TOPOLOGY OF CELL COMPLEXES

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1. Introduction

A topological space is called an n-cell, or e^n, if it is homeomorphic to the open n-disk $D^n - \partial D^n$. Recall that the open 0-disk is a single point, the open 1-disk is $(-1, 1)$, and so on.

The torus (colloquially, donut) is the most famous example in topology, and is typically constructed using a quotient map that identifies opposite sides of a square. In a sense, the interior of the square is a 2-cell, attached via quotient map to two circles obtained by gluing edges together. Furthermore, these circles may be viewed as 1-cells attached via quotient map to a single point, which itself is a 0-cell.

2. Construction of Cell Complexes

This construction naturally generalizes to higher dimensions, and using more unusual families of cells in each dimension. In particular, given a discrete space X^0, we construct a cell complex X having 0-cells from X^0 by the following inductive procedure:

(1) Suppose X^{n-1} is defined. Let $\{D^n_\alpha\}_\alpha$ be a collection of disjoint n-disks with boundaries $\{S^{n-1}_\alpha\}_\alpha$. For each α, define an attaching map $\phi_\alpha : S^{n-1}_\alpha \to X^{n-1}$. Extend and paste to obtain a quotient map $p : X^{n-1} \bigsqcup_\alpha D^n_\alpha \to X^{n-1} \bigsqcup_\alpha e^n_\alpha$ which is a homeomorphism between the interior of each D^n_α and e^n_α. We define the n-skeleton $X^n = X^{n-1} \bigsqcup_\alpha e^n_\alpha$ to be the quotient space induced by p.

(2) Either prematurely terminate the inductive procedure by setting $X = X^n$ for some n, or continue indefinitely by setting $X = \bigcup_{n \in \mathbb{N}} X^n$ and letting U be open in X if and only if $U \cap X^n$ is open in X^n for each n.

If $X = X^n$ for some n, then X is finite-dimensional and the dimension of X is the smallest such n, which is naturally the maximum dimension of any cell in X.

For each cell e^n_α, we obtain a characteristic map $\Phi_\alpha : D^n_\alpha \to X$ defined by the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \bigsqcup_\alpha D^n_\alpha \xrightarrow{p} X^n \hookrightarrow X.$$

In particular, Φ_α is continuous by composition of continuous functions (where $X^n \hookrightarrow X$ is continuous by (2)), extends ϕ_α, and is a homeomorphism $D^n_\alpha \to e^n_\alpha$. The characteristic map provides a nice criterion for the topology on X.

Theorem 2.1. *A set $A \subseteq X$ is open (closed) if and only if $\Phi^{-1}_\alpha(A)$ is open (closed) in D^n_α for each characteristic map Φ_α.*

Proof. ($\Longrightarrow$) Since each Φ_α is continuous, $\Phi^{-1}_\alpha(A)$ is open in D^n_α.

($\Longleftarrow$) By induction on n. Suppose $\Phi^{-1}_\alpha(A)$ is open in D^n_α for each Φ_α, and $A \cap X^{n-1}$ is open in X^{n-1}. By the first assumption, $A \cap X^n$ is open in X^n by definition of the quotient topology on X^n. By induction, A is open in X according to (3).

The statement for closed sets follows immediately by $\Phi_\alpha^{-1}(X - A) = D_\alpha^n - \Phi_\alpha^{-1}(A)$. $\square$

Corollary 2.2. *Suppose X is a cell complex, Y any topological space, and $f : X \to Y$. Then the following are equivalent:*

(1) *f is continuous.*
(2) *The restriction $f\big|_{X^n} : X^n \to Y$ is continuous for all n.*
(3) *The composition $f \circ \Phi_\alpha : D^n \to Y$ is continuous for every characteristic map Φ_α.*

Proof. (1) $\implies$ (2). This is well-known by point-set topology.

(2) $\implies$ (3). By definition of Φ_α, we may write $f \circ \Phi_\alpha$ as the composition

$$D_\alpha^n \hookrightarrow X^{n-1} \bigsqcup_\alpha D_\alpha^n \xrightarrow{p} X^n \hookrightarrow X \xrightarrow{f} Y = D_\alpha^n \hookrightarrow X^{n-1} \bigsqcup_\alpha D_\alpha^n \xrightarrow{p} X^n \xrightarrow{f\big|_{X^n}} Y.$$

This is clearly continuous.

