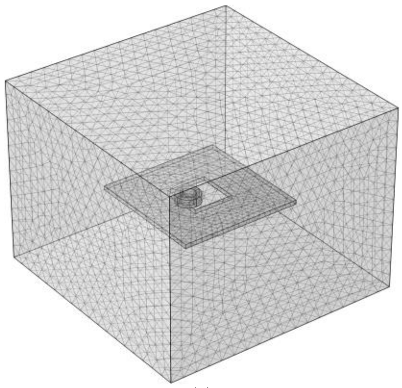
Finite element method and a modified method for modeling eddy current in magnetic conductors

李韋辰

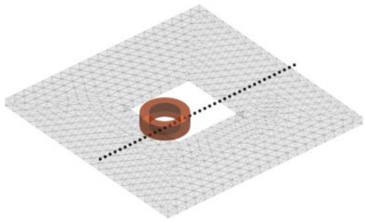
1. Introduction

Eddy current testing involves an electromagnet (EM) excited by a time-varying current, generating time-varying magnetic flux density (MFD) in its vicinity. The time-varying MFD induces eddy current inside targets, which generates a secondary MFD. The combined MFD due to EM current, eddy current, and material magnetization is conventionally sensed via sensing coils in the form of electromotive force or a change in impedance. Due to advancements in semiconductor technology, magnetic sensors are alternatives for measuring MFD, which allows higher sensitivity at low frequencies compared to sensing coils. Emerging eddy current technology utilizes multiple excitation EMs and magnetic sensors for extensive coverage of the target, and has been used in material characterization and defect imaging. For the analysis of the systems, modeling of eddy current and material magnetization is required. Thus, this project offers the following:

- Modeling eddy current problem using finite element method (FEM). Following the procedure of strong form, weak form, then discretization in the formulation. An example problem is shown in Fig. 1(a), both the magnetic conductor, EM, and the surrounding region need to be meshed.
- Modeling eddy current problem using a modified method. Only the magnetic conductor needs to be meshed, as illustrated in Fig. 1(b).



(a)



(b)

Fig. 1. An example eddy current problem containing a circular EM and a conductive plate with a square hole.
(a) Mesh for finite element method. (b) Mesh for the modified method, along with the EM and magnetic sensors.

2. Finite Element Method

Quasi-static analysis of magnetic and electric fields in eddy current problem is valid under the assumption $\partial \mathbf{D}/\partial t = 0$. This implies the maxwell equations (1)~(3) and the continuity equation (4).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{J} = 0 \tag{4}$$

$\mathbf{J}$ is the sum (5) of the current density due to electric field $\mathbf{E}$ and the external current density in the EMs $\mathbf{J}_e$, σ is the electrical conductivity. Assuming magnetic materials operate within the linear region where hysteresis can be neglected, magnetic flux density $\mathbf{B}$ and magnetic field intensity $\mathbf{H}$ are related by the constitutive relation (6), $\mathbf{M}$ is the magnetization vector and μ is the magnetic permeability.

$$\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_e \tag{5}$$

$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right) = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H} \tag{6}$$

Since $\mathbf{B}$ is solenoidal (3), it can be expressed as (7), $\mathbf{A}$ is the magnetic vector potential (MVP).

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{7}$$

Substituting (7) into (1) and by noticing $\nabla \times (-\nabla V) = 0$, (8) can be obtained.

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \mathbf{V} \tag{8}$$

Therefore, we have the governing equations (9) and (10), we refer to them as Ampère's law and continuity equation.

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = -\sigma \frac{\partial \mathbf{A}}{\partial t} - \sigma \nabla V + \mathbf{J}_e \tag{9}$$

$$\nabla \cdot \left(-\sigma \frac{\partial \mathbf{A}}{\partial t} - \sigma \nabla V + \mathbf{J}_e \right) = 0 \tag{10}$$

The current density $\mathbf{J}_e$ in the EMs is related to EM current I_e by (11), N_e is the number of turns of the EM, A_e is the cross-sectional area of the EM, and $\mathbf{t}_e$ is the unit-vector tangent to EM winding.

$$\mathbf{J}_e = \frac{N_e I_e}{A_e} \mathbf{t}_e \tag{11}$$

The magnetic flux linkage in an EM Φ_e is the line integral of the MVP along the EM winding (12), and can be represented as a volume integral over the EM volume. The time derivative of the magnetic flux linkage gives the electromotive force (EMF) in an EM. EM terminal voltage V_e and EM current I_e are thus related by (13), R_e is the resistance of the EM.

$$\Phi_e = \int_{C_e} \mathbf{A} \cdot \mathbf{t}_e dl = \int_{\Omega_e} \frac{N_e}{A_e} \mathbf{A} \cdot \mathbf{t}_e d\Omega \tag{12}$$

$$V_e - \frac{\partial}{\partial t} \Phi_e = R_e I_e \tag{13}$$

2.1 Finite element method formulation

2.1.1 Ampȇre's law

Ampȇre's law equation (9) is in strong form in the sense that $(\nabla \times \mathbf{A})/\mu$ may be discontinuous across material boundaries and its spacial derivative would not be numerically available. To convert into weak form, we first write the governing equation in indicial notation. The left-hand side of (9) can be written as (14).

