The Symmetry of Hopf Algebras

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1 Introduction

Assuming only a basic familiarity with linear algebra up to tensor products of vector spaces along with some ability to decode commutative diagrams, we build up to the definition of a Hopf algebra. The most famous example of a Hopf algebra is perhaps the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$, a q-deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. This quantum group inspired several constructions of knot invariants, solutions to integrable systems, and q-analogues of results from classical invariant theory. Other examples including group algebras, polynomial algebras, and tensor algebras have enjoyed similar benefits from their Hopf algebra structures.

Perhaps more unexpectedly, Hopf algebras have provided a useful structure for understanding certain combinatorial objects. We illustrate several examples of Hopf algebras that naturally arise in the study of symmetric functions.

2 Algebras, Bialgebras & Coalgebras

Let k be a field. All unadorned tensor products are taken over k.

2.1 Algebras

The reader may be familiar with the following definition of an algebra.

Definition 2.1. A *k-algebra* is a k-vector space A equipped with a k-bilinear map $M : A \times A \to A$.

This is sometimes taken in the more general commutative ring setting: let R be a commutative ring. An R-algebra is a ring A with an R-module structure, such that for all $r \in R$ and $a_1, a_2 \in A$, we have

$$r \cdot (a_1 a_2) = (r \cdot a_1)a_2 = a_1(r \cdot a_2).$$

However, we will stick to k-algebras. We expound upon Definition 2.1 via abstract nonsense. A k-bilinear map $M : A \times A \to A$ is the same as a k-linear map $\mu : A \otimes A \to A$, called multiplication. The k-algebra A is *associative* if the diagram

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \tag{2.1}$$

commutes, and *unital* if there exists a k-linear map $\eta : k \to A$ such that the diagram

$$\begin{tikzcd} k \otimes A \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd} \tag{2.2}$$

commutes. More explicitly, (2.1) states that the multiplication μ is associative, and (2.2) states that μ has a unit $1_A \coloneqq \eta(1_k)$, which satisfies $\mu(1_A \otimes a) = a = \mu(a \otimes 1_A)$ for all $a \in A$. Unfortunately, the map η is also called the unit of A. When the multiplication is unambiguous, we often write the product $\mu(a_1 \otimes a_2)$ more briefly as $a_1 a_2$, yielding the more familiar equation $1_A a = a = a 1_A$. In general, we denote $\eta(x)$ by x_A.

Example 2.2. k itself is a unital associative k-algebra with multiplication $\mu : k \otimes k \to k$ given by $x \otimes y \mapsto xy$ and unit $\eta : k \to k$ given by the identity map. ■

Example 2.3. The vector space $k[x]$ of polynomials in one indeterminate can be equipped with the usual (associative) polynomial multiplication and the unit map $\eta : k \to k[x]$ given by the canonical injection. This is called the polynomial algebra, and can be extended to up to countably many indeterminates. ■

Henceforth, we will denote a k-algebra by a pair (A, μ_A) or if unital, a triple (A, μ_A, η_A). If (A, μ_A, η_A) and (B, μ_B, η_B) are unital associative k-algebras, there is a natural k-algebra structure on the tensor product $A \otimes B$ of k-vector spaces. Multiplication $\mu_{A\otimes B} : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$ is defined by

$$\mu_{A\otimes B}((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)) = \mu_A(a_1 \otimes a_2) \otimes \mu_B(b_1 \otimes b_2).$$

More concisely, $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$. Or without the inputs, $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (\mathrm{id} \otimes \tau_{B,A} \otimes \mathrm{id})$, where $\tau_{B,A} : B \otimes A \to A \otimes B$ is the twist $\tau_{B,A}(b \otimes a) = a \otimes b$. The k-algebra $(A \otimes B, \mu_{A\otimes B})$ enjoys associativity, since

$$\begin{aligned} ((a_1 \otimes b_1)(a_2 \otimes b_2))\,(a_3 \otimes b_3) &= (a_1 a_2 \otimes b_1 b_2)(a_3 \otimes b_3) \\ &= a_1 a_2 a_3 \otimes b_1 b_2 b_3 \\ &= (a_1 \otimes b_1)(a_2 a_3 \otimes b_2 b_3) \\ &= (a_1 \otimes b_1)\,((a_2 \otimes b_2)(a_3 \otimes b_3))\,. \end{aligned}$$

Furthermore since A and B are unital we obtain a unit $\eta_{A\otimes B} : k \to A \otimes B$ given by $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ under the identification $k \cong k \otimes k$. More explicitly, $1_{A\otimes B} = 1_A \otimes 1_B$. Indeed,

$$(1_A \otimes 1_B)(a \otimes b) = 1_A a \otimes 1_B b = a \otimes b = a 1_A \otimes b 1_B = (a \otimes b)(1_A \otimes 1_B).$$

Therefore the tensor product of algebras $(A \otimes B, \mu_{A\otimes B}, \eta_{A\otimes B})$ is a unital associative k-algebra.

Let (A, μ, η) be a unital associative k-algebra. By defining $\mu^{\mathrm{op}} = \mu \circ \tau_{A,A}$, we obtain another unital associative k-algebra structure $A^{\mathrm{op}} \coloneqq (A, \mu^{\mathrm{op}}, \eta)$, called the *opposite algebra*. Indeed, associativity and unit follow immediately from the corresponding properties of μ and η.

As always, we are interested in maps between unital associative algebras that preserve their structure.

