Derived Functors & The Injective Revolution

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Contents

1 Introduction

1.1 A recollection of Abelian categories

Let $\mathfrak{A}$ be a *preadditive* category. That is, each hom set $\mathrm{Hom}_{\mathfrak{A}}(A,B)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$$

are bilinear; more explicitly given a diagram

$$A \xrightarrow{f} B \underset{g}{\overset{g'}{\rightrightarrows}} C \xrightarrow{h} D,$$

we have

$$h(g+g')f = hgf + hg'f \in \mathrm{Hom}(A,D).$$

We call $\mathfrak{A}$ an *additive* category if it is preadditive and finite products exist. In particular, it has a zero object (the empty product) and direct sums. This structure is more than sufficient to formulate categorical notions of kernels and cokernels, hence exact sequences and all that ensues. We call $\mathfrak{A}$ an *abelian* category if it is additive, all kernels and cokernels exist, and the natural map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism for any morphism f.

1.2 Roadmap

In the following $\mathfrak{A}$ and $\mathfrak{B}$ will be abelian categories. A functor $F\colon \mathfrak{A} \to \mathfrak{B}$ is said to be *left exact* if for any exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathfrak{A}$, the truncated sequence

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'')$$

is exact. Our goal is to define, for any left exact functor $F\colon \mathfrak{A} \to \mathfrak{B}$ to another abelian category, the right derived functors $R^iF\colon \mathfrak{A} \to \mathfrak{B}$. These should satisfy the following desiderata:

(I) $R^0F = F$.

(II) Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathfrak{A}$, there exists a natural morphism $\delta^i\colon R^iF(A'') \to R^{i+1}F(A')$ which gives a long exact sequence

$$\cdots \longrightarrow R^iF(A') \longrightarrow R^iF(A) \longrightarrow R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \longrightarrow R^{i+1}F(A) \longrightarrow \cdots$$

This should be functorial in the sense that a morphism to another short exact sequence

$$0 \to B' \to B \to B'' \to 0$$

gives a commutative diagram

$$\begin{tikzcd} R^iF(A'') \arrow[r, "\delta^i"] \arrow[d] & R^{i+1}F(A') \arrow[d] \\ R^iF(B'') \arrow[r, "\delta^i"] & R^{i+1}F(B'). \end{tikzcd}$$

We will define R^iF to be universal with respect to the above properties. We begin by considering injective objects and injective resolutions.

2 Having Enough Injectives

2.1 Injective objects

Definition 2.1

An object I of $\mathfrak{A}$ is *injective* if the functor $\operatorname{Hom}(\cdot, I)\colon \mathfrak{A}^{\mathrm{op}} \to \mathsf{Ab}$ is exact.

Namely for every short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

the sequence

$$0 \longrightarrow \mathrm{Hom}(A'', I) \longrightarrow \mathrm{Hom}(A, I) \longrightarrow \mathrm{Hom}(A', I) \to 0$$

is exact. Since $\mathrm{Hom}(\cdot, I)$ is a left exact functor irrespective of I, the nontrivial condition on I is to achieve right exactness. This may be concretely expanded as the following universal lifting property: given an injection $\iota\colon A \hookrightarrow B$ and any map $\alpha\colon A \to I$, there exists a lift $\beta\colon B \to I$ such that $\alpha = \beta \circ \iota$.

$$\begin{tikzcd} B \arrow[dr, dashed, "\exists!"] & \\ A \arrow[u, hook] \arrow[r] & I \end{tikzcd}$$

We say $\mathfrak{A}$ *has enough injectives* if every object is isomorphic to a subobject of an injective object.

2.2 Injective resolutions

Definition 2.2

An *injective resolution* of an object A of $\mathfrak{A}$ is a cochain complex $I^\bullet$ with a morphism $\epsilon\colon A \to I^0$ such that I^i is injective for all $i \geq 0$ and the sequence

$$0 \longrightarrow A \xrightarrow{\epsilon} I^0 \longrightarrow I^1 \longrightarrow \cdots$$

is exact.

The following lemma shows that every object has an injective resolution, and this injective resolution is unique up to homotopy:

Lemma 2.3

Let $\mathfrak{A}$ be an abelian category which has enough injectives.

(1) Any object of $\mathfrak{A}$ has an injective resolution.

(2) Let $A \to I^\bullet$ be an injective resolution of A and $f\colon A \to B$ a map in $\mathfrak{A}$. Then for every resolution $M \to J^\bullet$, there is a cochain map $F\colon J^\bullet \to I^\bullet$ lifting f, and F is unique up to cochain homotopy equivalence.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & J^0 \arrow[r] \arrow[d, "\exists"] & J^1 \arrow[r] \arrow[d, "\exists"] & J^2 \arrow[r] \arrow[d, "\exists"] & \cdots \\
0 \arrow[r] & N \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

Proof. (1) Let $\epsilon\colon A \to I^0$ be an injection into an injective object. Then choose an injection $\operatorname{coker}(\epsilon) \to I^1$ into an injective object, which induces a map $d^0\colon I^0 \to I^1$. Next, choose an injection $I^1/\operatorname{im}(d^0) \to I^2$ into an injective object, which induces a map $d^1\colon I^1 \to I^2$. Continuing in this way, we obtain an injective resolution for A.

