the contributions from a and b to find

$$2K^{\alpha\beta}_{\mu\nu} = \frac{1}{t_+} + \frac{\nu\beta\mu\alpha}{e_{tA} e_{tP} t_-} + \frac{\nu\beta}{e_{tP} t_P} + \frac{\mu\alpha}{e_{tA} t_A}.$$

As before this takes on only four distinct values that can be generated by choosing $\mu = \nu = +1$, so that

$$K^{\alpha\beta}_{\mu\nu} = K^{\bar{\alpha}\beta}_{\bar{\mu}\nu} = K^{\alpha\bar{\beta}}_{\mu\bar{\nu}} = K^{\bar{\alpha}\bar{\beta}}_{\bar{\mu}\bar{\nu}}$$

and

$$K^{(\alpha)(\beta)}_{\mu\nu} = K^{(\alpha\mu)(\beta\nu)}_{++}$$

can be used to generate all the other coefficients. Then write

$$2K^{\alpha\beta}_{++} = A^{\alpha\beta} + B^{\alpha\beta} + C^{\alpha\beta} + D^{\alpha\beta},$$

where

$$A^{\alpha\beta} = \frac{1}{t_+}$$

$$B^{\alpha\beta} = \frac{\beta\alpha}{e_{tA}e_{tP}t_-}$$

$$C^{\alpha\beta} = \frac{\beta}{e_{tP}t_P}$$

$$D^{\alpha\beta} = \frac{\alpha}{e_{tA}t_A}.$$

Then

$$4\sigma^2_{K^{\alpha\beta}_{++}} = \left(B^{\alpha\beta} + D^{\alpha\beta}\right)^2 \tilde{\sigma}^2_{e_{tA}} + \left(B^{\alpha\beta} + C^{\alpha\beta}\right)^2 \tilde{\sigma}^2_{e_{tP}} + \sum_{Y=\tilde{\tau}_A,\tilde{\tau}_P,P_A,P_P} W^2_Y \sigma^2_Y$$

where

$$W_Y = \sum_{X=A,B,C,D} X^{\alpha\beta} V^X_Y$$

and

$$V^{X=A,B}_Y = \bar{Y}\tanh^X\left(Y\bar{Y}\right) - \delta_{Y\tau}$$

$$V^{X=C,D}_Y = \bar{Y}\tanh^{-XY}\left(Y\bar{Y}\right) - \delta_{Y\tau}.$$

$\bar{Y}$ is the partner variable of Y in the pairs $\tilde{\tau}P_{He3}$. The A, B, C and D coefficients (X) are equivalent to $+1$, -1, $+1$ and -1 respectively, and the A and P subscripts (Y) of $\tilde{\tau}$ and P_{He3} are equivalent to -1 and $+1$, for the purposes of obtaining the exponent of $\tanh$ in these equations. From the equation for $NS^{\alpha\beta}$, the final error propagation is

$$\sigma^2_{NS^{\alpha\beta}} = \sum_{\mu\nu} \left(C^{\mu\nu}\right)^2 \sigma^2_{K^{\alpha\beta}_{\mu\nu}} + \left(K^{\alpha\beta}_{\mu\nu}\right)^2 \sigma^2_{C^{\alpha\beta}}.$$

4.3 single spin-flip cross-section only

Suppose that only $S^{+-} \neq 0$ or $S^{-+} \neq 0$, as might be the case for spin-wave scattering. Then the equations for count-rates as a function of the single cross-section S^{+-} are

$$C^{++}/\frac{N}{8} = (t_{sA} + e_{tA}t_{aA})\,(t_{sP} - e_{tP}t_{aP})\,S^{+-}$$

$$C^{--}/\frac{N}{8} = (t_{sA} - e_{tA}At_{aA})\,(t_{sP} + e_{tP}Pt_{aP})\,S^{+-}$$

$$C^{+-}/\frac{N}{8} = (t_{sA} + e_{tA}t_{aA})\,(t_{sP} + e_{tP}Pt_{aP})\,S^{+-}$$

$$C^{-+}/\frac{N}{8} = (t_{sA} - e_{tA}At_{aA})\,(t_{sP} - e_{tP}t_{aP})\,S^{+-},$$

where as usual $A = 2e_{FA} - 1$ and $P = 2e_{FP} - 1$. Then

$$r^{+}_{SW} = \frac{C^{+-}}{C^{++}} = \frac{1 + e_{tP}Pt_{aP}/t_{sP}}{1 - e_{tP}t_{aP}/t_{sP}} = \frac{C^{--}}{C^{-+}}.$$

This could be used to extract the polarizer transmission factor ratio t_{aP}/t_{sP} as

$$e_{tP}\frac{t_{aP}}{t_{sP}} = \frac{r^{+}_{SW} - 1}{r^{+}_{SW} + P},$$

where

$$\frac{t_{aP}}{t_{sP}} = \frac{C_{aP}}{C_{sP}}\tanh\left(\tilde{\tau}_P P_{He3P}\right).$$

Once the tanh is calculated, then sinh and coshcan be separately extracted to obtain the individual transmission factors t_{aP} and t_{sP},

$$\cosh\left(\tilde{\tau}_P P_{He3P}\right) = \frac{1}{\sqrt{1 - \tanh^2\left(\tilde{\tau}_P P_{He3P}\right)}}$$

$$\sinh\left(\tilde{\tau}_P P_{He3P}\right) = \frac{\tanh\left(\tilde{\tau}_P P_{He3P}\right)}{\sqrt{1 - \tanh^2\left(\tilde{\tau}_P P_{He3P}\right)}}$$

$$t_{aP} = 2t_{EP}C_{aP}\exp\left(-\tilde{\tau}_P\right)\sinh\left(\tilde{\tau}_P P_{He3P}\right)$$

$$t_{sP} = 2t_{EP}C_{sP}\exp\left(-\tilde{\tau}_P\right)\cosh\left(\tilde{\tau}_P P_{He3P}\right).$$

Similarly, if $S^{-+}$is the only non-zero cross-section contributing, then

$$r_{SW}^{-} = \frac{C^{-+}}{C^{++}} = \frac{1 + e_{tA} A t_{aA}/t_{sA}}{1 - e_{tA} t_{aA}/t_{sA}} = \frac{C^{--}}{C^{+-}}.$$

This could be used to extract the analyzer transmission factor ratio t_{aA}/t_{sA} as

$$e_{tP} \frac{t_{aA}}{t_{sA}} = \frac{r_{SW}^{-} - 1}{r_{SW}^{-} + A},$$

where

$$\frac{t_{aA}}{t_{sA}} = \frac{C_{aA}}{C_{sA}} \tanh\left(\tilde{\tau}_A P_{He3A}\right).$$

It should be noted that if $S^{+-}$and $S^{-+}$are from inelastic cross-sections (spin-waves) then the count rates may be too low to make this analysis possible. One must then rely on separate transmission measurements of the He-3 cells along with any correction for time dependence to obtain the individual transmission factors necessary to extract the cross-sections from the count rates.

