# SIR epidemic model

$S$ = # susceptible
$I$ = # infected
$R$ = # recovered or dead.

no distinction for mathematicians as long as they no longer spread.

**Model:** $\frac{dS}{dt} = -\beta IS$, $\frac{dI}{dt} = \beta IS - \nu I$, $\frac{dR}{dt} = \nu I$, $\beta, \nu > 0$.

- Susceptible are infected at rate $\propto IS$ (think NP in pop growth)
- infected either recover or die (const rate $\nu$)
- $N := S + I + R$ const.

**Trouble:** infected pop. increasing: $\frac{dI}{dt}(0) > 0 \iff \beta S(0) > \nu$.

originally, $S(0) \approx N$.

$$R_0 := \frac{\beta N}{\nu} \quad \text{"reproductive ratio"}.$$

by above, epidemic grows if $R_0 > 1$.

- COVID-19: $2 < R_0 < 5$
- polio: $4 < R_0 < 6$
- mumps: $10 < R_0 < 12$
- measles: $16 < R_0 < 18$.

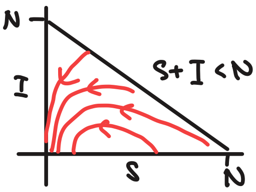
**Vaccination.** let $p$ be the fraction of pop. which is vaccinated.
then $S(0) = (1-p)N$
so effective reproductive ratio is $(1-p)R_0$.
to stop spread, vaccinate $p > (R_0 - 1)/R_0$ of pop (herd immunity)

Of course, the SIR model is WRONG. but it is not bad; in particular $R_0$ is robust in the sense that it remains "stable" under refinements of the model (hence the above ranges).

**Trajectories in phase plane:** $\frac{dI}{dS} = \frac{\frac{dI}{dt}}{\frac{dS}{dt}} = \frac{\beta IS - \nu I}{-\beta IS} = \frac{\nu}{\beta S} - 1 = \frac{N}{R_0 S} - 1$.

integrate

$$I(S) = \frac{N}{R_0} \log S - S + C.$$




max $I$ occurs when $S = N/R_0$.
indeed, effective reproductive ratio is $R_{eff} = \beta S/\nu$, epidemic recedes when $R_{eff} = 1$

**Good news:** we're not all dead ☺. $I(S) = 0$ at $S_0, S_\infty \rightsquigarrow \frac{N}{R_0}\log S_0 - S_0 = \frac{N}{R_0}\log S_\infty - S_\infty$

if $I_0 \ll N$, $S_0 \approx N$ then
$$S_\infty = \sigma N.$$
so if $R_0 < 1$, almost every is unscathed.
if $R_0$ large then $\sigma \approx e^{-R_0}$
- $R_0 \approx 10 \rightsquigarrow$ almost everyone infected!

**Limitations.**
- recovered are not immune forever, i.e. $S$ grows $\propto R$.
- people are born and die (of other causes)

say disease is nonfatal, then

$$\frac{dS}{dt} = -\beta IS + \underbrace{bN}_{\text{birth rate}} - \underbrace{\mu}_{\text{death rate}} S, \quad \frac{dI}{dt} = \beta IS - \nu I - \mu I, \quad \frac{dR}{dt} = \nu I - \mu R.$$

for simplicity $b = \mu$, so $N$ still const.

reproductive ratio : $S(0) = N$, so

$$I'(0) = \beta N - \nu - \mu \quad \text{and}$$

$$R_0 = \frac{\beta N}{\nu + \mu}.$$

dying reduces $R_0$ (:)) but usually $\nu \gg \mu$ so $R_0 \approx \frac{\beta N}{\nu}$ (:().

$\exists$ equilibrium solution (!)

$(S^*, I^*)$ with $S^* = \frac{\nu+\mu}{\beta} = \frac{N}{R_0}$,

$$I^* = \mu \frac{N - S^*}{\beta S^*} = \frac{\mu}{\beta}(R_0 - 1).$$

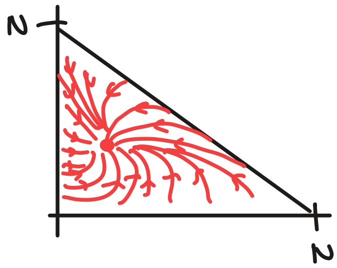
Check stability:

$$J = \begin{pmatrix} -\mu R_0 & -(\nu+\mu) \\ \mu(R_0-1) & 0 \end{pmatrix} \xrightarrow{\text{eigenvals}} \lambda = -\frac{\mu R_0}{2} \pm \frac{1}{2}\sqrt{\mu^2(R_0-2)^2 - 4\mu\nu(R_0-1)}.$$

if $\mu \ll \nu$ (usually) we have

$$\lambda \approx -\frac{\mu R_0}{2} \pm i\omega, \quad \omega = \sqrt{\mu\nu(R_0-1)}.$$

i.e. stable equilibrium




thus we observe transient oscillations with period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\mu\nu(R_0-1)}} \approx \frac{1}{\sqrt{\mu\nu}}.$$

e.g. measles. $R_0 \approx 20$, recovery period $\frac{1}{\nu} \approx 12$ days

say lifespan $\frac{1}{\mu} \approx 70$ years, then

$T \approx 2.2$ years.