1.4 Locally convex spaces

Recall that a topological vector space is *locally convex* if it admits a local basis of convex subsets at the origin. Seminorms are a useful tool for describing locally convex spaces.

Definition 1.4.1

A *seminorm* on a k-vector space X is a map $p\colon X \to \mathbb{R}$ such that

(i) $p(x+y) \le p(x) + p(y)$.

(ii) $p(\lambda x) = |\lambda| p(x)$.

We first collect some properties of seminorms.

Proposition 1.4.2

Let $k = \mathbb{R}$ or $\mathbb{C}$. Let $p\colon X \to \mathbb{R}$ be a seminorm. Then

(1) $p(0) = 0$.

(2) $|p(x) - p(y)| \le p(x-y)$.

(3) $p(x) \ge 0$.

(4) $p^{-1}(0) \subset X$ is a linear subspace.

(5) $p^{-1}[0,1) \subset X$ is convex and balanced.

(6) A seminorm with $p(x) \neq 0$ whenever $x \neq 0$ is a norm.

Proof. (1) Taking $\lambda = 0$, $p(0) = 0$.

(2) We have
$$p(x) = p(x-y+y) \le p(x-y) + p(y),$$
and similarly $p(y) - p(x) \le p(y-x) = |-1| p(x-y) = p(x-y)$.

(3) $y = 0$ in (2) gives $|p(x)| \le p(x)$.

(4) If $x, y \in p^{-1}(0)$ then by (3) we have
$$0 \le p(\lambda x + \mu y) \le |\lambda| p(x) + |\mu| p(y) = 0,$$
so $\lambda x + \mu y \in p^{-1}(0)$.

(5) B is clearly balanced by (ii). For convexity, let $x, y \in B$ and $t \in (0,1)$. Then
$$p(tx + (1-t)y) \le t p(x) + (1-t) p(y) < 1,$$
so $tx + (1-t)y \in B$.

(6) This follows from (1) and (3).

□

1.4.1 Locally convex spaces from seminorms

A sufficiently nice family of seminorms on a vector space determines a locally convex topological structure. More precisely, a family of seminorms $\mathscr{P}$ on X is *separating* if for each $x \neq 0 \in X$, there exists $p \in \mathscr{P}$ such that $p(x) \neq 0$.

Theorem 1.4.3

Let $\mathscr{P}$ be a separating family of seminorms on a vector space X. For each $p \in \mathscr{P}$ and $n \in \mathbb{N}$, define

$$V(p,n) = \{x \in X : p(x) < \tfrac{1}{n}\}.$$

Let β_0 be the set of all finite intersections of the $V(p,n)$. Then β_0 is a convex, balanced, local basis at 0, and it generates a locally convex topology such that

(1) Every $p \in \mathscr{P}$ is continuous.

(2) $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E.

Proof. Let $B_1 = \bigcap_{i=1}^N V(p_i, n_i)$, $B_2 = \bigcap_{j=1}^M V(q_j, m_j) \in \beta_0$. Then

$$B_3 := \bigcap_{i=1}^N V(p_i, n_i) \cap \bigcap_{j=1}^M V(q_i, m_i) \in \beta_0$$

is such that $B_3 \subset B_1 \cap B_2$. Thus β_0 is a local basis at 0, hence its translates generate a topology on X. Each $V(p,n)$ is convex and balanced by Proposition 1.4.2(5), so their finite intersections are convex and balanced.

To see X is T1, let $x, y \in X$. Since $x - y \neq 0$ and $\mathscr{P}$ is separating, there exists $p \in \mathscr{P}$ such that $p(x-y) > 0$. Let $n \in \mathbb{N}$ be sufficient large that $np(x-y) > 1$, so that $x \notin y + V(p,n)$. We have thus constructed an open neighbourhood of y disjoint from x.

