Topology by James Munkres – Chapter 2 Supplementary Exercises

William Gao

Summer 2024

A *topological group* G is a group that is also a T_1 topological space such that $f : G \times G \to G$ defined by $x \times y \mapsto x \cdot y$ and $g : G \to G$ defined by $x \mapsto x^{-1}$ are continuous.

1. Let H be a group and a T_1 topological space. Show that H is a topological group if and only if $h : H \times H \to H$ defined by $x \times y \mapsto x \cdot y^{-1}$ is continuous.

   SOLUTION. Suppose H is a topological group. Since f, g are continuous, $h : H \times H \to H$ defined by $x \times y \to x \cdot y^{-1} = f(x \times g(y))$ is continuous by the composition and product of continuous functions. Conversely, if h is continuous, then the restriction $h' : \{e\} \times H \to H$ defined by $e \times y \to e \cdot y^{-1} = y^{-1}$ is continuous; this is precisely g. Moreover $f(x \times y) = h(x \times g(y))$ is continuous. □

2. Show that the following are topological groups:

   (a) $(\mathbb{Z}, +)$.

   SOLUTION. Since $\{n\}$ is open in $\mathbb{Z}$ for all $n \in \mathbb{Z}^+$, $\mathbb{Z}$ has the discrete topology and is clearly T_1. Moreover, every function in $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ is continuous. □

   (b) $(\mathbb{R}, +)$.

   SOLUTION. $\mathbb{R}$ is Hausdorff and a group. $x^{-1} = -x$, so for any $x \times y \in \mathbb{R} \times \mathbb{R}$ and any neighbourhood V of $h(x \times y) = x - y$, let $B(x - y, \varepsilon) \subseteq V$. Then $B_\rho(x \times y, \frac{\varepsilon}{2})$ is a neighbourhood of $x \times y$ such that if $a \times b \in B_\rho(x \times y, \frac{\varepsilon}{2})$ then

   $$\max\{|x - a|, |y - b|\} < \frac{\varepsilon}{2},$$

   and thus

   $$|(x - y) - (a - b)| \leq |x - a| + |y - b| < \varepsilon,$$

   meaning $h(a \times b) = a - b \in B(x - y, \varepsilon) \subseteq V$. Thus h is continuous. □

   (c) $(\mathbb{R}^+, \cdot)$.

   SOLUTION. $x^{-1} = \frac{1}{x}$. h is simply the division operation, which we know is continuous on $\mathbb{R}^+$. □

   (d) $(S^1, \cdot)$ where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

   SOLUTION. Since $\mathbb{C}$ is Hausdorff and a group, so is S^1. Consider $h : S^1 \times S^1 \to S^1$ given by $x \times y \mapsto xy^{-1}$. Let $e^{i\alpha} \times e^{i\beta} \in S^1 \times S^1$ and let V be a neighbourhood of $h(e^{i\alpha} \times e^{i\beta}) = e^{i(\alpha - \beta)}$ in S^1. There exists $\varepsilon > 0$ such that $e^{i(\alpha-\beta)} \in \{e^{i\delta} : \delta \in (\alpha - \beta - \varepsilon, \alpha - \beta + \varepsilon)\} \subseteq V$. Then $U = \{e^{i\delta} : \delta \in (\alpha - \frac{\varepsilon}{2}, \alpha + \frac{\varepsilon}{2})\} \times \{e^{i\delta} : \delta \in (\beta - \frac{\varepsilon}{2}, \beta + \frac{\varepsilon}{2})\}$ is a neighbourhood of $x \times y$ in $S^1 \times S^1$ and if $e^{ix} \times e^{iy} \in U$, then

   $$|(x - y) - (\alpha - \beta)| \leq |x - \alpha| + |y - \beta| < \varepsilon,$$

hence $h(e^{ix} \times e^{iy}) = e^{i(x-y)} \in \{e^{i\delta} : \delta \in (\alpha - \beta - \varepsilon, \alpha - \beta + \varepsilon)\} \subseteq V$. Thus h is continuous. $\square$

(e) $\mathrm{GL}(n)$ as a subset of $\mathbb{R}^{n^2}$ under matrix multiplication.

