EXISTENCE WITHOUT UNIQUENESS

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1. Introduction

We will prove the following existence theorem of ordinary differential equations:

Theorem 1.1. *(Cauchy-Peano) Let $t_0, t_1 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Suppose $f : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bounded. Then there exists a solution $x : [t_0, t_1] \to \mathbb{R}^n$ to the initial value problem*

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

Uniqueness is notably not guaranteed under these assumptions, hence history has hawked this great theorem of its glory.

2. Function Spaces and Arzelà-Ascoli

We begin with some preliminary results about function spaces. Let $E \subseteq \mathbb{R}^n$ be compact. All functions we consider in this section will be $E \to \mathbb{R}^p$.

Definition 2.1. A family $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ of functions is *uniformly bounded* if there exists $M \geq 0$ such that for all $x \in E$ and $\alpha \in A$, $|f_\alpha(x)| \leq M$.

Definition 2.2. A family $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ of functions is *equicontinuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\alpha \in A$ and $x, y \in E$ with $|x - y| < \delta$, $|f_\alpha(x) - f_\alpha(y)| < \varepsilon$.

Definition 2.3. A sequence $(f_n)_{n \in \mathbb{N}}$ of functions *converges uniformly* to f, and we write $f_n \rightrightarrows f$, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in E$ and $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

The following is the reason why we care about uniform convergence.

Lemma 2.4. *Suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence of continuous functions and $f_n \rightrightarrows f$. Then f is continuous.*

Proof. For each $x \in E$, and any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f(y) - f_N(y)| < \frac{\varepsilon}{3}$$

for all $y \in E$. Since f_N is continuous, there exists $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}.$$

Then if $|y-x|<\delta$,

$$\begin{aligned}|f(y)-f(x)| &\leq |f(y)-f_N(y)|+|f_N(y)-f_N(x)|+|f_N(x)-f(x)| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$

□

The most natural way to study uniform convergence is in a function space. Let $\mathcal{C}(E,\mathbb{R}^p)$ be the space of continuous functions $E \to \mathbb{R}^p$.

Definition 2.5. We define the sup norm $\|\cdot\|_\infty : \mathcal{C}(E,\mathbb{R}^p) \to \mathbb{R}$ by

$$\|f\|_\infty = \sup_{x\in E} |f(x)|.$$

This is indeed a norm. First $\|f\|_\infty \geq |f(x)| \geq 0$ with $\|f\|_\infty = 0$ if and only $|f(x)| = 0$ for all $x \in E$, or $f = 0$. Second

$$\|cf\| = \sup_{x\in E} |cf(x)| = |c| \sup_{x\in E} |f(x)| = |c|\|f\|,$$

and third

$$\begin{aligned}\|f+g\| &= \sup_{x\in E} |f(x)+g(x)| \\ &\leq \sup_{x\in E} |f(x)|+|g(x)| \\ &\leq \sup_{x\in E} |f(x)| + \sup_{x\in E} |g(x)| \\ &= \|f\|+\|g\|.\end{aligned}$$

The sup norm naturally induces a metric $d(f,g) = \|f-g\|_\infty$. This metric yields a useful criterion for uniform convergence.

Lemma 2.6. *Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions. Then $f_n \rightrightarrows f$ if and only if $f_n \xrightarrow[\|\cdot\|_\infty]{} f$, or equivalently $\|f_n - f\|_\infty \to 0$.*

Proof. If $\|f_n - f\|_\infty \to 0$, then $\sup_{x\in E} |f_n(x)-f(x)| \to 0$. Hence there exists $N \in \mathbb{N}$ such that for $n \geq N$, $\sup_{x\in E} |f_n(x)-f(x)| < \varepsilon$, or equivalently, $|f_n(x)-f(x)| < \varepsilon$ for all $x \in E$.