4.4 saturated ferromagnet

The case of $S^{+-} = S^{-+} = 0$ and $S^{++} \neq S^{--}$ is treated in a following section

5 inelastic scattering

All of the analysis can be applied to inelastic scattering by simply scaling $\tilde{\tau}$ by λ/λ_0, where λ is the actual nominal wavelength of the incoming or scattered neutrons, and $\lambda_0$is the wavelength at which $\tilde{\tau}$ was originally calculated. $\tilde{\tau}$ occurs in all the He-3 transmission factors and also in the correction coefficients for the He-3 transmission.

6 flipping ratios and efficiency measurements

6.1 spin-flip and non-spin-flip cross-sections only

In order to examine the performance of a polarized beam setup, it is required that the cross-sections be known and fairly simple. One useful case is where $S^{++} = S^{--} = S^{nsf}$ and $S^{+-} = S^{-+} = S^{sf}$, so that the cross-section asymmetry can be defined as

$$s = \frac{S^{nsf} - S^{sf}}{S^{nsf} + S^{sf}}.$$

Note that in this case where the scattering matrix commutes with the transport and flipper matrices, e_A and e_P only appear as the product $e_A e_P$, and there is no way to separate the effects of the transport efficiency before the sample from the transport efficiency after the sample. To perform that separation would require $S^{++} \neq S^{--}$ or $S^{+-} \neq S^{-+}$. Examples of the S^{nsf}, S^{sf} case are pure non-spin-flip scattering, $s = 1$, pure spin-flip scattering, $s = -1$ and spin-incoherent scattering, $s = -1/3$. These cross-sections should be free of multiple scattering and produce count rates that are in the linear range of the detector electronics. Then expressions for the flipping ratios using the polarizer flipper or analyzer flipper can be used to determine transport and flipping efficiencies. These flipping ratios are given by $R_{P,A}(s) = CountsFlipperOFF^{++}/CountsFlipperON^{+-,-+}$. Thus

$$R_{P,A}(s) = \frac{t_+ + e_t s\, t_-}{t_+ - e_t(2e_{FP,A} - 1)s\, t_-} = \frac{1 + e_t s\, t_-/t_+}{1 - e_t(2e_{FP,A} - 1)s\, t_-/t_+}$$

Recalling the expressions for t_+ and t_-, the ratio t_-/t_+ is

$$\frac{t_-}{t_+} = P_n = \frac{C_{-\Delta}}{C_{+\Delta}} \tanh(\tilde{\tau}_P P_{He3P}) \tanh(\tilde{\tau}_A P_{He3A}) = \frac{R_{0,nsf} - 1}{R_{0,nsf} + 1}$$

($R_{0,nsf}$ is defined in the following) is approximately the product of the neutron polarizations produced by the two He3 cells. Here $e_{tF} = e_t(2e_F - 1)$ depends on the product of the transport and flipper efficiencies. If the transport and flipper efficiencies are unity then the expected flipping ratios are

$$\begin{aligned} R_0(s &= 1) = R_{0,nsf} = \frac{1 + t_-/t_+}{1 - t_-/t_+} = \frac{t_{++}}{t_{+-}} = \frac{t_{+A}t_{+P} + t_{-A}t_{-P}}{t_{+A}t_{-P} + t_{-A}t_{+P}} \\ &= \frac{\hat{C}_{+\Delta}}{\hat{C}_{-\Delta}} \frac{\cosh(\tilde{\tau}_A P_{He3A} + \tilde{\tau}_P P_{He3P})}{\cosh(\tilde{\tau}_A P_{He3A} - \tilde{\tau}_P P_{He3P})} \end{aligned}$$

$$\hat{C}_{\pm\Delta} = 1 + \left(\frac{\Delta_A}{\langle C_A \rangle} \pm \frac{\Delta_P}{\langle C_P \rangle} \right) \tanh(\tilde{\tau}_A P_{He3A} \pm \tilde{\tau}_P P_{He3P})$$

$$R_0(s = -1) = R_{0,sf} = \frac{1}{R_0(1)}$$

$$R_0(s = -1/3) = R_{0,inc} = \frac{R_0(1) + 2}{2R_0(1) + 1}.$$

If the flipping efficiency is unity then the expected flipping ratios are

$$R(e_F = 1, s) = \frac{1 + e_t s\, t_-/t_+}{1 - e_t s\, t_-/t_+},$$

and in particular

$$R(e_F = 1, s = 1) = \frac{1 + e_t t_-/t_+}{1 - e_t t_-/t_+} = \frac{R_{0,nsf} + \epsilon_t}{\epsilon_t R_{0,nsf} + 1}$$

where $\epsilon_t = (1 - e_t) / (1 + e_t)$ is the transport loss. Thus, when the flipping efficiency is assumed to be unity, then the transport efficiency can be determined as

$$e_t = \frac{1}{s}\left(\frac{R(s) - 1}{R(s) + 1}\right)\frac{t_+}{t_-} = \frac{1}{s}\frac{R_{0,nsf} + 1}{R_{0,nsf} - 1}\left(\frac{R(s) - 1}{R(s) + 1}\right).$$

or for nsf scattering the transport loss is given by

$$\epsilon_t = \frac{R_{0,nsf} - R_{nsf}}{R_{0,nsf} R_{nsf} - 1} \cong \frac{1}{R_{nsf}} - \frac{1}{R_{0,nsf}}$$

If both transport and flipping efficiencies are unknown then they cannot be determined separately by a single flipping ratio measurement. One of the efficiencies can be found in terms of the other for a single flipping ratio measurement as

$$e_t = \frac{1}{s}\frac{R(s) - 1}{R(s)\,(2e_{FP,A} - 1) + 1}\frac{t_+}{t_-} \tag{20}$$

