A RING OF GERMS

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1. Introduction

Let a be a point in a manifold M, let $f : M \to \mathbb{R}$ a smooth function, and let U a neighbourhood of a in M. Define an equivalence relation on pairs (f, U) by

$$(f, U) \sim (g, V) \iff f = g \text{ on some neighbourhood } W \subseteq U \cap V \text{ of } a.$$

Definition 1.1. The equivalence classes $[f, U]$ are called germs, and we denote by $\mathcal{C}_a^\infty(M)$ the ring of germs of smooth functions on M at a, under pointwise addition and multiplication.

For convenience, we will simply denote a germ by its function, although it remains true that we are only looking at its local behaviour.

2. A Local Ring

The ring of germs was a seminal example of a local ring; that is, a ring with a unique maximal ideal.

Proposition 2.1. *$\mathcal{C}_a^\infty(M)$ has unique maximal ideal $\mathfrak{m}_a = \{f \in \mathcal{C}_a^\infty(M) : f(a) = 0\}$.*

Proof. We first show $\mathfrak{m}_a$ is a subring of $\mathcal{C}_a^\infty(M)$. It surely contains 0; for any $f, g \in \mathfrak{m}_a$, $(f + g)(a) = f(a) + g(a) = 0$ and $(-f)(a) = -f(a) = 0$. To see that $\mathfrak{m}_a$ is an ideal, if $f \in \mathfrak{m}_a$ and $g \in \mathcal{C}_a^\infty(M)$ then $(fg)(a) = f(a)g(a) = 0$.

It remains to show $\mathfrak{m}_a$ is maximal, and uniquely so. If $[f, U] \notin \mathfrak{m}_a$, then $f(a) \neq 0$. By continuity, $f(a) \neq 0$ in some neighbourhood V of a. It follows that $f^{-1} = \frac{1}{f}$ exists smoothly on V, so $[f, U] = [f, V]$ is a unit in $\mathcal{C}_a^\infty(M)$. On other words, any ideal containing $[f, U]$ must be all of $\mathcal{C}_a^\infty(M)$. It follows that every proper ideal in $\mathcal{C}_a^\infty(M)$ must be contained in $\mathfrak{m}_a$, hence it is the unique maximal ideal in $\mathcal{C}_a^\infty(M)$. $\square$

Before taking a closer look $\mathfrak{m}_a$, we acquire the following lemma:

Lemma 2.2 (Hadamard)**.** *Let $a \in \mathbb{R}^n$ and let U be a star-convex neighbourhood of a. Let $f : U \to \mathbb{R}$ be smooth. Then for $i = 1, 2, \ldots, n$, there exist smooth functions $g_i : U \to \mathbb{R}$ such that*

$$f(x) = f(a) + \sum_{i=1}^{n} (x_i - a_i) g_i(x).$$

Proof. For $x \in U$, define $h : [0, 1] \to \mathbb{R}$ by $h(t) = f(a + t(x - a))$. By the chain rule,

$$h'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a + t(x - a))(x_i - a_i),$$

and by the fundamental theorem of calculus,

$$\begin{aligned} h(1) - h(0) &= \int_0^1 h'(t)\,dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a + t(x-a))(x_i - a_i)\,dt \\ &= \sum_{i=1}^n (x_i - a_i) \int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))\,dt. \end{aligned}$$

Since $h(1) - h(0) = f(x) - f(a)$, defining

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))\,dt$$

gives the desired result. □

Now we look at the unique maximal ideal $\mathfrak{m}_a$ in $\mathcal{C}_a^\infty(\mathbb{R}^n)$.

Theorem 2.3. *In $\mathcal{C}_a^\infty(\mathbb{R}^n)$, $\mathfrak{m}_a$ is generated by $x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n$.*

Proof. Clearly $x_i - a_i \in \mathfrak{m}_a$ for each i. Conversely, suppose $f \in \mathfrak{m}_a$. Then f is smooth on some star-convex neighbourhood U of a and $f(a) = 0$, so by Hadamard's Lemma there exist smooth functions $g_i : U \to \mathbb{R}$ such that

$$f(x) = \sum_{i=1}^n (x_i - a_i) g_i(x) \in \langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n \rangle.$$

□

While we will not assume M is $\mathbb{R}^n$ for the remaining sections, we can try to match the elegance of this result. In general, if M is an n-dimensional manifold and $a \in M$ then we have a chart (diffeomorphism) $\varphi : W \to V$ from a neighbourhood $W \subseteq \mathbb{R}^n$ of 0 into a neighbourhood $V \subseteq M$ of a.

