Chapter 1

Function Spaces

1.1 Topological vector spaces

We begin by recalling some facts about topological vector spaces. We are really only concerned with vector spaces over $\mathbb{R}$ or $\mathbb{C}$, so we may assume our scalar field carries a natural topology. Recall that a topological space is T1 if singletons are closed.

Definition 1.1.1

A *topological vector space* is a vector space endowed with a T1 topology with respect to which vector addition and scalar multiplication are continuous.

Our most important example will be normed vector spaces.

Proposition 1.1.2

A normed vector space, endowed with the metric topology, is a topological vector space.

Proof. Let X be a normed vector space. As a metric space, X is certainly T1. To show $+\colon X \times X \to X$ is continuous, let $U \subset X$ be open and $x + y \in U$. Let $\epsilon > 0$ be such that $B_\epsilon(x + y) \subset U$. We claim that

$$(x, y) \in B_{\epsilon/2}(x) \times B_{\epsilon/2}(y) \subset +^{-1}(U).$$

Indeed if $(w, z) \in B_{\epsilon/2}(x) \times B_{\epsilon/2}(y)$ then $\|x - w\| < \frac{\epsilon}{2}$ and $\|y - z\| < \frac{\epsilon}{2}$, so

$$\|(x + y) - (w + z)\| \leq \|x - w\| + \|y - z\| < \epsilon.$$

Hence $w + z \in B_\epsilon(x + y) \subset U$, so $(w, z) \in +^{-}(U)$, as desired.

To show $\cdot\colon k \times X \to X$ is continuous, let $x \in X$, $\lambda \in k$ be such that $\lambda x \in U$, and let $\epsilon > 0$ be such that $B_\epsilon(\lambda x) \subset U$. Let $\delta := \min\{1, \frac{\epsilon}{1+\|x\|+|\lambda|}\}$; we claim that

$$(\lambda, x) \in B_\delta(\lambda) \times B_\delta(x) \subset \cdot^{-1}(U).$$

Indeed if $(\mu, y) \in B_\delta(\lambda) \times B_\delta(x)$ then

$$\begin{aligned}
\|\mu y - \lambda x\| &= \|(\mu - \lambda)y + \lambda(y - x)\| \\
&\leq \|(\mu - \lambda)y\| + \|\lambda(y - x)\| \\
&= |\mu - \lambda|\|y - x + x\| + |\lambda|\|y - x\| \\
&\leq |\mu - \lambda| \left(\|y - x\| + \|x\|\right) + |\lambda|\|y - x\| \\
&< \delta(\|x\| + \delta) + \delta|\lambda| \\
&= \delta(\|x\| + |\lambda| + \delta) \\
&\leq \delta(\|x\| + |\lambda| + 1) \\
&\leq \epsilon.
\end{aligned}$$

Therefore $\mu y \in B_\epsilon(\lambda x) \subset U$, so $(\mu, y) \in \cdot^{-1}(U)$ as desired. □

A subset $E \subset X$ of a topological vector space if *bounded* if for every neighbourhood V of 0 there exists $s > 0$ such that $E \subset tV$ for $t > s$. For normed vector spaces, we recover a more familiar definition.

Proposition 1.1.3

Let X be a normed vector space. A set $E \subset X$ is bounded if and only if $\sup_{x \in E} \|x\| < \infty$.

Proof. Let $E \subset X$ be bounded. Then for the open neighbourhood $B_1(0)$ of 0, there exists $t > 0$ such that

$$E \subset tB_1(0) = B_t(0),$$

hence $\sup_{x \in E} \|x\| < t$. Conversely if $\sup_{x \in E} \|x\| = M < \infty$, let $B_\epsilon()$ be a basic open neighbourhood of 0 and let $s = \frac{M}{\epsilon} > 0$. Then

$$sB_\epsilon(0) = B_M(0),$$

so if $t > s$ then $E \subset s\overline{B}_\epsilon(0) \subset tB_\epsilon(0)$. □

Proposition 1.1.4

Let X be a topological vector space over k. For $a \in X$ and $\lambda \in k$, the maps

$$\begin{aligned}
T_a \colon X &\longrightarrow X \\
x &\longmapsto x + a, \\
M_\lambda \colon X &\longrightarrow X \\
x &\longmapsto \lambda x
\end{aligned}$$

are homeomorphisms.

Proof. They are clearly continuous with continuous inverses T_{-a} and $M_{\lambda^{-1}}$, respectively. □

In some sense, the topology on X is thus determined by its local structure near the origin. This is made precise in the following.

Proposition 1.1.5

Let X be a topological vector space and β_0 a local basis at 0. Then the collection of translates

$$\beta = \{a + B : a \in X, B \in \beta_0\}$$

is a basis for X.

Proof. β clearly consists of open sets which cover X. For any $U \subset X$ open and $x \in U$, $(-x) + U$ is a neighbourhood of 0 so there exists $B \in \beta_0$ such that $0 \in B \subset (-x) + U$. Then $x \in x + B \subset U$. $\square$

There is even something to say about convexity and balancedness. Recall that a subset $U \subset X$ of an $\mathbb{R}$-vector space is *convex* if for $x, y \in U$ and $t \in [0,1]$, $tx + (1-t)y \in U$. On the other hand, U is *balanced* if $\lambda U \subset U$ for all $\lambda \in k$ with $|\lambda| \leq 1$.

Proposition 1.1.6

Let X be a topological vector space. Then

(1) If $U \subset X$ is an open neighbourhood of 0 then U contains a balanced neighbourhood V of 0. Moreover, we may demand that $V + V \subset U$.

(2) If $U \subset X$ is a convex neighbourhood of 0 then U contains a convex balanced neighbourhood of 0.

Proof. (1) Firstly since scaling is continuous, there exists $\delta > 0$ and $V \subset X$ open such that $\lambda V \subset U$ for $|\lambda| < \delta$. Let

$$W := \bigcup_{|\lambda|<\delta} \lambda V.$$

Then W is balanced, open, and contained in U.

Furthermore, note $0 + 0 = 0$, so by continuity there exists an open neighbourhood $V_1 \times V_2$ of $(0,0)$ such that $V_1 + V_2 \subset U$. Then $V = V_1 \cap V_2$ satisfies $V + V \subset U$.

(2) If U is moreover convex, then

$$A := \bigcap_{|\lambda|=1} \lambda U$$

contains W because $|\lambda| = 1$ implies $\lambda^{-1}W = W$. In particular, A° is a neighbourhood of the origin, and $A^\circ \subset U$. Since U is convex, so are its scalar multiples λU, and so A is convex as an intersection of convex sets. As the interior of a convex set, A° is convex. To show A is balanced, it suffices to show that $r\beta A$ for $r \in [0,1]$ and $|\beta| = 1$. Now

$$r\beta A = \bigcap_{|\lambda|=1} r\beta\lambda U = \bigcap_{|\lambda|=1} r\lambda U.$$

Here λU is a convex neighbourhood of 0, so $r\lambda U \subset \lambda U$, showing that A is balanced. We conclude that A° is balanced, convex, open, and contains 0.

$\square$

Proposition 1.1.7

Let X be a topological vector space over k. Then

(1) X is Hausdorff.

(2) $\{x\}$ is bounded for each $x \in X$.

(3) If $E_1, E_2 \subset X$ are bounded, then so is $E_1 + E_2$.

(4) If $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X and $a_n \to 0$ in k, then $a_n x_n \to 0$.

Proof. (1) Let $x \neq y \in X$. By the T1 axiom, let U be a neighbourhood of x with $y \notin U$. Then $-x + w$ is a neighbourhood of 0, so by part (1) of the previous proposition there exists a balanced V with $V + V \subset -x + U$. Hence $x + V + V \subset W$, so $y \notin x + V + V$. If there existed $x + v_1 = y + v_2 \in (x + V) \cap (y + V)$, then $y = x + a - b$. But $a, -b \in V$ so $y \in x + U + U$, a contradiction. Thus $x + U$ and $y + U$ are disjoint open neighbourhoods of x and y.

(2) For $x \in X$, let $f_x\colon \mathbb{R} \to X$ be given by $f_x(\lambda) = \lambda x$. This is the restriction of the continuous scalar multiplication to $\mathbb{R} \times \{x\}$, so it is continuous. In particular for a neighbourhood V of 0, $f_x^{-1}(V)$ is an open neighbourhood of 0, so it contains $(-\epsilon, \epsilon)$ for small $\epsilon > 0$. In other words $\lambda x \in V$ for $\lambda \in (0, \epsilon)$, or $x \in tV$ for $t > \frac{1}{\epsilon}$.

