Chapter 1

Function Spaces

1.1 Topological vector spaces

We begin by recalling some facts about topological vector spaces. We are really only concerned with vector spaces over $\mathbb{R}$ or $\mathbb{C}$, so we may assume our scalar field carries a natural topology. Recall that a topological space is T1 if singletons are closed.

Definition 1.1.1

A *topological vector space* is a vector space endowed with a T1 topology with respect to which vector addition and scalar multiplication are continuous.

Our most important example will be normed vector spaces.

Proposition 1.1.2

A normed vector space, endowed with the metric topology, is a topological vector space.

Proof. Let X be a normed vector space. As a metric space, X is certainly T1. To show $+\colon X \times X \to X$ is continuous, let $U \subset X$ be open and $x + y \in U$. Let $\epsilon > 0$ be such that $B_\epsilon(x + y) \subset U$. We claim that

$$(x, y) \in B_{\epsilon/2}(x) \times B_{\epsilon/2}(y) \subset +^{-1}(U).$$

Indeed if $(w, z) \in B_{\epsilon/2}(x) \times B_{\epsilon/2}(y)$ then $\|x - w\| < \frac{\epsilon}{2}$ and $\|y - z\| < \frac{\epsilon}{2}$, so

$$\|(x + y) - (w + z)\| \leq \|x - w\| + \|y - z\| < \epsilon.$$

Hence $w + z \in B_\epsilon(x + y) \subset U$, so $(w, z) \in +^{-}(U)$, as desired.

To show $\cdot\colon k \times X \to X$ is continuous, let $x \in X$, $\lambda \in k$ be such that $\lambda x \in U$, and let $\epsilon > 0$ be such that $B_\epsilon(\lambda x) \subset U$. Let $\delta := \min\{1, \frac{\epsilon}{1+\|x\|+|\lambda|}\}$; we claim that

$$(\lambda, x) \in B_\delta(\lambda) \times B_\delta(x) \subset \cdot^{-1}(U).$$

Indeed if $(\mu, y) \in B_\delta(\lambda) \times B_\delta(x)$ then

$$\begin{aligned}
\|\mu y - \lambda x\| &= \|(\mu - \lambda)y + \lambda(y - x)\| \\
&\leq \|(\mu - \lambda)y\| + \|\lambda(y - x)\| \\
&= |\mu - \lambda|\|y - x + x\| + |\lambda|\|y - x\| \\
&\leq |\mu - \lambda| \left(\|y - x\| + \|x\|\right) + |\lambda|\|y - x\| \\
&< \delta(\|x\| + \delta) + \delta|\lambda| \\
&= \delta(\|x\| + |\lambda| + \delta) \\
&\leq \delta(\|x\| + |\lambda| + 1) \\
&\leq \epsilon.
\end{aligned}$$

Therefore $\mu y \in B_\epsilon(\lambda x) \subset U$, so $(\mu, y) \in \cdot^{-1}(U)$ as desired. ☐

A subset $E \subset X$ of a topological vector space if *bounded* if for every neighbourhood V of 0 there exists $s > 0$ such that $E \subset tV$ for $t > s$. For normed vector spaces, we recover a more familiar definition.

Proposition 1.1.3

Let X be a normed vector space. A set $E \subset X$ is bounded if and only if $\sup_{x \in E} \|x\| < \infty$.

Proof. Let $E \subset X$ be bounded. Then for the open neighbourhood $B_1(0)$ of 0, there exists $t > 0$ such that

$$E \subset tB_1(0) = B_t(0),$$

hence $\sup_{x \in E} \|x\| < t$. Conversely if $\sup_{x \in E} \|x\| = M < \infty$, let $B_\epsilon()$ be a basic open neighbourhood of 0 and let $s = \frac{M}{\epsilon} > 0$. Then

$$sB_\epsilon(0) = B_M(0),$$

so if $t > s$ then $E \subset s\overline{B}_\epsilon(0) \subset tB_\epsilon(0)$. ☐

Proposition 1.1.4

Let X be a topological vector space over k. For $a \in X$ and $\lambda \in k$, the maps

$$\begin{aligned}
T_a \colon X &\longrightarrow X \\
x &\longmapsto x + a, \\
M_\lambda \colon X &\longrightarrow X \\
x &\longmapsto \lambda x
\end{aligned}$$

are homeomorphisms.

Proof. They are clearly continuous with continuous inverses T_{-a} and $M_{\lambda^{-1}}$, respectively. ☐

In some sense, the topology on X is thus determined by its local structure near the origin. This is made precise in the following.

Proposition 1.1.5

Let X be a topological vector space and β_0 a local basis at 0. Then the collection of translates

$$\beta = \{a + B : a \in X, B \in \beta_0\}$$

is a basis for X.