(3) $\implies$ (1). Suppose V is open in Y. Then by continuity $(f \circ \Phi_\alpha)^{-1}(V) = \Phi_\alpha^{-1}(f^{-1}(V))$ is open in e_α^n for every α. By 2.1, $f^{-1}(V)$ is open in X, so f is continuous. $\square$

Example 2.3. A 1-dimensional cell complex $X = X^1$ is a graph. It consists of vertices (0-cells) to which edges (1-cells) are attached.

Example 2.4. Let X be the torus, a 2-dimensional cell complex constructed from the square $[-1,1]\times[-1,1]$. We start with $X^0 = \{-1\times-1\}$. We extend and paste the trivial maps $p_1, p_2 : \{-1,1\} \to X^0$ to obtain a quotient map $p : \{-1,1\} \sqcup [-1,1]_1 \sqcup [-1,1] \to \{-1,1\} \sqcup (-1,1)_1 \sqcup (-1,1)_2$, so that $X^1 = \{-1,1\} \sqcup (-1,1)_1 \sqcup (-1,1)_2$ is a union of two circles. Finally, the quotient map $q : X^1 \sqcup D^2 \to X^1 \sqcup e^2$ induces the 2-skeleton $X^2 = X^1 \sqcup e^2$.

Definition 2.5. The *Euler characteristic* of a finite-dimensional cell complex is the number of even-dimensional cells minus the number of odd-dimensional cells.

The Euler characteristic depends only on the homotopy type of the complex. A well-known result in graph theory states that the Euler characteristic of any connected planar (no crossing edges) graph is 2.

Example 2.6. The n-sphere S^n is often defined as the quotient space $D^n/\partial D^n$. This is equivalent to a cell complex with exactly two cells, e^0 and e^n, with $X^0 = X^1 = \cdots = X^{n-1} = e^0$ and $X^n = X^0 \sqcup e^n$ induced by the quotient map $X^0 \sqcup D^n \to X^0 \sqcup e^n$ taking S^{n-1} to e^0. A characteristic map for the n-cell is the quotient map $D^n \to S^n$ collapsing the boundary to a single point.

Example 2.7. Real projective n-space $\mathbb{R}P^n$ is defined as the vector space of all lines in $\mathbb{R}^{n+1}$ passing through the origin. Since each line is determined by a unique (up to scaling) nonzero vector in $\mathbb{R}^{n+1}$, we may topologize $\mathbb{R}P^n$ as the quotient of $\mathbb{R}^{n+1} - \{0\}$ under the equivalence relation $v \sim \lambda v$ for $\lambda \neq 0$. Equivalently, $\mathbb{R}P^n$ is the quotient of S^n by the equivalence relation $v \sim -v$, or $\mathbb{R}P^n = D^n/\mathbb{R}P^{n-1}$. Thus $\mathbb{R}P^n$ is simply obtained from $\mathbb{R}P^{n-1}$ by attaching a single n-cell by the quotient projection mapping $S^{n-1} \to \mathbb{R}P^{n-1}$. By induction, $\mathbb{R}P^n = \bigsqcup_{i=0}^n e^i$. This readily generalizes to the infinite-dimensional case: $\mathbb{R}P^\infty = \coprod_{n\in\mathbb{N}} e^n$. A characteristic map for each cell e^i is the quotient map $D^i \to \mathbb{R}P^i \subseteq \mathbb{R}P^n$ collapsing antipodal points of S^{i-1}.

Example 2.8. Similarly, complex projective n-space $\mathbb{C}P^n$ is the vector space of complex lines through the origin in $\mathbb{C}^{n+1}$. Topologically, $\mathbb{C}P^n$ is $\mathbb{C}^{n+1} - \{0\}$ modulo the equivalence relation $v \sim \lambda v$ for nonzero λ, or S^{2n+1} modulo $v \sim \lambda v$ for $|\lambda| = 1$.

