Integration

William Gao

December 28, 2025

1 Integrals

Let $(E, \mathcal{E}, \mu)$ be a measure space. Today we use it to define a notion of integration. We begin by considering nonnegative measurable functions $f\colon (E, \mathcal{E}) \to ([0, \infty), \mathcal{B}([0, \infty)))$. Just as the Riemann integral approximates functions by rectangles, we approximate nonnegative measurable functions by simple functions.

Definition 1.1

A nonnegative measurable function is *simple* if

$$f = \sum_{n=1}^{k} \alpha_n \mathbb{1}_{A_n} \quad \text{for} \quad \alpha_n \in \mathbb{R}, A_n \in \mathcal{E}$$

is a finite linear combination of characteristic functions.

For a nonnegative simple function we define

$$\int_E f \, d\mu = \mu(f) := \sum_{n=1}^{k} \alpha_n \mu(A_n).$$

We use the convention $0 \cdot \infty = 0$. To see that this is independent of the choice of linear combination, suppose $0 \leq \alpha_n, \beta_n < \infty$ and A_n, $B_n \in \mathcal{E}$ are such that

$$\sum_{n=1}^{k} \alpha_n \mathbb{1}_{A_n} = \sum_{n=1}^{\ell} \beta_n \mathbb{1}_{B_n}.$$

Then

$$\begin{aligned}\sum_{n=1}^{k}\alpha_n\mu(A_n) &= \sum_{n=1}^{k}\alpha_n\sum_{m=1}^{\ell}\mu(A_n\cap B_m)\\ &= \sum_{n=1}^{k}\alpha_n\sum_{m=1}^{\ell}\mu(A_n\cap B_m)\\ &= \sum_{m=1}^{\ell}\beta_m\sum_{n=1}^{k}\mu(A_n\cap B_m)\\ &= \sum_{m=1}^{\ell}\beta_m\mu(B_n),\end{aligned}$$

where we have used the assumption in the fourth equality. One can easily check that this definition of the integral is linear, preserves inequalities, and $\mu(f)=0$ if and only if $f=0$ almost everywhere. Now for a nonnegative measurable function f, we define its integral

$$\mu(f)=\int_E f\,d\mu=\sup\{\mu(g): g \text{ simple with } 0\le g\le f\}.$$

Once again this preserves inequalities, and for any $\epsilon>0$ there exists a simple function f_ϵ such that

$$\int_E |f-f_\epsilon|\,d\mu=\int_E (f-f_\epsilon)\,d\mu<\epsilon.$$

Moving onto any real-valued function, recall that if $f\colon E\to\mathbb{R}$ is measurable then so are the nonnegative functions f^+, f^-, and $|f|$. We say f is integrable if $\mu(|f|)<\infty$, in which case we define

$$\mu(f)=\mu(f^+)-\mu(f^-).$$

Since $f\le g$ if and only if $f^+\le g^+$ and $f^-\ge g^-$, we have $\mu(f)\le\mu(g)$ for $f\le g$. In particular, $|\mu(f)|\le\mu(|f|)$, and as before given $\epsilon>0$ there exists a simple function f_ϵ such that

$$\int_E |f-f_\epsilon| d\mu<\epsilon,$$

by applying the above approximation result to both f^+ and f^-.

If exactly one of $\mu(f^+)=\infty$ or $\mu(f^-)=\infty$ holds, then we can still define $\mu(f)$ by the same formula. However if they are both infinite, then there is no sensible value for $\mu(f)$.

More generally if $f\colon E\to\mathbb{R}$, we say f is integrable if each $f_i=\pi_i\circ f$ is integrable, and we define

$$\int_E f\,d\mu=\sum_{i=1}^{n}e_i\int_E f_i\,d\mu.$$

2 Convergence theorems

Theorem 2.1 (Monotone convergence)

