Chapter 1

Lie Algebras

Let k be a commutative ring with unit (usually a field) and A a k-algebra; that is, a k-module with a k-bilinear map

$$[\cdot,\cdot]\colon A \times A \to A,$$

or equivalently a k-linear map

$$[\cdot,\cdot]\colon A \otimes_k A \to A.$$

Definition 1.0.1

A *Lie algebra* over k is a k-algebra such that

(1) $[\cdot,\cdot]\colon A \otimes_k A \to A$ admits a factorization

$$A \otimes_k A \to \Lambda^2 A \to A.$$

Equivalently $[x,x] = 0$ for all $x \in A$, or $[x,y] = -[y,x]$ for all $x, y \in A$.

(2) The Jacobi identity

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$$

for all $x, y, z \in A$.

The set $\mathrm{Der}(A)$ of derivations of an algebra A is a Lie algebra with Lie bracket

$$[D, D'] = DD' - D'D.$$

Theorem 1.0.2

Let $\mathfrak{g}$ be a Lie algebra. For $x \in \mathfrak{g}$ define

$$\begin{aligned} \operatorname{ad} x \colon \mathfrak{g} &\longrightarrow \mathfrak{g} \\ y &\longmapsto [x, y]. \end{aligned}$$

Then

(1) $\operatorname{ad} x$ is a derivation of $\mathfrak{g}$.

(2) The map

$$\begin{aligned} \mathfrak{g} &\longrightarrow \operatorname{Der}(\mathfrak{g}) \\ x &\longmapsto \operatorname{ad} x \end{aligned}$$

is a Lie algebra homomorphism.

Proof. This is just a computation:

$$\begin{aligned} \operatorname{ad} x[y, z] &= [x, [y, z]] \\ &= -[y, [z, x]] - [z, [x, y]] \\ &= [y, [x, z]] + [[x, y], z] \\ &= [y, \operatorname{ad} x(z)] + [\operatorname{ad} x(y), z], \end{aligned}$$

so $\operatorname{ad} x$ is a derivation. Next

$$\begin{aligned} \operatorname{ad}[x, y](z) &= [[x, y], z] \\ &= -[[y, z], x] - [[z, x], y] \\ &= [x, [y, z]] - [y, [x, z]] \\ &= \operatorname{ad} x \circ \operatorname{ad} y(z) - \operatorname{ad} y \circ \operatorname{ad} x(z) \\ &= [\operatorname{ad} x, \operatorname{ad} y](z). \end{aligned}$$

□

1.1 The Lie algebra of an algebraic matrix group

Given polynomials $(P_\alpha)_\alpha \subset k[X_{11}, \ldots, X_{nn}]$, let $G(k)$ denote the set of matrices $(x_{ij})_{1 \leq i,j \leq n} \in \mathrm{GL}_n(k)$ such that $P_\alpha(x_{ij}) = 0$ for each α. More generally, for any associative commutative k-algebra k', we define $G(k')$ to be the set of common zeroes of $(P_\alpha)_\alpha$ in $\mathrm{GL}_n(k')$.

We have thus associated a set $G(k')$ to each k-algebra k', and this is functorial: a k-algebra homomorphism $\phi \colon k' \to k''$ induces a function

$$\begin{aligned} \phi_* \colon G(k') &\longrightarrow G(k'') \\ (x_{ij}) &\longmapsto (\phi(x_{ij})). \end{aligned}$$

Indeed since ϕ is a k-algebra homomorphism, we have

$$P_\alpha(\phi_*(x_{ij})) = \phi(P_\alpha(x_{ij})),$$

so if $x_{ij} \in G(k')$, then $\phi_*(x_{ij}) \in G(k'')$. Functoriality is clear from the definition.

If $G(k')$ happens to be a subgroup of $\mathrm{GL}_n(k')$ for all associative commutative k-algebras k', then this functor $G\colon \mathsf{Alg}_k \to \mathsf{Grp}$ is called an *algebraic group* over k.

The reader is likely familiar with certain algebraic groups:

Example 1.1.1. Consider the polynomials $P_{ij}(X) = (X^t X - I)_{ij} \in \mathbb{Z}[X_{11}, \ldots, X_{nn}]$, for $1 \leq i, j \leq n$. This exhibits the orthogonal group O_n as an algebraic group over $\mathbb{Z}$. Since $\mathbb{Z}$-algebras are precisely rings, for each ring R we get an orthogonal group $\mathrm{O}_n(R)$ over R.

Now let $k' := k[\epsilon]/(\epsilon^2)$.

Theorem 1.1.2

Let G be an algebraic group. Let $\mathfrak{g}$ be the set of matrices $X \in M_n(k)$ such that

$$1 + \epsilon X \in G(k[\epsilon]).$$

Then $\mathfrak{g}$ is a Lie subalgebra of $M_n(k)$, called the Lie algebra of G.