Proof. β clearly consists of open sets which cover X. For any $U \subset X$ open and $x \in U$, $(-x) + U$ is a neighbourhood of 0 so there exists $B \in \beta_0$ such that $0 \in B \subset (-x) + U$. Then $x \in x + B \subset U$. $\square$

There is even something to say about convexity and balancedness. Recall that a subset $U \subset X$ of an $\mathbb{R}$-vector space is *convex* if for $x, y \in U$ and $t \in [0,1]$, $tx + (1-t)y \in U$. On the other hand, U is *balanced* if $\lambda U \subset U$ for all $\lambda \in k$ with $|\lambda| \leq 1$.

Proposition 1.1.6

Let X be a topological vector space. Then

(1) If $U \subset X$ is an open neighbourhood of 0 then U contains a balanced neighbourhood V of 0. Moreover, we may demand that $V + V \subset U$.

(2) If $U \subset X$ is a convex neighbourhood of 0 then U contains a convex balanced neighbourhood of 0.

Proof. (1) Firstly since scaling is continuous, there exists $\delta > 0$ and $V \subset X$ open such that $\lambda V \subset U$ for $|\lambda| < \delta$. Let

$$W := \bigcup_{|\lambda| < \delta} \lambda V.$$

Then W is balanced, open, and contained in U.

Furthermore, note $0 + 0 = 0$, so by continuity there exists an open neighbourhood $V_1 \times V_2$ of $(0,0)$ such that $V_1 + V_2 \subset U$. Then $V = V_1 \cap V_2$ satisfies $V + V \subset U$.

(2) If U is moreover convex, then

$$A := \bigcap_{|\lambda| = 1} \lambda U$$

contains W because $|\lambda| = 1$ implies $\lambda^{-1} W = W$. In particular, A° is a neighbourhood of the origin, and $A^\circ \subset U$. Since U is convex, so are its scalar multiples λU, and so A is convex as an intersection of convex sets. As the interior of a convex set, A° is convex. To show A is balanced, it suffices to show that $r\beta A$ for $r \in [0,1]$ and $|\beta| = 1$. Now

$$r\beta A = \bigcap_{|\lambda|=1} r\beta\lambda U = \bigcap_{|\lambda|=1} r\lambda U.$$

Here λU is a convex neighbourhood of 0, so $r\lambda U \subset \lambda U$, showing that A is balanced. We conclude that A° is balanced, convex, open, and contains 0.

$\square$

Proposition 1.1.7

Let X be a topological vector space over k. Then

(1) X is Hausdorff.

(2) $\{x\}$ is bounded for each $x \in X$.

(3) If $E_1, E_2 \subset X$ are bounded, then so is $E_1 + E_2$.

(4) If $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X and $a_n \to 0$ in k, then $a_n x_n \to 0$.

Proof. (1) Let $x \neq y \in X$. By the T1 axiom, let U be a neighbourhood of x with $y \notin U$. Then $-x + w$ is a neighbourhood of 0, so by part (1) of the previous proposition there exists a balanced V with $V + V \subset -x + U$. Hence $x + V + V \subset W$, so $y \notin x + V + V$. If there existed $x + v_1 = y + v_2 \in (x + V) \cap (y + V)$, then $y = x + a - b$. But $a, -b \in V$ so $y \in x + U + U$, a contradiction. Thus $x + U$ and $y + U$ are disjoint open neighbourhoods of x and y.

(2) For $x \in X$, let $f_x \colon \mathbb{R} \to X$ be given by $f_x(\lambda) = \lambda x$. This is the restriction of the continuous scalar multiplication to $\mathbb{R} \times \{x\}$, so it is continuous. In particular for a neighbourhood V of 0, $f_x^{-1}(V)$ is an open neighbourhood of 0, so it contains $(-\epsilon, \epsilon)$ for small $\epsilon > 0$. In other words $\lambda x \in V$ for $\lambda \in (0, \epsilon)$, or $x \in tV$ for $t > \frac{1}{\epsilon}$.

(3) Let V be a neighbourhood of 0. By the previous proposition (1), let U be a neighbourhood of 0 such that $U + U \subset V$. Since E_1, E_2 are bounded there exist $s_1, s_1 > 0$ such that $E_1 \subset tU$ for $t > s_1$ and $E_2 \subset tU$ for $t > s_2$. So for $t > s := \max\{s_1, s_2\}$ we have

$$E_1 + E_2 \subset tU + tU \subset t(U + U) \subset tV.$$

(4) Let V be an open neighbourhood of 0. Let $U \subset V$ be a balanced open set. Since (x_n) is bounded, there exists $s > 0$ such that $(x_n) \subset tU$ for $t > s$. Since $a_n \to 0$, there exists N such that $|a_n| < s^{-1}$ for $n > N$. By balancedness of U, and the fact that $|ta_n| < 1$ for $n > N$, we have $a_n x_n \in U \subset V$ for $n > N$.