6.1.1 using two different cross-section asymmetries to measure efficiencies

One way to uniquely determine the efficiencies is to make flipping ratio measurements for two different types of cross-sections (different known s values). Then

$$e_t = \frac{f(s_1, s_2)}{s_1 s_2\,[R(s_1) - R(s_2)]}\frac{t_+}{t_-}$$

and

$$2e_F - 1 = \frac{s_2 R(s_1) - s_1 R(s_2) + (s_1 - s_2)}{f(s_1, s_2)}$$

where

$$f(s_1, s_2) = R(s_1)R(s_2)(s_1 - s_2) + s_2 R(s_2) - s_1 R(s_1).$$

Another measurement that can be made is the ratio of observed counts when both flippers are OFF to when both flippers are ON. This yields

$$R_{P+A}(s) = \frac{t_+ + e_t s\, t_-}{t_+ + e_t(2e_{FA} - 1)(2e_{FP} - 1)s\, t_-}$$

If $R_{P+A}(s) \equiv 1$, and it can be assumed that the transport is the same for either flipper state, then this is a good indication that the flipper efficiencies are unity

(Note that $R_{P+A}(s) \equiv 1$ if the flipper efficiency is zero also). In general it is expected that this flipping ratio is near unity. By measuring both $R_{P+A}(s)$ and $R_{P,A}(s)$ two equations are generated but the product of the two flipper efficiencies appears in one of the equations. If it can be assumed that the flipper efficiencies are equal (as might be suggested if $R_P(s) = R_A(s)$) then a quadratic equation can be found for the flipper efficiency,

$$\left(1 - \frac{1}{R_{P,A}}\right) X^2 + \left(1 - \frac{1}{R_{P+A}}\right) X - \left(\frac{1}{R_{P+A}} - \frac{1}{R_{P,A}}\right) = 0$$

where $X = 2e_F - 1$. Because $R_{P+A}(s)$ is near unity and $R_{P,A}$is not, an approximate solution is

$$X = 2e_F - 1 \cong 1 - \frac{R_{P+A} - 1}{1 - 1/R_{P,A}}$$

or

$$e_F \cong 1 - \frac{1}{2} R_{P,A} \frac{R_{P+A} - 1}{R_{P,A} - 1}$$

This solution for the flipper efficiency can then be used to solve for the transport efficiency13.

6.1.2 using polarizer and analyzer flippers to measure efficiencies

More commonly when both polarizer and analyzer spin flippers are available, the efficiencies can be determined by measuring all four polarized beam cross-sections and the He-3 cell transmissions. This is usually done with pure non-spin-flip scattering, although for any cross-section asymmetry, s, the observed counts for the four cross-sections are

$$C^{++} = K(1 + se_t t_-/t_+)$$

$$C^{+-} = K(1 - se_t P t_-/t_+)$$

$$C^{-+} = K(1 - se_t A t_-/t_+)$$

$$C^{--} = K(1 + se_t A P t_-/t_+)$$

where K is some proportionality constant, $e_t = e_{tA} e_{tP}$ is the total transport efficiency, $P = 2e_{FP} - 1$ and $A = 2e_{FA} - 1$. Note that if $S^{sf} = S^{nsf}$, then the counts for all four polarized beam cross-sections are identical and independent of beam transport and flipping efficiencies. For $s \neq 0$ it is easy to show that

$$P = 2e_{FP} - 1 = \frac{C^{--} - C^{+-}}{C^{++} - C^{-+}}$$

$$A = 2e_{FA} - 1 = \frac{C^{--} - C^{-+}}{C^{++} - C^{+-}}.$$

The error propagation produces

$$\tilde{\sigma}_P^2 = \frac{\sigma_P^2}{P^2} = \frac{\sigma_{C^{++}}^2 + \sigma_{C^{-+}}^2}{(C^{++} - C^{-+})^2} + \frac{\sigma_{C^{--}}^2 + \sigma_{C^{+-}}^2}{(C^{--} - C^{+-})^2}$$

$$\tilde{\sigma}_A^2 = \frac{\sigma_A^2}{A^2} = \frac{\sigma_{C^{++}}^2 + \sigma_{C^{+-}}^2}{(C^{++} - C^{+-})^2} + \frac{\sigma_{C^{--}}^2 + \sigma_{C^{-+}}^2}{(C^{--} - C^{-+})^2}.$$

and where $\sigma_{e_F} = \frac{1}{2}\sigma_{P,A}$. The transport efficiency can also be obtained from

$$se_t\frac{t_-}{t_+} = \frac{C^{++} - C^{+-}}{PC^{++} + C^{+-}} = \frac{C^{++} - C^{-+}}{AC^{++} + C^{-+}} = \frac{(C^{++} - C^{+-})(C^{++} - C^{-+})}{C^{++}C^{--} - C^{+-}C^{-+}},$$

which is symbolically

$$e_t = \frac{1}{sP_n}\frac{N^{+-}N^{-+}}{D}.$$

The transport efficiency can only be obtained as a function of the cross-section asymmetry and the He-3 transmission factor, t_-/t_+, where

$$P_n = \frac{t_-}{t_+} = \frac{C_{-\Delta}}{C_{+\Delta}} \tanh\left(\tilde{\tau}_A P_{He3A}\right) \tanh\left(\tilde{\tau}_P P_{He3P}\right)$$

and the correction coefficient is

$$\frac{C_{-\Delta}}{C_{+\Delta}} = C_R \cong 1 + \sum_{m=P,A} \frac{\Delta_m}{\langle C_m \rangle} \frac{2}{\sinh\left(2\tilde{\tau_m} P_{He3m}\right)}.$$

The error propagation for the beam transport efficiency measurement thus depends on uncertainties in $\tilde{\tau}$ and P_{He3}, in addition to the uncertainties in the measured count-rates, and is then given by

$$\tilde{\sigma}_{e_t}^2 = \sum_{\alpha\beta} W_{\alpha\beta}^2 \sigma_{C^{\alpha\beta}}^2 + \sum_{m=P,A} W_m^2 \left(\tilde{\sigma}_{P_{He}}^2 + \tilde{\sigma}_{\tilde{\tau}}^2\right)_m$$

where

$$W_{++} = \frac{1}{N^{+-}} + \frac{1}{N^{-+}} - \frac{C^{--}}{D}$$

$$W_{--} = \frac{-C^{++}}{D}$$

$$W_{+-} = \frac{-1}{N^{+-}} + \frac{C^{-+}}{D}$$

$$W_{-+} = \frac{-1}{N^{-+}} + \frac{C^{+-}}{D}$$

$$W_{m=P,A} = \left[\frac{2\tilde{\tau} P_{He3}}{\sinh\left(2\tilde{\tau} P_{He3}\right)} \right]_m .$$

See section ?? for an explanation of the calculation of the errors $\tilde{\sigma}^2_{P_{He}}$ and $\tilde{\sigma}^2_{\tilde{\tau}}$.

