Functoriality of $\theta \mapsto \theta_*$:

- $id_n : (0 \to \cdots \to n) \to (0 \to \cdots \to n)$

  $\Downarrow$

  $(id_n)_* = id_{|\Delta^n|}$ clear.

- $f : (0 \to \cdots \to n) \to (0 \to \cdots \to m)$, $g : (0 \to \cdots \to m) \to (0 \to \cdots \to p)$

$\rightsquigarrow (g \circ f)_* (t_0 \ldots t_n) = (u_0 \ldots u_p)$ where $u_i = \sum_{k \in (g \circ f)^{-1}(i)} t_k$

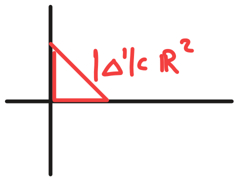
$= \sum_{k \in f^{-1}(g^{-1}(i))} t_k$

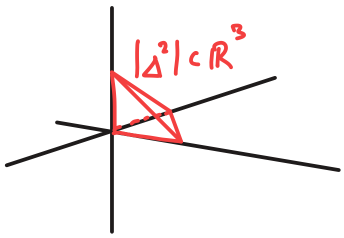
$= \sum_{j \in g^{-1}(i)} s_j$ where $s_j = \sum_{k \in f^{-1}(j)} t_k$

i.e. $f_* (t_0 \ldots t_n) = (s_0 \ldots s_m)$

$= g_* (s_0 \ldots s_m)$

$= g_* \circ f_* (t_0 \ldots t_n)$ $\square$.





T top. space $\rightsquigarrow$ singular set $S(T) : \Delta^{op} \to Sets$

given by $(0 \to \cdots \to n) \mapsto \hom_{Top}(|\Delta^n|, T)$

functoriality: recall singular homology.

important morphisms in Δ:
- $d^i : n-1 \to n$ $\quad 0 \le i \le n$. "coface" image is i^{th} face

  $(0 \to 1 \to \cdots \to n-1) \mapsto (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$.

- $s^j : n+1 \to n$ $\quad 0 \le j \le n$ "codegeneracy" image is j^{th} degen

  $(0 \to 1 \to \cdots \to n+1) \mapsto (0 \to 1 \to \cdots \to j \to j \to \cdots \to n)$.

"cosimplicial identities":
$$\begin{cases} d^j d^i = d^i d^{j-1} & i < j \quad \to \text{e.g. deleting two vertices} \\ s^j d^i = d^i s^{j-1} & i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & i > j+1 \\ s^j s^i = s^i s^{j+1} & i \le j. \end{cases}$$

Δ is wholly determined by the d^j, s^i subject to these relations.

So a simplicial set is determined by sets $(Y_n)_{n \in \mathbb{N}}$

with maps (faces) $d_i : Y_n \to Y_{n-1}$ s.t.

(degeneracies) $s_j : Y_n \to Y_{n+1}$

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & i > j+1 \\ s_i s_j = s_{j+1} s_i & i \le j \end{cases}$$

for $S(T)$, d_i is restriction to i^{th} face

s_j is collapsing two vertices together

Homology (!)

$Y : \Delta^{op} \to \mathrm{Sets}$ simplicial set.

$\rightsquigarrow$

$\mathbb{Z}Y : \Delta^{op} \to \mathrm{Ab}$ simplicial ab gp

by $\mathbb{Z}Y_n$ = free ab gp on Y_n

extend d_i, s_j by freeness.

$\rightsquigarrow$

chain complex "Moore".

$$\mathbb{Z}Y_0 \xleftarrow{\partial_1} \mathbb{Z}Y_1 \xleftarrow{\partial_2} \mathbb{Z}Y_2 \xleftarrow{\partial_3} \cdots$$

where $\partial_n = \sum_{i=1}^{n} (-1)^i d_i$. ($\partial^2 = 0$ by simplicial identities)

$\rightsquigarrow$

homology $H_*(Y;\mathbb{Z})$ of Y.

- if $T \in \mathrm{Top}$, $H_*(T;\mathbb{Z}) = H_*(S(T);\mathbb{Z})$.

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more generally, $\underbrace{\mathbb{Z}Y \otimes A}_{\text{Ab gp.}}$ (chain complex) $\rightsquigarrow$ $H_n(Y, A) := H_n(\mathbb{Z}Y \otimes A)$.