(3) Let V be a neighbourhood of 0. By the previous proposition (1), let U be a neighbourhood of 0 such that $U + U \subset V$. Since E_1, E_2 are bounded there exist $s_1, s_1 > 0$ such that $E_1 \subset tU$ for $t > s_1$ and $E_2 \subset tU$ for $t > s_2$. So for $t > s := \max\{s_1, s_2\}$ we have

$$E_1 + E_2 \subset tU + tU \subset t(U + U) \subset tV.$$

(4) Let V be an open neighbourhood of 0. Let $U \subset V$ be a balanced open set. Since (x_n) is bounded, there exists $s > 0$ such that $(x_n) \subset tU$ for $t > s$. Since $a_n \to 0$, there exists N such that $|a_n| < s^{-1}$ for $n > N$. By balancedness of U, and the fact that $|ta_n| < 1$ for $n > N$, we have $a_n x_n \in U \subset V$ for $n > N$.

□

Let X be a vector space with a metric $d\colon X \times X \to \mathbb{R}$. We say d is *invariant* if

$$d(x + z, y + z) = d(x, y)$$

for $x, y, z \in X$. In particular

$$d(nx, 0) \leq nd(x, 0). \tag{1.1.1}$$

Indeed, $n = 1$ is trivial, and by strong induction

$$\begin{aligned} d(kx, 0) &\leq d(kx, x) + d(x, 0) \\ &= d((k-1)x, 0) + d(x, 0) \\ &\leq (k-1)d(x, 0) + d(x, 0) \\ &= kd(x, 0). \end{aligned}$$

Proposition 1.1.8

Let X be a vector space with an invariant metric. Given a sequence $x_n \to 0$ in X, there exist scalars $a_n \to \infty$ such that $a_n x_n \to 0$.

Proof. For any $m \in \mathbb{N}$ there exists N_m such that

$$d(x_n, 0) < \frac{1}{m^2}$$

for $n > N_m$. If this choice of N_m is tight, then $N_m < N_{m+1}$. Define $a_n = m$ for $N_m < n \leq N_{m+1}$; clearly $a_n \to \infty$. But if $N_m < n \leq N_{m+1}$, we have by equation (1.1.1) that

$$d(a_n x_n, 0) \leq m d(x_n, 0) < \frac{1}{m}$$

so $a_n x_n \to 0$. □

1.2 Complete metric spaces

Let (X, d) be a metric space. Recall that a sequence (x_n) is *d-Cauchy* if for any $\epsilon > 0$ there exists N such that $d(x_n, x_m) < \epsilon$ for $n, m > N$. We say X is *complete* if every d-Cauchy sequence converges. In another setting, we have

Definition 1.2.1

Let (X, τ) be a topological vector space. A sequence (x_n) is *τ-Cauchy* if for any neighbourhood U of 0 there exists N such that $x_n - x_m \in U$ for $n, m > N$.

Proposition 1.2.2

Let X be a vector space with an invariant metric d which induces a topology τ. Then (x_n) is d-Cauchy if and only if τ-Cauchy.

Proof. If (x_n) is τ-Cauchy, then for any $\epsilon > 0$ there exists N such that $x_n - x_m \in B_\epsilon(0)$ for $n, m > N$. In other words,

$$d(x_n, x_m) = d(0, x_n - x_m) < \epsilon.$$

Conversely if (x_n) is d-Cauchy, let U be any neighbourhood of 0. Let $\epsilon > 0$ be such that $B_\epsilon(0) \subset V$. Since (x_n) is d-Cauchy, there exists N such that $d(x_n, x_m) < \epsilon$ for $n, m > N$, so $x_n - x_m \in B_\epsilon(0) \subset U$. □

1.3 Topological vector space zoo

Some rapidfire definitions: a topological vector space X is

(i) *locally convex* if there exists a local basis at 0 consisting of convex subsets.

(ii) *locally bounded* if 0 has a bounded neighbourhood.

(iii) *locally compact* if 0 has a relatively compact neighbourhood.

(iv) *metrizable* if its topology can be induced by a metric.

(v) an *F-space* if its topology is induced by a complete invariant metric.

(vi) *Fréchet* if a locally convex F-space.

(vii) *normable* if its topology is induced by a norm.

(viii) *Banach* if normable and complete with respect to the induced invariant metric.

(ix) *Heine–Borel* if every closed and bounded set is compact.

The converse of the Heine–Borel property is obtained for free in topological vector spaces:

Proposition 1.3.1

Let $K \subset X$ be a compact subset of a topological vector space. Then K is closed and bounded.

Proof. A compact subset of a Hausdorff space is closed. For boundedness, let U be a neighbourhood of 0. Let $V \subset U$ be a balanced open neighbourhood of 0. We claim that

$$\bigcup_{n \in \mathbb{N}} nV = X.$$

Indeed for $x \in X$, $f_x(\lambda) = \lambda x$ is continuous so $\{\lambda \in \mathbb{R} : \lambda x \in V\}$ is open in $\mathbb{R}$ and contains 0, so it contains $\frac{1}{n}$ for large n. This means $x \in nV$ for large n. By compactness of K, finitely many nV cover K, say for $n_1 < \cdots < n_N$. Since V is balanced, in fact $n_i < n_N$ implies

$$n_i V \subset n_N V \subset n_N U$$

so $K \subset n_N U$ is bounded. □

1.4 Locally convex spaces

Recall that a topological vector space is *locally convex* if it admits a local basis of convex subsets at the origin. Seminorms are a useful tool for describing locally convex spaces.

Definition 1.4.1

A *seminorm* on a k-vector space X is a map $p\colon X \to \mathbb{R}$ such that

(i) $p(x+y) \leq p(x) + p(y)$.

(ii) $p(\lambda x) = |\lambda| p(x)$.

We first collect some properties of seminorms.

Proposition 1.4.2

Let $k = \mathbb{R}$ or $\mathbb{C}$. Let $p\colon X \to \mathbb{R}$ be a seminorm. Then

(1) $p(0) = 0$.

(2) $|p(x) - p(y)| \leq p(x-y)$.

(3) $p(x) \geq 0$.

(4) $p^{-1}(0) \subset X$ is a linear subspace.

(5) $p^{-1}[0,1) \subset X$ is convex and balanced.

(6) A seminorm with $p(x) \neq 0$ whenever $x \neq 0$ is a norm.

Proof. (1) Taking $\lambda = 0$, $p(0) = 0$.

(2) We have
$$p(x) = p(x-y+y) \leq p(x-y) + p(y),$$
and similarly $p(y) - p(x) \leq p(y-x) = |-1|p(x-y) = p(x-y)$.

(3) $y = 0$ in (2) gives $|p(x)| \leq p(x)$.

(4) If $x, y \in p^{-1}(0)$ then by (3) we have
$$0 \leq p(\lambda x + \mu y) \leq |\lambda| p(x) + |\mu| p(y) = 0,$$
so $\lambda x + \mu y \in p^{-1}(0)$.

(5) B is clearly balanced by (ii). For convexity, let $x, y \in B$ and $t \in (0,1)$. Then
$$p(tx + (1-t)y) \leq tp(x) + (1-t)p(y) < 1,$$
so $tx + (1-t)y \in B$.

(6) This follows from (1) and (3).

□

1.4.1 Locally convex spaces from seminorms

A sufficiently nice family of seminorms on a vector space determines a locally convex topological structure. More precisely, a family of seminorms $\mathscr{P}$ on X is *separating* if for each $x \neq 0 \in X$, there exists $p \in \mathscr{P}$ such that $p(x) \neq 0$.

Theorem 1.4.3

Let $\mathscr{P}$ be a separating family of seminorms on a vector space X. For each $p \in \mathscr{P}$ and $n \in \mathbb{N}$, define
$$V(p, n) = \{x \in X : p(x) < \tfrac{1}{n}\}.$$
Let β_0 be the set of all finite intersections of the $V(p, n)$. Then β_0 is a convex, balanced, local basis at 0, and it generates a locally convex topology such that

(1) Every $p \in \mathscr{P}$ is continuous.

(2) $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E.

Proof. Let $B_1 = \bigcap_{i=1}^N V(p_i, n_i)$, $B_2 = \bigcap_{j=1}^M V(q_j, m_j) \in \beta_0$. Then
$$B_3 := \bigcap_{i=1}^N V(p_i, n_i) \cap \bigcap_{j=1}^M V(q_i, m_i) \in \beta_0$$
is such that $B_3 \subset B_1 \cap B_2$. Thus β_0 is a local basis at 0, hence its translates generate a topology on X. Each $V(p, n)$ is convex and balanced by Proposition 1.4.2(5), so their finite intersections are convex and balanced.

To see X is T1, let $x, y \in X$. Since $x - y \neq 0$ and $\mathscr{P}$ is separating, there exists $p \in \mathscr{P}$ such that $p(x - y) > 0$. Let $n \in \mathbb{N}$ be sufficient large that $np(x - y) > 1$, so that $x \notin y + V(p, n)$. We have thus constructed an open neighbourhood of y disjoint from x.