6.1.3 checking He-3 cell polarization during an experiment

The best way to keep track of the polarization of the He-3 cells is to use beam monitors as shown in the diagram at the start of this document, and measure the transmissions as a function of time. If this is not possible, measurements of the non-spin-slip flipping ratio can be used to monitor the polarized beam performance. Also, as will be shown in the following section, these flipping ratio measurements aid in correcting polarized beam data. Typically, previously measured values of transport and flipping efficiencies are assumed to remain in effect, and the flipping ratio measurement is used to check on the expected polarizing efficiency of the He-3 cells. The solution for the polarizing efficiency, P_n, in terms of the measured non-spin-flip flipping ratio and the transport efficiencies is

$$P_n = \frac{t_-}{t_+} = \frac{1}{e_t} \frac{R-1}{R(2e_F - 1) + 1}.$$

The error propagation for measuring this polarizing efficiency int terms of the flipping ratio is

$$\tilde{\sigma}^2_{P_n} = \tilde{\sigma}^2_{e_t} + \left(\frac{2e_F R}{R(2e_F - 1) + 1} \right)^2 \tilde{\sigma}^2_{e_F} + \left(\frac{2e_F R^2}{[R-1]\,[R(2e_F - 1) + 1]} \right)^2 \tilde{\sigma}^2_R$$

where

$$\tilde{\sigma}^2_R = \tilde{\sigma}^2_{Cnsf} + \tilde{\sigma}^2_{Csf},$$

and $Cnsf$ and Csf are the count rates that determine the flipping ratio. Recall that the expected value of P_n is

$$\bar{P}_n = \frac{t_-}{t_+} = C_R \tanh\left(\tilde{\tau}_A P_{He3A}\right) \tanh\left(\tilde{\tau}_P P_{He3P}\right),$$

and the error propagation for this expected value was calculated previously as

$$\tilde{\sigma}^2_{\bar{P}_n} = \sum_{m=P,A} \left[\frac{2\tilde{\tau}P_{He3}}{\sinh\left(2\tilde{\tau}P_{He3}\right)}\right]^2_m \left(\tilde{\sigma}^2_{P_{He}} + \tilde{\sigma}^2_{\tilde{\tau}}\right)_m .$$

The values for $\tilde{\sigma}^2_{P_{He}}(t)$ and $\tilde{\sigma}^2_{\tilde{\tau}}$ are given in section **??**.

Of course if there is already confidence in the expected value of $\bar{P}_n$, then the flipping ratio measurement can be used to check on the transport efficiency.

6.2 saturated ferromagnet

Another set of cross-sections that can be useful in characterizing a polarized beam setup, has the conditions that $S^{+-} = S^{-+} = S^{sf} = 0$ and $S^{++} \neq S^{--}$. For example, these cross-sections apply to a saturated ferromagnet. It is important that complete saturation is reached, otherwise there will be contributions from spin-flip scattering or beam depolarization from ferromagnetic domains. In this case the cross terms in the expression for the transfer matrix elements do not cancel. This cancellation had simplified these matrix elements in the case of spin-flip and non-spin-flip scattering symmetry, so that there was no dependence on solely the pre-sample or post-sample side of the beam path transport. Breaking this symmetry complicates the expressions, but does allow extraction of the separate beam transport efficiencies. The expressions for the expected count-rates are

$$C^{\alpha\beta}/\frac{N}{2} = c^{\alpha\beta}_{++}S^{++} + c^{\alpha\beta}_{--}S^{--}.$$

Explicitly writing these out

$$C^{++}/\frac{N}{8} = \left(t_{sA} + e_{tA}t_{aA}\right)\left(t_{sP} + e_{tP}t_{aP}\right)S^{++} + \left(t_{sA} - e_{tA}t_{aA}\right)\left(t_{sP} - e_{tP}t_{aP}\right)S^{--}$$

$$C^{--}/\frac{N}{8} = \left(t_{sA} - e_{tA}At_{aA}\right)\left(t_{sP} - e_{tP}Pt_{aP}\right)S^{++} + \left(t_{sA} + e_{tA}At_{aA}\right)\left(t_{sP} + e_{tP}Pt_{aP}\right)S^{--}$$

$$C^{+-}/\frac{N}{8} = \left(t_{sA} + e_{tA}t_{aA}\right)\left(t_{sP} - e_{tP}Pt_{aP}\right)S^{++} + \left(t_{sA} - e_{tA}t_{aA}\right)\left(t_{sP} + e_{tP}Pt_{aP}\right)S^{--}$$

$$C^{-+}/\frac{N}{8} = \left(t_{sA} - e_{tA}At_{aA}\right)\left(t_{sP} + e_{tP}t_{aP}\right)S^{++} + \left(t_{sA} + e_{tA}At_{aA}\right)\left(t_{sP} - e_{tP}t_{aP}\right)S^{--},$$

where recall that $A = 2e_{FA} - 1$ and $P = 2e_{FP} - 1$. Now define the following combinations of count rates,

$$d_P = C^{++} - C^{+-}$$

$$d_A = C^{--} - C^{-+}$$

$$s_P = PC^{++} + C^{+-}$$

$$s_A = C^{--} + PC^{-+},$$

and also the cross-section asymmetry,

$$s = \left(\frac{S^{++} - S^{--}}{S^{++} + S^{--}}\right).$$

Then if $P > 0$ (else the expected count-rate differences d_P and $d_A$would be zero) and $A > 0$ (else $d_P + d_A = 0$) and $e_{tA} > 0$,

$$\frac{1}{e_{tA}} \frac{t_{sA}}{t_{aA}} s = \frac{Ad_P - d_A}{d_P + d_A}. \tag{21}$$

However, even if $P = 0$ the following equation holds true provided $A > 0$ (else $s_P = s_A$),

$$e_{tA} \frac{t_{aA}}{t_{sA}} s = \frac{s_P - s_A}{As_P + s_A}. \tag{22}$$

Also if $P > 0$ and $A > 0$ there is the result

$$s^2 = \left(\frac{Ad_P - d_A}{d_P + d_A}\right)\left(\frac{s_P - s_A}{As_P + s_A}\right) = \frac{N_1 N_2}{D_1 D_2},$$

which is indpendent of transport efficiency (except that $e_{tA} > 0$) and independent of the time dependence of the He-3 transmission. These formulae allow determination of the beam transport efficiency on the analyzer side, or a measurement of the cross-section asymmetry, s, or a check on the He-3 transmission factor t_{aA}/t_{sA}. Note that the second equation, **??**, that holds true even if $P = 0$ indicates that s can be measured even with an unpolarized incident beam provided that $A > 0$ (otherwise $s_P = s_A$) and $e_{tA} > 0$. This is due to the fact that by the nature of the sample cross-sections, the scattered beam is polarized ($S^{++} \neq S^{--}$). For the error propagation on the analyzer-side beam-transport-efficiency, e_{tA}, the transmission factor, t_{aA}/t_{sA}, is required. Recall that this is

$$\frac{t_{aA}}{t_{sA}} = \frac{C_{aA}}{C_{sA}} \tanh\left(\tilde{\tau}_A P_{He3A}\right)$$

where the ratio of correction coefficents is

$$\frac{C_{aA}}{C_{sA}} = 1 + \frac{\Delta_A}{\langle C_A \rangle}\left[\frac{2}{\sinh\left(2\tilde{\tau}_A P_{He3A}\right)}\right].$$

