1.9 Nilpotent and solvable Lie algebras

Today k will be a field, often of characteristic 0, and all Lie algebras and modules are finite-dimensional over k.

1.9.1 Complements on $\mathfrak{g}$-modules

Let $\mathfrak{g}$ be a Lie algebra over k. A $\mathfrak{g}$-module is a k-vector space with a k-bilinear map

$$\mathfrak{g} \times V \to V$$

satisfying

$$[x, y]v = xyv - yxv.$$

This data is equivalent to a Lie homomorphism $\rho\colon \mathfrak{g} \to \operatorname{End}(V)$; in other words a linear representation of $\mathfrak{g}$. We call V the *representation space.*

> **Example 1.9.1.** For any V we have the trivial representation $xv = 0$ for all $x \in \mathfrak{g}$, $v \in V$. Unless stated otherwise, we will endow k with this trivial $\mathfrak{g}$-module structure.

Let V_1, V_2 be $\mathfrak{g}$-modules. Then there exists a unique $\mathfrak{g}$-module structure on $V_1 \otimes V_2$ such that

$$x(v_1 \otimes v_2) = (xv_1) \otimes v_2 + v_1 \otimes (xv_2),$$

called the *diagonal action.* This can be seen from the composition

$$U\mathfrak{g} \xrightarrow{\Delta} U\mathfrak{g} \otimes U\mathfrak{g} \xrightarrow{\rho_1 \otimes \rho_2} \operatorname{End} V_1 \otimes \operatorname{End} V_2 \longrightarrow \operatorname{End}(V_1 \otimes V_2).$$

Similarly, there is a natural $\mathfrak{g}$-module structure on $\operatorname{Hom}_k(V_1, V_2)$ given by

$$(xf)(v_1) = x(f(v_1)) - f(xv_1) \quad \text{for} \quad x \in \mathfrak{g}, v_1 \in V_1.$$

More generally, let $V_1, \ldots, V_n, V$ be $\mathfrak{g}$-modules. Then the space of k-multilinear maps

$$\prod_{i=1}^{n} V_i \to V$$

has a natural $\mathfrak{g}$-module structure

$$(xf)(v_1, \ldots, v_n) = x(f(v_1, \ldots, v_n)) - \sum_{i=1}^{n} f(v_1, \ldots, xv_i, \ldots, v_n).$$

We say $v \in V$ is $\mathfrak{g}$-invariant if $xv = 0$ for $x \in \mathfrak{g}$. One might expect this to mean $xv = v$, but the terminology really comes from the fact that $xv = 0$ is equivalent to $v = (1 + \epsilon x)v$. The $\mathfrak{g}$-invariant elements form a $\mathfrak{g}$-submodule of V, which is the largest submodule on which $\mathfrak{g}$ acts trivially.

> **Example 1.9.2.** $f \in \operatorname{Hom}_k(V_1, V_2)$ is $\mathfrak{g}$-invariant if $f(xv_1) = xf(v_1)$; in other words f is a $\mathfrak{g}$-module homomorphism.

Example 1.9.3. A bilinear form $B\colon V_1 \times V_2 \to k$ is invariant if

$$B(xv_1, v_2) + B(v_1, xv_2) = 0.$$

If $g = 1 + \epsilon x$, this means

$$B(gv_1, gv_2) = B(v_1, v_2).$$

Let V be a $\mathfrak{g}$-module and $\rho\colon \mathfrak{g} \to \operatorname{End} V$ the associated representation. Let $B_\rho(x, y) = \operatorname{Tr}_V(\rho(x)\rho(y))$.

Proposition 1.9.4

B_ρ is a symmetric bilinear form on $\mathfrak{g}$. If $\rho\colon \mathfrak{g} \to \operatorname{End} \mathfrak{g}$ is the adjoint representation, then B_ρ is $\mathfrak{g}$-invariant.

