WEAVING A WIDER NET: A FIRST COURSE IN TOPOLOGY

WILLIAM GAO

1. Introduction

This is an accessible introduction to topology with an emphasis on nets. We assume only a basic familiarity with sets, functions, and logic from the naive point of view. All this background may be recovered from [Munkres, Sections 1.1–1.2].

Readers already familiar with sequences should think of a nets as a generalization of sequences in which the indexing set, namely the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$, is replaced by any set with a sufficiently nice ordering.

Nets often admit a richer combinatorial structure than sequences, and as a result their properties align more closely with topological intuition. Topology, as viewed from the perspective of nets, is much more geometric and constructive than the traditional approach, which leans heavily on set theory. We hope this constitutes a more accessible first course as a result.

2. Relations and Partial Orders

In the introduction, we mentioned sets with "a sufficiently nice ordering". We begin by making this more precise.

Given a set A, recall that the Cartesian product

$$A \times A = \{(a_1, a_2) : a_1 \in A, a_2 \in A\}$$

of A with itself is the set of ordered pairs of elements in A. That is, (a_1, a_2) is not equal to (a_2, a_1) unless $a_1 = a_2$. If we take A to be $\mathbb{R}$ the set of real numbers, the Cartesian product $\mathbb{R} \times \mathbb{R}$, or more concisely $\mathbb{R}^2$, is the set of coordinates in the Cartesian plane.

Definition 2.1. A *relation* on A is a subset T of the Cartesian product $A \times A$.

This definition is fairly abstract, so let us unpack it. A relation seeks to capture a boolean outcome for each ordered pair of elements in A. Is a_1 equal to a_2? Is a_1 less than a_2? Is a_1 divisible by a_2? These are all True or False questions involving two elements of A with order, meaning that swapping the places of the two elements in the question may invert the answer. Now for any element $(a_1, a_2) \in A \times A$, exactly one of $(a_1, a_2) \in T$ or $(a_1, a_2) \notin T$ is true. This structure is perfect for storing our boolean outcomes. So given a reasonable enough question (no Berry paradoxes please!) in the form "is a_1 ______ a_2?", we can define T to be the set of ordered pairs of elements in A that answer this question in the affirmative. Then $(a_1, a_2) \in T$ is equivalent to the statement "a_1 is ______ a_2", and conversely $(a_1, a_2) \notin T$ is equivalent to the statement "a_1 is not ______ a_2".

Example 2.2. One subset of $A \times A$ we can always easily define is the 'diagonal'

$$\Delta = \{(a, a) : a \in A\}$$

consisting of those ordered pairs obtained by repeating the same element in A twice. If $a_1 \in A$ and $a_2 \in A$, we have $(a_1, a_2) \in \Delta$ exactly when $a_1 = a_2$. Thus this relation is the familiar "is equal to" relation.

If it is unclear how we thought of defining Δ to represent the question "Is a_1 equal to a_2?", we can also recover Δ from the question as follows: Let Δ be the subset of elements $(a_1, a_2) \in A \times A$ that answer "Is a_1 equal to a_2?" in the affirmative. Then $(a_1, a_2) \in \Delta$ if and only if a_1 is equal to a_2. We see this is the only restriction, so $\Delta = \{(a_1, a_1) : a_1 \in A\}$, and from here we can do without the subscript.

The more subtle relation "is less than or equal to", which we will be more interested in, is more difficult to come across in most sets. In general, we cannot explicitly construct this relation as we did for "is equal to". Instead, we abstract its properties with the following definition.

Definition 2.3. A *partial order relation* on A is a relation T which satisfies the following conditions:

(1) (reflexivity) $(\alpha, \alpha) \in T$ for each $\alpha \in A$.
(2) (antisymmetry) if $(\alpha, \beta) \in T$ and $(\beta, \alpha) \in T$, then $\alpha = \beta$.
(3) (transitivity) if $(\alpha, \beta) \in T$ and $(\beta, \gamma) \in T$, then $(\alpha, \gamma) \in T$.

In this case, we say A is partially ordered with respect to T. As a sanity check, we should verify that the $\leq$ relation we are trying to generalize actually satisfies these conditions.

Example 2.4. $\mathbb{R}$ is partially ordered by the relation $T = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$, or more explicitly $(x, y) \in T$ if and only if $x \leq y$. Indeed,

(1) $x \leq x$ for all $x \in \mathbb{R}$,
(2) if $x \leq y$ and $y \leq x$, then $x = y$, and
(3) if $x \leq y$ and $y \leq z$, then $x \leq z$.

The ensuing examples illustrate every way a relation can fail to be a partial order.

Example 2.5. The subset $\{(x, y) \in \mathbb{R}^2 : x < y\}$ of $\mathbb{R}^2$ defines a relation on $\mathbb{R}$, understood as "is less than". This is not a partial order relation because reflexivity fails; no real number is less than itself. Nonetheless it is antisymmetric (vacuously, the hypothesis $x < y$ and $y < x$ is never true) and transitive.

Example 2.6. $\mathbb{R}$ is not partially ordered by its relation $\{(x, y) \in \mathbb{R}^2 : x - y \in \mathbb{Z}\}$. This relation is reflexive as $x - x = 0 \in \mathbb{Z}$ for all $x \in \mathbb{R}$, transitive as $x - y \in \mathbb{Z}$ along with $y - z \in \mathbb{Z}$ implies $x - z = (x - y) + (y - z) \in \mathbb{Z}$, but not antisymmetric as $0, 1 \in \mathbb{R}$ satisfy $0 - 1 \in \mathbb{Z}$ and $1 - 0 \in \mathbb{Z}$, but $0 \neq 1$.

Example 2.7. The relation $\{(x, y) \in \mathbb{R}^2 : 0 \leq x - y \leq 1\}$ on $\mathbb{R}$ is not a partial order. It is reflexive as $x - x = 0$, and antisymmetric because $0 \leq x - y \leq 1$ along with $0 \leq y - x \leq 1$ implies $x - y = 0$. However this relation is not transitive: let $x = 3$, $y = 2$, $z = 1$. Then $x - y = 1$ and $y - z = 1$ show that the pairs (x, y) and (y, z) satisfy the relation, but $x - z = 2$ shows that (x, z) does not.

With our newfound appreciation for the three properties of a partial order, we provide some examples that will recur.

Example 2.8. Let S be any set whose elements are sets. A natural partial order relation on S is $\{(A, B) \in S \times S : A \supseteq B\}$, or colloquially "is a superset of".

(1) $A \supseteq A$ is true for all sets A,
(2) if $A \supseteq B$ and $B \supseteq A$, then $A = B$, and
(3) if $A \supseteq B$ and $B \supseteq C$, then $A \supseteq C$.

Unlike the $\leq$ partial order on $\mathbb{R}$, the $\supseteq$ relation permits the possibility that neither $A \supseteq B$ nor $B \supseteq A$ is true. In this case, we say A and B are not comparable. For example, the sets $\{1, 2, 3\}$ and $\{a, b, c\}$ are not comparable.

Henceforth, when we consider a partially ordered set without specifying the partial order relation T, we will use the shorthand $\alpha \preceq \beta$ to denote $(\alpha, \beta) \in T$, and we will interchangeably refer to $\preceq$ or T as the partial order.

Example 2.9. Let D be the set of words in the Oxford English Dictionary. Given two words $w_1 \in D$ and $w_2 \in D$, let $w_1 \preceq w_2$ precisely when w_1 appears no later than w_2 in the Oxford English Dictionary. For example, "continuity" $\preceq$ "convergence" and "net" $\preceq$ "sequence". This is reflexive, antisymmetric, and transitive by definition.

The following exercises may help consolidate readers' understanding of relations and partial orders.

Exercise 2.10. Give an example of a set with a relation that is:
(a) reflexive but neither antisymmetric nor transitive.
(b) antisymmetric but neither reflexive nor transitive.
(c) transitive but neither reflexive nor antisymmetric.
(d) not transitive, reflexive, nor antisymmetric.

Exercise 2.11. Define a relation $\preceq$ on $\mathbb{N}$ by $a_1 \preceq a_2$ if and only if a_1 divides a_2. Show that $\preceq$ is a partial order relation.

Exercise 2.12. Let A be a partially ordered set under $\preceq_A$ and B a partially ordered set under $\preceq_B$. Define a relation $\preceq$ on $A \times B$ by $(a_1, b_1) \preceq (a_2, b_2)$ if and only if $a_1 \preceq_A a_2$ and $b_1 \preceq_B b_2$. Show that $\preceq$ is partial order relation.

Exercise 2.13. Let A be any set, B a partially ordered set under $\preceq$, and $f : A \to B$ an injective function. (More specifically, if $a_1 \in A$ and $a_2 \in A$ are such that $f(a_1) = f(a_2)$, then $a_1 = a_2$.) Define a relation $\preceq_f$ on A by $a_1 \preceq_f a_2$ whenever $f(a_1) \preceq f(a_2)$ in B. Show that $\preceq_f$ is a partial order relation.

3. Directed Sets and Cofinal Subsets

To discuss convergence, a mere partial order relation will not suffice. In particular, we saw that a partially ordered set may contain elements that are not comparable, which renders it impossible to arrange a set of points to approach a unified limit. Yet demanding that each pair of elements be comparable is too strong a requirement. The following definition strikes a compromise.

Definition 3.1. A *directed set* J is a set with a partial order $\preceq$ such that for any elements $\alpha \in J$ and β of J, there exists an element $\gamma \in J$ which satisfies $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

This condition allows us to circumvent the difficulty of comparing α and β by passing through γ instead.

For visual intuition, readers should recall genealogy: consider the set of all humans plus God, partially ordered by $\alpha \preceq \beta$ if β is an ancestor of α. Perhaps unusually, we must require each person to be their own ancestor to respect reflexivity. God is an ancestor of every person. With this structure, we can draw a genealogical tree to picture the $\preceq$ relations. In particular, given any two persons α and β, we may find a common ancestor γ. For instance if α represents Abel and β represents Cain, we may take γ to be Adam, or Eve, or God.

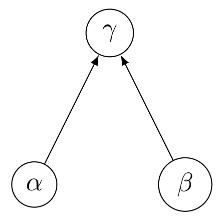

Figure 3.1. The directed set property, visualized as a tree. The relation $\alpha \preceq \gamma$ is represented by an arrow from α to γ.

We saw in Example 2.4 that $\mathbb{R}$ is partially ordered by the $\leq$ relation. We now show that it is moreover directed.

Example 3.2. $\mathbb{R}$ is a directed set with the $\leq$ partial order. Indeed, given any real numbers α and β, let $\gamma = \max\{\alpha, \beta\}$. Then

$$\alpha \leq \max\{\alpha, \beta\} = \gamma, \text{ and}$$
$$\beta \leq \max\{\alpha, \beta\} = \gamma.$$

This is exactly what we require in the definition of a directed set.

From this example, it is clear that the choice of γ is not unique; any real number even larger than $\max\{\alpha, \beta\}$ would work equally well.

Unfortunately, the case of Example 2.8 is not as simple.

Example 3.3. Let S be any set whose elements are sets, partially ordered by the $\supseteq$ relation. In general, S is not directed. For example, let $S = \{\{1, 2, 3\}, \{a, b, c\}\}$. Let $\alpha = \{1, 2, 3\}$ and $\beta = \{a, b, c\}$. Then there is no $\gamma \in S$ such that $\alpha \supseteq \gamma$ and $\beta \supseteq \gamma$, so S is not directed.

Example 3.4. Let S be as in the previous example, with the additional property that for all elements $A \in S$ and $B \in S$, the set intersection $A \cap B$ is also an element in S. Then S is directed. Indeed, given $\alpha \in S$ and $\beta \in S$, let $\gamma = \alpha \cap \beta$. By basic set theory, $\alpha \supseteq \alpha \cap \beta = \gamma$ and $\beta \supseteq \alpha \cap \beta = \gamma$, as desired.

Following a pattern that is all too common in mathematics, we seek subsets of directed sets which are themselves directed. The following definition will become indispensable.

Definition 3.5. A subset K of a directed set J under $\preceq$ is *cofinal* in J if for each $\alpha \in J$, there exists $\beta \in K$ such that $\alpha \preceq \beta$ as elements of J.

In words, we can find an element in K (weakly) above any element in J.

Example 3.6. The set $\mathbb{N}$ of natural numbers is cofinal in $\mathbb{R}$, directed by $\leq$. This is known as the Archimedean property, and often stated as an axiom in first-year calculus courses.

Example 3.7. The set P of prime numbers is cofinal in $\mathbb{R}$, directed by $\leq$ [Euclid, Book IX, Proposition 20].

Example 3.8. The set $[0, 1]$ of real numbers between 0 and 1 (inclusive) is not cofinal in $\mathbb{R}$ directed by $\leq$. Namely for the element $2 \in \mathbb{R}$, there exists no $\beta \in [0, 1]$ such that $2 \leq \beta$.

Our first proper result states being cofinal is sufficient for inheriting directedness.

Lemma 3.9. *If J is a directed set under $\preceq$ and K is cofinal in J, then K is a directed set under $\preceq$.*

Proof. Given elements $\alpha \in K$ and $\beta \in K$, since K is a subset of J we have $\alpha \in J$ and $\beta \in J$. Since J is directed, there exists $\gamma \in J$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Since K is cofinal, there exists $\delta \in K$ such that $\gamma \preceq \delta$. By transitivity of the partial order, $\alpha \preceq \delta$ and $\beta \preceq \delta$. We have thus shown that K is a directed set under $\preceq$. $\square$

Example 3.10. We saw that the set of natural numbers $\mathbb{N}$ is cofinal in $\mathbb{R}$ with $\leq$, so $\mathbb{N}$ is a directed set under $\leq$.

