EQUIVALENT FORMULATIONS OF A SUBMANIFOLD IN $\mathbb{R}^n$

WILLIAM GAO

Suppose U, V are open in $\mathbb{R}^n$. A diffeomorphism $h : U \to V$ is a $\mathcal{C}^\infty$ function with $\mathcal{C}^\infty$ inverse.

Theorem 0.1. *Suppose $M \subseteq \mathbb{R}^n$. The following are equivalent:*

(1) *(preimage) for all $a \in M$ there exists a neighbourhood U of a and a $\mathcal{C}^\infty$ function $f : U \to \mathbb{R}^{n-k}$ such that $M \cap U = f^{-1}(0)$ and $Df(x)$ has rank $n-k$ on $M \cap U$.*

(2) *(implicit function) for all $a \in M$ there exists a rectangular neighbourhood $V \times W \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ of a such that $M \cap (V \times W)$ is the graph of a $\mathcal{C}^\infty$ function $g : V \to W$ defined by $z = g(y)$.*

(3) *(diffeomorphism) for all $a \in M$ there exists a neighbourhood U of a, an open set V in $\mathbb{R}^n$, and a diffeomorphism $h : U \to V$ such that $h(M \cap U) = V \cap (\mathbb{R}^k \times \{0\}^{n-k})$.*

(4) *(coordinate charts) for all $a \in M$, there exists a neighbourhood U of a in $\mathbb{R}^n$, an open set $W \subseteq \mathbb{R}^k$, and an injective $\mathcal{C}^\infty$ function $\varphi : W \to \mathbb{R}^n$ such that $\varphi(W) = M \cap U$, φ has rank k on W, and for every Ω open in W, $\varphi(\Omega) = \varphi(W) \cap \Omega'$ for some Ω' open in $\mathbb{R}^n$.*

Proof. (1) $\Longrightarrow$ (2). $f : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ satisfies the hypotheses of the implicit function theorem, so there exist neighbourhoods V of $a_1 \in \mathbb{R}^k$ and W of $a_2 \in \mathbb{R}^{n-k}$ along with a $\mathcal{C}^\infty$ function $g : V \to W$ such that $z = g(y)$ satisfies $f(y, g(y)) = 0$; that is, $(y, g(y)) \in f^{-1}(0) = M \cap U$. Since $(y, g(y)) \in V \times W$, $M \cap (V \times W)$ is the graph of g.

(2) $\Longrightarrow$ (1). By the $\mathcal{C}^\infty$ coordinate change $f : V \times W \to \mathbb{R}^{n-k}$ defined by $f(y, z) = z - g(y)$, $M \cap (V \times W) = f^{-1}(o)$ and $Df(x)$ has rank $n-k$ on $M \cap (V \times W)$.

(2) $\Longrightarrow$ (3). Let $U = V \times W$. By the $\mathcal{C}^\infty$ coordinate change $h(y, z) = (y, z - g(y))$, we have $h(M \cap U) = V \cap (\mathbb{R}^k \times \{0\}^{n-k})$.

(3) $\Longrightarrow$ (1). Let $f(x) = (h_{k+1}(x), \cdots, h_n(x))$. $f : U \to \mathbb{R}^{n-k}$ is $\mathcal{C}^\infty$, $M \cap U = f^{-1}(0)$, and since $Dh(x)$ has rank n, $Df(x)$ has rank $n-k$ on $M \cap U$.

(3) $\Longrightarrow$ (4). Let $W = V \cap (\mathbb{R}^k \times 0)$ and let $\varphi = h^{-1} : W \to U$. Indeed, φ is $\mathcal{C}^\infty$, injective, has rank k on W because h has rank n, and if Ω is open in W then $\varphi(\Omega)$ is open in $\varphi(W)$ by continuity of h.

(4) $\Longrightarrow$ (3). Given $a \in M$, let $b \in W$ satisfy $\varphi(b) = a$. We may assume without loss of generality that $\frac{\partial(\varphi_1, \cdots, \varphi_k)}{\partial(y_1, \cdots, y_k)}$ has rank k on W. Define $\psi : W \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by

$$(y, z) \mapsto (\varphi_1(y), \cdots, \varphi_k(y), \varphi_{k+1}(y) + z_1, \cdots, \varphi_n(y) + z_{n-k}).$$

Now $D\psi(y, z)$ has block form

$$\begin{pmatrix} \frac{\partial(\varphi_1 \cdots \varphi_k)}{\partial(y_1 \cdots y_k)}(y, z) & O \\ * & I \end{pmatrix},$$

and thus has rank n. By the inverse function theorem, there exist neighbourhoods V'' of $(b, 0)$ and U'' of a such that $\psi : V'' \to U''$ is a diffeomorphism. Since U'' is open, $\varphi(U'') = \varphi(W) \cap U'$ for some U' open in $\mathbb{R}^n$. Let $U = U' \cap V''$ and $V = \psi^{-1}(U)$. Then $U \cap M = \{\varphi(a) : (a, 0) \in V\} = \{\psi(a, 0) : (a, 0) \in V\}$, so

$$\begin{aligned} h(U \cap M) &= \psi^{-1}(U \cap M), \\ &= \psi^{-1}(\{\psi(a, 0) : (a, 0) \in V\}, \\ &= V \cap (\mathbb{R}^k \times 0), \end{aligned}$$

as desired. □

We are now justified in calling a set M in $\mathbb{R}^n$ a submanifold if it satisfies any of the above conditions.