

# EXISTENCE WITHOUT UNIQUENESS

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## 1. INTRODUCTION

We will prove the following existence theorem of ordinary differential equations:

**Theorem 1.1.** (*Cauchy-Peano*) Let  $t_0, t_1 \in \mathbb{R}$  and  $x_0 \in \mathbb{R}^n$ . Suppose  $f : [t_0, t_1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous and bounded. Then there exists a solution  $x : [t_0, t_1] \rightarrow \mathbb{R}^n$  to the initial value problem

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

Uniqueness is notably not guaranteed under these assumptions, hence history has hawked this great theorem of its glory.

## 2. FUNCTION SPACES AND ARZELÀ-ASCOLI

We begin with some preliminary results about function spaces. Let  $E \subseteq \mathbb{R}^n$  be compact. All functions we consider in this section will be  $E \rightarrow \mathbb{R}^p$ .

**Definition 2.1.** A family  $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$  of functions is *uniformly bounded* if there exists  $M \geq 0$  such that for all  $x \in E$  and  $\alpha \in A$ ,  $|f_\alpha(x)| \leq M$ .

**Definition 2.2.** A family  $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$  of functions is *equicontinuous* if for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $\alpha \in A$  and  $x, y \in E$  with  $|x - y| < \delta$ ,  $|f_\alpha(x) - f_\alpha(y)| < \varepsilon$ .

**Definition 2.3.** A sequence  $(f_n)_{n \in \mathbb{N}}$  of functions *converges uniformly* to  $f$ , and we write  $f_n \rightrightarrows f$ , if for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $x \in E$  and  $n \geq N$ ,  $|f_n(x) - f(x)| < \varepsilon$ .

The following is the reason why we care about uniform convergence.

**Lemma 2.4.** Suppose  $(f_n)_{n \in \mathbb{N}}$  is a sequence of continuous functions and  $f_n \rightrightarrows f$ . Then  $f$  is continuous.

*Proof.* For each  $x \in E$ , and any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|f(y) - f_N(y)| < \frac{\varepsilon}{3}$$

for all  $y \in E$ . Since  $f_N$  is continuous, there exists  $\delta > 0$  such that  $|y - x| < \delta$  implies

$$|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}.$$

Then if  $|y - x| < \delta$ ,

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

□

The most natural way to study uniform convergence is in a function space. Let  $\mathcal{C}(E, \mathbb{R}^p)$  be the space of continuous functions  $E \rightarrow \mathbb{R}^p$ .

**Definition 2.5.** We define the sup norm  $\|\cdot\|_\infty : \mathcal{C}(E, \mathbb{R}^p) \rightarrow \mathbb{R}$  by

$$\|f\|_\infty = \sup_{x \in E} |f(x)|.$$

This is indeed a norm. First  $\|f\|_\infty \geq |f(x)| \geq 0$  with  $\|f\|_\infty = 0$  if and only if  $|f(x)| = 0$  for all  $x \in E$ , or  $f = 0$ . Second

$$\|cf\| = \sup_{x \in E} |cf(x)| = |c| \sup_{x \in E} |f(x)| = |c| \|f\|,$$

and third

$$\begin{aligned} \|f + g\| &= \sup_{x \in E} |f(x) + g(x)| \\ &\leq \sup_{x \in E} |f(x)| + |g(x)| \\ &\leq \sup_{x \in E} |f(x)| + \sup_{x \in E} |g(x)| \\ &= \|f\| + \|g\|. \end{aligned}$$

The sup norm naturally induces a metric  $d(f, g) = \|f - g\|_\infty$ . This metric yields a useful criterion for uniform convergence.

**Lemma 2.6.** Suppose  $(f_n)_{n \in \mathbb{N}}$  is a sequence of continuous functions. Then  $f_n \rightrightarrows f$  if and only if  $f_n \xrightarrow{\|\cdot\|_\infty} f$ , or equivalently  $\|f_n - f\|_\infty \rightarrow 0$ .

*Proof.* If  $\|f_n - f\|_\infty \rightarrow 0$ , then  $\sup_{x \in E} |f_n(x) - f(x)| \rightarrow 0$ . Hence there exists  $N \in \mathbb{N}$  such that for  $n \geq N$ ,  $\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$ , or equivalently,  $|f_n(x) - f(x)| < \varepsilon$  for all  $x \in E$ .

