

The Going-Up and Going-Down Theorems

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July 19, 2025

1 Introduction

We assume familiarity with the basic definitions of commutative rings and modules. The requisite background may be found in the first three chapters of Atiyah and Macdonald's book [AM18].

Commutative algebra is the study of commutative rings and their ideals. The prototypical commutative ring is \mathbb{Z} , which contains an ideal $(m) := m\mathbb{Z}$ for every integer m consisting of all multiples of m . Multiplying each integer by 2 gives a ring homomorphism $\phi : \mathbb{Z} \rightarrow 2\mathbb{Z}$. This sends the ideal (m) of \mathbb{Z} to the ideal $(2m)$ of $2\mathbb{Z}$. Going in the reverse direction we obtain, for each ideal $(2m)$ of $2\mathbb{Z}$, an ideal (m) of \mathbb{Z} by undoing the multiplication by 2, or simply by dividing by 2.

Likewise, embedding \mathbb{Z} as the constant polynomials in the polynomial ring $\mathbb{Z}[x]$ also gives a ring homomorphism. Yet the ideal (m) of \mathbb{Z} considered as the subset $m\mathbb{Z}$ of $\mathbb{Z}[x]$ is not an ideal unless $m = 0$. Nevertheless, it is easy to generate from $m\mathbb{Z}$ an ideal $m\mathbb{Z}[x]$ of $\mathbb{Z}[x]$. Conversely, for each ideal of $\mathbb{Z}[x]$ we obtain an ideal of \mathbb{Z} by undoing the embedding, which amounts to pulling out the constant term of each polynomial. We have thus illustrated that a ring homomorphism comes with a partial correspondence of ideals.

In these notes, we provide conditions under which this correspondence becomes complete.

2 Extension and Contraction

We begin by formalizing the correspondence of ideals under homomorphism suggested by the introduction. As a standing assumption, A and B will be commutative rings with unit.

Definition 2.1. Let $\phi : A \rightarrow B$ be a ring homomorphism. Let \mathfrak{a} be an ideal of A and \mathfrak{b} an ideal of B .

- (1) The *extension* of \mathfrak{a} under ϕ is $\mathfrak{a}_\phi^e := B\phi(\mathfrak{a})$.
- (2) The *contraction* of \mathfrak{b} under ϕ is $\mathfrak{b}_\phi^c := \phi^{-1}(\mathfrak{b})$.

We suppress the subscript when there is no ambiguity in the underlying map ϕ . The extension \mathfrak{a}^e is clearly an ideal of B ; in fact it is the ideal generated by $\phi(\mathfrak{a})$. Similarly, \mathfrak{b}^c is an ideal of A : the fact that \mathfrak{b}^c is an additive subgroup follows from the properties of \mathfrak{b} and ϕ ; if furthermore $x \in A$ and $y \in \mathfrak{b}^c$, then $\phi(y) \in \mathfrak{b}$ so $\phi(xy) = \phi(x)\phi(y) \in \mathfrak{b}$, hence $xy \in \mathfrak{b}^c$.

We hold a special interest in prime ideals, which behave nicely under contraction.

Lemma 2.2. Let $\phi : A \rightarrow B$ be a ring homomorphism and \mathfrak{q} a prime ideal of B . Then its contraction \mathfrak{q}^c is a prime ideal of A .

Proof. We already know \mathfrak{q}^c is an ideal. To see that it is moreover prime, let $xy \in \phi^{-1}(\mathfrak{q})$, so that $\phi(xy) = \phi(x)\phi(y) \in \mathfrak{q}$. Since \mathfrak{q} is a prime ideal, we must have $\phi(x) \in \mathfrak{q}$ or $\phi(y) \in \mathfrak{q}$. Thus $x \in \phi^{-1}(\mathfrak{q})$ or $y \in \phi^{-1}(\mathfrak{q})$. \square