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) &= \frac{\partial}{\partial x_l} \left(\frac{1}{\mu} \frac{\partial A_j}{\partial x_i} \right) \varepsilon_{ijk} \varepsilon_{lkm} \\ &= \frac{\partial}{\partial x_l} \left(\frac{1}{\mu} \frac{\partial A_j}{\partial x_i} \right) \left(\delta_{im} \delta_{jl} - \delta_{il} \delta_{jm} \right) \\ &= \frac{\partial}{\partial x_j} \left(\frac{1}{\mu} \frac{\partial A_j}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left(\frac{1}{\mu} \frac{\partial A_j}{\partial x_i} \right) \end{aligned} \tag{14}$$

Thus, (9) is equivalent to (15).

$$\frac{\partial}{\partial x_j} \left(\frac{1}{\mu} \frac{\partial A_j}{\partial x_i} \right) - \frac{\partial}{\partial x_j} \left(\frac{1}{\mu} \frac{\partial A_i}{\partial x_j} \right) = -\sigma \frac{\partial A_i}{\partial t} - \sigma \frac{\partial V}{\partial x_i} + J_{e,i} \tag{15}$$

Following the Galerkin method, we integrate (15) over the entire domain for a weight function $\mathbf{W}$. Equation (16) should satisfy for a variety of weight functions W_i.

$$\int_\Omega \left[\frac{\partial}{\partial x_j} \left(\frac{1}{\mu} \frac{\partial A_j}{\partial x_i} \right) - \frac{\partial}{\partial x_j} \left(\frac{1}{\mu} \frac{\partial A_i}{\partial x_j} \right) + \sigma \frac{\partial A_i}{\partial t} + \sigma \frac{\partial V}{\partial x_i} - J_{e,i} \right] W_i d\Omega = 0 \tag{16}$$

To apply integration by parts, we first expand (16) as (17). Utilizing the divergence theorem, we obtain (18).

$$\int_{\Omega}\left[\frac{\partial}{\partial x_j}\left(\frac{1}{\mu}\frac{\partial A_j}{\partial x_i}W_i\right)-\frac{1}{\mu}\frac{\partial A_j}{\partial x_i}\frac{\partial W_i}{\partial x_j}-\frac{\partial}{\partial x_j}\left(\frac{1}{\mu}\frac{\partial A_i}{\partial x_j}W_i\right)+\frac{1}{\mu}\frac{\partial A_i}{\partial x_j}\frac{\partial W_i}{\partial x_j}+\sigma\frac{\partial A_i}{\partial t}W_i+\sigma\frac{\partial V}{\partial x_i}W_i-J_{e,i}W_i\right]d\Omega=0 \tag{17}$$

$$\int_{\partial\Omega}\frac{1}{\mu}\left(\frac{\partial A_j}{\partial x_i}-\frac{\partial A_i}{\partial x_j}\right)W_i n_j dS+\int_{\Omega}\left[-\frac{1}{\mu}\frac{\partial A_j}{\partial x_i}\frac{\partial W_i}{\partial x_j}+\frac{1}{\mu}\frac{\partial A_i}{\partial x_j}\frac{\partial W_i}{\partial x_j}+\sigma\frac{\partial A_i}{\partial t}W_i+\sigma\frac{\partial V}{\partial x_i}W_i-J_{e,i}W_i\right]d\Omega=0 \tag{18}$$

Assume the domain boundary $\partial\Omega$ is far away from current sources and the MFD has zero tangential component on the domain boundary $\mathbf{n}\times\mathbf{B}=0$. $\mathbf{n}$ is the unit-normal of the domain boundary.

$$\begin{aligned}\mathbf{n}\times(\nabla\times\mathbf{A})&=\frac{\partial A_j}{\partial x_i}n_l\varepsilon_{ijk}\varepsilon_{lkm}\\&=\frac{\partial A_j}{\partial x_i}n_l\left(\delta_{im}\delta_{jl}-\delta_{il}\delta_{jm}\right)\\&=\frac{\partial A_j}{\partial x_i}n_j-\frac{\partial A_j}{\partial x_i}n_i=0\end{aligned} \tag{19}$$

Thus, the surface integral term in (18) equals zero. And the remaining volume integral in (18) is the weak form of the governing equation.