Proof. Symmetry is clear as $\operatorname{Tr}_V(\alpha\beta) = \operatorname{Tr}_V(\beta\alpha)$. Now let ρ be the adjoint representation; we wish to show that

$$\operatorname{Tr}_V(\rho([x, x_1])\rho(x_2) + \rho(x_1)\rho([x, x_2])) = 0.$$

We expand to obtain

$$\operatorname{Tr}_V(\rho(x)\rho(x_1)\rho(x_2) - \rho(x_1)\rho(x)\rho(x_2) + \rho(x_1)\rho(x)\rho(x_2) - \rho(x_1)\rho(x_2)\rho(x)),$$

which is zero by symmetry of trace. □

We call the invariant symmetric bilinear form $B(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$ on $\mathfrak{g}$ the *Killing form.*

1.9.2 Nilpotent Lie algebras

The *descending central series* of ideals for a finite-dimensional Lie algebra $\mathfrak{g}$ over k is defined by

$$C^1\mathfrak{g} = \mathfrak{g},$$
$$C^n\mathfrak{g} = [\mathfrak{g}, C^{n-1}\mathfrak{g}].$$

Clearly $[C^r\mathfrak{g}, C^s\mathfrak{g}] \subset C^{r+s}\mathfrak{g}$. We call $\mathfrak{g}$ *nilpotent* if any of the following equivalent conditions are satisfies:

Proposition 1.9.5

The following are equivalent:

(a) There exists n such that $C^n\mathfrak{g} = 0$.

(b) $\mathfrak{g}$ is a succession of central extensions of abelian Lie algebras; that is, there exists a chain of ideal $\mathfrak{g} = \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n = 0$ such that $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is the center of $\mathfrak{g}/\mathfrak{a}_{i+1}$ for each i, or equivalently $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$.

(c) There exists n such that

$$[x_1, [x_2, [x_3, \ldots, x_n] \cdots]] = (\operatorname{ad} x_1)(\operatorname{ad} x_2)\cdots(\operatorname{ad} x_{n-1})x_n = 0$$

for all (x_i) in $\mathfrak{g}$.

Proof. $(a) \implies (b)$. Let $\mathfrak{a}_i = C^n\mathfrak{g}$. By definition these are ideals satisfyinng

$$\mathfrak{g} = \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n = 0$$

and

$$[\mathfrak{g}, \mathfrak{a}_i] = [\mathfrak{g}, C^i\mathfrak{g}] \subset C^{i+1}\mathfrak{g} = \mathfrak{a}_{i+1}.$$

$(b) \implies (c)$. For any $x_1, \ldots, x_n \in \mathfrak{g}$ we have

$$(\operatorname{ad} x_1)\cdots(\operatorname{ad} x_{k-1})x_k \in \mathfrak{a}_k$$

for any k, so

$$(\operatorname{ad} x_1)\cdots(\operatorname{ad} x_{n-1})x_n \in \mathfrak{a}_n = 0.$$

$(c) \implies (a)$. Any $x \in C^n\mathfrak{g}$ takes the form

$$(\operatorname{ad} x_1)\cdots(\operatorname{ad} x_{n-1})x_n = 0.$$

□

Example 1.9.6. Let V be a vector space and $F = (V_i)$ a flag in V; that is a sequence of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

such that $\dim V_i = i$. Let

$$\mathfrak{u}(F) = \{u \in \operatorname{End} V : uV_i \subset V_{i-1} \text{ for all } i \geq 1\}.$$

This is an associative subalgebra of $\operatorname{End} V$, hence a Lie subalgebra under $[x, y] = xy - yx$. If $\beta = \{v_1, \ldots, v_n\}$ is a basis for V such that $V_i = \operatorname{span}\{v_1, \ldots, v_i\}$, then $\mathfrak{u}(F)$ consists of the endomorphisms with strictly upper-triangular matrices with respect to β. To show $\mathfrak{u}(F)$ is nilpotent, let

$$\mathfrak{u}_k(F) = \{u \in \operatorname{End} V : uV_i \subset V_{i-k}\}$$

and note that $\mathfrak{u}\mathfrak{u}_k \subset \mathfrak{u}_{k+1}$ and $\mathfrak{u}_k\mathfrak{u} \subset \mathfrak{u}_{k+1}$, so that $[\mathfrak{u}, \mathfrak{u}_k] \subset \mathfrak{u}_{k+1}$. But eventually $\mathfrak{u}_k = 0$, so $\mathfrak{u}$ is nilpotent.