(2) In degree 0, since $M \to J^0$ is injective and I^0 is an injective object, the composition

$$M \xrightarrow{f} N \longrightarrow I^0$$

induces a map $J^0 \to I^0$ making the first square commute. Inductively, we get a cochain map lifting f. Uniqueness is just another diagram chase by induction.

polish

$\square$

3 Derived functors

Let $\mathfrak{A}$ be an abelian category with enough injectives. Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a left exact functor. For each $A \in \operatorname{Obj}\mathfrak{A}$, let $I^\bullet$ be an injective resolution of A. Then we define the *right derived functors* R^iF by

$$R^iF(A) = H^iF(I^\bullet),$$

the ith cohomology of the chain complex $F(I^\bullet)$. For the sake of my sanity, I won't type out the proofs of the following results.

Theorem 3.1

Let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a left exact functor to another abelian category $\mathfrak{B}$.

(1) R^iF is an additive functor $\mathfrak{A} \to \mathfrak{B}$, independent up to natural isomorphism of the choice of injective resolution.

(2) There is a natural isomorphism $F \cong R^0F$.

(3) For a short exact sequence $0 \to A' \to A \to A'' \to 0$ any $i \geq 0$, there is a natural morphism $\delta^i\colon R^iF(A'') \to R^{i+1}F(a')$ which yields a long exact sequence

$$\cdots \longrightarrow R^iF(A') \longrightarrow R^iF(A) \longrightarrow R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \longrightarrow R^{i+1}F(A) \longrightarrow \cdots$$

(4) Given a morphism of the exact sequence in (c) to another $0 \to B' \to B \to B'' \to 0$, the following diagram commutes:

$$\begin{array}{ccc} R^iF(A'') & \xrightarrow{\delta^i} & R^{i+1}F(A') \\ \downarrow & & \downarrow \\ R^iF(B'') & \xrightarrow{\delta^i} & R^{i+1}F(B'). \end{array}$$

(5) For each injective I in $\mathfrak{A}$ and $i > 0$, we have $R^iF(I) = 0$.

Proof. (1) The fact that R^iF is an additive functor is straightforward. Any two injective resolutions are homotopy equivalent, so the cohomology groups will be isomorphic.

(2) Since $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact, we have

$$R^0F(A) \cong F(A).$$

(3) This is a diagram chase which we will not show.

(4) Another diagram chase.

(5) We can use the injective resolution $0 \to I \to I \to 0 \to 0$ which has $H^i = 0$ for $i > 0$. □

diagram chasing

An object J of $\mathfrak{A}$ satisfying the conclusion of (e) is called *F-acyclic*. An *F-acyclic resolution* of A is an exact sequence

$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$$

where each J^i is F-ayclic.

Proposition 3.2

Let $0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$ be an F-acyclic resolution of A. Then for each $i \geq 0$ there exists a natural isomorphism $R^iF(A) \cong H^i(F(J^\bullet))$.

Proof. Lemma 2.3 gives a map of cochain complexes which induces a map on cohomology. □

Everything we have mentioned thus far dualizes to projective objects (cf. projective modules), projective resolutions, having enough projectives, and left derived functors of right exact functors. Also, we can define the right derived functors of a left exact contravariant functor (using projective resolutions) and the left derived functors of a right exact contravariant functor (using injective resolutions).

We now investigate the universality of derived functors. In fact, we will generalize to the following:

Definition 3.3

A δ-*functor* from $\mathfrak{A}$ to $\mathfrak{B}$ is a collection of functors $(T^i)_{i\geq 0}$ satisfying desideratum (II).

A δ-functor $(T^i) : \mathfrak{A} \to \mathfrak{B}$ is said to be *universal* if, for any δ-functor $(S^i) : \mathfrak{A} \to \mathfrak{B}$ and any natural transformation $f^0 : T^0 \to S^0$, there exists a unique sequence of morphisms $f^i : T^i \to S^i$ starting with f^0 which commutes with the δ^i for each short exact sequence.

Note that if $F : \mathfrak{A} \to \mathfrak{B}$ is an additive functor, then there exists at most one universal δ-functor with $T^0 = F$.

Definition 3.4

An additive functor $F : \mathfrak{A} \to \mathfrak{B}$ is *effaceable* if for each A in $\mathfrak{A}$, there exists a monomorphism $u : A \to M$ such that $F(u) = 0$.

Theorem 3.5

Let (T^i) be a δ-functor from $\mathfrak{A}$ to $\mathfrak{B}$. If T^i is effaceable for $i > 0$, then T is universal.

Corollary 3.6

Let $\mathfrak{A}$ have enough injectives. Then for any left exact functor $F : \mathfrak{A} \to \mathfrak{B}$, the derived functors (R^iF) form a universal δ-functor with $F \cong R^0F$. Conversely if (T^i) is a universal δ-functor, then T^0 is left exact and the T^i are isomorphic to R^iT^0 for $i \geq 0$.

prove it