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Example 2.3. Let $\phi : \mathbb{Z} \rightarrow \mathbb{Z}[x]$ be the inclusion and consider the prime ideal $2\mathbb{Z}[x]$ of $\mathbb{Z}[x]$. The contraction of $2\mathbb{Z}[x]$ is $2\mathbb{Z}[x] \cap \mathbb{Z} = (2)$, which is a prime ideal of \mathbb{Z} . ■

The corresponding statement for extension is false. For example, let $A = \mathbb{Z}$ and consider the prime ideal (2) of \mathbb{Z} . Let $B = \mathbb{Q}$ and let $\phi : \mathbb{Z} \rightarrow \mathbb{Q}$ be the inclusion. The extension of (2) is $\mathbb{Q}(2) = \mathbb{Q}$, which is not a prime ideal.

It follows from the set-theoretic properties $\mathfrak{a} \subseteq \phi^{-1}(\phi(\mathfrak{a}))$ and $\phi(\phi^{-1}(\mathfrak{b})) \subseteq \mathfrak{b}$ that $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$ and $\mathfrak{b}^{ce} \subseteq \mathfrak{b}$, respectively. From these inclusions, we get

$$\mathfrak{a}^e = \mathfrak{a}^{ece} \text{ and } \mathfrak{b}^c = \mathfrak{b}^{cec}. \quad (2.1)$$

When is a prime ideal of A necessarily the contraction of a prime ideal of B ? This is the motivating question of our work, and we conclude this section with our first attempt at a satisfying answer.

Lemma 2.4. *Let $\phi : A \rightarrow B$ be a ring homomorphism and let \mathfrak{p} be a prime ideal of A . Then \mathfrak{p} is the contraction of a prime ideal of B if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.*

Proof. If there is a prime ideal \mathfrak{q} of B such that $\mathfrak{q}^c = \mathfrak{p}$, then by eq. (2.1),

$$\mathfrak{p}^{ec} = \mathfrak{q}^{cec} = \mathfrak{q}^c = \mathfrak{p}.$$

Conversely, suppose $\mathfrak{p}^{ec} = \mathfrak{p}$. Define a multiplicatively closed subset of B by $S = \phi(A \setminus \mathfrak{p})$. Indeed generic elements of S take the form $\phi(x), \phi(y)$ where $x, y \in A \setminus \mathfrak{p}$. Since \mathfrak{p} is a prime ideal this implies $xy \in A \setminus \mathfrak{p}$, and thus $\phi(x)\phi(y) = \phi(xy) \in S$. Hence we may consider the localization $S^{-1}B$ with canonical map $\psi : B \rightarrow S^{-1}B$ given by $x \mapsto \frac{x}{1}$. By assumption

$$A \setminus \mathfrak{p} = A \setminus \mathfrak{p}^{ec} = A \setminus \phi^{-1}(\mathfrak{p}^e) = \phi^{-1}(B \setminus \mathfrak{p}^e),$$

so $S = \phi(A \setminus \mathfrak{p})$ does not intersect \mathfrak{p}^e . Thus $\psi(\mathfrak{p}^e)$ contains no units in $S^{-1}B$ and $(\mathfrak{p}^e)_\psi^e$ is a proper ideal of $S^{-1}B$. By Zorn's lemma, $(\mathfrak{p}^e)_\psi^e$ is contained in some maximal ideal \mathfrak{m} of $S^{-1}B$.

Let $\mathfrak{q} := \mathfrak{m}_\psi^c$. Lemma 2.2 shows that \mathfrak{q} is a prime ideal of B . Moreover $(\mathfrak{p}^e)_\psi^e \subseteq \mathfrak{m}$ implies

$$\mathfrak{p}^e \subseteq (\mathfrak{p}^e)_\psi^{ec} \subseteq \mathfrak{m}^c = \mathfrak{q}$$

and thus $\mathfrak{p} = \mathfrak{p}^{ec} \subseteq \mathfrak{q}^c$. We also know that \mathfrak{q} does not intersect S , otherwise \mathfrak{m} would contain a unit in $S^{-1}B$. Therefore \mathfrak{q}^c does not intersect $A \setminus \mathfrak{p}$; that is $\mathfrak{q}^c \subseteq \mathfrak{p}$. □

3 The Cayley-Hamilton Theorem

The Cayley-Hamilton theorem is a fundamental result in linear algebra which states that a linear operator satisfies its own characteristic polynomial. We provide a generalization to finitely-generated modules. Its significance will become apparent in the next section, when we study polynomial dependence in more detail.

