1.8 Free Lie algebras

Today, all modules and algebras will be over k.

1.8.1 Free magmas

Definition 1.8.1

A set with a map $M \times M \to M$ is called a *magma*.

Let X be a set and by induction define $X_1 = X$,

$$X_n = \bigsqcup_{p+q=n} X_p \times X_q.$$

Let $M_X = \bigsqcup_{n=1}^{\infty} X_n$, and define

$$M_X \times M_X \to M_X$$

by

$$X_p \times X_q \longhookrightarrow X_{p+q}$$

where $\longhookrightarrow$ denotes the canonical inclusion from the definition of X_n. We call M_X the *free magma* on X. An element of M_X is called a non-associative word on W. If $w \in X_n$, then its length is $\ell(w) = n$. The free magma enjoys the following universal property:

Proposition 1.8.2

Let N be a magma and $f\colon X \to N$ a map. Then there exists a unique magma homomorphism $F\colon M_X \to N$ such that $f = F \circ \iota$, where $\iota\colon X = X_1 \longhookrightarrow M_X$ is the canonical inclusion.

Proof. Define F inductively by $F(u, v) = F(u) \cdot F(v)$ if $u, v \in X_p \times X_q$. □

1.8.2 Free algebras

Let A_X be the k-algebra of M_X. Concretely, each $\alpha \in A_X$ is a finite sum $\alpha = \sum_{m \in M_X} c_m m$ for $c_m \in k$, and multiplication in A_X is a linear extension of multiplication in M_X. We call A_X the *free algebra* on X. It enjoys a similar universal property:

Proposition 1.8.3

Let B be a k-algebra and $f\colon X \to B$ a map. Then there exists a unique k-algebra homomorphism $F\colon A_X \to B$ such that $f = F \circ \iota$.

Proof. By the universal property of the free magma, we have a magma homomorphism $f'\colon M_X \to B$ which extends f. Then extend f' linearly to a k-algebra homomorphism $F\colon A_X \to B$. □

Remark 1.8.4. A_X admits a natural grading by longest word length.

1.8.3 Free Lie algebras

Let $I \subset A_X$ be the ideal generated by the elements aa for $a \in A_X$ and $J(a, b, c)$, where $a, b, c \in A_X$ and J denotes the Jacobi product.

We call $L_X := A_X/I$ the *free Lie algebra* on X

Proposition 1.8.5

If $f\colon X \to X'$ is any map, then there exists a unique map $F\colon L_X \to L_{X'}$ which restricts to f.

Proof. L_X has basis $\{e_x\}_{x\in X}$, so we define $F(e_x) = e_{f(x)}$. Note that this is functorial. □

Corollary 1.8.6

Let $(X_\alpha, i_\alpha^\beta)$ be a directed system and $X = \varinjlim_\alpha X_\alpha$. Then

$$\varinjlim L_{X_\alpha} \cong L_X.$$

Proof. Let $i_\alpha\colon X_\alpha \to X$ be the canonical inclusions. By the previous proposition, we get $I_\alpha\colon L_{X_\alpha} \to L_X$ which is functorial so $L_X = \varinjlim L_{X_\alpha}$. □

Proposition 1.8.7

If $k \subset k'$ then $L_X(k') = L_X(k) \otimes_k k'$.

Proof. We have a natural isomorphism $e_x \otimes \lambda \mapsto \lambda e_x$. □

Proposition 1.8.8

I is a graded ideal of A_X, so L_X has a natural grading.

Proof. Let $I^\sharp$ be the ideal of $a \in A_X$ such that every homogeneous component of a belongs to I. Then $I^\sharp \subset I$. Conversely if $x = \sum x_n \in A_X$ then

$$x^2 = \sum x_n^2 + \sum_{n<m} (x_n x_m + x_m x_n),$$

but $x_n^2 \in I$ and

$$x_n x_m + x_m x_n = (x_n + x_m)^2 - x_n^2 - x_m^2 \in I,$$

so $x^2 \in I^\sharp$. Likewise

$$J(x, y, z) = \sum_{\ell,m,n} J(x_\ell, y_m, z_n) \in I^\sharp,$$

so $I^\sharp = I$. □

Proposition 1.8.9

L_X^1 has basis X and L_X^2 has basis $\{[x,y]\}_{x<y}$ where we have chosen a total order on X.