Definition 2.4. Let (A, μ_A, η_A), (B, μ_B, η_B) be unital associative k-algebras. A k-linear map $\varphi : A \to B$

is a *unital associative k-algebra morphism* if the following diagrams commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "\varphi\otimes\varphi"] \arrow[d, "\mu_A"'] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\varphi"] & & B \\ & k \arrow[ul, "\eta_A"] \arrow[ur, "\eta_B"'] & \end{tikzcd} \tag{2.3}$$

These diagrams express compatibility with the multiplication and unit, respectively. Thus we can define the category $\mathbf{Alg}_k$ of unital associative k-algebras and their morphisms.

2.2 Coalgebras

From what we have done so far, it may seem as though the arrow-theoretic formulations of a k-algebra obscure more than they clarify. However, the first fundamental strength of category theory is the possibility of reversing all the arrows.

Definition 2.5. A *counital coassociative k-coalgebra* is a k-vector space C equipped with k-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$, such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta\otimes\mathrm{id}"'] \\ A \otimes A \arrow[u, "\mathrm{id}\otimes\Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} k \otimes C & C \otimes C \arrow[l, "\epsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\epsilon"] & C \otimes k \\ & C \arrow[ul, "\cong"] \arrow[u, "\Delta"'] \arrow[ur, "\cong"'] & \end{tikzcd} \tag{2.4}$$

Δ is called the *comultiplication* and ε is called the *counit* of C. As with k-algebras, we capture the data of a counital coassociative k-coalgebra with a triple $(C, \Delta_C, \epsilon_C)$. However we will not omit the symbols for comultiplication and counit, since we will eventually deploy them simultaneously with the algebra structures. However, we will need the following notation for comultiplication, named after Sweedler. Let C be an n-dimensional vector space with basis $\{c_i\}_{i=1}^n$, so that $C \otimes C$ is n^2-dimensional with basis $\{c_i \otimes c_j\}_{i,j=1}^n$. If $c \in C$, then $\Delta(c) \in C \otimes C$ may be written in the form

$$\Delta(c) = \sum_{i,j=1}^{n} a_{ij} c_i \otimes c_j.$$

Formally, we will simply denote the right side by $c_{(1)} \otimes c_{(2)}$. However, the omission of the summation can be deceptive; there is no guarantee that the right side may be written as a simple tensor. In particular, $c_{(1)}$ and $c_{(2)}$ need not exist as vectors in C. To reiterate, the expression $c_{(1)} \otimes c_{(2)}$ is purely formal.

Example 2.6. As in the algebra setting, k is a counital coassociative k-coalgebra with comultiplication $x \mapsto 1_k \otimes x$ and counit id. ∎

Let $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ be counital coassociative k-coalgebras. We construct the tensor product of coalgebras, as we did with algebras. Consider the tensor product $C \otimes D$ of k-vector spaces. Define a comultiplication $\Delta_{C\otimes D} : C \otimes D \to (C \otimes D) \otimes (C \otimes D)$ by

$$\Delta_{C\otimes D}(c \otimes d) = (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)}) = \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{m} a_{ij} b_{k\ell} (c_i \otimes d_k) \otimes (c_j \otimes d_\ell), \tag{2.5}$$

where

$$\begin{aligned}
&\dim C = n, \dim D = m,\\
&\Delta_C(c) = \sum_{i,j=1}^{n} a_{ij} c_i \otimes c_j = c_{(1)} \otimes c_{(2)}, \text{ and}\\
&\Delta_D(d) = \sum_{k,\ell=1}^{m} b_{k\ell} d_k \otimes d_\ell = d_{(1)} \otimes d_{(2)}.
\end{aligned}$$

Omitting the inputs, this is equivalent to $\Delta_{C\otimes D} = (\mathrm{id} \otimes \tau_{C,D} \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$. The comultiplication is clearly k-linear, and coassociative by the following:

$$\begin{aligned}
((\mathrm{id} \otimes \Delta_{C\otimes D}) \circ \Delta_{C\otimes D})(c \otimes d) &= (\mathrm{id} \otimes \Delta_{C\otimes D}) \left(\left(c_{(1)} \otimes d_{(1)}\right) \otimes \left(c_{(2)} \otimes d_{(2)}\right) \right)\\
&= \left(c_{(1)} \otimes d_{(1)}\right) \otimes \left(\left(\left(c_{(2)}\right)_{(1)} \otimes \left(d_{(2)}\right)_{(1)} \right) \otimes \left(\left(c_{(2)}\right)_{(2)} \otimes \left(d_{(2)}\right)_{(2)} \right) \right)
\end{aligned}$$

and

$$\begin{aligned}
((\Delta_{C\otimes D} \otimes \mathrm{id}) \circ \Delta_{C\otimes D})(c \otimes d) &= (\Delta_{C\otimes D} \otimes \mathrm{id}) \left(\left(c_{(1)} \otimes d_{(1)}\right) \otimes \left(c_{(2)} \otimes d_{(2)}\right) \right)\\
&= \left(\left(\left(c_{(1)}\right)_{(1)} \otimes \left(d_{(1)}\right)_{(1)} \right) \otimes \left(\left(c_{(1)}\right)_{(2)} \otimes \left(d_{(1)}\right)_{(2)} \right) \right) \otimes \left(c_{(2)} \otimes d_{(2)}\right).
\end{aligned}$$

By coassociativity of Δ_C and Δ_D, the two must eventually coincide. Of course, we also have a counit $\epsilon_{C\otimes D} : C \otimes D \to k$ given by $\epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D$ followed by the identification $k \otimes k \cong k$. We verify

$$\begin{aligned}
((\epsilon_{C\otimes D} \otimes \mathrm{id}) \circ \Delta_{C\otimes D})(c \otimes d) &= \epsilon_{C\otimes D} \left(c_{(1)} \otimes d_{(1)}\right) \otimes \left(c_{(2)} \otimes d_{(2)}\right)\\
&= \left(\epsilon_C \left(c_{(1)}\right) \otimes \epsilon_D \left(d_{(1)}\right)\right) \otimes \left(c_{(2)} \otimes d_{(2)}\right)\\
&= \epsilon_C \left(c_{(1)}\right) c_{(2)} \otimes \epsilon_D \left(d_{(1)}\right) d_{(2)},
\end{aligned}$$

which by counitality of ϵ_C and ϵ_D is $c \otimes d$. The other triangle in the diagram defining the counit is similar. Therefore, the tensor product of coalgebras $(C \otimes D, \Delta_{C\otimes D}, \epsilon_{C\otimes D})$ is a counital coassociative k-coalgebra.