Theorem 3.1 (Cayley-Hamilton). *Let M be a finitely-generated A -module, \mathfrak{a} an ideal of A , and $\phi : M \rightarrow M$ an A -module homomorphism such that $\phi(M) \subseteq \mathfrak{a}M$. Then there exist $a_1, \dots, a_n \in \mathfrak{a}$ such that*

$$\phi^n + a_1\phi^{n-1} + \dots + a_n = 0.$$

Proof. Let x_1, \dots, x_n generate M . For $1 \leq i \leq n$ we have $\phi(x_i) \in \phi(M) \subseteq \mathfrak{a}M$, so we may write

$$\phi(x_i) = \sum_{j=1}^n a_{ij}x_j$$

for $a_{ij} \in \mathfrak{a}$. In other words if δ_{ij} is the Kronecker delta, then

$$\sum_{j=1}^n (\delta_{ij}\phi - a_{ij})x_j = 0.$$

If $A = (a_{ij})$, I is the identity matrix, and $x = (x_1, \dots, x_n)$, we can write this as $(I\phi - A)x = 0$. Now multiplying on the left by the adjugate matrix of $I\phi - A$,

$$\det(I\phi - A)x = \text{adj}(I\phi - A)(I\phi - A)x = 0.$$

Hence $\det(I\phi - A) = 0$, and we obtain the desired equation by expanding the determinant. \square

4 Integral Dependence

A classical goal of field theory was to determine, given a base field K , which polynomials in $K[x]$ (i.e. with coefficients in K) have roots in K , and to determine, for those polynomials with no roots in K , over which field extension of K we may find its roots. More formally, let L be a field extension of K . We say $\alpha \in L$ is *algebraic* over K if it is the root of some polynomial in $K[x]$.

The analogue of algebraic dependence in the commutative ring setting is integral dependence. In this case we impose the further requirement that the polynomial be monic.

Definition 4.1. Let A be a subring of B . An element $x \in B$ is *integral* over A if it is the root of a monic polynomial in $A[x]$; that is, there exist $a_1, \dots, a_n \in A$ such that

$$x^n + a_1x^{n-1} + \dots + a_n = 0. \quad (4.1)$$

Every $x \in A$ is integral over A by $x - x = 0$. The requirement that the polynomial in eq. (4.1) be monic is nontrivial. For example, every rational number $\frac{r}{s} \in \mathbb{Q}$ is algebraic over \mathbb{Z} since $s \cdot \frac{r}{s} - r = 0$, but only integers are integral over \mathbb{Z} : if $\frac{r}{s}$ is in lowest terms, we have

$$\left(\frac{r}{s}\right)^n + a_1\left(\frac{r}{s}\right)^{n-1} + \dots + a_{n-1} = 0,$$

and then

$$r^n + sa_1r^{n-1} + \dots + s^n a_{n-1} = 0,$$

so s divides r^n . Hence $s = \pm 1$, thus $\frac{r}{s} \in \mathbb{Z}$.

Henceforth, A will be a subring of B . The following characterization of integral elements will be useful, and notably features our first deployment of the Cayley-Hamilton theorem.

Theorem 4.2. *The following are equivalent:*

- (1) $x \in B$ is integral over A .
- (2) $A[x]$ is a finitely-generated A -module.
- (3) $A[x] \subseteq C$ for some subring C of B which is a finitely-generated A -module.
- (4) There exists a faithful $A[x]$ -module M which is finitely-generated as an A -module.

