Chapter 1

Riemann Surfaces

In our first course in complex analysis, we visited the phenomenon of analytic continuation. In particular, we saw that certain analytic functions take infinitely many values at any point, while other analytic functions simply cannot be continued very far. A more detailed study of these phenomena will lead us naturally to Riemann surfaces.

1.1 Natural boundaries

We will only investigate a very simple case. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a power series centered at 0 with radius of convergence 1.

Definition 1.1.1

A point $z_0 \in S^1$ is called a *regular point* for f if there exists an open neighbourhood U of z_0 and an analytic function $g\colon U \to \mathbb{C}$ such that $g = f$ on $U \cap \mathbb{D}$. Otherwise z_0 is a *singular point* for f.

In more down-to-earth terms, f admits an analytic continuation near regular points, but not singular points. By definition, regular points are dense.

Proposition 1.1.2

If f has radius of convergence 1, then it admits a singular point.

Proof. If not, then each $z \in S^1$ admits some $\epsilon_z > 0$ such that f extends analytically to $D_{\epsilon_z}(z)$. By compactness, finitely many $D_{\epsilon_{z_i}}(z_i)$ cover S^1, so by taking $\epsilon := \min_i \epsilon_{z_i} > 0$ we have an analytic extension of f to $D_{1+\epsilon}(0)$. This suggests the radius of convergence of f is at least $1 + \epsilon > 1$, a contradiction. $\square$

In the extreme case where every $z \in S^1$ is singular, we call S^1 a *natural boundary* for f. This behaviour is not uncommon:

Example 1.1.3. The power series

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

has radius of convergence 1 by the ratio test. Let $\omega = e^{2\pi ik/q}$ be a qth root of unity, for any q. If we can show ω is singular, then it will follow from the fact that roots of unity are dense in S^1 that S^1 is a natural boundary for f.
Indeed, for $r \in (0,1)$ we have

$$f(r\omega) = \sum_{n=0}^{q-1} r^{n!}\omega^{n!} + \sum_{n=q}^{\infty} r^{n!}.$$

The second term goes to ∞ as $r \to 1$ because

$$\lim_{r\to 1} \sum_{n=q}^{q+M} r^{n!} = M + 1$$

for any $M > 0$. But if ω were regular then $\lim_{r\to 1} f(r\omega) = f(\omega)$, so ω is singular.