THE ZARISKI TOPOLOGY: NAIVE TO NONSENSE

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1. Introduction

The Zariski topology is a topology on an algebraic variety which is a central instrument to algebraic geometry. It was originally defined on k^n from the polynomial ring $k[x_1, \ldots, x_n]$, and then extended to $\mathrm{Spec}(R)$ for general commutative rings R, and more recently formulated in terms of Galois connections induced by relations. We present these ideas in increasing generality.

2. The Naive Zariski Topology

Let k be an infinite field. The polynomial ring $k[x_1, \ldots, x_n]$ may be identified with the ring of polynomial functions $k^n \to k$. Throughout this section, we will use the shorthand $a = (a_1, \ldots, a_n)$.

A subset I of $k[x_1, \ldots, x_n]$ determines an affine algebraic subset of k^n by

$$Z(I) = \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

We remark that if (I) is the ideal generated by I, then $Z(I) = Z((I))$, so it suffices to consider the ideals I in $k[x_1, \ldots, x_n]$.

Let $X = Z(I)$. If $Y = Z(J)$ for some ideal J in $k[x_1, \ldots, x_n]$ and $Y \subseteq X$, we say Y is an algebraic subset of X. If X is not a union of smaller algebraic subsets, we say X is an algebraic variety.

Definition 2.1. The Zariski topology on k^n has as closed sets the algebraic subsets of k^n.

We make the necessary sanity checks below:

(1) $\varnothing$ and k^n are closed. Fortunately $\varnothing = Z(k[x_1, \ldots, x_n])$ and $X = Z(0)$ are algebraic.

(2) Intersections of closed sets are closed. Let $\{I_\alpha\}_{\alpha \in A}$ be a family of ideals in $k[x_1, \ldots, x_n]$. We have $\bigcap_{\alpha \in A} Z(I_\alpha) = Z\left(\bigcup_{\alpha \in A} I_\alpha\right)$, since $f(a) = 0$ for any $f \in I_\alpha$, for any $\alpha \in A$, if and only if $f(a) = 0$ for any $f \in \bigcup_{\alpha \in A} I_\alpha$.

(3) Finite unions of closed sets are closed. Let $\prod_{i=1}^n J_i$ be the set of all n-fold products of one function from each J_i. Then if $a \in \bigcup_{i=1}^n Z(J_i)$, say $a \in Z(J_i)$, then for any $f = f_1 \cdots f_n \in \prod_{i=1}^n J_i$, $f_i \in J_i$ implies $f_i(a) = 0$. hence $f(a) = 0$, so $a \in Z\left(\prod_{i=1}^n J_i\right)$. Conversely if $a \notin \bigcup_{i=1}^n Z(J_i)$, then for each i let $f_i \in J_i$ be such that $f_i(a) \neq 0$. Then $f = f_1 \cdots f_n \in \prod_{i=1}^n J_i$ is such that $f(a) \neq 0$, so $a \notin Z\left(\prod_{i=1}^n J_i\right)$. We have shown that $\bigcup_{i=1}^n Z(J_i) = Z\left(\prod_{i=1}^n J_i\right)$.

The following results on the separation axioms provide some intuition about the coarseness of the Zariski topology.

Lemma 2.2. *The Zariski topology is T_1.*

Proof. We show that singletons are closed. Given $(a_1, \ldots, a_n) \in k^n$, let $I = \{x_1 - a_1, \ldots, x_n - a_n\}$. Clearly $(a_1, \ldots, a_n) \in Z(I)$. Conversely if $b \in Z(I)$ then $b_1 - a_1 = 0, \ldots, b_n - a_n = 0$. This shows that $\{(a_1, \ldots, a_n)\} = Z(I)$, and thus is closed. $\square$

Lemma 2.3. *The Zariski topology on k^n is Hausdorff if and only if k is a finite field.*

Proof. Suppose k is a finite field; we show the Zariski topology is discrete. Since the Zariski topology is T_1, finite sets are closed. But k^n is finite, so every set is closed.