To see addition is continuous, let $U \subset X$ be open and $x + y \in U$. For some p_i and n_i, we have

$$x + y + \bigcap_{i=1}^N V(p_i, n_i) \subset U.$$

Let

$$V_1 = x + \bigcap_{i=1}^N V(p_i, 2n_i), \quad V_2 = y + \bigcap_{i=1}^N V(p_i, 2n_i);$$

we claim that $V_1 \times V_2 \subset +^{-1}(U)$. Indeed if $(v_1, v_2) \in V_1 \times V_2$ then for each i,

$$p_i(v_1 + v_2 - (x+y)) \leq p_i(v_1 - x) + p_i(v_2 - y) < \frac{1}{2n_i} + \frac{1}{2n_i} = \frac{1}{n_i},$$

so $(v_1, v_2) \in +^{-1}(U)$. To show that mutliplication is continuous, let $\lambda x \in U$. Once again we write

$$\lambda x + \bigcap_{i=1}^N V(p_i, n_i) \subset U.$$

Let $\delta_i = \frac{1}{2n_i p_i(x)}$, so that $\mu \in B_{\delta_i}(\lambda)$ implies

$$p_i((\mu-\lambda)x) = |\mu-\lambda| p_i(x) < \frac{1}{2n_i},$$

Let $\delta = \min_i \delta_i$, so that for $|\mu - \lambda| < \delta$, the above equation holds simultaneously for all i.

Let $\epsilon_i = \frac{1}{2n_i(\delta+|\lambda|)}$ and let $m_i > \frac{1}{\epsilon_i}$. If $y \in x + V(p_i, m_i)$, then $p_i(y-x) < \frac{1}{m_i} < \epsilon_i$, so for $\mu \in B_\delta(\lambda)$,

$$\begin{aligned} p_i(\lambda x - \mu y) &= p_i((\lambda-\mu)x + \mu(x-y)) \\ &\le p_i((\lambda-\mu)x) + p_i(\mu(x-y)) \\ &= |\lambda-\mu| p_i(x) + |\mu - \lambda + \lambda| p_i(x-y) \\ &< \delta_i p_i(x) + (|\mu-\lambda| + |\lambda|) p_i(x-y) \\ &< \frac{1}{2n_i} + (\delta_i + |\lambda|)\frac{1}{2n_i(\delta_i+|\lambda|)} \\ &= \frac{1}{n}, \end{aligned}$$

so $\mu y \in \lambda x + V(p_i, n_i)$. Thus

$$B_\delta(\lambda) \times x + \bigcap_{j=1}^{M} V(p_i, m_i) \subset \cdot^{-1}(U),$$

showing that scaling is continuous. We conclude that X is a locally convex topological vector space, and it remains to verify the two properties. Let $p \in \mathscr{P}$. If $x \in p^{-1}(a,b)$ then $p(x) \in (a,b)$ so for sufficiently large n we have

$$(p(x) - \tfrac{1}{n}, p(x) + \tfrac{1}{n}) \subset (a,b).$$

Thus $V(p,n) \subset p^{-1}(a,b)$; indeed if $p(y-x) < \frac{1}{n}$ then Proposition 1.4.2(2) implies that $p(y) \in (a,b)$.

Secondly, let $E \subset X$ is bounded an $p \in \mathscr{P}$. Since $V(p,1)$ is an open neighbourhood of 0, there exists $t < \infty$ such that $E \subset tV(p,1)$. But $x \in tV(p,1)$ if and only if $p(x) < t$, so p is bounded on E.

Conversely suppose every $p \in \mathscr{P}$ is bounded on E. Let U be an open neighbourhood of 0 and take

$$\bigcap_{i=1}^{N} V(p_i, n_i) \subset U.$$

Then for each i there exists $M_i < \infty$ such that $p_i < M_i$ on E; let $s := M_i n_i$. If $t > s$, then

$$p_i(x) < M_i < \frac{t}{n_i}$$

means $E \subset tV(p_i, n_i)$ for all i, hence $E \subset tU$. $\square$

If $\mathscr{P}$ is countable, we have another description of the topology it induces.