Conversely if $f_n \rightrightarrows f$, then there exists $N \in \mathbb{N}$ such that for all $x \in E$ and $n \geq N$, $|f_n(x)-f(x)| < \frac{\varepsilon}{2}$. This means for $n \geq N$, $\|f_n - f\|_\infty = \sup_{x\in E} |f_n(x)-f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$, and thus $\|f_n - f\|_\infty \to 0$. □

It is even easier to see that $A \subseteq \mathcal{C}(E,\mathbb{R}^p)$ is uniformly bounded if and only if A is bounded with respect to $\|\cdot\|_\infty$.

Theorem 2.7. *$(\mathcal{C}(E,\mathbb{R}^p), \|\cdot\|_\infty)$ is a complete metric space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}(E,\mathbb{R}^p)$. For each $x_0 \in E$, $(f_n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^p$, as

$$|f_n(x_0) - f_m(x_0)| \leq \sup_{x\in E} |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty.$$

Thus for each $x \in E$, $\lim_{n\to\infty} f_n(x)$ exists; we assign this value to $f(x)$. We claim that $f_n \rightrightarrows f$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies

$$\|f_n - f_m\|_\infty < \frac{\varepsilon}{2}.$$

Also by definition of $f(x) = \lim_{n\to\infty} f_n(x)$, for each x there exists $m \geq N$ such that

$$|f_m(x) - f(x)| < \frac{\varepsilon}{2}.$$

Hence if $n \geq N$ and $x \in E$, then

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq \|f_n - f_m\|_\infty + |f_m(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus $f_n \rightrightarrows f$. By 2.4, $f \in \mathcal{C}(E, \mathbb{R}^p)$. By 2.6, the Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ converges in $\mathcal{C}(E, \mathbb{R}^p)$. □

The next lemma will be necessary to our proof of the Cauchy-Peano theorem.

Lemma 2.8. *Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of Riemann integrable functions in $\mathcal{C}([a,b], \mathbb{R})$ and $f_n \rightrightarrows f$. Then f is Riemann integrable and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Since each f_n is integrable, it is continuous except possibly on a measure zero subset D_n of $[a,b]$. $\bigcup_{n\in\mathbb{N}} D_n$ is again measure zero, and on $[a,b] \setminus \bigcup_{n\in\mathbb{N}} D_n$, each f_n is continuous. By 2.4, f is continuous on $[a,b] \setminus \bigcup_{n\in\mathbb{N}} D_n$, and by 2.7, f is bounded on $[a,b]$. Thus f is integrable on $[a,b]$. Furthermore

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \int_a^b f_n\,dx \right| &= \left| \int_a^b f(x) - f_n(x)\,dx \right| \\ &\leq \int_a^b |(f - f_n)(x)\,dx| \\ &\leq \|f - f_n\|_\infty (b-a), \end{aligned}$$

which tends to 0 as $n \to \infty$. Therefore $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$. □

We now prove the Arzelà-Ascoli theorem, the main result needed for our existence theorem.

Lemma 2.9. *Suppose $(f_k)_{k\in\mathbb{N}}$ is a subsequence of $(g_n)_{n\in\mathbb{N}}$. Then for each k, $f_k = g_r$ for some $r \geq k$.*

Proof. By definition of a subsequence, $f_k = g_{n_k}$ for some n_k such that $1 \leq n_1 < n_2 < \cdots < n_k$. Thus $r = n_k \geq k$. □

Theorem 2.10. *(Arzelà-Ascoli) Every uniformly bounded and equicontinuous sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{C}(E, \mathbb{R}^p)$ has a uniformly convergent subsequence.*

Proof. Let $D = \{d_j\}_{j\in\mathbb{N}}$ be a countable dense set in E, for example $D = \mathbb{Q}^n \cap E$. By uniform boundedness of (f_n), let $M \geq 0$ be such that for all $x \in E$ and all $n \in \mathbb{N}$, $|f_n(x)| \leq M$. Then $(f_n(d_1))_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^p$, since $|f_n(d_1)| \leq M$ for all $n \in \mathbb{N}$. By the Bolzano-Weierstrass theorem, $(f_n(d_1))$ has a convergence subsequence, say

$$\lim_{k\to\infty} f_{1,k}(d_1) = y_1.$$

$(f_{1,k}(d_2))_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}^p$. Another invocation of Bolzano-Weierstrass produces a sub-subsequence $(f_{2,k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} f_{2,k}(d_2) = y_2.$$

