Submanifolds in Euclidean Space

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1 Introduction

We assume a certain proficiency with differential calculus in several variables. In particular, we rely heavily on the inverse and implicit function theorems, which we state without proof, although we deduce the latter from the former. These twin theorems encode the foundations upon which we erect the structures of smooth manifolds.

Roughly, a manifold is a space that locally resembles a Euclidean space. In different areas of math we demand varying degrees of resemblance. In differential topology we are interested in smooth manifolds, which demand to be locally identifiable with Euclidean space in a smooth fashion. In these notes, we focus on smooth manifolds that live naturally inside Euclidean space. In this special case, they possess four simple descriptions in the language of functions and their derivatives.

2 The Twin Theorems

For completeness, we state the two theorems that will be foundational to our formulation of submanifolds in $\mathbb{R}^n$. We omit the proof of the inverse function theorem, which may be found in Spivak [Spi65, Section 2.5]. Then we deduce the implicit function theorem from the inverse function theorem to emphasize their relationship.

2.1 The inverse function theorem

When is a function f invertible? That is, we would like to find a g such that $f(g(x)) = x = g(f(x))$ for all possible x. It turns out to be quite difficult to fulfill this dream globally on $\mathbb{R}^n$, but if we impose the condition that f be continuously differentiable, it is remarkably easy to obtain a local inverse. More precisely, we have the inverse function theorem.

Theorem 2.1 (Inverse function theorem)**.** *Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be continuously differentiable. If $a \in U$ is such that $\det f'(a) \neq 0$, then there exists a neighbourhood V of a and a neighbourhood W of $f(a)$ such that the restriction $f : V \to W$ has an inverse function $f^{-1} : W \to V$ which is continuously differentiable and for all $y \in W$,*

$$(f^{-1})'(y) = \left[f'(f^{-1}(y))\right]^{-1}.$$

2.2 The implicit function theorem

Now that we have more or less resolved the question of invertibility, a morally equivalent but originally different question asks when there exists a unique solution to a system of m equations depending on n parameters and m unknowns.

The most handy manifestation of this problem is as follows: if $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $f(a, b) = 0$, when can we find, for each $x \in \mathbb{R}^n$ near a, a unique $y \in \mathbb{R}^m$ near b that satisfies $f(x, y) = 0$?

Namely the m equations are given by

$$\begin{aligned} f_1(x_1, \dots, x_n, y_1, \dots, y_m) &= 0 \\ &\vdots \\ f_m(x_1, \dots, x_n, y_1, \dots, y_m) &= 0, \end{aligned}$$

and we can see they depend on n parameters $x_1, \dots, x_n$. We would like to solve for the m unknowns $y_1, \dots, y_n$ in terms of the parameters. We are given as initial data that when the parameters are $x_1 = a_1, \dots, x_n = a_n$, a solution is $y_1 = b_1, \dots, y_m = b_m$.

Since we have claimed that this question is morally equivalent, the reader should expect to receive a morally equivalent response. Indeed, we have the implicit function theorem.

Theorem 2.2 (Implicit function theorem)**.** *Let U be an open set in $\mathbb{R}^n \times \mathbb{R}^m$ and let $f : U \to \mathbb{R}^m$ be continuously differentiable. If $(a, b) \in U$ is such that $f(a, b) = 0$ and*

$$\det \left(\frac{\partial f_i}{\partial x_{n+j}}(a, b) \right)_{1 \le i,j \le m} \ne 0,$$

then there exists a neighbourhood $A \subseteq \mathbb{R}^n$ of a and a neighbourhood $B \subseteq \mathbb{R}^m$ of b, along with a continuously differentiable function $g : A \to B$ such that for each $x \in A$, there exists a unique $y \in B$ that satisfies $f(x, y) = 0$, given by $y \coloneqq g(x)$.

Proof. We deduce this from the inverse function theorem.

