Measurable functions

William Gao

December 27, 2025

Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces. We call a function $f\colon E \to G$ *measurable* if for all $A \in \mathcal{G}$, $f^{-1}A$ is in $\mathcal{E}$. It is worth noting the similarity to the definition of a continuous map between topological spaces.

If $\mathcal{E}$ is a Borel measure $\mathcal{B}(E)$ and $G = \mathbb{R}$ with the standard Borel measure, then a measurable function on $(E, \mathcal{E})$ is called a Borel function on E.

- Since $f^{-1}\varnothing = \varnothing$ and preimage commutes with union and complement, we can pull back a σ-algebra $\mathcal{G}$ on G along $f\colon E \to G$ to get a σ-algebra

  $$f^{-1}(\mathcal{G}) = \{f^{-1}(A) : A \in \mathcal{G}\}$$

  on E. Clearly f is measurable with respect to $f^{-1}\mathcal{G}$.

- Let $\mathcal{G} = \sigma(\mathcal{A})$. If $f\colon E \to G$ is such that $f^{-1}(A) \in \mathcal{E}$ for all $A \in \mathcal{A}$, then let

  $$\mathcal{M} = \{A \subset G : f^{-1}(A) \in \mathcal{E}\}$$

  This is a σ-algebra: $\varnothing \in \mathcal{M}$ as $f^{-1}\varnothing = \varnothing$ and $\mathcal{M}$ is closed under complement and countable union because these commute with f^{-1} and $\mathcal{E}$ is a σ-algebra. By assumption $\mathcal{M}$ contains $\mathcal{A}$, so it contains $\mathcal{G}$. In other words, f is measurable.

- We claim that $f\colon (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B})$ is measurable if and only if $f^{-1}(-\infty, \lambda) \in \mathcal{E}$ for all $\lambda \in \mathbb{R}$. The forward implication is obvious as these open rays lie in $\mathcal{E}$; conversely let $\mathcal{A}$ be the set of these open rays. By topology, we know that $\sigma(\mathcal{A}) = \mathcal{B}$, so by the previous bullet point f is measurable.

- Any continuous map $f\colon (E, \mathcal{B}(E)) \to (G, \mathcal{B}(G))$ is measurable. Indeed, this is another application of the second bullet point, as the open sets of G generate the σ-algebra $\mathcal{B}(G)$.

- If $f\colon E \to G$ is measurable and $g\colon G \to \mathbb{R}$ is continuous, then $g \circ f\colon E \to \mathbb{R}$ is measurable. Indeed, by the second bullet point again it suffices to show that the preimage of an open set in $\mathbb{R}$ under $g \circ f$ is measurable. Let $U \subset \mathbb{R}$ be open. By continuity $g^{-1}(U)$ is open in G, hence Borel. Then $(g \circ f)^{-1}(U)$ is in $\mathcal{E}$.

- $f\colon E \to \mathbb{R}^n$ is measurable if and only if each $\pi_i \circ f\colon E \to \mathbb{R}$ is measurable. The forward direction follows from the previous bullet point, as π_i is continuous. Conversely if each f_i is measurable, then it suffices to check f^{-1} of an open is measurable. This is clearly true.

- By similar arguments we can show that linear combinations, products, infima, suprema, limit infima, limit suprema, positive parts, negative parts, and absolute values of measurable functions are all measurable.