Using the first equation, **??**, to measure e_{tA}

$$e_{tA} = s\frac{t_{sA}}{t_{aA}}\frac{d_P + d_A}{Ad_P - d_A},$$

and the error propagation for e_{tA} is then

$$\tilde{\sigma}^2_{e_{tA}} = \sum_{\alpha\beta} W^2_{\alpha\beta}\sigma^2_{C^{\alpha\beta}} + W^2_A\tilde{\sigma}^2_A + W^2_t\left(\tilde{\sigma}^2_{P_{He}} + \tilde{\sigma}^2_{\tilde{\tau}}\right)_A + \tilde{\sigma}^2_s,$$

where

$$W_{\alpha\beta} = \frac{\alpha\beta}{d_P + d_A} - \frac{\beta A^{\delta_{\alpha+}}}{Ad_P - d_A}$$

$$W_A = \frac{Ad_P}{Ad_P - d_A}$$

$$W_t = \frac{2\tilde{\tau}_A P_{He3A}}{\sinh\left(2\tilde{\tau_A} P_{He3A}\right)}.$$

Using the first equation, **??**, to measure s

$$s = e_{tA}\frac{t_{aA}}{t_{sA}}\frac{Ad_P - d_A}{d_P + d_A},$$

the same error propagation applies so that

$$\tilde{\sigma}^2_s = \sum_{\alpha\beta} W^2_{\alpha\beta}\sigma^2_{C^{\alpha\beta}} + W^2_A\tilde{\sigma}^2_A + W^2_t\left(\tilde{\sigma}^2_{P_{He}} + \tilde{\sigma}^2_{\tilde{\tau}}\right)_A + \tilde{\sigma}^2_{e_{tA}}.$$

Similarly the error propagation for $s^2$is given by

$$\tilde{\sigma}^2_{s^2} = \sum_{\alpha\beta} W^2_{\alpha\beta}\sigma^2_{C^{\alpha\beta}} + \sum_{X=A,P} W^2_X\sigma^2_X,$$

where

$$W_{++} = \left(\frac{A}{N_1} - \frac{1}{D_1}\right) + \left(\frac{1}{N_2} - \frac{A}{D_2}\right)P$$

$$W_{--} = \left(\frac{1}{N_1} + \frac{1}{D_1}\right) + \left(\frac{1}{N_2} + \frac{1}{D_2}\right)$$

$$W_{+-} = \left(\frac{A}{N_1} - \frac{1}{D_1}\right) - \left(\frac{1}{N_2} - \frac{A}{D_2}\right)$$

$$W_{-+} = \left(\frac{1}{N_1} + \frac{1}{D_1}\right) - \left(\frac{1}{N_2} + \frac{1}{D_2}\right)P$$

$$W_A = \frac{d_P}{N_1} - \frac{s_P}{D_2}$$

$$W_P = \frac{C^{++} - C^{-+}}{N_2} - \frac{AC^{++} + C^{-+}}{D_2}.$$

7 wavelength and path-length variation of He-3 transmission

In order to account for wavelength dependence in the He-3 transmission, take

$$\tau_M = \tau(\lambda_M) = n\sigma_0(\lambda_0)L\frac{\lambda_M}{\lambda_0} = \tau_0\frac{\lambda_M}{\lambda_0}.$$

where λ_0 serves as a reference wavelength for σ_0. t_E is roughly wavelength independent, and is about 0.86 for the cells used at the NCNR made of GE180 glass. The neutron path-length through the He-3 may also vary due to beam divergence or variation in the separation of the cell walls. The beam divergence can be treated by assuming parallel cells walls so that the angle dependence of the path length is

$$L(\phi) = L/\cos(\phi) \cong L(1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2)$$

where L is the minimal He-3 thickness for a beam perpendicular to the cell flat walls (this is the value of L that goes into $\tau_\pm$), ϕ is the neutron path divergence angle with respect to the perpendicular to the cell walls, and γ and δ are the corresponding divergence angles in the scattering plane and perpendicular to the scattering plane respectively. The transmission in terms of these deviations is

$$t_\pm = t_E \exp\left(-\tau[1 \mp P_{He}]\right) = t_E \exp\left(-n\sigma_0\frac{\lambda}{\lambda_0}L(1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2)[1 \mp P_{He}]\right)$$

or

$$t_\pm = t_E \exp\left(-\tau_{\pm 0}\frac{\lambda}{\lambda_0}(1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2)\right)$$

where $\tau_{\pm 0} = n\sigma_0 L[1 \mp P_{He}]$.

Consider the case that the incident neutrons have been scattered by a monochromating crystal, so that the incident and outgoing deviation angles in the scattering plane, γ_0 and γ_1, are correlated according to Bragg's law. The transmission probability function (TPF) depends on the crystal mosaics and collimations before and after the crystal. The scattering plane TPF can be derived in terms of the deviation angles (measured positive with respect to nominal in the clockwise from above direction), collimations before and after the crystal, $\alpha_0$and α_1, and crystal scattering-plane mosaic, η_H, as

$$P_H(\gamma_0, \gamma_1) = N_H \exp\left\{-\frac{1}{2}\left[\left(\frac{\gamma_0}{\alpha_0}\right)^2 + \left(\frac{\gamma_0 + \gamma_1}{2\eta_H}\right)^2 + \left(\frac{\gamma_1}{\alpha_1}\right)^2\right]\right\} d\gamma_0 d\gamma_1.$$