Suppose k is an infinite field. Then for any $I \subseteq k[x_1, \ldots, x_n]$ the algebraic set $Z(I)$ is either finite or all of k^n, in the case $I = \varnothing$ or (0). Thus if $a, b \in k^n$ and U, V are neighbourhoods of a, b respectively, then $k^n - U, k^n - V$ are both closed. Since U, V are nonempty, they must be finite. Since k^n is infinite and $(k^n - U) \cup (k^n - V)$ is finite, there exists $c \in k^n - ((k^n - U) \cup (k^n - V))$; that is, $c \in U \cap V$. Thus k^n in the Zariski topology is not Hausdorff. $\square$

Although the previous results show that the Zariski topology is rather peculiar, the topological closure behaves quite tamely. As a partial converse to the definition of the algebraic set $Z(I)$ from $I \subseteq k[x_1, \ldots, x_n]$, given a subset X of k^n we define the vanishing set

$$I(X) = \{f \in k[x_1, \ldots, x_n] : f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in X\}.$$

Lemma 2.4. *Let k^n be given the Zariski topology and $A \subseteq k^n$. Then $\overline{A} = Z\left(I(A)\right)$.*

Proof. $\overline{A}$ is the intersection of all closed sets in k^n containing A. Closed sets in the Zariski topology are precisely in the form $Z(I)$ for $I \subseteq k[x_1, \ldots, x_n]$, so

$$\overline{A} = \bigcap_{\substack{I \subseteq k[x_1, \ldots, x_n] \\ A \subseteq Z(I)}} Z(I).$$

By the verification above that intersections of closed sets are closed,

$$\overline{A} = Z\left(\bigcup_{\substack{I \subseteq k[x_1, \ldots, x_n] \\ A \subseteq Z(I)}} I \right).$$

Now if $f \in \bigcup_{\substack{I \subseteq k[x_1, \ldots, x_n] \\ A \subseteq Z(I)}} I$, then $f \in I$ for some $I \subseteq k[x_1, \ldots, x_n]$ with $A \subseteq Z(I)$. Hence for each $a \in A$, $a \in Z(I)$ implies $f(a) = 0$, and thus $f \in I(A)$.

Conversely $I(A) \subseteq k[x_1, \ldots, x_n]$ satisfies $A \subseteq Z(I(A))$, since $a \in A$ implies that for any $f \in I(A)$, we have $f(a) = 0$. Hence the above equation becomes

$$\overline{A} = Z(I(A)).$$

$\square$

One can think of the symmetrical construction $I(Z(A)) \subseteq k[x_1, \ldots, x_n]$. In fact this is always an ideal in $k[x_1, \ldots, x_n]$, called the vanishing ideal. $I(Z(A))$ is clearly an additive subgroup. The fact that it is furthermore an ideal is clear from

$$(gf)(a_1, \ldots, a_n) = g(a_1, \ldots, a_n) f(a_1, \ldots, a_n) = 0$$

for any $g \in k[x_1, \ldots, x_n]$.

By idealizing an algebraic variety instead of an algebraic set, we can strengthen this relationship:

Theorem 2.5. *Let $A \subseteq k[x_1, \ldots, x_n]$. The algebraic set $Z(A)$ is an algebraic variety if and only if the vanishing ideal $I(Z(A))$ is prime.*

Proof. ($\Longrightarrow$) Suppose $Z(A)$ is an algebraic variety. Suppose $f, g \in k[x_1, \ldots, x_k]$ are such that $fg \in I(Z(A))$. Suppose $a \in Z(A)$. Since k is a field and $f(a)g(a) = 0$, we must have $f(a) = 0$ or $g(a) = 0$, so $a \in Z(f) \cup Z(g)$. It follows that $Z(A) \subseteq Z(f) \cup Z(g)$, or

$$Z(A) = (Z(f) \cap Z(A)) \cup (Z(g) \cap Z(A)).$$

Since $Z(f) \cap Z(A)$ and $Z(g) \cap Z(A)$ are algebraic sets, the assumption that $Z(A)$ is an algebraic variety implies that one of $Z(f) \cap Z(A)$ or $Z(g) \cap Z(A)$ is equal to $Z(A)$. Hence without loss of generality $Z(A) \subseteq Z(f)$. Now if $h \in I(Z(f))$, then $h(a) = 0$ for all $a \in Z(f)$, and in particular for all $a \in Z(A)$. So $h \in I(Z(A))$. Since $f(a) = 0$ for all $a \in Z(f)$ implies $f \in I(Z(f))$, we have $f \in I(Z(A))$, showing $I(Z(A))$ is prime.

