1 Motivation

Let $(X, \mathcal{O}_X)$ be a scheme. The global sections functor is left exact, meaning that for a short exact sequence
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$
of quasicoherent sheaves, the following sequence
$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H})$$
is exact, but the last map is not in general surjective. For example if $X = \mathbb{P}^1_k$, then applying global sections to the short exact sequence
$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(1)^{\oplus 2} \longrightarrow \mathcal{O} \longrightarrow 0$$
gives
$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow k,$$
and the last map is clearly not surjective. One of the raisons-d'être of sheaf cohomology is to measure to what extent $\Gamma(X, \cdot)$ fails to be right exact. The "shape" of the answer will be a sequence of additive functors $H^i \colon \mathsf{QCoh}(X) \to \mathsf{Ab}$ such that for every short exact sequence
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$
of quasicoherent sheaves, there exist $\delta^i \colon H^i(\mathcal{H}) \to H^{i+1}(\mathcal{F})$ producing a long exact sequence
$$\cdots \longrightarrow H^i(\mathcal{H}) \xrightarrow{\delta^i} H^{i+1}(\mathcal{F}) \longrightarrow H^{i+1}(\mathcal{G}) \longrightarrow \cdots$$
which is functorial, $H^0 \cong \Gamma$, and universal with respect to these properties. Why should we care about surjective morphisms of quasicoherent sheaves? We begin with some motivating examples.

1.1 Maps to projective space

Let X be a k-scheme and V be a k-vector space. Let $\mathbb{P}V = \operatorname{Proj} \operatorname{Sym} V^\vee$. Then by the universal property of Proj, a map $f \colon X \to \mathbb{P}V$ is the same as a surjective morphism $V \otimes \mathcal{O}_X \to \mathcal{L} \to 0$, where $\mathcal{L}$ is the pullback. For example if $V = \Gamma(X, \mathcal{L})$, we are interested in determining when the map $\operatorname{ev} \colon \Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. For a closed point $x \in X$, let $\mathcal{I}_x$ denote the ideal sheaf. We can check surjectivity on stalks, leading us to consider the sequence
$$0 \longrightarrow \mathcal{I}_x \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}\big|_x \longrightarrow 0$$
which becomes, under global sections,
$$0 \longrightarrow \Gamma(\mathcal{I}_x \otimes \mathcal{L}) \longrightarrow \Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L}\big|_x) \xrightarrow{\delta^0} H^1(\mathcal{I}_x \otimes \mathcal{L}).$$

So ev is surjective if $H^1(\mathcal{I}_x \otimes \mathcal{L}) = 0$ for all closed points $x \in X$.

Another interesting question is when $X \to \mathbb{P}\Gamma(X, \mathcal{L})$ injective. In terms of closed points, it is sufficient to demand that $\Gamma(X, \mathcal{L}) \longrightarrow \Gamma(\mathcal{L}_x \oplus \mathcal{L}_y)$ be surjective. That is,

$$0 \longrightarrow \mathcal{I}_{x \cup y} \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_x \oplus \mathcal{L}_y \longrightarrow 0$$

gives the long exact sequence

$$0 \longrightarrow \Gamma(\mathcal{I}_{x \cup y} \otimes \mathcal{L}) \longrightarrow \Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L}_x \oplus \mathcal{L}_y) \xrightarrow{\delta^0} H^1(\mathcal{I}_{x \cup y} \otimes \mathcal{L}).$$

A sufficient condition would then be $H^1(\mathcal{I}_{x \cup y} \otimes \mathcal{L}) = 0$ for all closed points $x, y \in X$.