SOLUTION. Since $\mathrm{GL}(n) \subseteq \mathbb{R}^{n^2}$, it is Hausdorff. f is continuous since each component of AB is a polynomial in the entries of A and B; g is continuous by Cramer's rule. $\square$

3. Let H be a subspace of G. Show that if H is a subgroup, then H and $\overline{H}$ are topological groups.

SOLUTION. Since H is a subspace, it is T_1. Moreover as H is a subgroup, it is closed under $\cdot$ and inverses, so $f\big|_H : H \times H \to H$ and $g\big|_H : H \to H$ are continuous.

$\overline{H}$ is similarly T_1. Since f, g are continuous, we have $f(\overline{H} \times \overline{H}) = f(\overline{H \times H}) \subseteq \overline{f(H \times H)} = \overline{H}$ and $g(\overline{H}) \subseteq \overline{g(H)} = \overline{H}$. Thus $f\big|_{\overline{H}} : \overline{H} \times \overline{H} \to \overline{H}$ and $g\big|_{\overline{H}} : \overline{H} \to \overline{H}$ are continuous. $\square$

4. Let $\alpha \in G$. Show that $f_\alpha, g_\alpha : G \to G$ defined by

$$f_\alpha(x) = \alpha \cdot x \text{ and } g_\alpha(x) = x \cdot \alpha$$

are homeomorphisms of G. Conclude that G is a homogeneous space.

SOLUTION. f_α, g_α are the product of a constant function and an identity function composed with $x \times y \mapsto x \cdot y$. Thus they are continuous. Similarly their inverses $(f_\alpha)^{-1} = f_{\alpha^{-1}}, (g_\alpha)^{-1} = g_{\alpha^{-1}}$ are continuous, so they are homeomorphisms. Given $x, y \in G$, we will construct a homeomorphism $G \to G$ that maps x to y by $f_{y \cdot x^{-1}}$. Indeed,

$$f_{y \cdot x^{-1}}(x) = (y \cdot x^{-1}) \cdot x = y \cdot (x^{-1} \cdot x) = y. \square$$

5. Let H be a subgroup of G. If $x \in G$, define the left coset of H in G by $xH = \{x \cdot h : h \in H\}$. Let G/H denote the collection of left cosets of H in G; it is a partition of G. Give G/H the quotient topology.

(a) Show that if $\alpha \in G$, f_α induces a homeomorphism of G/H carrying xH to $(\alpha \cdot x)H$. Conclude that G/H is a homogeneous space.

SOLUTION. We know $g = p \circ f_\alpha : G \to G/H$ defined by $x \mapsto (\alpha \cdot x)H$ is a quotient map since p, f_α are quotient maps. Moreover,

$$g^{-1}(\{xH\}) = \{y \in G : (\alpha \cdot y)H = xH\}$$

and $(\alpha \cdot y)H = xH$ implies for all $h_1 \in H$, $\alpha \cdot y \cdot h_1 = x \cdot h_2$ for some $h_2 \in H$. Then $y = \alpha^{-1} \cdot x \cdot h_2 \cdot h_1^{-1} \in (\alpha^{-1} \cdot x)H$. Conversely, if $y \in (\alpha^{-1} \cdot x)H$, then $\alpha \cdot y = x \cdot h$ for some $h \in H$. Then for any $h_1 \in H$, $\alpha \cdot y \cdot h_1 = x \cdot h \cdot h_1 \in xH$, and for any $h_2 \in H$, $x \cdot h_2 = x \cdot h \cdot h^{-1} \cdot h_2 = \alpha \cdot y \cdot h^{-1} \cdot h_2 \in (\alpha \cdot y)H$, so $(\alpha \cdot y)H = xH$. This means $g^{-1}(\{xH\}) = (\alpha^{-1} \cdot x)H$, and thus

$$G/H = \{g^{-1}(\{xH\}) : xH \in G/H\}.$$

By the corollary, g induces a homeomorphism $F_\alpha : G/H \to G/H$ satisfying $g = F_\alpha \circ p$, or $F_\alpha(xH) = (\alpha \cdot x)H$. To show that G/H is homogeneous, for any $xH, yH \in G/H$ we have $F_{y \cdot x^{-1}}(xH) = yH$. $\square$

(b) Show that if H is closed in G, then one-point sets are closed in G/H.