Define a continuously differentiable function $F : U \to \mathbb{R}^n \times \mathbb{R}^m$ by $F(x, y) = (x, f(x, y))$. Then

$$F'(a, b) = \begin{pmatrix} I_{n \times n} & 0_{n \times m} \\ \left(\frac{\partial f_i}{\partial x_j}(a, b) \right)_{1 \le i \le m, 1 \le j \le n} & \left(\frac{\partial f_i}{\partial x_{n+k}}(a, b) \right)_{1 \le i,k \le m} \end{pmatrix}.$$

Given that $\left(\frac{\partial f_i}{\partial x_{n+k}}(a, b) \right)_{1 \le i,k \le m}$ has rank m, we know that $F'(a, b)$ has rank $n + m$, so $\det F'(a, b) \ne 0$. By applying the inverse function theorem to F at (a, b), there exist neighbourhoods V of (a, b) and W of $F(a, b)$ such that $F : V \to W$ has a continuously differentiable inverse $F^{-1} : W \to V$. For any $(x, y) \in V$, we have

$$(x, y) = F^{-1}(F(x, y)) = F^{-1}(x, f(x, y)),$$

so F^{-1} must take the form $(x, y) \mapsto (x, h(x, y))$ for some continuously differentiable function $h : W \to V$. Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the projection onto $\mathbb{R}^m$. Then $f = \pi_2 \circ F$, so

$$\begin{aligned} f(x, h(x, y)) &= (f \circ F^{-1})(x, y) \\ &= ((\pi_2 \circ F) \circ F^{-1})(x, y) \\ &= \pi_2(x, y) \\ &= y. \end{aligned}$$

By possibly shrinking W, we may assume that it is in the rectangular form $W = A \times B \subseteq \mathbb{R}^n \times \mathbb{R}^m$. Since $A \times B$ is a neighbourhood of $F(a, b) = (a, f(a, b)) = (a, 0)$, we must have $0 \in B$. Define a continuously differentiable function $g : A \to B$ by $x \mapsto h(x, 0)$. By construction, it uniquely satisfies $f(x, g(x)) = 0$. $\square$

3 Smooth Submanifolds

In this section we describe smooth submanifolds of $\mathbb{R}^n$. After specifying a standard for equivalence, we state four equivalent conditions that may be imposed on a subset of $\mathbb{R}^n$. These will become our working definitions of a smooth submanifold of $\mathbb{R}^n$.

3.1 Four equivalent definitions

Let U and V be open sets in $\mathbb{R}^n$.

Definition 3.1 (Smooth map)**.** A function $f : U \to V$ is called a *smooth map* if its partial derivatives of any degree exist and are continuous.

There are incarnations of the inverse and implicit functions theorems obtained by replacing every occurrence of "continuously differentiable" by "smooth" in the statements of Theorems 2.1 and 2.2. They follow from the continuously differentiable case by induction, and they are the incarnations we will be using.

Definition 3.2 (Diffeomorphism)**.** A *diffeomorphism* $f : U \to V$ is a smooth bijection whose inverse is also smooth.

Diffeomorphisms are isomorphisms in the category of smooth manifolds and their smooth maps. To the uninitiated, this simply means we will use diffeomorphism as our standard for equivalence between spaces: two spaces are equivalent if there exists a diffeomorphism between them.

Theorem 3.3. *Let $M \subseteq \mathbb{R}^n$ and $a \in M$. The following are equivalent:*

(1) There is a neighbourhood U of a, an open set V in $\mathbb{R}^n$, and a diffeomorphism $h : U \to V$ such that

$$h(M \cap U) = V \cap \left(\mathbb{R}^k \times \{0\}^{n-k}\right).$$

(2) There is a neighbourhood U of a and a smooth map $f : U \to \mathbb{R}^{n-k}$ such that

$$M \cap U = f^{-1}(0)$$

and $f'(a)$ has rank $n-k$.

(3) There is a rectangular neighbourhood $V \times W \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ of a such that $M \cap (V \times W)$ is the graph of a smooth map $g : V \to W$. That is,

$$M \cap (V \times W) = \Gamma(g) \coloneqq \left\{(x, g(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x \in V\right\}.$$

(4) There is a neighbourhood Z of a, an open set W in $\mathbb{R}^k$, and a smooth map $\varphi : W \to \mathbb{R}^n$ such that

$$\varphi(W) = M \cap Z$$

and φ' has rank k on W.

Proof. Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be the projections onto the first k coordinates and the last $n-k$ coordinates, respectively.

(1) $\implies$ (2). Define a smooth map $f : U \to \mathbb{R}^{n-k}$ by $f = \pi_2 \circ h$. From

$$M \cap U = V \cap \left(\mathbb{R}^k \times \{0\}^{n-k}\right)$$

we have $M \cap U = f^{-1}(0)$. Since h is a diffeomorphism, h' must have rank n, and therefore $f'(a)$ has rank $n-k$.