To see addition is continuous, let $U \subset X$ be open and $x + y \in U$. For some p_i and n_i, we have
$$x + y + \bigcap_{i=1}^N V(p_i, n_i) \subset U.$$
Let
$$V_1 = x + \bigcap_{i=1}^N V(p_i, 2n_i), \quad V_2 = y + \bigcap_{i=1}^N V(p_i, 2n_i);$$
we claim that $V_1 \times V_2 \subset +^{-1}(U)$. Indeed if $(v_1, v_2) \in V_1 \times V_2$ then for each i,
$$p_i(v_1 + v_2 - (x + y)) \leq p_i(v_1 - x) + p_i(v_2 - y) < \frac{1}{2n_i} + \frac{1}{2n_i} = \frac{1}{n_i},$$
so $(v_1, v_2) \in +^{-1}(U)$. To show that mutliplication is continuous, let $\lambda x \in U$. Once again we write
$$\lambda x + \bigcap_{i=1}^N V(p_i, n_i) \subset U.$$

Let $\delta_i = \frac{1}{2n_i p_i(x)}$, so that $\mu \in B_{\delta_i}(\lambda)$ implies

$$p_i((\mu-\lambda)x) = |\mu-\lambda| p_i(x) < \frac{1}{2n_i},$$

Let $\delta = \min_i \delta_i$, so that for $|\mu - \lambda| < \delta$, the above equation holds simultaneously for all i.

Let $\epsilon_i = \frac{1}{2n_i(\delta+|\lambda|)}$ and let $m_i > \frac{1}{\epsilon_i}$. If $y \in x + V(p_i, m_i)$, then $p_i(y-x) < \frac{1}{m_i} < \epsilon_i$, so for $\mu \in B_\delta(\lambda)$,

$$\begin{aligned}
p_i(\lambda x - \mu y) &= p_i((\lambda-\mu)x + \mu(x-y)) \\
&\le p_i((\lambda-\mu)x) + p_i(\mu(x-y)) \\
&= |\lambda-\mu| p_i(x) + |\mu-\lambda+\lambda| p_i(x-y) \\
&< \delta_i p_i(x) + (|\mu-\lambda| + |\lambda|) p_i(x-y) \\
&< \frac{1}{2n_i} + (\delta_i + |\lambda|)\frac{1}{2n_i(\delta_i+|\lambda|)} \\
&= \frac{1}{n},
\end{aligned}$$

so $\mu y \in \lambda x + V(p_i, n_i)$. Thus

$$B_\delta(\lambda) \times x + \bigcap_{j=1}^{M} V(p_i, m_i) \subset \cdot^{-1}(U),$$

showing that scaling is continuous. We conclude that X is a locally convex topological vector space, and it remains to verify the two properties. Let $p \in \mathscr{P}$. If $x \in p^{-1}(a,b)$ then $p(x) \in (a,b)$ so for sufficiently large n we have

$$(p(x) - \tfrac{1}{n}, p(x) + \tfrac{1}{n}) \subset (a,b).$$

Thus $V(p,n) \subset p^{-1}(a,b)$; indeed if $p(y-x) < \frac{1}{n}$ then Proposition 1.4.2(2) implies that $p(y) \in (a,b)$.

Secondly, let $E \subset X$ is bounded an $p \in \mathscr{P}$. Since $V(p,1)$ is an open neighbourhood of 0, there exists $t < \infty$ such that $E \subset tV(p,1)$. But $x \in tV(p,1)$ if and only if $p(x) < t$, so p is bounded on E.

Conversely suppose every $p \in \mathscr{P}$ is bounded on E. Let U be an open neighbourhood of 0 and take

$$\bigcap_{i=1}^{N} V(p_i, n_i) \subset U.$$

Then for each i there exists $M_i < \infty$ such that $p_i < M_i$ on E; let $s := M_i n_i$. If $t > s$, then

$$p_i(x) < M_i < \frac{t}{n_i}$$

means $E \subset tV(p_i, n_i)$ for all i, hence $E \subset tU$. □

If $\mathscr{P}$ is countable, we have another description of the topology it induces.

Theorem 1.4.4

Let $\mathscr{P} = \{p_i\}_{i\in\mathbb{N}}$ be a countable separating family of seminorms on X. Then the locally convex topology on X is metrizable and equivalent to that induced by the invariant metric

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_i(x-y)}{1+p_i(x-y)}.$$

Remark 1.4.5. Somewhat counterintuitively, the metric balls $B_r(0)$ will not necessary be convex in this topology. Another counterintuitive property is that any subset has finite diameter (indeed $d(x,y) < 1$ for any $x, y \in X$) but they are not all bounded.

Proof. We begin by showing d is an invariant metric. Since $p(-x) = p(x)$, d is certainly symmetric. For positive-definiteness, note that

$$\begin{aligned} F\colon [0,\infty) &\longrightarrow [0,1) \\ t &\longmapsto \frac{t}{1+t} \end{aligned}$$

is a smooth, monotone increasing, concave function. Thus $d(x,y)$ is a sum of nonnegative terms so it is nonnegative, with equality if and only if $p_i(x-y) = 0$ for all i, which implies $x = y$ since $\mathscr{P}$ is separating. For the triangle inequality, convexity of F along with $F(0) = 0$ implies

$$F(\lambda t) = F(\lambda t + (1-\lambda)0) \geq \lambda F(t) + (1-\lambda)F(0) = \lambda F(t)$$

for $\lambda \in (0,1)$. Then if $t, s \geq 0$,

$$\begin{aligned} F(t) + F(s) &= F\left((t+s)\frac{t}{t+s}\right) + F\left((t+s)\frac{s}{t+s}\right) \\ &\geq \frac{t}{t+s}F(t+s) + \frac{s}{t+s}F(t+s) \\ &= F(t+s). \end{aligned}$$

Now since p is a seminorm,

$$p(x-y) \leq p(x-z) + p(z-y),$$

and since F is monotone increasing

$$F \circ p(x-y) \leq F(p(x-z) + p(z-y)),$$

and since F is subadditive

$$F \circ p(x-y) \leq F \circ p(x-z) + F \circ p(z-y).$$

Therefore

$$\begin{aligned} d(x,y) &= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_i(x-y)}{1+p_i(x-y)} \\ &\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{p_i(x-z)}{1+p_i(x-z)} + \frac{p_i(z-y)}{1+p_i(z-y)}\right) \\ &= d(x,z) + d(z,y), \end{aligned}$$

showing that d is a metric on X. The fact that it is invariant is clear by definition. To show that the metric topology τ_d induced by d agrees with the topology $\tau_{\mathscr{P}}$ induced by $\mathscr{P}$, the Weierstrass M-test implies that $d(x,y)$ converges uniformly. Since the p_i are $\tau_{\mathscr{P}}$-continuous, the metric $d\colon (X,\tau_{\mathscr{P}}) \times (X,\tau_{\mathscr{P}}) \to \mathbb{R}$ is continuous. In particular, $d_x\colon (X,\tau_{\mathscr{P}}) \to \mathbb{R}$ given by $d_x(y) = d(x,y)$ is continuous, so the basic open sets $B_r(x) = d_x^{-1}(-r,r)$ in τ_d are open in $\tau_{\mathscr{P}}$. This shows $\tau_{\mathscr{P}}$ is finer.

Now if W is open in $\tau_{\mathscr{P}}$ and $x \in W$, then there exists $B = \bigcap_{k=1}^{N} V(p_{i_k}, n_k)$ such that $x + B \subset W$. Let $M \geq i_k$ and $\epsilon < \frac{1}{2^{M+1} n_k}$ for all k. If $y \in B_\epsilon(x)$ then for each i we must have

$$\frac{p_i(x-y)}{1+p_i(x-y)} < 2^M \epsilon = \frac{1}{2n_k},$$

so

$$\begin{aligned} p_i(x-y) &< \frac{1}{2n_k} + \frac{1}{2n_k} p_i(x-y) \\ p_i(x-y) &< \frac{1}{2n_k} + \frac{1}{2} p_i(x-y) \\ \frac{1}{2} p_i(x-y) &< \frac{1}{2n_k} \\ p_i(x-y) &< \frac{1}{n_k}, \end{aligned}$$

so $y \in x + \bigcap_{k=1}^{N} V(p_{i_k}, n_k) \subset W$. This shows that $\tau_{\mathscr{P}}$ is finer, so the topologies coincide. □

1.4.2 The space of smooth functions

Let $\Omega \subset \mathbb{R}^n$ be open. We will endow $C^\infty(\Omega)$, the vector space of smooth real-valued functions on Ω under pointwise addition and multiplication, with a topology which makes it a Fréchet space with the Heine–Borel property. Let $K_1 \subset K_2 \subset \cdots \subset \Omega$ be a compact exhaustion. Define a countable family of seminorms

$$p_n(f) = \max\{|D^\alpha f(x)| : x \in K_n, |\alpha| \leq n\}.$$

$\mathscr{P} = \{p_n : n \in \mathbb{N}\}$ is separating. Indeed, if $f \neq 0$, then $f(x) \neq 0$ for some $x \in \Omega$. Then $x \in K_n$ for large n, so $p_n(f) \geq |f(x)| > 0$. By what we have done so far, $\mathscr{P}$ induces a topology on $C^\infty(\Omega)$ which is locally convex, metrizable, and has a local basis of sets

$$V_N = \{f \in C^\infty(\Omega) : p_N(f) < \tfrac{1}{N}\}$$

at 0. We will denote this locally convex space by $\mathscr{E}(\Omega)$.

