The L^p-spaces

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Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $f\colon E \to \mathbb{C}$ (or $\mathbb{R}$) be measurable. We say that f belongs to $\mathcal{L}^p(E, \mu)$ for some $1 \leq p < \infty$ if

$$\|f\|_{L^p} := \left(\int_E |f|^p \, d\mu\right)^{\frac{1}{p}} = (\mu(|f|^p))^{\frac{1}{p}} < \infty.$$

We say that $f \in \mathcal{L}^\infty(E, \mu)$ if f is bounded almost everywhere; that is, there exists $0 \leq K < \infty$ such that

$$\mu\left(\{|f(x)| > K\}\right) = 0.$$

In this case we define

$$\|f\|_{L^\infty} = \inf\{K : \mu\left(\{|f(x)| > K\}\right) = 0\}$$

1 The L^p-norm

Proposition 1.1

For $1 \leq p \leq \infty$, the function $f \mapsto \|f\|_{L^p}$ defines a seminorm on $\mathcal{L}^\infty(E, \mu)$. That is,

(1) $\|\cdot\|_{L^p}$ is nonnegative.

(2) $\|\cdot\|_{L^p}$ is homogeneous.

(3) $\|\cdot\|_{L^p}$ satisfies the triangle inequality.

Proof. Nonnegativity and homogeneneity follow from the nonnegativity and homogeneity of the absolute value. For example, we show homogeneity:

$$\|\lambda f\|_{L^p} = \left(\int_E |\lambda f|^p \, d\mu\right)^{\frac{1}{p}} = |\lambda|\left(\int_E |f|^p \, d\mu\right)^{\frac{1}{p}} = |\lambda|\|f\|_{L^p}.$$

The triangle inequality is more involved: first note that the triangle inequality holds in the extreme cases $p = 1,\infty$: for $p = 1$,

$$\begin{aligned}\|f+g\|_{L^1} &= \int_E |f+g|\,d\mu \\ &\le \int_E |f|+|g|\,d\mu \\ &= \|f\|_{L^1}+\|g\|_{L^1}.\end{aligned}$$

For $p = \infty$, let $A = \|f\|_{L^\infty}$, $B = \|g\|_{L^\infty}$. Given $\epsilon > 0$, let

$$N_f := \{|f(x)| > A+\epsilon\} \quad \text{and} \quad N_g := \{|g(x)| > B+\epsilon\}.$$

By definition,

$$\mu(N_f) = 0 \quad \text{and} \quad \mu(N_g) = 0.$$

For $x \notin N_f \cup N_g$ we have

$$|(f+g)(x)| \le |f(x)| + |g(x)| \le A + B + 2\epsilon,$$

so

$$\{|(f+g)(x)| > A+B+2\epsilon\} \subset N_f \cup N_g,$$

which has measure 0. We have shown that

$$\|f+g\|_{L^\infty} \le A + B + 2\epsilon,$$

and taking $\epsilon \to 0$ gives the triangle inequality for $p = \infty$. The case where $1 < p < \infty$ is Minkowski's inequality, stated and proven in several steps below. □

Lemma 1.2 (Young's inequality)

Let $a, b > 0$ and let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Let $t = \frac{1}{p} \in (0,1)$, so that $1 - t = \frac{1}{q} \in (0,1)$. Since the logarithm is concave on $\mathbb{R}_+$, we have

$$\begin{aligned}\log(ta^p + (1-t)b^q) &\ge t\log(a^p) + (1-t)\log(b^q) \\ &= tp\log(a) + (1-t)q\log(b) \\ &= \log a + \log b \\ &= \log(ab).\end{aligned}$$

Exponentiating both sides,

$$ab \le ta^p + (1-t)b^q = \frac{a^p}{p} + \frac{b^q}{q}.$$

□

Lemma 1.3 (Hölder's inequality)