The Bragg's law correlation gives

$$\gamma_1 = \gamma_0 + 2\frac{\Delta\lambda}{\lambda}\tan(\omega_M)$$

where ω_M is the Bragg angle of the crystal and $\Delta\lambda = \lambda - \lambda_M$, with $\lambda_M = 2d_M \sin(\omega_M)$. Of course, d_M is the crystal d-spacing for the reflecting atomic planes. Thus the in-plane TPF can be written in terms of γ_1 and $x = \frac{\Delta\lambda}{\lambda}$ as

$$P_H = N_H \exp\left\{-\frac{1}{2}\left[\left(\frac{\gamma_1 - 2x\tan(\omega_M)}{\alpha_0}\right)^2 + \left(\frac{\gamma_1 - x\tan(\omega_M)}{\eta_H}\right)^2 + \left(\frac{\gamma_1}{\alpha_1}\right)^2\right]\right\} d\gamma_1 dx$$

or

$$P_H(\gamma_1, x) = N_H \exp\left\{-\frac{1}{2}\left[A\gamma_1^2 - 2B\gamma_1 x + Cx^2\right]\right\} d\gamma_1 dx$$

where

$$N_H = \frac{1}{2\pi}\left(AC - B^2\right)^{1/2} = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\alpha_0^2 + \alpha_1^2 + 4\eta_H^2}}{\alpha_0\alpha_1}\tan(\omega_M)\frac{1}{\sqrt{2\pi}}\frac{1}{\eta_H}$$

$$A = \frac{1}{\eta_H^2}\frac{\alpha_0^2\alpha_1^2 + \left(\alpha_0^2 + \alpha_1^2\right)\eta_H^2}{\alpha_0^2\alpha_1^2}$$

$$B = \frac{1}{\eta_H^2}\frac{\alpha_0^2 + 2\eta_H^2}{\alpha_0^2}\tan(\omega_M)$$

$$C = \frac{1}{\eta_H^2}\frac{\alpha_0^2 + 4\eta_H^2}{\alpha_0^2}\tan^2(\omega_M).$$

If the crystal mosaic is zero, then$\Delta\lambda$ and γ are perfectly correlated so that

$$P_{H0}(\gamma_1, x) = N_{H0}\delta(\gamma_1 - x\tan(\omega_M))\exp\left\{-\frac{1}{2}C_0 x^2\right\} d\gamma_1 dx$$

where now

$$N_{H0} = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\alpha_0^2 + \alpha_1^2}}{\alpha_0\alpha_1}\tan(\omega_M)$$

$$C_0 = \frac{\alpha_0^2 + \alpha_1^2}{\alpha_0^2\alpha_1^2}\tan^2(\omega_M).$$

The TPF for deviation angles out of the scattering plane is

$$P_V(\delta_0, \delta_1) = N_V \exp\left\{-\frac{1}{2}\left[\left(\frac{\delta_0}{\beta_0}\right)^2 + \left(\frac{\delta_1 - \delta_1}{2\eta_V \sin(\omega_M)}\right)^2 + \left(\frac{\delta_1}{\beta_1}\right)^2\right]\right\} d\delta_0 d\delta_1$$

where δ_0 and $\delta_1$are the deviation angles before and after the crystal, β_0 and β_1 the corresponding vertical effective collimations and η_V the crystal mosaic in the out of scattering-plane direction. Integrating over δ_0 and normalizing gives

$$P_V(\delta_1) = N_V \exp\left\{-\frac{1}{2}A_V\delta_1^2\right\} d\delta_1$$

where

$$A_V = \frac{\beta_1^2 + \left[\beta_0^2 + \left(2\eta_V \sin(\omega_M)\right)^2\right]}{\beta_1^2\left[\beta_0^2 + \left(2\eta_V \sin(\omega_M)\right)^2\right]}$$

$$N_V = \sqrt{\frac{A_V}{2\pi}}.$$

Now the average transmission can be calculated. For this calculation use $\lambda = \lambda_M(1 + x)$ in the expression for the transmission so that

$$t_\pm = t_E \exp\left(-\tau_{\pm M}(1 + x)(1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2)\right)$$

where $\tau_{\pm M} = \tau_{\pm 0}\frac{\lambda_M}{\lambda_0}$. λ_M is the average wavelength produced by the monochromator and $\lambda_0$is the reference wavelength at which the He-3 absorption cross-section in $\tau_{\pm 0}$ is evaluated. The expansion of the transmission up to second order in the deviations is just

$$t_\pm = t_{\pm 0}\left\{1 - \tau_{\pm M}(x + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2) + \frac{1}{2}\tau_{\pm M}^2 x^2\right\}$$

where $t_{\pm 0} = t_E \exp\left(-\tau_{\pm M}\right)$ is the transmission for zero deviations. The average transmission requires the integrals

$$< t_\pm >= t_{\pm 0}\int P_H(\gamma, x)P_V(\delta)\left\{1 - \tau_{\pm M}(x + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2) + \frac{1}{2}\tau_{\pm M}^2 x^2\right\} d\gamma dx\, d\delta$$

The perfect crystal case with a delta function produces

$$< t_\pm >= t_{\pm 0}\left\{1 - \frac{1}{2}\tau_{\pm M}\left[\left(\tan^2(\omega_M) - \tau_{\pm M}\right)\sigma_\gamma^2 + \sigma_\delta^2\right]\right\} = \hat{C}_{\pm 0}t_{\pm 0},$$

where $\sigma_\gamma^2 = 1/C_0$ and $\sigma_\delta^2 = 1/A_V$. Note that the sign can change for the in-plane part of the correction for τ_{+M} and τ_{-M}.

The more general case requires tedious integration. The δ^2 integral term is simply

$$\int P_H(\gamma, x)P_V(\delta)\delta^2 d\gamma dx\, d\delta = \frac{1}{A_V} = \sigma_\delta^2.$$

The γ^2term integrated over γ yields

$$\int P_H(\gamma,x)P_V(\delta)\gamma^2 d\gamma dx\, d\delta = N_H\sqrt{\frac{2\pi}{A}}\int dx\ \exp\left[-\frac{1}{2}\left(C-\frac{B^2}{A}\right)x^2\right]\left[\frac{1}{A}+\left(\frac{B}{A}x\right)^2\right]$$

which is

$$\int P_H(\gamma,x)P_V(\delta)\gamma^2 d\gamma dx\, d\delta = \frac{C}{AC-B^2} = \frac{\alpha_1^2\left(\alpha_0^2+4\eta_H^2\right)}{\alpha_1^2+\alpha_0^2+4\eta_H^2} = \sigma_\gamma^2.$$

The integral of the linear x term can be shown to be zero. The integral of the x^2 term is

$$\int P_H(\gamma,x)P_V(\delta)x^2\, d\gamma dx\, d\delta = \frac{A}{AC-B^2} = \frac{\left(\alpha_1^2+\alpha_0^2\right)\eta_H^2+\alpha_0^2\alpha_1^2}{\alpha_1^2+\alpha_0^2+4\eta_H^2}\cot^2(\omega_M) = \sigma_x^2 = \left(\frac{\sigma_\lambda}{\lambda_M}\right)^2.$$