Theorem 1.9.7

$\mathfrak{g}$ is nilpotent if and only if $\operatorname{ad} x$ is nilpotent for all $x \in \mathfrak{g}$.

Theorem 1.9.8 (Engel)

Let $\rho\colon \mathfrak{g} \to \operatorname{End} V$ be a representation of $\mathfrak{g}$ such that $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$. Then there exists a flag $F = (V_i)$ in V such that $\rho(\mathfrak{g}) \subset \mathfrak{u}(F)$.

Remark 1.9.9. The converse to Engel's theorem is clear, since strictly upper-triangular matrices are nilpotent.

To unpack the statement of Engel's theorem, the hypothesis says that for each $x \in \mathfrak{g}$ there exists a flag $F_x = \{V_{x,i}\}$ such that $\rho(x)V_{x,i} \subset V_{x,(i-1)}$. The conclusion gives a single flag for which this holds simultaneously for all x. Equivalently,

Theorem 1.9.10

Let $\rho\colon \mathfrak{g} \to \operatorname{End} V$ be a representation of $\mathfrak{g}$ such that $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$. If $V \neq 0$, then there exists $v \neq 0 \in V$ such that $\rho(x)v = 0$ for all $x \in \mathfrak{g}$.

Indeed, this implies Engel's theorem by induction on $\dim V$. A flag $\overline{F}$ in $\overline{V} = V/kv$ lifts to a flag on V with this property if $\rho(\mathfrak{g})v = 0$.

Proof of Theorem 1.9.10. We proceed in 6 steps.

Step 1. We may replace $\mathfrak{g}$ be its image to assume $\mathfrak{g} \subset \operatorname{End} V$.

Step 2. $\operatorname{ad} x$ is nilpotent for each $x \in \mathfrak{g}$, so we can write

$$adx(y) = L_x y - R_x y$$

where L_x, R_x are k-linear endomorphisms of $\operatorname{End} V$ given by left and right multiplication by x, respectively. These are nilpotent by hypothesis and commute, so $L_x - R_x$ is nilpotent.

Step 3. We proceed by induction on $\dim \mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a proper Lie subalgebra. Then the normalizer $\mathfrak{u} = \{x \in \mathfrak{g} : \operatorname{ad} x(\mathfrak{h}) \subset \mathfrak{h}\}$ of $\mathfrak{h}$ is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal. We claim that $\mathfrak{a}$ is strictly larger than $\mathfrak{h}$. Indeed, $\mathfrak{h}$ acts nilpotently on $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$. Since $\dim \mathfrak{h} < \dim \mathfrak{g}$, there exists $\overline{x} = x + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ which is $\mathfrak{h}$-invariant. Then for $y \in \mathfrak{h}$ we have $\operatorname{ad} y(\overline{x}) = 0$, so

$$\operatorname{ad} x(y) = -\operatorname{ad} y(x) \in \mathfrak{h}.$$

Thus $x \in \mathfrak{u}$.

Step 4. Since $\mathfrak{g} \neq 0$ there exists an ideal $\mathfrak{h}$ of codimension 1. Indeed, let $\mathfrak{h}$ be a maximal proper Lie subalgebra of $\mathfrak{g}$. Then by the previous step, its normalizer is all of $\mathfrak{g}$ so $\mathfrak{h}$ is an ideal. The inverse image in $\mathfrak{g}$ of a line in $\mathfrak{g}/\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ strictly bigger than $\mathfrak{h}$, so it must be all of $\mathfrak{g}$, meaning $\mathfrak{g}/\mathfrak{h}$ is 1-dimensional.

Step 5. Let $W = \{v \in V : \mathfrak{h}v = 0\}$. Then W is stable by $\mathfrak{g}$, since $\mathfrak{h}$ is an ideal: for $x \in \mathfrak{g}$, $y \in \mathfrak{h}$ we have

$$yxv = xyv - [x, y]v = 0 \quad \text{for} \quad v \in W.$$

Step 6. By induction $W \neq 0$. If $y \in \mathfrak{g} \setminus \mathfrak{h}$ then it is nilpotent, so it kills some nonzero element in W. This element is thus killed by $\mathfrak{g} = \mathfrak{g} + ky$.