Proof. Let $X, Y \in \mathfrak{g}$. We wish to show that $\lambda X + \mu Y \in \mathfrak{g}$ for $\lambda, \mu \in k$ and $XY - YX \in \mathfrak{g}$. Firstly, note that $\epsilon^2 = 0$ implies that

$$(1 + \epsilon X_{ij})^n = 1 + n\epsilon X_{ij},$$

so

$$P_\alpha(1 + \epsilon X) = P_\alpha(1) + dP_\alpha(1)\epsilon X \quad \text{for} \quad X \in \mathfrak{g}.$$

Moreover $P_\alpha(1) = 0$ since $1 \in G(k)$ under the assumption that G is an algebraic group. It follows that for $\lambda, \mu \in k$,

$$P_\alpha(1 + \epsilon(\lambda X + \mu Y)) = dP_\alpha(1)\epsilon(\lambda X + \mu Y) = \lambda dP_\alpha(1)\epsilon X + \mu dP_\alpha(1)\epsilon Y = 0,$$

hence $\mathfrak{g}$ is a k-submodule of $M_n(k)$.

Now consider $k'' := k[\epsilon]/(\epsilon^2) \otimes_k k[\epsilon']/(\epsilon')^2$. We have

$$g := 1 + \epsilon X \in G(k[\epsilon]/(\epsilon^2)) \subset G(k'')$$
$$g' := 1 + \epsilon' Y \in G(k[\epsilon']/(\epsilon')^2) \subset G(k''),$$

Moreover

$$gg' = 1 + \epsilon X + \epsilon' Y + \epsilon\epsilon' XY$$
$$g'g = 1 + \epsilon X + \epsilon' Y + \epsilon\epsilon' YX,$$

so

$$gg' = g'g(1 + \epsilon\epsilon'[X, Y]).$$

This means $1 + \epsilon\epsilon'[X, Y] \in G(k'')$, once again by the assumption that G is an algebraic group. But $k[\epsilon\epsilon'] \subset k''$ is naturally identified with $k[\epsilon]$, so $1 + \epsilon[X, Y] \in G(k[\epsilon]/(\epsilon^2))$, showing that $[X, Y] \in \mathfrak{g}$. $\square$

Example 1.1.3. The Lie algebra of the orthogonal is the set of matrices X such that $(1+\epsilon X)(1+\epsilon X^t)=1$, or $X+X^t=0$. We have thus recovered the familiar definition of $\mathfrak{o}_n$.

1.2 Constructions of Lie algebras

1.2.1 Quotients by ideals

Let $\mathfrak{g}$ be a Lie algebra and $J \subset \mathfrak{g}$ an ideal, in the sense that $[X,Y] \in J$ for all $X \in \mathfrak{g}$, $Y \in J$. Then $\mathfrak{g}/J$ is a Lie algebra with bracket
$$[X+J, Y+J] = [X,Y]+J.$$

1.2.2 Products

Let $(\mathfrak{g}_i)_{i\in I}$ be Lie algebras. Then $\prod_{i\in I}\mathfrak{g}_i$ is a Lie algebra with componentwise bracket.

1.2.3 Semidirect products

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ an ideal, and $\mathfrak{a} \subset \mathfrak{g}$ a subalgebra. Then $\mathfrak{g}$ is a *semidirect product* of $\mathfrak{a}$ by $\mathfrak{h}$ if the canonical map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ induces an isomorphism $\mathfrak{a} \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h}$. In this case for $x \in \mathfrak{a}$, $\operatorname{ad} x(\mathfrak{h}) \subset \mathfrak{h}$ and thus

$$\begin{aligned}\theta\colon \mathfrak{a} &\longrightarrow \operatorname{Der}(\mathfrak{h})\\ x &\longmapsto \operatorname{ad}_{\mathfrak{h}} x.\end{aligned}$$

Theorem 1.2.1

$\mathfrak{g}$ is determined by $\mathfrak{a}$, $\mathfrak{h}$, and θ.

Remark 1.2.2. This statement is slightly vague, but the essence should be clear from the proof.

Proof. As k-modules, we have $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$. It remains to show that $[x,y]$ for $x, y \in \mathfrak{g}$ is determined by $\mathfrak{a}$, $\mathfrak{h}$, and θ. Firstly if $x, y \in \mathfrak{a}$ then $[x,y]$ is given by the bracket in $\mathfrak{a}$, and similarly if $x, y \in \mathfrak{h}$ Now if $x \in \mathfrak{a}$, $y \in \mathfrak{h}$ then
$$[x,y] = \operatorname{ad} x(y) = \theta(x)y.$$
Conversely, given Lie algebras $\mathfrak{a}$, $\mathfrak{h}$ and a Lie homomorphism $\theta\colon \mathfrak{a} \to \operatorname{Der}(\mathfrak{h})$, we can construct a semidirect product $\mathfrak{g}$ of $\mathfrak{a}$ by $\mathfrak{h}$ so that $\theta(x) = \operatorname{ad}_{\mathfrak{h}} x$. This amounts to checking the Jacobi identity. If $x, y, z \in \mathfrak{a}$ or $x, y, z \in \mathfrak{h}$ then the Jacobi identity is true. If $x, y \in \mathfrak{h}$ and $z \in \mathfrak{a}$ then the Jacobi identity is equivalent to $\theta(z)$ being a derivation of $\mathfrak{a}$. If $x \in \mathfrak{h}$ and $y, z \in \mathfrak{a}$ then the Jacobi identity is equivalent to

$$\theta([y,z]) = \theta(y)\theta(z) - \theta(z)\theta(y) = [\theta(y), \theta(z)],$$

which is true because θ is a Lie homomorphism. $\square$