Furthermore $\lim_{k\to\infty} f_{2,k}(d_1) = y_1$. Inductively on m, we define a nested family of subsequences $(f_{m,k})_{k\in\mathbb{N}}$ of (f_n) such that $(f_{m,k})$ is a subsequence of $(f_{m-1,k})$ and for all $j \le m$,

$$\lim_{k\to\infty} f_{m,k}(d_j) = y_j.$$

We claim that the diagonal subsequence $(g_m) = (f_{m,m})$ of (f_n) converges uniformly. For any $j \in \mathbb{N}$, and $m > j$, $(f_{m,k})$ is a subsequence of $(f_{m-1,k})$ so by 2.9, $f_{m,m} = f_{m-1,r_1}$ for some $r_1 \ge m$. Applying the lemma again, $f_{m-1,r_1} = f_{m-2,r_2}$ for some $r_2 \ge r_1 \ge m$, and by induction

$$f_{m,m} = f_{m-1,r_1} = f_{m-2,r_2} = \cdots = f_{j,r_{m-j}}$$

where $r_{m-j} \ge \cdots \ge r_2 \ge r_1 \ge m$. Since $r_{m-j} \ge m$, we have

$$\lim_{m\to\infty} g_m(d_j) = \lim_{m\to\infty} f_{m,m}(d_j) = \lim_{r\to\infty} f_{j,r}(d_j) = y_j.$$

To show that $g_m(x)$ converges for all $x \in E$, and that the convergence is uniform, it suffices to show (g_m) is Cauchy. Given $\varepsilon > 0$, let $\delta > 0$ be such that for all $n \in \mathbb{N}$ and $s, t \in E$ with $|s-t| < \delta$, $|f_n(s) - f_n(t)| < \frac{\varepsilon}{3}$, by equicontinuity of (f_n). In particular

$$|s-t| < \delta \implies |g_m(s) - g_m(t)| < \frac{\varepsilon}{3}.$$

By density of D in E, $\{B(d_j,\delta)\}_{j\in\mathbb{N}}$ is an open covering for E. By compactness of E, it has a finite subcovering $B(d_{j_1},\delta), \cdots, B(d_{j_\ell},\delta)$. Let $J = \max_{i=1}^{\ell} j_i$, so that for every $x \in E$ there exists $j \le J$ such that $x \in B(d_j,\delta)$.

For each $j \le J$, $(g_m(d_j))_{m\in\mathbb{N}}$ converges in $\mathbb{R}^p$, and thus is Cauchy. let $N_j \in \mathbb{N}$ be such that for all $m, n \ge N_j$, $|g_m(d_j) - g_n(d_j)| < \frac{\varepsilon}{3}$. Let $N = \max_{j=1}^{J} N_j$, so that for all $m, n \ge N$ and all $j \le J$, the fact that $m, n \ge N_j$ implies

$$|g_m(d_j) - g_n(d_j)| < \frac{\varepsilon}{3}.$$

Thus for any $x \in E$, let $j \le J$ be such that $|d_j - x| < \delta$. Then

$$\begin{aligned} |g_m(x) - g_n(x)| &\le |g_m(x) - g_m(d_j)| + |g_m(d_j) - g_n(d_j)| + |g_n(d_j) - g_n(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Therefore (g_m) is Cauchy in $\mathcal{C}(E,\mathbb{R}^p)$, so by 2.7 (g_m) converges uniformly. □

3. Existence

Proof of Cauchy-Peano. Without loss of generality $t_0 = 0$ and $t_1 = 1$. It suffices to find a solution to the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds.$$