Proof. Clearly X generates L_X^1 and $[X,X]$ generates L_X^2. Let $E = k^{(X)}$ and let $E \oplus \Lambda^2 E = \mathfrak{g}$ be the associated Lie algebra. The canonical map $X \to \mathfrak{g}$ yields a Lie homomorphism $L_X \to \mathfrak{g}$, and $L_X^1 \oplus L_X^2 \to L_X \to \mathfrak{g}$ is an isomorphism. □

1.8.4 Free associative algebras

Let $E = k^{(X)}$ be the free k-module with basis X. Then the free associative algebra on X, denoted by Ass_X, is the tensor algebra TE.

Theorem 1.8.10

Let $\phi\colon L_X \to \mathrm{Ass}_X$ and $\Phi\colon UL_X \to \mathrm{Ass}_X$ be the maps induced by $X \to \mathrm{Ass}_X$. Then

(1) Φ is an isomorphism.

(2) ϕ is an isomorphism onto the Lie subalgebra generated by X.

(3) L_X and its homogeneous components are free k-modules.

(4) If X is finite and $\#X = d$ then L_X^n is free of finite rank $\ell_d(n)$, which is determined by induction according to the formula

$$\sum_{m|n} m\ell_d(m) = d^m.$$

Proof. (1) is clear as $X \to UL_X$ gives a homomorphism $\Psi\colon \mathrm{Ass}_X \to UL_X$, which is seen to be invese to Φ.

Also, $\phi\colon L_X \to \mathrm{Ass}_X$ maps into the Lie subalgebra generated by X, so it suffices to show ϕ is injective. By the Poincaré–Birkhoff–Witt theorem, if (3) L_X is free over k then (2) $L_X \to UL_X$ is injective, and under the identification of UL_X with Ass_X we have (2).

To prove (3) and (4), first assume k is a field and X is finite. Let $(\gamma_i)_{i\in I}$ be a totally ordered homogeneous basis for L_X. Let $d_i = \deg(\gamma_i)$. The PBW theorem implies that the elements

$$\gamma^e = \gamma_{i_1}^{e_1}\cdots\gamma_{i_s}^{e_s} \quad \text{for} \quad i_1 < \cdots < i_s$$

form a basis of $UL_X \cong \mathrm{Ass}_X$ and $\deg(\gamma^e) = \sum e_{i_j} d_{i_j}$. Let $a(n)$ be the rank of Ass_X^n. Then $a(n)$ is the number of families (e_i) such that $n = \sum e_i d_i$. Equivalently, the power series

$$A(t) = \sum a(n)t^n$$

can be written in the form

$$A(t) = \prod_{i\in I} \frac{1}{1-t^{d_i}}$$

and the coefficient of t^n in

$$\prod_{i\in I} \frac{1}{1-t^{d_i}} = \prod_{i\in I}(1 + t^{d_i} + t^{2d_i} + \cdots)$$

is the number of (e_i) such that $\sum e_i d_i = n$. For any $m \in \mathbb{N}$ we have that the number of $d_i = m$ in the product $\prod_{i\in I} \frac{1}{1-t^{d_i}}$ is $\ell_d(m)$ the rank of L_X^m, so

$$A(t) = \prod_{m=1}^{\infty} \frac{1}{(1-t^m)^{\ell_d(m)}}.$$

On the other hand, Ass_X is the free associative algebra on X and so the monomials $x_{i_1}\cdots x_{i_n}$ form a basis. So $a(n) = d^n$, and thus

$$\begin{aligned} A(t) &= \sum d^n t^n \\ \prod_{m=1}^{\infty} &= \frac{1}{1-dt} \\ \sum_{m,\nu} \frac{1}{\nu}\ell_d(m)t^{m\nu} &= \sum_{n=1}^{\infty} \frac{1}{n} d^n t^n \\ \frac{1}{n} d^n &= \sum_{m\nu=n} \frac{1}{\nu}\ell_d(m) \\ d^n = \sum_{m|n} m\ell_d(m), \end{aligned}$$

proving (4).