Theorem 1.4.4

Let $\mathscr{P} = \{p_i\}_{i\in\mathbb{N}}$ be a countable separating family of seminorms on X. Then the locally convex topology on X is metrizable and equivalent to that induced by the invariant metric

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_i(x-y)}{1+p_i(x-y)}.$$

Remark 1.4.5. Somewhat counterintuitively, the metric balls $B_r(0)$ will not necessary be convex in this topology. Another counterintuitive property is that any subset has finite diameter (indeed $d(x,y)<1$ for any $x,y\in X$) but they are not all bounded.

Proof. We begin by showing d is an invariant metric. Since $p(-x)=p(x)$, d is certainly symmetric. For positive-definiteness, note that

$$\begin{aligned} F\colon [0,\infty) &\longrightarrow [0,1) \\ t &\longmapsto \frac{t}{1+t} \end{aligned}$$

is a smooth, monotone increasing, concave function. Thus $d(x,y)$ is a sum of nonnegative terms so it is nonnegative, with equality if and only if $p_i(x-y)=0$ for all i, which implies $x=y$ since $\mathscr{P}$ is separating. For the triangle inequality, convexity of F along with $F(0)=0$ implies

$$F(\lambda t)=F(\lambda t+(1-\lambda)0)\geq \lambda F(t)+(1-\lambda)F(0)=\lambda F(t)$$

for $\lambda\in(0,1)$. Then if $t,s\geq 0$,

$$\begin{aligned} F(t)+F(s) &= F\left((t+s)\frac{t}{t+s}\right)+F\left((t+s)\frac{s}{t+s}\right) \\ &\geq \frac{t}{t+s}F(t+s)+\frac{s}{t+s}F(t+s) \\ &= F(t+s). \end{aligned}$$

Now since p is a seminorm,

$$p(x-y)\leq p(x-z)+p(z-y),$$

and since F is monotone increasing

$$F\circ p(x-y)\leq F(p(x-z)+p(z-y)),$$

and since F is subadditive

$$F\circ p(x-y)\leq F\circ p(x-z)+F\circ p(z-y).$$

Therefore

$$\begin{aligned} d(x,y) &= \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{p_i(x-y)}{1+p_i(x-y)} \\ &\leq \sum_{i=1}^{\infty}\frac{1}{2^i}\left(\frac{p_i(x-z)}{1+p_i(x-z)}+\frac{p_i(z-y)}{1+p_i(z-y)}\right) \\ &= d(x,z)+d(z,y), \end{aligned}$$

showing that d is a metric on X. The fact that it is invariant is clear by definition. To show that the metric topology τ_d induced by d agrees with the topology $\tau_{\mathscr{P}}$ induced by $\mathscr{P}$, the Weierstrass M-test implies that $d(x,y)$ converges uniformly. Since the p_i are $\tau_{\mathscr{P}}$-continuous, the metric $d\colon (X,\tau_{\mathscr{P}})\times(X,\tau_{\mathscr{P}})\to\mathbb{R}$ is continuous. In particular, $d_x\colon (X,\tau_{\mathscr{P}})\to\mathbb{R}$ given by $d_x(y)=d(x,y)$ is continuous, so the basic open sets $B_r(x)=d_x^{-1}(-r,r)$ in τ_d are open in $\tau_{\mathscr{P}}$. This shows $\tau_{\mathscr{P}}$ is finer.

Now if W is open in $\tau_{\mathscr{P}}$ and $x \in W$, then there exists $B = \bigcap_{k=1}^N V(p_{i_k}, n_k)$ such that $x + B \subset W$. Let $M \geq i_k$ and $\epsilon < \frac{1}{2^{M+1} n_k}$ for all k. If $y \in B_\epsilon(x)$ then for each i we must have

$$\frac{p_i(x-y)}{1+p_i(x-y)} < 2^M \epsilon = \frac{1}{2n_k},$$

so

$$\begin{aligned}
p_i(x-y) &< \frac{1}{2n_k} + \frac{1}{2n_k} p_i(x-y) \\
p_i(x-y) &< \frac{1}{2n_k} + \frac{1}{2} p_i(x-y) \\
\frac{1}{2} p_i(x-y) &< \frac{1}{2n_k} \\
p_i(x-y) &< \frac{1}{n_k},
\end{aligned}$$

so $y \in x + \bigcap_{k=1}^N V(p_{i_k}, n_k) \subset W$. This shows that $\tau_{\mathscr{P}}$ is finer, so the topologies coincide. □