Let (C, Δ, ϵ) be a counital coassociative k-coalgebra. In analogy with the opposite algebra, we define $\Delta^{\mathrm{op}} = \tau_{C,C} \circ \Delta$. Then $C^{\mathrm{cop}} := (C, \Delta^{\mathrm{op}}, \epsilon)$ is a counital coassociative k-coalgebra, called the opposite coalgebra.

Definition 2.7. Let C, D be counital coassociative k-coalgebras endowed with comultiplications Δ_A, Δ_B and counits ϵ_A, ϵ_B, respectively. A k-linear map $\psi : C \to D$ is a *counital coassociative k-coalgebra morphism* if the following diagrams commute:

$$\begin{tikzcd}
C \otimes C \arrow[r, "\psi\otimes\psi"] & D \otimes D \\
C \arrow[u, "\Delta_C"] \arrow[r, "\psi"'] & D \arrow[u, "\Delta_D"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "\psi"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\
& k &
\end{tikzcd}
\tag{2.6}$$

From this, we may define the category $\mathbf{Coalg}_k$ of counital coassocitative k-algebras and their morphisms. Henceforth, all k-algebras will be unital and associative, and all k-coalgebras will be counital and coassociative.

2.3 Bialgebras

A k-bialgebra is both a k-algebra and a k-coalgebra such that the two structures are compatible. Namely, we require that Δ and ϵ are unital associative algebra morphisms, or equivalently that μ and η are counital coassociative coalgebra morphisms.

Lemma 2.8. *Let B be a k-vector space with a k-algebra structure (B, μ, η) as well as a k-coalgebra structure (B, Δ, ϵ). The following are equivalent:*

1. *μ and η are k-coalgebra morphisms.*
2. *Δ and ϵ are k-algebra morphisms.*

Proof. We consider $B \otimes B$ as the tensor product of k-algebras as well as the tensor product of k-coalgebras. Namely

$$\begin{aligned}
\mu_{B\otimes B} &= (\mu \otimes \mu) \circ (\mathrm{id} \otimes \tau_{B,B} \otimes \mathrm{id}),\\
\eta_{B\otimes B} &= \eta \otimes \eta,\\
\Delta_{B\otimes B} &= (\mathrm{id} \otimes \tau_{B,B} \otimes \mathrm{id}) \circ (\Delta \otimes \Delta),\\
\epsilon_{B\otimes B} &= \epsilon \otimes \epsilon.
\end{aligned}$$

The fact that $\mu : B \otimes B \to B$ is a k-coalgebra morphism is equivalent to the commutativity of the following two diagrams:

$$\begin{tikzcd}
(B\otimes B)\otimes(B\otimes B) \arrow[r, "\mu\otimes\mu"] & B\otimes B \\
(B\otimes B)\otimes(B\otimes B) \arrow[u, "\mathrm{id}\otimes\tau_{B,B}\otimes\mathrm{id}"] & \\
B\otimes B \arrow[u, "\Delta\otimes\Delta"] \arrow[r, "\mu"'] & B \arrow[uu, "\Delta"']
\end{tikzcd} \tag{2.7}$$

$$\begin{tikzcd}
B\otimes B \arrow[rr, "\mu"] \arrow[dr, "\epsilon\otimes\epsilon"'] & & B \arrow[dl, "\epsilon"] \\
& k &
\end{tikzcd} \tag{2.8}$$

Similarly, recall that k is a k-algebra with multiplication $x \otimes y \mapsto xy$ and unit id, as well as a k-coalgebra with comultiplication $\Delta_k : x \mapsto 1_k \otimes x$ and counit id. From this, the fact that $\eta : k \to B$ is a k-coalgebra morphism is equivalent to the commutativity of the following two diagrams:

$$\begin{tikzcd}
k\otimes k \arrow[r, "\eta\otimes\eta"] & B\otimes B \\
k \arrow[u, "\Delta_k"] \arrow[r, "\eta"'] & B \arrow[u, "\Delta"']
\end{tikzcd} \tag{2.9}$$

$$\begin{tikzcd} k \arrow[rr, "\eta"] \arrow[dr, "\mathrm{id}"'] & & B \arrow[dl, "\epsilon"] \\ & k & \end{tikzcd} \tag{2.10}$$

(2.6, left) and (2.7, left) are equivalent to $\Delta : B \to B \otimes B$ being a k-algebra morphism, and (2.6, right) and (2.7, right) are equivalent to $\epsilon : B \to k$ being a k-algebra morphism. □

This leads us to the following definition.

Definition 2.9. A *k-bialgebra* is a k-algebra as well as a k-coalgebra satisfying either equivalent condition in Lemma 2.5. A *k-bialgebra morphism* is a k-algebra morphism as well as a k-coalgebra morphism.

