Representations of $\mathfrak{sl}_n$

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Let k be an algebraically closed field of characteristic zero. All Lie algebras and modules will be finite-dimensional over k. Let $n \geq 2$, and let

$$\mathfrak{g} = \mathfrak{sl}_n := \{x \in M_n(k) : \operatorname{Tr}(x) = 0\}.$$

The center of $\mathfrak{g}$ is zero and k^n is irreducible as a $\mathfrak{g}$-module so $\mathfrak{g}$ is semisimple. Let $\mathfrak{h}$ denote the Lie algebra of trace zero diagonal matrices, let $\mathfrak{n}_+$ denote the Lie algebra of strictly upper triangular matrices, and $\mathfrak{n}_-$ the Lie algebra of strictly lower triangular matrices. We have a direct sum decomposition of vector spaces

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{g} \oplus \mathfrak{n}_+.$$

Moreover $\mathfrak{h}$ is abelian and $\mathfrak{n}_+$, $\mathfrak{n}_-$ are nilpotent. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$; this is a solvable subalgebra of $\mathfrak{g}$ and its derived algebra is $\mathfrak{n}_+$. An element $\chi \in \mathfrak{h}^*$ of the dual can be written as

$$\chi(\lambda_1, \ldots, \lambda_n) = u_1\lambda_1 + \cdots + u_n\lambda_n \quad \text{for} \quad u_i \in k.$$

Let R_+ be the subset of $\mathfrak{h}^*$ consisting of the linear forms $\lambda_i - \lambda_j$ for $i < j$ and let $R = R_+ \cup (-R_+)$. An element α of R $[R_+]$ is called a [positive] *root.* The roots

$$\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2 - \lambda_3, \ldots, \alpha_{n-1} = \lambda_{n-1} - \lambda_n$$

are called *fundamental roots*. Any positive root $\alpha = \lambda_i - \lambda_j$, i.e. $i < j$,is a sum of fundamental roots. Each roots $\alpha = \lambda_i - \lambda_j$ yields two elements X_α, H_α where $X_\alpha = (\delta_{ij}) \in \mathfrak{n}_+ \cup \mathfrak{n}_-$ and $H_\alpha \in \mathfrak{h}$ is diagonal with ith entry 1, jth entry -1, and others zero. We have $\alpha(H_\alpha) = 2$.

Proposition 0.1 (1) $(X_\alpha)_{\alpha \in R_+}$ is a basis for $\mathfrak{n}_+$ and $(X_{-\alpha})_{\alpha \in R_+}$ is a basis for $\mathfrak{n}_-$.

(2) If $H \in \mathfrak{h}$, then $[H, X_\alpha] = \alpha(H)X_\alpha$.

(3) $[X_\alpha, X_{-\alpha}] = H_\alpha$

Proof. (1) is clear. For (b), let $H = (\lambda_1, \ldots, \lambda_n)$. If $\alpha = \lambda_i - \lambda_j$, then

$$H \cdot X_\alpha = \lambda_i \cdot X_\alpha$$

and $X_\alpha \cdot H = \lambda_j \cdot X_\alpha$. Hence

$$[H, X_\alpha] = (\lambda_i - \lambda_j)X_\alpha = \alpha(X)X_\alpha.$$

The proof of (c) is similar. $\square$

1 Weights and primitive elements

Let V be a $\mathfrak{g}$-module. If $\chi \in \mathfrak{h}^*$, let

$$V_\chi = \{v \in V : Hv = \chi(X)v \text{ for all } H \in \mathfrak{h}\}.$$

We call $v \in V_\chi$ an *eigenvector* of $\mathfrak{h}$ of weight χ.

Proposition 1.1

Let $\alpha \in R$, $v \in V_\chi$. Then $X_\alpha \cdot v \in V_{\chi+\alpha}$.

Proof.

$$\begin{aligned} HX_\alpha v &= [H, X_\alpha]v + X_\alpha Hv \\ &= \alpha(H)X_\alpha v + X(H)X_\alpha v \\ &= (\chi + \alpha)(H)X_\alpha v, \end{aligned}$$

so $V_\alpha v$ is an eigenvector of weight $\chi + \alpha$. $\square$

Proposition 1.2

V is the direct sum of the V_χ for $\chi \in \mathfrak{h}^*$.