The sum of all terms yields

$$< t_\pm >= t_{\pm 0}\left\{1-\frac{1}{2}\tau_{\pm M}\left[\sigma_\gamma^2+\sigma_\delta^2\right]+\frac{1}{2}\tau_{\pm M}^2\sigma_x^2\right\} = \hat{C}_\pm t_{\pm 0}$$

Another case to consider is that λ and γ are uncorrelated so that the probability distribution of incident neutrons is just a product of independent Gaussians

$$P(\alpha,\beta,\lambda) = G(\delta;\sigma_\delta)G(\gamma;\sigma_\gamma)G(x;\sigma_x)dx\, d\gamma\, d\delta,$$

where for example

$$G(\gamma;\sigma_\gamma) = \frac{1}{\sqrt{2\pi}\sigma_\gamma}\exp\left\{-\frac{1}{2}\left(\frac{\gamma}{\sigma_\gamma}\right)^2\right\}.$$

This yields the same expression as in the general case above, except that now σ_γ, σ_δ and σ_x are measured quantities. Note that in general the wavelength variation increases the transmission while the angular distribution decreases the transmission. The effects are largest for the non-preferred spin-state.

If the glass walls through which the neutron beam passes are spherical sections instead of flat, then there is a correction to the transmission from the varying path lengths. In order to calculate this, take the coordinate system origin at the center of the He-3 cell with z-axis up out of the scattering plane and the beam travelling along the x-direction, so that in terms of the in-plane and out-of-plane deviation angles the neutron direction is

$$\hat{\mathbf{n}} = \cos(\delta)\left[\cos(\gamma)\,\hat{\mathbf{x}}+\sin(\gamma)\,\hat{\mathbf{y}}\right]+\sin(\delta)\,\hat{z}.$$

In the $x=0$ plane passing through the cell center, assume that the neutron passes through the point $\mathbf{P}=(0,Y,Z)$. Then the neutron path is along the line

$$\mathbf{r}_n = \left[\beta\cos(\delta)\cos(\gamma)\,, Y+\beta\cos(\delta)\sin(\gamma)\,, Z+\beta\sin(\delta)\right].$$

Find the intersections of this line with the front and back spherical faces of the He-3 cell in order to calculate the path length. If R is the radius of curvature for the spherical faces and L is the straight through diameter of the cell, then points on the front (beam entrance) face satisfy

$$\left[x_f - \left(R - \frac{L}{2}\right)\right]^2 + y_f^2 + z_f^2 = R^2,$$

and if it can be assumed that $y_f^2 + z_f^2 \ll R^2$, an approximate expression for x_f is

$$x_f = -\frac{L}{2} + \frac{1}{2}\frac{y_f^2 + z_f^2}{R}.$$

Now use the expression for the neutron path to find the intersection point

$$x_f = \beta_f \cos(\delta)\cos(\gamma)$$

$$y_f = Y + \beta_f \cos(\delta)\sin(\gamma)$$

$$z_f = Z + \beta_f \sin(\delta).$$

The result of substitutions is a quadratic equation for the beam-path intersection length parameter, β_f, (which must be negative for the front face)

$$A\beta_f^2 + B_f\beta_f - C = 0,$$

where

$$A = \frac{1}{2R}\left[\cos^2(\delta)\sin^2(\gamma) + \sin^2(\delta)\right]$$

$$B_f = \frac{1}{R}\left[Y\cos(\delta)\sin(\gamma) + Z\sin(\delta)\right] - \cos(\delta)\cos(\gamma)$$

$$C = \frac{L}{2} - \frac{Y^2 + Z^2}{2R}.$$

For small beam divergence angles, γ and δ, $B_f^2 \cong 1$ and $|AC| \ll 1$, so that the solution for β_f can be approximated as

$$\beta_f = -C\left\{1 + \frac{Y}{R}\gamma + \frac{Z}{R}\delta + \frac{1}{2}\left(\gamma^2 + \delta^2\right)\left(1 - \frac{C}{R}\right)\right\}.$$

The quadratic equation for β_b, the beam exit intersection path length parameter, has the same coefficients A and C, but there is a sign change in B_b

$$B_b = \frac{1}{R}\left[Y\cos(\delta)\sin(\gamma) + Z\sin(\delta)\right] + \cos(\delta)\cos(\gamma),$$

so that

$$\beta_b = C\left\{1 - \frac{Y}{R}\gamma - \frac{Z}{R}\delta + \frac{1}{2}\left(\gamma^2+\delta^2\right)\left(1-\frac{C}{R}\right)\right\}.$$

The total path length is then

$$L(\gamma,\delta,Y,Z) = \beta_b - \beta_f = \left(L - \frac{Y^2+Z^2}{R}\right)\left\{1 + \frac{1}{2}\left(\gamma^2+\delta^2\right)\left(1-\frac{C}{R}\right)\right\}.$$

Note that if $R \to \infty$ the expression for the path length in the flat wall case is recovered. The Y^2+Z^2 dependence can be handled by assuming that the probability distribution for beam divergence angles is independent of Y^2+Z^2 (which should be true for small enough Y^2+Z^2), and then replacing Y^2+Z^2 by its average over the effective beam cross sectional area in the $x=0$ plane. For example, if the effective beam cross sectional area is a disc of radius r then $\left\langle Y^2+Z^2\right\rangle = \left\langle \rho^2\right\rangle = \frac{1}{2}r^2$. The final result is that the previous expression for the transmission as a function of deviation angles and wavelength deviation, which was

$$t_\pm = t_E \exp\left(-\tau_{\pm M}(1+x)(1+\frac{1}{2}\gamma^2+\frac{1}{2}\delta^2)\right)$$

can be simply modified by scaling down $\tau_{\pm M}$

$$\tilde{\tau}_{\pm M} = \tau_{\pm M}\left(1 - \frac{\left\langle\rho^2\right\rangle}{LR}\right)$$

where L is the straight through path length of the cell. Also, the $\frac{1}{2}$ coefficients of γ^2 and $\delta^2$are scaled down

$$\frac{1}{2} \to \frac{1}{2}\left\{1 - \frac{L}{2R}\left[1 - \frac{\left\langle\rho^2\right\rangle}{LR}\right]\right\} = \frac{1}{2}P.$$