We represent a bialgebra as a quintuple when the operations are not obvious. As a culmination of the previous sections, we know that

- k is a k-bialgebra.
- The tensor product of two k-bialgebras is a k-bialgebra.
- If $(B, \mu, \eta, \Delta, \epsilon)$ is a k-bialgebra, then so are $B^{\mathrm{op}} = (B, \mu^{\mathrm{op}}, \eta, \Delta, \epsilon)$, $B^{\mathrm{cop}} = (B, \mu, \eta, \Delta^{\mathrm{op}}, \epsilon)$, and $B^{\mathrm{op\,cop}} = (B, \mu^{\mathrm{op}}, \eta, \Delta^{\mathrm{op}}, \epsilon)$.

3 Hopf Algebras

Let A be an algebra and C a coalgebra, always over k. We define a k-bilinear convolution $\mathrm{Hom}_k(C, A) \times \mathrm{Hom}_k(C, A) \to \mathrm{Hom}_k(C, A)$ as follows. Given $f, g \in \mathrm{Hom}_k(C, A)$, we define their convolution $f * g : C \to A$ by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A.$$

When H is a bialgebra, taking $C = H = A$ defines a convolution on $\mathrm{End}_k(H)$.

Definition 3.1. Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. An *antipode* of H is an endomorphism $S \in \mathrm{End}_k(H)$ such that

$$S * \mathrm{id}_H = \eta \circ \epsilon = \mathrm{id}_H * S.$$

Equivalently, the following diagram commutes:

$$\begin{tikzcd} & H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "\mu"] & \\ H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\eta"] & & H \\ & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "\mu"'] & \end{tikzcd} \tag{3.1}$$

The antipode is unique if it exists. First notice that for any $f \in \mathrm{End}_k(H)$ and $h \in H$,

$$\begin{aligned}(f * (\eta \circ \epsilon))(h) &= (\mu \circ (f \otimes (\eta \circ \epsilon)) \circ \Delta)(h) \\ &= (\mu \circ (f \otimes \eta) \circ (\mathrm{id} \otimes \epsilon) \circ \Delta)(h) \\ &= (\mu \circ (f \otimes \eta))(h \otimes 1_k) \\ &= \mu(f(h) \otimes 1_A) \\ &= f(h)1_A \\ &= f(h),\end{aligned}$$

where $(\mathrm{id} \otimes \epsilon) \circ \Delta$ takes h to $h \otimes 1_k$ by the defining property of ϵ. Similarly, $(\eta \circ \epsilon) * f = f$. Now if S and S' are two antipodes, then taking $f = S$ and later $f = S'$ gives

$$S = S * (\eta \circ \epsilon) = S * (\mathrm{id} * S') = (S * \mathrm{id}) * S' = (\eta \circ \epsilon) * S' = S'. \tag{3.2}$$

So the antipode is unique.

Definition 3.2. A *Hopf algebra* $(H, \mu, \eta, \Delta, \epsilon, S)$ is a bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ with an antipode S. A *Hopf algebra morphism* is a bialgebra morphism $\varphi : G \to H$ that commutes with the antipodes; that is, the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "S_H"] & H \\ G \arrow[u, "\varphi"] \arrow[r, "S_G"'] & G \arrow[u, "\varphi"'] \end{tikzcd} \tag{3.3}$$

At this point, we abandon algebraic territory and dive into the world of combinatorics.

4 Symmetric Functions

The purpose of this section is to exhibit several delightful and surprising Hopf algebra structures which arise in the study of symmetric functions and related phenomena.

4.1 The Shuffle Algebra

Let Σ be any set and let Σ^* be the set of words over the alphabet Σ. More formally,

$$\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$$

where $\Sigma^0 := \{\lambda\}$ consists of the empty word λ and for $n \geq 1$, Σ^n is the n-fold Cartesian product of Σ with itself. This may be viewed as the set of ordered lists of n letters in the alphabet Σ, and thus Σ^* is the set of arbitrary long but finite ordered lists in the alphabet. For brevity we omit the parentheses and commas when writing words.

Definition 4.1. Let w_1, $w_2 \in \Sigma^*$ be words. The *concatenation* $w_1 w_2$ or $w_1 \cdot w_2$ of w_1 with w_2 is obtained by appending the ordered list of letters w_2 to the end of w_1, as expected.

We define the concatenation of any word w with λ (and vice versa) by

$$w\lambda = w = \lambda w.$$

Example 4.2. For example, consider the alphabet $\Sigma = \{0, 1\}$. We have $\Sigma^0 = \{\lambda\}$, $\Sigma^1 = \{0, 1\}$, $\Sigma^2 = \{00, 01, 10, 11\}$, and so on. The concatenation of 00 with 10 is $0010 \in \Sigma^4$. ■

The k-vector space W having basis Σ^* under formal addition and scalar multiplication can be given a Hopf algebra structure. First, we define $ш : \Sigma^* \otimes \Sigma^* \to W$ by the recursion

$$\begin{aligned} ш(\lambda \otimes w) &= w = ш(w \otimes \lambda), \\ ш(aw_1 \otimes bw_1) &= a \cdot ш(w_1 \otimes bw_2) + b \cdot ш(aw_1 \otimes w_2) \end{aligned}$$

for any letters a, $b \in \Sigma$ and any words w, w_1, $w_2 \in \Sigma^*$. We extend linearly to obtain an algebra multiplication $ш : W \otimes W \to W$. Henceforth, we use the infix notation $w_1 ш w_2$ to denote $ш(w_1 \otimes w_2)$.