□

Proof of Theorem 1.9.7. If $\mathfrak{g}$ is nilpotent then $\operatorname{ad} x$ is nilpotent for all $x \in \mathfrak{g}$, by (c) in Proposition 1.9.5. Conversely if $\operatorname{ad} x$ is always nilpotent, then Engel's theorem for the adjoint representation gives a flag

$$0 \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{g}$$

such that $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i-1}$, fulfilling (b) in Proposition 1.9.5. □

1.9.3 Solvable Lie algebras

The *derived series* of ideals for a Lie algebra $\mathfrak{g}$ is

$$D^1\mathfrak{g} = \mathfrak{g}$$
$$D^n\mathfrak{g} = [D^{n-1}\mathfrak{g}, D^{n-1}\mathfrak{g}].$$

We call $\mathfrak{g}$ *solvable* if any of the following equivalent conditions are met:

Theorem 1.9.11

The following are equivalent:

(a) There exists n such that $D^n\mathfrak{g} = 0$.

(b) $\mathfrak{g}$ is a successive extension of abelian Lie algebras; that is, there exists a sequence of ideals

$$\mathfrak{g} = \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n = 0$$

such that $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is abelian for each i, or equivalently $[\mathfrak{a}_i, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$.

Proof. $(a) \implies (b)$. As with the conditions for nilpotent, let $\mathfrak{a}_i = D^i\mathfrak{g}$.
$(b) \implies (a)$. $D^i\mathfrak{g} \subset \mathfrak{a}_i$ for each i, so $D^n\mathfrak{g} = 0$. □

By conditions (b) in this definition and (b) in the definition of nilpotent, nilpotent implies solvable.

Example 1.9.12. Let $F = (V_i)$ be a flag in a finite-dimensional vector space V. Let

$$\mathfrak{b}(F) = \{x \in \operatorname{End} V : xV_i \subset V_i \text{ for all } i\}.$$

Then as with $\mathfrak{u}(F)$, with respective to an appropriate basis $\mathfrak{b}(F)$ consists of the upper triangular matrices. Since $\mathfrak{b}(F)/\mathfrak{a}(F)$ is abelian, $\mathfrak{b}(F)$ is solvable.

Theorem 1.9.13 (Lie)

Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field k of characteristic 0. Let $\rho\colon \mathfrak{g} \to \operatorname{End} V$ be a representation. Then there exists a flag $F = (V_i)$ in V such that $\rho(\mathfrak{g}) \subset \mathfrak{b}(F)$.

Similarly to Engel's theorem, by induction it suffices to show

Theorem 1.9.14

Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field k of characteristic 0. Let $\rho\colon \mathfrak{g} \to \operatorname{End} V$ be a representation. If $V \neq 0$, there exists $v \neq 0 \in V$ which is an eigenvector for each $\rho(x)$.

Our main lemma will be:

Lemma 1.9.15

Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic 0, $\mathfrak{h} \subset \mathfrak{g}$ an ideal, V a $\mathfrak{g}$-module, $v \neq 0 \in V$, $\chi\colon \mathfrak{h} \to k$ such that $hv = \chi(h)v$ for all $h \in \mathfrak{h}$. Then $\chi([x,h]) = 0$ for all $x \in \mathfrak{g}$.

Proof. Let $x \in \mathfrak{g}$. Let

$$V_i \coloneqq \operatorname{span}\{v, xv, \dots, x^{i-1}v\}.$$

Then

$$0 = V_0 \subset V_1 \subset \cdots \subset V_i \subset V_{i+1}.$$

Let n be minimally such that $V_n = V_{n+1}$. Then $\dim V = n$, $xV_n \subset V_n$, and $V_n = V_{n+k}$ for all $k \geq 0$.