Conversely if  $f_n \rightrightarrows f$ , then there exists  $N \in \mathbb{N}$  such that for all  $x \in E$  and  $n \geq N$ ,  $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ . This means for  $n \geq N$ ,  $\|f_n - f\|_\infty = \sup_{x \in E} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$ , and thus  $\|f_n - f\|_\infty \rightarrow 0$ . □

It is even easier to see that  $A \subseteq \mathcal{C}(E, \mathbb{R}^p)$  is uniformly bounded if and only if  $A$  is bounded with respect to  $\|\cdot\|_\infty$ .

**Theorem 2.7.**  $(\mathcal{C}(E, \mathbb{R}^p), \|\cdot\|_\infty)$  is a complete metric space.

*Proof.* Let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathcal{C}(E, \mathbb{R}^p)$ . For each  $x_0 \in E$ ,  $(f_n(x_0))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}^p$ , as

$$|f_n(x_0) - f_m(x_0)| \leq \sup_{x \in E} |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty.$$

Thus for each  $x \in E$ ,  $\lim_{n \rightarrow \infty} f_n(x)$  exists; we assign this value to  $f(x)$ . We claim that  $f_n \rightrightarrows f$ . Given  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $m, n \geq N$  implies

$$\|f_n - f_m\|_\infty < \frac{\varepsilon}{2}.$$

Also by definition of  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ , for each  $x$  there exists  $m \geq N$  such that

$$|f_m(x) - f(x)| < \frac{\varepsilon}{2}.$$

Hence if  $n \geq N$  and  $x \in E$ , then

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq \|f_n - f_m\|_\infty + |f_m(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus  $f_n \rightrightarrows f$ . By 2.4,  $f \in \mathcal{C}(E, \mathbb{R}^p)$ . By 2.6, the Cauchy sequence  $(f_n)_{n \in \mathbb{N}}$  converges in  $\mathcal{C}(E, \mathbb{R}^p)$ .  $\square$

The next lemma will be necessary to our proof of the Cauchy-Peano theorem.

**Lemma 2.8.** *Suppose  $(f_n)_{n \in \mathbb{N}}$  is a sequence of Riemann integrable functions in  $\mathcal{C}([a, b], \mathbb{R})$  and  $f_n \rightrightarrows f$ . Then  $f$  is Riemann integrable and*

$$\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n.$$

*Proof.* Since each  $f_n$  is integrable, it is continuous except possibly on a measure zero subset  $D_n$  of  $[a, b]$ .  $\bigcup_{n \in \mathbb{N}} D_n$  is again measure zero, and on  $[a, b] \setminus \bigcup_{n \in \mathbb{N}} D_n$ , each  $f_n$  is continuous. By 2.4,  $f$  is continuous on  $[a, b] \setminus \bigcup_{n \in \mathbb{N}} D_n$ , and by 2.7,  $f$  is bounded on  $[a, b]$ . Thus  $f$  is integrable on  $[a, b]$ . Furthermore

$$\begin{aligned} \left| \int_a^b f(x) dx - \int_a^b f_n dx \right| &= \left| \int_a^b f(x) - f_n(x) dx \right| \\ &\leq \int_a^b |(f - f_n)(x)| dx \\ &\leq \|f - f_n\|_\infty (b - a), \end{aligned}$$

which tends to 0 as  $n \rightarrow \infty$ . Therefore  $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$ .  $\square$

We now prove the Arzelà-Ascoli theorem, the main result needed for our existence theorem.

**Lemma 2.9.** *Suppose  $(f_k)_{k \in \mathbb{N}}$  is a subsequence of  $(g_n)_{n \in \mathbb{N}}$ . Then for each  $k$ ,  $f_k = g_r$  for some  $r \geq k$ .*

*Proof.* By definition of a subsequence,  $f_k = g_{n_k}$  for some  $n_k$  such that  $1 \leq n_1 < n_2 < \dots < n_k$ . Thus  $r = n_k \geq k$ .  $\square$

**Theorem 2.10.** (Arzelà-Ascoli) *Every uniformly bounded and equicontinuous sequence  $(f_n)_{n \in \mathbb{N}}$  of functions in  $\mathcal{C}(E, \mathbb{R}^p)$  has a uniformly convergent subsequence.*