Example 3.11. The set P of prime numbers is cofinal in $\mathbb{N}$ with $\leq$, so P is a directed set under $\leq$.

Example 3.12. Although $[0, 1]$ is not cofinal in $\mathbb{R}$ under $\leq$, it is still a directed set under $\leq$. In particular, for any $\alpha \in [0, 1]$ and $\beta \in [0, 1]$ we can take $1 \in [0, 1]$ to ensure that $\alpha \leq 1$ and $\beta \leq 1$. This shows that the converse of Lemma 3.9 is false: if J is directed by $\preceq$ and $K \subseteq J$ is directed by $\preceq$, it is not necessarily true that K is cofinal in J.

Henceforth, J will denote a directed set under an unspecified partial order $\preceq$.

Exercise 3.13. Let A be any set partially ordered by $\preceq$ and let $\gamma \in A$ be such that $\alpha \preceq \gamma$ for all $\alpha \in A$. Show that:

(a) $\gamma \in A$ is the unique element of A satisfying this property.
(b) A is directed by $\preceq$.
(c) $B \subseteq A$ is cofinal in A if and only if $\gamma \in B$.

Exercise 3.14. Give an example of a set J with two possible partial order relations $\preceq_1$ and $\preceq_2$. Give an example of a subset $K \subseteq J$ that is cofinal with respect to $\preceq_1$ but not $\preceq_2$.

Exercise 3.15. Let J be a directed set under $\preceq$. Show that for any $\alpha \in J$, $\beta \in J$, and $\gamma \in J$, there exists $\delta \in J$ such that $\alpha \preceq \delta$, $\beta \preceq \delta$, and $\gamma \preceq \delta$. Is the converse true? (Namely, if this three-element condition holds, is J directed? *Hint: there is no requirement that α, β, and γ be distinct.*)

Exercise 3.16. For any $n \in \mathbb{N}$, generalize the previous exercise to an equivalent condition on n elements in J.

4. Bases and Open Sets

Let X be a set. We would like to associate to X a structure that will provide a nice framework for understanding locality, which will lead to the fundamental concepts of continuity, sequences, and most importantly to us, nets. We will characterize locality by designating an interesting set of subsets of X to be 'open'.

Recall that the power set $\mathscr{P}(X)$ of X is the set of all subsets of X. For example if $X = \{a, b, c\}$ then

$$\mathscr{P}(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

With this, a set of subsets of X is simply a subset of $\mathscr{P}(X)$.

Definition 4.1. A subset $\mathscr{B}$ of $\mathscr{P}(X)$ is a *basis* if:

(1) for each $x \in X$, there is at least one $B \in \mathscr{B}$ such that $x \in B$.
(2) if $x \in X$, $B_1 \in \mathscr{B}$, and $B_2 \in \mathscr{B}$ are such that $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

A set X with a basis $\mathscr{B}$ is called a *topological space*. The elements of $\mathscr{B}$ are called basic open sets in X.

In more relaxed language, (1) the basic open sets entirely cover X, and (2) if two basic open sets overlap, then we can zoom in to a smaller basic open set contained in their overlap (see Figure 4.1).

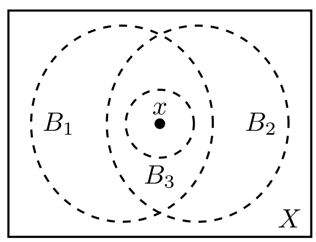

FIGURE 4.1. The intersection property of basis elements.

The above figure introduces many visual conventions of topology. The topological space X is drawn as a solid rectangle, elements of X are drawn as points, and basic open sets are drawn as dotted ellipses.

Example 4.2. Let $X = \{a, b, c\}$. Then $\mathscr{B}_1 = \{\{b\}, \{a, b\}, \{b, c\}\}$ is a basis. The covering condition (1) is clear, and the intersection property (2) is readily verified:

$$\begin{aligned} \{b\} \cap \{a, b\} &= \{b\} \\ \{b\} \cap \{b, c\} &= \{b\} \\ \{a, b\} \cap \{b, c\} &= \{b\}. \end{aligned}$$

These equations show that each intersection of two basic open sets is again a basic open set, namely $\{b\}$. In the scenario of Definition 4.1(2), for any $B_1 \in \mathscr{B}$ and $B_2 \in \mathscr{B}$ the only possible $x \in X$ is $x = b$, and in all three cases $B_3 = \{b\}$ satisfies $b \in \{b\}$ and $\{b\} \subseteq B_1 \cap B_2$.

Example 4.3. Again let $X = \{a, b, c\}$. The subset $\mathscr{B}_2 = \{\{a\}, \{b\}\}$ of $\mathscr{P}(X)$ is not a basis. Although it vacuously satisfies (2), it does not cover X as there is no basic open set containing $c \in X$.

Example 4.4. In our final example with $X = \{a, b, c\}$, we show that a subset of $\mathscr{P}(X)$ may satisfy (1) but not (2). Let $\mathscr{B}_3 = \{\{a, b\}, \{b, c\}\}$. (1) is readily verified, but (2) fails: $b \in X$, $\{a, b\} \in \mathscr{B}_2$, and $\{b, c\} \in \mathscr{B}_2$ are such that

$$b \in \{b\} = \{a, b\} \cap \{b, c\}.$$

However there is no basis element $B_3 \in \mathscr{B}_3$ such that $B_3 \subseteq \{b\}$.

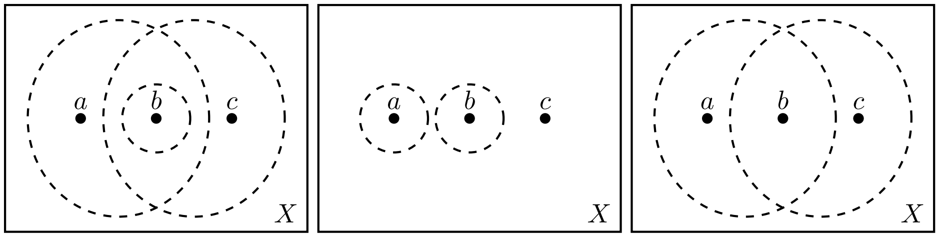

FIGURE 4.2. Example 4.2 (left), Example 4.3 (center), Example 4.4 (right).

The above examples (see Figure 4.2) where X is a small finite set are illustrative because we can see every possible subset of X. However, the next examples possess richer structures.

Example 4.5. Let $X = \mathbb{R}$. The collection of all open intervals (a, b), where a and b are real numbers such that $a < b$, is a basis, called the standard basis. Indeed (1) for each $x \in \mathbb{R}$, we have an open interval $(x-1, x+1)$ around x, and (2) if $x \in \mathbb{R}$ is contained in two open intervals (a, b) and (c, d), then $a < x < b$ and $c < x < d$. Together, this gives $\max\{a, c\} < x < \min\{b, d\}$, so $x \in (\max\{a, c\}, \min\{b, d\})$. Moreover

$$(\max\{a, c\}, \min\{b, d\}) \subseteq (a, b) \cap (c, d).$$

Indeed, if $y \in (\max\{a, c\}, \min\{b, d\})$, so $a \leq \max\{a, c\} < y$ and $y < \min\{b, d\} \leq b$, showing that $y \in (a, b)$. Similarly, $y \in (c, d)$. We have thus found a basic open set $(\max\{a, c\}, \min\{b, d\})$ containing x and contained in the intersection of the original two, so we have fulfilled (2).

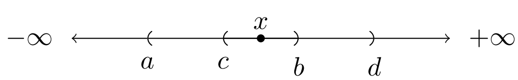

FIGURE 4.3. The intersection property in Example 4.5. In the case where $a < c < b < d$, the desired subinterval is (c, b).

To generalize this example to higher-dimensional Euclidean spaces $\mathbb{R}^n$, we will need a notion of distance.

Definition 4.6. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The *distance* between x and y is

$$|x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Lemma 4.7. *$|x - y|$ satisfies the following properties, for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$:*

(1) *(positive definiteness) $|x - y| \geq 0$ with equality if and only if $x = y$.*
(2) *(symmetry) $|x - y| = |y - x|$.*
(3) *(triangle inequality) $|x - y| + |y - z| \geq |x - z|$ for all $z \in \mathbb{R}^n$.*

Proof. The first two properties are straightforward computations, left as an exercise for the reader. The triangle inequality requires some cleverness; see [Axler, Theorem 6.17]. $\square$

These three properties carry extremely intuitive geometric interpretations, namely:

(1) distances are never negative, and the distance between two points is zero if and only if they are the exact same point.
(2) distance is intrinsic, or unchanged by swapping the start and end points.
(3) the shortest distance between two points is the straight line distance.

In fact, these properties determine a more general notion of distance, called a metric, in topology.

Example 4.8. In $X = \mathbb{R}^n$, the collection of all open balls $B(a, \varepsilon)$, defined as

$$B(a, \varepsilon) = \{x \in \mathbb{R}^n : |x - a| < \varepsilon\}$$

for any point $a \in \mathbb{R}^n$ and any $\varepsilon > 0$, is a basis, called the standard basis. We have (1) any $x \in \mathbb{R}^n$ is contained in $B(x, 1)$ for example, and (2) suppose $x \in B(a_1, \varepsilon_1) \cap B(a_2, \varepsilon_2)$. By definition, $|x - a_1| < \varepsilon_1$ and $|x - a_2| < \varepsilon_2$. Take $\delta = \min\{\varepsilon_1 - |x - a_1|, \varepsilon_2 - |x - a_2|\}$. We claim that $B(x, \delta) \subseteq B(a_1, \varepsilon_1) \cap B(a_2, \varepsilon_2)$. Indeed, if $y \in B(x, \delta)$, then

$$|x - y| < \delta \leq \varepsilon_1 - |x - a_1|$$
$$|x - y| < \delta \leq \varepsilon_2 - |x - a_2|.$$

Rearranging the inequalities via symmetry,

$$|y - x| + |x - a_1| < \varepsilon_1$$
$$|y - x| + |x - a_2| < \varepsilon_2.$$

By the triangle inequality,

$$|y - a_1| \leq |y - x| + |x - a_1| < \varepsilon_1$$
$$|y - a_2| \leq |y - x| + |x - a_2| < \varepsilon_2.$$

So $y \in B(a_1, \varepsilon_1) \cap B(a_2, \varepsilon_2)$. We have thus found a basic open set $B(x, \delta)$ such that $x \in B(x, \delta) \subseteq B(a_1, \varepsilon_1) \cap B(a_2, \varepsilon_2)$, so (2) holds.

Definition 4.9. Let X be a topological space with basis $\mathscr{B}$, and $x \in X$. We denote by $\mathscr{B}_x$ the set of basic open sets $B \in \mathscr{B}$ such that $x \in B$.

The elements of $\mathscr{B}_x$ are called basic neighbourhoods of x in X.

Lemma 4.10. *$\mathscr{B}_x$ is a directed set under the partial order relation where $B_1 \preceq B_2$ if and only if $B_1 \supseteq B_2$.*

Proof. The fact that $\supseteq$ defines a partial order was previously shown in Example 2.8. Showing that $\mathscr{B}_x$ is a directed set under this partial order relation is a straightforward application of the intersection property of a basis, left as an exercise for the reader. □

To define a topology on X, it suffices to specify a basis $\mathscr{B}$. However, we will eventually need the following more general construction.

Definition 4.11. Let X be a topological space with basis $\mathscr{B}$. A subset U of X is an *open set* in X if it is an arbitrary union of basic open sets. More precisely, for any $\mathscr{C} \subseteq \mathscr{B}$, the subset $U = \bigcup_{B \in \mathscr{C}} B$ of X is an open set in X.

In analogy with basic open sets, if U is an open set and $x \in U$, we call U a *neighbourhood* of x.

If X is a topological space with basis $\mathscr{B}$, then the *topology* on X is the set of all open sets in X.

Example 4.12. Let $X = \{a, b, c\}$ with the basis $\mathscr{B} = \{\{b\}, \{a, b\}, \{b, c\}\}$ as in Example 4.2. We compute

$$\begin{aligned}\{b\} \cup \{a, b\} &= \{a, b\}\\ \{b\} \cup \{b, c\} &= \{b, c\}\\ \{a, b\} \cup \{b, c\} &= \{a, b, c\}\\ \{b\} \cup \{a, b\} \cup \{b, c\} &= \{a, b, c\}.\end{aligned}$$

Additionally, $\varnothing$ is an open set, by taking $\mathscr{C} = \varnothing$ in Definition 4.9, and each basic open set is open by taking $\mathscr{C}$ to be the corresponding one-element subset of $\mathscr{B}$. Thus the topology on X induced by $\mathscr{B}$ is

$$\{\varnothing, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}.$$

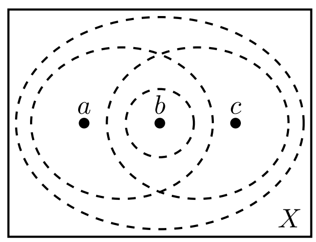

FIGURE 4.4. Example 4.10.

In this example, the only open sets which were not basic open sets are $\varnothing$ and X itself. This is no coincidence.

Example 4.13. $\varnothing$ is open in any topological space X. Indeed, we can always take $\mathscr{C}$ to be the empty subset of $\mathscr{B}$ in Definition 4.9.

Example 4.14. X itself is open in any topological space X. Indeed, we can always take $\mathscr{C}$ to be the entirety of $\mathscr{B}$ in Definition. Then $\bigcup_{B \in \mathscr{B}} B = X$ by property (1) of a basis.

Example 4.15. Every basic open set $B \in \mathscr{B}$ is an open set in any topological space X. Indeed, we can always take $\mathscr{C} = \{B\} \subseteq \mathscr{B}$.

