Semisimple Lie algebras

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Today, k will be a field of characteristic zero and all algebras or modules will be finite-dimensional over k.

If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals of a Lie algebra $\mathfrak{g}$, then $\mathfrak{a}+\mathfrak{b}$ is solvable, being an extension of $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$ by $\mathfrak{a}$. Hence there exists a largest solvable ideal in $\mathfrak{g}$, called the *radical* $\mathfrak{r}$.

1 Semisimple Lie algebras

A Lie algebra $\mathfrak{g}$ is *semisimple* if the radical is zero. Equivalently, $\mathfrak{g}$ contains no nonzero abelian ideal. Indeed if $\mathfrak{r} \neq 0$, then the last nonzero derived algebra of $\mathfrak{r}$ is a nonzero abelian ideal. The following criterion is often the most handy:

Theorem 1.1

$\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate.

Proof. Let $\mathfrak{u}$ be the ideal of $x \in \mathfrak{g}$ such that $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ for all $y \in \mathfrak{g}$. For $x \in \mathfrak{u}$ we have $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ for $y \in D\mathfrak{u}$, so by Cartan's criterion, $\operatorname{ad}_{\mathfrak{g}} \mathfrak{u}$ is a solvable Lie subalgebra of $\operatorname{End} \mathfrak{g}$. Since it is the quotient of $\mathfrak{u}$ by the center of $\mathfrak{g}$, it follows that $\mathfrak{u}$ is solvable, so if $\mathfrak{g}$ is semisimple, then $\mathfrak{u} = 0$.

Conversely, let $\mathfrak{a}$ be an abelian ideal; we claim that $\mathfrak{a} \subset \mathfrak{u}$. Indeed, $\operatorname{ad} x \operatorname{ad} y(\mathfrak{g}) \subset \mathfrak{a}$ and $\operatorname{ad} x \operatorname{ad} y(\mathfrak{a}) = 0$, so $(\operatorname{ad} x \operatorname{ad} y)^2 = 0$ which implies $\operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$. $\square$

Theorem 1.2

Let $\mathfrak{g}$ be semisimple and $\mathfrak{a}$ and ideal in $\mathfrak{g}$. Then $\mathfrak{a}^\perp$ is an ideal and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ is a direct sum.

Proof. The invariance of the Killing form shows that $\mathfrak{a}^\perp$ is an ideal. Then $\mathfrak{a}\cap\mathfrak{a}^\perp$ is solvable by Cartan's criterion, by similar arguments as the preceding proof. Since $\mathfrak{g}$ is semisimple, $\mathfrak{a}\cap\mathfrak{a}^\perp = 0$. $\square$

Definition 1.3

A Lie algebra $\mathfrak{s}$ is *simple* if it is nonabelian and it has no ideal other than 0 and itself.

By induction on $\dim \mathfrak{g}$ we see that every semisimple Lie algebra is isomorphic to a product of simple Lie algebras.

If $\mathfrak{s}$ is simple, then $D\mathfrak{s} = \mathfrak{s}$. It follows that

Corollary 1.4

If $\mathfrak{g}$ is semisimple, then $D\mathfrak{g} = \mathfrak{g}$.

Corollary 1.5

If $\mathfrak{g} = \bigoplus \mathfrak{a}_\alpha$ is a direct sum decomposition of $\mathfrak{g}$ into simple ideals, then any ideal of $\mathfrak{g}$ is some sum of the $\mathfrak{a}_\alpha$.

Example 1.6. $\mathfrak{sl}(V)$, the trace zero endomorphisms of V, is simple for $\dim V \geq 2$.

Example 1.7. $\mathfrak{sp}(V)$, the endomorphisms which fix a nondegenerate alternating form, is simple for $\dim V = 2n$, $n \geq 1$.

Example 1.8. $\mathfrak{o}(V)$, the endomorphisms of V fixing a nondegenerate symmetric form, is semisimple for $\dim V \geq 3$, and simple unless $\dim V = 4$ and the discriminant of the symmetric form is square.

2 Complete reducibility

Definition 2.1

A representation V of $\mathfrak{g}$ is *simple* if $V \neq 0$ and V has no submodules other than 0 and V. It is called *semisimple* or *completely reducible* if V is the direct sum of simple submodules.

By induction, complete reducibility is equivalent to every submodule of V having a supplementary submodule.