1.4.2 The space of smooth functions

Let $\Omega \subset \mathbb{R}^n$ be open. We will endow $C^\infty(\Omega)$, the vector space of smooth real-valued functions on Ω under pointwise addition and multiplication, with a topology which makes it a Fréchet space with the Heine–Borel property. Let $K_1 \subset K_2 \subset \cdots \subset \Omega$ be a compact exhaustion. Define a countable family of seminorms

$$p_n(f) = \max\{|D^\alpha f(x)| : x \in K_n, |\alpha| \leq n\}.$$

$\mathscr{P} = \{p_n : n \in \mathbb{N}\}$ is separating. Indeed, if $f \neq 0$, then $f(x) \neq 0$ for some $x \in \Omega$. Then $x \in K_n$ for large n, so $p_n(f) \geq |f(x)| > 0$. By what we have done so far, $\mathscr{P}$ induces a topology on $C^\infty(\Omega)$ which is locally convex, metrizable, and has a local basis of sets

$$V_N = \{f \in C^\infty(\Omega) : p_N(f) < \tfrac{1}{N}\}$$

at 0. We will denote this locally convex space by $\mathscr{E}(\Omega)$.

Proposition 1.4.6

A sequence (f_n) in $\mathscr{E}(\Omega)$ converges to f if and only if $D^\alpha f_n \rightrightarrows D^\alpha f$ on compact subsets for each multiindex α.

Proof. After translating, we may assume $f = 0$. By definition $f_n \to 0$ if and only if for each N there exists m_N such that $n > m_N$ implies $f_n \in V_N$.

If $f_n \to 0$, then fix α. It suffices to show $D^\alpha f_n \rightrightarrows 0$ on each K_i. For any $\epsilon > 0$, let $N > \max\{|\alpha|, \frac{1}{\epsilon}\}$. If $n > m_N$ then $f_n \in V_N$, so

$$\sup_{K_i} |D^\alpha f_n| \leq p_N(f_n) \leq \frac{1}{N} < \epsilon,$$

so $D^\alpha f_n \rightrightarrows 0$ on K_i.

Conversely if $D^\alpha f_n \rightrightarrows 0$ on compact subsets, fix N. If $|\alpha| \leq N$ then $D^\alpha f_n \rightrightarrows 0$ on K_N. So for each of the finitely many α with $|\alpha| \leq N$, there exists m_α such that $n > m_\alpha$ implies

$$\sup_{K_N} |D^\alpha f_n| < \frac{1}{N}.$$

Let $m = \max_{|\alpha| \le N} m_\alpha$; then for $n > m$ we have $f_n \in V_N$ so $f_n \to 0$. $\square$

Theorem 1.4.7

$\mathscr{E}(\Omega)$ is a Fréchet space with the Heine–Borel property.

Proof. We know $\mathscr{E}(\Omega)$ is locally convex and its topology is induced by an invariant metric, so it remains to show it is complete with respect to this metric. A sequence (f_n) in $\mathscr{E}(\Omega)$ is Cauchy if for any N, there exists M such that $i, j > M$ implies $f_i - f_j \in V_N$, or

$$\sup_{K_N} |D^\alpha f_i - D^\alpha f_j| < \frac{1}{N}$$

for $|\alpha| \le N$. Since the K_N exhaust Ω, there exist continuous g^α such that $D^\alpha f_n \rightrightarrows g^\alpha$ on compact subsets. Then there exists a smooth function f such that $f_n \rightrightarrows f$ and $D^\alpha f_n \rightrightarrows D^\alpha f$ on compact subsets, so $\mathscr{E}(\Omega)$ is complete.

For the Heine–Borel property, suppose $E \subset \mathscr{E}(\Omega)$ is closed and bounded. We will show that every sequence in E has a convergent subsequence.