*Proof.* Let  $D = \{d_j\}_{j \in \mathbb{N}}$  be a countable dense set in  $E$ , for example  $D = \mathbb{Q}^n \cap E$ . By uniform boundedness of  $(f_n)$ , let  $M \geq 0$  be such that for all  $x \in E$  and all  $n \in \mathbb{N}$ ,  $|f_n(x)| \leq M$ . Then  $(f_n(d_1))_{n \in \mathbb{N}}$  is a bounded sequence in  $\mathbb{R}^p$ , since  $|f_n(d_1)| \leq M$  for all  $n \in \mathbb{N}$ . By the Bolzano-Weierstrass theorem,  $(f_n(d_1))$  has a convergence subsequence, say

$$\lim_{k \rightarrow \infty} f_{1,k}(d_1) = y_1.$$

$(f_{1,k}(d_2))_{k \in \mathbb{N}}$  is bounded in  $\mathbb{R}^p$ . Another invocation of Bolzano-Weierstrass produces a sub-subsequence  $(f_{2,k})_{k \in \mathbb{N}}$  such that

$$\lim_{k \rightarrow \infty} f_{2,k}(d_2) = y_2.$$

Furthermore  $\lim_{k \rightarrow \infty} f_{2,k}(d_1) = y_1$ . Inductively on  $m$ , we define a nested family of subsequences  $(f_{m,k})_{k \in \mathbb{N}}$  of  $(f_n)$  such that  $(f_{m,k})$  is a subsequence of  $(f_{m-1,k})$  and for all  $j \leq m$ ,

$$\lim_{k \rightarrow \infty} f_{m,k}(d_j) = y_j.$$

We claim that the diagonal subsequence  $(g_m) = (f_{m,m})$  of  $(f_n)$  converges uniformly. For any  $j \in \mathbb{N}$ , and  $m > j$ ,  $(f_{m,k})$  is a subsequence of  $(f_{m-1,k})$  so by 2.9,  $f_{m,m} = f_{m-1,r_1}$  for some  $r_1 \geq m$ . Applying the lemma again,  $f_{m-1,r_1} = f_{m-2,r_2}$  for some  $r_2 \geq r_1 \geq m$ , and by induction

$$f_{m,m} = f_{m-1,r_1} = f_{m-2,r_2} = \cdots = f_{j,r_{m-j}}$$

where  $r_{m-j} \geq \cdots \geq r_2 \geq r_1 \geq m$ . Since  $r_{m-j} \geq m$ , we have

$$\lim_{m \rightarrow \infty} g_m(d_j) = \lim_{m \rightarrow \infty} f_{m,m}(d_j) = \lim_{r \rightarrow \infty} f_{j,r}(d_j) = y_j.$$

To show that  $g_m(x)$  converges for all  $x \in E$ , and that the convergence is uniform, it suffices to show  $(g_m)$  is Cauchy. Given  $\varepsilon > 0$ , let  $\delta > 0$  be such that for all  $n \in \mathbb{N}$  and  $s, t \in E$  with  $|s - t| < \delta$ ,  $|f_n(s) - f_n(t)| < \frac{\varepsilon}{3}$ , by equicontinuity of  $(f_n)$ . In particular

$$|s - t| < \delta \implies |g_m(s) - g_m(t)| < \frac{\varepsilon}{3}.$$

By density of  $D$  in  $E$ ,  $\{B(d_j, \delta)\}_{j \in \mathbb{N}}$  is an open covering for  $E$ . By compactness of  $E$ , it has a finite subcovering  $B(d_{j_1}, \delta), \dots, B(d_{j_\ell}, \delta)$ . Let  $J = \max_{i=1}^\ell j_i$ , so that for every  $x \in E$  there exists  $j \leq J$  such that  $x \in B(d_j, \delta)$ .

For each  $j \leq J$ ,  $(g_m(d_j))_{m \in \mathbb{N}}$  converges in  $\mathbb{R}^p$ , and thus is Cauchy. let  $N_j \in \mathbb{N}$  be such that for all  $m, n \geq N_j$ ,  $|g_m(d_j) - g_n(d_j)| < \frac{\varepsilon}{3}$ . Let  $N = \max_{j=1}^J N_j$ , so that for all  $m, n \geq N$  and all  $j \leq J$ , the fact that  $m, n \geq N_j$  implies

$$|g_m(d_j) - g_n(d_j)| < \frac{\varepsilon}{3}.$$

Thus for any  $x \in E$ , let  $j \leq J$  be such that  $|d_j - x| < \delta$ . Then

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq |g_m(x) - g_m(d_j)| + |g_m(d_j) - g_n(d_j)| + |g_n(d_j) - g_n(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Therefore  $(g_m)$  is Cauchy in  $\mathcal{C}(E, \mathbb{R}^p)$ , so by 2.7  $(g_m)$  converges uniformly.  $\square$