($\Longleftarrow$) Suppose $I(Z(A))$ is a prime ideal, and $Z(A) = Z(A_1) \cup Z(A_2)$. We wish to show that $Z(A) = Z(A_1)$ or $Z(A_2)$. If not, then neither of $Z(A_1), Z(A_2)$ may contain the other, and thus neither of $I(Z(A_1)), I(Z(A_2))$ may contain the other. In particular there exist $f \in I(Z(A_1) - I(Z(A_2))$ and $g \in I(Z(A_2)) - I(Z(A_1))$. Since $I(Z(A_1)), I(Z(A_2))$ are ideals, this implies $fg \in I(Z(A_1)) \cap I(Z(A_2)) = I(Z(A))$. Since this is a prime ideal, we may assume without loss of generality that $f \in I(V(A)) = I(V(A_1)) \cap I(V(A_2))$. But this contradicts $f \notin I(Z(A_2))$. $\square$

The preceding theorem is the basis for our first generalization of the Zariski topology in the next section.

3. The Ideal Zariski Topology

We now give a slightly more general treatment of the previous section. Let R be a commutative ring. Let $\mathrm{Spec}(R)$ denote the set of prime ideals in R. For any subset X of R, let

$$Z(X) = \{P \in \mathrm{Spec}(R) : X \subseteq P\}$$

be the set of prime ideals containing X. As in the naive case, it suffices to consider ideals $X \subseteq R$.

Definition 3.1. The Zariski topology on $\mathrm{Spec}(R)$ has as closed sets the subsets $Z(I)$, for all ideals I in R.

We make the necessary sanity checks below:

(1) $\varnothing$ and $\mathrm{Spec}(R)$ are closed. Once again $\varnothing = Z(R)$ and $\mathrm{Spec}(R) = Z(0)$.
(2) Intersections of closed sets are closed. Let $\{I_\alpha\}_{\alpha \in A}$ be a family of ideals in R. We claim that

$$\bigcap_{\alpha \in A} Z(I_\alpha) = Z\left(\sum_{\alpha \in A} I_\alpha\right).$$

$P \in \bigcap_{\alpha \in A} Z(I_\alpha)$ is equivalent to P being a prime ideal such that for each $\alpha \in A$, $I_\alpha \subseteq P$, and thus to $\sum_{\alpha \in A} I_\alpha \subseteq P$, or $P \in Z\left(\sum_{\alpha \in A} I_\alpha\right)$.

(3) Finite unions of closed sets are closed. We claim that
$$\bigcup_{i=1}^{n} Z(J_i) = Z\left(\prod_{i=1}^{n} J_i\right).$$
If $P \in \bigcup_{i=1}^{n} Z(J_i)$, then $P \in Z(J_i)$ for some i, so $J_i \subseteq P$. It follows that $\prod_{i=1}^{n} J_i \subseteq J_i \subseteq P$, so $P \in Z\left(\prod_{i=1}^{n} J_i\right)$.

Conversely if $P \in Z\left(\prod_{i=1}^{n} J_i\right)$, then P is a prime ideal such that $\prod_{i=1}^{n} J_i \subseteq P$. If, for each i, there existed $r_i \in J_i$ such that $r_i \notin P$, then $r_1 \cdots r_n \notin P$, since P is a prime ideal. But $r_1 \cdots r_n \in \prod_{i=1}^{n} J_i$, so by contraposition there must be some i such that $J_i \subseteq P$. We conclude that $P \in Z(J_i) \subseteq \bigcup_{i=1}^{n} Z(J_i)$, as desired.

Lemma 3.2. *Let $P \in Spec(R)$. Then the topological closure of $\{P\}$ is $\overline{P} = Z(P)$.*

Proof. Suppose $Q \in \overline{P}$. Since $Z(P)$ is a closed set containing P, we have $Q \in Z(P)$.

Conversely suppose $Q \in Z(P)$, and consider any closed set $Z(I)$ containing P, meaning $I \subseteq P$. Since $P \subseteq Q$, transitivity implies $I \subseteq Q$, or $Q \in Z(I)$. Since $Z(I)$ was arbitrary, $Q \in \overline{P}$. □

Theorem 3.3. *The closed points in $Spec(R)$ are precisely the maximal ideals in R.*

Proof. By Lemma 3.2, it suffices to show that $Z(\mathfrak{m}) = \{\mathfrak{m}\}$ if and only if $\mathfrak{m}$ is a maximal ideal in R.

($\Longleftarrow$) Suppose $\mathfrak{m} \in \mathrm{Spec}(R)$ is maximal. Then by definition $\mathfrak{m}$ is the only prime ideal containing $\mathfrak{m}$, so $Z(\mathfrak{m}) = \{\mathfrak{m}\}$.