Finally, when is $X \to \mathbb{P}\Gamma(X, \mathcal{L})$ an immersion? Using homogeneous coordinates, we need $\bigoplus \Gamma(X, \mathcal{L}^{\otimes n})$ to be generated in degree 1. That is, $\operatorname{Sym} \Gamma(X, \mathcal{L}) \to \bigoplus \Gamma(X, \mathcal{L}^{\otimes n})$ should be surjective. Since $\mathcal{L}$ is locally free, tensoring the sequence

$$0 \longrightarrow \ker \longrightarrow \Gamma(X, \mathcal{L}) \otimes \mathcal{O} \longrightarrow \mathcal{L} \longrightarrow 0$$

gives

$$0 \longrightarrow \ker \otimes \mathcal{L} \longrightarrow \Gamma(X, \mathcal{L}) \otimes \mathcal{L} \longrightarrow \mathcal{L}^{\otimes 2} \longrightarrow 0$$

and then applying global sections,

$$0 \longrightarrow \Gamma(\ker \otimes \mathcal{L}) \longrightarrow \Gamma(X, \mathcal{L})^{\otimes 2} \longrightarrow \Gamma(\mathcal{L}^{\otimes 2}) \xrightarrow{\delta^0} H^1(\ker \otimes \mathcal{L}).$$

Once again, a sufficient condition is $H^1(\ker \otimes \mathcal{L}) = 0$.

1.2 A source of invariants for the classification of varities

Let X be a proper k-scheme. In this case, it will be a theorem that $H^i(X, \mathcal{F})$ is finite-dimensional for $\mathcal{F}$ coherent. Cohomology is a common source of invariants. For example,

- Let X be a smooth curve. Then its genus is $\dim H^1(X, \mathcal{O}_X)$.
- In general, $\dim H^i(X, \mathcal{O}_X)$ is a birational invariant for any smooth X.
- Let X be smooth. Then $\dim H^0(X, (\Lambda^{\dim X} \Omega^1_X)^{\otimes n})$ are birational invariants called plurigenera.
- Let X be smooth. Then $\dim H^p(X, (\Lambda^{\dim X} \Omega^1_X)^{\otimes n})$ are called Hodge numbers.
- Let $\mathcal{F}$ be a coherent sheaf. Then $\chi(X, \mathcal{F}) = \sum_i (-1)^i \dim H^i(X, \mathcal{F})$ is the Euler characteristic.

1.3 Intersection theory

Let X be a smooth projective surface and C, D curves in X. Then $C \cdot D$, which can be thought of as the points of intersection, is defined as

$$\chi(\mathcal{I}_C \mathcal{I}_D) - \chi(\mathcal{I}_C) - \chi(\mathcal{I}_D) + \chi(\mathcal{O}_X).$$

1.4 Hodge theory

Let X be a smooth projective complex variety. There are deep relationships between the cohomology of coherent sheaves on X and the singular cohomology of the complex manifold naturally associated to X.

2 Derived Functors

2.1 Abelian categories

A category $\mathcal{C}$ is called *abelian* if

(a) Each hom set possesses the structure of an abelian group with respect to which composition is bilinear.

(b) It has a zero object.

(c) It has a biproduct; that is, finite coproducts and products exist and the natural map $A \sqcup B \to A \times B$ is an isomorphism.

(d) Kernels and cokernels exist.

(e) For $f \colon A \to B$, the natural map

$$\operatorname{coker}(\ker f \to A) \longrightarrow \ker(B \to \operatorname{coker} f)$$

is an isomorphism.

It turns out that the group structure on hom sets can be recovered from the biproduct, so being abelian is a property, not an additional datum.

Example 2.1.1. R-Mod and $\mathcal{O}_X$-Mod are abelian categories.

Counterexample 2.1.2. Free abelian groups are not an abelian category.

Counterexample 2.1.3. Hausdorff topological groups are not an abelian category. For example $\mathbb{Q} \to \mathbb{R}$ has image $\mathbb{Q}$ but coimage $\mathbb{R}$. Also $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ has image $2\mathbb{Z}$ but coimage $\mathbb{Z}$.

2.2 δ-functors

Let F be a left exact additive functor; that is, F preserves left exact sequences and F is a group homomorphism on hom sets.

Definition 2.2.1

A δ-functor is a sequence of additive functors $T^i\colon A \to B$ such that for any short exact sequence $0 \to U \to V \to W \to 0$ in A, there exist morphisms $\delta^i\colon T^i(W) \to T^{i+1}(U)$ such that

$$\cdots \longrightarrow T^i(W) \xrightarrow{\delta^i} T^{i+1}(U) \longrightarrow T^{i+1}(V) \longrightarrow \cdots$$

is a long exact sequence in B and δ^i is functorial in the following sense: given a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & U' \arrow[r] & V' \arrow[r] & W' \arrow[r] & 0, \end{tikzcd}$$

the following square commutes:

$$\begin{tikzcd} T^i(W) \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(U) \arrow[d] \\ T^i(W') \arrow[r, "\delta^i"] & T^{i+1}(U'). \end{tikzcd}$$

We say T^i extends F if $T^0 = F$.