Let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. THen

$$\int_E |fg| \, d\mu \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. By replacing f with $\frac{f}{\|f\|_{L^p}}$ and g with $\frac{g}{\|g\|_{L^q}}$, it suffices by homogeneity to show that if $\|f\|_{L^p} = 1 = \|g\|_{L^q}$, then

$$\int_E |fg| \, d\mu \leq 1.$$

By Young's inequality for $a = |f(x)|$, $b = |g(x)|$, we have

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}.$$

Integrating over E,

$$\int_E |fg| \, d\mu \leq \frac{1}{p} \int_E |f|^p \, d\mu + \frac{1}{q} \int_E |g|^q \, d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

$\square$

Theorem 1.4 (Minkowski's inequality)

Let $1 < p < \infty$ and let $f, g \in \mathcal{L}^p(E, \mu)$. Then

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Proof. We have

$$\begin{aligned}
\|f + g\|_{L^p}^p &= \int_E |f + g|^p \, d\mu \\
&= \int_E (|f| + |g|)|f + g|^{p-1} \, d\mu \\
&= \int_E |f||f + g|^{p-1} \, d\mu + \int_E |g||f + g|^{p-1} \, d\mu.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
\int_E |f||f + g|^{p-1} \, d\mu + \int_E |g||f + g|^{p-1} \, d\mu &\leq \|f\|_{L^p} \|f + g\|_{L^p}^{p-1} + \|g\|_{L^p} \|f + g\|_{L^p}^{p-1} \\
&= (\|f\|_{L^p} + \|g\|_{L^p}) \|f + g\|_{L^p}^{p-1},
\end{aligned}$$

and dividing by $\|f + g\|_{L^p}^{p-1}$ completes the proof. $\square$

To get a norm, we would need positivity; $\|f\|_{L^p} = 0$ if and only if $f = 0$. However, we only have that $\|f\|_{L^p} = 0$ if and only if $f = 0$ almost everywhere. To obtain a normed vector space, we will quotient by the elements of $\mathcal{L}^p(E, \mu)$ with $\|f\|_{L^p} = 0$. In essence, we define an equivalence relation

$$f \sim g \iff f - g = 0 \text{ almost everywhere.}$$

It is clear that this is an equivalence relation and since

$$\|f\|_{L^p} - \|g\|_{L^p} \leq \|f - g\|_{L^p} = 0,$$

we have $\|f\|_{L^p} = \|g\|_{L^p}$ for $f \sim g$. Thus

$$L^p(E, \mu) := \mathcal{L}^p(E, \mu)/\sim$$

inherits a norm from the seminorm on $\mathcal{L}^p$. By abuse of notation we make implicit the equivalence class and simply speak of functions in $L^p(E, \mu)$.

2 Completeness

In this section we show L^p-spaces are complete.

Lemma 2.1

Let $1 \leq p < \infty$. Let $(g_n)_{n=1}^\infty$ be a sequence of functions in $L^p(E, \mu)$ such that

$$\sum_{n=1}^\infty \|g_n\|_{L^p} < \infty.$$

Then there exists $f \in L^p(E, \mu)$ such that

$$\sum_{n=1}^\infty g_n = f,$$

where the sum converges pointwise almost everywhere.

Proof. Pick representatives $\tilde{g_n} \in \mathcal{L}^p(E, \mu)$. Define $h_n, h \colon E \to [0, \infty]$ by

$$h_n = \sum_{k=1}^n |\tilde{g}_k| \quad \text{and} \quad h = \sum_{k=1}^\infty |\tilde{g}_k|.$$

Then $(h_n)_{n=1}^\infty$ is a monotone increasing sequence of nonnegative measurable functions converging pointwise to h, so by monotone convergence

$$\int_E h^p \, d\mu = \lim_{n\to\infty} \int_E h_n^p \, d\mu.$$

By Minkowski's inequality,

$$\|h_n\|_{L^p} \le \sum_{k=1}^{n} \|g_k\|_{L^p} \le \sum_{k=1}^{\infty} \|g_k\|_{L^p} =: K.$$