We claim that for $h \in \mathfrak{h}$,

$$hx^i v \equiv \chi(h)x^i v \pmod{V_i} \quad \text{for} \quad i \geq 0.$$

For $i = 0$ this is the definition of χ, and by induction

$$hx^i v = hxx^{i-1}v = xhx^{i-1}v - [x,h]x^{i-1}v.$$

Now write $hx^{i-1}v = \chi(h)x^{i-1}v + v'$ where $v' \in V_{i-1}$. Since $xV_{i-1} \subset V_i$ and $\mathfrak{h}V_i \subset V_i$, the result follows. Thus using the basis $v, xv, \dots, x^{n-1}v$, the endomorphisms of V_n arising from elements of $\mathfrak{h}$ are upper triangular matrices with diagonal entries $\chi(h)$. Thus

$$\operatorname{Tr}_{V_n}(h) = n\chi(h).$$

By replacing h with $[x,h]$ we conclude that

$$n\chi([x,h]) = \operatorname{Tr}_{V_n}([x,h]) = \operatorname{Tr}_{V_n}(xh - hx) = 0.$$

□

Proof of Theorem 1.9.14. By induction on $\dim \mathfrak{g}$. The base case $\dim \mathfrak{g} = 0$ is clear. For $\dim \mathfrak{g} > 0$, we know that $D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ have codimension 1 and contain $D\mathfrak{g}$, as in the proof of Engel's theorem. Then $\mathfrak{h}/D\mathfrak{g}$ is an ideal of $\mathfrak{g}/D\mathfrak{g}$ because the latter is abelian, hence $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. By induction we may find $v \neq 0 \in V$ and $\chi\colon \mathfrak{h} \to k$ such that $hv = \chi(h)v$ for all $h \in fh$. Let

$$W = \{w \in V : hw = \chi(h)w \text{ for all } h \in \mathfrak{h}\}.$$

This is a nonzero linear subspace of V, so by the main lemma we show W is $\mathfrak{g}$-stable. If $w \in W$, $x \in \mathfrak{g}$, then for any $h \in \mathfrak{h}$ we have

$$hxw = xhw - [x,h]w = \chi(h)xw - \chi([x,h])w = \chi(h)xw,$$

hence $xw \in W$. If $x \in \mathfrak{g} \setminus \mathfrak{h}$, then x mapsto W into W, and since k is algebraically closed there exists an eigenvector $v_0 \in W$ for x. Then v_0 is an eigenvector for $kx + \mathfrak{h} = \mathfrak{g}$. ☐

Counterexample 1.9.16. A counterexample in positive characteristic is $_2(\mathbb{F})$. This is nilpotent of dimension 3, but its standard representation on $\mathbb{F}_2^2$ has no eigenvector.

Corollary 1.9.17

Under the hypotheses of Lie's theorem, $\mathfrak{g}$ admits a flag of ideals.

Proof. Apply Lie's theorem to the adjoint representation. ☐

Corollary 1.9.18

If $\mathfrak{g}$ is solvable over k of characteristic zero, then $[\mathfrak{g},\mathfrak{g}]$ is nilpotent.

Proof. After a base change, we may assume k is algebraically closed. By the previous corollary we have a flag $(\mathfrak{g}_i)$ of ideals. Let $x \in [\mathfrak{g},\mathfrak{g}]$. Then $\operatorname{ad} x\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ as $\operatorname{End}(\mathfrak{g}_i/\mathfrak{g}_{i+1}) \cong k$ is commutative. This shows $\operatorname{ad} x$ is nilpotent on $\mathfrak{g}$, hence on $[\mathfrak{g},\mathfrak{g}]$. By Engel's theorem $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. ☐

Remark 1.9.19. If $[\mathfrak{g},\mathfrak{g}]$ is nilpotent then clearly $\mathfrak{g}$ is solvable.

1.10 Lemmas on endomorphisms

Let k be algebraically closed of characteristic zero. Let V be finite-dimensional over k. An endomorphism $u \in \operatorname{End}(V)$ is *semisimple* if its eigenvectors span V, or equivalently if there exists a system of coordinates in which it is a diagonal matrix.

Lemma 1.10.1

For $u \in \operatorname{End} V$ there exist unique semisimple s and nilpotent n in $\operatorname{End} V$ such that $sn = ns$ and $u = s + n$. Moreoever, there exist polynomials S and N such that $S(0) = 0 = N(0)$ and $s = S(u)$ and $n = N(u)$.

Proof. Take your favourite linear algebra and enjoy. ☐

Corollary 1.10.2

As in the conclusion of the previous lemma, if $A \subset B \subset V$ with $uB \subset A$, then $sB \subset A$ and $nB \subset A$.