Example 4.3. Let $\Sigma = \{0, 1\}$ and consider $1, 10 \in \Sigma^*$. We have

$$\begin{aligned} 1 ш 10 &= 1(\lambda ш 10) + 1(1 ш 0) \\ &= 110 + 1(1(\lambda ш 0) + 0(1 ш \lambda)) \\ &= 110 + 110 + 101. \end{aligned}$$

■

Associativity is recursively verified:

$$\begin{aligned} (aw_1 ш bw_2) ш cw_3 &= (a(w_1 ш bw_2) + b(aw_1 ш w_2)) ш cw_3 \\ &= a(w_1 ш bw_2) ш cw_3 + b(aw_1 ш w_2) ш cw_3 \\ &= a((w_1 ш bw_2) ш cw_3) + c(a(w_1 ш bw_2) ш w_3) + \\ &\quad + b((aw_1 ш w_2) ш cw_3) + c(b(aw_1 ш w_2) ш w_3), \\ &= a(w_1 ш (bw_2 ш cw_3)) + b((aw_1 ш w_2) ш cw_3) + \\ &\quad + c(a(w_1 ш bw_2) ш w_3) + c(b(aw_1 ш w_2) ш w_3), \\ &= a(w_1 ш (b(w_2 ш cw_3) + c(bw_2 ш w_3))) + b((aw_1 ш w_2) ш cw_3) + \\ &\quad + c((a(w_1 ш bw_2) + b(aw_1 ш w_2)) ш w_3), \\ &= a(w_1 ш b(w_2 ш cw_3)) + a(w_1 ш c(bw_2 ш w_3)) + b(aw_1 ш (w_2 ш cw_3)) + \\ &\quad + c((aw_1 ш bw_2) ш w_3), \\ &= aw_1 ш b(w_2 ш cw_3) + a(w_1 ш c(bw_2 ш w_3)) + c(aw_1 ш (bw_2 ш w_3)) \\ &= aw_1 ш b(w_2 ш cw_3) + aw_1 ш c(bw_2 ш w_3) \\ &= aw_1 ш (b(w_2 ш cw_3) + c(bw_2 ш w_3)) \\ &= aw_1 ш (bw_2 ш cw_3). \end{aligned}$$

The unit map is given by $1_k \mapsto \lambda$; we verify that for any $w \in \Sigma^*$,

$$\lambda ш w = w = w ш \lambda,$$

and by linearity λ is a unit on all of W. Let the comultiplication be given by

$$\Delta(a_1 a_2 \ldots a_n) = \sum_{i=0}^{n} a_1 a_2 \ldots a_i \otimes a_{i+1} a_{i+2} \ldots a_n \tag{4.1}$$

for a nonempty basis vector $a_1 a_2 \ldots a_n$; it is not hard to convince oneself that this is compatible with the shuffle product. It is compatible with unit as $\Delta(\lambda) = \lambda \otimes \lambda$. Coassociativity follows from associativity of the tensor product:

$$\begin{aligned}(\Delta \otimes \mathrm{id})(\Delta(a_1a_2\dots a_n)) &= \sum_{i=0}^{n} \Delta(a_1a_2\dots a_i) \otimes a_{i+1}a_{i+2}\dots a_n\\ &= \sum_{i=0}^{n}\sum_{j=0}^{i}(a_1a_2\dots a_j \otimes a_{j+1}a_{j+2}\dots a_i) \otimes a_{i+1}a_{i+2}\dots a_n\\ &= \sum_{j=0}^{n}\sum_{i=j+1}^{n} a_1a_2\dots a_j \otimes (a_{j+1}a_{j+2}\dots a_i \otimes a_{i+1}a_{i+2}\dots a_n)\\ &= \sum_{j=0}^{n} a_1a_2\dots a_j \otimes \Delta(a_{j+1}a_{j+2}\dots a_n)\\ &= (\mathrm{id}\otimes\Delta)(\Delta(a_1a_2\dots a_n)).\end{aligned}$$

The counit is defined by $\epsilon(\lambda) = 1_k$ and $\epsilon(w) = 0$ for a nonempty word w, then extended to an algebra morphism. We verify:

$$\begin{aligned}(\epsilon \otimes \mathrm{id})(\Delta(a_1a_2\dots a_n)) &= (\epsilon\otimes\mathrm{id})\left(\sum_{i=0}^{n} a_1a_2\dots a_i \otimes a_{i+1}a_{i+2}\dots a_n\right)\\ &= (\epsilon\otimes\mathrm{id})\,(\lambda \otimes a_1a_2\dots a_n)\\ &= 1_k \otimes a_1a_2\dots a_n,\end{aligned}$$

and symmetrically for $(\mathrm{id}\otimes\epsilon)\circ\Delta$. Since Δ and ϵ have been constructed as algebra morphisms, W is a bialgebra. Finally, we define the antipode $S : W \to W$ on the basis by

$$S(a_1a_2\dots a_n) = (-1)^n a_na_{n-1}\dots a_1.$$

We compute

$$\begin{aligned}(S * \mathrm{id})(a_1a_2\dots a_n) &= (S \sqcup\!\sqcup\, \mathrm{id})(\Delta(a_1a_2\dots a_n))\\ &= \sum_{i=0}^{n} S(a_1a_2\dots a_i) \sqcup\!\sqcup\, a_{i+1}a_{i+2}\dots a_n\\ &= \sum_{i=0}^{n} (-1)^i a_ia_{i-1}\dots a_1 \sqcup\!\sqcup\, a_{i+1}a_{i+2}\dots a_n\\ &= \lambda \sqcup\!\sqcup\, a_1a_2\dots a_n - a_1 \sqcup\!\sqcup\, a_2\dots a_n + \dots + (-1)^n a_na_{n-1}\dots a_1 \sqcup\!\sqcup\, \lambda\\ &= a_1a_2\dots a_n - a_1(\lambda \sqcup\!\sqcup\, a_2\dots a_n) - a_2(a_1 \sqcup\!\sqcup\, a_3\dots a_n) + \dots + (-1)^n a_na_{n-1}\dots a_1\\ &= a_1a_2\dots a_n - a_1a_2\dots a_n - a_2(a_1 \sqcup\!\sqcup\, a_3\dots a_n) + \dots + (-1)^n a_na_{n-1}\dots a_1\\ &= \begin{cases}\lambda & \text{if } a_1a_2\dots a_n = \lambda,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This coincides with $\eta\circ\epsilon$. The computation of $\mathrm{id} * S$ is identical. Therefore, W is a Hopf algebra.