($\Longrightarrow$) Suppose $Z(\mathfrak{m}) = \{\mathfrak{m}\}$. Let I be any proper (not necessarily prime) ideal in R and $\mathfrak{m} \subseteq I$. It follows from Zorn's lemma that $I \subseteq \mathfrak{n}$ for some maximal ideal $\mathfrak{n}$ in R. Then $\mathfrak{n} \in Z(\mathfrak{m})$, so $\mathfrak{n} = \mathfrak{m}$, showing that $I = \mathfrak{m}$, and therefore $\mathfrak{m}$ is maximal. □

For the set-theoretically inclined, Zorn's lemma is actually stronger than we need. The weaker ultrafilter principle implies that every ideal is contained in some prime ideal, from which the above arguments can be applied.

Definition 3.4. Given an ideal I in R, we define its radical as
$$\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}.$$

We say I is radical if $I = \sqrt{I}$. To characterize radicals, we will need the following lemma about ideals in localizations:

Lemma 3.5. *Let S be a multiplicatively closed subset of R. Let $S^{-1}R$ be the localization of R at S and $\varphi : R \to S^{-1}R$ the canonical ring homomorphism $a \mapsto \frac{a}{1}$. Then*

(1) *If I is an ideal in R, then the ideal generated by $\varphi(I)$ is $I^e = \{\frac{a}{s} : a \in I, s \in S\}$, and is called the extension of I in $S^{-1}R$.*
(2) *If J is an ideal in $S^{-1}R$ then $J^c = \varphi^{-1}(J)$ is an ideal in R, called the contraction of J to R. Furthermore $(J^c)^e = J$.*
(3) *Contraction and extension by φ give a one-to-one correspondence between prime ideals in $S^{-1}R$ and prime ideals in R which do not intersect S.*

Proof. (1) For each $s \in S$, $\frac{1}{s} \in S^{-1}R$, so the ideal I^e generated by $\varphi(I)$ must contain $\frac{1}{s} \cdot \varphi(a) = \frac{a}{s}$ for all $a \in I$, so $\{\frac{a}{s} : a \in I, s \in S\} \subseteq I^e$. The reverse containment follows from the fact that this is an ideal in $S^{-1}R$. The additive subgroup properties are inherited from I, and for any $\frac{r}{t} \in S^{-1}R$ we have $\frac{r}{t} \cdot \frac{a}{s} = \frac{ra}{st}$, where $ra \in I$ and $ts \in S$.

(2) The additive subgroup properties are inherited from J, and for any $r \in R$ and $a \in \varphi^{-1}(J)$ we have $\varphi(ra) = \varphi(r)\varphi(a) \in J$ since $\varphi(a) \in J$. Hence $ra \in \varphi^{-1}(J)$.

Now by (1), $(J^c)^e = \{\frac{a}{s} : a \in \varphi^{-1}(J), s \in S\}$. If $\frac{a}{s} \in J$ then $\frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s} \in J$, so $a \in \varphi^{-1}(J)$, and thus $\frac{a}{s} \in (J^c)^e$. Conversely if $\frac{a}{s} \in (J^c)^e$ then $\frac{a}{1} \in J$, so $\frac{a}{s} = \frac{a}{1} \cdot \frac{1}{s} \in J$.

(3) We must show that $J \mapsto J^c$ is well-defined on the specified domain; that is, if J is a prime ideal in $S^{-1}R$ then J^c is a prime ideal in R with $J^c \cap S = \varnothing$. Suppose $x, y \in R$ are such that $xy \in J^c$. Then $\varphi(xy) = \varphi(x)\varphi(y) \in J$. Since J is prime, either $\varphi(x) \in J$ or $\varphi(y) \in J$, thus $x \in J^c$ or $y \in J^c$. Since J is a proper ideal in $S^{-1}R$, it contains no units. Since $\varphi(s) \in S^{-1}R$ is a unit for all $s \in S$, J^c must not intersect S.

We already showed $(J^c)^e = J$ in (2).

In the other direction, we must show that $I \mapsto I^e$ is well-defined; if I is a prime ideal in R with $I \cap S = \varnothing$ then I^e is a prime ideal in $S^{-1}R$. If $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ are such that $\frac{a}{s} \cdot \frac{b}{t} \in I^e$, then $\frac{ab}{st} = \frac{c}{u}$ for some $c \in I$ and $u \in S$. By the equivalence relation of $S^{-1}R$, there exists $v \in S$ such that

$$v(abu - stc) = 0,$$

or $uvab = stc \in S$. Since $u, v \in S$, we have $u, v \notin I$, and since I is prime, $uv \notin I$. Hence $ab \in I$, and thus $a \in I$ or $b \in I$. We conclude that $\frac{a}{s} \in I^e$ or $\frac{b}{t} \in I^e$.