Remark 2.2. $\mathfrak{g}$ may be semisimple as a $\mathfrak{g}$-module without being semisimple as a Lie algebra. Consider $\mathfrak{g} = k$.

However, the converse is not possible.

Theorem 2.3 (Weyl)

Let $\mathfrak{g}$ be semisimple. Then every finite-dimensional $\mathfrak{g}$-module is semisimple.

Proof. We provide Weyl's geometric proof in the case where $k = \mathbb{C}$. Let G be a connected and simply connected complex Lie group for $\mathfrak{g}$ and let K be a maximal compact subgroup of G. Any complex group submanifold of G containing K is equal to G, so the G-submodules are the same as K-submodules. By compactness of K there exists a K-invariant definite Hermitian form on V, with respect to which we can construct orthogonal supplementary subspaces. □

Corollary 2.4

Let $\mathfrak{g}$ be a semisimple ideal of a Lie algebra $\mathfrak{h}$. Then there exists a unique ideal $\mathfrak{a}$ in $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$.

Proof. Since $\mathfrak{h}$ is semisimple as an $\mathfrak{g}$-module, there exists a k-subspace $\mathfrak{a}$ of $\mathfrak{h}$ supplementary to $\mathfrak{g}$ and table under $\operatorname{ad} x$ for $x \in \mathfrak{g}$. We claim that $[\mathfrak{g}, \mathfrak{a}] = 0$. Indeed, $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{g}$ since $\mathfrak{g}$ is an ideal and $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$ since $\mathfrak{a}$ is $\mathfrak{g}$-stable. Therefore $\mathfrak{a}$ consists precisely of the $y \in \mathfrak{h}$ such that $[\mathfrak{g}, y] = 0$, which is 0 because the center of $\mathfrak{g}$ is zero. Therefore $\mathfrak{a}$ is unique as a $\mathfrak{g}$-module, and as the annihilator of $\mathfrak{g}$, it is an ideal of $\mathfrak{h}$. □

Corollary 2.5

Let $\mathfrak{g}$ be semisimple. Then every derivation $\mathfrak{g}$ takes the form $\operatorname{ad} x$ for $x \in \mathfrak{g}$.

Proof. Applying the previous corollary with $\mathfrak{h} = \operatorname{Der} \mathfrak{g}$, $\mathfrak{g}$ is an ideal in $\operatorname{Der} \mathfrak{g}$ because $x \in \mathfrak{g}$ and $D \in \operatorname{Der} \mathfrak{g}$ implies $[D, \operatorname{ad} x] = \operatorname{ad}(Dx)$. Hence we have a decomposition $\operatorname{Der} \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{a}$, where $\mathfrak{a}$ consists of the derivations which commute with $\operatorname{ad} \mathfrak{g}$. If $D \in \mathfrak{a}$, then $\operatorname{ad}(Dx) = [D, \operatorname{ad} x] = 0$ implies $Dx = 0$, since the center of $\mathfrak{g}$ is 0. Hence $\mathfrak{a} = 0$. □

3 Levi's theorem

Theorem 3.1 (Levi)

Let $\phi\colon \mathfrak{g} \to \mathfrak{s}$ be a surjective homomorphism of $\mathfrak{g}$ onto a semisimple Lie algebra $\mathfrak{s}$. Then there exists a homomorphism $\epsilon\colon \mathfrak{s} \to \mathfrak{g}$ such that $\phi \circ \epsilon = \mathrm{id}_{\mathfrak{s}}$.

Proof. Let $\mathfrak{a} = \ker \phi$, so that $\mathfrak{s} = \mathfrak{g}/\mathfrak{a}$. We reduce to the case where $\mathfrak{a}$ is abelian and simple as a $\mathfrak{g}$-module with a nontrivial action. Let $\mathfrak{a}_1$ be any ideal in $\mathfrak{g}$ with $0 \subset \mathfrak{a}_1 \subset \mathfrak{a}$. If there exists a supplementary subalgebra $\mathfrak{s}_1 = \mathfrak{g}_1/\mathfrak{a}_1$ to $\mathfrak{a}/\mathfrak{a}_1$ in $\mathfrak{a}/\mathfrak{a}_1$ and a supplementary subalgebra $\mathfrak{s}_2$ to $\mathfrak{a}_1$ in $\mathfrak{g}_1$, then $\mathfrak{s}_2$ is supplementary to $\mathfrak{a}$ in $\mathfrak{g}$. By induction on $\dim\mathfrak{a}$, we may assume $\mathfrak{a}$ is a simple $\mathfrak{g}$-module. Then $\mathfrak{a}$ contains the radical of $\mathfrak{g}$, and if $\mathfrak{r}=0$ then $\mathfrak{g}$ is semisimple and we are done. Otherwise if $\mathfrak{r}=\mathfrak{a}$ then $\mathfrak{a}$ is solvable,