4.2 Symmetric Functions

The polynomial algebra $k[x_1, x_2, \dots]$ in countably many indeterminates has an important subalgebra generated by the elementary symmetric functions

$$\begin{aligned} e_1 &= \sum_{i=1}^{\infty} x_i \\ e_2 &= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} x_i x_j \\ &\vdots \\ e_n &= \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \dots x_{i_n}. \end{aligned}$$

It is clear that these are symmetric; that is, invariant under swapping indeterminates. It turns out that they generate the entire algebra of symmetric functions in $k[x_1, x_2, \dots]$, or *Sym*. More explicitly, *Sym* is the subalgebra of all $f \in k[x_1, x_2, \dots]$ such that $f(x_1, \dots, x_i, \dots, x_j, \dots) = f(x_1, \dots, x_j, \dots, x_i, \dots)$ for all $i < j$.

Define a comultiplication $\Delta : Sym \to Sym \otimes Sym$ by

$$e_n \mapsto \sum_{k=0}^{n} e_k \otimes e_{n-k}$$

where $e_0 := 1$ by convention, and extending with respect to multiplication. This respects unit as $\Delta(1) = \Delta(e_0) = e_0 \otimes e_0 = 1 \otimes 1$. Coassociativity is immediate from associativity of the tensor product:

$$\begin{aligned} (\Delta \otimes \mathrm{id})(\Delta(e_n)) &= \sum_{k=0}^{n} \Delta(e_k) \otimes e_{n-k} \\ &= \sum_{k=0}^{n} \sum_{\ell=0}^{k} (e_\ell \otimes e_{k-\ell}) \otimes e_{n-k} \\ &= \sum_{\ell=0}^{n} \sum_{k=\ell}^{n} e_\ell \otimes (e_{k-\ell} \otimes e_{n-k}) \\ &= \sum_{\ell=0}^{n} e_\ell \otimes \Delta(e_\ell) \\ &= (\mathrm{id} \otimes \Delta)(\Delta(e_n)). \end{aligned}$$

Define a counit $\epsilon : Sym \to k$ by $e_0 \mapsto 1_k$, $e_n \mapsto 0$ for $n \geq 1$, and then extending to an algebra morphism. We compute

$$\begin{aligned} (\epsilon \otimes \mathrm{id})(\Delta(e_n)) &= \sum_{k=0}^{n} \epsilon(e_k) \otimes e_{n-k} \\ &= \epsilon(e_0) \otimes e_n \\ &= 1_k \otimes e_n, \end{aligned}$$

and conversely for $(\mathrm{id} \otimes \epsilon) \circ \Delta$. Thus we have made *Sym* into a bialgebra, and it remains to find an antipode. We define $S : Sym \to Sym$ by

$$e_n \mapsto \sum_{\substack{\alpha\in\mathbb{N}^* \\ |\alpha|=n}} (-1)^{\mathrm{len}(\alpha)} e_\alpha,$$

where, if $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, then $|\alpha| = a_1 + a_2 + \cdots + a_n$, $\mathrm{len}(\alpha) = n$, $e_\alpha = e_{a_1} e_{a_2} \ldots e_{a_n}$. We extend this to products of the elementary symmetric functions, so it suffices to verify the antipode property on the elementary symmetric functions:

$$\begin{aligned}
(S * \mathrm{id})(e_n) &= \sum_{k=0}^{n} S(e_k) e_{n-k} \\
&= \sum_{k=0}^{n} \sum_{|\alpha|=k} (-1)^{\mathrm{len}(\alpha)} e_\alpha e_{n-k} \\
&= \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases} \\
&= (\eta \circ \epsilon)(e_n).
\end{aligned}$$

For $n \geq 1$, we evaluated $\sum_{k=0}^{n} \sum_{|\alpha|=k} (-1)^{\mathrm{len}(\alpha)} e_\alpha e_{n-k}$ as 0 because there is a 1-1 correspondence of words $e_\alpha e_{n-k}$ for $|\alpha| = k < n$ and words e_α with $|\alpha| = n$ by appending e_{n-k}. Due to the factor $(-1)^{\mathrm{len}(\alpha)}$, these precisely cancel. This completes the Hopf algebra structure on Sym.

4.3 Noncommutative Symmetric Functions

Let $X = \{x_1, x_2, \ldots\}$ be a countable set of indeterminates and let X^* be the set of words in the alphabet X, including the empty word λ. Consider the vector space $k\{X\}$ having basis X^*, just as we defined the vector space underlying the shuffle algebra. However, we define a multiplication on $k\{X\}$ by concatenation:

$$(x_{i_1} x_{i_2} \ldots x_{i_p})(x_{i_{p+1}} x_{i_{p+2}} \ldots x_{i_q}) = x_{i_1} x_{i_2} \ldots x_{i_p} x_{i_{p+1}} \ldots x_{i_q}.$$

As before, $1_k \mapsto \lambda$ defines the unit map. The following theorem shows the resulting algebra is free on X in the category of algebras.