Further discretizing the governing equation, choosing the Lagrangian shape functions as the basis for both the independent variables $\mathbf{A}$, V and the weight function $\mathbf{W}$, we obtain (20).

$$\begin{aligned}&\int_{\Omega}\left[-\frac{1}{\mu}\frac{\partial N^a}{\partial x_i}A_j^a\frac{\partial N^b}{\partial x_j}W_i^b+\frac{1}{\mu}\frac{\partial N^a}{\partial x_j}A_i^a\frac{\partial N^b}{\partial x_j}W_i^b+\sigma N^a\frac{\partial A_i^a}{\partial t}N^bW_i^b+\sigma\frac{\partial N^a}{\partial x_i}V^aN^bW_i^b-J_{e,i}N^bW_i^b\right]d\Omega\\&=\left[\begin{matrix}\left(\int_{\Omega}-\frac{1}{\mu}\frac{\partial N^a}{\partial x_i}\frac{\partial N^b}{\partial x_j}d\Omega\right)A_j^a+\left(\int_{\Omega}\frac{1}{\mu}\frac{\partial N^a}{\partial x_j}\frac{\partial N^b}{\partial x_j}d\Omega\right)A_i^a+\\\left(\int_{\Omega}\sigma N^aN^bd\Omega\right)\frac{\partial A_i^a}{\partial t}+\left(\int_{\Omega}\sigma\frac{\partial N^a}{\partial x_i}N^bd\Omega\right)V^a-\left(\int_{\Omega}J_{e,i}N^bd\Omega\right)\end{matrix}\right]W_i^b=0\end{aligned} \tag{20}$$

Which is a matrix-vector system that can be expressed in the form

$$\left[C_{1A}\right]_{aibj}A_j^a+\left[C_{1Ad}\right]_{ab}\frac{\partial A_i^a}{\partial t}+\left[C_{1V}\right]_{abi}V^a+\left[C_{1e}\right]_{bi}I_e=0 \tag{21}$$

where

$$\begin{aligned}\left[C_{1A}\right]_{aibj}&=\int_{\Omega}-\frac{1}{\mu}\frac{\partial N^a}{\partial x_i}\frac{\partial N^b}{\partial x_j}d\Omega+\int_{\Omega}\frac{1}{\mu}\frac{\partial N^a}{\partial x_k}\frac{\partial N^b}{\partial x_k}\delta_{ij}d\Omega\\\left[C_{1Ad}\right]_{ab}&=\int_{\Omega}\sigma N^aN^bd\Omega\\\left[C_{1V}\right]_{abi}&=\int_{\Omega}\sigma\frac{\partial N^a}{\partial x_i}N^bd\Omega\\\left[C_{1e}\right]_{bi}&=-\frac{N_e}{A_e}\int_{\Omega_e}t_{e,i}N^bd\Omega\end{aligned}$$

The integrals can be evaluated using Gaussian quadrature.

2.1.2 Continuity Equation

The continuity equation (10) again needs to be converted to weak form. Written in indicial notation gives (22).

$$\frac{\partial}{\partial x_i}\left(-\sigma\frac{\partial A_i}{\partial t}-\sigma\frac{\partial V}{\partial x_i}+J_{e,i}\right)=0 \tag{22}$$

Again, following the Galerkin method and integrating over the entire domain for a weight function W (23).

$$\int_{\Omega}\frac{\partial}{\partial x_i}\left(-\sigma\frac{\partial A_i}{\partial t}-\sigma\frac{\partial V}{\partial x_i}+J_{e,i}\right)Wd\Omega=0 \tag{23}$$

Performing integration by parts, we obtain (24).

$$\int_{\partial\Omega}\left(-\sigma\frac{\partial A_i}{\partial t}-\sigma\frac{\partial V}{\partial x_i}+J_{e,i}\right)n_i W dS+\int_{\Omega}\left[\sigma\frac{\partial A_i}{\partial t}\frac{\partial W}{\partial x_i}+\sigma\frac{\partial V}{\partial x_i}\frac{\partial W}{\partial x_i}-J_{e,i}\frac{\partial W}{\partial x_i}\right]d\Omega=0 \tag{24}$$

The surface integral in (24) automatically equals zero, since no current flows out of the domain boundary $\partial\Omega$. The remaining volume integral can be discretized, giving (25).

$$\left[\left(\int_{\Omega}\sigma N^a\frac{\partial N^b}{\partial x_i}d\Omega\right)\frac{\partial A_i^a}{\partial t}+\left(\int_{\Omega}\sigma\frac{\partial N^a}{\partial x_i}\frac{\partial N^b}{\partial x_i}d\Omega\right)V^a-\left(\int_{\Omega}J_{e,i}\frac{\partial N^b}{\partial x_i}d\Omega\right)\right]W^b=0 \tag{25}$$