Let $(E, \mathcal{E}, \mu)$ be a measure space and $(f_n)_{n=1}^{\infty}$ an increasing sequence of nonnegative measurable functions $E \to [0, \infty)$. Then for each $x \in E$,

$$f(x) = \lim_{n \in \infty} f_n(x)$$

exists in $[0, \infty]$. We know f is measurable; the monotone convergence theorem states that

$$\int_E f \, d\mu = \lim \int_E f_n \, d\mu.$$

Proof. Let $M = \sup_n \mu(f_n)$. We claim that $M = \mu(f)$. Since the f_n are increasing, we have $f_n \le f$ so $\mu(f_n) \le \mu(f)$ for all n, thus

$$M \le \mu(f) = \sup\{\mu(g) : g \text{ simple with } g \le f\}.$$

Let $0 \le g \le f$ be a simple function. We wish to show $\mu(g) \le M$. If $g = \sum_{i=1}^{m} a_k \mathbb{1}_{A_k}$, then without loss of generality assume the A_k are disjoint. Let $g_n(x) = \min\{g(x), \frac{1}{2^n} \lfloor 2^n f_n \rfloor\}$. Then the g_n form an increasing sequence of simple functions such that $g_n \le f_n \le f$ and $g_n \to g$. For $\epsilon \in (0, 1)$ let

$$A_{k,n} := \{x \in A_k : g_n(x) \ge (1 - \epsilon) a_k\}.$$

Since the g_n are increasing, we have $A_{k,n} \subset A_{k,n+1}$. By countable additivity of μ, we have $\mu(A_{k,n}) \to \mu(A_k)$ as $n \to \infty$, and by construction

$$\mathbb{1}_{A_k} g_n \ge (1 - \epsilon) a_k \mathbb{1}_{A_{k,n}}$$

which implies

$$\mu(\mathbb{1}_{A_k} g_n) \ge (1 - \epsilon) a_k \mu(A_{k,n}).$$

By linearity $g = \sum_{k=1}^{m} \mathbb{1}_{A_k} g_n$ we have

$$\mu(g_n) \ge (1 - \epsilon) \sum_{k=1}^{m} a_k \mu(A_{k,n}) \to (1 - \epsilon) \sum_{k=1}^{m} a_k \mu(A_k) = (1 - \epsilon) \mu(g).$$

Now the desired inequality $\mu(g) \le M$ follows from

$$\mu(g_n) \le \mu(f_n) \le M,$$

taking $\epsilon \to 0$. ∎

Corollary 2.2

Let (f_n) be a sequence of nonnegative measurable functions on E. Then

$$\int_E \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu.$$

Proof. Apply the monotone convergence theorem to the increasing sequence $(\sum_{n=1}^{N} f_n)$. ☐

Corollary 2.3

Let (f_n) be a decreasing sequence of bounded measurable functions. Then

$$\int_E f\,d\mu = \lim \int_E f_n\,d\mu.$$

Proof. We take $g_n = f_1 - f_n$. Since the f_n are bounded we avoid $\infty - \infty$ problems, so this is well-defined. Then apply the monotone convergence theorem to the increasing sequence (g_n). ☐

The monotone convergence theorem allows us to generalize the following facts from simple functions to arbitrary measurable functions:

- $\int_E (af + bg)\,d\mu = a\int_E f\,d\mu + b\int_E g\,d\mu$.
- If $f \le g$ then $\int_E f\,d\mu \le \int_E g\,d\mu$.
- $\int_E f\,d\mu = 0$ if and only if $f = 0$ almost everywhere.

As a result, we can integrate over a restriction $(A, \mathcal{E}\big|_A, \mu\big|_A)$ of our measure space, so that

$$\int_A f\,d\mu = \int_E f \cdot \mathbb{1}_A\,d\mu = \int_A f\big|_A\,d\mu\big|_A.$$

Moreover if A, B are disjoint then

$$\int_A f\,d\mu + \int_B f\,d\mu = \int_{A\cup B} f\,d\mu$$

and if $|f| \le K$ almost everywhere and $\mu(E) < \infty$, then f is integrable and

$$\left|\int_E f\,d\mu\right| \le K\mu(E).$$

A more interesting result connects the Lebesgue integral to the Riemann integral.

