## 1.2 Riemann surfaces

A Riemann surface is the 1-dimensional complex equivalent of a manifold. In a conformal structure we demand that transition maps be holomorphic.

**Definition 1.2.1**

A Riemann surface $R$ is a connected Hausdorff topological space with a maximal conformal structure.

*Remark* 1.2.2. Unlike for real manifolds, we will not demand the point-set requirement that $R$ be second countable. This is because a theorem of Radó gives second countable for free!

The fact that every atlas is contained in a unique maximal atlas is shown in the same way as smooth structures. There is a canonical conformal structure on $\mathbb{C}$ given by extending the atlas $\{\mathrm{id}_{\mathbb{C}}\}$. The resulting Riemann surface will simply be denoted by $\mathbb{C}$.

A different conformal structure is obtained by extending the atlas $\{z \mapsto \overline{z}\}$. The resulting Rieman surface will be denoted $\overline{\mathbb{C}}$. The same considerations hold for any connected open set in $\mathbb{C}$ by restricting charts.

**Example 1.2.3.** Our first nontrivial example is the Riemann sphere $\mathbb{C}_\infty \cong S^2$, which may be identified $\mathbb{C} \cup \{\infty\}$ via stereographic projection. We define an atlas with two charts: first $\mathrm{id}\colon \mathbb{C} \to \mathbb{C}$ and second

$$\begin{aligned} \psi\colon \mathbb{C}_\infty \setminus 0 &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{1}{z} \\ \infty &\longmapsto 0. \end{aligned}$$

Both transition functions are $\psi\colon C^\times \to C^\times$, which is analytic.

Just as with real manifolds, a continuous map $f\colon R \to S$ between Riemann surfaces is *holomorphic* if $\phi_\beta^{-1} \circ f \circ \phi_\alpha$ is holomorphic for all charts. It suffices to check this condition near each point. Compositions of holomorphic maps are holomorphic. A *biholomorphism* or *conformal* map is defined just as in complex analysis, and acts as an isomorphism in the category of Riemann surfaces and holomorphic maps.

**Example 1.2.4.** Complex conjugation

$$\begin{aligned} \mathbb{C} &\longrightarrow \overline{\mathbb{C}} \\ z &\longmapsto \overline{z} \end{aligned}$$

is a biholomorphism, being its own inverse.

### 1.2.1 Covering maps

Covering maps are a useful tool for pulling back conformal structure from the base space to the covering space. We will use the term covering map to denote a local homeomorphism, and we say a covering map is *regular* if preimages are discrete (this is what most people simply call a covering map).

**Proposition 1.2.5**

Let $\pi\colon E \to R$ be a covering map, where $R$ is a Riemann surface. Then there exists a unique conformal structure on $E$ such that $\pi$ is analytic.

*Proof.* Each $p \in E$ has an open neighbourhood $U_p$ such that $\pi|_{U_p}$ is a homeomorphism onto its image. By possibly shrinking $\pi(U_p)$, we assume we have a chart $\phi_p\colon V_p \to \pi(U_p)$ around $\pi(p)$, where $V_p \subset \mathbb{C}$. Then $\psi_p = (\pi|_{U_p})^{-1} \circ \phi_p\colon V_p \to U_p$ is a chart for $E$ near $p$. It remains to show the transition maps are analytic. Given $p, q \in E$, $\pi|_{U_p \cap U_q}$ is invertible. Thus the transition map is

$$\psi_q^{-1} \circ \psi_p = \phi_q^{-1} \circ \pi|_{U_p \cap U_q} \circ (\pi|_{U_p \cap U_q})^{-1} \circ \phi_p = \phi_q^{-1} \circ \phi_p,$$

which is analytic since $R$ is a Riemann surface. Also, for $p \in E$ we can take charts $(\psi_p, U_p)$ and $(\phi_p, \pi(U_p))$ around $p$ and $\pi(p)$, respectively. We verify that

$$\psi_p^{-1} \circ \pi \circ \phi_p = \mathrm{id}$$

is analytic.

For uniqueness, if $\psi'_p\colon V'_p \to U'_p$ is another chart near $p$ making $\pi$ analytic, then

$$\psi_p^{-1} \circ \psi'_p = \phi_p^{-1} \circ \pi \circ \psi'_p,$$

which is analytic. So $\psi'_p$ is compatible with the atlas we constructed, and they represent the same conformal structure. $\square$

### 1.2.2 Analytic functions

**Proposition 1.2.6**

Let $f\colon R \to \mathbb{C}$ be a nonconstant analytic function and let $p \in R$ be a zero of $f$. Then there exists a chart $\phi_\alpha\colon U_\alpha \to V_\alpha$ around $p$ with $\phi_\alpha(0) = p$ and

$$f \circ \phi_\alpha(z) = z^m$$

for some $m \in \mathbb{N}$.

*Proof.* Let $\phi\colon U_\alpha \to V_\alpha$ be any chart around $p$ with $\phi_\alpha(0) = p$. Then $f \circ \phi_\alpha(0) = 0$ so there exists a nonvanishing holomorphic function $g\colon U_\alpha \to \mathbb{C}$ such that

$$f \circ \phi_\alpha(z) = z^m g(z).$$

We claim that $g$ is an $m$th root of a holomorphic function. Indeed, $g(0) \neq 0$, so by continuity there exists $\delta > 0$ such that $g(D_\delta(0)) \subset D_{|g(0)|}(g(0))$. There is a holomorphic $m$th root function on $D_{|g(0)|}(g(0))$, so we may write

$$f \circ \phi_\alpha(z) = (h(z))^m$$

for $h$ holomorphic. Since $h'(0) = \sqrt[m]{g(0)} \neq 0$, the inverse function theorem implies $h$ is conformal, so by replacing $\phi_\alpha$ with $\phi_\alpha \circ h^{-1}$ we have the desired result. $\square$

### 1.2.3 Complex tori

Let $\tau_1$, $\tau_2 \in \mathbb{C}$ be $\mathbb{R}$-linearly independent. This means they are nonzero and $\tau_2/\tau_1 \notin \mathbb{R}$. They generate an additive subgroup $\Lambda \coloneqq \langle \tau_1, \tau_2 \rangle \subset \mathbb{C}$. Consider the quotient map $\pi \colon \mathbb{C} \to \mathbb{C}/\Lambda =: T$.

We claim that $T$ is homeomorphic to the 2-torus $S^1 \times S^1$. Indeed, consider the fundamental parallelogram with corners $0, \tau, \tau_2, \tau_1 + \tau_2$. Then $T$ is obtained by identifying opposite sides, which is equivalent to one construction of the torus. In particular, $T$ is compact Hausdorff.

We claim that $\pi$ is a regular covering map. Indeed, let $\epsilon < \frac{1}{2} \min\{|\lambda| : \lambda \in \Lambda \setminus 0\}$. Then if $z \in \mathbb{C}$, we have

$$D_\epsilon(z) \cap \{z + \Lambda\} = \{z\}.$$

If $p = \pi(z_0) \in T$, then $U = \pi(D_\epsilon(z_0))$ is open with preimage

$$\pi^{-1}(U) = \bigsqcup_{\lambda \in \Lambda} D_\epsilon(z_0) + \lambda \cong D_\epsilon(z_0) \times \Lambda,$$

as desired. Finally, we define an atlas on $T$. Let $z_0 \in \mathbb{C}$ and consider the open set $U = \pi(D_\epsilon(z_0)) \subset T$. Then $\pi \colon D_\epsilon(z_0) \to U$ is a homeomorphism, so it is a good chart for $T$. Transition functions are given by translation by an element of $\Lambda$, which is certainly analytic.