Theorem 4.4. *Given an algebra A and a set-theoretic function $f : X \to A$, there exists a unique algebra morphism $\tilde{f} : k\{X\} \to A$ such that $\tilde{f}\Big|_X = f$.*

Proof. It suffices to define $\tilde{f}$ on the basis X^* of $k\{X\}$, then we extend linearly. Let $\tilde{f}(\lambda) = 1_A$, and

$$\tilde{f}(x_{i_1} x_{i_2} \ldots x_{i_p}) = f(x_{i_1}) f(x_{i_2}) \ldots f(x_{i_p}).$$

$\tilde{f} : k\{X\} \to A$ is clearly an algebra morphism and restricts to f on X. For uniqueness, we see that any algebra morphism $g : k\{X\} \to A$ is uniquely determined by its restriction to X, because the definition of an algebra morphism requires that $g(x_{i_1} x_{i_2} \ldots x_{i_p}) = g(x_{i_1}) g(x_{i_2}) \ldots g(x_{i_p})$. □

Thus we are justified in calling $k\{X\}$ the *free algebra* on X. From now on, we call $k\{X\}$ the algebra of noncommutative symmetric functions or $NSym$. Analogously to Sym, we define a comultiplication $\Delta : NSym \to NSym \otimes NSym$ by

$$x_n \mapsto \sum_{k=0}^{n} x_k \otimes x_{n-k},$$

where $x_0 = \lambda$; a counit $\epsilon : NSym \to k$ by $\epsilon(\lambda) = 1$ and $\epsilon(x_n) = 0$ for $n \geq 1$; and an antipode $S : NSym \to NSym$ by

$$S(x_n) = \sum_{|\alpha|=n} (-1)^{\text{len}(\alpha)} e_\alpha.$$

4.4 Quasi-Symmetric Functions

Recall that a polynomial is symmetric if it is invariant under swapping indeterminates. Equivalently, $f \in k[x_1, x_2, \dots]$ is symmetric for any pair of finite sequences $x_{i_1}, x_{i_2}, \dots, x_{i_p}$ and $x_{j_1}, x_{j_2}, \dots, x_{j_p}$ of indeterminates and any exponents $a_1, a_2, \dots, a_p \in \mathbb{N}$, the coefficients of $x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_p}^{a_p}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \dots x_{j_p}^{a_p}$ in f are equal. A slightly weaker condition follows.

Definition 4.5. A polynomial $f \in k[x_1, x_2, \dots]$ is *quasi-symmetric* if for any pair of finite increasing sequences $x_{i_1} < x_{i_2} < \cdots < x_{i_p}$ and $x_{j_1} < x_{j_2} < \cdots < x_{j_p}$ of indeterminates and any exponents $a_1, a_2, \dots, a_p \in \mathbb{N}$, the coefficients of $x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_p}^{a_p}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \dots x_{j_p}^{a_p}$ in f are equal.

The above definition is the most wieldly, but it may be useful to think of quasi-symmetry as invariance under order-preserving permutations of indeterminates, which are ordered by their indices.

Example 4.6. The polynomial $x_1x_2^2 + x_2x_3^2 + x_1x_3^2$ is quasi-symmetric but not symmetric. ■

We will find a basis for the algebra of quasi-symmetric functions or $QSym$ as we did for Sym. For each word $\alpha = a_1a_2 \dots a_m \in \mathbb{N}^*$, define $M_\lambda = 1$, and otherwise

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_m}^{a_m}.$$

We define $QSym$ to be the k-vector space spanned by $\{M_\alpha\}_{\alpha \in \mathbb{N}^*}$. The multiplication on $QSym$ is quite tricky to describe.

It suffices to describe the product $M_\alpha M_\beta$ of two basis vectors. Let $\alpha = a_1a_2 \dots a_m$ and $\beta = b_1b_2 \dots b_n$. For $i = 0, 1, \dots, \min\{m, n\}$, we will construct words of length $m + n - i$. To do so, we start with $m + n - i$ empty slots in which we can add letters. Choose m slots from the $m + n - i$ and add the letters of α in order. Choose i slots from these m. Add the letters of β in order to these i slots and the remaining $n - i$ empty slots. For the i nonempty slots, simply add the letter of β to the current letter of α occupying the slot. For the $n - i$ empty slots, simply insert the letter of β. The result is a word in $\mathbb{N}^{m+n-i}$. We denote the *overlapping shuffle*, defined as the set of all words in $\mathbb{N}^*$ constructible via this process, by $\alpha \uplus \beta$.

Example 4.7. Let $\alpha = 1$ and $\beta = 23$. For $i = 0$, there are no overlapping slots so the constructible words are 123, 213, 231. For $i = 1$, there will be one overlapping slot so we can construct 33 or 24. Therefore $1 \uplus 23 = \{123, 213, 231, 33, 24\}$. ■

Now we define our multiplication $\mu : QSym \otimes QSym \to QSym$ by

$$M_\alpha M_\beta = \sum_{\gamma \in \alpha \uplus \beta} M_\gamma,$$

and extending linearly. The unit map simply takes $1_k \mapsto 1$. Luckily the coproduct is more straightforward:

$$\Delta(M_\alpha) = \sum_{\alpha = \beta\gamma} M_\beta \otimes M_\gamma,$$

where $\beta\gamma$ is the concatenation. And as one might expect by now, the counit is $\epsilon(M_\alpha) = 1$ if $\alpha = \lambda$ and 0 if α is nonempty. The counit and coassociative properties are verified as we have already done several times. Lastly, we have antipode

$$S(M_\alpha) = (-1)^{|\alpha|} \sum_{\beta \leq \alpha^T} M_\beta,$$

where $\alpha^T = a_m a_{m-1} \dots a_1$, and $b_1 b_2 \dots b_n \leq c_1 c_2 \dots c_k$ holds if and only if there is a partition $\{P_1, P_2, \dots, P_\ell\}$ of $\{1, 2, \dots, k\}$ that respects the order and satisfies

$$b_j = \sum_{p \in P_j} c_j$$

for each $j = 1, 2, \dots, \ell$. By respecting order, we mean $i < j$ implies $p_i < p_j$ for all $p_i \in P_i$ and $p_j \in P_J$.