Which is a matrix-vector system that can be expressed in the form

$$\left[C_{2Ad}\right]_{abi}\frac{\partial A_i^a}{\partial t}+\left[C_{2V}\right]_{ab}V^a+\left[C_{2e}\right]_b I_e=0 \tag{26}$$

where

$$\left[C_{2Ad}\right]_{abi}=\int_{\Omega}\sigma N^a\frac{\partial N^b}{\partial x_i}d\Omega$$

$$\left[C_{2V}\right]_{ab}=\int_{\Omega}\sigma\frac{\partial N^a}{\partial x_k}\frac{\partial N^b}{\partial x_k}d\Omega$$

$$\left[C_{2e}\right]_b=-\frac{N_e}{A_e}\int_{\Omega_e}t_{e,i}\frac{\partial N^b}{\partial x_i}d\Omega$$

2.1.3 EMF in EM

EMF in EM (12) can be discretized as (27).

$$\Phi_e=\frac{N_e}{A_e}\int_{\Omega_e}N^a A_i^a t_{e,i}d\Omega \tag{27}$$

Combined with (13), we once again have a matrix-vector system

$$\left[C_{3Ad}\right]_{ai}\frac{\partial A_i^a}{\partial t}+R_e I_e=V_e \quad \text{where} \quad \left[C_{3Ad}\right]_{ai}=\frac{N_e}{A_e}\int_{\Omega_e}N^a t_{e,i}d\Omega \tag{28}$$

2.2 Solution to finite element method

The coupled vector-matrix equations (21), (26), and (28) can be used to simultaneously solve for A_i^a, V^a, and I_e (a=1~#of nodes, i=1~3).

For a time-harmonic excitation, all physical quantities can be expressed as $Ce^{j\omega t}$, ω is the frequency and C is a complex number representing in-phase and quadrature components. Substituting $\partial/\partial t$ for $j\omega$, we obtain the time-harmonic solution (29).

$$\begin{bmatrix}[C_{1A}]+j\omega[C_{1Ad}] & [C_{1V}] & [C_{1e}]\\ j\omega[C_{2Ad}] & [C_{2V}] & [C_{2e}]\\ j\omega[C_{3Ad}] & [0] & R_e\end{bmatrix}\begin{bmatrix}[A]\\ [V]\\ I_e\end{bmatrix}=\begin{bmatrix}[0]\\ [0]\\ V_e\end{bmatrix} \tag{29}$$

For a time-domain simulation, at each time instant, the time derivative of $[A]$ can be solved from (30). The problem can thus be solved as an initial value problem using for example Runge-Kutta methods.

$$\begin{bmatrix}[C_{1Ad}] & [C_{1V}] & [C_{1e}]\\ [C_{2Ad}] & [C_{2V}] & [C_{2e}]\\ [C_{3Ad}] & [0] & R_e\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial t}[A]\\ [V]\\ I_e\end{bmatrix}=\begin{bmatrix}-[C_{1A}][A]\\ [0]\\ V_e\end{bmatrix} \tag{30}$$

3. Modified Method for Eddy Current Problem

In the modified method, the magnetization $\mathbf{M}$ in the magnetic conductor is equivalently modeled [1] as volume current density $\mathbf{J_M}$ and surface current density $\mathbf{K_M}$, $\mathbf{n}$ is the unit-normal of the magnetic conductor surface.

$$\begin{aligned}\mathbf{J_M}&=\nabla\times\mathbf{M}\\ \mathbf{K_M}&=\mathbf{M}\times\mathbf{n}\end{aligned} \tag{31}$$

Under this framework, equation (2) becomes (32).

$$\begin{aligned}\nabla\times\mathbf{B} &= \mu_0\mathbf{J}\\ &= \nabla\times\nabla\times\mathbf{A} = \nabla\left(\nabla\cdot\mathbf{A}\right) - \nabla^2\mathbf{A}\end{aligned} \tag{32}$$

And by using the Coulomb gauge $\nabla\cdot\mathbf{A}=0$, we have $\nabla^2\mathbf{A}=-\mu_0\mathbf{J}$. The solution to the PDE is (33). Taking the curl of it gives the Biot-Savart law (34).

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\mathbf{J}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega \tag{33}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\mathbf{J}(\mathbf{r}')\times\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d\Omega \tag{34}$$

3.1 Modified method formulation

3.1.1 MVP equivalence

The MVP due to EM current at a point $\mathbf{r}$ in space is given by (35), which integrates over the EM volume. For a circular EM, the integral can be evaluated efficiently using Forbes' formula [2].

$$\mathbf{A}_e(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{N_e}{A_e}I_e\int_{\Omega_e} \frac{\mathbf{t}_e(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega \tag{35}$$

Eddy current and magnetization in magnetic conductors also contribute to MVP as given by (36), which integrates over the conductor volume and/or surface.