Hence $h \in \mathcal{L}^p(E,\mu)$ satisfies $\|h\|_{L^p} \le K$, so h is finite almost everywhere. Whenever h is finite at x, $\sum_{k=1}^{\infty} \tilde{g}_k(x)$ converges absolutely, hence converges by the completeness of $\mathbb{C}$. So $\sum_{k=1}^{\infty} \tilde{g}_k$ converges pointwise almost everywhere and by defining f to take this value when it converges and 0 otherwise, we have $|f| \le h$, hence $\|f\|_{L^p} \le \|h\|_{L^p} \le K$ and

$$\left| f - \sum_{k=1}^{n} \tilde{g}_k \right|^p \le \left(|f| + \sum_{k=1}^{n} |\tilde{g}_k| \right)^p \le (2h)^p.$$

Now h^p is integrable, so the dominated convergence theorem implies

$$\int_E \left| f - \sum_{k=1}^{n} \tilde{g}_k \right|^p \, d\mu \to 0$$

as $n \to \infty$, so $\sum_{k=1}^{\infty} \tilde{g}_k$ converges to f in L^p. □

Theorem 2.2 (Riesz–Fischer)

$L^p(E,\mu)$ is complete.

Proof. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence with respect to the L^p-norm. First suppose $1 \le p < \infty$. Then the Cauchy property means that we can find a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that

$$\left\| f_{n_{k+1}} - f_{n_k} \right\|_{L^p} < \frac{1}{2^k}.$$

Let $g_k = f_{n_{k+1}} - f_{n_k}$. By construction

$$\sum_{k=1}^{\infty} \|g_k\|_{L^p} < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

so by the preceding lemma there exists $g \in L^p(E,\mu)$ such that $\sum_{k=1}^{\infty} g_k = g$, converging pointwise almost everywhere and in L^p. Since $f_{n_{j+1}} = f_{n_1} + \sum_{k=1}^{j} g_k$, we deduce that $(f_{n_k})_{k=1}^{\infty}$ converges in L^p. Since $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence, the entire sequence must converge in L^p.

Now suppose $p = \infty$. Since (f_n) is Cauchy in $L^{\infty}(E,\mu)$, for each m there exists N such that $j, k > N$ implies

$$|f_j(x) - f_k(x)| < \frac{1}{m} \quad \text{for} \quad x \in N_{j,k,m}^c$$

for some measure zero set $N_{j,k,m}$. In particular $N = \bigcup_{j,k,m} N_{j,k,m}$ has measure zero, and for any m there exists N such that $j, k > N$ implies

$$\sup_{x \in N^c} |f_j(x) - f_k(x)| < \frac{1}{m},$$

so by completeness of $\mathbb{C}$, (f_n) converges pointwise to some f on N^c. By defining $f = 0$ on N, we see that

$$\sup_{x \in N^c} |f_j(x) - f(x)| < \frac{1}{m}$$

so $\|f\|_{L^\infty} < \infty$ and $f_n \to f$ in L^∞. □

3 A density theorem

Theorem 3.1

Let $1 \leq p < \infty$ and let S be the set of all complex measurable simple functions s on E such that

$$\mu(\{x : s(x) \neq 0\}) < \infty.$$

Then S is dense in $L^p(E, \mu)$

Proof. Clearly $S \subset L^p(E, \mu)$. If $f \in L^p(E, \mu)$ and $f \geq 0$, then let

$$f_n(x) = \min\left\{2^n \left\lfloor 2^{-n} f(x) \right\rfloor, n\right\}.$$

Then $f_n \in S$ and $0 \leq f_n \leq f$, so that $f_n \in L^p(E, \mu)$. Furthermore, $f_n(x) \to f(x)$ and $|f - f_n|^p \leq |f|^p$, so by dominated convergence we deduce

$$\int_E |f - f_n|^p \, d\mu \to 0,$$

and thus $f_n \to f$ in L^p. In general, write $f = f_r^+ - f_r^- + i(f_i^+ - f_i^-)$, where each $f_r^\pm$, $f_i^\pm$ is nonnegative so the result follows by linearity. □