Cobordism and the Generalized Poincaré Conjecture

William Gao

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Contents

1 Introduction

Notes for a talk given at the University of Toronto Math Union Colloquium on September 4, 2025. They are based heavily on [LM24] and [Mil65].

1.1 Smooth manifolds

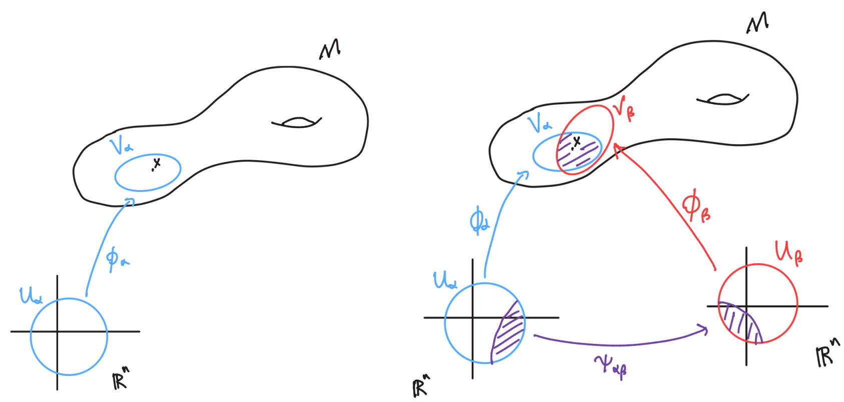
We begin with an imprecise definition of a smooth manifold. To make it rigorous we should really worry about compatibility of charts, which can be resolved by demanding that our collection of charts be maximal.

Definition 1.1

A smooth n-dimensional *manifold* is a Hausdorff second-countable topological space M such that for any $x \in M$, there exists an open neighbourhood V_α of x, an open set U_α of $\mathbb{R}^n$, and a homeomorphism $\phi_\alpha : U \to V$. We call ϕ_α a *chart*, and we demand that the *transition maps*

$$\psi_{\alpha\beta} \coloneqq \phi_\beta^{-1} \circ \phi_\alpha : \mathbb{R}^n \supseteq \phi_\alpha^{-1}(V_\beta) \to \phi_\beta^{-1}(V_\alpha) \subseteq \mathbb{R}^n$$

between charts are smooth.

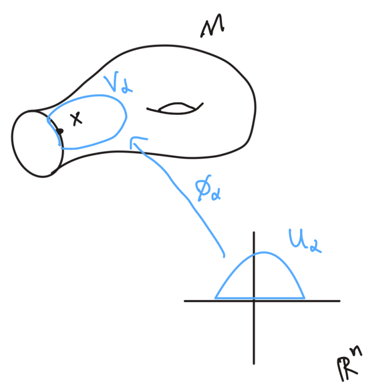


1.2 Smooth manifolds with boundary

Definition 1.2

A *smooth manifold with boundary* is the same as a smooth manifold except the charts may start with domain U_α open in the half-space

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}.$$



The set of points $x \in M$ covered by a chart $\phi_\alpha : \mathbb{H}^n \supseteq U_\alpha \to V_\alpha \subseteq M$ such that $\phi_\alpha(x_1, \ldots, x_{n-1}, 0) = x$ are called the *boundary*, and denoted ∂M. If M is a smooth k-dimensional manifold with boundary, ∂M is a smooth $(k-1)$-dimensional manifold.

Note that the boundary may be empty, hence every smooth manifold is a smooth manifold with boundary.