$$\mathbf{A}_c(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\Omega_c} \frac{\mathbf{J}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega + \frac{\mu_0}{4\pi}\int_{\Omega_c} \frac{\mathbf{J}_\mathbf{M}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega + \frac{\mu_0}{4\pi}\int_{\partial\Omega_c} \frac{\mathbf{K}_\mathbf{M}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}dS \tag{36}$$

In particular, $\mathbf{J}$ is the eddy current density due to the electric field (37). And $\mathbf{J_M}$ is a scalar multiple of $\mathbf{J}$ (38).

$$\mathbf{J} = \sigma\mathbf{E} = -\sigma\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla V\right) \tag{37}$$

$$\mathbf{J_M} = \nabla\times\mathbf{M} = \nabla\times\left(\mu_r-1\right)\mathbf{H} = \left(\mu_r-1\right)\mathbf{J} \tag{38}$$

Additionally, $\mathbf{K_M}$ at the magnetic conductor surface is related to $\mathbf{B}$ by (39), $\mathbf{n}$ is the unit-normal of the magnetic conductor surface.

$$\mathbf{K_M} = \nu\mathbf{B}\times\mathbf{n} \quad \text{where} \quad \nu = \frac{\mu_r-1}{\mu_0\mu_r} \tag{39}$$

Thus, (36) can be alternatively expressed as (40) or in indicial notation (41).

$$\mathbf{A}_c(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\Omega_c} \frac{-\sigma\mu_r}{\left|\mathbf{r}-\mathbf{r}'\right|}\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla V\right)d\Omega + \frac{\mu_0}{4\pi}\int_{\partial\Omega_c} \frac{\nu}{\left|\mathbf{r}-\mathbf{r}'\right|}\left(\nabla\times\mathbf{A}\right)\times\mathbf{n}dS \tag{40}$$

$$A_{c,i}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\Omega_c} \frac{-\sigma\mu_r}{\left|\mathbf{r}-\mathbf{r}'\right|}\left(\frac{\partial A_i}{\partial t} + \frac{\partial V}{\partial x_i}\right)d\Omega + \frac{\mu_0}{4\pi}\int_{\partial\Omega_c} \frac{\nu}{\left|\mathbf{r}-\mathbf{r}'\right|}\left(\frac{\partial A_i}{\partial x_j}n_j - \frac{\partial A_j}{\partial x_i}n_j\right)dS \tag{41}$$

Discretizing (41) by using Lagrangian shape function as the interpolation function gives (42).

$$\begin{aligned}A_{c,i}(\mathbf{r}) = &\left(\frac{\mu_0}{4\pi}\int_{\Omega_c} \frac{-\sigma\mu_r}{\left|\mathbf{r}-\mathbf{r}'\right|}N^a d\Omega\right)\frac{\partial A_i^a}{\partial t} + \left(\frac{\mu_0}{4\pi}\int_{\Omega_c} \frac{-\sigma\mu_r}{\left|\mathbf{r}-\mathbf{r}'\right|}\frac{\partial N^a}{\partial x_i}d\Omega\right)V^a\\ &+\left(\frac{\mu_0}{4\pi}\int_{\partial\Omega_c} \frac{\nu}{\left|\mathbf{r}-\mathbf{r}'\right|}\frac{\partial N^a}{\partial x_j}n_j dS\right)A_i^a + \left(\frac{\mu_0}{4\pi}\int_{\partial\Omega_c} \frac{-\nu}{\left|\mathbf{r}-\mathbf{r}'\right|}\frac{\partial N^a}{\partial x_i}n_j dS\right)A_j^a\end{aligned} \tag{42}$$

The MVP A_i^b at node $\mathbf{r}^b$ (43) equals the sum of (35) and (42).

$$A_i^b \equiv A_i(\mathbf{r}^b) = A_{e,i}(\mathbf{r}^b) + A_{c,i}(\mathbf{r}^b) \tag{43}$$

Which is a matrix-vector system that can be expressed in the form

$$\left[C_{1A}\right]_{aibj}A_j^a + \left[C_{1Ad}\right]_{ab}\frac{\partial A_i^a}{\partial t} + \left[C_{1V}\right]_{abi}V^a + \left[C_{1e}\right]_{bi}I_e = 0 \tag{44}$$

where

$$[C_{1A}]_{aibj} = \frac{\mu_0}{4\pi}\int_{\partial\Omega_c}\frac{\nu}{\left|\mathbf{r}^b-\mathbf{r}'\right|}\left(\frac{\partial N^a}{\partial x_k}n_k\delta_{ij}-\frac{\partial N^a}{\partial x_i}n_j\right)dS-\delta^{ab}\delta_{ij}$$

$$[C_{1Ad}]_{ab} = \frac{\mu_0}{4\pi}\int_{\Omega_c}\frac{-\sigma\mu_r}{\left|\mathbf{r}^b-\mathbf{r}'\right|}N^a d\Omega$$

$$[C_{1V}]_{abi} = \frac{\mu_0}{4\pi}\int_{\Omega_c}\frac{-\sigma\mu_r}{\left|\mathbf{r}^b-\mathbf{r}'\right|}\frac{\partial N^a}{\partial x_i}d\Omega$$

$$[C_{1e}]_{bi} = \frac{\mu_0}{4\pi}\frac{N_e}{A_e}I_e\int_{\Omega_e}\frac{t_{e,i}(\mathbf{r}')}{\left|\mathbf{r}^b-\mathbf{r}'\right|}d\Omega$$