Proof. If $P(T)$ is any polynomial in T with constant term 0, then $P(u)B \subset A$. □

Let V^* be the dual space of V and

$$V_{p,q} = V^{\otimes p} \otimes (V^*)^{\otimes q}.$$

The diagonal action defines an $\operatorname{End} V$-module structure on $V_{p,q}$; we denote by $u_{p,q}$ the endomorphism of $V_{p,q}$ corresponding to $u \in \operatorname{End} V$. For example

$$u_{12} = u \otimes 1 \otimes 1 - 1 \otimes u^* \otimes 1 - 1 \otimes 1 \otimes u^*,$$

where $u^* \in \operatorname{End} V^*$ is the familiar transpose of u. When $p = q = 1$, there is a canonical isomorphism

$$\begin{aligned} V_{1,1} &\longrightarrow \operatorname{End} V \\ x \otimes y &\longmapsto (x' \mapsto x\langle y, x'\rangle). \end{aligned}$$

A simple computation will show that $u_{1,1} \in \operatorname{End} V_{1,1}$ corresponds to $\operatorname{ad} u \in \operatorname{End}(\operatorname{End} V)$.

Proposition 1.10.3

If $u = s + n$ is the canonical decomposition of u then $u_{p,q} = s_{p,q} + n_{p,q}$ is the canonical decomposition of $u_{p,q}$.

Proof. We have

$$[s_{p,q}, n_{p,q}] = [s, n]_{p,q} = 0$$

so if (x_i) is a basis of eigenvectors of s for V, then the dual basis (x_i^*) consists of eigenvectors of s^*, so the basis $(x_{i_1} \otimes \cdots \otimes x_{i_p} \otimes x_{j_1}^* \otimes \cdots \otimes x_{j_q}^*)$ of $V_{p,q}$ consists of eigenvectors of $s_{p,q}$, which is then semisimple. $n_{p,q}$ is a sum of endomorphisms of the form

$$1 \otimes \cdots \otimes n \otimes \cdots \otimes 1$$

or

$$1 \otimes \cdots \otimes n^* \otimes \cdots \otimes 1,$$

which are both nilpotent and commute, so $n_{p,q}$ is nilpotent. Then uniqueness of the canonical decomposition gives the desired result. □

Let $s \in \operatorname{End} V$ be semisimple and $V = \oplus V_i$ the corresponding eigenspace decomposition; that is $s\big|_{V_i} = \lambda_i$. Let $\phi\colon k \to k$ be a $\mathbb{Q}$-linear map.

Definition 1.10.4

We define $\phi(s)$ to be the semisimple endomorphism of V such that $\phi(s)\big|_{V_i} = \phi(\lambda_i)$.

If s is represented by a diagonal matrix, the matrix representing $\phi(s)$ is obtained by applying ϕ to the entries of s. There is a polynomial $P(T)$ such that $P(0) = 0$ and $P(s) = \phi(s)$.

Lemma 1.10.5

$(\phi(s))_{p,q} = \phi(s_{p,q})$.

Proof. $V_{p,q}$ is a direct sum of subspaces of the form

$$V_{i_1} \otimes \cdots \otimes V_{i_p} \otimes V_{j_1}^* \otimes \cdots \otimes V_{j_q}^*.$$

On such a subspace, $s_{p,q}$ acts as scalar multiplication by

$$\lambda_{i_1} + \cdots + \lambda_{i_p} - \lambda_{j_1} - \cdots - \lambda_{j_q},$$

$\phi(s_{p,q})$ acts as scalar multiplication by

$$\phi(\lambda_{i_1} + \cdots + \lambda_{i_p} - \lambda_{j_1} - \cdots - \lambda_{j_q}),$$

and $(\phi(s))_{p,q}$ by scalar multiplication by

$$\phi(\lambda_{i_1}) + \cdots + \phi(\lambda_{i_p}) - \phi(\lambda_{j_1}) - \cdots - \phi(\lambda_{j_q}).$$

$\square$

Let $u = s + n$ be the canonical decomposition of $u \in \operatorname{End} V$. Let $A \subset B \subset V_{p,q}$ be such that $u_{p,q} B \subset A$. Then for each $\mathbb{Q}$-linear $\phi \colon k \to k$ we have

$$\phi(s)_{p,q} B \subset A.$$

Indeed, we have $s_{p,q} B \subset A$. Now since $\phi(s)_{p,q} = \phi(s_{p,q})$ is a polynomial in $s_{p,q}$ without constant term, we have the desired result.