SOLUTION. Suppose H is closed in G. Then since $f_x : G \to G$ is a homeomorphism, $f_x(H) = xH = p^{-1}(\{xH\})$ is closed in G, and since p is a quotient map, this means $\{xH\}$ is closed in G/H. $\square$

(c) Show that the quotient map $p : G \to G/H$ is open.

SOLUTION. If U is open in G then for any $h \in H$, $g_h(U)$ is open in G as g_h is a homeomorphism. Then

$$p(U) = \bigcup_{x \in U} \{xH\} = \bigcup_{h \in H} g_h(U)$$

is open in G/H. $\square$

(d) Show that if H is closed in G and is a normal subgroup, then G/H is a topological group.

SOLUTION. Suppose H is closed in G and a normal subgroup of G. Then G/H is a group as the operation $\cdot : G/H \times G/H \to G/H$ defined by $(xH) \cdot (yH) = (x \cdot y)H$ is well defined for normal H, is associative, has identity element eH, and has inverses $(xH)^{-1} = x^{-1}H$. By (b), G/H is moreover T_1.

$p : G \to G/H$ is open by (c), so $p \times p : G \times G \to G/H \times G/H$ is open, and thus is a quotient map.

Let $f : G \times G \to G$ be defined by $x \times y \mapsto x \cdot y^{-1}$; since G is a topological group, f is a continuous surjection. Then $p \circ f : G \times G \to G/H$ defined by $x \times y \mapsto (x \cdot y^{-1})H$ is a composition of continuous surjections. For any $xH \times yH \in G/H \times G/H$ and $w \times z \in (p \times p)^{-1}(\{xH \times yH\})$, we have $wH = xH$ and $zH = yH$, so

$$(p \circ f)(w \times z) = (w \cdot z^{-1})H = (wH) \cdot (zH)^{-1} = (xH) \cdot (yH)^{-1} = (x \cdot y^{-1})H,$$

so $p \circ f$ is constant on $(p \times p)^{-1}(\{xH \times yH\})$. Thus $p \circ f$ induces a continuous map $h : G/H \times G/H \to G/H$ such that $h \circ (p \times p) = p \circ f$, meaning $h(xH \times yH) = (x \cdot y^{-1})H$ for all $x, y \in G$. Then by Exercise 1 and the continuity of h, G/H is a topological group. □

6. $\mathbb{Z}$ is a normal subgroup of $(\mathbb{R}, +)$. What is the quotient $\mathbb{R}/\mathbb{Z}$?

SOLUTION. Let $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map and let $g : \mathbb{R} \to S^1$ be defined by $g(t) = e^{2\pi i t}$. g is a quotient map by ε-δ arguments. Moreover

$$\mathbb{R}/\mathbb{Z} = \{\{xn : n \in \mathbb{Z}\} : x \in \mathbb{R}\} = \{\{x \in \mathbb{R} : e^{2\pi i x} = z\} : z \in S^1\} = \{g^{-1}(\{z\}) : z \in S^1\}.$$

Thus by the corollary, g induces a homeomorphism $f : \mathbb{R}/\mathbb{Z} \to S^1$, meaning $\mathbb{R}/\mathbb{Z}$ may be viewed as the topological group $(S^1, \cdot)$. □

7. If $A, B \subseteq G$ let $A \cdot B = \{a \cdot b : a \in A, b \in B\}$. Let $A^{-1} = \{a^{-1} : a \in A\}$.

(a) A neighbourhood V of e is symmetric if $V = V^{-1}$. If U is a neighbourhood of e, show that there exists a symmetric neighbourhood V of e such that $V \cdot V \subseteq U$.