Note that for a completely spherical cell ($R = L/2$) and a beam that must pass through the cell center ($\left\langle\rho^2\right\rangle = 0$) the dependence on angular deviation becomes zero, as it should. The scaling of γ^2 and δ^2translate directly into scaling of σ_γ^2 and σ_δ^2 in the results for the averaged transmission. Disregarding angle and wavelength deviations the basic transmission is modified to

$$\tilde{t}_{\pm 0} = t_E \exp\left[-\left(1 - \frac{\left\langle\rho^2\right\rangle}{LR}\right)n\sigma_0\frac{\lambda_M}{\lambda_0}L\left(1 \mp P_{He}\right)\right] = t_E\exp\left(-\tilde{\tau}_{\pm M}\right).$$

The general form for the averaged transmission for all corrections is

$$< t_\pm >= \tilde{t}_{\pm 0}\left\{1 - \frac{1}{2}\tilde{\tau}_{\pm M}P\left[\sigma_\gamma^2 + \sigma_\delta^2\right] + \frac{1}{2}\tilde{\tau}_{\pm M}^2\sigma_x^2\right\} = \hat{C}_\pm \tilde{t}_{\pm 0}$$

Also consider the effect of higher order wavelength contamination of the neutron beam. In this case the wavelength probability distribution is a sum of probability distributions centered at each higher order wavelength, $\lambda_n = \lambda_1/n$, so that

$$P(\lambda) = \sum_{n=1} a_n P_n(\lambda_n),$$

where the sum of wavelength fractions is unity

$$\sum_{n=1} a_n = 1.$$

All the wavelength fractions are at the same settings for angles and angle distribution parameters so that the transmission correction factor, $\hat{C}_\pm$, should be approximately wavelength order independent. The averaged transmission factor is then

$$\langle t_\pm \rangle = \hat{C}_\pm t_E \sum_{n=1} a_n \exp\left(-\tau_{\pm n}\right),$$

where $\tau_{\pm n} = \tau_{\pm 1}\lambda_n/\lambda_1 = \tau_\pm/n$. Thus

$$\langle t_\pm \rangle = \hat{C}_\pm t_E \sum_{n=1} a_n \exp\left(-\frac{1}{n}\tau_\pm\right) = \widetilde{C}_\pm t_\pm,$$

where the correction factor is now

$$\widetilde{C}_\pm = \hat{C}_\pm\left\{1 + \sum_{n=2} a_n K_{\pm n}\right\}$$

and

$$K_{\pm n} = \exp\left[\left(1 - \frac{1}{n}\right)\tau_\pm\right] - 1$$

For example, take $\tau_m = 1.8662$, $P_{He} = 0.7$ and the primary wavelength as 1.77 Angstroms. For the uncorrelated beam correction, using $\frac{\sigma_\lambda}{\lambda_m} = 0.05$, $\sigma_\alpha = 0.01$ and $\sigma_\beta = 0.04$, $C_+ = 1.0001$ and $C_- = 1.01$. For the correlated beam correction with $\cot(\theta_m) = 1$ and the same σ_α and σ_β values, $C_+ = 0.9998$ and $C_- = 1.0017$. The second order wavelength contamination factors (which still have to be multiplied by a_n) are $K_{+2} = 0.323$ and $K_{-2} = 3.885$. This means that the corrections to the transmission factors due to second order wavelength contamination can be significant (depending on the fraction a_2).

8 monitoring He-3 polarization and neutron polarization

If the transmission, t_{00}, through the unpolarized He-3 cell is measured ($P_{He} = 0$), then measurements of $t_0(P_{He})$ can be used to monitor the He-3 polarization, P_{He}, of the He-3 cell, assuming $\tilde{\tau}$ has been determined by a transmission measurement of the unpolarized cell. This is most conveniently done when there are no higher order wavelength contaminations, so that

$$r(P_{He}) = \frac{t_0(P_{He})}{t_{00}} = \cosh(\tilde{\tau} P_{He}) + \Delta \sinh(\tilde{\tau} P_{He}). \tag{23}$$

Neglecting the correction term in Δ, the coshfunction can be inverted to give

$$\tilde{\tau} P_{He} \cong x_0 = \ln\left(r + \sqrt{r^2 - 1}\right).$$

If the correction coefficient is known then

$$\tilde{\tau} P_{He} \cong x_0 - \Delta.$$

The outgoing neutron polarization, $-1 \leq P_n \leq 1$, after an incident unpolarized beam passes through the cell is

$$P_n = \frac{n_+ - n_-}{n_+ + n_-} = \tanh\left(\tilde{\tau}_M P_{He}\right) + \frac{\Delta}{\cosh^2(\tilde{\tau}_M P_{He})}. \tag{24}$$

As in the example above, using a 7 cm gas-thickness He-3 cell at 2 bars has $\tau_M = 1.8662$. With $P_{He} = 0.7$ He-3 polarization and $t_E = 0.86$, the cell transmits an uncorrected $t_0 = 0.2636$ of an incident unpolarized beam at 1.77 Angstroms and produces an outgoing beam that is $P_n = 0.8633$ polarized ($n_-/n_+ = 0.0733$). Making the corrections as in the example above, for the uncorrelated beam case, $t_0 = 0.2638$, and for the correlated beam case, $t_0 = 0.2637$. The corrections to the polarization for these two cases yield $P_n = 0.8621$ and $P_n = 0.8631$.

The best way to keep track of the polarization of the He-3 cells is to use beam monitors as shown in the diagram at the start of this document, and measure the transmissions as a function of time. If this is not possible, the remaining handle on the polarized beam performance is the flipping ratio, preferably measured with a non-spin-flip cross section. Recall that this flipping ratio is

$$R_{nsf} = \frac{t_+ + e_t t}{t_+ - e_{tF}(2e_{FP,A} - 1)t_-}$$

with $e_{tF} = e_t(2e_{FP,A} - 1)$. Now it is assumed that the correction factors for the He-3 transmission factors are unity. When the transport and flipping efficiencies are unity this simplifies to

$$R_{0,nsf} = \frac{\cosh(\tau_{M1} P_{He1} + \tau_{M2} P_{He2})}{\cosh(\tau_{M1} P_{He1} - \tau_{M2} P_{He2})}$$

and in terms of this ideal flipping ratio

$$R_{nsf} = R_{0,nsf}\frac{1+e_t}{1+e_{tF}}\left(1+\epsilon_t/R_{0,nsf} - \epsilon_{tF}R_{0,nsf}\right)$$

where the transport loss is $\epsilon_t = (1-e_t)/(1+e_t)$ and transport-flipper loss is $\epsilon_{tF} = (1-e_{tF})/(1+e_{tF})$. If the cell parameters τ_{M1} and $\tau_{M2}$are known, as well as the cell He-3 polarizations (through transmission measurements and known time dependences) and beam efficiencies, then the calculated R_{nsf}can be compared to measured values.