2 Cobordism

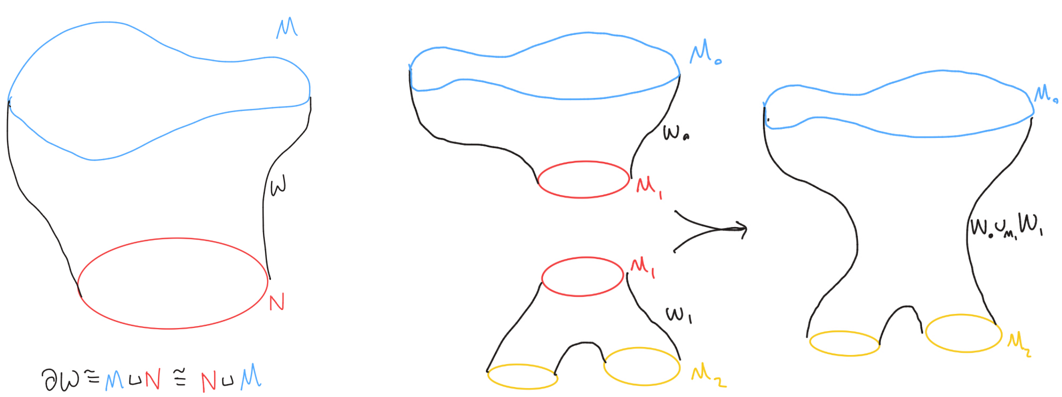
2.1 The cobordism relation

For the sake of simplicification, we will only talk about unoriented cobordism of compact smooth manifolds. Here is the motivating example. Let M be a compact smooth n-dimensional manifold. Then $M \times [0,1]$ is a compact smooth $(n+1)$-dimensional manifold with boundary $\partial(M \times [0,1]) \cong M \sqcup M$. If $M \cong N$, then equally $\partial(M \times [0,1]) \cong M \sqcup N$. This motivates the following definition.

Definition 2.1

Compact smooth k-dimensional manifolds M and N are *cobordant* if there exists a compact smooth $(k+1)$-dimensional manifold with boundary W such that $\partial W = M \sqcup N$.

We call W a cobordism for $M \sim N$. We have shown that diffeomorphic implies cobordant; in particular $M \sim M$. Since $\partial W = M \sqcup N$ can be equally interpreted as $N \sqcup M$, cobordism is symmetric. Finally, transitivity is seen by a picture, or alternatively if W_0 is a cobordism $M_0 \sim M_1$ and W_1 is a cobordism $M_1 \sim M_2$, then $W_0 \cup_{M_1} W_1$ is a cobordism $M_0 \sim M_2$. So cobordism is an equivalence relation.



2.2 The cobordism group

Definition 2.2

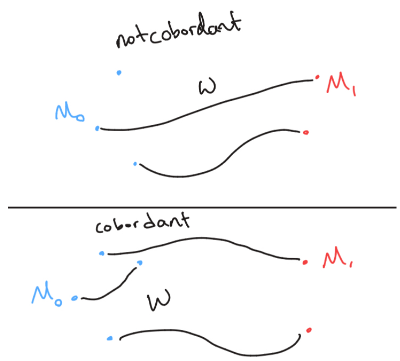
The k-th unoriented cobordism group Ω_k^O is the set of cobordism classes of compact smooth k-dimensional manifolds, under disjoint union.

We first verify this is well-defined, that is disjoint union is independent of choice of representative. Suppose $M \sim M'$ and $N \sim N'$. We want $[M] + [N] = [M \sqcup N] = [M' \sqcup N'] = [M'] + [N']$. Simply take the disjoint union of a cobordism $M \sim M'$ with a cobordism $N \sim N'$ to obtain a cobordism $M \sqcup N \sim M' \sqcup N'$.

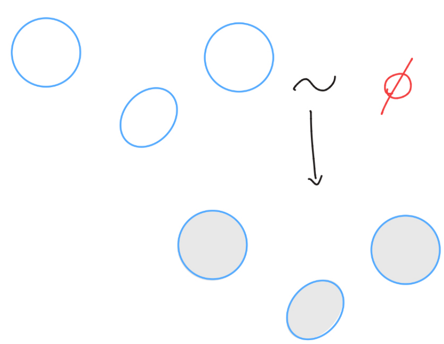
Associativity and commutativity are immediate. Now we have identity in $[\varnothing]$ as $[M]+[\varnothing] = [M \sqcup \varnothing] = [M]$. For inverses, we interpret $M \times [0, 1]$ as a cobordism $M \sqcup M \sim \varnothing$, so $[M] + [M] = [M \sqcup M] = [\varnothing]$. In other words, Ω_k^O is a 2-torsion abelian group.

2.3 Computing some cobordism groups

We first compute Ω_0^O. The compact smooth 0-dimensional manifolds are simply finite discrete sets. The compact smooth 1-dimensional manifolds with boundary are disjoint unions of bounded smooth curves, which always have a finite even number of boundary points. So $\Omega_0^O \to \mathbb{Z}/2\mathbb{Z}$ given by $\{r \text{ points}\} \mapsto r \bmod 2$ is an isomorphism.



Now we do Ω_1^O. The compact smooth 1-dimensional manifolds are just finite disjoint unions of circles, which are boundaries of finite disjoint unions of disks. These disks are thus a cobordism to the empty manifold, so $\Omega_1^O = 0$.