3.1.2 Continuity equation

The continuity equation in the magnetic conductor is (45).

$$\nabla\cdot\left(-\sigma\frac{\partial\mathbf{A}}{\partial t}-\sigma\nabla V\right)=0 \tag{45}$$

Following the procedures in section *2.1.2* allows us to write the matrix-vector system

$$[C_{2Ad}]_{abi}\frac{\partial A_i^a}{\partial t}+[C_{2V}]_{ab}V^a=0 \tag{46}$$

where

$$[C_{2Ad}]_{abi}=\int_\Omega \sigma N^a\frac{\partial N^b}{\partial x_i}d\Omega$$

$$[C_{2V}]_{ab}=\int_\Omega \sigma\frac{\partial N^a}{\partial x_k}\frac{\partial N^b}{\partial x_k}d\Omega$$

3.1.3 EMF in EM

EMF in EM can be represented as (47).

$$\Phi_e=\int_{\Omega_e}\frac{N_e}{A_e}\mathbf{A}\cdot\mathbf{t}_e d\Omega=\int_{\Omega_e}\frac{N_e}{A_e}\left(A_{e,i}+A_{c,i}\right)t_{e,i}d\Omega \tag{47}$$

Substituting in (35) and (42), we once again have a matrix-vector system

$$[C_{3A}]_{ai}A_i^a+[C_{3Ad}]_{ai}\frac{\partial A_i^a}{\partial t}+[C_{3V}]_a V^a+[C_{3e}]I_e=V_e \tag{48}$$

3.1.4 MFD at magnetic sensor

Since only the magnetic conductor is meshed, the Biot-Savart law (34) is required to calculate the MFD in space. The MFD due to EM current is given by (49), and can be evaluated efficiently using Forbes' formula [2] for a circular EM.

$$\mathbf{B}_e(\mathbf{r})=\frac{\mu_0}{4\pi}\frac{N_e}{A_e}I_e\int_{\Omega_e}\frac{\mathbf{t}_e(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d\Omega \tag{49}$$

Similar to (40), the MFD due to eddy current and magnetization in magnetic conductors is given by (50).

$$\mathbf{B}_c(\mathbf{r})=\frac{\mu_0}{4\pi}\int_{\Omega_c}\sigma\mu_r\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\times\left(\frac{\partial\mathbf{A}}{\partial t}+\nabla V\right)d\Omega+\frac{\mu_0}{4\pi}\int_{\partial\Omega_c}-\nu\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\times\left((\nabla\times\mathbf{A})\times\mathbf{n}\right)dS \tag{50}$$

The sum of $\mathbf{B}_e$(r) and $\mathbf{B}_c(\mathbf{r})$ gives the MFD at a point $\mathbf{r}$ in space.

3.2 Solution to the modified method

The coupled vector-matrix equations (44), (46), and (48) can be used to simultaneously solve for A_i^a, V^a, and I_e (a=1~#of nodes, i=1~3).

For a time-harmonic excitation with ω as the excitation frequency, the solution is (51).

$$\begin{bmatrix} [C_{1A}]+j\omega[C_{1Ad}] & [C_{1V}] & [C_{1e}] \\ j\omega[C_{2Ad}] & [C_{2V}] & [0] \\ [C_{3A}]+j\omega[C_{3Ad}] & [C_{3V}] & [C_{3e}] \end{bmatrix} \begin{bmatrix} [A] \\ [V] \\ I_e \end{bmatrix} = \begin{bmatrix} [0] \\ [0] \\ V_e \end{bmatrix} \tag{51}$$

For a time-domain simulation, at each time instant, the time derivative of $[A]$ can be solved from (52). The problem can thus be solved as an initial value problem.

$$\begin{bmatrix} [C_{1Ad}] & [C_{1V}] & [C_{1e}] \\ [C_{2Ad}] & [C_{2V}] & [C_{2e}] \\ [C_{3Ad}] & [C_{3V}] & [C_{3e}] \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t}[A] \\ [V] \\ I_e \end{bmatrix} = \begin{bmatrix} -[C_{1A}][A] \\ [0] \\ V_e-[C_{3A}][A] \end{bmatrix} \tag{52}$$

4. Results and Discussion

The configuration shown in Fig. 1 is used to test the FEM and the modified method. The metal plate has a side length of 60 mm and a thickness of 2 mm. A through-hole with a side length of 20 mm is centered at the metal plate. The FEM and the modified method uses the same mesh for the metal plate. The FEM has a total of 91586 linear tetrahedron elements and 16266 nodes. The modified method has 3089 linear tetrahedron elements and 1117 nodes.