$\square$

Let X be a vector space with a metric $d \colon X \times X \to \mathbb{R}$. We say d is *invariant* if

$$d(x + z, y + z) = d(x, y)$$

for $x, y, z \in X$. In particular

$$d(nx, 0) \leq nd(x, 0). \tag{1.1.1}$$

Indeed, $n = 1$ is trivial, and by strong induction

$$\begin{aligned} d(kx, 0) &\leq d(kx, x) + d(x, 0) \\ &= d((k-1)x, 0) + d(x, 0) \\ &\leq (k-1)d(x, 0) + d(x, 0) \\ &= kd(x, 0). \end{aligned}$$

Proposition 1.1.8

Let X be a vector space with an invariant metric. Given a sequence $x_n \to 0$ in X, there exist scalars $a_n \to \infty$ such that $a_n x_n \to 0$.

Proof. For any $m \in \mathbb{N}$ there exists N_m such that

$$d(x_n, 0) < \frac{1}{m^2}$$

for $n > N_m$. If this choice of N_m is tight, then $N_m < N_{m+1}$. Define $a_n = m$ for $N_m < n \leq N_{m+1}$; clearly $a_n \to \infty$. But if $N_m < n \leq N_{m+1}$, we have by equation (1.1.1) that

$$d(a_n x_n, 0) \leq m d(x_n, 0) < \frac{1}{m}$$

so $a_n x_n \to 0$. $\square$

1.2 Complete metric spaces

Let (X, d) be a metric space. Recall that a sequence (x_n) is *d-Cauchy* if for any $\epsilon > 0$ there exists N such that $d(x_n, x_m) < \epsilon$ for $n, m > N$. We say X is *complete* if every d-Cauchy sequence converges. In another setting, we have

Definition 1.2.1

Let (X, τ) be a topological vector space. A sequence (x_n) is *τ-Cauchy* if for any neighbourhood U of 0 there exists N such that $x_n - x_m \in U$ for $n, m > N$.

Proposition 1.2.2

Let X be a vector space with an invariant metric d which induces a topology τ. Then (x_n) is d-Cauchy if and only if τ-Cauchy.

Proof. If (x_n) is τ-Cauchy, then for any $\epsilon > 0$ there exists N such that $x_n - x_m \in B_\epsilon(0)$ for $n, m > N$. In other words,

$$d(x_n, x_m) = d(0, x_n - x_m) < \epsilon.$$

Conversely if (x_n) is d-Cauchy, let U be any neighbourhood of 0. Let $\epsilon > 0$ be such that $B_\epsilon(0) \subset V$. Since (x_n) is d-Cauchy, there exists N such that $d(x_n, x_m) < \epsilon$ for $n, m > N$, so $x_n - x_m \in B_\epsilon(0) \subset U$. $\square$

1.3 Topological vector space zoo

Some rapidfire definitions: a topological vector space X is

(i) *locally convex* if there exists a local basis at 0 consisting of convex subsets.

(ii) *locally bounded* if 0 has a bounded neighbourhood.

(iii) *locally compact* if 0 has a relatively compact neighbourhood.

(iv) *metrizable* if its topology can be induced by a metric.

(v) an *F-space* if its topology is induced by a complete invariant metric.

(vi) *Fréchet* if a locally convex F-space.

(vii) *normable* if its topology is induced by a norm.

(viii) *Banach* if normable and complete with respect to the induced invariant metric.

(ix) *Heine–Borel* if every closed and bounded set is compact.

The converse of the Heine–Borel property is obtained for free in topological vector spaces:

Proposition 1.3.1

Let $K \subset X$ be a compact subset of a topological vector space. Then K is closed and bounded.

Proof. A compact subset of a Hausdorff space is closed. For boundedness, let U be a neighbourhood of 0. Let $V \subset U$ be a balanced open neighbourhood of 0. We claim that

$$\bigcup_{n \in \mathbb{N}} nV = X.$$

Indeed for $x \in X$, $f_x(\lambda) = \lambda x$ is continuous so $\{\lambda \in \mathbb{R} : \lambda x \in V\}$ is open in $\mathbb{R}$ and contains 0, so it contains $\frac{1}{n}$ for large n. This means $x \in nV$ for large n. By compactness of K, finitely many nV cover K, say for $n_1 < \cdots < n_N$. Since V is balanced, in fact $n_i < n_N$ implies

$$n_i V \subset n_N V \subset n_N U$$

so $K \subset n_N U$ is bounded. $\square$