The natural inclusion $\mathcal{C}_a^\infty(V) \to \mathcal{C}_a^\infty(M)$ is an isomorphism, because it has an inverse in the restriction $\mathcal{C}_a^\infty(M) \to \mathcal{C}_a^\infty(V)$ given by $[f, U] \mapsto [f\big|_{U \cap V}, U \cap V]$. Moreover we have a pullback

$$\begin{aligned} \varphi_0^* : \mathcal{C}_a^\infty(V) &\to \mathcal{C}_0^\infty(W) \\ [f, U] &\mapsto [f \circ \varphi, \varphi^{-1}(U)], \end{aligned}$$

which is an isomorphism, having inverse $(\varphi^{-1})_a^*$. Another inclusion $W \to \mathbb{R}^n$ completes the following sequence of isomorphisms:

$$\mathcal{C}_0^\infty(\mathbb{R}^n) \hookleftarrow \mathcal{C}_0^\infty(W) \xleftarrow{\varphi_0^*} \mathcal{C}_a^\infty(U) \hookrightarrow \mathcal{C}_a^\infty(M).$$

3. A Tangent Space

We endow $\mathcal{C}_a^\infty(M)$ with the obvious $\mathbb{R}$-algebra structure. We immediately get:

Proposition 3.1. *$\mathcal{C}_a^\infty(M)/\mathfrak{m}_a$ is a 1-dimensional $\mathbb{R}$-vector space.*

Proof. Let $\varphi : \mathcal{C}_a^\infty(M) \to \mathbb{R}$ be defined by $f \mapsto f(a)$. Clearly φ is surjective, and $\ker(\varphi) = \mathfrak{m}_a$. The First Isomorphism Theorem yields

$$\mathcal{C}_a^\infty(M)/\mathfrak{m}_a \cong \mathbb{R}.$$

□

Let M be an n-dimensional manifold.

Proposition 3.2. *$\mathfrak{m}_a/\mathfrak{m}_a^2$ is an n-dimensional $\mathbb{R}$-vector space.*

Proof. Since $\mathcal{C}_a^\infty(M) \cong \mathcal{C}_0^\infty(\mathbb{R}^n)$, it suffices to prove the result for the latter ring. We show that $x_1, x_2, \ldots, x_n$ is a basis for $\mathfrak{m}_0/\mathfrak{m}_0^2$. By Theorem 2.3, $\mathfrak{m}_0 \subseteq \mathcal{C}_0^\infty(\mathbb{R}^n)$ is generated by $x_1, x_2, \ldots, x_n$. Thus given $f \in \mathfrak{m}_0$, write

$$f = \sum_{i=1}^n x_i g_i$$

for some $g_i \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Defining $b_i = g_i(0)$, we now have $g_i - b_i \in \mathfrak{m}_0$, so

$$f + \mathfrak{m}_0^2 = \sum_{i=1}^n b_i x_i + x_i(g_i - b_i) + \mathfrak{m}_0^2 = \sum_{i=1}^n b_i x_i + \mathfrak{m}_0^2.$$

This shows the x_i's span $\mathfrak{m}_0/\mathfrak{m}_0^2$. To show linear independence, we will use the linear functionals

$$\begin{aligned} \left.\frac{\partial}{\partial x_j}\right|_0 : \mathfrak{m}_0/\mathfrak{m}_0^2 &\to \mathbb{R} \\ f + \mathfrak{m}_0^2 &\mapsto \frac{\partial f}{\partial x_j}(0), \end{aligned}$$

which will in fact define a basis of the dual space $(\mathfrak{m}_a/\mathfrak{m}_a^2)^*$. To see that this functional is well-defined on $\mathfrak{m}_0/\mathfrak{m}_0^2$, suppose $f, g \in \mathfrak{m}_0$. By the product rule,

$$\left.\frac{\partial}{\partial x_j}\right|_0 (fg) = \frac{\partial f}{\partial x_j}(0) \cdot g(0) + f(0) \cdot \frac{\partial g}{\partial x_j}(0) = 0,$$

so $\mathfrak{m}_0^2 \mapsto 0$ as desired. We remark that

$$\left.\frac{\partial}{\partial x_j}\right|_0 (x_i) = \delta_{ij},$$

showing that the x_i's are linearly independent. Therefore they form a basis of $\mathfrak{m}_0/\mathfrak{m}_0^2$ with the above functionals forming its dual basis. □

While defining the dual basis in the above proof may appear overkill, it becomes useful after our definition of the tangent space below.