It remains to show $(I^e)^c = I$. If $a \in I$ then $\frac{a}{1} \in I^e$ shows that $a \in (I^e)^c$. Conversely if $a \in (I^e)^c$, then $\varphi(a) = \frac{a}{1} \in I^e$. By definition, $\frac{a}{1} = \frac{b}{s}$ for some $b \in I, s \in S$, and by the equivalence relation there exists $u \in S$ such that

$$u(as - b) = 0.$$

Equivalently, $sua = b \in I$, and since I is prime, $su \in I$ or $a \in I$. But $s, u \in S$ implies $s, u \notin I$ and thus $su \notin I$, so we conclude that $a \in I$.

□

From this, we obtain a useful characterization of radicals.

Lemma 3.6. *Let I be an ideal in R. Then $\sqrt{I}$ is the intersection of all prime ideals containing I.*

Proof. Suppose P is a prime ideal in R containing I. Let $r \in \sqrt{I}$; that is, $r^n \in I$ for some n. If $n = 1$ then $r \in I \subseteq P$. If $n > 1$, then $r^n = r \cdot r^{n-1} \in I \subseteq P$ implies $r \in P$ or $r^{n-1} \in P$. By induction, $r \in P$, so $\sqrt{I}$ is contained in every prime ideal containing I.

Conversely, suppose $r \in P$ for every prime ideal P containing I. Suppose, for the sake of contradiction, that $r^n \notin I$ for all $n \in \mathbb{N}$. $S = \{1, r, r^2, r^3, \cdots\}$ is multiplicatively closed and does not intersect I. Consider $S^{-1}R$, the localization of R at S, and the canonical ring homomorphism $\varphi : R \to S^{-1}R$ given by $a \mapsto \frac{a}{1}$.

By Lemma 3.5(1), I^e is an ideal in $S^{-1}R$. By Zorn's lemma (again, the ultrafilter principle suffices), there exists a prime ideal Q in $S^{-1}R$ containing I^e. By Lemma 3.5(3), Q^c is a prime ideal in R which does not intersect S. In particular $r \notin Q^c$, but $I = (I^e)^c \subseteq Q^c$, a contradiction. □

Theorem 3.7. *An ideal I in R is prime if and only if I is radical and $Z(I)$ is an irreducible closed set in Spec(R).*

Proof. ($\Longrightarrow$) Suppose P is a prime ideal, and consider $r \in \sqrt{P}$. By Lemma 3.6, $\sqrt{P} = P$.

Suppose $Z(P) = Z(I) \cup Z(J)$ for some ideals I, J in R. By our verification of finite unions in the Zariski topology, $P = IJ$, which is contained in I and J. If $P \subsetneq I$ and $P \subsetneq J$, then there exist $f \in I - P$ and $g \in J - P$. Then $fg \in IJ = P$, contradicting the assumption that P is prime.

($\Longleftarrow$) Suppose $Z(I)$ is irreducible in Spec(R) and I is radical. Let $f, g \in R$ be such that $fg \in I$. If $P \in Z(I)$ then $I \subseteq P$. Since P is prime, $fg \in P$ implies $f \in P$ or $g \in P$; that is, $P \in Z(f)$ or $P \in Z(g)$. It follows that $Z(I) \subseteq Z(f) \cup Z(g)$, or

$$Z(I) = (Z(I) \cap Z(f)) \cup (Z(I) \cap Z(g)).$$

Since $Z(I)$ is irreducible, we obtain $Z(I) \subseteq Z(f)$, without loss of generality. Thus every prime ideal containing I must also contain (f), thus by Lemma 3.6, $(f) \subseteq \sqrt{I}$. Since I is radical, $f \in I$, and we conclude that I is prime. $\square$

4. The Nonsense Zariski Topology

For sets X, Y and a relation $E \hookrightarrow X \times Y$, we denote $(x, y) \in E$ by $E(x, y)$.