(2) $\implies$ (3). Since $a \in M \cap U$, we know that $f(a) = 0$. Since $f'(a)$ has rank $n-k$, we may assume without loss of generality that $\left(\frac{\partial f_i}{\partial x_{k+j}}(a)\right)_{1\le i,j\le n-k}$ has rank $n-k$. Viewing U as an open set in $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and $a = (a_1, a_2)$ as a point in $\mathbb{R}^k \times \mathbb{R}^{n-k}$, f satisfies the hypotheses of the smooth implicit function theorem at (a_1, a_2). Therefore there exist neighbourhoods V of a_1 and W of a_2, along with a smooth map $g : V \to W$ that uniquely satisfies $f(x, g(x)) = 0$ for each $x \in V$.

By replacing V with $\pi_1(U) \cap V$ and W with $\pi_2(U) \cap W$ as necessary, we may assume $V \times W \subseteq U$. We claim that $V \times W \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ is the desired neighbourhood of a. Namely,

$$M \cap (V \times W) = \Gamma(g).$$

In one direction, if $(x, y) \in M \cap (V \times W)$ then $(x, y) \in M \cap U$. By assumption, $f(x, y) = 0$, so by uniqueness of g we must have $y = g(x)$. Conversely if $(x, g(x)) \in \Gamma(g)$ then of course $(x, g(x)) \in V \times W$, and moreover $f(x, g(x)) = 0$ implies that $(x, g(x)) \in M$.

(3) $\implies$ (4). Let $Z \coloneqq V \times W$ and define a smooth map $\varphi : W \to \mathbb{R}^n$ by $w \mapsto (w, g(w))$. This is a homeomorphism onto its image since it has continuous inverse $\pi_1 : \varphi(W) \to W$. Moreover, we have $\varphi(W) = \Gamma(g) = M \cap Z$ by construction. Finally,

$$\varphi'(w) = \begin{pmatrix} I_{k\times k} \\ g'(w) \end{pmatrix}$$

clearly has rank k.

(4) $\implies$ (1). Since φ' has rank k on W, we may assume without loss of generality that $\left(\frac{\partial \varphi_i}{\partial x_j}\right)_{1\le i,j\le k}$ has rank k on W. Since $a \in M \cap Z$, there exists $b \in W$ with $\varphi(b) = a$. Define $\psi : W \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by

$$(w, z) \mapsto \varphi(w) + (0, z).$$

Then $\psi'(w, z)$ takes the form

$$\psi'(w,z) = \begin{pmatrix} \left(\frac{\partial \varphi_i}{\partial x_j}(w,z)\right)_{1\le i,j\le k} & O_{k\times(n-k)} \\ \left(\frac{\partial \varphi_{k+i}}{\partial x_j}(w,z)\right)_{1\le i\le n-k, 1\le j\le k} & I_{(n-k)\times(n-k)} \end{pmatrix},$$

and given that the top left block has rank k and the bottom right block has rank $n-k$, it has rank n. In particular, $\det \psi'(b, 0) \neq 0$. Applying the smooth inverse function theorem to ψ at $(b, 0)$, there exist neighbourhoods V_0 of $(b, 0)$ and W_0 of $\psi(b, 0) = a$ such that $\psi : V_0 \to W_0$ is a diffeomorphism.

Let $U \coloneqq Z \cap W_0$ and $V \coloneqq \psi^{-1}(U)$. We claim that

$$M \cap U = \psi\left(V \cap \mathbb{R}^k \times \{0\}^{n-k}\right). \tag{3.1}$$

We already guaranteed that $U = \psi(V)$, so it only remains to show that every point in $M \cap U$ may be written in the form $\psi(w, 0)$ for $(w, 0) \in \mathbb{R}^k \times \{0\}^{n-k}$, or equivalently $M \cap U \subseteq \varphi(W)$. Indeed,

$$M \cap U = M \cap Z \cap W_0 = \varphi(W) \cap W_0.$$

We have thus shown eq. (3.1), so we obtain a diffeomorphism $h \coloneqq \psi^{-1} : U \to V$ such that

$$h(M \cap U) = \psi^{-1}(M \cap U) = V \cap \left(\mathbb{R}^k \times \{0\}^{n-k}\right).$$

□

We are now justified in making the following definition.