Example 4.16. Let $X = \mathbb{R}$ be a topological space with the standard basis consisting of open intervals. By taking unions of these open intervals, we get many open sets which are not basic open sets, such as $(0, 1) \cup (1, 2)$, $\mathbb{R} \setminus \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n + 1)$, $(-\infty, 0) = \bigcup_{x<0} (x, 0)$.

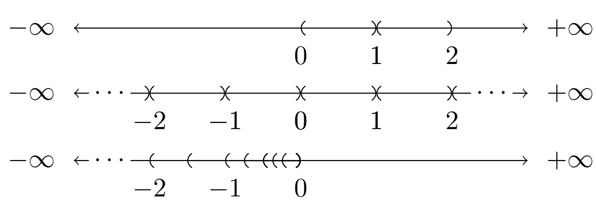

FIGURE 4.5. Open sets in $\mathbb{R}$ from Example 4.14: $(0, 1) \cup (1, 2)$ (top), $\mathbb{R} \setminus \mathbb{Z}$ (middle), $(-\infty, 0)$ (bottom).

This example shows that the "arbitrary union" of basic open sets in Definition 4.9 may be finite, countably infinite, or uncountably infinite.

Example 4.17. As a special case of Example 4.8, let $X = \mathbb{R}^2$ with the standard basis consisting of open balls (drawn as dotted circles in the plane). We can construct even more irregularly-shaped open sets than in one-dimension, such as two balls almost touching uniquely at the origin $B((1,0),1) \cup B((-1,0),1)$, a tube around the x-axis $\bigcup_{x\in\mathbb{R}} B((x,0),1)$, or a small annulus around the unit circle $\bigcup_{\theta\in[0,2\pi]} B((\cos\theta,\sin\theta),0.1)$.

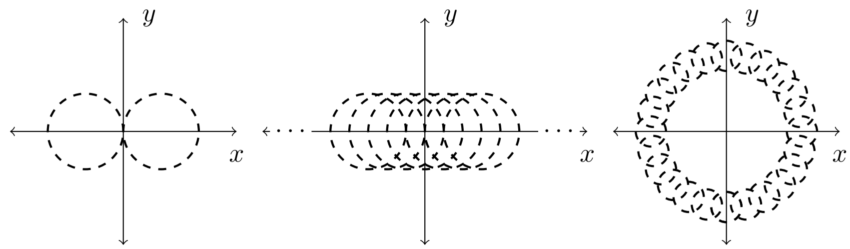

FIGURE 4.6. Open sets in $\mathbb{R}^2$ from Example 4.15: two balls (left), a tube (middle), and an annulus (bottom).

Although these open sets are already quite intricate, they possess a very simple description as a union of basic open sets. There will be open sets where it is much less clear how to express it as a union of basic open sets, and so the following criterion will be invaluable.

Lemma 4.18. *A subset U of X is open if and only if for each $x \in U$, there exists a basic neighbourhood $B_x \in \mathscr{B}_x$ of x such that $B_x \subseteq U$.*

Proof. ($\Longrightarrow$) Let $U = \bigcup_{B\in\mathscr{C}} B$ be an arbitrary open set. Given any $x \in U$, let $B_x \in \mathscr{C}$ be such that $x \in B_x$. Then $B_x \in \mathscr{B}_x$, and of course $B_x \subseteq \bigcup_{B\in\mathscr{C}} B = U$.

($\Longleftarrow$) Suppose for each $x \in U$, there exists a basis neighbourhood B_x of x such that $B_x \subseteq U$. We claim that $U = \bigcup_{x\in U} B_x$. Indeed, for any $x \in U$, we have $x \in B_x$, so $x \in \bigcup_{x\in U} B_x$. Conversely, $B_x \subseteq U$ for each $x \in U$ by definition, so the union $\bigcup_{x\in U} B_x$ is contained in U. Therefore $U = \bigcup_{x\in U} B_x$ is a union of basic open sets, so U is open. $\square$

With this tool we can identify even more unexpected open sets.

Example 4.19. Let $X = \mathbb{R}^2$ in the standard basis. Then the Cartesian product

$$(a,b)\times(c,d) = \{(x,y)\in\mathbb{R}^2 : x\in(a,b), y\in(c,d)\}$$

of two open intervals in $\mathbb{R}$ is an open set. This may be visualized as an open-boundary rectangle in the plane (see Figure 4.7). Indeed given $(x,y)\in(a,b)\times(c,d)$, let

$$\varepsilon = \min\{x-a, b-x, y-c, d-y\}.$$

Since $a < x < b$ and $c < y < d$, we know that $\varepsilon > 0$, so $(x,y) \in B((x,y),\varepsilon)$. We claim that $B((x,y),\varepsilon) \subseteq (a,b)\times(c,d)$. The intuition behind this choice of ε is that ε is the perpendicular distance from (x,y) to the nearest side of the rectangle. A ball of this radius should just touch this nearest side, but it will never extend beyond the rectangle, which is exactly the form of basic open set we desire.

More formally if $(z, w) \in B((x, y), \varepsilon)$, then $|(z, w) - (x, y)| < \varepsilon$. That is,

$$\sqrt{(z-x)^2 + (w-y)^2} < \min\{x-a, b-x, y-c, d-y\}.$$

We wish to show that $a < z$, $z < b$, $c < w$, and $w < d$. These all follow from the above equation:

$$\begin{aligned}
x - z \leq |x-z| &\leq \sqrt{(z-x)^2 + (w-y)^2} < \min\{x-a, b-x, y-c, d-y\} \leq x-a \\
z - x \leq |z-x| &\leq \sqrt{(z-x)^2 + (w-y)^2} < \min\{x-a, b-x, y-c, d-y\} \leq b-x \\
y - w \leq |y-w| &\leq \sqrt{(z-x)^2 + (w-y)^2} < \min\{x-a, b-x, y-c, d-y\} \leq y-c \\
w - y \leq |w-y| &\leq \sqrt{(z-x)^2 + (w-y)^2} < \min\{x-a, b-x, y-c, d-y\} \leq d-y.
\end{aligned}$$

Removing the x's and y's, we conclude that $a < z < b$ and $c < w < d$, so $(z, w) \in (a, b) \times (c, d)$ and thus $B((x, y), \varepsilon) \subseteq (a, b) \times (c, d)$, as desired. Since $(x, y) \in (a, b) \times (c, d)$ was arbitrary, we have shown that $(a, b) \times (c, d)$ is open by Lemma 4.18.

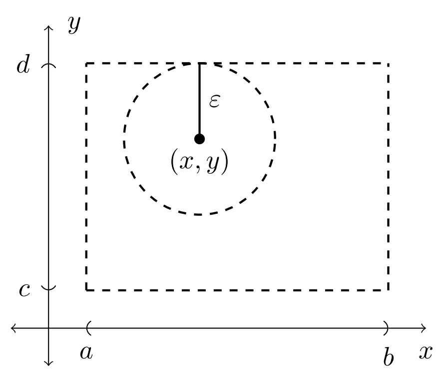

FIGURE 4.7. The open set $(a, b) \times (c, d)$, with a permissibly small open ball around an arbitrary point (x, y).

As we can see from the ($\Longleftarrow$) direction of the proof of Lemma 4.18, we can express $(a, b) \times (c, d)$ as the union of all open balls $B((x, y), \varepsilon)$ where $(x, y) \in (a, b) \times (c, d)$ and ε is chosen as above depending on (x, y). This would be extremely difficult to conclude without first passing through Lemma 4.18. We will almost always use the more wieldy criterion provided by this lemma when determining open sets.

From Definition 4.11, it seems very difficult to decide which sets are *not* open, but the lemma saves us once again.

Remark 4.20. The logical negation of Lemma 4.18 follows. $U \subseteq X$ is *not* open if *there exists* $x \in U$ such that *for every* basic neighbourhood B_x of x, B_x is *not contained* in U.

Example 4.21. Let $X = \mathbb{R}$ in the standard topology. Then $A = [0, 1]$ is *not* an open set. Indeed, we deploy Remark 4.20 at the point $0 \in A$. Any basic neighbourhood of 0 is an open interval containing 0. We may write this in the form $(-a, b)$ where $a > 0$ and $b > 0$. Then $-\frac{a}{2} \in (-a, b)$ but $-\frac{a}{2} < 0$, showing that $(-a, b)$ is not a subset of $[0, 1]$. Therefore A must not be open.

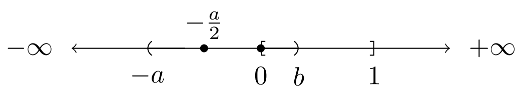

FIGURE 4.8. A neighbourhood $(-a, b)$ of 0 must not be contained in $[0, 1]$.

We can easily generalize our argument to any closed interval $[a, b]$, and furthermore any closed ball or closed rectangle

$$\overline{B}(a, \varepsilon) = \{x \in \mathbb{R}^n : |x - a| \leq \varepsilon\} \text{ or } [a_1, b_1] \times \cdots \times [a_n, b_n]$$

in $\mathbb{R}^n$. This vindicates our hope that open sets in Euclidean spaces $\mathbb{R}^n$ are exactly the subsets with open boundaries; that is, sets defined by strict ($<$ or $>$) inequalities rather than weak ($\leq$ or $\geq$) ones.

As with basic open sets, if $x \in X$ and U is an open set containing x, we say U is a neighbourhood of x.

Recall that open sets are intended as a template for discussing 'locality'. Though it is not always possible with our definition of a topology, it is often desirable to localize our search to a single point. We formalize this additional condition, and then explore its possibility in various topological spaces we have seen thus far.

As a reminder, we say two sets A and B are *disjoint* if $A \cap B = \varnothing$. If $A \cap B$ is nonempty, we say A and B *intersect*.

Definition 4.22. A topological space X is *Hausdorff* if for each pair of distinct points $x \in X$ and $y \in X$, there exist disjoint neighbourhoods U and V of x and y, respectively.

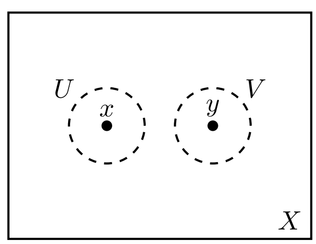

FIGURE 4.9. The Hausdorff condition must be satisfied for any pair of distinct points x and y.

Example 4.23. $X = \mathbb{R}$ with the standard topology is Hausdorff. Indeed, let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be distinct. Without loss of generality, we may assume that $x < y$. Since $\mathbb{R}$ is a continuum, there exists $z \in (x, y)$. Define neighbourhoods $U = (x - 1, z)$ and $V = (z, y + 1)$ of x and y, respectively. It is immediate from this construction that U and V are open (in fact basic open) sets, $x \in U$ and $y \in V$, and lastly $U \cap V = \varnothing$. Since x and y were arbitrary, the Hausdorff condition is satisfied.

To decide when the Hausdorff condition fails, we will need the logical negation.

Remark 4.24. X is *not* Hausdorff if *there exists* a pair of distinct points $x \in X$ and $y \in X$ such that *for any* neighbourhoods U of x and V of y, $U \cap V$ is *nonempty*.

Even better, we claim that it suffices to show that for any *basic* neighbourhoods B_x of x and B_y of y, $B_x \cap B_y$ is nonempty. More rigorously, if there exist no disjoint *basic* neighbourhoods of x and y, then there exist no disjoint neighbourhoods of x and y. We prove the contrapositive: if there exist disjoint neighbourhoods U of x and V of y, then there exist disjoint *basic* neighbourhoods B_x of x and B_y of y.

Let U be a neighbourhood of x and V a neighbourhood of y, such that $U \cap V = \varnothing$. By Lemma 4.18 we may find basic neighbourhoods B_x of x and B_y of y such that $B_x \subseteq U$ and $B_y \subseteq V$. Then $B_x \cap B_y \subseteq U \cap V = \varnothing$, so there exist disjoint *basic* neighbourhoods of x and y.

Example 4.25. $X = \{a, b, c\}$ with the basis $\mathscr{B} = \{\{b\}, \{a, b\}, \{b, c\}\}$ is not Hausdorff. To see this, consider the distinct points a and b in X.

By the remark above, we only need to check basic neighbourhoods of a and b. The only basic neighbourhood of a is $\{a, b\}$. However, this contains b, so it will certainly intersect every basic neighbourhood of b. No disjoint basic neighbourhoods exist, so by the remark, no disjoint neighbourhoods exist, meaning X is not Hausdorff.

Henceforth, X will denote a topological space with basis $\mathscr{B}$. If $X = \mathbb{R}$ or $\mathbb{R}^n$, we will always use the standard topology.

Exercise 4.26. Let $X = \mathbb{R}$. In this exercise we give a complete description of a topology on $\mathbb{R}$ different from the standard topology, called the *lower limit* topology. Let $\mathscr{B}$ denote the set of all half-open intervals $[a, b)$, where a and b are real numbers such that $a < b$.

(a) Show that $\mathscr{B}$ is a basis. We call the topology generated by $\mathscr{B}$ the lower limit topology.
(b) Show that every open set in the standard topology is also open in the lower limit topology. (*Hint: it suffices to show this for basic open sets in the standard topology. Why?*)
(c) Provide an example of an open set in the lower limit topology that is not open in the standard topology.
(d) Show that the lower limit topology is Hausdorff.

Exercise 4.27. We describe another different topology on $\mathbb{R}$, called the *finite complement* topology. Let $\mathscr{A} = \{\mathbb{R} \setminus \{x\} : x \in \mathbb{R}\}$ and $\mathscr{B} = \{\mathbb{R} \setminus \{x_1, \ldots, x_n\} : n \in \mathbb{N}, x_1 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}$.