Proof. Nonzero eigenvectors of distinct eigenvalues are linearly independent, by the usual proof. Hence $\sum_{\chi\in\mathfrak{h}^*} V_\chi$ is a direct sum. The previous proposition shows this decomposition is stable by X_α, and thus stable by $\mathfrak{g}$. By complete reducibility, V is the direct sum of W and another submodule V'. If $V' \neq 0$, then the fact that $\mathfrak{h}$ is abelian and k algebraically closed implies that there exists a nonzero eigenvector $v \in V'$ of $\mathfrak{h}$. This is contained in some V_χ, contradicting $V' \cap W = 0$. Hence $V' = 0$. $\square$

The χ for which $V_\chi \neq 0$ are called the *weights* of V. The *multiplicity* of χ is $\dim V_\chi$.

Since $\mathfrak{n}_+ = [\mathfrak{b}, \mathfrak{b}]$ and the X_α for $\alpha \in R_+$ form a basis for $\mathfrak{n}_+$, we know that $v \in V$ is an eigenvector for $\mathfrak{h}$ with $X_\alpha v = 0$ if and only if it is an eigenvector for $\mathfrak{b}$. We call such a nonzero v *primitive.*

By Lie's theorem for the $\mathfrak{b}$-module V, every nonzero $\mathfrak{g}$-module contains a primitive element.

2 Irreducible $\mathfrak{g}$-modules

Theorem 2.1

Let V be a $\mathfrak{g}$-module and let $v \in V$ be a primitive element of weight χ. Let $V_1 = (U\mathfrak{g}) \cdot v$ be the $\mathfrak{g}$-submodule of V generated by v. Then

(1) V_1 is irreducible.

(2) The weights of V_1 are of the form $\chi - \sum_{i=1}^{n-1} m_i \alpha_i$ with $m_i \geq 0$.

(3) Any element of V_1 of weight χ is a multiple of v

Proof. Write $U\mathfrak{g} = U\mathfrak{n}_- \otimes U\mathfrak{b}$. Since v is an eigenvector of $\mathfrak{b}$, $(U\mathfrak{b})v = kv$ hence $V_1 = (U\mathfrak{g})v = (U\mathfrak{n}_-)v$. By the Birkhoff–Witt theorem, V_1 is generated by elements of the form Mv where M is a monomial in the $X_{-\alpha}$ for $\alpha \in R_+$. THen Mv are eigenvectors of $\mathfrak{h}$ of weight $\chi - \sum_{\alpha>0} q_\alpha \alpha$ with $q_\alpha \geq 0$, which implies (2). Claim (3) follows from the fact that the q_α are all zero only if M has degree 0, so $Mv = v$. For (1), suppose $V_1 = V' \oplus V''$ and write $v = v' + v''$. Since

$$(V_1)_\chi = V'_\chi \oplus V''_\chi,$$

v' and v'' are both of weight χ; (3) then shows that they are multiples of v so one of the must be zero. Since v generates V_1, one of V', V'' must be 0. □

Theorem 2.2

Let V be an irreducible $\mathfrak{g}$-module. Then V contains a unique primitive element up to scaling; its weight is called the highest weight of V. Two irreducible $\mathfrak{g}$-modules with the same highest weight are isomorphic.

Proof. V contains at least one primitive element v; let χ be its weight. Let v' be another primitive with weight χ'. Since V is irreducible, v generates V, so

$$\chi - \chi' = \sum_{i=1}^{n-1} m_i \alpha_i.$$

Similarly the right hand side is equal to $\chi' - \chi$, which must then be 0, showing (1) by part (3) of the previous theorem.

Now let V_1, V_2 be two irreducible $\mathfrak{g}$-moduels with primitives v_1, v_2 of weight χ. Then $v = (v_1, v_2) \in V_1 \oplus V_2$ is primitive of weight χ, so the $\mathfrak{g}$-submodule W it generates is irreducible. The projection $\pi_i \colon W \to V_i$ is nonzero, hence an isomorphism by irreducibility. Thus $V_1 \cong W \cong V_2$. □

3 Determination of the highest weights

The last results show that to classify irreducible $\mathfrak{g}$-modules, it suffices to determine the elements $\chi \in \mathfrak{h}^*$ which are highest weights.