We will not fully explain Ω_2^O, but we will show that the real projective plane $\mathbb{R}P^2$ is not cobordant to the empty manifold. It will turn out that $\mathbb{R}P^2$ represents the only nontrivial cobordism class in dimension 2, hence $\Omega_2^O \cong \mathbb{Z}/2$.

If $\mathbb{R}P^2$ were cobordant to $\varnothing$, then it would be the boundary of some compact smooth 3-dimensional manifold with boundary. There is an easy way to see this is impossible via Euler characteristic, but we won't go into the details.

2.4 The graded cobordism ring

With multiplication as the cartesian product

$$[M] \cdot [N] = [M \times N],$$

$\Omega_*^O := \bigcup_{k \geq 0} \Omega_k^O$ becomes a commutative graded ring with unit $[*]$.

3 The Generalized Poincaré Conjecture

3.1 A quick introduction to homotopy

Recall that two functions are homotopic if one can be continuously translated into the other. More precisely,

Definition 3.1

Let X, Y be topological spaces. Two maps $f, g : X \to Y$ are homotopic, and we write $f \simeq g$, if there exists a continuous map $H : X \times [0,1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. We call H a homotopy from f to g.

We will not show it, but this is an equivalence relation.

Definition 3.2

X is simply connected if it is path-connected and every map $f : S^1 \to X$ is homotopic to a constant map.

Intuitively, we say every loop can be continuously contracted to a point.

Definition 3.3

Let X, Y be topological spaces. A homotopy equivalence $f : X \to Y$ is a continuous map such that there exists a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Definition 3.4

Let A be a subspace of X with the inclusion $\iota : A \hookrightarrow X$. A deformation retraction of X onto A is a map $r : X \to A$ such that $r\big|_A = \mathrm{id}_A$ and $\iota \circ r \simeq \mathrm{id}_X$.

X is called contractible if it deformation retracts onto a point.

3.2 The Generalized Poincaré Conjecture

Theorem 3.5 (Generalized Poincaré Conjecture)
Let $n \geq 6$, and let M be a compact smooth n-dimensional manifold that is homotopy equivalent to S^n. Then M is homeomorphic to S^n.

3.3 h-cobordism

Ordinary cobordism will not be enough for the Generalized Poincaré Conjecture. However, it's not immediately clear how to strengthen cobordism. We can start by looking at the nicest kind of cobordism. The prototypical example of a cobordism is the cylinder $M \times [0, 1]$ for $M \sim M$. We want to know when a cobordism is 'the same' as the cylinder. We start with a notion of sameness.

Definition 3.6
Two cobordisms (W, M, N) and (W, M, N') are diffeomorphic relative M if there is a diffeomorphism $F : W \to W'$ such that

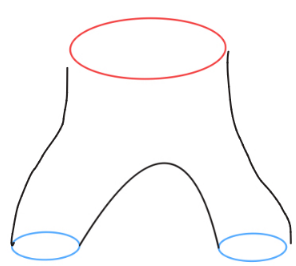
$$F \circ f = g,$$

where $f : M \hookrightarrow \partial W$ and $g : M \hookrightarrow \partial W'$ are the inclusions.

Now, we define the nicest kind of cobordism.

Definition 3.7
A cobordism W for $M \sim N$ is trivial if it is diffeomorphic relative M to the cylinder $M \times [0, 1]$.

The pair of pants is an example of a cobordism that is not trivial. The pair of pants is a cobordism for $S^1 \sqcup S^1 \sim S^1$, but it is not diffeomorphic relative S^1 to the cylinder $S^1 \times [0, 1]$, because they are not diffeomorphic at all.



Under what conditions is a cobordism trivial? We can notice that the cylinder deformation retracts onto both ends. This motivates the following definition.

Definition 3.8

A cobordism W for $M \sim N$ is called an h-cobordism if W deformation retracts onto M and N.

An example of a cobordism that is not an h-cobordism is the pair of pants P. P has one path component while one boundary component $S^1 \sqcup S^1$ has two, so P cannot deformation retract onto it.

Even if one thinks real hard about h-cobordism, it is difficult to construct any nontrivial examples. However, they do exist.

