FUSRP Project 007

Quantized Weyl Algebras and Representations of i-Quantum Groups

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August 14, 2025

Fundamental definitions

Definition

A **algebra** over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space A equipped with a $\mathbb{K}$-bilinear multiplication $A \times A \to A$.

Definition

An **action** of a group G on an algebra A is a group homomorphism $G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of algebra automorphisms of A.

Definition

The **Lie algebra** $\mathfrak{sl}_2(\mathbb{C})$ is the set of 2×2 complex matrices with trace zero:

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{C} \right\},$$

with the Lie bracket defined by the commutator:

$$[X, Y] = XY - YX.$$

What is a quantization?

Definition

A **quantization** of an algebra is a variant of the algebra created by modifying its relations to depend on a parameter q such that at $q = 1$, we recover the original algebra.

Example

A quantization of the usual polynomial ring $\mathbb{C}[x, y]$ is the algebra over $\mathbb{C}(q)$ generated by x and y with the relation:

$$xy = q^{-1}yx.$$

For instance, the binomial expansion becomes:

$$(x + y)^2 = x^2 + (1 + q)xy + y^2.$$

Goal

We will study a particular quantization of the Weyl algebra.

The Classical Weyl algebra

Definition

The **Weyl algebra** $\mathcal{PD}$ is the algebra over $\mathbb{C}$ consisting of linear operators on $\mathcal{P} = \mathbb{C}[t_1, \ldots, t_n]$. Multiplication in $\mathcal{PD}$ is composition of functions. $\mathcal{PD}$ is generated by

$$t_1, \ldots, t_n, \partial_1, \ldots, \partial_n,$$

which act on $\mathcal{P}$ by left multiplication and differentiation, respectively. Here ∂_i denotes $\frac{\partial}{\partial t_i}$.

- For example, $t_2\partial_1$ acts on t_1^2 by
$$(t_2\partial_1) \cdot t_1^2 = t_2 \cdot (2t_1) = 2t_1 t_2.$$
- Generally, we consider $\mathcal{PD}$ as a subalgebra of $\mathrm{End}(\mathcal{P})$, the endomorphisms of $\mathcal{P}$.

The Weyl algebra: generators and relations

Abstractly, we can view the Weyl algebra as the algebra over $\mathbb{C}$ generated by

$$t_1, \ldots, t_n, \partial_1, \ldots, \partial_n$$

subject to the relations

$$\begin{aligned}
t_j t_i &= t_i t_j \\
\partial_j \partial_i &= \partial_i \partial_j \\
\partial_i t_j &= t_j \partial_i \\
\partial_j t_i &= t_i \partial_j \\
\partial_i t_i &= 1 + t_i \partial_i \quad \text{(product rule)}
\end{aligned}$$

for $1 \leq i < j \leq n$.

Classical theory: $O(n)$ action on $\mathcal{PD}$

Definition

The **orthogonal group**

$$O(n) = \{A \in GL_n(\mathbb{C}) : A^T = A^{-1}\}$$

is the group of transformations of $\mathbb{C}^n$ preserving the symmetric bilinear form $(x, y) \mapsto x^T y$.

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- $O(n)$ acts on the polynomial algebra $\mathcal{P}$ by

$$(A \cdot p)\left(\vec{\mathbf{t}}\right) := p\left(A^{-1}\vec{\mathbf{t}}\right) \qquad \text{for } A \in O(n), p \in \mathcal{P}.$$

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$$(A \cdot p)\left(\vec{\mathbf{t}}\right) := p\left(A^{-1}\vec{\mathbf{t}}\right) \qquad \text{for } A \in O(n), p \in \mathcal{P}.$$
- $O(n)$ also acts on $\mathcal{D} = \mathbb{C}[\partial_1, \ldots, \partial_n]$ by
$$(A \cdot D)\left(p\left(\vec{\mathbf{t}}\right)\right) := A \cdot D\left(p\left(A\vec{\mathbf{t}}\right)\right) \qquad \text{for } A \in O(n), D \in \mathcal{D}.$$
- Altogether, we have an action of $O(n)$ on $\mathcal{PD}$.

