Topology by James Munkres – Chapter 3 Supplementary Exercises

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A partial order $\preceq$ on a set X is a relation that satisfies:

(1) $\alpha \preceq \alpha$ for all $\alpha \in X$.

(2) If $\alpha \preceq \beta$ and $\beta \preceq \alpha$, then $\alpha = \beta$.

(3) If $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$.

A *directed set* J is a set with a partial order $\preceq$ such that for any $\alpha, \beta \in J$, there exists $\gamma \in J$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

1. Show that the following are directed sets:

   (a) Any simply ordered set under $\leq$.

   SOLUTION. The fact that $\leq$ is a partial order is trivial. Given $x, y \in X$, either $x \leq y$ or $y \leq x$; without loss of generality assume $x \leq y$. Then $y \in X$ satisfies $x \leq y$ and $y \leq y$. □

   (b) The set of all subsets of S, ordered by inclusion.

   SOLUTION. Every subset is contained in itself. If $A, B \subseteq S$ are such that $A \subseteq B$ and $B \subseteq A$, then $A = B$. If $A \subseteq B \subseteq C$, then $A \subseteq C$. Now for any $A, B \subseteq S$ clearly S is the desired element. □

   (c) A collection $\mathcal{A}$ of subsets of S that is closed under finite intersection, under reverse inclusion.

   SOLUTION. This is a partial order by symmetrical arguments as (b). For any $A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$ is clearly contained in both. □

   (d) The collection of all closed subsets of a space X, under inclusion.

   SOLUTION. This is identical to (c); closed subsets are closed under finite union so the desired set is $A \cup B$. □

2. $K \subseteq J$ is *cofinal* in J if for every $\alpha \in J$, there exists $\beta \in K$ with $\alpha \preceq \beta$. Show that if J is directed and K is cofinal in J, then K is directed.

   SOLUTION. Given $k_1, k_2 \in K$, we also have $k_1, k_2 \in J$ so there exists $\alpha \in J$ with $k_1 \preceq \alpha, k_2 \preceq \alpha$. Then since K is cofinal there exists $\beta \in K$ with $\alpha \preceq \beta$, so by transitivity $k_1 \preceq \beta$ and $k_2 \preceq \beta$. □

3. Let X be a topological space. A *net* in X is a function from a directed set J to X. The net $(x_\alpha)_{\alpha \in J}$ converges to $x \in X$ if for every neighbourhood U of x, there exists $\alpha \in J$ such that $\alpha \preceq \beta$ implies $x_\beta \in U$.

   Show that these definitions reduce to the usual ones when $J = \mathbb{N}$.

   SOLUTION. The net $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ if and only if for every neighbourhood U of x, there exists $\alpha \in \mathbb{N}$ such that $\alpha \preceq \beta$ implies $x_\beta \in U$. $\alpha \preceq \beta$ may be written as $\alpha \leq \beta$. This may be rewritten as the usual criterion that there exists $N \in \mathbb{N}$ such that $N \leq n$ implies $x_n \in U$. □

4. Suppose $(x_\alpha)_{\alpha\in J} \to x \in X$ and $(y_\alpha)_{\alpha\in J} \to y \in Y$. Show that $(x_\alpha \times y_\alpha)_{\alpha\in J} \to x \times y \in X \times Y$.

   SOLUTION. For any neighbourhood $U\times V$ of $x\times y$, U is a neighbourhood of x and V is a neighbourhood of y. By convergence, there exist α_x, α_y such that $\alpha_x \preceq \beta$ implies $x_\beta \in U$ and $\alpha_y \preceq \beta$ implies $y_\beta \in V$. Let $\alpha \in J$ be such that $\alpha_x \preceq \alpha$ and $\alpha_y \preceq \alpha$. Then $\alpha \preceq \beta$ implies $\alpha_x \preceq \beta$ and $\alpha_y \preceq \beta$ by transitivity, so $x_\beta \times y_\beta \in U \times V$. $\square$

5. Show that if X is Hausdorff, then nets converge uniquely.

   SOLUTION. Suppose X is Hausdorff and $(x_\alpha)_{\alpha\in J}$ converges to distinct points x and y. Let U, V be disjoint neighbourhoods of x, y. By convergence, there exist $\alpha_x, \alpha_y \in J$ such that $\alpha_x \preceq \beta$ implies $x_\beta \in U$, and $\alpha_y \preceq \beta$ implies $x_\beta \in V$. If $\alpha \in J$ is such that $\alpha_x \preceq \alpha$ and $\alpha_y \preceq \alpha$, then for $\alpha \preceq \beta$ we have $x_\beta \in U \cap V$, a contradiction. $\square$

6. *Theorem.* Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if there is a net of points in A converging to x.