### 3. EXISTENCE

*Proof of Cauchy-Peano.* Without loss of generality  $t_0 = 0$  and  $t_1 = 1$ . It suffices to find a solution to the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s)) ds.$$

For in this case  $x(0) = x_0$  and we differentiate by the fundamental theorem of calculus to get

$$x'(t) = f(t, x(t)).$$

For each  $k \in \mathbb{N}$ , define  $x_k : [0, 1] \rightarrow \mathbb{R}^n$  by the recursion

$$x_k(t) = \begin{cases} x_0 & \text{if } 0 \leq t \leq \frac{1}{k} \\ x_0 + \int_0^{t-\frac{1}{k}} f(s, x_k(s)) ds & \text{if } \frac{j}{k} \leq t \leq \frac{j+1}{k}, \text{ for } j = 1, \dots, k-1. \end{cases}$$

It is necessary to define  $x_k$  inductively on each interval  $[\frac{j}{k}, \frac{j+1}{k}]$ . We claim that  $(x_k)_{k \in \mathbb{N}}$  is a uniformly bounded and equicontinuous sequence in  $\mathcal{C}([0, 1], \mathbb{R}^n)$ . Since  $f$  is bounded on  $[0, 1] \times \mathbb{R}^n$ , let  $M \geq 0$  be such that  $|f(t, x)| \leq M$  for all  $(t, x) \in [0, 1] \times \mathbb{R}^n$ .

For all  $k \in \mathbb{N}$  and  $t \in [0, 1]$ , we have  $|x_k(t)| \leq |x_0| + M$ . Indeed if  $0 \leq t \leq \frac{1}{k}$  then  $|x_k(t)| = |x_0|$ , and if  $\frac{1}{k} \leq t \leq 1$  then

$$\begin{aligned} |x_k(t)| &\leq |x_0| + \left| \int_0^{t-\frac{1}{k}} f(s, x_k(s)) ds \right| \\ &\leq |x_0| + \int_0^{t-\frac{1}{k}} |f(s, x_k(s))| ds \\ &\leq |x_0| + M \left( t - \frac{1}{k} \right) \\ &\leq |x_0| + M. \end{aligned}$$

Now suppose  $t' \leq t \in [0, 1]$ . For any  $k \in \mathbb{N}$ , if  $0 \leq t' \leq t \leq \frac{1}{k}$  then

$$|x_k(t) - x_k(t')| = |x_0 - x_0| = 0 \leq M|t - t'|.$$

Similarly if  $0 \leq t' \leq \frac{1}{k} \leq t \leq 1$ , then

$$\begin{aligned} |x_k(t) - x_k(t')| &= \left| x_0 + \int_0^{t-\frac{1}{k}} f(s, x_k(s)) ds - x_0 \right| \\ &\leq \int_0^{t-\frac{1}{k}} |f(s, x_k(s))| ds \\ &\leq M \left( t - \frac{1}{k} \right) \\ &\leq M|t - t'|. \end{aligned}$$

Finally if  $\frac{1}{k} \leq t' \leq t \leq 1$ , then

$$\begin{aligned} |x_k(t) - x_k(t')| &= \left| x_0 + \int_0^{t-\frac{1}{k}} f(s, x_k(s)) ds - \left( x_0 + \int_0^{t'-\frac{1}{k}} f(s, x_k(s)) ds \right) \right| \\ &= \left| \int_{t'-\frac{1}{k}}^{t-\frac{1}{k}} f(s, x_k(s)) ds \right| \\ &\leq \int_{t'-\frac{1}{k}}^{t-\frac{1}{k}} |f(s, x_k(s))| ds \\ &\leq M \left( t - \frac{1}{k} - \left( t' - \frac{1}{k} \right) \right) \\ &= M|t - t'|. \end{aligned}$$

Thus for every  $\varepsilon > 0$ ,  $\delta = \frac{\varepsilon}{2M}$  is sufficient to ensure that for all  $k \in \mathbb{N}$  and  $t, t' \in [0, 1]$  with  $|t - t'| < \delta$ , we have  $|x_k(t) - x_k(t')| \leq M \cdot \frac{\varepsilon}{2M} < \varepsilon$ . We have thereby shown that  $(x_k)_{k \in \mathbb{N}}$  is a uniformly bounded and equicontinuous sequence in  $\mathcal{C}([0, 1], \mathbb{R}^n)$ .