Slightly more involved is the quotient of D^{2n} by $v \sim \lambda v$ for $v \in \partial D^{2n}$. Vectors in $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ with real nonnegative $n+1$-th coordinate are precisely in the form $(w, \sqrt{1-|w|^2}) \in \mathbb{C}^n \times \mathbb{C}$, where $|w| \leq 1$. Thus the set of such vectors is precisely the graph of the function $D^{2n} \to \mathbb{R}$ defined by $w \mapsto \sqrt{1-|w|^2}$. This graph is a disk D^{2n}_+ bounded by S^{2n-1} (and thus S^{2n+1}) consisting of vectors in the form $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$, where $|w| = 1$. Moreover, every vector in S^{2n+1} is equivalent to a vector $v \in D^{2n}_+$ under the relation $v \sim \lambda v$, and v is unique provided that its last coordinate is nonzero (otherwise $v \in S^{2n-1}$). Now $\mathbb{C}P^n$ is the quotient of D^{2n}_+ under the equivalence relation $v \sim \lambda v$ for $v \in S^{2n-1}$, so $\mathbb{C}P^n$ may be obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell by the quotient map $S^{2n-1} \to \mathbb{C}P^{n-1}$. By induction, $\mathbb{C}P^n = \coprod_{i=0}^n e^{2i}$ and $\mathbb{C}P^\infty = \coprod_{n\in\mathbb{N}} e^{2n}$. As in the real case, a characteristic map for e^{2i} is the quotient map collapsing S^{2n-1} according to the equivalence relation of D^{2n}_+.

Definition 2.9. A closed subspace $A \subseteq X$ of a cell complex X is called a *subcomplex* if it is a union of cells in X.

Since A is closed, the characteristic maps land in A, so A itself is a cell complex. Given that $A^n = A \cap X^n$, we can equivalently say $A \subseteq X$ is a subcomplex if A is a union of cells such that the closure of each cell is contained in A.

For example, each skeleton X^n of a cell complex X is clearly a subcomplex. In fact, the only subcomplexes of $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are its skeletons, in the form $\mathbb{R}P^k$ and $\mathbb{C}P^k$ for $k \leq n$.

Viewing S^n as $e^0 \sqcup e^n$, S^k for $k \leq n$ is not a subcomplex of S^n. However, we may define an alternative cell structure on S^n with respect to which S^k is a subcomplex; namely by obtaining S^k from S^{k-1} by attaching two k-cells to the equatorial S^{k-1}. In this construction, $S^\infty = \bigcup_{n\in\mathbb{N}} S^n$ is also a cell complex. Notably, the quotient map $S^n \to \mathbb{R}P^n$ is two-to-one, and for each i identifies the two i-cells of S^n with the unique n-cell of $\mathbb{R}P^n$.

3. Topology of Cell Complexes

Given a subset A of a cell complex X, we define an open set $N_\varepsilon(A)$ in X containing A, associated with a function ε mapping each cell e^n_α to a number $\varepsilon_\alpha > 0$, by the following inductive procedure:

(1) Let $N^0_\varepsilon(A) = A \cap X^0$. $N^0_\varepsilon(A)$ is open in X^0 since X^0 is discrete.
(2) Given an open set $N^n_\varepsilon(A)$ in X^n containing $A \cap X^n$, we define $N^{n+1}_\varepsilon(A)$ by determining its preimage under $\Phi_\alpha : D^{n+1} \to X$ for every e^{n+1}_α. Indeed, let $\Phi_\alpha^{-1}(N^{n+1}_\varepsilon(A))$ be the union of:
   (a) an ε_α-ball around $\Phi_\alpha^{-1}(A) - \partial D^{n+1}$ in the interior of D^{n+1}, and
   (b) a product $(1-\varepsilon_\alpha, 1] \times \Phi_\alpha^{-1}(N^n_\varepsilon(A))$ in D^{n+1} written in polar coordinates.

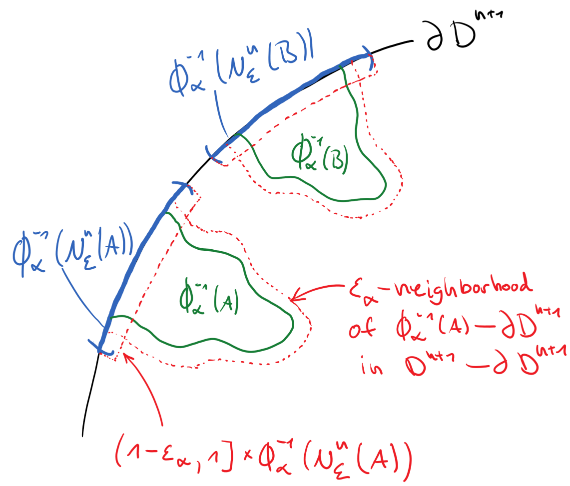

FIGURE 1. Inductive construction of $N_\varepsilon(A)$. Courtesy of Christoph.