so $\mathfrak{a}\neq[\mathfrak{a},\mathfrak{a}]$. But the latter is an ideal, and thus 0 as $\mathfrak{a}$ is abelian. If $\mathfrak{g}$ acts trivially on $\mathfrak{a}$, then $\mathfrak{a}$ is in the center of $\mathfrak{g}$, so $\mathfrak{g}$ acts through $\mathfrak{g}/\mathfrak{a}\cong\mathfrak{s}$, so $\mathfrak{g}$ is completely reducible as an $\mathfrak{s}$-module. We conclude that there exists an ideal supplementary to $\mathfrak{a}$.

Now for the case where $\mathfrak{a}$ is abelian and simple as a $\mathfrak{s}$-module with nontrivial action, then we will use the following lemma. □

Lemma 3.2

Let W be a $\mathfrak{g}$-module which contains some w such that $a\mapsto aw$ is a bijection $\mathfrak{a}\to\mathfrak{a}w$ and $\mathfrak{g}w=\mathfrak{a}w$. Let $i_w=\{x\in\mathfrak{g}:xw=0\}$ be the stabilizer of w. Then i_w is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{a}\oplus i_w$.

It remains to construct a suitable w. Let $W=\operatorname{End}\mathfrak{g}$ as a $\mathfrak{g}$-module, the representation being

$$\begin{aligned}\sigma\colon\mathfrak{g}&\longrightarrow\operatorname{End}\operatorname{End}\mathfrak{g}\\ x&\longmapsto(\phi\mapsto\operatorname{ad}x\circ\phi-\phi\circ\operatorname{ad}x).\end{aligned}$$

Define $P\subset Q\subset R\subset W$ by

$$\begin{aligned}P&=\{\operatorname{ad}_{\mathfrak{g}}a:a\in\mathfrak{a}\}\\ Q&=\{\phi\in W:\phi\mathfrak{g}\subset\mathfrak{a}\quad\text{and}\quad\phi\mathfrak{a}=0\}\\ R&=\{\phi\in W:\phi\mathfrak{g}\subset\mathfrak{a}\quad\text{and}\quad\phi\mid\mathfrak{a}\text{ is a homothety}\}.\end{aligned}$$

We have an exact sequence of $\mathfrak{g}$-modules

$$0\to Q\to R\xrightarrow{\rho}k\to0,$$

where $Q\to R$ is the inclusion and ρ sends $r\in R$ to the scalar by which r multiplies an element of $\mathfrak{a}$. For $x\in\mathfrak{a}$ and $\phi\in R$, we have

$$\sigma(x)\phi=\operatorname{ad}x\circ\phi-\phi\circ\operatorname{ad}x=-\lambda\operatorname{ad}x,$$

where $\lambda=\rho(\phi)$. Thus $\sigma(x)R\subset P$, so we get an exact sequence of $\mathfrak{s}$-modules

$$0\to Q/P\to R/P\to k\to0.$$

By lifting invariants, there exists $\overline{w}\in R/P$ such that $\overline{\rho}(\overline{w})=1$ and which is $\mathfrak{s}$-invariant. Let w be a representative of $\overline{w}$ in R. We verify:

(i) For $a\in\mathfrak{a}$, $\sigma(a)w=-\operatorname{ad}a$. If $\sigma(a)w=0$, then $\operatorname{ad}_{\mathfrak{g}}a=0$, so $[a,x]=0$ for all $x\in\mathfrak{g}$. Then $a=0$ since $\mathfrak{a}$ is simple, so $\mathfrak{g}$ acts nontrivially.

(ii) Let $x\in\mathfrak{g}$. We wish to show $\sigma(x)w$ takes the form $\sigma(a)w$ for some $a\in\mathfrak{a}$. Since $\sigma(a)w=-\operatorname{ad}_{\mathfrak{g}}a$, it suffices to show $\sigma(x)w\in P$. This holds since $\overline{w}$ is $\mathfrak{s}$-invariant.

