Topological K-Theory and Hopf Invariant One

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The Hopf Invariant One Problem

$$\mathbb{R}^n \text{ is a division } \mathbb{R}\text{-algebra}$$

$$\Downarrow$$

$$S^{n-1} \text{ is parallelizable}$$

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$$\text{There exists a map } f : S^{2n-1} \to S^n \text{ of Hopf invariant one}$$

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  Definition. An H-space is a topological space with a "nice" multiplication.
- There exists a map $f : S^{2n-1} \to S^n$ of Hopf invariant one.
  Definition. We will use K-Theory.

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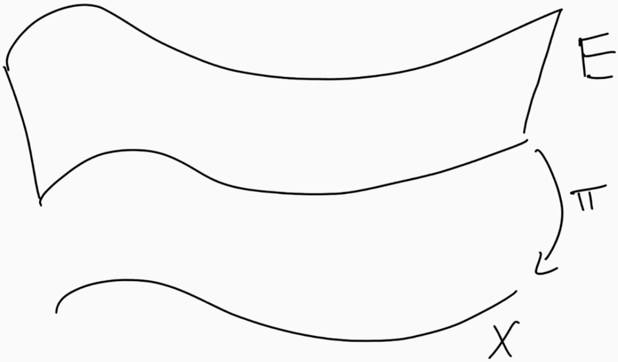
- Cohomology solution: Adams (1960), 85 pages.
- K-Theory solution: Adams & Atiyah (1966), 8 pages.

Vector Bundles

Vector Bundles

Let X be a topological space.

Definition. A vector bundle (E, π, X) over X is

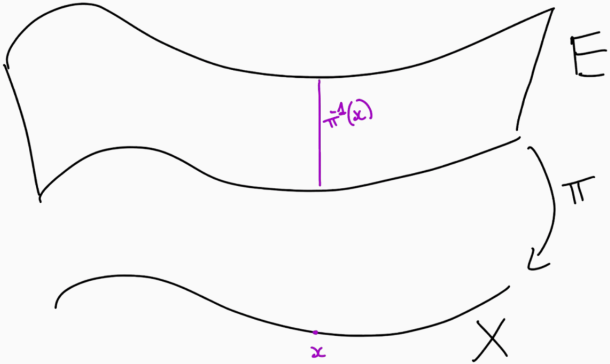


A total space E, a base space X, and a projection $\pi : E \to X$.

Vector Bundles

Definition. A vector bundle (E, π, X) over X is

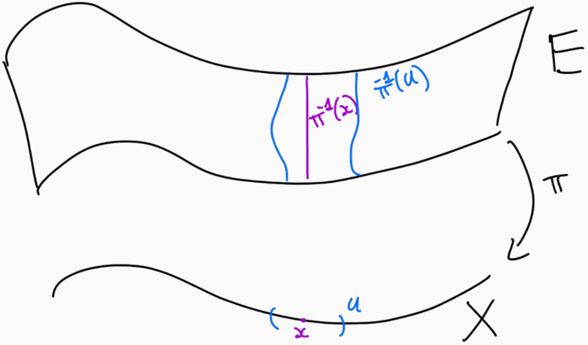
- a finite dimensional $\mathbb{R}$- or $\mathbb{C}$-vector space E_x for each $x \in X$,
- a topology on $E := \bigsqcup_{x \in X} E_x$ such that the projection $\pi : E \to X$, defined so that $\pi^{-1}(x) = E_x$ for each $x \in X$, is continuous.



$\pi^{-1}(x) = E_x$ is called the fiber over x.

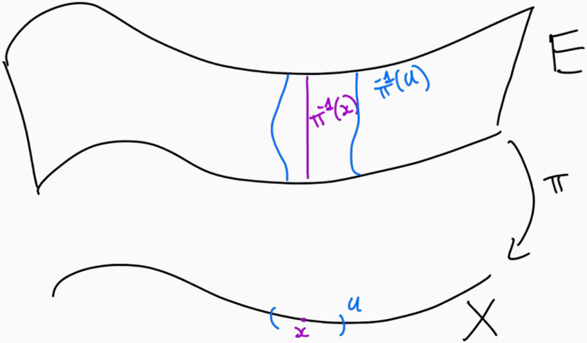
Definition. A vector bundle (E, π, X) over X is

- such that for each $x \in X$, there is a neighbourhood U of x and a vector space V such that $\pi^{-1}(U)$ is isomorphic (as vector bundles) to $U \times V$.



Trivial Bundles

A vector bundle over X in the form $X \times V$ is called the trivial bundle of rank $\dim(V)$. The previous condition says vector bundles are locally trivial.



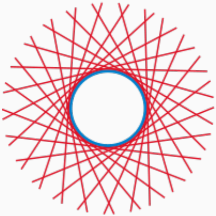
The Tangent Bundle

Example. Let X be the sphere $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$.

For each $x \in S^1$, let E_x be the tangent space $(S^1)_x$.

$E = \bigsqcup_{x \in S^1} (S^1)_x$ is a subspace of $S^1 \times \mathbb{R}^2$, so $\pi : E \to S^1$ is the projection onto S^1. With the subspace topology on E, π is continuous.

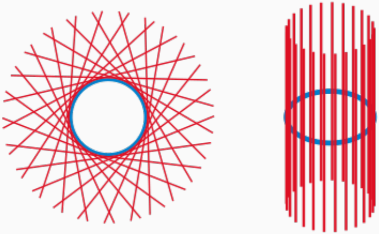
The vector bundle (E, π, S^1) is called the tangent bundle.



The Tangent Bundle Trivialization

Claim. The tangent bundle over S^1 is trivial.