Theorem 2.4

Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a rectangle in $\mathbb{R}^n$ and let $f\colon A \to \mathbb{R}^n$ be bounded. Then f is Riemann integrable if and only if f is continuous almost everywhere. In this case f is Lebesgue integrable and the two integrals agree:

$$\int_A f(x)\,dx = \int_A f\,d\mu.$$

Proof. The first part is well-known. We show f is Lebesgue integrable and the integrals agree. Since f is bounded, without loss of generality $0 \le f \le K$. Let $\mathcal{P}_n$ be a sequence of partitions of A such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ and the mesh of $\mathcal{P}_n$ goes to 0 as $n \to \infty$. Let

$$\underline{f}_n := \sum_{\pi \in \mathcal{P}_n} \inf_\pi f \cdot \mathbb{1}_\pi \quad \text{and} \quad \overline{f}_n := \sum_{\pi \in \mathcal{P}_n} \sup_\pi f \cdot \mathbb{1}_\pi,$$

so that

$$0 \le \underline{f}_n \le \underline{f}_{n+1} \le f \le \overline{f}_{n+1} \le \overline{f}_n \le K.$$

Since each $\pi \in \mathcal{P}_n$ is a rectangle, it is Lebesgue measurable and $\underline{f}_n$, $\overline{f}_n$ are simple functions. Moreover

$$\int_A \underline{f}_n \, d\mu = L(f, \mathcal{P}_n) \quad \text{and} \quad \int_A \overline{f}_n \, d\mu = U(f, \mathcal{P}_n).$$

Now $(\underline{f}_n)$ is a monotone increasing sequence bounded above by f, so its limit (equivalently supremum) is a bounded measurable function $\underline{f} \le f$. Similarly, the limit or infimum of $\overline{f}_n$ is a bounded measurable function $\overline{f}$. By monotone convergence,

$$\lim \int_A \underline{f}_n \, d\mu = \int_A \underline{f} \, d\mu \le \int_A \overline{f} \, d\mu = \lim \int_A \overline{f}_n \, d\mu.$$

As we can see these correspond to the upper and lower Riemann integrals, and their difference becomes 0. By squeezing, the integrals agree. $\square$

Lemma 2.5 (Fatou)

Let $(f_n)_{n=1}^\infty$ be a sequence of nonnegative measurable functions $E \to \mathbb{R}$. Then

$$\int_E \liminf f_n \, d\mu \le \liminf \int_E f_n \, d\mu.$$

Proof. Apply monotone convergence theorem to the increasing sequence $g_n = \inf_{m \ge n} f_m$ of nonnegative measurable functions, which goes to $\liminf f_n$. Indeed, $g_n \le f_k$ for $k \ge n$ so

$$\int_E g_n \, d\mu \le \int_E f_k \, d\mu \quad \text{for} \quad k \ge n.$$

Hence

$$\int_E g_n \, d\mu \le \inf_{k \ge n} \int_E f_k \, d\mu$$

and we are done as $n \to \infty$. $\square$

Theorem 2.6 (Dominated convergence)
Let (f_n) be a sequence of measurable functions such that

(a) There exists an integrable function g such that $|f_n| \leq g$.

(b) $f_n(x) \to f(x)$ for all x.

Then f is integrable and
$$\int_E f_n \, d\mu \to \int_E f \, d\mu.$$

Proof. We know f is measurable and $absf \leq g$ implies f is integrable. Note
$$0 \leq g \pm f_n \to g \pm f,$$
so $\liminf g \pm f_n = g \pm f$. By Fatou's lemma, we have
$$\begin{aligned}\int_E g \, d\mu + \int_E f \, d\mu &= \int_E \liminf (g + f_n) \, d\mu \\ &\leq \liminf \int_E g + f_n \, d\mu \\ &= \int_E g \, d\mu + \liminf \int_E f_n \, d\mu.\end{aligned}$$
Similarly
$$\int_E g \, d\mu - \int_E f \, d\mu \leq \int_E g \, d\mu - \limsup \int_E f_n \, d\mu.$$
Rearranging,
$$\int_E f \, d\mu \leq \liminf \int_E f_n \, d\, u \leq \limsup \int_E f_n \, d\mu \leq \int_E f \, d\mu,$$
hence $\liminf \int_E f_n \, d\, u = \limsup \int_E f_n \, d\mu$. □