(a) Is $\mathscr{A}$ a basis on $\mathbb{R}$?
(b) Show that $\mathscr{B}$ is a basis on $\mathbb{R}$. We call the topology on $\mathbb{R}$ generated by $\mathscr{B}$ the finite complement topology.
(c) Show that every open set in the finite complement topology is also open in the standard topology.
(d) Provide an example of an open set in the standard topology that is not open in the finite complement topology.
(e) Show that the finite complement topology is not Hausdorff.

Exercise 4.28. Let $\{U_\alpha\}_{\alpha \in I}$ be a family of open sets in X.

(a) Show that $\bigcup_{\alpha \in I} U_\alpha$ is open.
(b) Is $\bigcap_{\alpha \in I} U_\alpha$ necessarily open? What if I is finite?

Exercise 4.29. Is $\{(x, y) \in \mathbb{R}^2 : \sin(xy) \in (0, 1)\}$ an open subset of $\mathbb{R}^2$? (*Feel free to skip some details for now and revisit this exercise after Theorem 7.10.*)

5. Nets

In this section, we turn our attention to nets. We start with sequences, then broaden our scope.

Intuitively, a sequence can be considered a set of points in a topological space numbered by the natural numbers $\mathbb{N}$, say $\{x_1, x_2, x_3, \dots\}$. This numbering acts as an ordering on this set of points, by comparing two points according to their numbers, via the partial order $\leq$ on $\mathbb{N}$. More formally, we define it as a function.

Definition 5.1. A sequence of points in X is a function $x_\bullet : \mathbb{N} \to X$.

We denote the output of this function when evaluated at a number $n \in \mathbb{N}$ by x_n, rather than the usual function notation $x_\bullet(n)$. To emphasize its duality as both a function into X and a subset of X, we represent a sequence by the concise and special notation $(x_n)_{n\in\mathbb{N}}$.

To refine the remark at the beginning of this section, we can make specifications such as "the set of all points x_m in the sequence $(x_n)_{n\in\mathbb{N}}$ such that $N \leq m$", where $N \in \mathbb{N}$ is some fixed threshold.

The natural numbers $\mathbb{N}$ are just one example of a directed set. Adopting this generalization, we obtain the following:

Definition 5.2. A *J-net* of points in X is a function $x_\bullet : J \to X$.

As with sequences, we denote the output of a J-net when evaluated at an index $\alpha \in J$ by x_α, and we denote the entire J-net by $(x_\alpha)_{\alpha\in J}$.

Example 5.3. Taking $J = \mathbb{N}$, we see that sequences are exactly $\mathbb{N}$-nets.

As with sequences, the set J, directed by $\preceq$, induces a notion of order on a J-net; we can consider "the set of all points x_β in the J-net $(x_\alpha)_{\alpha\in J}$ such that $\gamma \preceq \beta$", where $\gamma \in J$ is given.

When discussing J-nets for arbitrary indexing sets J, we simply call them nets.

Definition 5.4. A net $(x_\alpha)_{\alpha\in J}$ of points in X *converges* to a point $x \in X$ if for every neighbourhood U of x, there exists $\alpha \in J$ such that for all $\beta \in J$ with $\alpha \preceq \beta$, we have $x_\beta \in U$.

In words, every neighbourhood of x has a threshold $\alpha \in J$, such that any further point (x_β where $\alpha \preceq \beta$) along the net must remain within the neighbourhood.

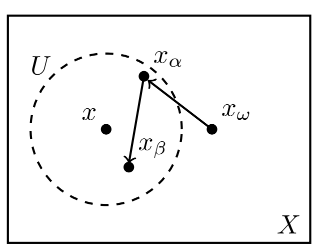

Figure 5.1. A net $(x_\alpha)_{\alpha\in J}$ converging to $x \in X$. The arrows signify that $\omega \preceq \alpha$ and $\alpha \preceq \beta$.

In this case, we call x the limit of $(x_\alpha)_{\alpha \in J}$, and we interchangeably write $x_\alpha \to x$, $\lim_{\alpha \in J} x_\alpha = x$, and $\lim_{\alpha \to \infty} x_\alpha = x$.

Example 5.5. Consider $J = \mathbb{N}$ as a directed set under $\leq$ and $X = \mathbb{R}$ as a topological space with the standard basis. Define an $\mathbb{N}$-net of points in $\mathbb{R}$ by $(\frac{1}{n})_{n \in \mathbb{N}}$, which we may alternatively write as the function $x_\bullet : \mathbb{N} \to \mathbb{R}$ given by $x_n = \frac{1}{n}$.

This $\mathbb{N}$-net converges to $x = 0$, because $\frac{1}{n}$ grows arbitrarily close to 0 as n grows arbitrarily large. More formally, let U be an arbitrary neighbourhood of 0. By Lemma 4.18, there exists a basic neighbourhood B of 0 such that $B \subseteq U$. If we can confine the eventual x_β's to B, then of course they will be confined to U. Now B is by definition in the form $(-a, b)$ where $a > 0$ and $b > 0$. Let $N \in \mathbb{N}$ be sufficiently large that $\frac{1}{b} < N$. This is possible for any $b \in \mathbb{R}$ since $\mathbb{N}$ is cofinal in $\mathbb{R}$. Rearranging, we get $\frac{1}{N} < b$, so for any $m \in \mathbb{N}$ with $N \leq m$, we have

$$-a < 0 < \frac{1}{m} \leq \frac{1}{N} < b.$$

Thus $\frac{1}{m} \in (-a, b)$, or equivalently $x_m \in B$, and thus $x_m \in U$, as desired.

Cross-referencing with our definition of convergence, we used N in place of α and m in place of β. This will be our convention when treating $\mathbb{N}$-nets.

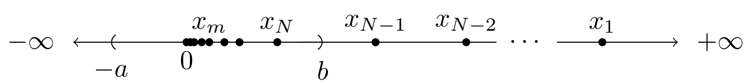

FIGURE 5.2. The $\mathbb{N}$-net $(\frac{1}{n})_{n \in \mathbb{N}}$ in $\mathbb{R}$ converges to 0.

In this example, we started with a general neighbourhood, but refined our work to a basic neighbourhood contained in it. This is a useful trick because basic open sets often yield a more wieldy description. In particular, to show convergence we must show that the net eventually remains within a neighbourhood U of x, so it certainly does no harm to ensure that the net remains within a smaller neighbourhood $B \subseteq U$. When it does not risk confusion, we may take this shortcut without mentioning, by starting with an arbitrary basic neighbourhood.

Remark 5.6. The logical negation of our definition of convergence is the following: a net $(x_\alpha)_{\alpha \in J}$ of points in X *does not* converge to $x \in X$ if *there exists a* neighbourhood $B \in \mathscr{B}_x$ of x such that for all $\alpha \in J$, there exists $\beta \in J$ such that $\alpha \preceq \beta$ *but* $x_\beta \notin B$.

We say $(x_\alpha)_{\alpha \in J}$ diverges if it fails to converge to any point in X.

We now put this formulation of divergence to practice.

Example 5.7. Consider the $\mathbb{N}$-net of points in $\mathbb{R}$ by $(n)_{n \in \mathbb{N}}$, or as a function, $x_\bullet : \mathbb{N} \to \mathbb{R}$ given by $x_n = n$.

This $\mathbb{N}$-net does not converge to any point $x \in \mathbb{R}$. It is intuitively clear that this sequence never approaches any particular point. More formally, let $x \in \mathbb{R}$ be arbitrary. We will show that $(n)_{n \in \mathbb{N}}$ does not converge to x by the formulation in the above remark. Since it suffices to fulfill the statement for a single neighbourhood, we will consider the open interval $B = (x - \frac{1}{2}, x + \frac{1}{2})$ around x, which is a basic open set in the standard basis. Let $N \in \mathbb{N}$ be arbitrary. We have $N \leq N$ and

$N \leq N+1$; we claim that either $x_N \notin B$ or $x_{N+1} \notin B$. Indeed, suppose $x_N \in B$; more explicitly $N \in (x - \frac{1}{2}, x + \frac{1}{2})$, or

$$x - \frac{1}{2} < N < x + \frac{1}{2}.$$

Adding 1,

$$x + \frac{1}{2} < N + 1 < x + \frac{3}{2}.$$

But $x_{N+1} = N+1$, so $x_{N+1} > x + \frac{1}{2}$, and thus $x_{N+1} \notin B$. This shows that either $x_N \notin B$ or $x_{N+1} \notin B$, so in either case we can find $m \in \mathbb{N}$ such that $N \leq m$ but $x_m \notin B$. Since $N \in \mathbb{N}$ was arbitrary, $(n)_{n \in \mathbb{N}}$ does not converge to x. Since $x \in \mathbb{R}$ was arbitary, the sequence $(n)_{n \in \mathbb{N}}$ does not converge to any point in $\mathbb{R}$.

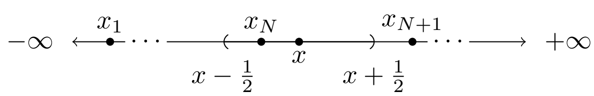

FIGURE 5.3. The $\mathbb{N}$-net $(n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ diverges.

The next example is rather trivial.

Example 5.8. Let X be a topological space containing some point x, and J any directed set. Let $(x_\alpha)_{\alpha \in J}$ be the constant J-net given by $x_\alpha = x$ for all $\alpha \in J$. Clearly, $(x_\alpha)_{\alpha \in J}$ converges to x. Indeed, for any neighbourhood B of x, take any $\alpha \in J$. If $\alpha \preceq \beta$, then $x_\beta = x \in B$ because B is a neighbourhood of x.

The next example is rather pathological.

Example 5.9. Let $X = \{a, b, c\}$, with the basis $\{\{b\}, \{a, b\}, \{b, c\}\}$ from Example 4.2. Let J be any directed set, and let $(x_\alpha)_{\alpha \in J}$ be the constant net $x_\alpha = b$ for all $\alpha \in J$. From the previous example, we know $(x_\alpha)_{\alpha \in J}$ converges to b.

However, we claim that $(x_\alpha)_{\alpha \in J}$ also converges to a and c. Indeed, the only basic neighbourhood of a in X is $\{a, b\}$. Take any $\alpha \in J$; if $\alpha \preceq \beta$, then $x_\beta = b \in \{a, b\}$, so the sequence converges to a. The proof of $x_\alpha \to c$ is identical.

The previous example shows that a net may convergence to more than one point, provided they are topologically indistinguishable. Thus if we demand our space to have no topologically indistinguishable points, we can enforce that nets convergence to a unique limit.

Lemma 5.10. *If X is Hausdorff, then a net in X converges to at most one point.*

Proof. Suppose, for the sake of contradiction, that some net $(x_\alpha)_{\alpha \in J}$ in X converges to distinct points x and y in X. By the Hausdorff condition, let B_1, B_2 be disjoint basis neighbourhoods of x, y, respectively. By definition of convergence, let $\alpha_x, \alpha_y \in J$ be such that $\alpha_x \preceq \beta$ implies $x_\beta \in B_1$, and $\alpha_y \preceq \beta$ implies $x_\beta \in B_2$. Since J is directed, there exists $\alpha \in J$ such that $\alpha_x \preceq \alpha$ and $\alpha_y \preceq \alpha$. Then if $\alpha \preceq \beta$, then by transitivity $\alpha_x \preceq \beta$ and $\alpha_y \preceq \beta$, so $x_\beta \in B_1 \cap B_2$, contradicting the fact that B_1 and B_2 are disjoint. $\square$

Of course, this does not exclude the possibility that a net converges to no point at all, as in Example 5.7 in the Hausdorff space $\mathbb{R}$.

Example 5.11. Let $X = \mathbb{R}$ with the standard topology. We saw that the $\mathbb{N}$-net $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to 0 in Example 5.5. We know $\mathbb{R}$ is Hausdorff by Example 4.23, so by Lemma 5.10 the net $(\frac{1}{n})_{n\in\mathbb{N}}$ converges uniquely to 0.

Henceforth, $(x_\alpha)_{\alpha\in J}$ will always refer to a net of points in X.

Exercise 5.12. In this exercise we explore how the specific topology we assign affects net convergence. Let $X = \mathbb{R}$.

(a) Show that the $\mathbb{N}$-net $(-\frac{1}{n})_{n\in\mathbb{N}}$ does not converge to 0 in the lower limit topology.

(b) Show that the $\mathbb{N}$-net $(n)_{n\in\mathbb{N}}$ converges to every point in $\mathbb{R}$ with the finite complement topology.

Exercise 5.13. Show that the $\mathbb{R}$-net $(\arctan(x))_{x\in\mathbb{R}}$ in $\mathbb{R}$ with the standard topology converges to $\frac{\pi}{2}$.

Exercise 5.14. Show that the $\mathbb{R}$-net $(\sin(x))_{x\in\mathbb{R}}$ in $\mathbb{R}$ with the standard topology diverges.

Exercise 5.15. Let X be an arbitrary topological space with basis $\mathscr{B}$. Let $x \in X$. Recall that the set $\mathscr{B}_x$ of basic neighbourhoods of x is a directed set under $\supseteq$ or reverse containment. For each $B \in \mathscr{B}_x$, let $x_B \in X$ be any point in B. Show that the $\mathscr{B}_x$-net $(x_B)_{B\in\mathscr{B}_x}$ in X converges to x.

6. Closure and Closed Sets

Now that we are sufficiently familiar with open sets and their role in net convergence, we temporarily shut the door on open sets and turn our attention to closed sets.

If open sets are about searching inward, closure is about reaching outward, but only slightly.

Definition 6.1. Let $A \subseteq X$, not necessarily open. The *closure* $\overline{A}$ of A is the set of all points $x \in X$ such that there exists a net of points in A converging to x.

We have yet to formally define a net of points in a subset A of a topological space X. By this, we simply mean a function $x_\bullet : J \to A$, or equivalently a net in X such that $x_\alpha \in A$ for every $\alpha \in J$. In words, $\overline{A}$ is the set of attainable limits of nets in A.