Proof. (1) \implies (2). Since $x \in B$ is integral over A we have

$$x^n + a_1x^{n-1} + \dots + a_n = 0$$

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for some $a_i \in A$. We claim that $1, x, \dots, x^{n-1}$ generates $A[x]$ as an A -module. For each $r \geq 0$, we have

$$x^{n+r} = -(a_1 x^{n+r-1} + \dots + a_n x^r).$$

By induction on r , each power of x is in the A -module generated by $1, x, \dots, x^{n-1}$, as desired.

(2) \implies (3). Let $C := A[x]$. This is a subring of B and a finitely-generated A -module by assumption.

(3) \implies (4). Let $M := C$. This is a faithful $A[x]$ -module because if $yM = 0$ then in particular $y1 = 0$, and a finitely-generated A -module by assumption.

(4) \implies (1). We use Theorem 3.1 (Cayley-Hamilton) with ϕ being multiplication by x and $\mathfrak{a} = A$. ϕ is an A -module homomorphism such that $xM \subseteq M$ since M is an $A[x]$ -module. We conclude that there exist $a_1, \dots, a_n \in A$ such that

$$x^n + a_1 x^{n-1} + \dots + a_n$$

annihilates M ; since M is faithful it must be zero. \square

As a corollary of the equivalence of (1) and (2), if $x, y \in B$ are integral over A , then $A[x]$ is a finitely-generated A -module. Furthermore y is integral over $A[x]$, so $A[x, y] = A[x][y]$ is a finitely-generated $A[x]$ -module, thus a finitely-generated A -module. Now $x + y \in A[x, y]$ so $A[x + y] \subseteq A[x, y]$, where $A[x, y]$ is a subring of B and a finitely-generated A -module. By the equivalence of (3) and (1), $x + y \in B$ is integral over A . Similarly $xy \in A[x, y]$ so $xy \in B$ is integral over A . This justifies the following definition.

Definition 4.3. The *integral closure* of A in B is the subring \overline{A} of elements in B which are integral over A .

If $A = \overline{A}$ we say A is integrally closed in B , if $\overline{A} = B$ we say B is integral over A . The latter is a property that is preserved under quotients and localizations.

Lemma 4.4. Let B be integral over A . Let $\iota : A \hookrightarrow B$ be the inclusion map.

- (1) Let \mathfrak{b} be an ideal of B and $\mathfrak{a} := \mathfrak{b}_\iota^c = A \cap \mathfrak{b}$ its contraction under the inclusion. Then B/\mathfrak{b} is integral over A/\mathfrak{a} .
- (2) Let S is a multiplicatively closed subset of A , and thus B . Then $S^{-1}B$ is integral over $S^{-1}A$.

Proof. (1) Given $\bar{x} \in B/\mathfrak{b}$, since $x \in B$ and B is integral over A we have

$$x^n + a_1 x^{n-1} + \dots + a_n = 0$$

for some $a_i \in A$. Reducing modulo \mathfrak{b} ,

$$\bar{x}^n + \bar{a}_1 \bar{x}^{n-1} + \dots + \bar{a}_n = \bar{0},$$

where $\bar{a}_i \in A/(A \cap \mathfrak{b}) = A/\mathfrak{a}$.

- (2) Given $\frac{x}{s} \in S^{-1}B$, since $x \in B$ and B is integral over A we have

$$x^n + a_1 x^{n-1} + \dots + a_n = 0$$

for some $a_i \in A$. Then

$$\left(\frac{x}{s}\right)^n + \left(\frac{a_1}{s}\right)\left(\frac{x}{s}\right)^{n-1} + \dots + \frac{a_n}{s^n} = 0,$$

where $\frac{a_i}{s} \in S^{-1}A$. \square

Recall that a ring is an integral domain if it has no zero divisors. This strengthens the notion of integral dependence to provide the following dependence of fields.