For in this case $x(0) = x_0$ and we differentiate by the fundamental theorem of calculus to get

$$x'(t) = f(t, x(t)).$$

For each $k \in \mathbb{N}$, define $x_k : [0,1] \to \mathbb{R}^n$ by the recursion

$$x_k(t) = \begin{cases} x_0 & \text{if } 0 \le t \le \frac{1}{k} \\ x_0 + \int_0^{t-\frac{1}{k}} f(s, x_k(s))\, ds & \text{if } \frac{j}{k} \le t \le \frac{j+1}{k}, \text{ for } j = 1, \dots, k-1. \end{cases}$$

It is necessary to define x_k inductively on each interval $\left[\frac{j}{k}, \frac{j+1}{k}\right]$. We claim that $(x_k)_{k\in\mathbb{N}}$ is a uniformly bounded and equicontinuous sequence in $\mathcal{C}([0,1], \mathbb{R}^n)$. Since f is bounded on $[0,1] \times \mathbb{R}^n$, let $M \ge 0$ be such that $|f(t,x)| \le M$ for all $(t,x) \in [0,1] \times \mathbb{R}^n$.

For all $k \in \mathbb{N}$ and $t \in [0,1]$, we have $|x_k(t)| \le |x_0| + M$. Indeed if $0 \le t \le \frac{1}{k}$ then $|x_k(t)| = |x_0|$, and if $\frac{1}{k} \le t \le 1$ then

$$\begin{aligned} |x_k(t)| &\le |x_0| + \left| \int_0^{t-\frac{1}{k}} f(s, x_k(s))\, ds \right| \\ &\le |x_0| + \int_0^{t-\frac{1}{k}} |f(s, x_k(s))|\, ds \\ &\le |x_0| + M\left(t - \frac{1}{k}\right) \\ &\le |x_0| + M. \end{aligned}$$

Now suppose $t' \le t \in [0,1]$. For any $k \in \mathbb{N}$, if $0 \le t' \le t \le \frac{1}{k}$ then

$$|x_k(t) - x_k(t')| = |x_0 - x_0| = 0 \le M|t - t'|.$$

Similarly if $0 \le t' \le \frac{1}{k} \le t \le 1$, then

$$\begin{aligned} |x_k(t) - x_k(t')| &= \left| x_0 + \int_0^{t-\frac{1}{k}} f(s, x_k(s))\, ds - x_0 \right| \\ &\le \int_0^{t-\frac{1}{k}} |f(s, x_k(s))|\, ds \\ &\le M\left(t - \frac{1}{k}\right) \\ &\le M\,|t - t'|\,. \end{aligned}$$

Finally if $\frac{1}{k} \le t' \le t \le 1$, then

$$\begin{aligned} |x_k(t) - x_k(t')| &= \left| x_0 + \int_0^{t-\frac{1}{k}} f(s, x_k(s))\, ds - \left(x_0 + \int_0^{t'-\frac{1}{k}} f(s, x_k(s))\, ds \right) \right| \\ &= \left| \int_{t'-\frac{1}{k}}^{t-\frac{1}{k}} f(s, x_k(s))\, ds \right| \\ &\le \int_{t'-\frac{1}{k}}^{t-\frac{1}{k}} |f(s, x_k(s))|\, ds \\ &\le M\left(t - \frac{1}{k} - \left(t' - \frac{1}{k}\right)\right) \\ &= M\,|t - t'|\,. \end{aligned}$$

Thus for every $\varepsilon > 0$, $\delta = \frac{\varepsilon}{2M}$ is sufficient to ensure that for all $k \in \mathbb{N}$ and $t, t' \in [0,1]$ with $|t - t'| < \delta$, we have $|x_k(t) - x_k(t')| \leq M \cdot \frac{\varepsilon}{2M} < \varepsilon$. We have thereby shown that $(x_k)_{k\in\mathbb{N}}$ is a uniformly bounded and equicontinuous sequence in $\mathcal{C}([0,1], \mathbb{R}^n)$.

By Arzelà-Ascoli, $(x_k)_{k\in\mathbb{N}}$ has a subsequence $(x_\ell)_{\ell\in\mathbb{N}}$ such that $x_\ell \rightrightarrows x$ for some $x \in \mathcal{C}([0,1], \mathbb{R}^n)$. We claim that the sequence $(F_\ell)_{\ell\in\mathbb{N}}$ in $\mathcal{C}([0,1], \mathbb{R}^n)$ defined by $F_\ell(s) = f(s, x_\ell(s))$ converges uniformly to F, defined by $F(s) = f(s, x(s))$.