Proposition 1.4.6

A sequence (f_n) in $\mathscr{E}(\Omega)$ converges to f if and only if $D^\alpha f_n \rightrightarrows D^\alpha f$ on compact subsets for each multiindex α.

Proof. After translating, we may assume $f = 0$. By definition $f_n \to 0$ if and only if for each N there exists m_N such that $n > m_N$ implies $f_n \in V_N$.

If $f_n \to 0$, then fix α. It suffices to show $D^\alpha f_n \rightrightarrows 0$ on each K_i. For any $\epsilon > 0$, let $N > \max\{|\alpha|, \frac{1}{\epsilon}\}$. If $n > m_N$ then $f_n \in V_N$, so

$$\sup_{K_i} |D^\alpha f_n| \leq p_N(f_n) \leq \frac{1}{N} < \epsilon,$$

so $D^\alpha f_n \rightrightarrows 0$ on K_i.

Conversely if $D^\alpha f_n \rightrightarrows 0$ on compact subsets, fix N. If $|\alpha| \leq N$ then $D^\alpha f_n \rightrightarrows 0$ on K_N. So for each of the finitely many α with $|\alpha| \leq N$, there exists m_α such that $n > m_\alpha$ implies

$$\sup_{K_N} |D^\alpha f_n| < \frac{1}{N}.$$

Let $m = \max_{|\alpha|\leq N} m_\alpha$; then for $n > m$ we have $f_n \in V_N$ so $f_n \to 0$. $\square$

Theorem 1.4.7

$\mathscr{E}(\Omega)$ is a Fréchet space with the Heine–Borel property.

Proof. We know $\mathscr{E}(\Omega)$ is locally convex and its topology is induced by an invariant metric, so it remains to show it is complete with respect to this metric. A sequence (f_n) in $\mathscr{E}(\Omega)$ is Cauchy if for any N, there exists M such that $i, j > M$ implies $f_i - f_j \in V_N$, or

$$\sup_{K_N} |D^\alpha f_i - D^\alpha f_j| < \frac{1}{N}$$

for $|\alpha| \leq N$. Since the K_N exhaust Ω, there exist continuous g^α such that $D^\alpha f_n \rightrightarrows g^\alpha$ on compact subsets. Then there exists a smooth function f such that $f_n \rightrightarrows f$ and $D^\alpha f_n \rightrightarrows D^\alpha f$ on compact subsets, so $\mathscr{E}(\Omega)$ is complete.

For the Heine–Borel property, suppose $E \subset \mathscr{E}(\Omega)$ is closed and bounded. We will show that every sequence in E has a convergent subsequence.

Since E is bounded, for each N there exists M_N such that $p_N(f) < M_N$ for all $f \in E$. In particular

$$|D^\alpha f| < M_N$$

on K_N for $|\alpha| = N$ and $f \in E$. So for $|\beta| < N_1$, the set $\{D^\beta f : f \in E\}$ is equicontinuous on K_{N-1}, and it is pointwise bounded since $p_N(f) < M_N$. Now if (f_n) is any sequence in E, then the Arzelà–Ascoli theorem plus an additional diagonalization argument gives a subsequence (F_k) such that $D^\alpha F_k$ converges uniformly on compact subsets, so (f_n) has a convergent subsequence in E. $\square$

1.4.3 Compactly-support smooth functions

Let $K \subset \Omega$ be compact. Then the space $\mathscr{D}_K$ of $f \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(f) \subset K$ is a linear subspace of $C^\infty(\Omega)$. In fact, it is closed in $\mathscr{E}(\Omega)$. Indeed, the evaluation map

$$\begin{aligned} \delta_x \colon \mathscr{E}(\Omega) &\longrightarrow \mathbb{R} \\ f &\longmapsto f(x) \end{aligned}$$

is continuous, so $\delta_x^{-1}(0)$ is closed. Thus

$$\mathscr{D}_K = \bigcap_{x\in\Omega\setminus K} \delta_x^{-1}(0)$$

is closed. In particular, it is a Fréchet space in the subspace topology, denoted τ_K. Let

$$\mathscr{D}(\Omega) = \bigcup_{K\subset\Omega} \mathscr{D}_K$$

be the space of compactly-support smooth functions on Ω. This is clearly closed under addition and multiplication. We will endow $\mathscr{D}(\Omega)$ with a topology which makes it a complete locally convex topological space such that each subspace $\mathscr{D}_K$ inherits τ_K.

A natural candidate would use the seminorms

$$\|f\|_k = \max\{|D^\alpha f(x)| : x \in \Omega, |\alpha| \leq k\}.$$

In fact, this satisfies all our desirata except completeness, which suggests we should find a finer topology. We will use the finest locally convex topology such that the subspace topology on $\mathscr{D}_K$ agreees with τ_K.

Let β be the collection of all convex balanced sets $W \subset \mathscr{D}(\Omega)$ such that $\mathscr{D}_K \cap W \in \tau_K$ for every compact $K \subset \Omega$.

Theorem 1.4.8

β is a local basis at 0, and the resulting topology τ makes $\mathscr{D}(\Omega)$ a locally convex topological space.

Proof. Let $W_1, W_2 \in \beta$. We claim that $W_1 \cap W_2 \in \beta$. This is clearly convex, balanced, and for $K \subset \Omega$ compact,

$$\mathscr{D}_K \cap W_1 \cap W_2 \in \tau_K.$$

It remains to show the axioms of a topological vector space are satisfied. For the T1 axiom, let $\phi_1, \phi_2 \in \mathscr{D}(\Omega)$. Consider

$$W = \{\phi \in \mathscr{D}(\Omega) : \|\phi\|_0 \leq \|\phi_1 - \phi_2\|_0\}.$$

As a metric ball, this is convex and balanced. Moreover, its intersection with each $\mathscr{D}_K$ is

$$\{\phi \in \mathscr{D}_K : \|\phi\|_0 < r\},$$

which is open, so $W \in \beta$. But $\phi_1 \notin \phi_2 + W$, so $\{\phi_1\}$ is closed.

To show that addition is continuous, let U be open and $\phi_1 + \phi_2 \in U$. Then $\phi_1 + \phi_2 + W \subset U$ for some $W \in \beta$. We claim that

$$(\phi_1 + \tfrac{1}{2}W) \times (\phi_2 \times \tfrac{1}{2}W) \subset +^{-1}(U).$$

Indeed by convexity of W,

$$(\phi_1 + \tfrac{1}{2}W) + (\phi_2 \times \tfrac{1}{2}W) = \phi_1 + \phi_2 + W \subset U.$$

To show that mutliplication is continuous, let U be open and $\lambda\phi \in U$. Then $\lambda\phi + W \subset U$ for some $W \in \beta$. Let $\mathrm{supp}(\phi) \subset K$. Then scalar multiplication is continuous on $\mathscr{D}_K$, so there exists $\delta > 0$ such that $\delta\phi \in \frac{1}{2}W$. Let $\epsilon = \frac{1}{2(|\lambda|+\delta)}$. Let $(\mu, \psi) \in B_\delta(\lambda) \times (\phi + \epsilon W)$, so that $|\mu| \leq |\lambda| + \delta$. Since W is balanced and convex,

$$\mu\psi - \lambda\phi = \mu(\psi - \phi) + (\mu - \lambda)\phi \in (|\lambda| + \delta)\epsilon W + \frac{1}{2}W = W,$$

so scaling is continuous. □

For the rest of this section we characterize convergence and continuity in $\mathscr{D}(\Omega)$.

Proposition 1.4.9 (1) A convex balanced subset V of $\mathscr{D}(\Omega)$ is open if and only if $V \in \beta$.

(2) The Fréchet topology τ_K of any $\mathscr{D}_K$ coincides with the subspace topology inherited from $\mathscr{D}(\Omega)$.

(3) If $E \subset \mathscr{D}(\Omega)$ is bounded then $E \subset \mathscr{D}_K$ for some $K \subset \Omega$, and there exist real numbers $M_N < \infty$ such that each $\phi \in E$ satisfies $\|\phi\|_N \leq M_N$ for all N.

(4) $\mathscr{D}(\Omega)$ has the Heine–Borel property.

(5) If (ϕ_i) is a Cauchy sequence in $\mathscr{D}(\Omega)$ then $(\phi_i) \subset \mathscr{D}_K$ for some K, and it is Cauchy with respect to $\|\cdot\|_N$ for each N.

(6) If $\phi_i \to 0$ in $\mathscr{D}(\Omega)$, then there is a compact subset $K \subset \Omega$ containing the support of every ϕ_i, and $D^\alpha \phi_i \rightrightarrows 0$ as $i \to \infty$ for all α.

(7) $\mathscr{D}(\Omega)$ is complete.