Theorem 3.1

Let $\chi \in \mathfrak{h}^*$ be given by

$$\chi(\lambda_1, \dots, \lambda_n) = u_1\lambda_1 + \cdots + u_n\lambda_n.$$

There exists an irreducible $\mathfrak{g}$-module with highest weight χ if and only if $u_i - u_j$ is a positive integer for $i < j$.

($\Longrightarrow$) If α is the positive root $\lambda_i - \lambda_j$, then $u_i - u_j = \chi(H_\alpha)$. We claim that $\chi(H_\alpha)$ is a positive integer if χ is a highest weight. We proceed as follows:

Proposition 3.2

Let v be a primitive element with weight χ and let $v_m^\alpha = (X_{-\alpha})^m v/m!$, where $(X_{-\alpha})^m$ is the mth iterate. Then

(1) $X_{-\alpha}v_m^\alpha = (m+1)v_{m+1}^\alpha$.

(2) $Hv_m^\alpha = (\chi - m\alpha)(H)v_m^\alpha$ for $H \in \mathfrak{h}$.

(3) $X_\alpha v_m^\alpha = (\chi(H_\alpha) - m + 1)v_{m-1}^\alpha$.

Proof. (1) is clear, (ii) holds as v_m^α has weight $\chi - m\alpha$, (iii) is by induction on m: the base case $m = 0$ is trivial. Then

$$mX_\alpha v_m^\alpha = X_\alpha X_{-\alpha} v_{m-1}^\alpha = H_\alpha v_{m-1}^\alpha + X_{-\alpha}X_\alpha v_{m-1}^\alpha = \lambda v_{m-1}^\alpha$$

with

$$\lambda = \chi(H_\alpha) - (m-1)\alpha(H_\alpha) + (m-1)(\chi(H_\alpha) - m + 2).$$

Since $\alpha(H_\alpha) = 2$, we have $\lambda = m(\chi(H_\alpha) - m + 1)$. □

Corollary 3.3

There exists m such that $v_m^\alpha \neq 0$, $v_{m+1}^\alpha = 0$. In particular $\chi(H_\alpha) = m$.

Proof. The v_m^α have weight $\chi - m\alpha$, and the number of possible weights is finite, so v_m^α for large m, and the first statement is true. Then applying (3) from the previous proposition,

$$0 = X_\alpha v_{m+1}^\alpha = (\chi(H_\alpha) - m)v_m^\alpha$$

which implies $\chi(H_\alpha) = m$. □

($\Longleftarrow$) Let $\pi_1, \dots, \pi_{n-1}$ be the linear forms λ, $\lambda_1 + \lambda_2, \dots, \lambda_1 + \cdots + \lambda_{n-1}$. The condition that $u_i - u_j$ is a positive integer is equivalent to being able to write χ as $\chi = \sum_{i=1}^{n-1} m_i\pi_i$.

Proposition 3.4

If χ, χ' are highest weights of V, V' then $\chi + \chi'$ is highest weight of $V \otimes V'$.

Proof. If v, v' are primitives of V, V' then $v \otimes v'$ is a primitive of $V \otimes V'$ and its weight is $\chi + \chi'$. It generates an irreducible $\mathfrak{g}$-submodule with weight $\chi + \chi'$. $\square$

Hence to prove that χ is a highest weight, it is enough to prove π_i are. We exhibit this explicitly:

Proposition 3.5

View k^n as a $\mathfrak{g}$-module. For $1 \leq i \leq n-1$, the ith exterior power V_i of V is an irreducible $\mathfrak{g}$-module of highest weight π_i

Proof. Given the canonical basis $e_1, \ldots, e_n$ for V, let $v_i = e_1 \wedge \cdots \wedge e_i$. This is a primitive element of V_i of weight π_i, and by applying a monomial in $X_{-\alpha}$ we obtain any term $e_{m_1} \wedge \cdots \wedge e_{m_i}$. Hence V_i is irreducible. $\square$