Proposition 1.10.6

Let $u = s + n$ be the canonical decomposition. If $\operatorname{Tr}(u\phi(s)) = 0$ for all $\phi \in \operatorname{Hom}_{\mathbb{Q}}(k, k)$, then u is nilpotent.

Proof. We have

$$\operatorname{Tr}(u\phi(s)) = \sum m_i \lambda_i \phi(\lambda_i) = 0$$

for $\phi \in \operatorname{Hom}_{\mathbb{Q}}(k, k)$. For ϕ such that $\phi(k) \subset \mathbb{Q}$, another application of ϕ gives

$$0 = \sum m_i (\phi(\lambda_i))^2,$$

showing that $\phi(\lambda_i) = 0$. Thus $\lambda_i = 0$ for each i, so $s = 0$ and $u = n$. $\square$

We will not prove the following two results:

Theorem 1.10.7

Let $s, s' \in \operatorname{End} V$ be semisimple. Then there exists ϕ such that $s' = \phi(s)$ if and only if for each p, q, an element of $V_{p,q}$ is killed by s' whenever killed by s.

Theorem 1.10.8

Let $\mathfrak{g}$ be the set of $\phi(s)$ for $\phi \in \operatorname{Hom}_{\mathbb{Q}}(k, k)$. Then $\mathfrak{g}$ is the Lie algebra of the smallest algebraic subgroup G of $\mathrm{GL}(V)$ whose Lie algebra contains s.

1.10.1 Cartan's criterion

Theorem 1.10.9

Let k be a field of characteristic zero, V a finite-dimensional k-vector space, and $\mathfrak{g}$ a Lie subalgebra of $\operatorname{End} V$. The following are equivalent:

(a) $\mathfrak{g}$ is solvable.

(b) $\operatorname{Tr}(xy) = 0$ for all $x \in \mathfrak{g}$, $y \in D\mathfrak{g}$.

Proof. By base change, we reduce to the case where k is algebraically closed.

$(a) \implies (b)$ By Lie's theorem there exists a flag (V_i) in V stable under $\mathfrak{g}$. Then

$$\operatorname{Tr}_V(xy) = \sum_i \operatorname{Tr}_{V_i/V_{i+1}}(xy) = 0$$

because $y \in D\mathfrak{g}$ annihilates the one-dimensional $\mathfrak{g}$-modules V_i/V_{i+1}.

$(b) \implies (a)$. Let $u \in D\mathfrak{g}$. By Engel's theorem it suffices to show that u is nilpotent. By the proposition above, it suffices to show $\operatorname{Tr}(u\phi(s)) = 0$ for all $\phi \in \operatorname{Hom}_{\mathbb{Q}}(k, k)$. However the $\phi(s)$ need not belong to $\mathfrak{g}$. Write

$$u = \sum c_\alpha [x_\alpha, y_\alpha]$$

where $c_\alpha \in k$, $x_\alpha, y_\alpha \in \mathfrak{g}$. The fact that $\operatorname{Tr}([a, b]c) = \operatorname{Tr}(b[c, a])$ implies

$$\operatorname{Tr}(u\phi(s)) = \sum c_\alpha \operatorname{Tr}([x_\alpha, y_\alpha]\phi(s)) = \sum c_\alpha \operatorname{Tr}(y_\alpha[\phi(s), x_\alpha]).$$

It remains to show $[\phi(s), x_\alpha] \in D\mathfrak{g}$. Recall the canonical isomorphism $\operatorname{End} V \cong V_{1,1}$; we have

$$u_{1,1}(x) = ux - xu = [u, x]$$

so $u_{1,1}\mathfrak{g} \subset D\mathfrak{g}$. By an earlier remark $\phi(s)_{1,1}\mathfrak{g} \subset D\mathfrak{g}$, so $[\phi(s), x] \in D\mathfrak{g}$. □