Lemma 4.5. *Let $A \subseteq B$ be integral domains, and let B be integral over A . Then B is a field if and only if A is a field.*

Proof. (\implies) Suppose B is a field. Let $x \in A$ be nonzero. Then $x^{-1} \in B$, so it is integral over A ; that is,

$$x^{-m} + a_1 x^{-m+1} + \cdots + a_m = 0$$

for some $a_i \in A$. Multiplying by x^{m-1} ,

$$x^{-1} = -(a_1 + a_2 x + \cdots + a_m x^{m-1}) \in A.$$

(\impliedby) Suppose A is a field. Let $y \in B$ be nonzero. Since y is integral over A , we have

$$y^n + a'_1 y^{n-1} + \cdots + a'_n = 0 \tag{4.2}$$

for $a'_i \in A$. We may assume that this has minimal degree among all equations of integral dependence, so that $a'_n \neq 0$. Otherwise, the equation eq. (4.2) could be written as

$$y(y^{n-1} + a'_1 y^{n-2} + \cdots + a'_{n-1}) = 0.$$

Then since B is an integral domain, $y^{n-1} + a'_1 y^{n-2} + \cdots + a'_{n-1} = 0$, contradicting minimality in eq. (4.2). We conclude that

$$y^{-1} = -(a'_n)^{-1}(y^{n-1} + a'_1 y^{n-2} + \cdots + a'_{n-1}) \in B.$$

□

Along with the natural behaviour of quotients, the previous statement about fields unsurprisingly yields a statement about maximal ideals.

Lemma 4.6. *Let B be integral over A . Let \mathfrak{q} be a prime ideal of B and let $\mathfrak{p} := \mathfrak{q}^c = \mathfrak{q} \cap A$ be its contraction under the inclusion $A \hookrightarrow B$. Then \mathfrak{q} is maximal if and only if \mathfrak{p} is maximal.*

Proof. We know that \mathfrak{q} is a prime ideal of B and $\mathfrak{p} = \mathfrak{q} \cap A$ is a prime ideal of A and, so B/\mathfrak{q} and A/\mathfrak{p} are integral domains with $A/\mathfrak{p} \subseteq B/\mathfrak{q}$. By Lemma 4.4, B/\mathfrak{q} is integral over A/\mathfrak{p} . By Lemma 4.5, B/\mathfrak{q} is a field if and only if A/\mathfrak{p} is a field, or equivalently \mathfrak{q} is a maximal ideal of B if and only if \mathfrak{p} is a maximal ideal of A . □

We can make the definition of integral dependence without A being a subring. The only other case we are interested in is integral dependence over an ideal, which is completely characterized by the following lemma.

Lemma 4.7. *Let \mathfrak{a} be an ideal of A , let $A \subseteq B$, and let $\iota : A \hookrightarrow \overline{A}$ be the inclusion map. Then the integral closure of \mathfrak{a} in B is $\sqrt{\mathfrak{a}_t^e}$.*

Proof. If $x \in B$ is integral over \mathfrak{a} , then there exist $a_1, \dots, a_n \in \mathfrak{a}$ such that

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

In particular $a_1, \dots, a_n \in A$ implies $x \in \overline{A}$, so $x^n = -(a_1 x^{n-1} + \cdots + a_n) \in \overline{A}\mathfrak{a} = \mathfrak{a}_t^e$ implies $x \in \sqrt{\mathfrak{a}_t^e}$.

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Conversely if $x \in \sqrt{\mathfrak{a}_t^e}$ then we may write

$$x^k = \sum_{i=1}^m c_i x_i \quad (4.3)$$

for some k , $c_i \in \mathfrak{a}$ and $x_i \in \overline{A}$. Since each x_i is integral over A , the A -module $M := A[x_1, \dots, x_m]$ is finitely-generated, and eq. (4.3) shows that $x^k M \subseteq \mathfrak{a}M$. By taking ϕ to be multiplication by x^k in Theorem 3.1, there exist $a_1, \dots, a_n \in \mathfrak{a}$ such that

$$(x^k)^n + a_1(x^k)^{n-1} + \dots + a_n = 0.$$

Thus x is integral over \mathfrak{a} . □

Now looking at integral elements over an ideal, we can characterize its minimal polynomial.