Secondly, assume $k = \mathbb{Z}$ and X is finite. It is a fact from the structure theorem of abelian groups that if E is a finitely-generated $\mathbb{Z}$-module and $\dim_{\mathbb{F}_p}(E \otimes_{\mathbb{Z}} \mathbb{F}_p)$ is independent of p, then E is a free $\mathbb{Z}$-module with rank $\dim_{\mathbb{F}_p}(E \otimes_{\mathbb{Z}} \mathbb{F}_p)$. Now since $L_X^n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p$ and $\dim(L_X^n(\mathbb{F}_p)) = \ell_d(n)$ is independent of p, L_X^n is a free $\mathbb{Z}$-module of rank $\ell_d(n)$.

Now for $k = \mathbb{Z}$ and dropping the finiteness assumption on X, let (Y_α) be the finite subsets of X, so that $X = \varinjlim_\alpha Y_\alpha$. We first prove (2). By the previous case

$$\phi_\alpha \colon L_{Y_\alpha} \to \mathrm{Ass}_{Y_\alpha}$$

is injective. Now $\phi = \varinjlim_\alpha \phi_\alpha$ is injective as the colimit of injective maps, proving (2). Then (2) implies that L_X and L_X^n are $\mathbb{Z}$-submodules of Ass_X, which is free, so L_X and L_X^n are free.

Finally, the result for arbitrary k follows from the equality

$$L_X^n(k) = L_X^n(\mathbb{Z}) \otimes_{\mathbb{Z}} k.$$

This shows L_X^k is free (3), which implies (2). Also $\operatorname{rank} L_X^n(k) = \operatorname{rank} L_X^n(\mathbb{Z})$ so we have (4) when X is finite. $\square$

1.8.5 P. Hall families

Definition 1.8.11

Let X be a set. A *P. Hall family* in M_X is a totally ordered subset $H \subset M_X$ such that

(i) $X \subset H$.

(ii) If $u, v \in H$ with $\ell(u) < \ell(v)$ then $u < v$.

(iii) Let $u \in M_X \setminus X$ and write $u = vw$. Then $u \in H$ if and only if (a) $v, w \in H$ and $v < w$, and (b) either $w \in X$ or $w = w'w''$ with $w', w'' \in H$ and $w' \leq v$.

Proposition 1.8.12

Every set admits a P. Hall family.

Proof. By induction, we define $H^n = H \cap X_n$. Let $H^1 = X$, and let X be totally ordered. If $H^1, \dots, H^{n-1}$ are defined and totally ordered so that (i), (ii), (iii) hold for elements of length $\leq n-1$, then H^n is defined by (iii): we order H^n and put $u < v$ if $u \in H^i$ and $v \in H^n$. By induction, $H = \bigcup H^n$ is a P. Hall family. $\square$

We will not prove the following result.

Theorem 1.8.13

Let H be a P. Hall family in M_X. Then the canonical image of H is a basis of L_X.

1.8.6 Free groups

Let $k = \mathbb{Z}$. Let X be a set and F_X the free group on X. Let F_X^n be the descending central series of F_X, defined by $F_X^1 = F_X$ and $F_X^n = (F_X, F_X^{n-1})$.

The associated graded group

$$\operatorname{gr} F_X = \sum_{n=1}^{\infty} \operatorname{gr}_n F_X$$

where

$$gr_n F_X = F_X^n / F_X^{n+1}$$

is a Lie algebra; in particular $\operatorname{gr}_1 F_X = F_X/(F_X, F_X)$ is the free abelian group on X.

Theorem 1.8.14

The canonical map $X \to \operatorname{gr}_1 F_X$ induces a Lie isomorphism

$$\phi_1 \colon L_1 \xrightarrow{\sim} \operatorname{gr} F_X.$$

Corollary 1.8.15

The groups $\operatorname{gr}_n F_X$ are free $\mathbb{Z}$-modules. If X is finite with $\#X = d$, then $\operatorname{rank} \operatorname{gr}_n F_X = \ell_d(n)$.