By Arzelà-Ascoli,  $(x_k)_{k \in \mathbb{N}}$  has a subsequence  $(x_\ell)_{\ell \in \mathbb{N}}$  such that  $x_\ell \rightrightarrows x$  for some  $x \in \mathcal{C}([0, 1], \mathbb{R}^n)$ . We claim that the sequence  $(F_\ell)_{\ell \in \mathbb{N}}$  in  $\mathcal{C}([0, 1], \mathbb{R}^n)$  defined by  $F_\ell(s) = f(s, x_\ell(s))$  converges uniformly to  $F$ , defined by  $F(s) = f(s, x(s))$ .

$(x_\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded by  $|x_0| + M$ . Naturally, so is  $x$ . Since  $f$  is continuous on  $[0, 1] \times \mathbb{R}^n$ , it is uniformly continuous on the compact set

$$K = [0, 1] \times [-(|x_0| + M), |x_0| + M]^n.$$

Given  $\varepsilon > 0$ , let  $\delta > 0$  be such that for all  $(s, x), (t, y) \in K$ ,

$$|(s, x) - (t, y)| < \delta \implies |f(s, x) - f(t, y)| < \varepsilon.$$

Using  $x_\ell \rightrightarrows x$ , let  $L \in \mathbb{N}$  be sufficiently large that for all  $\ell \geq L$  and  $s \in [0, 1]$ ,  $|x_\ell(s) - x(s)| < \delta$ . In particular  $(s, x_\ell(s)), (s, x(s)) \in K$  are such that

$$|(s, x_\ell(s)) - (s, x(s))| < \delta,$$

and thus

$$|F_\ell(s) - F(s)| = |f(s, x_\ell(s)) - f(s, x(s))| < \varepsilon,$$

showing that  $F_\ell \rightrightarrows F$ . Now we may take the pointwise limit of  $x_\ell$  to determine  $x$ :

$$\begin{aligned} \lim_{\ell \rightarrow \infty} x_\ell(t) &= \lim_{\ell \rightarrow \infty} \left( x_0 + \int_0^t f(s, x_\ell(s)) ds - \int_{t-\frac{1}{\ell}}^t f(s, x_\ell(s)) ds \right) \\ &= x_0 + \lim_{\ell \rightarrow \infty} \int_0^t f(s, x_\ell(s)) ds - \lim_{\ell \rightarrow \infty} \int_{t-\frac{1}{\ell}}^t f(s, x_\ell(s)) ds. \end{aligned}$$

We compute

$$\left| \int_{t-\frac{1}{\ell}}^t f(s, x_\ell(s)) ds \right| \leq \int_{t-\frac{1}{\ell}}^t |f(s, x_\ell(s))| ds \leq \frac{M}{\ell} \rightarrow 0$$

as  $\ell \rightarrow \infty$ , and by 2.8

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \int_0^t f(s, x_\ell(s)) ds &= \lim_{\ell \rightarrow \infty} \int_0^t F_\ell(s) ds \\ &= \int_0^t \lim_{\ell \rightarrow \infty} F_\ell(s) ds \\ &= \int_0^t F(s) ds \\ &= \int_0^t f(s, x(s)) ds. \end{aligned}$$

We conclude that

$$x(t) = x_0 + \int_0^t f(s, x(s)) ds.$$

As  $x(0) = x_0$ , we have constructed a solution in  $x : [0, 1] \rightarrow \mathbb{R}^n$ . □

## 4. UNIQUENESS, OR A LACK THEREOF

In general, the assumptions of the Cauchy-Peano theorem are insufficient to ensure uniqueness of the solution. For example, define  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(t, x) = \begin{cases} 2\sqrt{|x|} & \text{if } |x| \leq 1 \\ 2 & \text{if } |x| > 1. \end{cases}$$

$f$  is clearly continuous and bounded by 2. Consider the following initial value problem:

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(0) = 0, \end{cases}$$

where  $t \in [0, 1]$ .  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and bounded by 1. A trivial solution is  $y(t) = 0$ , since  $y(0) = 0$  and

$$y'(t) = 0 = f(t, 0) = f(t, y(t)).$$

Another solution is  $z(t) = t^2$ , since  $z(0) = 0$  and  $|z(t)| \leq 1$  on  $[0, 1]$ , so

$$z'(t) = 2t = 2\sqrt{t^2} = 2\sqrt{|z(t)|} = f(t, z(t)).$$

Therefore the solution given by Cauchy-Peano is not unique.

## REFERENCES

1. C. Pugh. *Real Mathematical Analysis*. Springer, 2015.