Lemma 4.8. *Let $A \subseteq B$ be integral domains, let A be integrally closed in its field of fractions $K(A)$, let \mathfrak{a} be an ideal of A , and let $x \in B$ be integral over \mathfrak{a} . Then x is algebraic over $K(A)$, and if its minimal polynomial over $K(A)$ is $t^m + b_1 t^{m-1} + \dots + b_m$, then $b_1, \dots, b_m \in \sqrt{\mathfrak{a}}$.*

Proof. Since x is integral over \mathfrak{a} , it satisfies an equation of integral dependence

$$x^n + a_1 x^{n-1} + \dots + a_n = 0, \quad (4.4)$$

where $a_1, \dots, a_n \in \mathfrak{a}$. Since $\mathfrak{a} \subseteq A \subseteq K(A)$, x is certainly algebraic over $K(A)$. Let

$$t^m + b_1 t^{m-1} + \dots + b_m \quad (4.5)$$

be the minimal polynomial of x over $K(A)$. In particular eq. (4.5) divides the left side of eq. (4.4). Let L be the algebraic closure of $K(A)$, so that L contains all the roots x_1, \dots, x_n of eq. (4.5). Then each x_i satisfies eq. (4.4), so each x_i is integral over \mathfrak{a} . If \overline{A}_L is the integral closure of A in L and $\iota : A \hookrightarrow \overline{A}_L$ is the inclusion map, then by Lemma 4.7, $x_i \in \sqrt{\mathfrak{a}_t^e}$.

By Vieta's formulas, the coefficients b_1, \dots, b_m of eq. (4.5) are polynomials in the x_i , so they also lie in $\sqrt{\mathfrak{a}_t^e}$ and thus are integral over \mathfrak{a} . Since A is integrally closed in $K(A)$, Lemma 4.7 states that the integral closure of \mathfrak{a} in $K(A)$ is simply $\sqrt{\mathfrak{a}}$. Therefore $b_1, \dots, b_m \in \sqrt{\mathfrak{a}}$. □

5 Lying-Over, Going-Up, and Going-Down

The three procedures of interest are introduced in the following definition.

Definition 5.1. Let $\phi : A \rightarrow B$ be a ring homomorphism.

- (1) Let \mathfrak{p} be a prime ideal of A . A prime ideal \mathfrak{q} of B *lies over* \mathfrak{p} if $\mathfrak{p} = \mathfrak{q}^c$.
- (2) ϕ has the *going-up property* if given any chain $\mathfrak{p}_1 \subseteq \dots \subseteq \mathfrak{p}_n$ of prime ideals of A and any chain $\mathfrak{q}_1 \subseteq \dots \subseteq \mathfrak{q}_m$ of prime ideals of B such that $m < n$ and \mathfrak{q}_j lies over \mathfrak{p}_j for $1 \leq j \leq m$, the latter chain may be extended to a chain $\mathfrak{q}_1 \subseteq \dots \subseteq \mathfrak{q}_n$ such that each \mathfrak{q}_j lies over \mathfrak{p}_j .
- (3) ϕ has the *going-down property* if given any chain $\mathfrak{p}_1 \supseteq \dots \supseteq \mathfrak{p}_n$ of prime ideals of A and any chain $\mathfrak{q}_1 \supseteq \dots \supseteq \mathfrak{q}_m$ of prime ideals of B such that $m < n$ and each \mathfrak{q}_j lies over \mathfrak{p}_j , the latter chain may be extended to a chain $\mathfrak{q}_1 \supseteq \dots \supseteq \mathfrak{q}_n$ such that each \mathfrak{q}_j lies over \mathfrak{p}_j .

From now on, we will implicitly extend and contract ideals under the inclusion $A \hookrightarrow B$, meaning $\mathfrak{a}^e = B\mathfrak{a}$ and $\mathfrak{b}^c = \mathfrak{b} \cap A$.