   SOLUTION. ($\implies$) Consider the collection $\mathcal{U}$ of all neighbourhoods of x, ordered by reverse inclusion. This is directed by 1(c). If $x \in \overline{A}$, then every neighbourhood U of x intersects A at some point x_U. We claim that $(x_U)_{U\in\mathcal{U}}$ is a net of points in A converging to x. For every neighbourhood U of x, $U \in \mathcal{U}$ is such that if $U \preceq V$, then $U \supseteq V$, so $x_V \in V \subseteq U$, as desired.

   ($\impliedby$) Suppose $(x_\alpha)_{\alpha\in J}$ is a net of points in A converging to x. By definition, for every neighbourhood U of x there exists $\alpha \in J$ such that $\alpha \preceq \beta \implies x_\beta \in U$. Since $\alpha \preceq \alpha$, we have $x_\alpha \in U \cap A$, so U intersects A, showing that $x \in \overline{A}$. $\square$

7. *Theorem.* $f : X \to Y$ is continuous if and only if for every net $(x_\alpha)_{\alpha\in J}$ in X converging to x, the net $(f(x_\alpha))_{\alpha\in J}$ converges to $f(x)$.

   SOLUTION. ($\implies$) Suppose $(x_\alpha)_{\alpha\in J}$ converges to x. Let V be a neighbourhood of $f(x)$; $f^{-1}(V)$ is a neighbourhood of x. Let $\alpha \in J$ be such that $\alpha \preceq \beta$ implies $x_\beta \in f^{-1}(V)$. Then $f(x_\beta) \in V$, showing that $(f(x_\alpha))_{\alpha\in J} \to f(x)$.

   ($\impliedby$) Given $A \subseteq X$, suppose $x \in \overline{A}$. By 6, there exists a net of points $(x_\alpha)_{\alpha\in J}$ in A converging to x. By assumption, $(f(x_\alpha))_{\alpha\in J}$ is a net of points in $f(A)$ converging to $f(x)$. By 6, $f(x) \in \overline{f(A)}$, showing that $f(\overline{A}) \subseteq \overline{f(A)}$, and thus f is continuous. $\square$

8. Let $f : J \to X$ be a net in X, and $f(\alpha) = x_\alpha$. If K is a directed set and $g : K \to J$ is a function satisfying

   (1) $i \preceq j \implies g(i) \preceq g(j)$,

   (2) $g(K)$ is cofinal in J,

   then $f \circ g : K \to X$ is a subnet of (x_α). Show that if (x_α) converges to x, then so does any subnet.

   SOLUTION. For every neighbourhood U of x, there exists $\alpha \in J$ such that $\alpha \preceq \beta \implies f(\beta) \in U$. Since $g(K)$ is cofinal in J, there exists $g(\gamma) \in g(K)$ such that $\alpha \preceq g(\gamma)$. If $\gamma \preceq \beta$, then $g(\gamma) \preceq g(\beta)$, and by transitivity $\alpha \preceq g(\beta)$. Then $f(g(\beta)) \in U$, showing that $f \circ g$ converges to x. $\square$

9. Let $(x_\alpha)_{\alpha\in J}$ be a net in X. x is an *accumulation point* of (x_α) if for every neighbourhood U of x, the set of all α for which $x_\alpha \in U$ is cofinal in J.

   Lemma. The net $(x_\alpha)_{\alpha\in J}$ has x as accumulation point if and only if some subnet of (x_α) converges to x.

   SOLUTION. ($\implies$) Let K be the set of all pairs (α, U) where $\alpha \in J$ and U is a neighbourhood of x containing x_α. Define a partial order $\preceq$ on K by $(\alpha, U) \preceq (\beta, V)$ if $\alpha \preceq \beta$ and $U \supseteq V$. Given $(\alpha, U), (\beta, V) \in K$, let $\gamma \in J$ be such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Applying the definition of x as an accumulation point to the neighbourhood $U \cap V$ of x,, there exists $\gamma \preceq \delta$ such that $x_\delta \in U \cap V$. This means $(\delta, U \cap V) \in K$. By transitivity, $\alpha \preceq \delta$ and $\beta \preceq \delta$, showing that $(\alpha, U) \preceq (\delta, U \cap V)$ and $(\beta, V) \preceq (\delta, U \cap V)$, and thus K is directed.

Define $g : K \to J$ by $(\alpha, U) \mapsto \alpha$. We verify that if $(\alpha, U) \preceq (\beta, V)$, then $g(\alpha, U) = \alpha \preceq \beta = g(\beta, V)$, and $g(K)$ is cofinal in J because for any $\alpha \in J$, $\alpha \preceq g(\alpha, X)$.