Classical theory: $O(n)$ action on $\mathcal{PD}$

Let's look at an example. We have

$$\left\langle \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \right\rangle = O(2).$$

We claim that $t_1^2 + t_2^2 \in \mathcal{P}$ is **invariant** under the action of every element of $O(2)$. Note

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -t_1 \\ t_2 \end{bmatrix}$$

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Hence,

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot (t_1^2 + t_2^2) = (-t_1)^2 + t_2^2 = t_1^2 + t_2^2$$

Classical theory: $O(n)$ action on $\mathcal{PD}$

Now we look at the action by rotation matrices.

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^{-1} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta)t_1 + \sin(\theta)t_2 \\ -\sin(\theta)t_1 + \cos(\theta)t_2 \end{bmatrix}$$

Hence,

$$\begin{aligned} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \cdot (t_1^2 + t_2^2) &= (\cos^2(\theta) + \sin^2(\theta))t_1^2 \\ &\quad + (\sin^2(\theta) + \cos^2(\theta))t_2^2 \\ &\quad + 2\cos(\theta)\sin(\theta)t_1t_2 \\ &\quad - 2\sin(\theta)\cos(\theta)t_1t_2 \\ &= t_1^2 + t_2^2 \end{aligned}$$

Thus, $t_1^2 + t_2^2$ is invariant under the action of $O(2)$.

Classical result: $O(n)$-invariants in $\mathcal{P}$ and $\mathcal{D}$

Definition

An operator $\psi \in \mathcal{P}\mathcal{D}$ is $O(n)$-**invariant** if for all $A \in O(n)$, $A \cdot \psi = \psi$.

Classical result: $O(n)$-invariants in $\mathcal{P}$ and $\mathcal{D}$

Definition

An operator $\psi \in \mathcal{PD}$ is $O(n)$-**invariant** if for all $A \in O(n)$, $A \cdot \psi = \psi$.

Proposition (background)

The space $\mathcal{P}^{O(n)}$ of $O(n)$-invariants in $\mathcal{P}$ is equal to $\mathbb{C}[r^2]$, where

$$r^2 = t_1^2 + \cdots + t_n^2.$$

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Proposition (background)

The space $\mathcal{D}^{O(n)}$ of $O(n)$-invariants in $\mathcal{D}$ is equal to $\mathbb{C}[\Delta]$, where

$$\Delta = \partial_1^2 + \cdots + \partial_n^2 \text{ is the Laplacian.}$$

Classical result: harmonic decomposition

Definition

A polynomial $p \in \mathcal{P}$ is **harmonic** if $\Delta(p) = 0$. The space of harmonic polynomials is $\mathcal{H} = \ker(\Delta)$.

- For $n = 2$, the polynomial $t_1^2 - t_2^2$ is harmonic as

  $$\Delta(t_1^2 - t_2^2) = (\partial_1^2 + \partial_2^2)(t_1^2 - t_2^2) = 2\partial_1 t_1 - 2\partial_2 t_2 = 2 - 2 = 0.$$

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Theorem (background)

The $\mathbb{C}$-algebra $\mathcal{P} = \mathbb{C}[t_1, \ldots, t_n]$ admits the following decomposition:

$$\mathcal{P} = \mathbb{C}[r^2] \cdot \mathcal{H}.$$

Proof sketch.

The $O(n)$-action on $\mathcal{P}$ gives the decomposition $\mathcal{P} = \mathcal{P}^{O(n)} \cdot \mathcal{H}$. Then apply $\mathcal{P}^{O(n)} = \mathbb{C}[r^2]$. □

Classical result: $O(n)$-invariants in $\mathcal{PD}$

Theorem (background)

The space $\mathcal{PD}^{O(n)}$ of $O(n)$-invariants in $\mathcal{PD}$ is generated by

$$r^2,$$

$$\Delta, \text{ and}$$

$$\frac{n}{2} + \sum_{i=1}^{n} t_i \partial_i.$$

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- $\sum_{i=1}^{n} t_i \partial_i$ is called the **Euler operator**.
- With the commutator $[X, Y] := XY - YX$, these operators span a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

What is a quantization?