Example 3.9 (Akbulut's cork)

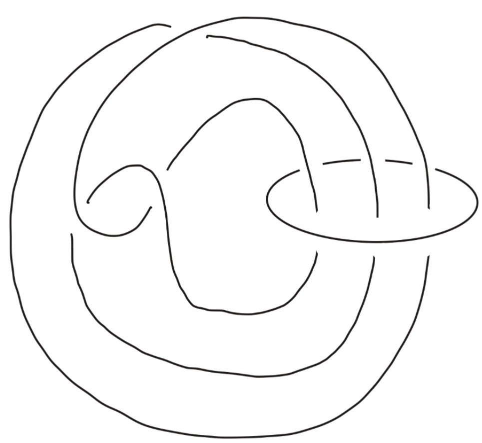
We begin by constructing Mazur's manifold:

(1) Start with D^4.

(2) Attach a 1-handle $D^1 \times D^3$ to D^4, producing $D^3 \times S^1$.

(3) The boundary $S^2 \times S^1$ of $D^3 \times S^1$ can be split into two halves $D^2 \times S^1$. The boundary of a 2-handle $D^2 \times D^2$ also splits as $S^1 \times D^2$ and $D^2 \times S^1$. We can thus attach a 2-handle to $D^3 \times S^1$ by gluing the $D^2 \times S^1$ boundary component of $D^2 \times D^2$ to the $D^2 \times S^1$ half of $S^2 \times S^1$, according to the curve shown below. The result is Mazur's manifold W.

Mazur's manifold is a compact contractible smooth 4-dimensional manifold with boundary the Brieskorn sphere $\Sigma(2, 5, 7)$. The Brieskorn sphere $\Sigma(2, 5, 7)$ is a homology 3-sphere. That is, it has the same homology groups as S^3, but it is not homeomorphic to S^3.
Define an involution $\tau : \partial W \to \partial W$ by swapping the three crossings in the figure below.
It is possible to embed W into a compact smooth 4-dimensional manifold M. Let $N := (M \setminus W) \cup_\tau W$ be the result of cutting out W and regluing it via τ. It turns out that τ extends to a homeomorphism $M \to N$, but M and N are not diffeomorphic.
Define a cobordism X for $M \sim N$ by gluing $(M \setminus \mathrm{Int}(W)) \times [0, 1]$ to $W \times [0, 1]$, identifying $\partial W \times \{0\}$ according to the identity and $\partial W \times \{1\}$ according to τ. Since W is contractible, this is an h-cobordism. However, it is not trivial, because if it were then M and N would be diffeomorphic.



Example 3.10 (Lens spaces)

Let $L^7(p; q_1, q_2, q_3, q_4)$ be the 7-dimensional lens space defined as the quotient of $S^7 \subseteq \mathbb{C}^4$ by the action of $\mathbb{Z}/p\mathbb{Z}$ given by

$$(z_1, z_2, z_3, z_4) \mapsto (e^{2\pi i q_1/p} z_1, e^{2\pi i q_2/p} z_2, e^{2\pi i q_3/p} z_3, e^{2\pi i q_4/p} z_4).$$

Let $M = L^7(7; 1, 1, 1, 2)$ and $N = L^7(7; 1, 1, 1, 3)$. Then M and N are homotopy equivalent but not homeomorphic. Notably, $pi_1(M) = \mathbb{Z}/7 = \pi_1(N)$. M and N turn out to be h-cobordant, but since they are not diffeomorphic, the h-cobordism cannot be trivial.

We have seen two examples of nontrivial h-cobordisms. The first was 4-dimensional, while the second was not simply connected. The following theorem shows that these are the only obstructions to triviality.

Theorem 3.11 (h-cobordism; Smale 1962)

Let $n \geq 5$. Let M be a simply connected compact smooth n-dimensional manifold. Then any h-cobordism W for $M \sim N$ is trivial.

Proof sketch. Some Morse theory shows that W is diffeomorphic to the cylinder $M \times [0, 1]$ with some handles attached, called a handlebody decomposition. From here, we can simplify the handlebody decomposition by cancelling handles until no handles remain. □

3.4 Solving the Generalized Poincaré Conjecture

Theorem 3.12

Let $n \geq 6$, and let M be a compact smooth n-dimensional manifold that is homotopy equivalent to S^n. Then M is homeomorphic to S^n.

Proof. Pick two disjoint embedded n-disks $D_0^n, D_1^n \subseteq M$. Then $W \coloneqq M \setminus (\mathrm{Int}(D_0) \sqcup \mathrm{Int}(D_1))$ is a cobordism between S_0^{n-1} and S_1^{n-1}. In fact W is an h-cobordism. This fact requires more background than we have provided.