For any neighbourhood U of x and any $\alpha \in J$, the definition of x as an accumulation point implies that there exists $\beta \in J$ such that $\alpha \preceq \beta$ and $x_\beta \in U$. Now $(\beta, U) \in K$, and if $(\gamma, V) \in K$ is such that $(\beta, U) \preceq (\gamma, V)$ then $x_{g(\gamma,V)} = x_\gamma \in V \subseteq U$. Hence the subnet $(x_{g(\alpha,U)})_{(\alpha,U)\in K}$ converges to x.

($\Longleftarrow$) Suppose some subnet $(x_{g(i)})_{i\in K}$ of $(x_\alpha)_{\alpha\in J}$ converges to x. For every neighbourhood U of x, there exists $i \in K$ such that $i \preceq j$ implies $x_{g(j)} \in U$. Let A be the set of indices $\alpha \in J$ for which $x_\alpha \in U$. We show that A is cofinal in J. Given $\alpha \in J$, if $\alpha \preceq g(i)$ then $g(i) \in A$ is the desired successor. Otherwise since $g(K)$ is cofinal in J, there exists $g(j) \in g(K)$ such that $\alpha \preceq g(j)$. If $j \preceq i$, then $g(j) \preceq g(i)$ and by transitivity $\alpha \preceq g(i)$, so we are in the first case. Otherwise let $k \in K$ be such that $i \preceq k$ and $j \preceq k$; we have $\alpha \preceq g(j) \preceq g(k)$ and $i \preceq k$ implies $g(k) \in A$, so $g(k)$ is the desired successor. $\square$

10. *Theorem.* X is compact if and only if every net has a convergent subnet.

    SOLUTION. ($\Longrightarrow$) Suppose X is compact and suppose $(x_\alpha)_{\alpha\in J}$ is a net of points in X with no convergent subnet. By exercise 9, (x_α) has no accumulation point, so for every $x \in X$, there exists a neighbourhood U_x of x and $\alpha_x \in J$ such that $\alpha_x \preceq \beta$ implies $x_\beta \notin U$.

    Clearly $\{U_x\}_{x\in X}$ is an open covering of X, so by compactness it admits a finite subcover $U_{x_1}, \cdots, U_{x_n}$. If $\beta \in J$ is such that $\alpha_{x_i} \preceq \beta$ for each i, then $x_\beta \notin \bigcup_{i=1}^{n} U_{x_i}$, a contradiction.

    ($\Longleftarrow$) Suppose every net has a convergent subnet. Let $\mathcal{A}$ be any collection of closed sets having the finite intersection property. Let $\mathcal{B}$ be the set of finite intersections of elements in $\mathcal{A}$, ordered under reverse inclusion. $\mathcal{B}$ is directed by 1(c). For each $B \in \mathcal{B}$, B is nonempty; let $x_B \in B$. $(x_B)_{B\in\mathcal{B}}$ admits a subnet $(x_{g(i)})_{i\in K}$ converging to some point x. We claim that $x \in \bigcap_{A\in\mathcal{A}} A$, which shows that $\mathcal{A}$ has the finite intersection property. Indeed, for any $A \in \mathcal{A}$, we naturally have $A \in \mathcal{B}$, so that $x_A \in A$. Now there exists $g(i) \in g(K)$ such that $A \supseteq g(i)$, and thus for any $i \preceq j$, we have $g(i) \supseteq g(j)$. By transitivity, $A \supseteq g(j)$, so $x_{g(j)} \in g(j) \subseteq A$. Hence the tail of $(x_{g(i)})_{i\in K}$ is a sequence of points in A converging to x, so $x \in \overline{A} = A$. Since A was arbitrary, $x \in \bigcap_{A\in\mathcal{A}} A$, so X is compact. $\square$

11. *Corollary.* Let G be a topological group; let $A, B \subseteq G$. If A is closed and B is compact, then $A \cdot B$ is closed in G.

    SOLUTION. We will show $G - A \cdot B$ is open. Suppose $x \in G - A \cdot B$. Recall that $\varphi : G \to G$ defined by $x \times y \mapsto xy^{-1}$ is continuous. We observe that $\{x\} \times B \subseteq \varphi^{-1}(G - A)$; indeed if $x \times b \in \{x\} \times B$ suppose $xb^{-1} \in A$. Then $x \in Ab \subseteq A \cdot B$, a contradiction.

    Since B is compact, the tube lemma states that there exists a neighbourhood W of x such that $W \times B \subseteq \varphi^{-1}(G - A)$; that is, for any $u \times v \in W \times B$, $uv^{-1} \notin A$, so $u \notin Av \subseteq AB$. Therefore $x \in W \subseteq G - AB$, showing that AB is closed. $\square$