For the first example, we choose the metal plate to be an Aluminum plate, which has an electrical conductivity of 37.74 MS/m and a relative magnetic permeability of 1. The excitation frequency is 1kHz. Fig. 2(a) and 2(b) show the eddy current density at various positions across the Aluminum plate. The results for the FEM and the modified method highly agree. In a general sense, the EM induces eddy current tangential to its winding, and the eddy current is obstructed and flows around the through-hole. Fig. 2(c) and 2(d) show the MFD at positions illustrated as black cubes in Fig. 1(b). The two method yields similar MFD components, but the modified method has smoother results.

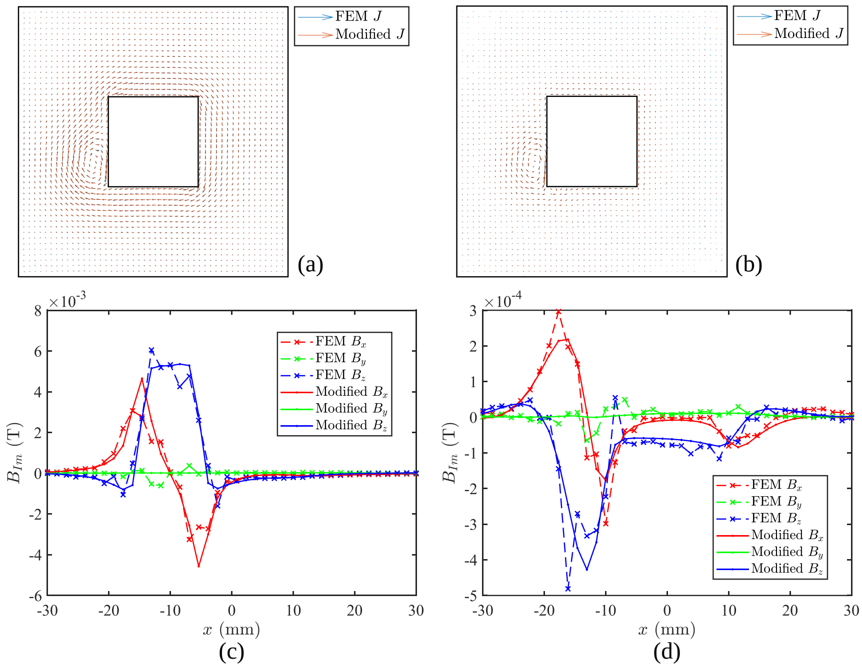


Fig. 2. Numerical result for Aluminum plate with σ=37.74 MS/m and μ_r=1 at 1 kHz. (a) Real part eddy current. (b) Imaginary part eddy current. (c) Real part MFD. (d) Imaginary part MFD.

For the second example, we choose the metal plate to be ferromagnetic with an electrical conductivity of 5 MS/m and relative magnetic permeability of 10. The excitation frequency is 100 Hz. Fig. 3 shows the eddy current density and MFD for the FEM and the modified method. The results between the FEM and modified method once again agrees, with the modified method having smoother MFD components.

From the two examples above, the modified method yields smoother and more accurate MFD results. This is because the computation of MFD in the modified method is based on analytical expressions (Biot-Savart law), while the FEM requires the interpolation of nodal values and the accuracy would be strongly dependent on mesh quality. Another advantage of the modified method is the availability of separating MFD due to EM current, eddy current, and material magnetization since each source contributes to a term in (49) and (50). Fig. 4 shown an example, the MFD for each contributing source can be separated.