We have $\pi_1(S_0^{n-1}) = 0$. Moreover we have the following exact sequence for the relative homotopy of the pair (W, S_0^{n-1}):

$$\cdots \to \pi_1(S_0^{n-1}) \xrightarrow{\iota_*} \pi_1(W) \to \pi_1(W, S_0^{n-1}) \to \pi_0(S_0^{n-1}) \to \pi_0(W),$$

where the first two objects are 0 and the last map is injective. Hence $\ker(\pi_1(W, S_0^{n-1}) \to \pi_0(S_0^{n-1})) = 0$ and $\mathrm{im}(\pi_1(W, S_0^{n-1}) \to \pi_0(S_0^{n-1})) = 0$, so $\pi_1(W, S_0^{n-1}) = 0$.

By excision $H_*(W, S_0^{n-1}) \cong H_*(M \setminus D_1^n, D_0^n)$, and by some more sequence chasing with the pair $(M \setminus D_1^n, D_0^n)$, $H_*(W, S_0^{n-1}) = 0$.

The relative Hurewicz theorem now gives $\pi_k(W, S_0^{n-1}) = 0$ for $k \geq 0$. Now since (W, S_0^{n-1}) is a triangulable pair (that is, W is homeomorphic to a simplicial complex and S_0^{n-1} is a subcomplex), W deformation retracts onto S_0^{n-1}. By symmetry, W also deformation retracts onto S_1^{n-1}. Hence W is an h-cobordism.

Moreover, S_0^{n-1} has dimension $n - 1 \geq 5$ and it is simply connected and compact. Hence the h-cobordism theorem implies W is trivial, so we have a diffeomorphism $F : W \to S_0^{n-1} \times [0, 1]$ that restricts to the identity $S_0^{n-1} \to S_0^{n-1} \times \{0\} \cong S_0^{n-1}$. Since F is a diffeomorphism of manifolds with boundary, it restricts to a diffeomorphism

$$f : S_1^{n-1} \to S_0^{n-1} \times \{1\} \cong S_0^{n-1}$$

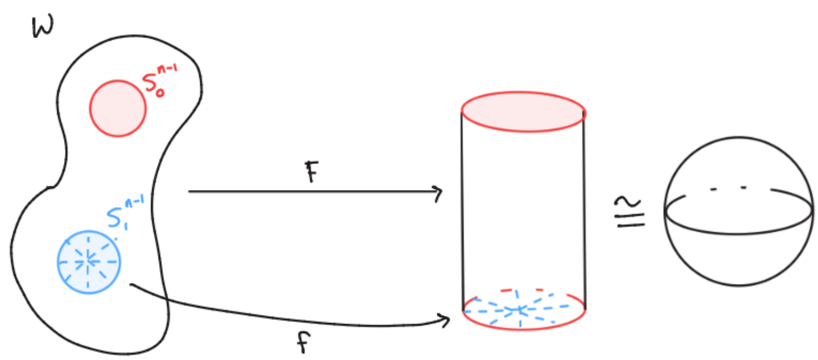
on the other half of the boundary.

We extend f to a homeomorphism $\overline{f} : D_1^n \to D_0^n$ via the Alexander trick. That is, for $tx \in D_1^n$ where $t \in [0, 1]$ and $x \in S_1^{n-1}$, let

$$\overline{f}(tx) = tf(x).$$

Now let X be $D_0^n \times \{0\} \cup S_0^{n-1} \times [0, 1] \cup D_0^n \times \{1\}$ be the hollow cylinder. This is homeomorphic to S^n by bending the middle section S_0^{n-1}.

We have a homeomorphism $h : X \to M$ given by $\mathrm{id} : D_0^n \times \{0\} \to D_0^n$, $F : S_0^{n-1} \times [0, 1] \to M$, and $\overline{f} : D_0^n \times \{1\} \to D_1^n$. Essentially, the h-cobordism work was needed to show this is a proper gluing. We are done. □



3.5 Historical notes

Strictly speaking, this statement of the Generalized Poincaré Conjecture is not entirely faithful to the original Poincaré Conjecture. We assume M is a *smooth* manifold, but only showed it is *homeomorphic* to S^n. The Generalized Poincaré Conjecture in the topological category should really characterize topological manifolds. Smale's proof can be adapted to work for piecewise-linear manifolds, but not for topological manifolds.

Stallings and Zeeman used a method called "engulfing" to provide a different proof with the piecewise-linear assumption. In 1966, Newman extended these engulfing techniques to show the veritable topological Generalized Poincaré Conjecture for $n \geq 5$.

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[LM24] Wolfgang Lück and Tibor Macko. *Surgery Theory.* Springer, Cham, 2024.

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