Applying Levi's theorem to $\mathfrak{g} \to \mathfrak{g}/\mathfrak{r}$, we get:

Corollary 3.3

Any Lie algebra $\mathfrak{g}$ is the semidirect product of its radical with a semisimple subalgebra.

If $\mathfrak{g} \neq D\mathfrak{g}$ and $\mathfrak{a} \subset \mathfrak{g}$ has codimension 1 and contains $D\mathfrak{g}$, then $\mathfrak{a}$ is an ideal, so $\mathfrak{g} = \mathfrak{a} \oplus kx$ for any $x \in \mathfrak{g} \setminus \mathfrak{a}$. kx is by construction a Lie subalgebra, so:

Corollary 3.4

A nonzero Lie algebra which is neither simple nor one-dimensional abelian is a semidirect product of two lower-dimensional Lie algebras.

4 Complete reducibility II

The next theorem characterizes complete reducibility of a representation:

Theorem 4.1

Let k be algebraically closed. Let V be a vector space and $\mathfrak{g}$ a Lie subalgebra of $\operatorname{End} V$. Then V is a completely reducible $\mathfrak{g}$-module if and only if (a) $\mathfrak{g}$ is a product $\mathfrak{c} \times \mathfrak{s}$ with $\mathfrak{c}$ abelian and $\mathfrak{s}$ semisimple, and (b) the elements of $\mathfrak{c}$ are diagonalizable.

Proof. If $V = 0$ there is nothing to show. If $V \neq 0$ is completely reducible, then there exists a line in V stable under $\mathfrak{r}$, or equivalently a linear form $\chi\colon \mathfrak{r} \to k$ with nonzero eigenspace V_χ. By the lemma used in Lie's theorem, V_χ is $\mathfrak{g}$-stable. B complete reducibility, there exist characters χ_i of $\mathfrak{r}$ such that

$$V = V_{\chi 1} \oplus V_{\chi 2} \oplus \cdots \oplus V_{\chi m}.$$

It is then clear that $\mathfrak{r}$ acts diagonally and commutes with the action of $\mathfrak{g}$, so $\mathfrak{c} = \mathfrak{r}$ is the center of $\mathfrak{g}$. We get $\mathfrak{s}$ from Levi's theorem.

Conversely if (a), (b) hold then (b) gives a decomposition of V in the form

$$V = V_{\chi 1} \oplus \cdots \oplus V_{\chi m}$$

where the χ_i are linear forms on $\mathfrak{c}$, which lies in the center of $\mathfrak{g}$. Thus the V_{χ_i} are $\mathfrak{g}$-stable, so we have reduced to $V = V_\chi$. In this case $\mathfrak{g}$-submodules are equivalent to $\mathfrak{s}$-submodules, so Weyl's theorem gives the desired result. □

Corollary 4.2

If $\mathfrak{g} = \mathfrak{c} \times \mathfrak{s}$ with $\mathfrak{c}$ abelian and $\mathfrak{s}$ semisimple, then a $\mathfrak{g}$-module is semisimple if and only if $\mathfrak{c}$ acts diagonally on it.

Corollary 4.3

If a $\mathfrak{g}$-module V is completely reducible, then so are the $V_{p,q}$.

Corollary 4.4

The tensor product of completely reducible $\mathfrak{g}$-modules is completely reducible.

Theorem 4.5

Let V be a finite-dimensional k-vector space. Let $g \subset \operatorname{End} V$ be a Lie algebra. If $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ is determined by its tensor invariants; that is, there exist some $v_\alpha \in V_{p,q}$ such that $\mathfrak{g} = \{x \in \operatorname{End} V : xv_\alpha = 0 \text{ for all } \alpha\}$.

Proof. As usual, we reduce to the case where k is algebraically closed. Let $\mathfrak{h}$ be the set of $x \in \operatorname{End} V$ such that $xv = 0$ for all $v \in V_{p,q}$ with $\mathfrak{g}v = 0$. Clearly $\mathfrak{g} \subset \mathfrak{h} \subset \operatorname{End} V$ and $\mathfrak{h}$ is a Lie algebra. We wish to show that $\mathfrak{h} = \mathfrak{g}$.