Theorem 5.2 (Lying-over). *Let B be integral over A and \mathfrak{p} a prime ideal of A . Then there exists a prime ideal \mathfrak{q} of B that lies over \mathfrak{p} .*

Proof. The following diagram commutes:

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \alpha \downarrow & & \downarrow \beta \\ A_{\mathfrak{p}} & \hookrightarrow & B_{\mathfrak{p}} \end{array} \quad (5.1)$$

By Lemma 4.4, $B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$. Let \mathfrak{n} be a maximal ideal of $B_{\mathfrak{p}}$. Then $\mathfrak{m} := \mathfrak{n}^c = \mathfrak{n} \cap A_{\mathfrak{p}}$ is a maximal ideal of $A_{\mathfrak{p}}$ by Lemma 4.6. In fact $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{p}_{\alpha}^e := \mathfrak{p}A_{\mathfrak{p}}$, since every element not contained in $\mathfrak{p}A_{\mathfrak{p}}$ is a unit. Let $\mathfrak{q} := \mathfrak{n}_{\beta}^c = \beta^{-1}(\mathfrak{n})$. Then \mathfrak{q} is a prime ideal of B and we claim that

$$\mathfrak{q} \cap A = \alpha^{-1}(\mathfrak{m}) = \alpha^{-1}(\mathfrak{p}A_{\mathfrak{p}}) = \mathfrak{p}.$$

We have $x \in \mathfrak{q} \cap A$ if and only if $x \in A$ and $\beta(x) \in \mathfrak{n}$. Since eq. (5.1) commutes, this is equivalent to $\alpha(x) \in \mathfrak{m}$, hence $\mathfrak{q} \cap A = \alpha^{-1}(\mathfrak{m}) = \alpha^{-1}(\mathfrak{p}A_{\mathfrak{p}})$. Now $x \in \alpha^{-1}(\mathfrak{p}A_{\mathfrak{p}})$ if and only if $\frac{x}{1} \in \mathfrak{p}A_{\mathfrak{p}}$, or $x \in \mathfrak{p}$. Therefore \mathfrak{q} lies over \mathfrak{p} . \square

The going-up property is now a direct consequence of the lying-over theorem.

Theorem 5.3 (Going-up). *Let B be integral over A . Then $A \hookrightarrow B$ has the going-up property.*

Proof. By induction, it suffices to show that given prime ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ of A and a prime ideal \mathfrak{q}_1 of B lying over \mathfrak{p}_1 , there exists a prime ideal \mathfrak{q}_2 of B that lies over \mathfrak{p}_2 .

We know that $A/\mathfrak{p}_1 \subseteq B/\mathfrak{q}_1$ are rings, B/\mathfrak{q}_1 is integral over A/\mathfrak{p}_1 by Lemma 4.4, and $\mathfrak{p}_2/\mathfrak{p}_1$ is a prime ideal of A/\mathfrak{p}_1 . By Theorem 5.2, there exists a prime ideal of B/\mathfrak{q}_1 that lies over $\mathfrak{p}_2/\mathfrak{p}_1$. By the fourth isomorphism theorem this prime ideal takes the form $\mathfrak{q}_2/\mathfrak{q}_1$ for some prime ideal \mathfrak{q}_2 of B . In particular $(\mathfrak{q}_2/\mathfrak{q}_1) \cap (A/\mathfrak{p}_1) = \mathfrak{p}_2/\mathfrak{p}_1$, hence $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$ meaning \mathfrak{q}_2 lies over \mathfrak{p}_2 . \square

With stronger assumptions and a more delicate proof, we pick up the going-down property as well.