We define $N_\varepsilon(A)$ as $\bigcup_{n\in\mathbb{N}} N_\varepsilon^n(A)$; our construction ensures that $A \subseteq N_\varepsilon(A)$. Also for each α, $\Phi_\alpha^{-1}(N_\varepsilon(A)) = \bigcup_{n\in\mathbb{N}} \Phi_\alpha^{-1}(N_\varepsilon^n(A))$ is open in D_α^n, so $N_\varepsilon(A)$ is open in X by 2.1.

Theorem 3.1. *Cell complexes are normal.*

Proof. Suppose A and B are disjoint closed sets in X. We show by induction that for sufficiently small choices of ε_α, the open sets $N_\varepsilon(A), N_\varepsilon(B)$ are disjoint. Initially, $N_\varepsilon^0(A) = A \cap X^0$ and $N_\varepsilon^0(B) = B \cap X^0$ are disjoint. Suppose $N_\varepsilon^n(A)$ and $N_\varepsilon^n(B)$ are disjoint; we will choose ε_α such that $N_\varepsilon^{n+1}(A), N_\varepsilon^{n+1}(B)$ remain disjoint.

In particular, using the labels of the construction above, we must show that for each characteristic map $\Phi_\alpha : D^{n+1} \to X$, the subsections of $\Phi_\alpha^{-1}(N_\varepsilon^{n+1}(A))$ and $\Phi_\alpha^{-1}(N_\varepsilon^{n+1}(B))$ of type (a) remain disjoint, their subsections of type (b) remain disjoint, and each one's subsection of type (a) is disjoint from the other's subsection of type (b).

For the first part, $\Phi_\alpha^{-1}(A)$ and $\Phi_\alpha^{-1}(B)$ are disjoint closed subsets of the compact space D^{n+1}, so $d(\Phi_\alpha^{-1}(A), \Phi_\alpha^{-1}(B)) > 0$. Otherwise there exist sequences (a_n) and (b_n) of points in $\Phi_\alpha^{-1}(A)$ and $\Phi_\alpha^{-1}(B)$, respectively, converging to the same point. Since both $\Phi_\alpha^{-1}(A)$ and $\Phi_\alpha^{-1}(B)$ are closed, the limit of this sequence is in $\Phi_\alpha^{-1}(A) \cap \Phi_\alpha^{-1}(B)$, contradicting the fact that A and B are disjoint. Now provided that $\varepsilon_\alpha < \frac{1}{2} d(\Phi_\alpha^{-1}(A), \Phi_\alpha^{-1}(B))$, the ε_α-balls around $\Phi_\alpha^{-1}(A)$ and $\Phi_\alpha^{-1}(B)$ will not intersect.

For the second part, it suffices to take $\varepsilon_\alpha < 1$, because the fact that $\Phi_\alpha^{-1}(N_\varepsilon^n(A))$ and $\Phi_\alpha^{-1}(N_\varepsilon^n(B))$ are disjoint implies that $(1-\varepsilon_\alpha, 1] \times \Phi_\alpha^{-1}(N_\varepsilon^n(A))$ and $(1-\varepsilon_\alpha, 1] \times \Phi_\alpha^{-1}(N_\varepsilon^n(B))$ may only intersect at a point with radial coordinate 0.

For the last part, it suffices to show that $d(\Phi_\alpha^{-1}(N_\varepsilon^n(A)), \Phi_\alpha^{-1}(B)) > 0$. Suppose otherwise. Then there exists sequences (a_n) and (b_n) of points in $\Phi_\alpha^{-1}(N_\varepsilon^n(A))$ and $\Phi_\alpha^{-1}(B)$ converging to the same point $x \in D^{n+1}$. Since (a_n) is a sequence of points in the closed set ∂D^{n+1} and $\Phi_\alpha^{-1}(B)$ is closed, $x \in \partial D^{n+1} \cap \Phi_\alpha^{-1}(B)$. Now $\Phi_\alpha^{-1}(N_\varepsilon^n(B))$ is an open set containing $\Phi_\alpha^{-1}(B) \cap \partial D^{n+1}$ (in particular containing x), contained in ∂D^{n+1}, by construction. There must thereby exist $\delta > 0$ such that