If $u\colon V_{p,q} \to V_{r,s}$ is a $\mathfrak{g}$-linear homomorphism, then it is $\mathfrak{h}$-linear, as $\operatorname{Hom}_k(V_{p,q}, V_{r,s}) \cong V_{q+r,p+s}$ as $\operatorname{End} V$-modules, and a k-linear map is a $\mathfrak{g}$-linear homomorphism if and only if killed by $\mathfrak{g}$.

If $W \subset V_{p,q}$ is $\mathfrak{g}$-stable, then it is $\mathfrak{h}$-stable. Indeed, $V_{p,q}$ is completely reducible as a $\mathfrak{g}$-module, so there exists a $\mathfrak{g}$-endomorphism projecting $V_{p,q}$ onto W. Since u is also an $\mathfrak{h}$-endomorphism, its image is $\mathfrak{h}$-stable.

Taking $W = \mathfrak{g}$ and $p = q = 1$, we see that $\mathfrak{g}$ is an ideal in $\mathfrak{h}$, so by a corollary of Weyl's theorem $\mathfrak{h} = \mathfrak{g} \times \mathfrak{c}$ where $\mathfrak{c}$ commutes with $\mathfrak{g}$. Thus $\mathfrak{c}$ lies in the center of $\mathfrak{h}$.

Now let W be an irreducible $\mathfrak{g}$-submodule of V. Then W is $\mathfrak{c}$-stable, and by Schur's lemma the elements of $\mathfrak{c}$ act as homotheties on W. It remains to show they are zero, since V is the direct sum of W's this will show $\mathfrak{c} = 0$. Since we are in characterstic zero, it is enough to show the trace of each homothety is 0.

By the following lemma, we have

$$\Lambda^m W \subset W^{\otimes m} \subset V^{\otimes m} = V_{m,0}.$$

$\mathfrak{g}$, being semisimple, has no nontrivial one-dimensional module, so $\Lambda^m W$ is killed by $\mathfrak{g}$, hence killed by $\mathfrak{c}$, hence $\operatorname{Tr}_W(x) = 0$ for $x \in \mathfrak{g}$. □

Lemma 4.6

Let $\mathfrak{g}$ be a Lie algebra and W a $\mathfrak{g}$-module of dimension m. Then $\Lambda^m W$ is $\mathfrak{g}$-stable as a quotient space of $W^{\otimes m}$ and $x \in \mathfrak{g}$ acts on $\Lambda^m W$ by scaling by $\operatorname{Tr}_W(x)$.

Corollary 4.7

Let $\mathfrak{g} \subset \operatorname{End} V$ be semisimple. Let $x \in \mathfrak{g}$ and write $x = n + s$ the canonical decomposition. Then $n, s \in \mathfrak{g}$ and for any $\phi \in \operatorname{Hom}_{\mathbb{Q}}(k, k)$, $\phi(s) \in \mathfrak{g}$.

Proof. Any element in $V_{p,q}$ which is killed by $\mathfrak{g}$ is also killed by x so also by $n, s, \phi(s)$. $\square$

Definition 4.8

Let $\mathfrak{g}$ be a semisimple Lie algebra. $x \in \mathfrak{g}$ is called semisimple if $\operatorname{ad} x$ is semisimple.

Theorem 4.9

If $\mathfrak{g}$ is semisimple, then every $x \in \mathfrak{g}$ has a canonical decomposition $x = n + s$ with $n \in \mathfrak{g}$ nilpotent, $s \in \mathfrak{g}$ semisimple, and $[n, s] = 0$.

Theorem 4.10

Let $\phi \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie homomorphism between semisimple Lie algebras. If $x \in \mathfrak{g}_1$ is semisimple, then so is $\phi(x)$.

Proof. $\mathfrak{g}_2$ may be viewed as a $\mathfrak{g}_1$-module, so if V is the product of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as $\mathfrak{g}_1$-modules, then any $x \in \mathfrak{g}_1$ can be written as $x = n + s$. If x is semisimple then $n = 0$, so $\phi(x)$ is semisimple. $\square$

5 Compact Lie groups

Theorem 5.1

Let G be a connected compact complex Lie group. Then $\mathfrak{g}$ is a complex torus; that is, of the form $\mathbb{C}^n/\Gamma$ where Γ is a discrete rank $2n$ subgroup of $\mathbb{C}^n$.