Theorem 5.4 (Going-down). *Let $A \subseteq B$ be integral domains, A integrally closed in its field of fractions $K(A)$, and B integral over A . Then $A \hookrightarrow B$ has the going-down property.*

Proof. As in the proof of going-up, it suffices to show that given prime ideals $\mathfrak{p}_1 \supseteq \mathfrak{p}_2$ of A and a prime ideal \mathfrak{q}_1 of B lying over \mathfrak{p}_1 , \mathfrak{p}_2 is the contraction of a prime ideal $\mathfrak{q}_2 \subseteq \mathfrak{q}_1$. Note that $\mathfrak{q}_2 \subseteq \mathfrak{q}_1$ if and only if \mathfrak{q}_2 is a prime ideal in $B_{\mathfrak{q}_1}$. Let $\beta : A \rightarrow B_{\mathfrak{q}_1}$ be the restriction of the canonical homomorphism $x \mapsto \frac{x}{1}$ to A . By Lemma 2.4, \mathfrak{p}_2 is the contraction of a prime ideal of $B_{\mathfrak{q}_1}$ if and only if

$$\mathfrak{p}_2 = ((\mathfrak{p}_2)_{\beta}^e)_{\beta}^c = (B_{\mathfrak{q}_1}\mathfrak{p}_2)_{\beta}^c = B_{\mathfrak{q}_1}\mathfrak{p}_2 \cap A.$$

The \subseteq inclusion is immediate since $1 \in B_{\mathfrak{q}_1}$. It remains to show that $B_{\mathfrak{q}_1}\mathfrak{p}_2 \cap A \subseteq \mathfrak{p}_2$. Every element of $B_{\mathfrak{q}_1}\mathfrak{p}_2$ may be written as $\frac{y}{s}$, where $y \in B\mathfrak{p}_2$ and $s \in B \setminus \mathfrak{q}_1$. In particular $y \in \mathfrak{p}_2^e \subseteq \sqrt{\mathfrak{p}_2^e}$, where the extension is under the inclusion $A \hookrightarrow B$ as usual. Since B is precisely the integral closure of A in $K(A)$, Lemma 4.7 shows that y is integral over \mathfrak{p}_2 . Now applying Lemma 4.8, y is algebraic over $K(A)$ and its minimal equation over $K(A)$ takes the form

$$y^r + u_1 y^{r-1} + \cdots + u_r = 0, \quad (5.2)$$

where $u_1, \dots, u_r \in \mathfrak{p}_2$. Let $x = \frac{y}{s} \in B_{\mathfrak{q}_1}\mathfrak{p}_2 \cap A$, so that $s = yx^{-1}$ for $x^{-1} \in K(A)$. Then the minimal equation for s over $K(A)$ is obtained by multiplying eq. (5.2) by x^{-r} , yielding

$$s^r + v_1 s^{r-1} + \cdots + v_r = 0 \quad (5.3)$$

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where $v_i = u_i x^{-i}$. Hence $x^i v_i = u_i \in \mathfrak{p}_2$ for $1 \leq i \leq r$. But $s \in B$ is integral over A , so by applying Lemma 4.8 with A itself as the ideal, $v_i \in A$ for $1 \leq i \leq r$.

If we had $x \notin \mathfrak{p}_2$, then as \mathfrak{p}_2 is a prime ideal of A and $x^i v_i \in \mathfrak{p}_2$, we would have $v_i \in \mathfrak{p}_2$. Then

$$s^r = -(v_1 s^{r-1} + \cdots + v_r) \in B\mathfrak{p}_2 \subseteq B\mathfrak{p}_1 \subseteq \mathfrak{q}_1,$$

where the last containment holds because $B\mathfrak{p}_1 \subseteq B\mathfrak{q}_1 = \mathfrak{q}_1$. Since \mathfrak{q}_1 is a prime ideal of B , this implies $s \in \mathfrak{q}_1$, which is a contradiction. Therefore $x \in \mathfrak{p}_2$, and $B_{\mathfrak{q}_1}\mathfrak{p}_2 \cap A = \mathfrak{p}_2$, as desired. \square

References

- [AM18] M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra*. CRC Press, Taylor & Francis Group, 2018. Reprint of the 1969 edition.