It follows that $A \subseteq \overline{A}$, because for any $x \in A$, the constant net equal to x is a net of points in A that converges to x. The reverse containment $\overline{A} \subseteq A$ is not true in general (otherwise this would be a useless definition).

Example 6.2. Let $X = \{a, b, c\}$ with the basis $\mathscr{B} = \{\{b\}, \{a, b\}, \{b, c\}\}$ from Example 4.2. Let $A = \{b\}$. We claim that $\overline{A} = \{a, b, c\}$. Indeed, in Example 5.9 we saw that the constant net $(b)_{\alpha\in J}$ converges to a, b, and c.

Example 6.3. Consider $X = \mathbb{R}$ as a topological space with the standard basis. The closure of $A = (0, 1)$ is $\overline{A} = [0, 1]$.

Indeed, we first show that $0 \in \overline{A}$ by considering the $\mathbb{N}$-net $(\frac{1}{n+1})_{n\in\mathbb{N}}$ of points in A. This is the exact same sequence as Example 5.5, only shifted by 1 to ensure that the net lies in $A = (0, 1)$. Regardless, the exact same arguments show that $\frac{1}{n+1} \to 0$, and thus $0 \in \overline{A}$. Identically, the $\mathbb{N}$-net $(1 - \frac{1}{n+1})_{n\in\mathbb{N}}$ of points in A converges to 1, showing that $1 \in \overline{A}$. Thus $[0, 1] \subseteq \overline{A}$.

For the reverse inclusion $\overline{A} \subseteq [0,1]$, suppose $x \in (-\infty, 0)$; we show that $x \notin \overline{A}$. To do so, we must show that no net of points in A converges to x, or equivalently, that every net of points in A does not converge to x. Let $(x_\alpha)_{\alpha \in J}$ be an arbitrary net of points in A. Consider the neighbourhood $B = (x-1, 0)$ of x. Let $\alpha \in J$ be arbitrary. Then $\alpha \preceq \alpha$ and $x_\alpha \in (0,1)$. Thus $x_\alpha \notin B$, showing that the sequence does not converge to x. Similarly if $x \in (1, \infty)$ then same arguments applied to the neighbourhood $B = (1, x+1)$ show that $x \notin \overline{A}$. We conclude that $\overline{A} = [0,1]$.

By slightly generalizing these arguments, we see that the closure of any open interval (a,b) in $\mathbb{R}$ is the corresponding closed interval $[a,b]$. In the more general $\mathbb{R}^n$ setting, it is possible to show that the closure of any open ball $B(a, \varepsilon)$ is the closed ball

$$\overline{B}(a, \varepsilon) = \{x \in \mathbb{R}^n : |x - a| \leq \varepsilon\},$$

where the inequality is modified to be weak.

These examples hint at an interpretation of the closure of a set A being the set of points that come arbitrarily near A. This is an indispensable tool in topology, especially in topological spaces with no global notion of distance.

The following theorem characterizes the closure without relying on nets.

Theorem 6.4. *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

Proof. We have two implications to show.

($\Longrightarrow$) Suppose $x \in \overline{A}$. By definition, there exists a net $(x_\alpha)_{\alpha \in J}$ of points in A converging to x. We wish to show that every neighbourhood of x intersects A. Let U be an arbitrary neighbourhood of x. Because $x_\alpha \to x$, there exists $\alpha \in J$ such that $\alpha \preceq \beta$ implies $x_\beta \in U$. In particular $x_\alpha \in U$. Also, the net $(x_\alpha)_{\alpha \in J}$ is a net of points in A, meaning $x_\alpha \in A$. So $x_\alpha \in U \cap A$, showing that U intersects A. Since U was an arbitrary neighbourhood of x, this completes the proof of the forward direction.

($\Longleftarrow$) Suppose every neighbourhood of x intersects A. We wish to show that $x \in \overline{A}$, by constructing a net of points in A that converges to x. First, Lemma 4.10 shows that the set $J = \mathscr{B}_x$ of basic neighbourhoods of x is a directed set under reverse inclusion.

Now we construct a J-net of points in A that converges to x. By assumption, every neighbourhood of x intersects A. So for each $B \in J$, since B is a neighbourhood of x, we may pick x_B to be any point in $B \cap A$. Then $(x_B)_{B \in J}$ is a J-net of points in A, and it remains to show that $x_B \to x$.

Let B be an arbitrary neighbourhood of x; we wish to find $\alpha \in J$ such that for all $\beta \in J$ with $\alpha \preceq \beta$, we have $x_\beta \in B$. Indeed, we may simply take α to be B. If $\beta \in J$ is such that $B \preceq \beta$, then $B \supseteq \beta$. By construction of $(x_B)_{B \in J}$, we have $x_\beta \in \beta$, so $x_\beta \in B$, as desired. We have constructed a net $(x_B)_{B \in J}$ of points in A that converges to x, and thus we conclude that $x \in \overline{A}$. $\square$

Remark 6.5. Equivalently to "every neighbourhood of x intersects A", we can say "every *basic* neighbourhood of x intersects A". Indeed, the first condition immediately implies the second, since basic neighbourhoods are neighbourhoods. Conversely, if the second condition holds, then for every neighbourhood U of x, take a basic neighbourhood B of x such that $B \subseteq U$. By assumption, $B \cap A$ is nonempty, so its superset $U \cap A$ is nonempty. Thus to show that $x \in \overline{A}$, it suffices to check the criterion of Theorem 6.4 for basic neighbourhoods.

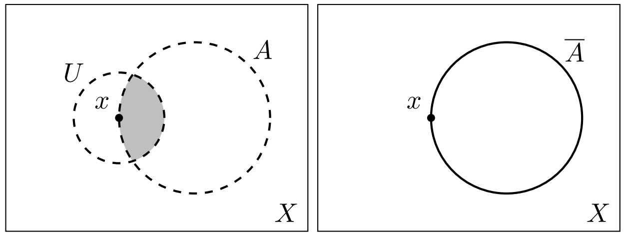

FIGURE 6.1. The criterion of Theorem 6.4 is exceedingly geometric. U is an arbitrary neighbourhood of $x \in \overline{A}$, intersecting A along the shaded region (left). $\overline{A}$ contains every point along the dashed boundary of A (right).

We draw closed sets as solid round shapes, to show that they contain the points along their boundary.

Remark 6.6. With this modification, the negation of Theorem 6.4 is as follows: $x \notin \overline{A}$ if and only if *there exists* a basic neighbourhood B of x such that $B \cap A = \varnothing$.

Example 6.7. The argument in Example 6.2 may be rephrased in the language of Theorem 6.4. Recall that in $X = \{a, b, c\}$ with basis $\{\{b\}, \{a, b\}, \{b, c\}\}$, the subset $A = \{b\}$ has closure $\overline{A} = \{a, b, c\}$. The only basic neighbourhood of a is $\{a, b\}$, which intersects A at b. Certainly $b \in \overline{A}$. The only basic neighbourhood of c is $\{b, c\}$, which intersects A at b. Thus $\overline{A} = \{a, b, c\}$.

Example 6.8. Example 6.3 may also be rephrased in the language of Theorem 6.4. Recall that in $X = \mathbb{R}$ with the standard topology, the closure of $A = (0, 1)$ is $\overline{A} = [0, 1]$. Basic neighbourhoods of 0 take the general form $(-a, b)$ for real numbers $a > 0$ and $b > 0$. Since $0 < \frac{b}{2} < b$, every basic neighbourhood $(-a, b)$ of 0 intersects $A = (0, 1)$ at $\frac{b}{2}$, among many other points. Thus $0 \in \overline{A}$. Similarly, every basic neighbourhood of 1 intersects A. Finally for points $x \in (-\infty, 0)$, the basic neighbourhood $(x - 1, 0)$ does not intersect A, and for points $x \in (1, \infty)$, the basic neighbourhood $(1, x + 1)$ does not intersect A. By Remark 6.6, no point in $(-\infty, 0) \cup (1, \infty)$ lies in $\overline{A}$. Thus $\overline{A} = [0, 1]$.

Recall that in general $A \subseteq \overline{A}$ but the reverse containment fails. The sets for which equality holds deserve special mention.

Definition 6.9. A subset A of X is closed if $\overline{A} = A$.

With the picture of Theorem 6.4, we can say closed sets contain all their limits.

Example 6.10. In $X = \{a, b, c\}$ with basis $\{\{b\}, \{a, b\}, \{b, c\}\}$, the closure of $C = \{a\}$ is $\overline{C} = \{a\}$. Indeed, the basic neighbourhood $\{b, c\}$ of both b and c does not intersect C, so by Remark 6.6 $b \notin \overline{C}$ and $c \notin \overline{C}$. Since $\overline{C} = C$, the subset C is closed.

The subset $A = \{b\}$ is not closed, because $\overline{A} = \{a, b, c\}$ strictly contains A.

Example 6.11. $\varnothing$ is closed in any topological space X. Vacuously, there are no nets in $\varnothing$, so these nets have no limits, and thus $\overline{\varnothing} = \varnothing$, so $\varnothing$ is closed.

Example 6.12. Any topological space X itself is closed. Indeed, every point $x \in X$ may be realized as the limit of a net in X, namely the constant net $(x)_{\alpha \in J}$. This shows that $\overline{X} = X$.

Example 6.13. In $X = \mathbb{R}$ with the standard topology, the closure of $C = [0, 1]$ is $\overline{C} = [0, 1]$. Indeed, the same arguments as in Example 6.3 or Example 6.8 show that no point in $(-\infty, 0) \cup (1, \infty)$ lies in $\overline{C}$. Thus $\overline{C} = C$, so C is closed.

The subset $A = (0, 1)$ is not closed, as $\overline{A} = [0, 1]$ is not equal to A.

The sharp reader might have noticed a pattern in the previous examples. $\{a\}$ is closed in $X = \{a, b, c\}$, and its complement $\{b, c\}$ is open. Similarly $[0, 1]$ is closed in $X = \mathbb{R}$, and its complement $(-\infty, 0) \cup (1, \infty)$ is open. In light of Example 4.13 and Example 4.14, $\varnothing$ and X also follow this pattern. These facts were all proven in more or less equivalent fashion, which is summarized by the following lemma.

Lemma 6.14. *$A \subseteq X$ is closed if and only if $X \setminus A$ is open.*

Proof. ($\Longrightarrow$) Suppose $\overline{A} = A$. We show that $X \setminus A$ is open using Lemma 4.18. Let $x \in X \setminus A$. Then $x \notin \overline{A}$. By negating Theorem 6.4, there exists a neighbourhood U of x that does not intersect A. That is, $U \subseteq X \setminus A$. So by Lemma 4.18, $X \setminus A$ is open.

($\Longleftarrow$) Suppose $X \setminus A$ is open. Suppose, for the sake of contradiction, that $A \subsetneq \overline{A}$. In particular, take $x \in \overline{A} \setminus A$. Since $X \setminus A$ is open by assumption, there exists a basic neighbourhood B of x such that $B \subseteq X \setminus A$, by Lemma 4.18. But $B \subseteq X \setminus A$ means $B \cap A = \varnothing$, which contradicts the fact that $x \in \overline{A}$, by Theorem 6.4. Thus $\overline{A} = A$, so A is closed. $\square$

Just as Lemma 4.18 allowed us to identify more complex open sets, Lemma 6.14 allows us to construct more complex closed sets.

Example 6.15. From Example 4.16, we know that $\mathbb{R} \setminus \mathbb{Z}$ is open in $\mathbb{R}$. Thus its complement $\mathbb{Z}$ is closed in $\mathbb{R}$.

Example 6.16. From Example 4.17, we know that a small annulus of inner radius 0.9 and outer radius 1.1 around the unit circle in $\mathbb{R}^2$ is open. Its complement is a closed set, shown in Figure 6.2.

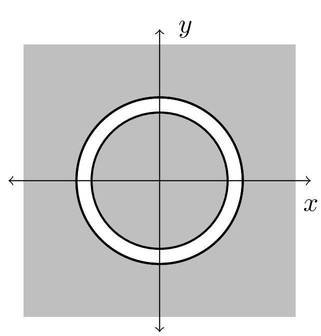

FIGURE 6.2. The complement of an annulus in $\mathbb{R}^2$, represented by the shaded region, is closed.

Example 6.17. From Example 4.19, we know that an open rectangle $(a, b) \times (c, d)$ in $\mathbb{R}^2$ is open. Its complement, shown in Figure 6.3, is thus closed.

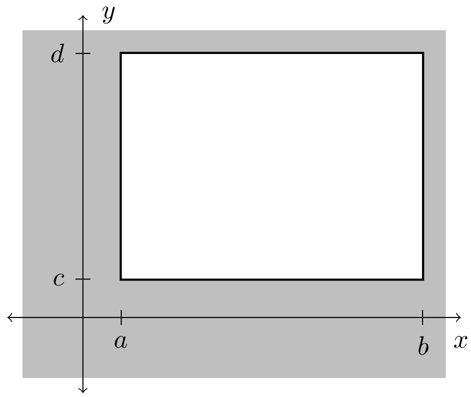

FIGURE 6.3. The complement of a rectangle $(a, b) \times (c, d)$ in $\mathbb{R}^2$, represented by the shaded region, is closed.

Theorem 6.18. *Let $A \subseteq X$. Then $\overline{A}$ is the intersection of all closed subsets of X containing A.*

Proof. We use the equivalent characterization of $\overline{A}$ from Theorem 6.4 to show that $x \notin \overline{A}$ if and only if some closed subset of Y containing A does not contain x.

($\supseteq$) We show the contrapositive: if $x \notin \overline{A}$, then x is not in the intersection of all closed subsets of X containing A. Suppose $x \notin \overline{A}$. By Theorem 6.4, there exists a neighbourhood U neighbourhood of x such that $U \cap A = \varnothing$. Then $X \setminus U$ is a closed set containing A, so the intersection of all such closed sets is contained in it. But $x \notin X \setminus U$, so x is not in the intersection of all closed sets containing A.