We define the category $\delta\text{-}\mathsf{Fun}(A,B)$ to be the category whose objects are δ-functors and whose morphisms are natural transformations $t^i\colon T^i \to S^i$ such that for any short exact sequence

$$0 \to U \to V \to W \to 0$$

in A, the following square commutes:

$$\begin{tikzcd} T^i(W) \arrow[r, "\delta^i"] \arrow[d, "t^i"] & T^{i+1}(U) \arrow[d, "t^{i+1}"] \\ S^i(W) \arrow[r, "\delta^i"] & S^{i+1}(U). \end{tikzcd}$$

We call a δ-functor T^i *universal* if $\mathrm{Hom}_{\delta\text{-}\mathsf{Fun}(A,B)}(T^i, S^i) \cong \mathrm{Hom}_{\mathsf{Fun}(A,B)}(T^0, S^0)$, and *effaceable* if for each $U \in A$, there exists a monomorphism $0 \longrightarrow U \xrightarrow{u} V$ such that $T^i(u) = 0$ for $i > 0$.

Proposition 2.2.2

Let T^i be an effaceable δ-functor. Then T^i is a universal δ-functor.

Proof. Let S^i be a δ-functor and $t^0\colon T^0 \to S^0$ a natural transformation. By induction on n, we construct natural transformations t^i for $i \leq n$ which are compatible with the δ^i

for $i < n$. Given $U \in A$, pick a monomorphism $u\colon U \to V$. Let $W = \operatorname{coker} u$. The short exact sequence

$$0 \longrightarrow U \xrightarrow{u} V \longrightarrow \operatorname{coker} u \longrightarrow 0$$

gives a long exact sequence

$$\cdots \longrightarrow T^n(U) \longrightarrow T^n(V) \longrightarrow T^n(W) \xrightarrow{\delta^n} T^{n+1}(U) \xrightarrow{0} T^{n+1}(V) \longrightarrow \cdots,$$

which shows that $T^{n+1}(U) \cong \operatorname{coker}(T^n(V) \longrightarrow T^n(W))$. Consider the commutative diagram

$$\begin{tikzcd}
T^n(V) \arrow[r] \arrow[d, "t^n"] & T^n(W) \arrow[r, "\delta^n"] \arrow[d, "t^n"] & T^{n+1}(U) \cong \operatorname{coker}(T^n(V) \to T^n(W)) \\
S^n(V) \arrow[r] & S^n(W) \arrow[r, "\delta^n"] & S^{n+1}(U) \cong \operatorname{coker}(S^n(V) \to S^n(W)).
\end{tikzcd}$$

By the universal property of $\operatorname{coker}(T^n(V) \to T^n(W))$, there exists a unique vertical arrow making the following diagram commute:

$$\begin{tikzcd}
T^n(W) \arrow[r, "\delta^n"] \arrow[d, "t^n"] & T^{n+1}(U) \arrow[d, dashed, "\exists!"] \\
S^n(W) \arrow[r, "\delta^n"] & S^{n+1}(U).
\end{tikzcd}$$

We define $t_U^{n+1}\colon T^{n+1}(U) \to S^{n+1}(U)$ to be this vertical arrow, which is compatible with δ^n by construction. It remains to show that t^{n+1} defines a natural transformation $T^{n+1} \to S^{n+1}$. A morphism $U \to U'$ yields a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U' \arrow[r] & V' \arrow[r] & W' \arrow[r] & 0,
\end{tikzcd}$$

since $U' \to V'$ is mono. The following cube commutes:

$$\begin{tikzcd}
& T^n(W) \arrow[rr] \arrow[dl, "\delta^n"'] \arrow[dd, "t^n" near start] & & T^n(W') \arrow[dl, "\delta^n"'] \arrow[dd, "t^n"] \\
T^{n+1}(U) \arrow[rr, crossing over] \arrow[dd, "t^{n+1}"'] & & T^{n+1}(U') & \\
& S^n(W) \arrow[rr] \arrow[dl, "\delta^n"] & & S^n(W') \arrow[dl, "\delta^n"] \\
S^{n+1}(U) \arrow[rr] & & S^{n+1}(U') \arrow[from=uu, crossing over, "t^{n+1}" near start] &
\end{tikzcd}$$