$B(x,\delta) \cap \partial D^{n+1} \subseteq \Phi_\alpha^{-1}(N_\varepsilon^n(B))$. Since $\Phi_\alpha^{-1}(N_\varepsilon^n(B))$ is contained in ∂D^{n+1} and disjoint from $\Phi_\alpha^{-1}(N_\varepsilon^n(A))$, it follows that

$$d(x, \Phi_\alpha^{-1}(N_\varepsilon^n(A))) \geq \delta > 0,$$

contradicting the fact that $a_n \to x$. Therefore, if $\varepsilon_\alpha < \frac{1}{2} d(\Phi_\alpha^{-1}(N_\varepsilon^n(A)), \Phi_\alpha^{-1}(B))$, then the ε_α-ball around $\Phi_\alpha^{-1}(B)$ will not intersect $(1-\varepsilon_\alpha, 1] \times \Phi_\alpha^{-1}(N_\varepsilon^n(A))$.

Symmetrically, we may choose ε_α sufficiently small that the ε_α ball around $\Phi_\alpha^{-1}(A)$ and $(1-\varepsilon_\alpha, 1] \times \Phi_\alpha^{-1}(N_\varepsilon^n(B))$ remain disjoint. Altogether, we may ensure that $\Phi_\alpha^{-1}(N_\varepsilon^{n+1}(A))$ and $\Phi_\alpha^{-1}(N_\varepsilon^{n+1}(B))$ are disjoint for every α, which implies that $N_\varepsilon^{n+1}(A)$ and $N_\varepsilon^{n+1}(B)$ are disjoint. By induction, $N_\varepsilon(A)$ and $N_\varepsilon(B)$ are disjoint open sets containing A and B, respectively. □

Corollary 3.2. *Cell complexes are Hausdorff.*

Proof. Suppose $x_0 \in X$; we show by induction on n that for each characteristic map $\Phi_\alpha : D^n \to X$, $\Phi_\alpha^{-1}(\{x_0\})$ is closed in D^n. D^0 is discrete, so the base case is trivial. Now suppose for each characteristic map $\Phi_\alpha : D^{n-1} \to X$, $\Phi_\alpha^{-1}(\{x_0\})$ is closed in D^{n-1}. If $x_0 \in X^{n-1}$, then by the inductive hypothesis $\Phi_\alpha^{-1}(\{x_0\}) = \phi_\alpha^{-1}(\{x_0\})$ is closed in $D^{n-1} = \partial D^n$. ∂D^n is a closed subspace of D^n, so $\Phi_\alpha^{-1}(\{x_0\})$ is closed in D^n. Otherwise if $x_0 \notin X^{n-1}$, then it lies in the interior of some D_α^n, on which Φ_α is a homeomorphism. Thus $\Phi_\alpha^{-1}(\{x_0\})$ contains one or no points, and either the T_1 axiom on D^n or its emptyness implies it is closed in D^n. This completes the induction, and by 2.1, shows that $\{x_0\}$ is closed in X. It follows that X is T_1.

Since X is normal and satisfies the T_1 axiom, it is Hausdorff. □

A finite (having finitely many cells; this is stronger than finite-dimensional) cell complex is compact because attaching a single cell to a compact set preserves compactness. Somewhat conversely,

Theorem 3.3. *A compact subspace of a cell complex is contained in a finite subcomplex.*

Proof. Suppose C is a compact subset of a cell complex X. We first show that C intersects only finitely many cells of X. Suppose otherwise; then there is an infinite sequence of points $(x_i)_{i\in\mathbb{N}}$ in C lying in distinct cells. We show by induction that $S = \{x_i\}_{i\in\mathbb{N}}$ is closed in X. Assuming $S \cap X^{n-1}$ is closed in X^{n-1}, for each cell e_α^n of X, $\phi_\alpha^{-1}(S)$ is closed in ∂D_α^n. Moreover, Φ_α extends ϕ_α and is homeomorphic on the interior of D_α^n. Thus $\Phi_\alpha^{-1}(S)$ is simply $\phi_\alpha^{-1}(S)$ in addition to at most one point in D_α^n, so $\Phi_\alpha^{-1}(S)$ is closed in D_α^n. This means $S \cap X^n$ is closed in X^n, so by induction S is closed in X. Similarly, any subset of S is closed in X, meaning S is discrete. But a compact discrete set is finite, which contradicts the definition of S.