We define the completion $\hat{\operatorname{Ass}}_X$ of Ass_X to be the product $\prod_{n=0}^{\infty} \operatorname{Ass}_X^n$. An element $f \in \hat{\operatorname{Ass}}_X$ is represented by a formal series $f = \sum_{n=0}^{\infty} f_n$ with $f_n \in \mathrm{Ass}_X^n$. We have a natural homomorphism

$$\begin{aligned}\theta\colon F_X &\longrightarrow \hat{\mathrm{Ass}_X}^* \\ x &\longmapsto 1 + x.\end{aligned}$$

Let

$$\hat{\mathfrak{m}}^n = \{\sum_{m=0}^{\infty} f_m \in \hat{\mathrm{Ass}}_X : f_0 = f_1 = \cdots = f_{n-1} = 0\},$$

and let ${}'F_X^n = \theta^{-1}(1 + \hat{\mathfrak{m}}^n)$. Then $g \in F_X$ is in ${}'F_X^n$ if and only if

$$\theta(g) = 1 + \sum_{m \geq n} \psi_n.$$

Theorem 1.8.16

${}'F_X^n = F_X^n$.

Proof of Theorem 1.8.14 and Theorem 1.8.16. It is clear that ϕ_1 is surjective.

We claim that ${}'F_X^n$ is a filtration of F_X. It suffices to show

$$({}'F_X^m, {}' F_X^p) \subset' F_X^{m+p}.$$

Indeed, pick $g \in' F_X^m$, $h \in' F_X^p$ with $\theta(g) = 1 + G$ for $G \in \hat{\mathfrak{m}}^m$, $\theta(h) = 1 + H$ for $H \in \hat{\mathfrak{m}}^p$. Then

$$\begin{aligned} gh = hg(g,h), \theta(gh) &= 1 + G + H + GH, \\ \theta(hg) = 1 + G + H + HG. \end{aligned}$$

Since θ is a homomorphism we get $\theta(gh) = \theta(hg)\theta((g,h))$, so

$$\theta((g,h)) = 1 + (GH - HG) + \text{ higher terms}.$$

Hence $(g,h) \in' F_X^{m+p}$.

Now there is a natural map $\eta\colon {}' \operatorname{gr} F_X \to \mathrm{Ass}_X$: given $\xi \in' \operatorname{gr}_n F_X$ let $g \in' F_X^n$ be a representative and let

$$\theta(g) = 1 + G_n + G_{n+1} + \cdots,$$

where $G_p \in \mathrm{Ass}_X^p$. Define

$$\eta(\xi) = G_n.$$

This clearly doesn't depend on g, and the above equation for $\theta((g,h))$ shows that η is a Lie homomorphism.

Now since ${}'F_X^n$ is a filtration we automatically have $F_X^n \subset' F_X^n$, which yields a homomorphism $\psi\colon \operatorname{gr} F_X \to' \operatorname{gr} F_X$. Consider the composition

$$L_X \xrightarrow{\phi_1} \operatorname{gr} F_X \xrightarrow{\psi} {}' \operatorname{gr} F_X \xrightarrow{\eta} \mathrm{Ass}_X,$$

where ϕ_1 is surjective and η is injective. This composition coincides with the map $\phi\colon L_X \to \mathrm{Ass}_X$ we saw earlier, which is injective. Hence ϕ_1 is injective, proving Theorem 1.8.14. This implies ψ is injective. By induction, we claim that $F_X^n =' F_X^n$. For $n = 1$ this is by definition, and if $n > 1$ then

$$F_X^n \subset' F_X^n \subset F_X^{n-1} \subset' F_X^{n-1},$$

and the injection

$$\operatorname{gr}_{n-1} F_X \to' \operatorname{gr}_{n-1} F_X$$

is the canonical map

$$F_X^{n-1}/F_X^n \to F_X^{n-1}/{}'F_X^n,$$

so $F_X^n =' F_X^n$. ☐

1.8.7 The Campbell–Hausdorff formula

In this section let k be a $\mathbb{Q}$-algebra, for example a field of characteristic 0.

Theorem 1.8.17

Let X be a set. Then the free Lie algebra L_X is the set of primitive elements of Ass_X.