Indeed, the rear face commutes by naturality of t^n, the top and bottom faces commute by functoriality of δ^n, the left and right faces commute by compatibility of t^n with δ^n, and the δ^n are all epimorphisms. The front face is the desired naturality of t^{n+1}. $\square$

We have thus reduced our search for universal δ-functors to a search for effaceable δ-functors. This line of inquiry leads us to injective objects. We say $I \in A$ is *injective* if it satisfies the following universal lifting property: given an injection $X \to Y$ and a map $X \to I$, there exists a lift $Y \to I$.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[dl, dashed, "\exists"] \\ & I. & \end{tikzcd}$$

Proposition 2.2.3

I is satisfies the universal lifting property if and only if $\operatorname{Hom}(\cdot, I)$ is exact.

Proof. ($\implies$) Let $0 \to U \xrightarrow{u} V \xrightarrow{v} W \to 0$ be a short exact sequence. We want to show

$$0 \to \operatorname{Hom}(W, I) \to \operatorname{Hom}(V, I) \to \operatorname{Hom}(U, I) \to 0$$

is exact. Since $\operatorname{Hom}(\cdot, I)$ is always left exact, it remains to show $\operatorname{Hom}(V, I) \to \operatorname{Hom}(U, I)$ given by precomposing with u is surjective. This is just the lifting property.

($\impliedby$) Let $u \colon U \to V$ and let $\alpha \colon U \to I$ be given. Since $f \mapsto f \circ u$ is a surjection $\operatorname{Hom}(V, I) \to \operatorname{Hom}(U, I)$, there exists $\beta \colon V \to I$ such that $\beta \circ u = \alpha$. $\square$

Example 2.2.4. The divisible abelian groups; that is, $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, are injective. In fact, every injective in Ab is a direct sum of these.

Proposition 2.2.5

Let T^i be effaceable. Let I be injective. Then $T^i(I) = 0$ for $i > 0$.

Proof. Since T^i is effaceable, there exists a monomorphism $0 \longrightarrow I \xrightarrow{u} V$ such that $T^i(u) = 0$ for $i > 0$. Write a short exact sequence

$$0 \longrightarrow I \xrightarrow{u} V \longrightarrow W \longrightarrow 0.$$

The identity $I \to I$ has a lift $s \colon V \to I$, so for $i > 0$,

$$0 = T^i(s) \circ T^i(u) = T^i(s \circ u) = T^i(\mathrm{id}_I) = \mathrm{id}_{T^i(I)}\,.$$

Hence $T^i(I) = 0$ for $i > 0$. $\square$

We say A has enough injectives if for any $A \in U$, there exists a monomorphism $0 \to U \to I$. An *injective resolution* of U is an exact sequence $0 \to U \to I^0 \to I^1 \to \cdots$. We denote the chain complex by $I^\bullet = (I^n)_{n \in \mathbb{N}}$. If A has enough injectives, then every object has an injective resolution. Indeed, pick a monomorphism $0 \to U \to I^0$, then complete the exact sequence

$$0 \longrightarrow U \longrightarrow I^0 \longrightarrow W^0 \longrightarrow 0.$$

Then repeat the process with W^0 to get

$$0 \longrightarrow W^0 \longrightarrow I^1 \longrightarrow W^1 \longrightarrow 0,$$

and so on. This will be our recipe for computing effaceable δ-functors. More precisely, given a left exact functor $F\colon A \to B$, let $R^iF(U) = H^i(F(I^\bullet))$, where $I^\bullet$ is an injective resolution of U. We first show this is well-defined.

Theorem 2.2.6

R^iF is a universal δ-functor such that $R^0F \cong F$.