We will now show that any cell e_α^n in a cell complex X is contained in a finite subcomplex. Let ϕ_α be the associated attaching map; $\phi_\alpha(S^{n-1})$ is compact. By induction on dimension in this theorem, $\phi_\alpha(S^{n-1})$ is contained in a finite subcomplex $A \subseteq X^{n-1}$. Clearly e_α^n is contained in $A \cup e_\alpha^n$, which is a finite subcomplex by the attaching ϕ_α.

Now C is contained in a finite union of cells, and each of these cells is contained in a finite subcomplex of X. Since a finite union of finite subcomplexes is again a finite subcomplex, it follows that C is contained in a finite subcomplex of X. □

It follows that cell complexes necessarily satisfy the next two properties:

Corollary 3.4. *A cell complex is closure-finite; that is, the closure of each cell intersects only finitely many other cells.*

Proof. The closure of any cell is the image of the compact set S^{n-1} under the continuous characteristic map Φ_α, and thus is compact. By the previous theorem, it intersects only finitely many other cells. $\square$

Corollary 3.5. *A cell complex is in the weak topology; that is, a set is closed if and only if it intersects the closure of each cell in a closed set.*

Proof. ($\Longrightarrow$) Suppose $C \subseteq X$ is closed. Since the closure of each cell is closed, so is its intersection with the closed set C.

($\Longleftarrow$) Suppose $C \cap \overline{e_\alpha^n}$ is closed. Then $\Phi_\alpha^{-1}(C) = p^{-1}(C \cap \overline{e_\alpha^n})$ is closed in D_α^n by continuity of p. By 2.1, C is closed in X. $\square$

In fact, closure-finite weakness is also a sufficient condition to construct cell complexes:

Theorem 3.6. *Suppose X is Hausdorff and admits an indexed family of maps $\Phi_\alpha : D_\alpha^n \to X$. Then the family of maps Φ_α are characteristic maps of a cell complex structure on X if and only if the following conditions all hold:*

- (i) *Each Φ_α restricts to a homeomorphism from the interior of D_α^n onto its image, which we identify as a cell e_α^n of X. Moreover these cells are disjoint and their union is X.*
- (ii) *For each cell e_α^n, $\Phi_\alpha(\partial D_\alpha^n)$ is contained in a finite union of cells of dimension less than n.*
- (iii) *A subset of X is closed if and only if it meets the closure of each cell in a closed set.*

Proof. We have already shown the forward direction: (i) is by construction, (ii) is 3.4 because $\Phi_\alpha(\partial D_\alpha^n) = \overline{e_\alpha^n} - e_\alpha^n$, and (iii) is 3.5. In the reverse direction, suppose by induction that X^{n-1}, the union of all cells of dimension less than n, is a cell complex with characteristic maps from the corresponding subfamily of Φ_α's. We may use $X^{-1} = \varnothing$ as the base case. Define a continuous surjection $p : X^{n-1} \bigsqcup_\alpha D_\alpha^n \to X^n$ as the inclusion on X^{n-1} and Φ_α on each D_α^n. We will show p is a quotient map.

Suppose $C \subseteq X^n$ is such that $p^{-1}(C)$ is closed in $X^{n-1} \bigsqcup_\alpha D_\alpha^n$; we will show that for each cell e_β^m of X, $C \cap \overline{e_\beta^m}$ is closed in X^n. If $m < n$, then $p : X^{n-1} \to X^n$ is the inclusion map from a closed set, and thus is a closed map. It follows that $p(p^{-1}(C)) = C \cap X^{n-1}$ is closed in

$p^{-1}(C) \cap X^{n-1}$ is closed in $X^{n-1} \bigsqcup_\alpha D_\alpha^n$, and thus its image under implies $C \cap X^{n-1}$ is closed in $X^{n-1} \bigsqcup_\alpha D_\alpha^n$, since p is the inclusion on X^{n-1}. Thus $(C \cap X^{n-1}) \cap \overline{e_\beta^m}$ is closed, and since $\overline{e_\beta^m} \subseteq X^{n-1}$, this is simply $C \cap \overline{e_\beta^m}$.