Proof. This follows from the fact that the analogous result about the universal enveloping algebra under the identification $\mathrm{Ass}_X \cong UL_X$. □

As with the associative algebra, define the completion of L_X by

$$\hat{L}_X = \prod_{n=0}^{\infty} L_X^n$$

and define the *completed tensor product*

$$\hat{\mathrm{Ass}}_X \hat{\otimes} \hat{\mathrm{Ass}}_X = \prod_{p,q} \mathrm{Ass}_X^p \otimes \mathrm{Ass}_X^q .$$

The diagonal $\Delta\colon \mathrm{Ass}_X \to \mathrm{Ass}_X \otimes \mathrm{Ass}_X$ thus extends to

$$\Delta\colon \hat{\mathrm{Ass}}_X \to \hat{\mathrm{Ass}}_X \hat{\otimes} \hat{\mathrm{Ass}}_X,$$

and the above theorem about primitive elements remains true when we pass to completions. Let $\hat{\mathfrak{m}} \subset \hat{\mathrm{Ass}}_X$ be the ideal generated by X as before. Define

$$\begin{aligned} \exp\colon \hat{\mathfrak{m}} &\longrightarrow 1 + \hat{\mathfrak{m}} \\ x &\longmapsto \sum \frac{x^n}{n!}, \\ \log\colon 1 + \hat{\mathfrak{m}} &\longrightarrow \hat{\mathfrak{m}} \\ 1 + x &\longmapsto \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}. \end{aligned}$$

Theorem 1.8.18

$\exp \circ \log = \mathrm{id}$ and $\log \circ \exp = \mathrm{id}$.

Proof. Let $y \in \hat{\mathfrak{m}}$, we show that $\exp(\log(1+y)) = 1+y$. For any indeterminate T, the formula

$$\exp(\log(1+T)) = 1+T$$

is true in $\mathbb{Q}[[T]]$. But $y \in \hat{\mathfrak{m}}$ so there exists a well-defined and continuous homomorphism $f\colon \mathbb{Q}[[T]] \to \hat{\mathrm{Ass}}_X$ sending T to y, so we are done. The other formula is similar. □

Corollary 1.8.19

exp is a bijection of the set of primitive elements in $\hat{\mathfrak{m}}$ onto the set of $\beta \in 1+\hat{\mathfrak{m}}$ with $\Delta\beta = \beta \otimes \beta$.

Proof. Let $\alpha \in \hat{\mathfrak{m}}$ and $\beta \in e^\alpha$. Since Δ commutes with exp and $\alpha \otimes 1$ commutes with $1 \otimes \alpha$,

$$\Delta\beta = \Delta e^\alpha = e^{\Delta\alpha} = e^{\alpha\otimes 1 + 1\otimes\alpha} = e^{\alpha\otimes 1} e^{1\otimes\alpha} = (\beta \otimes 1)(1 \otimes \beta) = \beta \otimes \beta.$$

□

Now for our second big theorem:

Theorem 1.8.20 (Campbell–Hausdorff)
Let $X = \{x, y\}$ where $x \neq y$. Then $e^x e^y = e^z$ whith $z \in \hat{L}_X$.

Proof. Since $e^x, e^y \in 1 + \hat{\mathfrak{m}}$ we have $e^x e^y \in 1 + \hat{\mathfrak{m}}$. Since exp is a bijection, there exists a unique $z \in \hat{\mathfrak{m}}$ with $e^z = e^x e^y$. In particular

$$\begin{aligned} \Delta e^z &= \Delta(e^x e^y) \\ &= \Delta(e^x)\Delta(e^y) \\ &= (e^x \otimes e^x)(e^y \otimes e^y) \\ e^z \otimes e^z. \end{aligned}$$

By the previous proposition this implies z is a primitive element, and by the correspondence between primitive elements of $\hat{\mathrm{Ass}}_X$ and $\hat{L}_X$, we have $z \in \hat{L}_X$. □

If X is arbitrary and $z(x, y)$ is the element of $\hat{L}_{\{x,y\}} \subset \hat{L}_X$ such that $e^x e^y = e^{z(x,y)}$, then write

$$z(x, y) = \sum_{n=1}^{\infty} z_n(x, y)$$

for $z_n(x, y) \in L_X^n$. Explicitly, we may compute the first few homogeneous components?

$$\begin{aligned} z_1(x, y) &= x + y \\ z_2(x, y) &= \frac{1}{2}[x, y] \\ z_3(x, y) &= \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]]. \end{aligned}$$

It is also clear that

$$\begin{aligned} z(x, 0) &= x \\ z(0, y) &= y \\ z(z(w, x), y) &= z(w, z(x, y)). \end{aligned}$$