Definition 3.4 (Submanifold of $\mathbb{R}^n$)**.** $M \subseteq \mathbb{R}^n$ is a *smooth k-dimensional submanifold of* $\mathbb{R}^n$ if for each $a \in M$, any of the four equivalent conditions in Theorem 3.3 is satisfied.

3.2 Four illustrative examples

We conclude by giving one example of a submanifold per definition given by Theorem 3.3.

Approximately, (1) states that M is locally identifiable with a k-dimensional slice of $\mathbb{R}^n$. We call this the *slice condition*, and we now use it to show that the sphere is a submanifold of $\mathbb{R}^3$.

Example 3.5. Let S^2 be the 2-sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. First pick the north pole $a = (0, 0, 1) \in S^2$. Let $U \coloneqq B^2 \times (0, \infty)$ where B^2 is the open unit ball centered at the origin in $\mathbb{R}^2$, and let $V \coloneqq B^2 \times (0, \infty)$. Define a smooth map $h : U \to V$ by

$$(x, y, z) \mapsto (x, y, z - \sqrt{1 - x^2 - y^2}).$$

We observe that h has an inverse $h^{-1} : V \to U$ given by

$$(u, v, w) \mapsto (u, v, w + \sqrt{1 - u^2 - v^2}),$$

which is also smooth on V. Thus h is a diffeomorphism, and since $M \cap U$ is the northern hemisphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z > 0\}$, we have

$$z - \sqrt{1 - x^2 - y^2} = 0$$

for all $(x, y, z) \in S^2 \cap U$. Therefore

$$h(S^2 \cap U) = V \cap \left(\mathbb{R}^2 \times \{0\}\right),$$

as desired. The above construction of the diffeomorphism h fulfills condition (1) at any point $a \in S^2 \cap U$ on the northern hemisphere. Considering the intrinsic symmetry of the sphere, this should convince the reader that (1) may be fulfilled at each point in S^2, and thus S^2 is a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

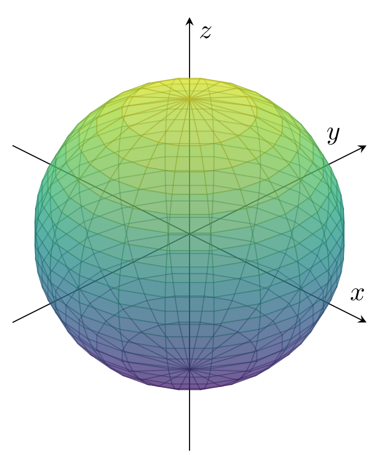

Figure 3.1: The 2-sphere S^2, a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

For the skeptical reader, we have a symmetrical diffeomorphism $B^2 \times (-\infty, 0) \to B^2 \times (-\infty, 0)$ given by

$$(x, y, z) \mapsto (x, y, \sqrt{1 - x^2 - y^2} - z).$$

As above, this fulfills (1) at any point on the southern hemisphere, so all that remains is the equator $\{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. To cover it with local diffeomorphisms, we repeat the above process for the half-spheres determined by $y > 0$ and $y < 0$, respectively. By this point the only points left unaccounted for are $(1, 0, 0)$ and $(-1, 0, 0)$. To fulfill (1) at these two points, we simply repeat the process for the half-spheres with $x > 0$ and $x < 0$, respectively. ■

Somewhat dually, (2) states that M is locally identifiable with a level set of a graph with values in $\mathbb{R}^{n-k}$. We call this the *level set condition*, and we use it to show that the set of solutions to a sufficiently nice system of equations forms a manifold.

Example 3.6. Let M be the set points $(x, y, z, w) \in \mathbb{R}^4$ satisfying the system of equations

$$\begin{aligned} x^2 + y^2 + z^2 + w^2 &= 1 \\ z + w &= 2. \end{aligned}$$

We need not even specify a point $a \in M$ because we can identify M with $\mathbb{R}^2$ globally; that is, we can take $U = \mathbb{R}^4$ in (2). Define a smooth map $f : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$f(x, y, z, w) = (x^2 + y^2 + z^2 + w^2 - 1, z + w - 2).$$

Clearly $M = f^{-1}(0)$, and the matrix of interest is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\ \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial w} \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z & 2w \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

which has rank 2 unless $x = y = 0$ and $z = w$. Indeed on M, we cannot even have $z = w$; otherwise $z + w = 2$ implies $z = w = 1$, whence there are no solutions (x, y) to the first equation. Therefore (2) states that M is a smooth 2-dimensional submanifold of $\mathbb{R}^4$. ■

Now (3) provides perhaps the most explicit description of a manifold's local structure as the graph of a function. We call this the *graph condition*, and among the many immediate examples we show that the paraboloid is a submanifold of $\mathbb{R}^3$.