If $m = n$, then $p^{-1}(C)$ and D_β^n being closed in $X^{n-1} \bigsqcup_\alpha D_\alpha^n$ implies the intersection $p^{-1}(C) \cap D_\beta^n$ is closed in $X^{n-1} \bigsqcup_\alpha D_\alpha^n$. Since $X^{n-1} \bigsqcup_\alpha D_\alpha^n$ is compact, $p^{-1}(C) \cap D_\beta^n$ is compact, so its image $p(p^{-1}(C) \cap D_\beta^n)$ under p is compact in X^n. Since p is surjective, $p(p^{-1}(C) \cap D_\beta^n) = C \cap \overline{e_\beta^n}$. Therefore by the Hausdorff property of X^n, $C \cap \overline{e_\beta^n}$ is closed in X^n.

Finally if $m > n$, then $C \subseteq X^n$ implies $C \cap \overline{e_\beta^m} \subseteq \Phi_\beta(\partial D_\beta^m)$. By (ii), the latter set is contained in a finite union of cell closures $\overline{e_\gamma^\ell}$, where each $\ell < m$. By induction on m, we may assume each $C\cap\overline{e^\ell_\gamma}$ is closed, so $C\cap\bigcup\overline{e^\ell_\gamma}$ is closed because the union is finitely indexed. It follows that $C\cap\overline{e^m_\beta}=(C\cap\bigcup\overline{e^\ell_\gamma})\cap\overline{e^m_\beta}$ is closed in X^n.

Since C intersects the closure of each cell e^m_β in a closed set, it is closed by (iii). This shows that p is a quotient map, and thus X^n is constructed from X^{n-1} by attaching the n-cells e^n_α by the quotient map p.

It remains to verify that X is in the weak topology with respect to the n-skeletons X^n; that is, $C\subseteq X$ is closed if and only if $C\cap X^n$ is closed for each n. The argument above with $C=X^n$ shows that X^n itself is closed, so the forward implication is obvious. Conversely, if $C\cap X^n$ is closed for all n, then $C\cap\overline{e^n_\alpha}$ is closed for each cell e^n_α, and thus C is closed in X by (iii). □

Due to this result, cell complexes are often called CW (closure-finite weak) complexes.

We now examine some more interesting topological properties of cell complexes. Recall that a space X is contractible if the identity map $i:X\to X$ is nulhomotopic, and locally contractible if for each point $x\in X$ and each neighbourhood U of x, there exists a neighbourhood V of x contained in U such that the inclusion map $j:V\to U$ is nulhomotopic.

Theorem 3.7. *Each point in a cell complex has arbitrarily small contractible neighbourhoods; in particular, cell complexes are locally contractible.*

Proof. Given $x\in X$ and a neighbourhood U of x, we may choose the ε_α's sufficiently small that $N_\varepsilon(x)\subseteq U$. Indeed, simply ensure that $\overline{N^n_\varepsilon(x)}\subseteq U$ in each step of the inductive construction. It thereby suffices to show that $N_\varepsilon(x)$ is contractible.

Let m be such that $x\in X^m-X^{m-1}$. Since $N^m_\varepsilon(x)$ is an open ball around x, it suffices to construct a deformation retraction of $N_\varepsilon(x)$ onto $N^m_\varepsilon(x)$. For each $n>m$ we construct a deformation retraction of $N^n_\varepsilon(x)$ onto $N^{n-1}_\varepsilon(x)$ by sliding outward along radial segments in cells e^n_β, which are the images of radial segments D^n under the characteristic maps. We paste the deformation retractions of $N^n_\varepsilon(x)$ onto $N^{n-1}_\varepsilon(x)$ together, each occupying the interval $\left[\frac{1}{2^n},\frac{1}{2^{n-1}}\right]$, to obtain a deformation retraction of $N_\varepsilon(x)$ onto $N^m_\varepsilon(x)$. □

It follows that cell complexes are locally path-connected. Moreover, a cell complex is connected if and only if path-connected.

Theorem 3.8. *Suppose A is a subcomplex of a cell complex X. If $\varepsilon_\alpha<1$ for each α, then there exists a deformation retraction of $N_\varepsilon(A)$ onto A.*

Proof. In every cell of $X-A$, $N_\varepsilon(A)$ is a product neighbourhood of its boundary, so for each n we may clearly retract $N^n_\varepsilon(A)$ onto $N^{n-1}_\varepsilon(A)$ as in the previous theorem, and then paste together to obtain a deformation retraction of $N_\varepsilon(A)$ onto A. □

References

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