Proof. (1) The if direction is by definition, now suppose V is convex balanced and open. Let $K \subset \Omega$ and let $\phi \in \mathscr{D}_K \cap V$. Then there exists $W \in \beta$ such that $\phi + W \subset V$, and then

$$\phi + (\mathscr{D}_K \cap W) \subset \mathscr{D}_K \cap V.$$

In particular $\mathscr{D}_K \cap W \in \tau_K$ means $\mathscr{D}_K \cap V$ is open in $\mathscr{D}_K$, hence $V \in \beta$.

(2) The previous part shows that any open set in the subspace topology on $\mathscr{D}_K \subset \mathscr{D}(\Omega)$ is open in τ_K. Conversely if $E \subset \tau_K$ is open, then we wish to show that $E = \mathscr{D}_K \cap U$ for U open in $\mathscr{D}(\Omega)$. Let $\phi \in E$, so that there exists N, δ such that

$$\{\psi \in \mathscr{D}_K : \|\psi - \phi\|_N < \delta\} \subset E.$$

Let

$$W_\phi := \{\psi \in \mathscr{D}(\Omega) : \|\psi\|_N < \delta\}.$$

By definition $W_\phi \in \beta$, and

$$\mathscr{D}_K \cap (\phi + W_\phi) = \phi + \mathscr{D}_{\mathscr{K}} \cap W_\phi \subset E.$$

So

$$U = \bigcup_{\phi \in E} (\phi + W_\phi) \in \tau$$

is the desired open set with $\mathscr{D}_K \cap U = E$.

(3) By contraposition, suppose E is not contained in any $\mathscr{D}_K$. For each K_m in a compact exhaustion of Ω, there exists $\phi_m \in E$ with $\operatorname{supp}(\phi_m) \not\subset K_m$. In particular, there exists $x_m \in \Omega \setminus K_m$ with $\phi_m(x_m) \neq 0$. Let

$$W = \{\phi \in \mathscr{D}(\Omega) : |\phi(x_m)| < \frac{1}{m}|\phi_m(x_m)|\}.$$

It is not difficult to see that W is convex and balanced. Moreover, for any $K \subset K_M$ we have $x_m \notin K$ for $m \geq M$, so $\mathscr{D}_K \cap W$ is the intersection of finitely many open sets, meaning $W \in \beta$. So W is an open neighbourhood of 0 such that $\phi_m \notin mW$ for any m, so E is not bounded. In summary, any bounded subset of $\mathscr{D}(\Omega)$ belongs to some $\mathscr{D}_K$, and by (2) it will be bounded in $\mathscr{D}_K$. We previously saw that this means that there exist real numbers $M_N < \infty$ such that $\|\phi\|_N \leq M_N$ for all N and $\phi \in E$.

(4) This follows from (3), because $\mathscr{D}_K$ has the Heine–Borel property and every closed and bounded subset is contained in some $\mathscr{D}_K$, where it is again closed and bounded.

(5) Since Cauchy sequences are bounded, this also follows from (3) and the characterization of Cauchy sequences in $\mathscr{D}_K$.

(6) Same as (5).

(7) This follows from (5) and (2), since $\mathscr{D}_K$ is complete.

□

The next result concerns linear maps from $\mathscr{D}(\Omega)$ into another locally convex space. Recall that a linear map $\Lambda\colon X \to Y$ is bounded if $\Lambda(E) \subset Y$ is bounded whenever $E \subset X$ is bounded.

Proposition 1.4.10

Let Y be a locally convex topological vector space. Let $\Lambda\colon \mathscr{D} \to Y$ be a linear map. The following are equivalent:

(a) Λ is continuous.

(b) Λ is bounded.

(c) If $\phi_i \to 0$ in $\mathscr{D}(\Omega)$ then $\Lambda(\phi_i) \to 0$ in Y.

(d) For all $K \subset \Omega$ compact, $\Lambda\big|_{\mathscr{D}_K}$ is continuous.

Proof. $(a) \implies (b)$. Let E be bounded and let W be a neighbourhood of 0 in Y. By continuity, there exists a neighbourhood V of 0 in $\mathscr{D}(\Omega)$ such that $\Lambda(V) \subset W$. Let $s > 0$ be such that for $t > s$, $E \subset tV$. By linearity

$$\Lambda(E) \subset \Lambda(tV) = t\Lambda(V) \subset tW,$$

as desired.

$(b) \implies (c)$. By the previous proposition (5), there exists $K \subset \Omega$ such that $\phi_i \to 0$ in $\mathscr{D}_K$. Since $\mathscr{D}_K$ is metrizable, there exist scalars $a_i \to \infty$ with $a_i\phi_i \to 0$ in $\mathscr{D}_K$, and the same holds in $\mathscr{D}(\Omega)$. By linearity

$$\Lambda(\phi_i) = a_i^{-1}\Lambda(a_i\phi_i).$$

Since $\mathscr{D}(\Omega)$ is complete, $(a_i\phi_i)$ is Cauchy, hence bounded. Since Λ is bounded, $\{\Lambda(a_i\phi_i)\}$ is bounded. Since $a_i^{-1} \to 0$, we have $\Lambda\phi_i \to 0$.

$(c) \implies (d)$. Since $\mathscr{D}_K$ is a metric space, this is well-known.

$(d) \implies (a)$. First let U be a convex, balanced, open neighbourhood of 0 in Y. Then $V = \Lambda^{-1}(U)$ is convex and balanced by linearity, and to show it is open in $\mathscr{D}(\omega)$ it suffices to show $\mathscr{D}_K \cap V$ is open in $\mathscr{D}_K$ for all K. But this is true by continuity of $\Lambda\big|_{\mathscr{D}_K}$.

More generally, let W be open in Y and $\phi \in \Lambda^{-1}(W)$. Since Y is locally convex, there exists a convex balanced neighbourhood U of 0 such that $\Lambda\phi + U \subset W$. Since Λ is linear,

$$\phi + \Lambda^{-1}(U) \subset \Lambda^{-1}(W),$$

and $\phi + \Lambda^{-1}(U)$ is open in $\mathscr{D}(\Omega)$, so we win. □

1.5 Sigma algebras and measure

Given a set E, the purpose of measure theory is to assign to certain subsets $A \subset E$ a value $\mu(A)$ which records in some reasonable sense the 'size' of A. If E is finite countable and $A \subset E$ it makes sense to define $\mu(A)$ to be the number of elements in A, so μ is defined on the entire power set 2^E. We call this the *counting measure*.

For $E = \mathbb{R}$, a natural start would be to define $\mu(A)$ as the 'length' of A. If A is an interval, this certainly works but in general, 'length' is a rather ill-defined notion. Thus we will not define μ on all subsets; instead we will restrict our attention to a smaller collection of sets.

Definition 1.5.1

Let E be a set. A collection $\mathcal{E}$ of subsets of E is called a σ-algebra if

(i) $\varnothing \in \mathcal{E}$.

(ii) If $A \in \mathcal{E}$, then $A^c = E \setminus A \in \mathcal{E}$.

(iii) If $(A_n)_{n=1}^{\infty}$ is a sequence of subsets in $\mathcal{E}$, then

$$\bigcup_{n=1}^{\infty} \in \mathcal{E}.$$

The pair $(E, \mathcal{E})$ is called a *measurable space*. A *measure* on a measurable space $(E, \mathcal{E})$ is a function $\mu\colon \mathcal{E} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and for a sequence $(A_n)_{n=1}^{\infty}$ of disjoint subsets in $\mathcal{E}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The triple $(E, \mathcal{E}, \mu)$ is called a *measure space*.

Example 1.5.2. The power set 2^E is always a σ-algebra, but it is highly possible that there exists no reasonable measure on the measurable space $(E, 2^E)$.

If $A \subset E$, then a measure μ on $(E, \mathcal{E})$ induces a measure $\mu\big|_A$ on $(A, \mathcal{E}\big|_A)$ where

$$\mathcal{E}\big|_A = \{B \in \mathcal{E} : B \subset A\}$$

and

$$\mu\big|_A(B) = \mu(B) \quad \text{for} \quad B \in \mathcal{E}\big|_A.$$

Example 1.5.3. Let E be finite or countable and μ the counting measure. Then $(E, 2^E, \mu)$ is a measure space. Indeed, it suffices to show μ is countably additive. But this is clear by definition.

Example 1.5.4. Let $m\colon E \to [0,\infty]$ be a mass function. This yields a set function on $(E,\mathcal{E})$ by

$$\mu_m(A) = \sum_{x\in A} m(x).$$

Clearly $\mu_m(\varnothing) = 0$ and

$$\mu_m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{x\in\bigcup_{n=1}^{\infty} A_n} m(x) = \sum_{n=1}^{\infty}\sum_{x\in A_n} m(x) = \sum_{n=1}^{\infty}\mu_m(A_n).$$

In general, constructing σ-algebras and measure is not as straightforward as the previous examples suggest, so we will collect some result to streamline the process.