($\subseteq$) We show the contrapositive: if there exists a closed subset C of X containing A but not containing x, then $x \notin \overline{A}$. By Lemma 6.14, $X \setminus C$ is open. Moreover, $A \subseteq C$ implies $X \setminus C \subseteq X \setminus A$, so $(X \setminus C) \cap A = \varnothing$. By the negation of Theorem 6.4, $x \notin \overline{A}$. □

Example 6.19. Consider, once again, $A = (0, 1)$ in $\mathbb{R}$. Suppose we already know that $[0, 1]$ is closed, but we don't yet know that $\overline{A} = [0, 1]$. Then Theorem 6.18 provides an alternate proof than Example 6.3. Let C be any closed set containing A. Since $\overline{C} = C$, the net of points $(\frac{1}{n+1})_{n\in\mathbb{N}}$ in C converges to 0, so $0 \in \overline{C} = C$. Similarly, $1 \in C$. So $[0, 1]$ is contained in every closed set containing A, and thus $[0, 1] \subseteq \overline{A}$.

Conversely $[0, 1]$ is a closed set containing A, so the intersection of all such sets much be contained inside $[0, 1]$; that is, $\overline{A} \subseteq [0, 1]$. We conclude that $\overline{A} = [0, 1]$.

Exercise 6.20. Let $\{C_\alpha\}_{\alpha\in I}$ be a family of closed sets in X.

(a) Show that $\bigcap_{\alpha\in I} C_\alpha$ is closed. (*With this, the intersection of all closed subsets of X containing a subset A is a closed set, so by Theorem 6.18 we can say $\overline{A}$ is the smallest closed set containing A. Moreover, this gives an easy proof that the closure of a set is closed.*)

(b) Is $\bigcup_{\alpha\in I} C_\alpha$ necessarily closed? What if I is finite?

Exercise 6.21. Find the closure of the following sets in the given topological spaces:

(a) $\mathbb{Q}$ in $\mathbb{R}$. What if we give $\mathbb{R}$ the lower limit topology?

(b) $\{\frac{1}{n} : n \in \mathbb{N}\}$ in $\mathbb{R}$. What if we give $\mathbb{R}$ the finite complement topology?
(c) $\{(x, 0) : x \in \mathbb{R}\}$ in $\mathbb{R}^2$.
(d) $\{(x, x^2) : x \in [0, 1]\}$ in $\mathbb{R}^2$.
(e) $\{(x, \frac{1}{x}) : x \in (0, 1]\}$ in $\mathbb{R}^2$.
(f) $\{(x, \sin(\frac{1}{x})) : x \in (0, 1]\}$ in $\mathbb{R}^2$. (*Hint: draw a picture.*)

Exercise 6.22. Let X be a Hausdorff space, and let $A \subseteq X$ be a finite subset.
(a) Show that A is closed.
(b) Show that the converse (if every finite subset A of X is closed, then X is Hausdorff) is false in general, using the finite complement topology on $\mathbb{R}$ from Exercise 4.27 as a counterexample.

Exercise 6.23. We have seen that $\varnothing$ and X are both open and closed in any topological space X. More concisely, we say $\varnothing$ and X are *clopen*.
(a) Let $X = \mathbb{R}$. Show that there are no other clopen sets.
(b) Let $X = \{a, b\}$ with the basis $\mathscr{B} = \{\{a\}, \{b\}\}$. Show that $\{a\}$ and $\{b\}$ are clopen.

A topological space in which the only clopen sets are $\varnothing$ and X is called *connected*. Connectedness is the property that makes the intermediate value theorem possible.

7. Continuity

Topology is often considered as the study of continuous functions. At last, we can explore this key concept.

Definition 7.1. A function $f : X \to Y$ is *continuous at* $x \in X$ if for every net $(x_\alpha)_{\alpha \in J}$ in X converging to x, the net $(f(x_\alpha))_{\alpha \in J}$ in Y converges to $f(x)$.

This definition is very geometric. If f is continuous at x, we should be able to determine the behaviour of f at x using the behaviour of f near x. This "nearness" is built into the net $(x_\alpha)_{\alpha \in J}$, so the net $(f(x_\alpha))_{\alpha \in J}$ exhibits the corresponding behaviour of f.

We say $f : X \to Y$ is *continuous* if it is continuous at every point $x \in X$.

Example 7.2. Let X be any set, $Y = X$, and let $\mathrm{id} : X \to X$ be the identity map given by $x \mapsto x$ for all $x \in X$. For any $x \in X$ and any net $(x_\alpha)_{\alpha \in J}$ of points in X converging to x, the net $(\mathrm{id}(x_\alpha))_{\alpha \in J}$ is simply the same net $(x_\alpha)_{\alpha \in J}$, which indeed converges to $\mathrm{id}(x) = x$. Thus the identity map is continuous on X.

Remark 7.3. The logical negation of Definition 7.1 is the following. $f : X \to Y$ is *not* continuous at x if *there exists* a net $(x_\alpha)_{\alpha \in J}$ of points in X converging to x such that the corresponding net $(f(x_\alpha))_{\alpha \in J}$ of points in Y *does not* converge to $f(x)$.

Example 7.4. Let $X = \mathbb{R}$ and $Y = \mathbb{R}$ be both given the standard basis, and let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Readers may recall from first-year calculus that f is not continuous because it has an unexplainable 'jump' at $x = 0$. We will prove this fact using our topological definition, or more specifically the preceding remark. Consider our favourite net $(\frac{1}{n})_{n \in \mathbb{N}}$ of points in $\mathbb{R}$ converging to 0. If f were continuous, then we would expect the corresponding net $(f(\frac{1}{n}))_{n\in\mathbb{N}}$ to converge to $f(0) = 0$. But $(f(\frac{1}{n}))_{n\in\mathbb{N}}$ is the constant net equal to 1, because $\frac{1}{n} > 0$ for each $n \in \mathbb{N}$, and thus $f(\frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. Since we know how constant nets converge, $(f(\frac{1}{n}))_{n\in\mathbb{N}}$ converges to 1. Since $\mathbb{R}$ with the standard basis is Hausdorff, $(f(\frac{1}{n}))_{n\in\mathbb{N}}$ has a unique limit, so it does not converge to $0 = f(0)$. Therefore f is not continuous.

Unfortunately Definition 7.1 is a highly inconvenient definition of continuity; verifying this condition for *all* nets converging to *any* point in X is often intractable. We will formulate several equivalent definitions, and together they will provide a more powerful arsenal for characterizing continuity. The next definition will be crucial.

Definition 7.5. Let $f : A \to B$ be any function and $C \subseteq B$. We define the *preimage of C under f* to be the subset $f^{-1}(C) = \{a \in A : f(a) \in C\}$ of A.

We provide some examples to clarify.

Example 7.6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $C \subseteq \mathbb{R}$ be the set $C = [1, 4]$. Then the preimage of C under f is $f^{-1}(C) = [-2, -1] \cup [1, 2]$, because $x^2 \in [1, 4]$ if and only if $x \in [-2, -1] \cup [1, 2]$.

Example 7.7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin(x)$. Let $C \subseteq \mathbb{R}$ be the set $C = \{1\}$. Then the preimage of C under f is $f^{-1}(C) = \{\frac{\pi}{2} + 2\pi k : k \in \mathbb{Z}\}$, because $\sin(x) = 1$ if and only if $x = \frac{\pi}{2} + 2\pi k$ for some integer k.

Example 7.8. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be given by $f(x) = \frac{1}{x}$. Let $C \subseteq \mathbb{R}$ be the set $C = \mathbb{Q}$. Then the preimage of C under f is $f^{-1}(C) = \mathbb{Q}$, because $\frac{1}{x} \in \mathbb{Q}$ if and only if $x \in \mathbb{Q}$.

The following set-theoretic properties follow from the definition.

Lemma 7.9.
1. *$f(f^{-1}(C)) \subseteq C$.*
2. *$f^{-1}(Y \setminus A) \subseteq X \setminus f^{-1}(A)$.*

Proof.
1. Suppose $x \in f^{-1}(C)$. Then $f(x) \in C$, so $f(f^{-1}(C)) \subseteq C$.
2. We expand the definition:

$$\begin{aligned} f^{-1}(Y \setminus A) &= \{x \in X : f(x) \in Y \setminus A\} \\ &= \{x \in X : f(x) \in Y\} \setminus \{x \in X : f(x) \in A\} \\ &= f^{-1}(Y) \setminus f^{-1}(A) \\ &= X \setminus f^{-1}(A). \end{aligned}$$

□

With these properties, we are prepared to give our many alternative criteria for continuity.

Theorem 7.10. *Let $f : X \to Y$ be a function between topological spaces. The following are equivalent:*

1. *f is continuous.*
2. *For any subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$.*
3. *For any closed subset C of Y, $f^{-1}(C)$ is a closed subset of X.*
4. *For any open set V in Y, $f^{-1}(V)$ is an open set in X.*

(5) *For each $x \in X$ and every neighbourhood V of $f(x)$ in Y, there exists a neighbourhood U of x in X such that $f(U) \subseteq V$.*

Proof. (1) $\implies$ (2). Suppose f is continuous as in Definition 7.1. Let $A \subseteq X$ be arbitrary; we wish to show that $f(\overline{A}) \subseteq \overline{f(A)}$. That is, for any $x \in \overline{A}$, we want to show that $f(x) \in \overline{f(A)}$. Given $x \in \overline{A}$, Definition 5.11 states that there exists a net $(x_\alpha)_{\alpha \in J}$ of points in A that converges to x. Since f is continuous, the net $(f(x_\alpha))_{\alpha \in A}$ converges to $f(x)$. But since each x_α is a point in A, each $f(x_\alpha)$ is a point in $f(A)$. So $(f(x_\alpha))_{\alpha \in A}$ is a net of points in $f(A)$ converging to $f(x)$, showing that $f(x) \in \overline{f(A)}$, as desired. Since $A \subseteq X$ was arbitrary, we are done this implication.

(2) $\implies$ (3). Let C be an arbitrary closed subset of Y, and let $A = f^{-1}(C)$. Then by Lemma 7.9(1), $f(A) = f(f^{-1}(C)) \subseteq C$. Since C is closed and $\overline{f(A)}$ is the intersection of all closed sets containing $f(A)$, $\overline{f(A)} \subseteq C$. Now if $x \in \overline{A}$, then $f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq C$, so $x \in f^{-1}(C) = A$. We have shown that $\overline{A} \subseteq A$, so $A = f^{-1}(C)$ is a closed subset of X.

(3) $\implies$ (4). Let B be any open set in Y. Then $Y \setminus B$ is closed, so by (3), $f^{-1}(Y \setminus B)$ is closed. By Lemma 7.9(2),

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$$

is closed, so $f^{-1}(B)$ is open.

(4) $\implies$ (5). Let $x \in X$ and any neighbourhood V of $f(x)$ in Y be given. Since V is open, (4) implies that $f^{-1}(V)$ is open. Moreover, $f(x) \in V$ implies $x \in f^{-1}(V)$, so $U = f^{-1}(V)$ is a neighbourhood of x in X. By Lemma 7.9(1),

$$f(U) = f(f^{-1}(V)) \subseteq V,$$

as desired.

(5) $\implies$ (1). Suppose $(x_\alpha)_{\alpha \in J}$ is a net of points in X that converges to $x \in X$. We wish to show that the net $(f(x_\alpha))_{\alpha \in J}$ of points in Y converges to $f(x)$. Given any neighbourhood V of $f(x)$ in Y, let U be a neighbourhood of x in X such that $f(U) \subseteq V$, by (5). Since $x_\alpha \to x$, there exists $\alpha \in J$ such that $\alpha \preceq \beta$ implies $x_\beta \in U$, for all $\beta \in J$. Then the same α works for $(f(x_\alpha))_{\alpha \in J}$, because for all $\beta \in J$ with $\alpha \preceq \beta$, $f(x_\beta) \in f(U) \subseteq V$. Since V was an arbitrary neighbourhood of $f(x)$, we conclude that $f(x_\alpha) \to f(x)$, as desired. $\square$

Remark 7.11. The previous theorem discusses only the continuity of $f : X \to Y$ on the entire domain X, rather than continuity at a single point as originally formulated in Definition 7.1. In analysis, we are more often interested in continuity at a particular point, but in topology, we often care to have continuity everywhere. Nonetheless, all five conditions can be adapted to describe continuity at a specific point. Most obviously, f is continuous at x if and only if x satisfies the statement in condition (5) immediately following "for each $x \in X$ and ...".

Remark 7.12. Readers who have taken a first-year undergraduate course in calculus should recognize (5) as the ε-δ definition of continuity when $X = \mathbb{R}$ and $Y = \mathbb{R}$. Namely, by taking a basic neighbourhood of $f(x)$ within V, it suffices to consider V in the form $(f(x) - \varepsilon, f(x) + \varepsilon)$. Once the desired U is found, we may shrink U to a basic neighbourhood of x in the form $(x - \delta, x + \delta)$.

With this, we give two examples for each criterion: one function that is continuous, and one that is not. The exact logical negations will have to be filled in by the reader.