$(x_\ell)_{\ell\in\mathbb{N}}$ is uniformly bounded by $|x_0| + M$. Naturally, so is x. Since f is continuous on $[0,1] \times \mathbb{R}^n$, it is uniformly continuous on the compact set

$$K = [0,1] \times [-(|x_0| + M), |x_0| + M]^n.$$

Given $\varepsilon > 0$, let $\delta > 0$ be such that for all $(s,x), (t,y) \in K$,

$$|(s,x) - (t,y)| < \delta \implies |f(s,x) - f(t,y)| < \varepsilon.$$

Using $x_\ell \rightrightarrows x$, let $L \in \mathbb{N}$ be sufficiently large that for all $\ell \geq L$ and $s \in [0,1]$, $|x_\ell(s) - x(s)| < \delta$. In particular $(s, x_\ell(s)), (s, x(s)) \in K$ are such that

$$|(s, x_\ell(s)) - (s, x(s))| < \delta,$$

and thus

$$|F_\ell(s) - F(s)| = |f(s, x_\ell(s)) - f(s, x(s))| < \varepsilon,$$

showing that $F_\ell \rightrightarrows F$. Now we may take the pointwise limit of x_ℓ to determine x :

$$\begin{aligned}
\lim_{\ell\to\infty} x_\ell(t) &= \lim_{\ell\to\infty} \left(x_0 + \int_0^t f(s, x_\ell(s))\, ds - \int_{t-\frac{1}{\ell}}^t f(s, x_\ell(s))\, ds \right) \\
&= x_0 + \lim_{\ell\to\infty} \int_0^t f(s, x_\ell(s))\, ds - \lim_{\ell\to\infty} \int_{t-\frac{1}{\ell}}^t f(s, x_\ell(s))\, ds.
\end{aligned}$$

We compute

$$\left| \int_{t-\frac{1}{\ell}}^t f(s, x_\ell(s))\, ds \right| \leq \int_{t-\frac{1}{\ell}}^t |f(s, x_\ell(s))|\, ds \leq \frac{M}{\ell} \to 0$$

as $\ell \to \infty$, and by 2.8

$$\begin{aligned}
\lim_{\ell\to\infty} \int_0^t f(s, x_\ell(s))\, ds &= \lim_{\ell\to\infty} \int_0^t F_\ell(s)\, ds \\
&= \int_0^t \lim_{\ell\to\infty} F_\ell(s)\, ds \\
&= \int_0^t F(s)\, ds \\
&= \int_0^t f(s, x(s))\, ds.
\end{aligned}$$

We conclude that

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds.$$

As $x(0) = x_0$, we have constructed a solution in $x : [0,1] \to \mathbb{R}^n$. □

4. Uniqueness, or a Lack Thereof

In general, the assumptions of the Cauchy-Peano theorem are insufficient to ensure uniqueness of the solution. For example, define $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,x) = \begin{cases} 2\sqrt{|x|} & \text{if } |x| \leq 1 \\ 2 & \text{if } |x| > 1. \end{cases}$$

f is clearly continuous and bounded by 2. Consider the following initial value problem:

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(0) = 0, \end{cases}$$

where $t \in [0,1]$. A trivial solution is $y(t) = 0$, since $y(0) = 0$ and

$$y'(t) = 0 = f(t,0) = f(t, y(t)).$$

Another solution is $z(t) = t^2$, since $z(0) = 0$ and $|z(t)| \leq 1$ on $[0,1]$, so

$$z'(t) = 2t = 2\sqrt{t^2} = 2\sqrt{|z(t)|} = f(t, z(t)).$$

Therefore the solution given by Cauchy-Peano is not unique.

References

1. C. Pugh. *Real Mathematical Analysis*. Springer, 2015.