3 Product measures

Given measure spaces $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \nu)$ we will construct a measure space on $E \times F$. A *rectangle* is a subset of $E \times F$ of the form $A \times B$ with $A \in \mathcal{E}$, $B \in \mathcal{F}$. Let $\mathcal{E} \boxtimes \mathcal{F}$ denote the collection of finite disjoint unions of rectangles. Then for $A_i \in \mathcal{E}$ and $B_i \in \mathcal{F}$,
$$\begin{aligned}(A_1 \times B_1) \cap (A_2 \times B_2) &= (A_1 \cap A_2) \times (B_1 \cap B_2) \\ (A_1 \times B_1) \cup (A_2 \times B_2) &= (A_1 \times B_1 \setminus B_2) \cup (A_1 \cup A_2 \times B_1 \cap B_2) \cup (A_2 \times B_2 \setminus B_1) \\ (A_1 \times B_1)^c &= (E \times B_1^c) \cup (A_1^c \times F),\end{aligned}$$
so $\mathcal{E} \boxtimes \mathcal{F}$ is an algebra. Let $\mathcal{E} \otimes \mathcal{F}$ be the σ-algebra it generates and define a set function
$$\begin{aligned}\pi \colon \mathcal{E} \boxtimes \mathcal{F} &\longrightarrow [0, \infty] \\ \bigcup_{i=1}^N A_i \times B_i &\longmapsto \sum_{i=1}^N \mu(A_i)\nu(B_i).\end{aligned}$$

We claim that for a sequence of disjoint rectangles $(A_j \times B_j)$ with $\bigcup_{j=1}^{\infty} A_j \times B_j = A \times B \in \mathcal{E} \times \mathcal{F}$, we have

$$\mu(A)\mu(B) = \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j).$$

Indeed, note that

$$\mathbb{1}_A(x)\mathbb{1}_B(y) = \mathbb{1}_{A\times B}(x,y) = \sum_{j=1}^{\infty} \mathbb{1}_{A_j\times B_j}(x,y) = \sum_{j=1}^{\infty} \mathbb{1}_{A_j}(x)\mathbb{1}_{B_j}(y).$$

Integrating with respect to x, we get

$$\mu(A)\mathbb{1}_B(y) = \sum_{j=1}^{\infty} \mathbb{1}_{B_j}(y) \int \mathbb{1}_{A_j}\, d\mu = \sum_{j=1}^{\infty} \mathbb{1}_{B_j}(y)\mu(A_j).$$

Integrating with respect to y,

$$\mu(A)\mu(B) = \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j).$$

This shows that $\pi(C)$ is well-defined for $C \in \mathcal{E} \boxtimes \mathcal{F}$. Moreover this satisfies the hypothesisof Carathéodory's theorem, so we get an outer measure π^* on $E \times F$ whose restriction to $\mathcal{E} \otimes \mathcal{F}$ gives a measure which agrees with π on $\mathcal{E} \boxtimes \mathcal{F}$. We call this measure the product measure $\mu \times \nu$. We now consider integration of a measurable function $f\colon E \times F \to \mathbb{R}$. If $x \in E$, $y \in F$ we define

$$f_x(y) = f(x,y) = f^y(x).$$

Also

$$A_x = \pi_2(\pi_1^{-1}(x)) \subset F \quad \text{and} \quad A^-y = \pi_1(\pi_2^{-1}(y)) \subset E.$$

Note that

$$(\mathbb{1}_A)_x = \mathbb{1}_{A_x} \quad \text{and} \quad (\mathbb{1}_A)^y = \mathbb{1}_{A^y}.$$

Also, if f is $\mathcal{E} \otimes \mathcal{F}$-measurable, then f_x is $\mathcal{F}$-measurable and f^y is $\mathcal{E}$-measurable. Indeed

$$(f_x)^{-1}(S) = (f^{-1}(S))_x \quad \text{and} \quad (f^y)^{-1}(S) = (f^{-1}(S))^y.$$

We use use this to prove special cases of the Tonelli and Fubini theorems. First, a lemma:

Lemma 3.1

Let μ, ν be finite and $A \in \mathcal{E} \otimes \mathcal{F}$. Then

$$x \mapsto \nu(A_x) \quad \text{and} \quad y \mapsto \mu(A^y)$$

are measurable functions and

$$(\mu \times \nu)(A) = \int_E \nu(A_x)\, d\mu(x) = \int_E \mu(A^y)\, d\nu(y).$$

Proof. This is clearly true for rectangles A. Let $\mathcal{C}$ consist of all the sets $A \in \mathcal{E} \otimes \mathcal{F}$ for which the conclusion holds. Since rectangles form a π-system, if we can show that $\mathcal{C}$ is a d-system then the result will follow. Clearly $E \times F \in \mathcal{C}$. Now suppose $B \subset A \in \mathcal{C}$. Then

$$(A \setminus B)_x = A_x \setminus B_x$$

so

$$\nu((A \setminus B)_x) = \nu(A_x) - \nu(B_x)$$

so $x \mapsto \nu((A \setminus B)_x)$ is measurable and

$$\begin{aligned}(\mu \times \nu)(A \setminus B) &= (\mu \times \nu)(A) - (\mu \times \nu)(B) \\ &= \int_E \nu(A_x)\, d\mu(x) - \int_E \nu(B_x)\, d\mu(x) \\ &= \int_E \nu((A \setminus B)_x)\, d\mu(x).\end{aligned}$$

Similarly for $(A \setminus B)^y$, we see that $A \setminus B \in \mathcal{C}$. Now if $(A_n) \in \mathcal{C}$ is an increasing sequence with union A, then by countable addivity

$$(x \mapsto \nu((A_n)_x)_{n=1}^{\infty})$$

is a monotone increasing sequence of functions with limit $\nu(A_x)$ so by monotone convergence $\nu(A_x)$ is measurable and

$$\int_E \nu(A_x)\, d\mu(x) = \lim \int_E \nu((A_n)_x)\, d\mu(x) = \lim (\mu \times \nu)(A_n) = \mu \times \nu(A).$$

Similarly for $\mu(A^y)$, we see that $A \in \mathcal{C}$ and we are done. □

Theorem 3.2 (Tonelli)

Let $(E, \mathcal{E}, \mu)$, $(F, \mathcal{F}, \nu)$ be σ-finite measure spaces. If $f \colon E \times F \to [0, \infty]$ is a nonnegative measurable function, then

$$h(x) = \int_F f_x(y)\, d\nu(y) \quad \text{and} \quad g(y) = \int_E f^y(x)\, d\mu(x)$$

are nonnegative measurable functions with

$$\int_{E \times F} f\, d(\mu \times \nu) = \int_E h\, d\mu = \int_F g\, d\nu.$$

Proof. Let (f_n) be a monotone increasing sequence of nonnegative simple functions converging to f. Let

$$h_n(x) = \int_F (f_n)_x(y)\, d\nu(y) \quad \text{and} \quad g_n(y) = \int_E (f_n)^y(x)\, d\mu(x),$$

so that

$$\int_{E \times F} f_n\, d(\mu \times \nu) = \int_E h_n\, d\mu = \int_F g_n\, d\nu$$

by the previous lemma and linearity. For each $x \in E$, the monotone increasing sequence $(f_n)_x$ converges to f_x and similarly for $(f_n)^y$. Thus (h_n) is increasing and converges to h, and similarly for g_n. By passing to the limit we obtain the desired result. $\square$

Theorem 3.3 (Fubini)

Let $(E, \mathcal{E}, \mu)$, $(F, \mathcal{F}, \nu)$ be σ-finite measure spaces. Let $f\colon E \times F \to \mathbb{R}$ be integrable. Then f_x is integrable for μ-almost all $x \in E$ and f^y is integrable for ν-almost all y. Thus

$$h(x) = \int_F f_x(y)\,d\nu(y) \quad \text{and} \quad g(y) = \int_E f^y(x)\,d\mu(x)$$

are defined almost everywhere.$h\colon E \to \mathbb{R}$ and $g\colon F \to \mathbb{R}$ are integrable and

$$\int_{E\times F} d(\mu \times \nu) = \int_E h\,d\mu = \int_F g\,d\nu.$$

Proof. Write $f = f^+ - f^-$; Tonelli's theorem implies

$$\int_{E\times F} f^\pm\,d(\mu \times \nu) = \int_E h^\pm\,d\mu = \int_F g^\pm\,d\nu.$$

The first integral is finite, so $h^\pm$ and $g^\pm$ must be finite almost everywhere and integrable. The result follows from combining the above equations for $h = h^+ - h^-$ and $g = g^+ - g^-$. $\square$