Proposition 1.5.5

Let $(\mathcal{E}_i)_{i\in I}$ be a family of σ-algebras of E. Then $\mathcal{E} = \bigcap_{i\in I}\mathcal{E}_i$ is a σ-algebra.

Proof. Clearly $\varnothing in \mathcal{E}$. If $A \in \bigcap_{i\in I}\mathcal{E}_i$ then $A \in \mathcal{E}_i$ for all i so $A^c \in \bigcap_{i\in I}\mathcal{E}_i$. Similarly for countable unions. □

Definition 1.5.6

Let $\mathcal{A}$ be a collection of subsets of E. Then the σ-algebra generated by $\mathcal{A}$, denoted $\sigma(\mathcal{A})$, is the intersection of all σ-algebras on E containing A.

This is well-defined as 2^E is always a σ-algebra containing $\mathcal{A}$. If τ is the collection of open sets in a topological space E, we call $\sigma(\tau)$ the *Borel algebra*, denoted by $\mathcal{B}(E)$. When $E = \mathbb{R}$ with the standard topology, we simply write $\mathcal{B} := \mathcal{B}(\mathbb{R})$. A measure on the measurable space $(E, \mathcal{B}(E))$ is called a *Borel measure*. A Borel measure μ such that $\mu(K) < \infty$ for K compact is called a *Radon measure*.

We know how to generate a σ-algebra from a smaller collection of subsets $\mathcal{A}$, but this procedure is not constructive and thus $\sigma(\mathcal{A})$ may not admit a particularly nice description. It is thus favourable to be able to define a measure according to its values on $\mathcal{A}$, then extend it to $\sigma(\mathcal{A})$. To establish the existence and uniqueness for extensions of measures, we introduce π-systems and d-systems.

Definition 1.5.7

Let $\mathcal{A}$ be a collection of subsets of E.

(a) We call $\mathcal{A}$ a *π-system* if

- $\varnothing \in \mathcal{A}$.
- if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

(b) We call $\mathcal{A}$ a *d-system* if

- $E \in \mathcal{A}$.
- if $A \subset B \in \mathcal{A}$ then $B \setminus A \in \mathcal{A}$.
- if $(A_n)_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{A}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Lemma 1.5.8

If $\mathcal{A}$ is both a π-system and a d-system, then it is a σ-algebra.

Proof. We have $\varnothing \in \mathcal{A}$ from the π-system axiom, and for any $A \in \mathcal{A}$, the first two d-system axioms imply $A^c \in \mathcal{A}$. Finally if $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$, then define an increasing sequence $B_k = \bigcup_{n=1}^{k} A_n$. Since

$$A_1 \cup A_2 = E \setminus ((E \setminus A_1) \cap (E \setminus A_2)),$$

$\mathcal{A}$ is closed under finite unions so the B_k lie in $\mathcal{A}$. Then by the last d-system axiom $\bigcup_{k=1}^{\infty} B_k = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. $\square$

More generally,

Lemma 1.5.9 (Dynkin's π-system lemma)

Let $\mathcal{A}$ be a π-system. Then any d-system containing $\mathcal{A}$ also contains $\sigma(\mathcal{A})$.

Proof. Let $\mathcal{D}$ be the intersection of all d-systems containing $\mathcal{A}$. Then $\mathcal{D}$ is a d-system after some definition chasing. We claim that $\sigma(\mathcal{A}) \subset \mathcal{D}$. We will show that $\mathcal{D}$ is a π-system, hence by Lemma 1.5.8 it will be a σ-algebra containing $\mathcal{A}$, hence it will contain $\sigma(\mathcal{A})$. Consider

$$\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \text{ for all } A \in \mathcal{A}\}.$$

Claerly $\mathcal{A} \subset \mathcal{D}'$ by the second π-system axiom. We claim that $\mathcal{D}'$ is a d-system, and by minimality we will conclude that $\mathcal{D} = \mathcal{D}'$. Clearly $E \in \mathcal{D}'$. Secondly if $B_1 \subset B_2 \in \mathcal{D}'$, then for $A \in \mathcal{A}$ we have

$$(B_2 \setminus B_1) \cap A = (B_2 \cap A) \setminus B_1 \cap A \in \mathcal{D},$$

since $\mathcal{D}$ is a d-system. So $B_2 \setminus B_1 \in \mathcal{D}$. Third, if (B_n) is an increasing sequence in $\mathcal{D}'$ then for all $A \in \mathcal{A}$, $C_n = B_n \cap A$ is an increasing sequence in $\mathcal{D}$ so $\bigcup_{n=1}^{\infty} C_n = A \cap \bigcup_{n=1}^{\infty} B_n$ is in $\mathcal{D}$. Hence $\mathcal{D}'$ is a d-system containing $\mathcal{A}$, which implies $\mathcal{D}' = \mathcal{D}$.

By similar arguments, we can show that the π-system

$$\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \text{ for all } A \in \mathcal{D}\}$$

is a d-system containing A, so $\mathcal{D}'' = \mathcal{D}$ is a π-system. $\square$

1.5.1 Constructing measures

We say a function $\mu\colon \mathcal{A} \to [0, \infty]$ with $\mu(\varnothing) = 0$ (called a set function) is

- *increasing* if $A \subset B$ implies $\mu(A) \leq \mu(B)$.

- *additive* if for A, B disjoint we have

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

- *countably additive* if the previous holds for a sequence of disjoint sets.
- *countably subadditive* if in the same setting we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Also, we say $\mathcal{A}$ is a *ring* on E is $\varnothing \in \mathcal{A}$, and for all $A, B \in \mathcal{A}$ we have

$$B \setminus A \in \mathcal{A} \quad \text{and} \quad A \cup B \in \mathcal{A}.$$

We say $\mathcal{A}$ is an *algebra* if $\varnothing \in \mathcal{A}$ and for all $A, B \in \mathcal{A}$ we have

$$A^c \in \mathcal{A} \quad \text{and} \quad A \cup B \in \mathcal{A}.$$

If $\mathcal{A}$ is a ring of subsets of E with a countably additive set function $\mu\colon \mathcal{A} \to [0, \infty]$, then we define the *outer measure*

$$\mu^*(B) := \inf \sum_{n=1}^{\infty} \mu(A_n)$$

where the infimum is taken over all sequences $(A_n)_{n=1}^{\infty}$ of sets in $\mathcal{A}$ with $B \subset \bigcup_{i=1}^{\infty} A_n$. By convention $\mu^*(B) = \infty$ is no such sequence exists. By taking the constant sequence $\varnothing$, we see that $\mu^*(\varnothing) = 0$, and on 2^E, μ^* is increasing. However, μ^* need not be a measure on $(E, 2^E)$, so we must restrict to a smaller σ-algebra. Namely say $A \subset E$ is μ^*-measurable if for all $B \subset E$,

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Let $\mathcal{M}$ denote the collection of μ^*-measurable sets. Our first big theorem is

Theorem 1.5.10 (Carathéodory)

Let $\mathcal{A}$ be a ring of subsets of E and $\mu\colon \mathcal{A} \to [0, \infty]$ a countably additive set function. Let μ^* be the outer measure and $\mathcal{M}$ the collection of μ^*-measurable sets. Then $\mathcal{M}$ is a σ-algebra containing $\mathcal{A}$ and μ^* is a measure on $(E, \mathcal{M})$.

The proof will span several steps.

Lemma 1.5.11

$\mu^*\colon 2^E \to [0, \infty]$ is countably subadditive.

Proof. Let $B = \bigcup_{n=1}^{\infty} B_n$. If $\mu^*(B_n) = \infty$ for some n then $\mu^*(B) = \infty$, so suppose $\mu^*(B_n) < \infty$ for all n. Let $\epsilon > 0$. For each n let $(A_{n,m})_{m=1}^{\infty}$ be a sequence in $\mathcal{A}$ with $B_n \subset \bigcup_{m=1}^{\infty} A_{n,m}$ and

$$\sum_{m=1}^{\infty} \mu(A_{n,m}) \leq \mu^*(B_n) + \frac{\epsilon}{2^n}.$$

Then $B \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m}$ implies

$$\mu^*(B) \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(A_{n,m}) \leq \sum_{n=1}^{\infty} \mu(B_n) + \epsilon,$$

and the result follows as $\epsilon \to 0$. $\square$

Lemma 1.5.12

Let $A \in \mathcal{A}$. Then $\mu^*(A) = \mu(A)$.