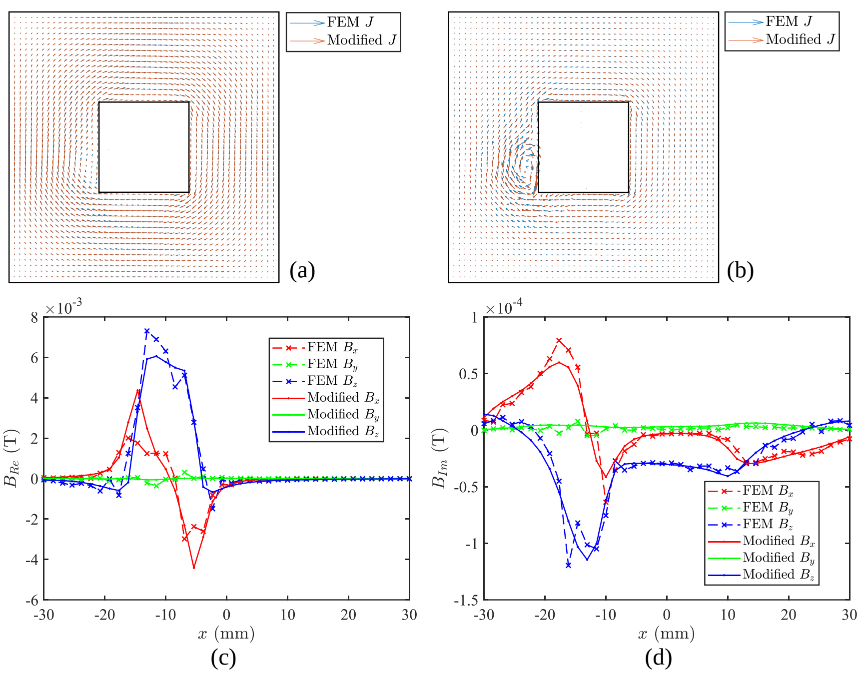


Fig. 3. Numerical result for ferromagnetic plate with σ=5 MS/m and μ_r=10 at 100 Hz. (a) Real part eddy current. (b) Imaginary part eddy current. (c) Real part MFD. (d) Imaginary part MFD.

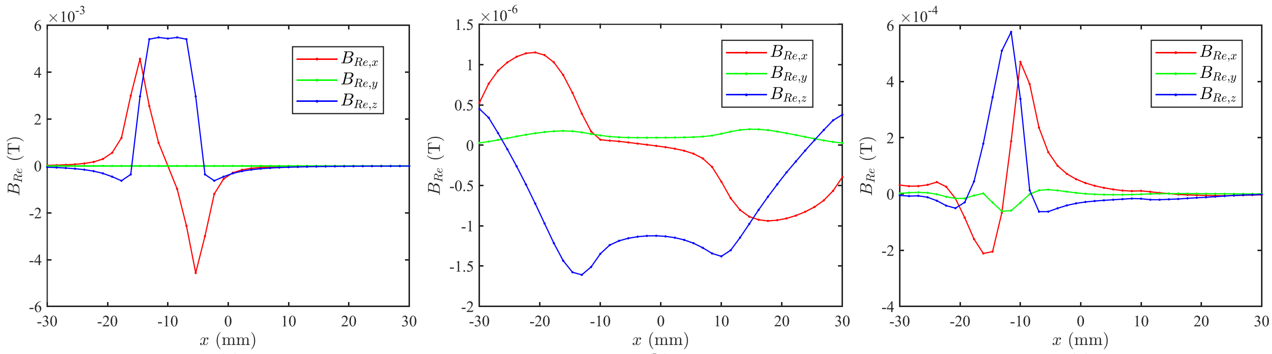


Fig. 4. Real part of MFD for ferromagnetic plate with σ=5 MS/m and μ_r=10 at 100 Hz. (a) Due to EM current. (b) Due to eddy current. (c) Due to magnetization.

We summarize the time and space complexity of the two methods in Table I. N_e is the number of elements and N_n is the number of nodes. Note that in general, the modified method has lesser elements and nodes since it only requires meshing the magnetic conductor. The total degrees of freedom to be solved include three MVP components and one electric potential at each node. During the construction of governing matrices, the FEM loops over all elements and enforces a local approximation of the PDE. Thus, the time and storage are proportional to N_e. For the modified method, one of the governing matrices is a square dense matrix with a size of $3N_n$. During its construction, the calculation of MVP at a point takes time proportional to N_e, and a total of N_n evaluations is required, thus the N_eN_n time complexity. The time complexity for solving the system of linear equations is the cubic of the solution length when using Gaussian-elimination methods, and approximately squared if the linear system is sparse and iterative methods can be applied. In the above examples, the FEM is faster than the modified method in the construction of the governing matrices, while similar in time for solving the linear system.

Table I. Time and space complexity for FEM and modified method

	FEM	Modified method
Degree of freedom to be solved	4 N_n	4 N_n
Space for storing governing matrices	$\Theta(N_e)^*$	$\Theta(N_n^2)$
Time for computing governing matrices	$\Theta(N_e)$	$\Theta(N_eN_n)$
Time for solving linear system	$\Omega(N_n^2)^*$	$\Theta(N_n^3)$

*Assuming using sparse matrices and iterative solvers.

The code for this project is available on [GitHub]().

References

[1] D. K. Cheng, "Magnetization and Equivalent Current Densities," in *Field and Wave Electromagnetics*, 2 ed., Addison-Wesley, 1989, pp. 243-247.

[2] L. K. Forbes, S. Crozier and D. M. Doddrell, "Rapid computation of static fields produced by thick circular solenoids," *IEEE Transactions on Magnetics,* vol. 33, no. 5, pp. 4405-4410, 1997.

[3] C.-K. Wang, "ME7112 Finite Element Method lecture notes," 2022.