Definition 3.3. The tangent space TM_a of M at a is the dual space $(\mathfrak{m}_a/\mathfrak{m}_a^2)^*$.

With this definition, we construct a tangent mapping.

Theorem 3.4. *A smooth function $\varphi : M \to N$ between manifolds induces a linear transformation $\partial_a\varphi : TM_a \to TN_{\varphi(a)}$ satisfying*

$$\partial_a(\psi \circ \varphi) = \partial_{\varphi(a)}\psi \circ \partial_a\varphi.$$

Proof. Recall that we have a pullback

$$\begin{aligned} \varphi_a^* : \mathcal{C}_{\varphi(a)}^\infty(N) &\to \mathcal{C}_a^\infty(M) \\ [f, U] &\mapsto [f \circ \varphi, \varphi^{-1}(U)]. \end{aligned}$$

Given $f \in \mathfrak{m}_{\varphi(a)}$, $(f \circ \varphi)(a) = 0$ implies $\varphi_a^*(f) \in \mathfrak{m}_a$. From this, we obtain the restriction $\varphi_a^* : \mathfrak{m}_{\varphi(a)} \to \mathfrak{m}_a$. Moreover given $f \in \mathfrak{m}_{\varphi(a)}^2$, write $f = \sum_{i=1}^n g_i h_i$ for $g_i, h_i \in \mathfrak{m}_{\varphi(a)}$. Then

$$\varphi_a^*(f) = \varphi_a^* \left(\sum_{i=1}^n g_i h_i \right) = \sum_{i=1}^n \varphi_a^*(g_i) \varphi_a^*(h_i) \in \mathfrak{m}_a^2.$$

Thus $\varphi_a^* : \mathfrak{m}_{\varphi(a)} \to \mathfrak{m}_a$ descends to a linear map $\varphi_a^* : \mathfrak{m}_{\varphi(a)}/\mathfrak{m}_{\varphi(a)}^2 \to \mathfrak{m}_a/\mathfrak{m}_a^2$.

Given another smooth function $\psi : N \to K$, the composition of pullbacks is given by

$$\begin{aligned} \varphi_a^* \circ \psi_{\varphi(a)}^* : \mathcal{C}_{(\psi \circ \varphi)(a)}^\infty(K) &\to \mathcal{C}_a^\infty(M) \\ [f, U] &\mapsto [f \circ \psi \circ \varphi, \varphi^{-1}(\psi^{-1}(U))], \end{aligned}$$

which may be written as $[f \circ (\psi \circ \varphi), (\psi \circ \varphi)^{-1}(U)]$, or simply $(\psi \circ \varphi)_a^*([f, U])$. In other words, $\varphi_a^* \circ \psi_{\varphi(a)}^* = (\psi \circ \varphi)_a^*$.

Define $\partial_a \varphi : (\mathfrak{m}_a/\mathfrak{m}_a^2)^* \to (\mathfrak{m}_{\varphi(a)}/\mathfrak{m}_{\varphi(a)}^2)^*$ as the dual of $\varphi_a^* : \mathfrak{m}_{\varphi(a)}/\mathfrak{m}_{\varphi(a)}^2 \to \mathfrak{m}_a/\mathfrak{m}_a^2$. Under the identification of Definition 3.3, $\partial_a \varphi : TM_a \to TN_{\varphi(a)}$. Now

$$\begin{aligned} \partial(\psi \circ \varphi) &= ((\psi \circ \varphi)_a^*)^* \\ &= \left(\varphi_a^* \circ \psi_{\varphi(a)}^* \right)^* \\ &= \left(\psi_{\varphi(a)}^* \right)^* \circ (\varphi_a^*)^* \\ &= \partial_{\varphi(a)} \psi \circ \partial_a \varphi, \end{aligned}$$

as desired. $\square$