Definition

A **quantization** of an algebra is a variant of the algebra created by modifying its relations to depend on a parameter q such that at $q = 1$, we recover the original algebra.

Simple q-analogues

Quantization often replaces familiar objects with q-versions:

- **Quantum integers:** $[n]_q = 1 + q + \cdots + q^{n-1} = \dfrac{1 - q^n}{1 - q}$
- q**-factorials:** $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$
- q**-binomial coefficients:** $\dbinom{n}{k}_q = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}$

These reduce to the classical ones as $q \to 1$.

Goal

We will study a particular quantization of the Weyl algebra.

The quantized Weyl algebra

Letzter, Sahi & Salmasian [LSS24] proposed a quantization of $\mathcal{PD}$, denoted $\mathscr{P}\mathscr{D}$. The previous relations are quantized as:

Classical	Quantum
$t_j t_i = t_i t_j$	$t_j t_i = q^{-1} t_i t_j$
$\partial_j \partial_i = \partial_i \partial_j$	$\partial_j \partial_i = q \partial_i \partial_j$
$\partial_i t_j = t_j \partial_i$	$\partial_i t_j = q t_j \partial_i$
$\partial_j t_i = t_i \partial_j$	$\partial_j t_i = q t_i \partial_j$
$\partial_i t_i = 1 + t_i \partial_i$	$\partial_i t_i = 1 + q^2 t_i \partial_i + (q^2 - 1) \sum_{j>i} t_j \partial_j$

for $1 \leq i < j \leq n$.

Our goal is to justify this choice of quantization by proving analogues of results about the classical $\mathcal{PD}$ for $\mathscr{P}\mathscr{D}$.

Quantum analogue: $\mathcal{U}_q'(\mathfrak{o}_n)$

We have a quantum analogue of $O(n)$.

Definition

The "i-quantum group" $\mathcal{U}_q'(\mathfrak{o}_n)$ is a $\mathbb{C}(q)$-algebra with generators $B_1, \ldots, B_{n-1}$ subject to the relations

$$\begin{aligned} B_j B_i &= B_i B_j \qquad\qquad j \notin \{i, i+1\} \\ -B_{i+1} &= B_i^2 B_{i+1} - (q+q^{-1}) B_i B_{i+1} B_i + B_{i+1} B_i^2 \\ -B_i &= B_{i+1}^2 B_i - (q+q^{-1}) B_{i+1} B_i B_{i+1} + B_i B_{i+1}^2 \end{aligned}$$

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$\mathcal{U}'_q(\mathfrak{o}_n)$ acts on $\mathscr{P}\mathscr{D}$. For example, if $j \notin \{i, i+1\}$

$$\begin{aligned} B_i(t_i) &= -q^{-1} t_{i+1} & \qquad B_i(\partial_i) &= -q^{-2} \partial_{i+1} \\ B_i(t_{i+1}) &= q^{-1} t_i & \qquad B_i(\partial_{i+1}) &= \partial_i \end{aligned}$$

Example

When $n = 2$, we can calculate $B_1(q^{-1} t_1^2 + q^{-2} t_2^2) = 0$.

Quantum result: $\mathcal{U}_q'(\mathfrak{o}_n)$-invariants in $\mathscr{P}$ and $\mathscr{D}$

Definition

An operator $\psi \in \mathscr{P}\mathscr{D}$ is $\mathcal{U}_q'(\mathfrak{o}_n)$-invariant if $B_i(\psi) = 0$ for all $1 \leq i \leq n-1$.