Since E is bounded, for each N there exists M_N such that $p_N(f) < M_N$ for all $f \in E$. In particular

$$|D^\alpha f| < M_N$$

on K_N for $|\alpha| = N$ and $f \in E$. So for $|\beta| < N_1$, the set $\{D^\beta f : f \in E\}$ is equicontinuous on K_{N-1}, and it is pointwise bounded since $p_N(f) < M_N$. Now if (f_n) is any sequence in E, then the Arzelà–Ascoli theorem plus an additional diagonalization argument gives a subsequence (F_k) such that $D^\alpha F_k$ converges uniformly on compact subsets, so (f_n) has a convergent subsequence in E. $\square$

1.4.3 Compactly-support smooth functions

Let $K \subset \Omega$ be compact. Then the space $\mathscr{D}_K$ of $f \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(f) \subset K$ is a linear subspace of $C^\infty(\Omega)$. In fact, it is closed in $\mathscr{E}(\Omega)$. Indeed, the evaluation map

$$\begin{aligned} \delta_x \colon \mathscr{E}(\Omega) &\longrightarrow \mathbb{R} \\ f &\longmapsto f(x) \end{aligned}$$

is continuous, so $\delta_x^{-1}(0)$ is closed. Thus

$$\mathscr{D}_K = \bigcap_{x \in \Omega \setminus K} \delta_x^{-1}(0)$$

is closed. In particular, it is a Fréchet space in the subspace topology, denoted τ_K. Let

$$\mathscr{D}(\Omega) = \bigcup_{K \subset \Omega} \mathscr{D}_K$$

be the space of compactly-support smooth functions on Ω. This is clearly closed under addition and multiplication. We will endow $\mathscr{D}(\Omega)$ with a topology which makes it a complete locally convex topological space such that each subspace $\mathscr{D}_K$ inherits τ_K.

A natural candidate would use the seminorms

$$\|f\|_k = \max\{|D^\alpha f(x)| : x \in \Omega, |\alpha| \le k\}.$$

In fact, this satisfies all our desirata except completeness, which suggests we should find a finer topology. We will use the finest locally convex topology such that the subspace topology on $\mathscr{D}_K$ agreees with τ_K.

Let β be the collection of all convex balanced sets $W \subset \mathscr{D}(\Omega)$ such that $\mathscr{D}_K \cap W \in \tau_K$ for every compact $K \subset \Omega$.

Theorem 1.4.8

β is a local basis at 0, and the resulting topology τ makes $\mathscr{D}(\Omega)$ a locally convex topological space.

Proof. Let $W_1, W_2 \in \beta$. We claim that $W_1 \cap W_2 \in \beta$. This is clearly convex, balanced, and for $K \subset \Omega$ compact,

$$\mathscr{D}_K \cap W_1 \cap W_2 \in \tau_K.$$

It remains to show the axioms of a topological vector space are satisfied. For the T1 axiom, let $\phi_1, \phi_2 \in \mathscr{D}(\Omega)$. Consider

$$W = \{\phi \in \mathscr{D}(\Omega) : \|\phi\|_0 \leq \|\phi_1 - \phi_2\|_0\}.$$

As a metric ball, this is convex and balanced. Moreover, its intersection with each $\mathscr{D}_K$ is

$$\{\phi \in \mathscr{D}_K : \|\phi\|_0 < r\},$$

which is open, so $W \in \beta$. But $\phi_1 \notin \phi_2 + W$, so $\{\phi_1\}$ is closed.

To show that addition is continuous, let U be open and $\phi_1 + \phi_2 \in U$. Then $\phi_1 + \phi_2 + W \subset U$ for some $W \in \beta$. We claim that

$$(\phi_1 + \tfrac{1}{2}W) \times (\phi_2 \times \tfrac{1}{2}W) \subset +^{-1}(U).$$

Indeed by convexity of W,

$$(\phi_1 + \tfrac{1}{2}W) + (\phi_2 \times \tfrac{1}{2}W) = \phi_1 + \phi_2 + W \subset U.$$