Example 4.8. If $\beta = (3, 7, 11)$ and $\gamma = (1, 2, 3, 4, 5, 6)$, then $\beta \leq \gamma$, by grouping each consecutive pair of letters in γ: $1 + 2 = 3$, $3 + 4 = 7$, $5 + 6 = 11$. ■

For the antipode property, we see that

$$(\mu \circ (S \otimes \mathrm{id}) \circ \Delta)(1) = S(M_\lambda)M_\lambda = 1,$$

and if $\alpha \neq \lambda$, then

$$\begin{aligned}
(\mu \circ (S \otimes \mathrm{id}) \circ \Delta)(M_\alpha) &= \sum_{\alpha = \beta\gamma} S(M_\beta)M_\gamma \\
&= \sum_{\alpha = \beta\gamma} (-1)^{|\beta|} \sum_{\delta \leq \beta^T} M_\delta M_\gamma \\
&= \sum_{\alpha = \beta\gamma} (-1)^{|\beta|} \sum_{\delta \leq \beta^T} \sum_{\omega \in \delta \uplus \gamma} M_\omega.
\end{aligned}$$

There is a fairly involved combinatorial argument that shows this is 0; see [GR14, Therorem 5.1.11].

4.5 The Malvenuto–Reutenauer Algebra

Let S_n be the n-th symmetric group, or the set of all permutations of the set $\{1, \dots, n\}$. By convention $S_0 = \{\varnothing\}$. Let $\mathfrak{S} = \bigcup_{n=0}^\infty S_n$. Denote by $\mathfrak{S}Sym$ the k-vector space having basis $\mathfrak{S}$ under formal addition and scaling.

We define a multiplication on $\mathfrak{S}Sym$ by the shifted shuffle product. Given basis vectors $\sigma \in S_m$ and $\tau \in S_n$, we represent σ by the word $(\sigma(1), \dots, \sigma(m)) \in \mathbb{N}^*$, and likewise for τ.

Example 4.9. This must not be conflated with cycle notation, in which no commas appear. If $\sigma \in S_3$ is given by $1 \mapsto 3 \mapsto 2 \mapsto 1$, then it is written in cycle notation as $(1\ 3\ 2)$ and in word notation as $(3, 1, 2)$. ■

The product of basis vectors σ and τ is the shuffle product

$$\sigma \otimes \tau \mapsto (\sigma(1), \dots, \sigma(m)) \sqcup\!\sqcup (\tau(1) + m, \dots, \tau(n) + m).$$

We saw in Section 4.1 that the shuffle is associative, and has unit $1_k \mapsto \varnothing$, where $\varnothing$ is the empty permutation in S_0 which is represented by the empty word.

To construct a comultiplication, we will first need a way to transform a bijection from $\{1, \dots, n\}$ to some n-element subset of $\mathbb{N}$ into a permutation of $\{1, \dots, n\}$ that preserves the order of the bijection.

This is quite straightforward: given such a bijection $f : \{1, \ldots, n\} \to \{a_1, \ldots, a_n\}$, let $\tau \in S_n$ be the permutation of the indices such that

$$a_{\tau(1)} < a_{\tau(2)} < \cdots < a_{\tau(n)}.$$

Then $\tau^{-1} \in S_n$ is such that if $a_i < a_j$, then $\tau^{-1}(i) < \tau^{-1}(j)$. Now given any word $(a_1, \ldots, a_n) \in \mathbb{N}^*$ with the a_i's distinct, we may view this word as a bijection $\{1, \ldots, n\} \mapsto \{a_1, \ldots, a_n\}$ given by $i \mapsto a_i$. By the above process, we obtain a permutation $\sigma \in S_n$ such that $a_i < a_j$ implies $\sigma(i) < \sigma(j)$. We denote the word representation of σ by $\mathrm{std}(a_1, \ldots, a_n)$.

Example 4.10. $\mathrm{std}(8, 2, 7) = (3, 1, 2)$. Indeed, $\tau \coloneqq (1\ 2\ 3) \in S_3$ satisfies

$$a_{\tau(1)} = 2 < a_{\tau(2)} = 7 < a_{\tau(3)} = 8.$$

Then $\tau^{-1} = (1\ 3\ 2)$, or in word notation $\mathrm{std}(8, 2, 7) = (3, 1, 2)$. ■

The coproduct of a basis vector $\sigma = (\sigma(1), \ldots, \sigma(n)) \in S_n$ is

$$(\sigma(1), \ldots, \sigma(n)) \mapsto \sum_{i=0}^{n} \mathrm{std}(\sigma(1), \ldots, \sigma(i)) \otimes \mathrm{std}(\sigma(i+1), \ldots, \sigma(n)).$$

The proof of coassociativity is identical to the many we have already shown. Ditto for the counit $\varnothing \mapsto 1$ and $\sigma \mapsto 0$ if $\sigma \in S_n$ for $n \geq 1$.

Finally, the antipode is given by

$$(\sigma(1), \ldots, \sigma(n)) \mapsto (-1)^n (\sigma^{-1}(n), \ldots, \sigma^{-1}(1)).$$

By following the combinatorial argument of Section 4.1, this should coincide with $\eta \circ \epsilon$.

References

[GR14] Darij Grinberg and Victor Reiner. Hopf algebras in combinatorics. *arXiv preprint*, arXiv:1409.8356, 2014.