Proof. Clearly $\mu^*(A) \leq \mu(A)$ from the sequence $A_1 = A$, $A_n = \varnothing$ for $n > 1$. Conversely since μ is countably additive and $\mathcal{A}$ is a ring, for $A \subset B \in \mathcal{A}$ we have

$$\mu(B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A) \geq \mu(A),$$

so μ is increasing. If (A_n) is a sequence in $\mathcal{A}$, then

$$B_n = \bigcup_{k=1}^{n} A_k \setminus \bigcup_{k=1}^{n-1} A_k$$

is a disjoint sequence, $B_n \subset A_n$, and $B_n \in \mathcal{A}$ as it is a ring. Thus

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

so μ is countably subadditive. Now for $A \in \mathcal{A}$, let $(A_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{A}$ with $A \subset \bigcup_{n=1}^{\infty} A_n$. Then $A \cap A_n = A \setminus ((A \cup A_n) \setminus A)$ implies $A \cap A_n \in \mathcal{A}$. Therefore

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} (A \cap A_n)\right) \leq \sum_{n=1}^{\infty} \mu(A \cap A_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

We conclude that $\mu(A) \leq \mu^*(A)$. $\square$

Lemma 1.5.13

$\mathcal{M} \supset \mathcal{A}$

Proof. Let $A \in \mathcal{A}$ and $B \subset E$. We wish to show that

$$\mu^*(B) = \mu^(B \cap A) + \mu^*(B \cap A^c).$$

By subadditivity of μ^*, we immediately have

$$\mu^*(B) \leq \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

On the other hand if $\mu^*(B)$ then the other inequality is trivial, so suppose $\mu^*(B) < \infty$. For $\epsilon > 0$, there exists a sequence (A_n) in $\mathcal{A}$ with $B \subset \bigcup_{n=1}^{\infty} A_n$ and

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(B) + \epsilon.$$

Then

$$B \cap A \subset \bigcup_{n=1}^{\infty} (A_n \cap A) \quad \text{and} \quad B \cap A^c \subset \bigcup_{n=1}^{\infty} (A_n \cap A^c).$$

Since $A_n \cap A \in \mathcal{A}$ and $A_n \cap A^c = (A \cup A_n) \setminus A \in \mathcal{A}$, we conclude that

$$\begin{aligned}
\mu^*(B \cap A) + \mu^*(B \cap A^c) &\leq \sum_{n=1}^{\infty} \mu(A_n \cap A) + \sum_{n=1}^{\infty} \mu(A_n \cap A^c) \\
&= \sum_{n=1}^{\infty} \mu(A_n) \\
&\leq \mu^*(B) + \epsilon.
\end{aligned}$$

As $\epsilon \to 0$, we win. □

Lemma 1.5.14

$\mathcal{M}$ is an algebra.

Proof. $E \in \mathcal{M}$ and closure under complement is clear. It remains to show $\mathcal{M}$ is closed under finite union, which by de Morgan's law is equivalent to finite intersection. Let $A_1, A_2 \in \mathcal{M}$ and $B \subset E$. This is just a double application of the definition of μ^*-measurability:

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap A_1 \cap (A_1 \cap A_2)^c) + \mu^*(B \cap A_1^c \cap (A_1 \cap A_2)^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap (A_1 \cap A_2)^c),
\end{aligned}$$

showing $A_1 \cap A_2 \in \mathcal{M}$. □

Now we prove Carathéodory's theorem:

Proof of Theorem 1.5.10. We know $\mathcal{M}$ is an algebra containing $\mathcal{A}$, so it remains to show that it is closed under countably disjoint union and μ^* is countably additive. Let (A_n) be a sequence of disjoint sets in $\mathcal{M}$ and $A = \bigcup_{n=1}^{\infty} A_n$. To show $A \in \mathcal{M}$, let $B \subset E$. Since the A_n are disjoint we have

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap A_1 \cap A_2^c) + \mu^*(B \cap A_1^c \cap A_2) + \mu^*(B \cap A_1^c + A_2^c) \\
&= \mu^*(B \cap A_1) + \mu^*(B \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c) \\
&\vdots \\
&= \mu^*(B \cap A_1^c \cap \cdots \cap A_n^c) + \sum_{k=1}^{n} \mu^*(B \cap A_k).
\end{aligned}$$

Now $B \cap A^c \subset B \cap A_1^c \cap \cdots \cap A_n^c$ for some n, and since μ^* is increasing we know $\mu^*(B \cap A_1^c \cap \cdots \cap A_n^c) \geq \mu^*(B \cap A^c)$. Thus as $n \to \infty$,

$$\mu^*(B) \geq \mu^*(B \cap A^c) + \sum_{k=1}^{\infty} \mu^*(B \cap A_k) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

By subadditivity the reverse inequality is immediate, so we have $A \in \mathcal{M}$. When we take $B = A$, we recover

$$\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n),$$

showing μ^* is countably additive. $\square$

Thus we can extend a countably additive set function on a ring to a measure on $\sigma(\mathcal{A})$ by restricting the outer measure from $\mathcal{M}$ to $\sigma(\mathcal{A})$. We now consider when this extension is unique.

Theorem 1.5.15

Let μ_1, μ_2 be measures on $(E, \mathcal{E})$ with $\mu_1(E) = \mu_2(E) < \infty$. If $\mu_1 = \mu_2$ on some π-system $\mathcal{A}$ which generates $\mathcal{E}$, then $\mu_1 = \mu_2$ on $\mathcal{E}$.

Proof. Let $\mathcal{D} = \{A \in \mathcal{E} : \mu_1(A) = \mu_2(A)\}$ be the collection of sets on which the measures agree. We know $E \in \mathcal{D}$ and $\mathcal{A} \subset \mathcal{D}$. We will show $\mathcal{D}$ is a d-system, whence Dynkin's π-system lemma will imply $\mathcal{E} = \sigma(\mathcal{A}) \subset \mathcal{D}$.

For $A \subset B \in \mathcal{D}$, additivity implies

$$\mu_1(A) + \mu_1(B \setminus A) = \mu_1(B) < \infty \quad \text{and} \quad \mu_2(A) + \mu_2(B \setminus A) = \mu_2(B) < \infty,$$

so $B \setminus A \in \mathcal{D}$.

Next if $(A_n)_{n=1}^{\infty}$ with $A_n \in \mathcal{D}$ is an increasing sequence with $A = \bigcup_{n=1}^{\infty} A_n$, then $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ yields a disjoint sequence with the same union, so

$$\mu_1(A) = \sum_{n=1}^{\infty} \mu_1(B_n) = \sum_{n=1}^{\infty} \mu_2(B_n) = \mu_2(A),$$

showing $A \in \mathcal{D}$. Therefore $\mathcal{D}$ is a d-system as desired. $\square$

The above assumption that E has finite measure is more restrictive than we would like. Fortunately, we can improve our result:

Corollary 1.5.16

Let μ_1, μ_2 be measures on $(E, \mathcal{E})$. Suppose $\mu_1 = \mu_2$ on a π-system $\mathcal{A}$ which generates $\mathcal{E}$. Suppose $E = \bigcup_{i=1}^{\infty} B_i$, where $B_i \in \mathcal{A}$ are disjoint with $\mu_1(B_i) = \mu_2(B_i) < \infty$. Then $\mu_1 = \mu_2$ on $\mathcal{E}$.

Proof. For each i and $A \in \mathcal{E}$, let $\mu_1^i(A) = \mu_1(A \cap B_i)$, $\mu_2^i(A) = \mu_2(A \cap B_i)$. Then $\mu_1^i(E) = \mu_2^i(E) < \infty$ and $\mu_1^i(A) = \mu_2^i(A)$ for $A \in \mathcal{A}$ by assumption. By the previous result $\mu_1^i = \mu_2^i$ on $\mathcal{E}$. Now if $A \in \mathcal{E}$ is any measurable set, then

$$\mu_1(A) = \mu_1\left(\bigcup_{i=1}^{\infty}(B_i \cap A)\right) = \sum_{i=1}^{\infty} \mu_1(B_i \cap A) = \sum_{i=1}^{\infty} \mu_2(B_i \cap A) = \mu_2\left(\bigcup_{i=1}^{\infty}(B_i \cap A)\right) = \mu_2(A).$$

$\square$

1.5.2 Complete measures

Definition 1.5.17

Let $(E, \mathcal{E}, \mu)$ be a measure space. We say μ is *complete* if for any $A \in \mathcal{E}$ with $\mu(A) = 0$, every subset of A is also in $\mathcal{E}$.

Proposition 1.5.18

A measure space $(E, \mathcal{M}, \mu)$ obtained by Carathéodory's theorem is complete.

Proof. Let μ^* be the outer measure on E which restricts to μ on $\mathcal{M}$. If $A \in \mathcal{M}$ has measure zero and $N \subset A$ is a null set, then since μ^* is increasing we have $\mu^*(N) \leq \mu(A) = 0$, so $\mu^*(N) = 0$. Thus for any $B \subset E$,

$$\mu^*(B \cap N) + \mu^*(B \cap N^c) \leq \mu^*(N) + \mu^*(T) = \mu^*(T),$$

and the reverse inequality holds by subadditivity, so $N \in \mathcal{M}$. □

1.5.3 Lebesgue measure

The most important measure will be the Lebesgue measure, which gives us the standard notion of volume for sets in $\mathbb{R}^n$. First, consider rectangles in $\mathbb{R}^n$ to be sets of the form

$$R = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n]$$

with $a_i < b_i$. Let $\mathcal{A}_R$ be the collection of finite unions of disjoint rectangles. $\mathcal{A}_R$ is a π-system: $\varnothing$ is the empty union and the intersection of two rectangles is again a rectangle.

