

# 1 Introduction

## 1.1 Definition of a representation

Let  $k$  be a field.

### Definition 1.1.1

A  $G$ -representation is a  $k$ -vector space  $V$  with any of the following equivalent data:

- (I) A group homomorphism  $\rho: G \rightarrow \mathrm{GL}(V)$ .
- (II) A group action  $G \curvearrowright V$  such that  $g \cdot (\lambda u + v) = \lambda g \cdot u + g \cdot v$ .
- (III) A left  $k[G]$ -module structure on  $V$ . Here  $k[G]$  denotes the group algebra, which is a  $k$ -vector space with basis  $\{e_g\}_{g \in G}$  and multiplication  $e_g \cdot e_h = e_{gh}$  extended by linearity.

Concretely, for each  $g \in G$  we associate an invertible linear map  $\rho(g): V \rightarrow V$  such that  $\rho(gh) = \rho(g)\rho(h)$ .

### Proposition 1.1.2

The three definitions are equivalent.

*Proof.* (I)  $\implies$  (II). Let  $\rho: G \rightarrow \mathrm{GL}_n(k)$  be a group homomorphism. Define a group action  $G \times V \rightarrow V$  by  $g \cdot v = \rho(g)v$ . We have

$$(gh) \cdot v = \rho(gh)v = \rho(g)(\rho(h)v) = g \cdot (h \cdot v),$$

and

$$g \cdot (\lambda u + v) = \rho(g)(\lambda u + v) = \lambda \rho(g)u + \rho(g)v = \lambda g \cdot u + g \cdot v.$$

(II)  $\implies$  (III). Extend the action of  $G$  by linearity.

(III)  $\implies$  (I). Define  $\rho(g): V \rightarrow V$  by  $v \mapsto e_g \cdot v$ . Then  $\rho(gh)v = e_{gh} \cdot v = (e_g e_h) \cdot v = e_g \cdot (e_h \cdot v) = \rho(g)\rho(h)v$ , so this is indeed a group homomorphism.  $\square$

The motivation for studying representation theory, in particular of finite groups, is to transform difficult group theory questions into linear algebra questions. Moreover, representations often arise very naturally, as we will see from the following examples.

### 1.1.1 Examples of representations

- We always have the trivial representation  $g \mapsto \text{id}_V$  for all  $g \in G$ .
- Let  $G$  be a group and  $k^G$  the set of functions  $G \rightarrow k$ . There is a natural action  $G \curvearrowright k^G$  by  $g \cdot f = f(g^{-1} \cdot)$ . Here the inverse is needed to get a left action. If  $G$  is finite, then this is precisely  $k[G]$ , viewed as a  $k[G]$ -module. Indeed, for  $f: G \rightarrow k$  let  $v_f = \sum_{g \in G} f(g)e_g \in k[G]$ . We call this the left regular representation.
- As a special case, let  $G = S_n$ . This acts naturally on  $k^{\{1,2,\dots,n\}}$  by  $\sigma \cdot f = f(\sigma^{-1} \cdot)$ . We observe that the constant functions are invariant under this action, and so is the subspace

$$\{f \in k^{\{1,2,\dots,n\}} : \sum_{i=1}^n f(i) = 0\}.$$

Indeed, we have

$$\sum_{i=1}^n f(\sigma^{-1}(i)) = \sum_{\sigma^{-1}(i)=1}^n f(\sigma^{-1}(i)) = \sum_{j=1}^n f(j) = 0.$$

- Let us do a more explicit example. Let  $G = S_3$  and  $V = k^{\{1,2,3\}}$ . Then

$$\begin{aligned} \text{id} &\mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, & (1\ 2) &\mapsto \begin{pmatrix} & 1 \\ 1 & & \\ & & 1 \end{pmatrix}, \\ (1\ 2\ 3) &\mapsto \begin{pmatrix} & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, & (2\ 3) &\mapsto \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \\ (1\ 3\ 2) &\mapsto \begin{pmatrix} & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, & (1\ 3) &\mapsto \begin{pmatrix} & 1 \\ 1 & & \\ & 1 & \end{pmatrix}. \end{aligned}$$

- Let  $G = \mathbb{Z}$ . Since

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix},$$

we have a group homomorphism  $\mathbb{Z} \rightarrow \text{GL}_2(k)$  sending  $n$  to  $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ .

- Let  $G = \mathbb{Z}/m\mathbb{Z}$ . By exponent rules, we have a group homomorphism  $\mathbb{Z}/m\mathbb{Z} \rightarrow \text{GL}_1(\mathbb{C})$  sending  $n$  to  $e^{2\pi i n/m}$ .

### 1.1.2 Operations on representations

Given  $G$ -representations  $V$  and  $W$ , we can perform the following constructions.

- $V \oplus W$  is a  $G$ -representation with  $g \cdot (v, w) = (g \cdot v, g \cdot w)$ .
- If  $H \leq G$ , then  $V|_H$  is an  $H$ -representation where the group homomorphism  $\rho|_H: H \rightarrow \text{GL}(V)$  is obtained by restricting that of  $G$ .

- $V^*$  is a  $G$ -representation where  $g \cdot f = f(g^{-1} \cdot)$ .
- $V \otimes_k W$  is a  $G$ -representation where  $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$ . Note that  $V \otimes_k W$  is a priori a  $k[G] \otimes_k k[G]$ -module, and we need a map  $k[G] \rightarrow k[G] \otimes_k k[G]$  called a *coproduct* to define a  $k[G]$ -module structure. We will explore this next lecture.
- $\underline{\text{Hom}}_k(V, W) = \mathcal{L}(V, W)$  is a  $G$ -representation where  $g \cdot f = g \cdot f(g^{-1} \cdot)$ .

We use an underline to distinguish  $\text{Hom}_k(V, W)$  from  $\text{Hom}_G(V, W)$ , the morphisms of  $G$ -representations, or equivalently  $k[G]$ -modules. More explicitly,

$$\text{Hom}_G(V, W) = \{f \in \text{Hom}_k(V, W) : f(gv) = gf(v) \quad \text{for } v \in V, g \in G\}.$$

This leads naturally to the definition of invariants. A vector  $v \in V$  is called  *$G$ -invariant* if  $gv = v$  for all  $g \in G$ . The subspace of  $G$ -invariants is denoted  $V^G$ . With this language,

$$\text{Hom}_G(V, W) = \underline{\text{Hom}}_k(V, W)^G.$$

Given a group  $G$ , our goal will be to classify all  $G$ -representations up to isomorphism. We will do this for  $G$  finite and  $k$  of characteristic 0. But what do we mean by classify? It will be convenient to call a linear subspace  $W \subset V$  a *subrepresentation* if  $GW \subset W$ . We say  $V$  is simple or irreducible if its only subrepresentations are 0 and itself. We will see next lecture that arbitrary representations are build quite easily from simple representations, and then we will turn our attention towards classifying the simple representations of finite groups.