Example 3.7. Let P be the standard paraboloid $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z\}$. Once again, we can globally identify P with the graph of the smooth map $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$(x, y) \mapsto x^2 + y^2.$$

By (3), P is a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

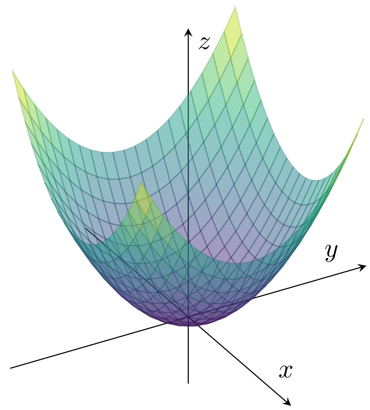

Figure 3.2: The paraboloid P, a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

■

Finally, (4) provides a direct local correspondence between $\mathbb{R}^k$ and M. We call such smooth maps φ coordinate charts, hence we call this criterion the *coordinate chart condition.* We will now exhibit a coordinate chart covering a large part of the torus.

Example 3.8. Let T be the torus of major radius 2 and minor radius 1 obtained by taking the circle in the xz-plane of radius 1 centered at $(2, 0, 0)$ and rotating it about the z-axis. We define via parametrization a coordinate chart ψ that covers almost the entire torus. Let $W \coloneqq (0, 2\pi) \times (0, 2\pi)$, and define $\psi : W \to \mathbb{R}^3$ by

$$(\theta, \phi) \mapsto ((2 + \cos\phi)\cos\theta, (2 + \cos\phi)\sin\theta, \sin\phi).$$

This is smooth on W and the matrix of interest

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial \theta} & \frac{\partial \psi_1}{\partial \phi} \\ \\ \frac{\partial \psi_2}{\partial \theta} & \frac{\partial \psi_2}{\partial \phi} \\ \\ \frac{\partial \psi_3}{\partial \theta} & \frac{\partial \psi_3}{\partial \phi} \end{pmatrix} = \begin{pmatrix} -(2+\cos\phi)\sin\theta & -\sin\phi\cos\theta \\ (2+\cos\phi)\cos\theta & -\sin\phi\sin\theta \\ 0 & \cos\phi \end{pmatrix}$$

has rank 2 on W, a fact that may be tediously verified by taking the cross product of the two columns and arriving at a nonzero vector, implying that the columns are linearly independent. $\psi(W)$ covers all of the torus but the circles

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 9, z = 0\}$$

and

$$\{(x, y, z) \in \mathbb{R}^3 : (x - 2)^2 + z^2 = 1, y = 0\}.$$

These are shown in fig. 3.3 as the horizontal gap around the torus's greatest circumference and the vertical gap near the x-axis label, respectively. Their complement in $\mathbb{R}^3$ is an open set Z, so $\varphi(W) = T \cap Z$, as desired.

As with the 2-sphere, it is possible to exploit the symmetry of the torus to construct the remaining coordinate charts. Therefore T is a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

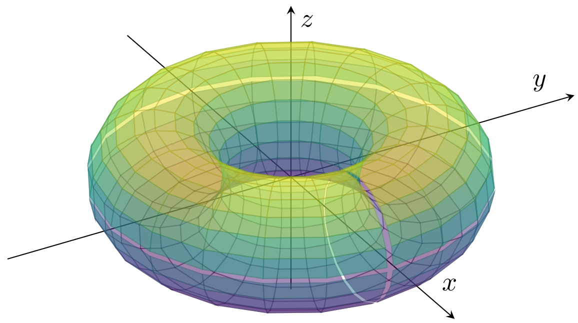

Figure 3.3: The torus T, a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

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References

[Spi65] Michael Spivak. *Calculus on Manifolds*. Addison-Wesley, 1965.