In fact, $\mathcal{A}_R$ is a ring as complements and unions of rectangles can be expressed as finite unions of disjoint rectangles. Moreover, by considering the product topology, $\mathcal{A}_R$ clearly generates $\mathcal{B}(\mathbb{R}^n)$.

Theorem 1.5.19

There exists a unique Borel measure μ on $\mathbb{R}^n$ such that for all rectangles $R = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n]$ we have

$$\mu(R) = \prod_{i=1}^{n} (b_i - a_i).$$

We call μ the *Lebesgue measure* on $\mathbb{R}^n$.

Proof. For any $A \in \mathcal{A}_R$ let us write $A = \bigcup_{i=1}^{N} R_i$ for disjoint rectangles $R_i = (a_1^i, b_1^i] \times \cdots \times (a_n^i, b_n^i]$. We define

$$\mu(A) := \sum_{i=1}^{n} (b_1^i - a_1^i) \cdots (b_n^i - a_n^i).$$

Although the decomposition of A into rectangles is not unique, this is well-defined and additive. We claim that μ is countably additive.

Let (A_n) be a sequence of disjoint sets in $\mathcal{A}_R$, and $A = \bigcup_{n=1}^\infty A_n$. Let $B_n = \bigcup_{k=n}^\infty A_k$ so that $\bigcap_{n=1}^\infty B_n = \varnothing$ since the A_n are disjoint. Since $\mathcal{A}_R$ is a ring, $B_n \in \mathcal{A}_R$. By finite additivity

$$\mu(A) = \sum_{k=1}^{n-1} \mu(A_k) + \mu(B_n),$$

so it is enough to show that $\mu(B_n) \to 0$ as $n \to \infty$. If not, then let $\epsilon > 0$ be such that $\mu(B_n) \geq 2\epsilon$ for all n. For each n let $C_n \in \mathcal{A}$ be such that $\overline{C_n} \subset B_n$ and $\mu(C_n \setminus B_n) \leq \frac{\epsilon}{2^n}$. Then

$$\mu(B_n \setminus (C_1 \cap \cdots \cap C_n)) \leq \mu((B_1 \setminus C_1) \cup \cdots \cup (B_n \setminus C_n)) \leq \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \epsilon.$$

Since $\mu(B_n) \geq 2\epsilon$, this implies $\mu(C_1 \cap \cdots \cap C_n) \geq \epsilon$, so $C_1 \cap \cdots \cap C_n \neq \varnothing$ and thus $K_n = \overline{C}_1 \cap \cdots \cap \overline{C}_n \neq \varnothing$. Now K_n is a nseted sequence of nonempty compact sets so $\bigcap K_i \subset \bigcap B_i$ is nonempty; a contradiction.

We conclude by the Carathéodory theorem that a Borel measure μ exists on $\mathbb{R}^n$ with the desired property on rectangles. For uniqueness, we invoke the earlier uniqueness theorem, noting that the set of rectangles is a π-system and that $\mathbb{R}^n$ is a countable disjoint union of rectangles. $\square$

A useful property of Lebesgue measure is translation invariance:

$$\mu(B + x) = \mu(B)$$

for $x \in \mathbb{R}^n$, $B \in \mathcal{B}(\mathbb{R}^n)$. Indeed, fix $x \in \mathbb{R}^n$ and let $\mu_x(B) = \mu(B)$. If B is a rectangle, then $\mu_x(R) = \mu(R)$ so $\mu_x = \mu$.

Note that Carathéodory's theorem actually defines the Lebesgue measure on $\mathcal{M}$, which is strictly larger than the Borel algebra $\mathcal{B}(\mathbb{R}^n)$. We call $\mathcal{M}$ the algebra of Lebesgue measurable sets. The Lebesgue measure is complete with respect to $\mathcal{M}$ but not $\mathcal{B}(\mathbb{R}^n)$.

Proposition 1.5.20

Let $A \in \mathcal{M}$ be Lebesgue measurable. Then for $\epsilon > 0$ there exist an open set O and a closed set C such that $C \subset A \subset O$ and

$$\mu(O \setminus A) < \epsilon \quad \text{and} \quad \mu(A \setminus C) < \epsilon.$$

If $\mu(A) < \infty$, then we may take C compact.

Proof. First if $\mu(A) < \infty$, then by definition

$$\mu(A) = \mu^*(A) = \inf \sum_{n=1}^\infty \mu(A_n)$$

where (A_n) is a sequence in $\mathcal{A}_R$ with $\bigcup A_n \supset A$. Since each A_n is a finite union of disjoint rectangles, we may simply assume each A_n is a rectangle. Fix $\epsilon > 0$. Let (A_n) be such that

$$\inf \sum_{n=1}^\infty \mu(A_n) < \mu(A) + \frac{\epsilon}{2}.$$

For each A_n, we can find a rectangle $\tilde{A}_n$ with $A_n \subset \tilde{A}_n^\circ$ and $\mu(\tilde{A}_n) < \mu(A_n) + \frac{\epsilon}{2^{n+1}}$. Let $O = \bigcup_{n=1}^\infty \tilde{A}_n^\circ$. Clearly O is open and contains A. Moreover

$$\mu(O) \leq \sum_{n=1}^\infty \mu(\tilde{A}_n) \leq \sum_{n=1}^\infty \mu(A_n) + \frac{\epsilon}{2}\sum_{n=1}^\infty \frac{1}{2^n} < \mu(A) + \epsilon,$$

so $\mu(O \setminus A) < \epsilon$. Now assume $\mu(A) = \infty$. Let $A_k = A \cap \overline{B}_k(x)$ so that $\mu(A_k) < \infty$, and thus there exists O_k open with $\mu(O_k \setminus A_k) < \frac{\epsilon}{2^k}$. Let $O = \bigcup_{k=1}^\infty O_k$. Then O is open, contains, A, and

$$O \setminus A = \bigcup_{k=1}^\infty O_k \setminus A = \bigcup_{k=1}^\infty O_k \setminus A \subset \bigcup_{k=1}^\infty (O_k \setminus A_k),$$

hence

$$\mu(O \setminus A) \leq \sum_{k=1}^\infty \mu(O_k \setminus A_k) < \epsilon.$$

For the second part of the proof, note that A^c is also measurable, so there exists O open with $A^c \subset O$ and $\mu(O \setminus A^c) < \epsilon$. Let $C = O^c$, which is closed, contains A, and

$$A \setminus C = C^c \setminus A^c = O \setminus A^c,$$

hence $\mu(A \setminus C) < \epsilon$. Finally if $\mu(A) < \infty$, then since A_k is an increasing sequence with $\bigcup A_k = A$, we have $\lim \mu(A_k) = \mu(A) < \infty$, so for large enough k

$$\mu(A \setminus A_k) = \mu(A) - \mu(A_k) < \frac{\epsilon}{2}.$$

Let $C \subset \mu(A_k \setminus C) < \frac{\epsilon}{2}$. Then $\mu(A \setminus C) = \mu((A \setminus A_k) \cup (A_k \setminus C)) < \epsilon$ and C is bounded. □

Proposition 1.5.21

Let $A \subset \mathbb{R}^n$. Suppose there exists O open and C closed with $C \subset A \subset O$ and

$$\mu(O \setminus C) < \epsilon.$$

Then $A = B_1 \cup N$ for $N \subset B_2$, $B_1, B - 2 \in \mathcal{B}(\mathbb{R}^n)$ with $\mu(B_2) = 0$.

Proof. For each i, let O_i be open and C_i be closed such that $C_i \subset A \subset O_i$ and

$$\mu(O_i \setminus C_i) < \frac{1}{2^i}.$$

Then $B_1 = \bigcup_{i=1}^\infty C_i \in \mathcal{B}(\mathbb{R}^n)$. Let $B_2 = \bigcap_{i=1}^\infty O_i \setminus C_i \in \mathcal{B}(\mathbb{R}^n)$. Then

$$\mu(B_2) \leq \mu\left(\bigcap_{i=1}^n O_i \setminus C_i\right) \leq \frac{1}{2^{n-1}},$$

so $\mu(B_2) = 0$ by taking $n \to \infty$. Since $A \setminus B_1 \subset B_2$, we are done. □

Since null sets have Lebesgue measure zero by completeness, the union of a Borel set with a null set is Lebesgue measurable. We have established:

Theorem 1.5.22

Let $A \subset \mathbb{R}^n$. The following are equivalent:

(1) A is Lebesgue measurable.

(2) For any $\epsilon > 0$ there exists O open and C closed with $C \subset A \subset O$ and $\mu(O \setminus C) < \epsilon$.

(3) $A = B_1 \cup N$ where $N \subset B_2$ and $B_1, B_2 \in \mathcal{B}(\mathbb{R}^n)$ with $\mu(B_2) = 0$.