To show that mutliplication is continuous, let U be open and $\lambda\phi \in U$. Then $\lambda\phi + W \subset U$ for some $W \in \beta$. Let $\mathrm{supp}(\phi) \subset K$. Then scalar multiplication is continuous on $\mathscr{D}_K$, so there exists $\delta > 0$ such that $\delta\phi \in \frac{1}{2}W$. Let $\epsilon = \frac{1}{2(|\lambda|+\delta)}$. Let $(\mu, \psi) \in B_\delta(\lambda) \times (\phi + \epsilon W)$, so that $|\mu| \leq |\lambda| + \delta$. Since W is balanced and convex,

$$\mu\psi - \lambda\phi = \mu(\psi - \phi) + (\mu - \lambda)\phi \in (|\lambda| + \delta)\epsilon W + \frac{1}{2}W = W,$$

so scaling is continuous. □

For the rest of this section we characterize convergence and continuity in $\mathscr{D}(\Omega)$.

Proposition 1.4.9 (1) A convex balanced subset V of $\mathscr{D}(\Omega)$ is open if and only if $V \in \beta$.

(2) The Fréchet topology τ_K of any $\mathscr{D}_K$ coincides with the subspace topology inherited from $\mathscr{D}(\Omega)$.

(3) If $E \subset \mathscr{D}(\Omega)$ is bounded then $E \subset \mathscr{D}_K$ for some $K \subset \Omega$, and there exist real numbers $M_N < \infty$ such that each $\phi \in E$ satisfies $\|\phi\|_N \leq M_N$ for all N.

(4) $\mathscr{D}(\Omega)$ has the Heine–Borel property.

(5) If (ϕ_i) is a Cauchy sequence in $\mathscr{D}(\Omega)$ then $(\phi_i) \subset \mathscr{D}_K$ for some K, and it is Cauchy with respect to $\|\cdot\|_N$ for each N.

(6) If $\phi_i \to 0$ in $\mathscr{D}(\Omega)$, then there is a compact subset $K \subset \Omega$ containing the support of every ϕ_i, and $D^\alpha \phi_i \rightrightarrows 0$ as $i \to \infty$ for all α.

(7) $\mathscr{D}(\Omega)$ is complete.

Proof. (1) The if direction is by definition, now suppose V is convex balanced and open. Let $K \subset \Omega$ and let $\phi \in \mathscr{D}_K \cap V$. Then there exists $W \in \beta$ such that $\phi + W \subset V$, and then

$$\phi + (\mathscr{D}_K \cap W) \subset \mathscr{D}_K \cap V.$$

In particular $\mathscr{D}_K \cap W \in \tau_K$ means $\mathscr{D}_K \cap V$ is open in $\mathscr{D}_K$, hence $V \in \beta$.

(2) The previous part shows that any open set in the subspace topology on $\mathscr{D}_K \subset \mathscr{D}(\Omega)$ is open in τ_K. Conversely if $E \subset \tau_K$ is open, then we wish to show that $E = \mathscr{D}_K \cap U$ for U open in $\mathscr{D}(\Omega)$. Let $\phi \in E$, so that there exists N, δ such that

$$\{\psi \in \mathscr{D}_K : \|\psi - \phi\|_N < \delta\} \subset E.$$

Let

$$W_\phi := \{\psi \in \mathscr{D}(\Omega) : \|\psi\|_N < \delta\}.$$

By definition $W_\phi \in \beta$, and

$$\mathscr{D}_K \cap (\phi + W_\phi) = \phi + \mathscr{D}_{\mathscr{K}} \cap W_\phi \subset E.$$

So

$$U = \bigcup_{\phi \in E} (\phi + W_\phi) \in \tau$$

is the desired open set with $\mathscr{D}_K \cap U = E$.

