1 Baby's First Functor

In this chapter we define the fundamental group π_1, a functor from the category of based topological spaces Top_* to the category of groups Grp. Its underlying set admits a relatively easy description:

$$\pi_1(X, x_0) := \mathrm{Hom}_{h\mathsf{Top}_*}((S^1, 1), (X, x_0))$$

where $h\mathsf{Top}_*$ is the category of based topological spaces with morphisms up to homotopy relative basepoint. However, we will spend a few lectures defining all these terms very concretely. All maps will be continuous.

1.1 Homotopy of loops

We view based maps $(S^1, 1) \to (X, x_0)$ as paths $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0 = \gamma(1)$.

Definition 1.1.1

A map $\gamma_0\colon [0,1] \to X$ is *homotopic* relative $\{0,1\}$ to $\gamma_1\colon [0,1] \to X$ if there exists a map $h\colon [0,1] \times [0,1] \to X$ such that for all $s, t \in [0,1]$,

$$\begin{aligned} h(0,s) &= \gamma_0(s) \\ h(1,s) &= \gamma_1(s) \\ h(t,0) &= x_0 = h(t,1). \end{aligned}$$

Whenever we consider paths we will consider homotopy relative $\{0,1\}$, often without saying so.

Lemma 1.1.2

Homotopy is an equivalence relation.

Proof. Any map $\gamma\colon [0,1] \to X$ is homotopic to itself via the constant homotopy $h(t,s) = \gamma(s)$. If γ_0 is homotopic to γ_1 via h, then $h'(t,s) = h(1-t,s)$ exhibits γ_1 as homotopic to γ_0. Finally if γ_0 is homotopic to γ_1 via h_1 and γ_1 is homotopic to γ_2 via h_2, then

$$h(t,s) = \begin{cases} h_1(2t, s) & t \in [0, \frac{1}{2}] \\ h_2(2t-1, s) & t \in [\frac{1}{2}, 1] \end{cases}$$

is continuous by the gluing lemma and exhibits γ_0 as homotopic to γ_2. $\square$

If γ_0 is homotopic to γ_1, we denote this by $\gamma_0 \simeq \gamma_1$. Now we can define, as a set,

$$\pi_1(X, x_0) := \{\gamma\colon [0,1] \to X : \gamma(0) = x_0 = \gamma(1)\}/\simeq .$$

The group structure is concatenation: $[\gamma_0] \cdot [\gamma_1] := [\gamma_0 * \gamma_1]$, where

$$\gamma_0 * \gamma_1(s) = \begin{cases} \gamma_1(2s) & s \in [0, \frac{1}{2}] \\ \gamma_0(2s-1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

To see this is well-defined, suppose $\gamma_0 \simeq \gamma_0'$ via h_0 and $\gamma_1 \simeq \gamma_1'$ via h_1. Then

$$h(t,s) = \begin{cases} h_1(t, 2s) & s \in [0, \frac{1}{2}] \\ h_0(t, 2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

is a homotopy between $\gamma_0 * \gamma_1$ and $\gamma_0' * \gamma_0'$.

Theorem 1.1.3

Concatenation $*\colon \pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$ makes $\pi_1(X, x_0)$ into a group.

Proof. The homotopy class of the constant path $e(s) = x_0$ is the identity. Indeed, for any $[\gamma] \in \pi_1(X, x_0)$,

$$h(t,s) = \begin{cases} \gamma((1-t)s) & s \in [0, \frac{1}{2}] \\ \gamma(st - t + s) & s \in [\frac{1}{2}, 1] \end{cases}$$

is a homotopy between γ and $\gamma * e$, and similarly $[\gamma] = [e * \gamma] = [e] \cdot [\gamma]$. Now the inverse of $[\gamma]$ is the homotopy class of the reverse path

$$\gamma^{-1}(s) = \gamma(1-s),$$

as we have the homotopy

$$h(t,s) = \begin{cases} \gamma(2s) & s \in [0, \frac{1}{2}(1-t)] \\ \gamma(\frac{2(1-t)}{1+t}(1-s)) & s \in [\frac{1}{2}(1-t), 1] \end{cases}$$

between $\gamma^{-1} * \gamma$ and e, and similarly $[\gamma] \cdot [\gamma^{-1}] = [\gamma * \gamma^{-1}] = [e]$. Finally, for associativity we must show that

$$\gamma_0 * (\gamma_1 * \gamma_2) \simeq (\gamma_0 * \gamma_1) * \gamma_2.$$

Indeed,

$$h(t,s) = \begin{cases} \gamma_2(2st + 4s(1-t)) & s \in [0, \frac{t}{2} + \frac{1-t}{4}] \\ \gamma_1(4s-1) & s \in [\frac{t}{2} + \frac{1-t}{4}, \frac{3t}{4} + \frac{1-t}{2}] \\ \gamma_0((4s-3)t + (2s-1)(1-t)) & s \in [\frac{3t}{4} + \frac{1-t}{2}, 1] \end{cases}$$

is the desired homotopy. ∎