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Proposition (FUSRP 007, 2025)

The space $\mathscr{P}^{\mathcal{U}_q'(\mathfrak{o}_n)}$ of $\mathcal{U}_q'(\mathfrak{o}_n)$-invariants in $\mathscr{P}$ is $\mathbb{C}(q)[r_q^2]$, where

$$r_q^2 = q^{-1}t_1^2 + q^{-2}t_2^2 + \cdots + q^{-n}t_n^2.$$

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Proposition (FUSRP 007, 2025)

The space $\mathscr{D}^{\mathcal{U}'_q(\mathfrak{o}_n)}$ of $\mathcal{U}'_q(\mathfrak{o}_n)$-invariants in $\mathscr{D}$ is $\mathbb{C}(q)[\Delta_q]$, where

$$\Delta_q = q^{-1}\partial_1^2 + q^{-2}\partial_2^2 + \cdots + q^{-n}\partial_n^2.$$

Quantum result: harmonic decomposition

Theorem (FUSRP 007, 2025)

$$\mathscr{P} = \mathbb{C}(q)[r_q^2] \cdot \mathscr{H}$$

where $\mathscr{H} = \ker(\Delta_q)$.

Example

Let $n = 2$ and consider $t_1^2 + t_2 \in \mathscr{P}$. We have

$$r_q^2 = t_1^2 - q^2 t_2^2$$

$$t_2 \quad \text{and} \quad q^{-1} t_1^2 - q^{-2} t_2^2 \quad \in \mathscr{H}$$

and

$$t_1^2 + t_2 = \frac{1}{1+q^3}(t_1^2 - q^2 t_2^2) + \frac{q^4}{1+q^3}(q^{-1} t_1^2 + q^{-2} t_2^2) + t_2$$

Quantum result: $\mathcal{U}_q'(\mathfrak{o}_n)$-invariants in $\mathscr{P}\mathscr{D}$

Theorem (FUSRP 007, 2025)

The space $\mathscr{P}\mathscr{D}^{\mathcal{U}_q'(\mathfrak{o}_n)}$ of $\mathcal{U}_q'(\mathfrak{o}_n)$-invariants in $\mathscr{P}\mathscr{D}$ is generated by

$$r_q^2,$$

$$\Delta_q, \text{ and}$$

$$\sum_{i=1}^{n} t_i \partial_i.$$

Quantum result: $\mathcal{U}'_q(\mathfrak{o}_n)$-invariants in $\mathscr{PD}$

Theorem (FUSRP 007, 2025)

The space $\mathscr{PD}^{\mathcal{U}'_q(\mathfrak{o}_n)}$ of $\mathcal{U}'_q(\mathfrak{o}_n)$-invariants in $\mathscr{PD}$ is generated by

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- Noumi, Umeda, and Wakayama studied an analogous quantum group action [NUW96]. In their construction, the space of invariants contains a homomorphic image of $\mathcal{U}_{q^2}(\mathfrak{sl}_2)$.

Quantum result: $\mathcal{U}'_q(\mathfrak{o}_n)$-invariants in $\mathscr{P}\mathscr{D}$

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- Noumi, Umeda, and Wakayama studied an analogous quantum group action [NUW96]. In their construction, the space of invariants contains a homomorphic image of $\mathcal{U}_{q^2}(\mathfrak{sl}_2)$.
- In our construction, a subalgebra of $\mathcal{U}_{q^2}(\mathfrak{sl}_2)$ is generated by

$$r_q^2, \Delta_q, \text{ and } 1 + (q^2 - 1) \sum_{i=1}^{n} t_i \partial_i.$$

Applications

- Restricted to the unit sphere, the harmonic decomposition is related to Fourier analysis. More specifically, a function f on the unit sphere can be expanded as a Fourier series

$$f(x) = \sum_{d=0}^{\infty} \operatorname{proj}_d f(x),$$

where $\operatorname{proj}_d f(x)$ is the orthogonal projection of f onto $\mathcal{H}^d$.

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- **Dream.** Invariant quantum differential operators and their spectra can be connected to the combinatorial theory of Macdonald polynomials.

Acknowledgements

It is our pleasure to thank the Fields Institute and NSERC for funding this project, the University of Ottawa for hosting us, and Professor Hadi Salmasian for his guidance.

FIELDS

NSERC CRSNG

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