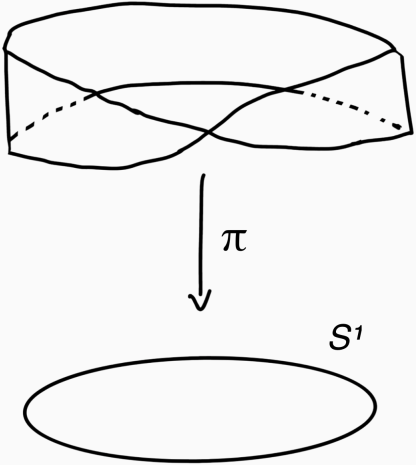
90% Proof. Rotate the fibers by $\frac{\pi}{2}$ to be perpendicular to the plane containing the circle. This transformation preserves the vector bundle structure, and the result is the trivial bundle $S^1 \times \mathbb{R}$.



We have shown S^1 is parallelizable!

The Möbius Bundle

Example. Another vector bundle over S^1 is the (infinite) Möbius band.



Vector Bundle Addition

Let E, E' be vector bundles over X, their fibers denoted by E_x, E'_x.

Definition. We define $E \oplus E'$ as the vector bundle over X having fibers

$$(E \oplus E')_x := E_x \oplus E'_x.$$

Topological K-Theory

The Group $K(X)$

From now on, we consider only complex vector bundles.

Definition. $K(X)$ is the group of formal differences $E - E'$ of vector bundles over X, modulo the equivalence relation $E_1 - E_2 \sim E_3 - E_4$ if

$$E_1 \oplus E_4 \cong E_2 \oplus E_3.$$

Vector Bundle Multiplication

Let E, E' be vector bundles over X.

Definition. We define $E \otimes E'$ as the vector bundle over X having fibers

$$(E \otimes E')_x := E_x \otimes E'_x.$$

As the tensor product distributes over the direct sum, this makes $K(X)$ a commutative ring.

The Adams Operations

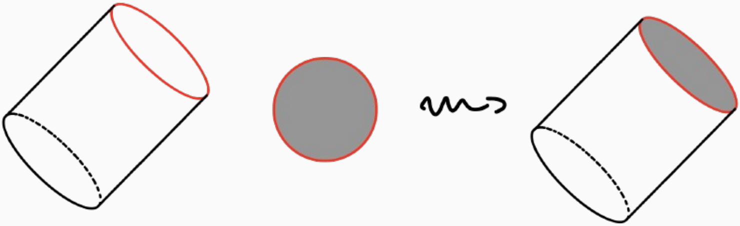
Theorem. There exists a family of operations $\psi^k : K(X) \to K(X)$, called the Adams operations, such that

1. ψ^k is a ring homomorphism.
2. $\psi^k(L) = L^k$ whenever L is a rank 1 vector bundle.
3. $\psi^k \circ \psi^\ell = \psi^\ell \circ \psi^k$.

The Hopf Invariant One Problem — Solution

The Hopf Invariant

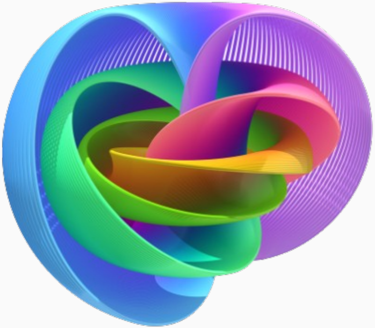
Given $f : S^{2n-1} \to S^n$, glue the open $2n$-dimensional unit ball B^{2n} to S^n via f along the boundary S^{2n-1}.



Call the resulting space X_f.

The Hopf Fibration

Another example of these maps from $S^{2n-1} \to S^n$ where $n = 2$.



The Hopf Invariant

$K(X_f)$ turns out to be free on two generators α and β, satisfying

$$\beta^2 = d\alpha$$

for some $d \in \mathbb{Z}$, which we define as the Hopf invariant $H(f) := d$.

The Hopf Invariant One Problem

For which n does there exist a map $f : S^{2n-1} \to S^n$ with $H(f) = 1$?

- If n is odd, only $n = 1$.
- If $n = 2m$ is even, the Adams operations require

$$\psi^3 \circ \psi^2(\beta) = \psi^2 \circ \psi^3(\beta).$$

This simplifies as $2^m \mid 3^m - 1$, thus by number theory, $m = 1, 2$, or 4.

Conclusion

If $n = 1, 2, 4, 8$, we have division algebras: $\mathbb{R}$, $\mathbb{C}$, quaternions, octonions.

$$\begin{gathered}\mathbb{R}^n \textbf{ is a division } \mathbb{R}\textbf{-algebra} \\ \Downarrow \\ S^{n-1} \textbf{ is parallelizable} \\ \Downarrow \\ S^{n-1} \textbf{ is an } H\textbf{-space} \\ \Downarrow \\ \textbf{There exists a map } f : S^{2n-1} \to S^n \textbf{ of Hopf invariant one}\end{gathered}$$

If there is a map of Hopf invariant one, then $n = 1, 2, 4$, or 8.

Conclusion

$$\mathbb{R}^n \text{ is a division } \mathbb{R}\text{-algebra}$$

$$\Updownarrow$$

$$S^{n-1} \text{ is parallelizable}$$

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References

1. J.F. Adams & M.F. Atiyah (1966). *K-Theory and the Hopf Invariant*. Quarterly Journal of Mathematics, Volume 17, Issue 1.
2. I. Banerjee (2016). *The Hopf invariant one problem*.
3. M. Karoubi (1978). *K-Theory: An Introduction*. Springer-Verlag.