Definition 4.1. The Galois connection induced by a relation $E \hookrightarrow X \times Y$ is a pair of functions $Z_E : P(X) \to P(Y)$ and $I_E : P(Y) \to P(X)$ defined by

$$Z_E(S) = \{y \in Y : E(x, y) \text{ for all } x \in S\}$$
$$I_E(T) = \{x \in X : E(x, y) \text{ for all } y \in T\}.$$

The Zariski topology on k^n may be viewed as the Galois connection induced by the relation $E \hookrightarrow k[x_1, \ldots, x_n] \times k^n$ defined by

$$E(f, a) \iff f(a) = 0.$$

Under this relation $Z_E(S)$ is the algebraic set determined by S, and $I_E(T)$ is the vanishing set determined by T.

Likewise, the Zariski topology on Spec(R) may be viewed as the Galois connection induced by the relation $E \hookrightarrow R \times \text{Spec}(R)$ defined by

$$E(x, P) \iff x \in P.$$

Then $Z_E(S)$ is the set of prime ideals containing S, and $I_E(T)$ is the intersection of all prime ideals in T.

From this point onward, the proofs are mostly routine logical manipulations given familiarity with the two special cases above, so we will omit them. The Galois connection satisfies the following properties:

(1) Z_E, I_E are contravariant order-preserving; that is, if $S \subseteq S'$ then $Z_E(S') \subseteq Z_E(S)$, and if $T \subseteq T'$, then $I_E(T') \subseteq I_E(T)$.
(2) The adjunction law; $T \subseteq Z_E(S)$ if and only if $S \subseteq I_E(T)$.
(3) $\bigcup_{\alpha \in A} Z_E(S_\alpha) = Z_E\left(\bigcap_{\alpha \in A} S_\alpha\right)$ and $\bigcup_{\beta \in B} I_E(T_\beta) = I_E\left(\bigcap_{\beta \in B} T_\beta\right)$.

Moreover, the compositions $I_E \circ Z_E$ and $Z_E \circ I_E$ are closure operators; namely

(1) For all $S \in P(X)$, $S \subseteq I_E \circ Z_E(S)$;
(2) For all $S \in P(X)$, $Z_E(S) = Z_E \circ I_E \circ Z_E(S)$;
(3) $I_E \circ Z_E$ is idempotent and covariant,

and

(1) For all $T \in P(Y)$, $T \subseteq Z_E \circ I_E(T)$;
(2) For all $T \in P(Y)$, $I_E(T) = I_E \circ Z_E \circ I_E(T)$;
(3) $Z_E \circ I_E$ is idempotent and covariant.

From this we can make the following definition:

Definition 4.2. Given a Galois connection, we define the closure of $S \in P(X)$ as $\overline{S} = I_E \circ Z_E(S)$ and the closure of $T \in P(Y)$ as $\overline{T} = Z_E \circ I_E(T)$. We say S and T are closed, respectively, if $S = \overline{S}$ and $T = \overline{T}$.

The following is immediate from the second property of $I_E \circ Z_E$ and $Z_E \circ I_E$ as closure operators:

Theorem 4.3. *The closed elements of $P(X)$ are precisely those in the image of I_E. The closed elements of $P(Y)$ are precisely those in the image of V_E.*

We conclude by verifying the topological requirement on intersections of closed sets, a property which descends to the familiar examples.

Theorem 4.4. *The sets of closed elements in $P(X)$ and $P(Y)$ are closed under intersections.*

Proof. We prove the result for $P(X)$; the proof for $P(Y)$ is identical. Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of closed elements in $P(X)$. Since $S_\alpha \subseteq I_E \circ Z_E(S_\alpha)$ for all $\alpha \in A$, we have

$$\bigcap_{\alpha \in A} S_\alpha \subseteq \bigcup_{\alpha \in A} I_E \circ Z_E(S_\alpha) = I_E \circ Z_E \left(\bigcap_{\alpha \in A} S_\alpha \right).$$

Conversely, since $\bigcap_{\alpha \in A} S_\alpha \subseteq S_\beta$ for any $\beta \in A$, $I_E \circ Z_E$ is covariant, and S_β is closed, we have

$$I_E \circ Z_E \left(\bigcap_{\alpha \in A} S_\alpha \right) \subseteq I_E \circ Z_E(S_\beta) = S_\beta$$

for each $\beta \in A$, and thus $I_E \circ Z_E \left(\bigcap_{\alpha \in A} S_\alpha \right) \subseteq \bigcap_{\alpha \in A} S_\alpha$. Therefore $\bigcap_{\alpha \in A} S_\alpha$ is again closed in $P(X)$. □

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