Proof. By the maximum principle, there is no nonconstant analytic function on G, so no nonconstant analytic map $G \to \operatorname{End}_{\mathbb{C}} \mathfrak{g} \cong \mathbb{C}^{n^2}$ where $n = \dim \mathfrak{g}$ is the dimension of the Lie algebra of G. The inner automorphism $x \mapsto gxg^{-1}$ induced by $g \in G$ induces an automorphism $\operatorname{Ad} g \in \operatorname{Aut} \mathfrak{g}$. Then $g \mapsto \operatorname{Ad} g \in \mathbb{C}^{n^2}$ is analytic, hence constant, so $\operatorname{Ad} g = \operatorname{Ad} 1 = 1$. For x near zero in $\mathfrak{g}$,

$$g(\exp x)g^{-1} = \exp(\operatorname{Ad} g(x)),$$

and since the exponential map is a homeomorphism near 0, we conclude that G is locally abelian. By connectedness, it is abelian, so its universal covering is $\mathbb{C}^n$ and $G \cong \mathbb{C}^n/\Gamma$ as desired. $\square$

Theorem 5.2

Let G be a compact Lie group over $\mathbb{R}$ with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g} \cong \mathfrak{c} \times \mathfrak{s}$ where $\mathfrak{c}$ is abelian and $\mathfrak{s}$ is semisimple with negative definite Killing form.

Proof. As above, G acts on $\mathfrak{g}$ by Ad and since G is compact there exists a positive definite quadratic form on $\mathfrak{g}$ which is left-fixed by G, hence by $\mathfrak{g}$. This shows $\mathfrak{g}$ is completely reducible as a $\mathfrak{g}$-module, so it is the direct sum of minimal nonzero ideals $\mathfrak{a}_i$ and thus isomorphic to the product of the $\mathfrak{a}_i$. Each $\mathfrak{a}_i$ is either simple or one-dimensional abelian, so $\mathfrak{g} \cong \mathfrak{c} \times \mathfrak{s}$. It remains to show $\mathfrak{s}$ has negative definite Killing form. Let (x, y) be the Euclidean inner product on $\mathfrak{g}$. For $x \in \mathfrak{s}$ let $u = \operatorname{ad}_{\mathfrak{s}} x$. Then for $y, z \in \mathfrak{s}$,

$$(uy, z) + (y, uz) = 0$$

so taking $z = uy$ we find that $(y, u^2 y) = -(uy, uy)$. If (y_i) is an orthonormal basis for $\mathfrak{s}$ then

$$\operatorname{Tr}_{\mathfrak{s}}(u^2) = \sum_i (y_i, u^2 y_i) = -\sum_i |uy_i|^2,$$

so if $x \neq 0$ then $u = \operatorname{ad} x \neq 0$, so $\operatorname{Tr}_{\mathfrak{s}} < 0$. $\square$

We also have a converse:

Theorem 5.3

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ and $\mathfrak{g} \cong \mathfrak{c} \times \mathfrak{s}$ with $\mathfrak{c}$ abelian and $\mathfrak{s}$ semisimple with definite Killing form. Then there exists a compact Lie group over $\mathbb{R}$ giving $\mathfrak{g}$. If $\mathfrak{c} = 0$, then any such connected G is compact.

Proof. As a compact Lie group over $\mathbb{R}$ giving $\mathfrak{c}$ we can use a torus $(\mathbb{R}/\mathbb{Z})^n$. To obtain one giving $\mathfrak{s}$, we take $\operatorname{Aut} \mathfrak{s}$, a closed subgroup of the orthogonal group fixing the Killing form of $\mathfrak{s}$. Since the Killing form is definite, $\operatorname{Aut} \mathfrak{s}$ is compact. Its Lie algebra is $\operatorname{Der} \mathfrak{s}$ which is isomorphic to $\mathfrak{s}$ by Weyl's theorem.

If $\mathfrak{c} = 0$, so $\mathfrak{g}$ is semisimple, let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We have a canonical homomorphism

$$\operatorname{Ad}\colon G \to \operatorname{Aut} \mathfrak{g}.$$

We have seen that $\operatorname{Aut} \mathfrak{g}$ is a compact Lie group with Lie algebra $\mathfrak{g}$, so Ad is étale. Then $H = \mathfrak{Im}(\operatorname{Ad})$ is a connected component of $\operatorname{Aut} \mathfrak{g}$, and

$$G / \ker(\operatorname{Ad}) = H$$

where Z is discrete, H is compact, and (H, H) is dense in H. Hence G is compact. $\square$