Example 7.13. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x+1$. This function is continuous, as the reader may know from the ε-δ definition. Alternatively, we may use criterion (2) of Theorem 7.10. Let $A \subseteq \mathbb{R}$, and let $x \in \overline{A}$. We wish to show that $f(x) \in \overline{f(A)}$. We use Theorem 6.4. Since $x \in \overline{A}$, every neighbourhood of x intersects A. Let V be any neighbourhood of $f(x)$; we will show that V intersects $f(A)$. By possibly shrinking V, we may assume that V is a basic neighbourhood of the form $(f(x)-\varepsilon, f(x)+\varepsilon)$ for some $\varepsilon > 0$. Let $U = (\frac{1}{2}(f(x)-\varepsilon-1), \frac{1}{2}(f(x)+\varepsilon-1))$. Then U is a neighbourhood of x, because $f(x) = 2x+1$ may be rewritten as $x = \frac{1}{2}(f(x)-1)$, which clearly lies in U. So U intersects A, say at some point $a \in A$. Then $\frac{1}{2}(f(x)-\varepsilon-1) < a < \frac{1}{2}(f(x)+\varepsilon-1)$ implies $f(x) - \varepsilon < 2a+1 < f(x)+\varepsilon$, and thereby $f(a) \in V$. Of course $f(a) \in f(A)$, so V intersects $f(A)$, as desired. Since V was an arbitrary neighbourhood of $f(x)$, we conclude that $f(\overline{A}) \in \overline{f(A)}$, so f is continuous.

Example 7.14. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We show f is not continuous on $\mathbb{R}$ using criterion (2). Let $A = \mathbb{Q}$, which has closure $\overline{A} = \mathbb{R}$, by Exercise 6.21. Then $f(\overline{A}) = f(\mathbb{R}) = \{0, 1\}$, but $\overline{f(A)} = \overline{\{1\}} = \{1\}$ because the only net in $\{1\}$ is the constant net. Since $f(\overline{A})$ is not contained in $\overline{f(A)}$, f is not continuous by the negation of criterion (2).

Example 7.15. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. We show that f is continuous using criterion (3). Let $C \subseteq \mathbb{R}$ be any closed set. We wish to show that $f^{-1}(C)$ is closed, and it suffices to show that $\overline{f^{-1}(C)} \subseteq f^{-1}(C)$. Namely let $x \in \overline{f^{-1}(C)}$. By Theorem 6.4, every neighbourhood of x intersects $f^{-1}(C)$. Let V be a neighbourhood of x^3, which we may assume takes the form $(x^3 - \varepsilon, x^3 + \varepsilon)$. We obtain a neighbourhood $U = (\sqrt[3]{x^3 - \varepsilon}, \sqrt[3]{x^3 + \varepsilon})$ of x, because $x = \sqrt[3]{x^3}$. Then U must intersect $f^{-1}(C)$ at some point $a \in f^{-1}(C)$. It follows from $\sqrt[3]{x^3 - \varepsilon} < a < \sqrt[3]{x^3 + \varepsilon}$ that $x^3 - \varepsilon < a^3 < x^3 + \varepsilon$, and thereby $a^3 \in V$. Moreover as $a \in f^{-1}(C)$, we know that $a^3 \in C$, so V intersects C. Since V was an arbitrary neighbourhood of x^3, we conclude that $f(x) = x^3 \in \overline{C}$ Since C is closed, $\overline{C} = C$, so $f(x) \in C$, or equivalently $x \in f^{-1}(C)$. We have shown that $\overline{f^{-1}(C)} \subseteq f^{-1}(C)$, so $f^{-1}(C)$ is closed, and thus f is continuous.

Example 7.16. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \lfloor x \rfloor$, the floor function. Namely $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x. We show that f is not continuous by negating criterion (3). Let $C = \{0\}$, which is closed in $\mathbb{R}$. Then

$$f^{-1}(C) = \{x \in \mathbb{R} : \lfloor x \rfloor = 0\} = [0, 1).$$

But $[0, 1)$ is not closed in $\mathbb{R}$, because it does not contain its limit point 1. Since C is closed but $f^{-1}(C)$ is not, f is not continuous.

Example 7.17. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x+y$. We show that f is continuous using criterion (4). Let V be any open set in $\mathbb{R}$. As in the previous examples, we may assume V is in the basic form $(f(x,y)-\varepsilon, f(x,y)+\varepsilon)$. We wish to show that $f^{-1}(V)$ is open in $\mathbb{R}^2$. Let $(z,w) \in f^{-1}(V)$, so $f(z,w) = z+w \in V$. Then

$$x+y-\varepsilon < z+w < x+y+\varepsilon,$$

or equivalently

$$|(z+w)-(x+y)| < \varepsilon.$$

Let $\delta = \varepsilon - |(z+w)-(x+y)|$, and let $U = (z-\frac{\delta}{2}, z+\frac{\delta}{2}) \times (w-\frac{\delta}{2}, w+\frac{\delta}{2})$. From Example 4.19, we know that U is a neighbourhood of (z,w) in $\mathbb{R}^2$. Let B be a basic neighbourhood of (x,y) in $\mathbb{R}^2$ such that $B \subseteq U$. Then for any $(a,b) \in B$, we have $(a,b) \in U$, so $|a-z| < \frac{\delta}{2}$ and $|b-w| < \frac{\delta}{2}$. Then

$$\begin{aligned}
|(a+b)-(x+y)| &= |(a+b)-(z+w)+(z+w)-(x+y)| \\
&\leq |a-z| + |b-w| + |(z+w)-(x+y)| \\
&< \delta + |(z+w)-(x+y)| \\
&= \varepsilon.
\end{aligned}$$

So $f(a,b) = a+b \in V$, showing that $B \subseteq f^{-1}(V)$. This shows $f^{-1}(V)$ is open in $\mathbb{R}^2$, and therefore f is continuous.

Example 7.18. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) = \begin{cases} (\frac{x}{y}, 1) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

We show that f is not continuous by negating criterion (4). Consider the open set $V = B((0,0),1) = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 1\}$. We show that $f^{-1}(V)$ is not open in $\mathbb{R}^2$; in particular, $(0,0) \in f^{-1}(V)$ because $f(0,0) = (0,0)$, but there is no basic neighbourhood of $(0,0)$ contained in $f^{-1}(V)$. Suppose there were such a basic neighbourhood $B((a,b),\delta)$ (we can assume without loss of generality that the ball is centered at $(0,0)$) of $(0,0)$. Then $B((a,b),\delta)$ must contain some point (z,w) with $w \neq 0$. It follows that $f(z,w) = (\frac{z}{w}, 1)$, which is not in V because $\sqrt{(\frac{z}{w})^2 + 1^2} \geq 1$. Thus $B((a,b),\delta)$ cannot be contained in $f^{-1}(V)$, so $f^{-1}(V)$ is not open. Since V is open but $f^{-1}(V)$ is not, f is not continuous.

Example 7.19. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (x^2+y^2, 0)$. We show that f is continuous using criterion (5). Let $(x,y) \in \mathbb{R}^2$ and let V be a neighbourhood of $f(x,y) = (x^2+y^2,0)$ in $\mathbb{R}^2$. Shrinking V if necessary, we may assume that V is a basic neighbourhood of the form $B(f(x,y),\varepsilon)$ for some $\varepsilon > 0$. We wish to find a neighbourhood U of (x,y) such that $f(U) \subseteq V$. Let $\delta = \varepsilon$. Then $U = B((x,y),\delta)$ is a neighbourhood of (x,y) in $\mathbb{R}^2$. For any point $(a,b) \in U$, we have $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, so

$$\begin{aligned}
|f(a,b) - f(x,y)| &= |(a^2+b^2,0) - (x^2+y^2,0)| \\
&= \sqrt{(x-a)^2 + (y-b)^2} \\
&< \delta \\
&= \varepsilon.
\end{aligned}$$

Thus $f(a,b) \in V$, showing that $f(U) \subseteq V$. Since V was an arbitrary neighbourhood of $f(x,y)$, we conclude that f is continuous.

Example 7.20. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 < 1 \\ 0 & \text{if } x^2+y^2 \geq 1. \end{cases}$$

We show that f is not continuous by negating criterion (5). Consider the point $(x,y) = (1,0)$ and the neighbourhood $V = (-\frac{1}{2}, \frac{1}{2})$ of $f(1,0) = 0$. We wish to show that there exists no neighbourhood U of $(1,0)$ such that $f(U) \subseteq V$. Let U be any neighbourhood of $(1,0)$. By possibly shrinking U, we may assume it is a basic neighbourhood in the form $B((1,0),\delta)$.

But $B((1,0),\delta)$ contains some point to the left of $(1,0)$, say $(x,y) = (1-\frac{\delta}{2}, 0)$. In particular $f(1-\frac{\delta}{2}, 0) = 1$, because $(1-\frac{\delta}{2})^2 + 0^2 < 1$. This means that $1 \in f(B((1,0),\delta))$, but clearly $1 \notin V = (-\frac{1}{2}, \frac{1}{2})$. So $f(U)$ is not a subset of V. We conclude that f is not continuous.

Readers may notice that most of the above examples inadvertently use an ε-δ style argument, even though we are not directly using the ε-δ definition of continuity. This is both a testament to their equivalence and an unfortunate concealment of how strange continuity can be in general topological spaces. This strangeness is left for the reader to explore in the exercises.

Exercise 7.21. Let X, Y, and Z be topological spaces.

(a) Show that the constant function $g : X \to Y$ given by $g(x) = c$ for all $x \in X$, where $c \in Y$, is continuous.
(b) Let $f : Y \to Z$ and $g : X \to Y$ be continuous. Show that $f \circ g : X \to Z$ is also continuous.

Exercise 7.22. Let X be a topological space. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be real-valued functions. Show that the following functions are continuous:

(a) The projections $\pi_i : \mathbb{R}^n \to \mathbb{R}$ given by $\pi_i(x_1, x_2, \ldots, x_n) = x_i$ for $i = 1, 2, \ldots, n$.
(b) $f + g : X \to \mathbb{R}$, given by $(f+g)(x) = f(x) + g(x)$.
(c) $f \cdot g : \mathbb{R}^n \to \mathbb{R}$, given by $(f \cdot g)(x) = f(x) \cdot g(x)$.

Exercise 7.23. Let $f : X \to Y$ be a bijection of sets. We call f a *homeomorphism* if both f and f^{-1} are continuous.

Before proceeding with this exercise, we need to mention how to obtain a topology on a subset Y of $\mathbb{R}$. The standard basis on $\mathbb{R}$ consists of all open intervals (a,b) in the real number line. So we define the standard basis on Y to consist of all sets of the form $Y \cap (a,b)$, where (a,b) is an open interval in $\mathbb{R}$. We will use the resulting topology for all parts of this exercise.

(a) Show that the tangent function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a homeomorphism.
(b) Show that the function $f : [0,1) \to \mathbb{R}^2$ defined by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is a continuous bijection, but not a homeomorphism.

Exercise 7.24. We saw in Exercise 5.12 that net convergence depends on the underlying topology of the set, and in Exercise 6.21 that closure depends on the underlying topology. In this exercise, we investigate how continuity depends on the underlying topologies.

(a) Let X be any topological space and Y a set with basis $\{Y\}$. Show that every function $f : X \to Y$ is continuous.

(b) Let X be a set with basis $\mathscr{P}(X)$ and Y any topological space. Show that every function $f : X \to Y$ is continuous.
(c) Let $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ be identity map from Example 7.2. We saw that id is continuous in the standard topology. Show that it is not continuous if we give the domain the finite complement topology.
(d) Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

This is almost identical to the function in Example 7.4. Show that f is not continuous in the standard topology on $\mathbb{R}$, but is continuous if we give the domain the lower limit topology.

8. Subnets and Compactness

Long ago, we defined a cofinal subset of a directed set in a way that guaranteed a directed set structure on these cofinal subsets. In this section, we use this inheritance to study nets.

This section, particularly the eventual treatment of compactness, is significantly more difficult than the preceding sections.

Definition 8.1. Let $(x_\alpha)_{\alpha \in J}$ be a net in X. Let K be a directed set and let $g : K \to J$ be a function such that

(1) if $i \in K$ and $j \in K$ are such that $i \preceq j$, then $g(i) \preceq g(j)$ in J.
(2) $g(K)$ is cofinal in J.

Then $(x_{g(\beta)})_{\beta \in K}$ is a net in X, called a *subnet* of $(x_\alpha)_{\alpha \in J}$.

Alternatively if we write our original net as $x_\bullet : J \to X$, then $x_\bullet \circ g$ is the described subnet.

Intuitively, a subnet is obtained from a net by selecting a few points, and omitting the rest. The few selected points must be (1) selected in an order that respects their order in the original net (see Example 8.4 for a counterexample), and (2) abundant enough that the subnet always continues with the original net, rather than stalling unexplainably before its limit (see Example 8.4).

Example 8.2. Let $X = \mathbb{R}$, and consider our favourite net $(\frac{1}{n})_{n \in \mathbb{N}}$. Let $K = \mathbb{N}$ and define $g : K \to J$ by $g(n) = 2n$. Then

(1) If $m \in \mathbb{N}$ and $n \in \mathbb{N}$ are such that $m \leq n$, then $g(m) = 2m \leq 2n = g(n)$.
(2) $g(\mathbb{N}) = \{2n : n \in \mathbb{N}\} = 2\mathbb{N}$ is cofinal in $\mathbb{N}$, because for any $k \in \mathbb{N}$, $2k \in 2\mathbb{N}$ satisfies $k \leq 2k$.

So the net $(\frac{1}{2n})_{n \in \mathbb{N}}$ is a subnet of $(\frac{1}{n})_{n \in \mathbb{N}}$.

With the same example $(\frac{1}{n})_{n \in \mathbb{N}}$, we explore how (1) and (2) can fail.

