simplicial identities:

$$\begin{tikzcd} \Delta^{n-2} \arrow[r, hook, "d_i"] \arrow[d, hook, "d_j"] & \Delta^{n-1} \arrow[d, hook, "d_{j-1}"] \\ \Delta^{n-1} \arrow[r, hook, "d_i"] & \Delta^n \end{tikzcd} \qquad (i<j)$$

we have a coequalizer

$$\coprod_{0\le i<j\le n} \Delta^{n-2} \underset{d_j}{\overset{d_i}{\rightrightarrows}} \coprod_{i=0}^{n} \Delta^{n-1} \xrightarrow{d_i} \partial\Delta^n . \qquad (*)$$

Now we show $|X|$ is CW.

Let $sk_n X = \bigcup_{i\le n} X_i$.

Then $X = \bigcup sk_n X$, and we have pushouts

$$\begin{tikzcd} \coprod_{x\in NX_n} \partial\Delta^n \arrow[r] \arrow[d, hook] & sk_{n-1} X \arrow[d, hook] \\ \coprod_{x\in NX_n} \Delta^n \arrow[r] & sk_n X \end{tikzcd}$$

where $NX_n = \{\text{nondegenerate } n\text{-simplices}\}$
$= \{x\in X_n : x \ne s_i y \text{ for } 0\le i\le n-1,\ y\in X_{n-1}\}$

Clearly the realization of Δ^n is the usual topological simplex $|\Delta^n|$, since $\Delta\downarrow\Delta^n$ has terminal obj $1:\Delta^n\to\Delta^n$. By $(*)$, we get coequalizer in Top

$$\coprod_{0\le i<j\le n} |\Delta^{n-2}| \rightrightarrows \coprod_{i=0}^{n} |\Delta^{n-1}| \to |\partial\Delta^n|$$

So $|\partial\Delta^n| \to |\Delta^n|$ maps into the $(n-1)$-sphere bounding $|\Delta^n|$.

So $|X|$ is filtered colim of $|sk_n X|$, which is obtained from $|sk_{n-1}X|$ by attaching n-cells according to

$$\begin{tikzcd} \coprod_{x\in NX_n} |\partial\Delta^n| \arrow[r] \arrow[d, hook] & |sk_{n-1} X| \arrow[d, hook] \\ \coprod_{x\in NX_n} |\Delta^n| \arrow[r] & |sk_n X| \end{tikzcd}$$

$\square$