(3) By contraposition, suppose E is not contained in any $\mathscr{D}_K$. For each K_m in a compact exhaustion of Ω, there exists $\phi_m \in E$ with $\mathrm{supp}(\phi_m) \not\subset K_m$. In particular, there exists $x_m \in \Omega \setminus K_m$ with $\phi_m(x_m) \neq 0$. Let

$$W = \{\phi \in \mathscr{D}(\Omega) : |\phi(x_m)| < \frac{1}{m}|\phi_m(x_m)|\}.$$

It is not difficult to see that W is convex and balanced. Moreover, for any $K \subset K_M$ we have $x_m \notin K$ for $m \geq M$, so $\mathscr{D}_K \cap W$ is the intersection of finitely many open sets, meaning $W \in \beta$. So W is an open neighbourhood of 0 such that $\phi_m \notin mW$ for any m, so E is not bounded. In summary, any bounded subset of $\mathscr{D}(\Omega)$ belongs to some $\mathscr{D}_K$, and by (2) it will be bounded in $\mathscr{D}_K$. We previously saw that this means that there exist real numbers $M_N < \infty$ such that $\|\phi\|_N \leq M_N$ for all N and $\phi \in E$.

(4) This follows from (3), because $\mathscr{D}_K$ has the Heine–Borel property and every closed and bounded subset is contained in some $\mathscr{D}_K$, where it is again closed and bounded.

(5) Since Cauchy sequences are bounded, this also follows from (3) and the characterization of Cauchy sequences in $\mathscr{D}_K$.

(6) Same as (5).

(7) This follows from (5) and (2), since $\mathscr{D}_K$ is complete.

$\square$

The next result concerns linear maps from $\mathscr{D}(\Omega)$ into another locally convex space. Recall that a linear map $\Lambda\colon X \to Y$ is bounded if $\Lambda(E) \subset Y$ is bounded whenever $E \subset X$ is bounded.

Proposition 1.4.10

Let Y be a locally convex topological vector space. Let $\Lambda\colon \mathscr{D} \to Y$ be a linear map. The following are equivalent:

(a) Λ is continuous.

(b) Λ is bounded.

(c) If $\phi_i \to 0$ in $\mathscr{D}(\Omega)$ then $\Lambda(\phi_i) \to 0$ in Y.

(d) For all $K \subset \Omega$ compact, $\Lambda\big|_{\mathscr{D}_K}$ is continuous.

Proof. $(a) \implies (b)$. Let E be bounded and let W be a neighbourhood of 0 in Y. By continuity, there exists a neighbourhood V of 0 in $\mathscr{D}(\Omega)$ such that $\Lambda(V) \subset W$. Let $s > 0$ be such that for $t > s$, $E \subset tV$. By linearity

$$\Lambda(E) \subset \Lambda(tV) = t\Lambda(V) \subset tW,$$

as desired.

$(b) \implies (c)$. By the previous proposition (5), there exists $K \subset \Omega$ such that $\phi_i \to 0$ in $\mathscr{D}_K$. Since $\mathscr{D}_K$ is metrizable, there exist scalars $a_i \to \infty$ with $a_i\phi_i \to 0$ in $\mathscr{D}_K$, and the same holds in $\mathscr{D}(\Omega)$. By linearity

$$\Lambda(\phi_i) = a_i^{-1}\Lambda(a_i\phi_i).$$

Since $\mathscr{D}(\Omega)$ is complete, $(a_i\phi_i)$ is Cauchy, hence bounded. Since Λ is bounded, $\{\Lambda(a_i\phi_i)\}$ is bounded. Since $a_i^{-1} \to 0$, we have $\Lambda\phi_i \to 0$.

$(c) \implies (d)$. Since $\mathscr{D}_K$ is a metric space, this is well-known.

$(d) \implies (a)$. First let U be a convex, balanced, open neighbourhood of 0 in Y. Then $V = \Lambda^{-1}(U)$ is convex and balanced by linearity, and to show it is open in $\mathscr{D}(\omega)$ it suffices to show $\mathscr{D}_K \cap V$ is open in $\mathscr{D}_K$ for all K. But this is true by continuity of $\Lambda\big|_{\mathscr{D}_K}$.

More generally, let W be open in Y and $\phi \in \Lambda^{-1}(W)$. Since Y is locally convex, there exists a convex balanced neighbourhood U of 0 such that $\Lambda\phi + U \subset W$. Since Λ is linear,

$$\phi + \Lambda^{-1}(U) \subset \Lambda^{-1}(W),$$

and $\phi + \Lambda^{-1}(U)$ is open in $\mathscr{D}(\Omega)$, so we win. $\square$