SOLUTION. Given a neighbourhood U of e, there exists a neighbourhood W of e such that $W \cdot W \subseteq U$ by continuity of $\cdot : G \times G \to G$. By Exercise 1, there exists a neighbourhood Z of e such that $Z \cdot Z^{-1} \subseteq W$ by continuity of $x \times y \mapsto x \cdot y^{-1}$. $Z \cdot Z^{-1}$ is clearly a symmetric neighbourhood of e such that $(Z \cdot Z^{-1}) \cdot (Z \cdot Z^{-1}) \subseteq W \subseteq U$. □

(b) Show that G is Hausdorff; in fact show that if $x \neq y$ then there exists a neighbourhood V of e such that $V \cdot x$ and $V \cdot y$ are disjoint.

SOLUTION. Given $x \neq y \in G$, $\{x \cdot y^{-1}\}$ is closed in G by the T_1 axiom. Then $G - \{x \cdot y^{-1}\}$ is open. By (a), there exists a symmetric neighbourhood V of e such that $V \cdot V \subseteq G - \{x \cdot y^{-1}\}$. Suppose $z \in (V \cdot x) \cap (V \cdot y)$. Then $z = v \cdot x = w \cdot y$ for some $v, w \in V$. This means $x \cdot y^{-1} = v^{-1} \cdot w \in V^{-1} \cdot V = V \cdot V \subseteq G - \{x \cdot y^{-1}\}$, a contradiction. Hence $V \cdot x, V \cdot y$ are disjoint, and thus G is Hausdorff. □

(c) Show that G satisfies the regularity axiom: Given a closed set A and $x \notin A$, there exist disjoint open sets containing A and x, respectively.

SOLUTION. Given A closed in G and $x \in G - A$, $g_{x^{-1}}(A) = A \cdot x^{-1}$ is closed in G, where g_α is defined in Exercise 4. Since $G - A \cdot x^{-1}$ is open in G, there exists a symmetric neighbourhood V of e such that $V \cdot V \subseteq G - A \cdot x^{-1}$. Suppose $z \in (V \cdot A) \cap (V \cdot x)$ and write $z = v \cdot a = w \cdot x$ for some $v, w \in V, a \in A$. Since $a \cdot x^{-1} = v^{-1} \cdot w$, $A \cdot x^{-1}$ intersects $V \cdot V$, a contradiction. Hence $V \cdot A, V \cdot x$ are disjoint open sets containing A, x, respectively. □

(d) Let H be a subgroup of G that is closed in G; let $p : G \to G/H$ be the quotient map. Show that G/H satisfies the regularity axiom.

SOLUTION. Given A be closed in G/H and $xH \in G/H - A$, $p^{-1}(A)$ is closed and saturated in G, satisfying $p(p^{-1}(A)) = A$ since p is surjective and $x \notin p^{-1}(A)$. By (c), there exists a symmetric neighbourhood V of e such that $V \cdot p^{-1}(A)$ and $V \cdot x$ are disjoint open sets containing $p^{-1}(A)$ and x, respectively. Since p is an open map, $p(V \cdot p^{-1}(A))$ and $p(V \cdot x)$ are open in G/H and contain $p(p^{-1}(A)) = A$ and $p(x) = xH$, respectively.

Suppose $z \in p(V \cdot p^{-1}(A)) \cap p(V \cdot x)$. Then there exist $v, w \in V$, $a \in p^{-1}(A)$ such that

$$z = (v \cdot a)H = (w \cdot x)H,$$

and thus for some $h \in H$,

$$w \cdot x = v \cdot a \cdot h.$$

Then $p(a \cdot h) = (a \cdot h)H = aH \in A$, meaning $a \cdot h \in p^{-1}(A)$. This implies $w \cdot x = v \cdot (a \cdot h) \in (V \cdot x) \cap (V \cdot p^{-1}(A))$, a contradiction. Hence $p(V \cdot p^{-1}(A))$ and $p(V \cdot x)$ are disjoint open sets in G/H containing A and xH, respectively, so G/H satisfies the regularity axiom. $\square$