Example 8.3. Let $K = \mathbb{N}$ and define $g : K \to J$ so that $g(n)$ is the smallest exponent of 2 that no longer divides n, for all $n \in \mathbb{N}$. For example, $g(1) = 1$ because 2^0 divides 1 but 2^1 does not, $g(2) = 2$, $g(3) = 1$, $g(4) = 3$. Then $g(K) = \mathbb{N}$, because for any $m \in \mathbb{N}$, 2^m satisfies $g(2^m) = m$. Since $\mathbb{N}$ is trivially cofinal in itself, (2) is satisfied. However (1) fails, as we see from $g(2) = 1$ and $g(3) = 0$. Thus, although $(\frac{1}{g(n)})_{n \in \mathbb{N}}$ is still a net, it is not a subnet of $(\frac{1}{n})_{n \in \mathbb{N}}$.

Example 8.4. Let $K = \mathbb{N}$ and let $g : K \to J$ be the constant function equal to 1. Then (1) is vacuously true; $g(m) \leq g(n)$ for all natural numbers m and n. However, (2) fails, because $g(K) = \{1\}$ is not cofinal in $\mathbb{N}$; there is no number greater than 2. So the net $(\frac{1}{g(n)})_{n\in\mathbb{N}} = (1)_{n\in\mathbb{N}}$ is not a subnet of $(\frac{1}{n})_{n\in\mathbb{N}}$.

Lemma 8.5. *If $(x_\alpha)_{\alpha\in J}$ converges to x, then so does every subnet.*

Proof. Let $(x_{g(\beta)})_{\beta\in K}$ be a subnet of $(x_\alpha)_{\alpha\in J}$, and let V be any neighbourhood of x. Since $(x_\alpha)_{\alpha\in J}$ converges to x, there exists $\alpha \in J$ such that $\alpha \preceq \beta$ implies $x_\beta \in V$, for all $\beta \in J$. Since $g(K)$ is cofinal in J, there exists $\gamma \in K$ such that $\alpha \preceq g(\gamma)$. Then for all $\beta \in K$ such that $\gamma \preceq \beta$, condition (1) of a subnet implies $g(\gamma) \preceq g(\beta)$ and along with $\alpha \preceq g(\gamma)$, the transitive property implies $\alpha \preceq g(\beta)$. By definition of α, $x_{g(\beta)} \in V$. Thus $(x_{g(\beta)})_{\beta\in K}$ converges to x, as desired. □

Example 8.6. The subnet $(\frac{1}{2n})_{n\in\mathbb{N}}$ of $(\frac{1}{n})_{n\in\mathbb{N}}$ from Example 8.2 converges to 0, by the exact same arguments that we initially used to show that $\frac{1}{n} \to 0$.

Example 8.7. The net $(\frac{1}{g(n)})_{n\in\mathbb{N}}$ from Example 8.3 fails to be a subnet of $(\frac{1}{n})_{n\in\mathbb{N}}$, and in fact it diverges. To demonstrate that it is not a subnet, we will simply show it does not converge to 0. To do so, it suffices to show that $\{n \in \mathbb{N} : \frac{1}{g(n)} = 1\}$ is cofinal in $\mathbb{N}$. In this case, we can never guarantee that the net falls into the neighbourhood $(-1, 1)$.

Indeed, $\frac{1}{g(n)} = 1$ is equivalent to $g(n) = 1$, which by definition of g is true if and only if n is not divisible by 2. The odd numbers are obviously cofinal in $\mathbb{N}$, because for any $k \in \mathbb{N}$, $2k + 1$ is an odd number greater than k. Therefore $(\frac{1}{g(n)})_{n\in\mathbb{N}}$ does not converge to 0, so it cannot be a subnet of $(\frac{1}{n})_{n\in\mathbb{N}}$.

Example 8.8. The net $(1)_{n\in\mathbb{N}}$ from Example 8.4 is not a subnet of $(\frac{1}{n})_{n\in\mathbb{N}}$, and of course it does not converge to 0; it is constant, thus converges to 1. By the Hausdorff property of $\mathbb{R}$, it cannot converge to 0.

Definition 8.9. x is an accumulation point of a net $(x_\alpha)_{\alpha\in J}$ in X if for each neighbourhood U of x, the set of all $\alpha \in J$ such that $x_\alpha \in U$ is cofinal in J.

Example 8.10. Let $X = \mathbb{R}$ and consider the net $((-1)^n)_{n\in\mathbb{N}}$. Then -1 and 1 are both accumulation points. Indeed, the set $\{\alpha \in J : x_\alpha = -1\}$ is the set of odd numbers, so for each neighbourhood U of -1 the set $\{\alpha \in J : x_\alpha \in U\}$ contains all odd numbers, and thus is cofinal in $\mathbb{N}$. The argument for 1 is similar.

Example 8.11. Let $X = \mathbb{R}$ and consider the net $(\frac{1}{n})_{n\in\mathbb{N}}$. Then 1 is not an accumulation point, because for the neighbourhood $U = (\frac{1}{2}, \frac{3}{2})$ of 1, the set $\{\alpha \in J : x_\alpha \in U\}$ is exactly $\{1\}$, which is obviously not cofinal.

Lemma 8.12. *x is an accumulation point of $(x_\alpha)_{\alpha\in J}$ if and only if some subnet of (x_α) converges to x.*

Proof. ($\implies$) Let K be the set of all pairs (α, U) where $\alpha \in J$ and U is a neighbourhood of x containing x_α. Define a partial order $\preceq$ on K by $(\alpha, U) \preceq (\beta, V)$ if $\alpha \preceq \beta$ and $U \supseteq V$. Given $(\alpha, U), (\beta, V) \in K$, let $\gamma \in J$ be such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Applying the definition of x as an accumulation point to the neighbourhood $U \cap V$ of x,, there exists $\gamma \preceq \delta$ such that $x_\delta \in U \cap V$. This means $(\delta, U \cap V) \in K$.

By transitivity, $\alpha \preceq \delta$ and $\beta \preceq \delta$, showing that $(\alpha, U) \preceq (\delta, U \cap V)$ and $(\beta, V) \preceq (\delta, U \cap V)$, and thus K is directed.

Define $g : K \to J$ by $(\alpha, U) \mapsto \alpha$. We verify that if $(\alpha, U) \preceq (\beta, V)$, then $g(\alpha, U) = \alpha \preceq \beta = g(\beta, V)$, and $g(K)$ is cofinal in J because for any $\alpha \in J$, $\alpha \preceq g(\alpha, X)$.

For any neighbourhood U of x and any $\alpha \in J$, the definition of x as an accumulation point implies that there exists $\beta \in J$ such that $\alpha \preceq \beta$ and $x_\beta \in U$. Now $(\beta, U) \in K$, and if $(\gamma, V) \in K$ is such that $(\beta, U) \preceq (\gamma, V)$ then $x_{g(\gamma,V)} = x_\gamma \in V \subseteq U$. Hence the subnet $(x_{g(\alpha,U)})_{(\alpha,U)\in K}$ converges to x.

($\Longleftarrow$) Suppose some subnet $(x_{g(i)})_{i\in K}$ of $(x_\alpha)_{\alpha\in J}$ converges to x. For every neighbourhood U of x, there exists $i \in K$ such that $i \preceq j$ implies $x_{g(j)} \in U$. Let A be the set of indices $\alpha \in J$ for which $x_\alpha \in U$. We show that A is cofinal in J. Given $\alpha \in J$, if $\alpha \preceq g(i)$ then $g(i) \in A$ is the desired successor. Otherwise since $g(K)$ is cofinal in J, there exists $g(j) \in g(K)$ such that $\alpha \preceq g(j)$. If $j \preceq i$, then $g(j) \preceq g(i)$ and by transitivity $\alpha \preceq g(i)$, so we are in the first case. Otherwise let $k \in K$ be such that $i \preceq k$ and $j \preceq k$; we have $\alpha \preceq g(j) \preceq g(k)$ and $i \preceq k$ implies $g(k) \in A$, so $g(k)$ is the desired successor. $\square$

Example 8.13. Let $X = \mathbb{R}$ and consider the net $((-1)^n)_{n\in\mathbb{N}}$. Recall that 1 is an accumulation point, so there must be a subnet that converges to 1. Indeed, let $K = \mathbb{N}$ and let $g : K \to J$ be given by $n \mapsto 2n$. g is order-preserving and $g(K) = 2\mathbb{N}$ is cofinal, so we have constructed a subnet $((-1)^{2n})_{n\in\mathbb{N}} = (1)_{n\in\mathbb{N}}$ converges to 1.

Example 8.14. Let $X = \mathbb{R}$ and consider the net $(\frac{1}{n})_{n\in\mathbb{N}}$. Recall that 1 is not an accumulation point, so we expect no subnet to converge to 1. Indeed, by Lemma 8.5, every subnet of $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to 0, and by the Hausdorff condition, does not converge to 1.

Definition 8.15. X is compact if every net in X has a convergent subnet in X.

Compact spaces are quite common, but their compactness is difficult to justify. We leave it as a remark that if $X = \mathbb{R}^n$, then compactness is equivalent to X being closed and bounded. This equivalence is known as the Bolzano-Weierstrass Theorem, and a proof may be found in [Spivak, Section 21].

To recover the traditional formulation of compactness in general topology, we first need the following definitions.

Definition 8.16. An *open covering* of X is a set $\{U_\alpha\}_{\alpha\in I}$ of open sets in X such that $X = \bigcup_{\alpha\in I} U_\alpha$. Given an open covering $\{U_\alpha\}_{\alpha\in I}$, a *finite subcovering* is a finite subset $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ of the open covering such that $X = \bigcup_{i=1}^n U_{\alpha_i}$.

Definition 8.17. A collection $\mathcal{C}$ of subsets of X has the *finite intersection property* if for every finite subset $\{C_1, \dots, C_n\}$ of $\mathcal{C}$, the intersection $\bigcap_{i=1}^n C_i$ is nonempty.

The following theorem gives a mostly complete description of compactness.

Theorem 8.18. *The following are equivalent:*

(1) *Every open covering has a finite subcovering.*
(2) *X is compact.*
(3) *For every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, $\bigcap_{C\in\mathcal{C}}$ is nonempty.*

Proof. (1) $\implies$ (2). Suppose X is compact and suppose $(x_\alpha)_{\alpha\in J}$ is a net of points in X with no convergent subnet. By Lemma 8.12, (x_α) has no accumulation point, so for every $x \in X$, there exists a neighbourhood U_x of x and $\alpha_x \in J$ such that $\alpha_x \preceq \beta$ implies $x_\beta \notin U$.

Clearly $\{U_x\}_{x\in X}$ is an open covering of X, so by compactness it admits a finite subcover $U_{x_1}, \ldots, U_{x_n}$. If $\beta \in J$ is such that $\alpha_{x_i} \preceq \beta$ for each i, then $x_\beta \notin \bigcup_{i=1}^n U_{x_i}$, a contradiction.

(2) $\implies$ (3). Suppose every net has a convergent subnet. Let $\mathcal{C}$ be any collection of closed sets having the finite intersection property. Let $\mathcal{B}$ be the set of finite intersections of elements in $\mathcal{C}$, ordered under reverse inclusion. $\mathcal{B}$ is directed, because the intersection of any two elements will always be contained in those two elements. For each $B \in \mathcal{B}$, B is nonempty; let $x_B \in B$. The net $(x_B)_{B\in\mathcal{B}}$ admits a subnet $(x_{g(i)})_{i\in K}$ converging to some point x. We claim that $x \in \bigcap_{C\in\mathcal{C}} C$, which shows that the latter set is nonempty. Indeed, for any $C \in \mathcal{C}$, we naturally have $C \in \mathcal{B}$, so that $x_C \in C$. Now there exists $g(i) \in g(K)$ such that $C \supseteq g(i)$, and thus for any $i \preceq j$, we have $g(i) \supseteq g(j)$. By transitivity, $C \supseteq g(j)$, so $x_{g(j)} \in g(j) \subseteq C$. Hence the tail of $(x_{g(i)})_{i\in K}$ is a sequence of points in C converging to x, so $x \in \overline{C} = C$. Since C was arbitrary, $x \in \bigcap_{C\in\mathcal{C}} C$, as desired.

(3) $\implies$ (1). We prove the contraposition of the open covering condition; namely if $\{U_\alpha\}_{\alpha\in I}$ is a set of open sets in X such that no finite subset covers X, then $\{U_\alpha\}_{\alpha\in I}$ does not cover X.

Let $\{U_\alpha\}_{\alpha\in I}$ be such a collection of open sets in X. Since each U_α is open, $X \setminus U_\alpha$ is closed. Hence $\mathcal{C} = \{X \setminus U_\alpha\}_{\alpha\in I}$ is a collection of closed sets in X. Since no finite subset covers X, for any finite subset $\{C_1 = X \setminus U_{\alpha_1}, \ldots, C_n = X \setminus U_{\alpha_1}\}$ of $\mathcal{C}$, we have

$$\bigcap_{i=1}^n C_i = \bigcap_{i=1}^n X \setminus U_{\alpha_i} = X \setminus \bigcup_{i=1}^n U_{\alpha_i},$$

and since $\bigcup_{i=1}^n U_{\alpha_i} \subsetneq X$ (otherwise the U_{α_i}'s would constitute a finite subset covering X), this is nonempty. Thus $\mathcal{C}$ has the finite intersection property, so by assumption $\bigcap_{C\in\mathcal{C}} C$ is nonempty. Now

$$\varnothing = \bigcap_{C\in\mathcal{C}} C = \bigcap_{\alpha\in I} X \setminus U_\alpha = X \setminus \bigcup_{\alpha\in I} U_\alpha,$$

showing that $\bigcup_{\alpha\in I} U_\alpha \subsetneq X$. So $\{U_\alpha\}_{\alpha\in I}$ does not cover X, as desired. □

References

[Axler] S. Axler. *Linear Algebra Done Right.* Springer, 2024.
[Euclid] Euclid. *Elements.* c. 300 BC.
[Munkres] J. Munkres. *Topology.* Prentice Hall, 2000.
[Spivak] M. Spivak. *Calculus.* Cambridge University Press, 2006.