SHEAVING ON THE PRIME SPECTRUM

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1. Introduction

All rings in these present notes will be commutative and unital; let A be one such. Let $X = \operatorname{Spec}(A)$ be endowed with the Zariski topology. Our objective is to assign to each open set U in X a ring $A(U)$ and to each inclusion $V \subseteq U$ of open sets a ring homomorphism $\rho^U_V : A(U) \to A(V)$ satisfying:

(1) $\rho^U_U = \mathrm{id}_{A(U)}$ for all open sets U,
(2) $\rho^U_W = \rho^V_W \circ \rho^U_V$ for all open sets $W \subseteq V \subseteq U$.

This is known as a presheaf of rings on X. We call $A(U)$ the section over U, and ρ^U_V the restriction of U to V. Finally, we will collect an additional property that will turn this presheaf into a sheaf.

2. Localization

First, we recover some commutative algebra background, starting with localization. Let S be a multiplicatively closed subset of A. The localization $S^{-1}A$ of A at S consists of equivalence classes of fractions $\frac{a}{s}$, where $a \in A$ and $s \in S$. Two fractions $\frac{a}{s}$ and $\frac{b}{t}$ are equivalent if there exists $u \in S$ such that $u(at - bs) = 0$. We have a canonical ring homomorphism $\phi : A \to S^{-1}A$ given by $\phi(a) = \frac{a}{1}$. We will really only need the following universal property:

Theorem 2.1. *Let $\psi : A \to B$ be a ring homomorphism such that $\psi(s)$ is a unit in B for all $s \in S$. Then there exists a unique ring homomorphism $\tilde{\psi} : S^{-1}A \to B$ such that $\psi = \tilde{\psi} \circ \phi$.*

Proof. (Existence) Define $\tilde{\psi}(\frac{a}{s}) = \psi(a)\psi(s)^{-1}$. To see this is well-defined, suppose $\frac{a}{s} = \frac{b}{t}$. Then there exists $u \in S$ such that $u(at - bs) = 0$, which implies

$$\psi(u)(\psi(a)\psi(t) - \psi(b)\psi(s)) = 0.$$

Since $\psi(s), \psi(t), \psi(u)$ are all units, this means $\psi(a)\psi(s)^{-1} = \psi(b)\psi(t)^{-1}$, so $\tilde{\psi}$ is well-defined. To see it is moreover a ring homomorphism, we verify that

$$\begin{aligned}
\tilde{\psi}\left(\frac{a}{s} + \frac{b}{t}\right) &= \tilde{\psi}\left(\frac{at + bs}{st}\right) \\
&= \psi(at + bs)\psi(st)^{-1} \\
&= (\psi(a)\psi(t) + \psi(b)\psi(s))\left(\psi(s)\psi(t)\right)^{-1} \\
&= \psi(a)\psi(s)^{-1} + \psi(b)\psi(t)^{-1} \\
&= \tilde{\psi}\left(\frac{a}{s}\right) + \tilde{\psi}\left(\frac{b}{t}\right),
\end{aligned}$$

and

$$\begin{aligned}\tilde{\psi}\left(\frac{a}{s}\cdot\frac{b}{t}\right) &= \tilde{\psi}\left(\frac{ab}{st}\right)\\ &= \psi(ab)\psi(st)^{-1}\\ &= \psi(a)\psi(b)\left(\psi(s)\psi(t)\right)^{-1} = \tilde{\psi}\left(\frac{a}{s}\right)\cdot\tilde{\psi}\left(\frac{b}{t}\right).\end{aligned}$$

(Uniqueness) If $\tilde{\psi}$ satisfies the universal property, then for any $a \in A$, we have $\tilde{\psi}(\frac{a}{1}) = \tilde{\psi}\circ\phi(a) = \psi(a)$, and for any $s \in S$, we have

$$\tilde{\psi}\left(\frac{1}{s}\right) = \tilde{\psi}\left(\left(\frac{s}{1}\right)^{-1}\right) = \tilde{\psi}\left(\frac{s}{1}\right)^{-1} = (\tilde{\psi}\circ\phi(s))^{-1} = \psi(s)^{-1}.$$

So $\tilde{\psi}$ is uniquely determined by ψ. □

Example 2.2. Each $f \in A$ naturally generates a multiplicatively closed subset $S_f = \{f^n : n \in \mathbb{N}\}$ of A. In this case, we call $S_f^{-1}A$ the localization A_f of A at f, and we denote the canonical homomorphism by ϕ_f.

Example 2.3. Given a prime ideal $\mathfrak{p}$, another case of special interest is the multiplicatively closed subset $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$. We call $S_{\mathfrak{p}}^{-1}A$ the localization $A_{\mathfrak{p}}$ of A away from $\mathfrak{p}$, and we denote the canonical homomorphism by $\phi_{\mathfrak{p}}$.

3. Radicalization

Our second piece of requisite background is the radical of an ideal.

Definition 3.1. The *radical* $\sqrt{\mathfrak{a}}$ of an ideal $\mathfrak{a}$ in A is defined as

$$\sqrt{\mathfrak{a}} = \{a \in A : a^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}.$$

Equivalently [Gao25, Lemma 3.6], $\sqrt{\mathfrak{a}}$ is the intersection of all prime ideals in A containing $\mathfrak{a}$. From this viewpoint, it becomes clear that the radical is again an ideal. We will henceforth alternate between these two definitions without much emphasis.

In the special case where $\mathfrak{a} = (0)$, $\sqrt{(0)}$ is the ideal of nilpotents in A, called the nilradical $\mathfrak{N}$. Equivalently, $\mathfrak{N}$ is the intersection of all prime ideals in A.

The following two lemmas will eventually prove useful.

Lemma 3.2. $\sqrt{\mathfrak{a}+\mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}}+\sqrt{\mathfrak{b}}}$.

Proof. Suppose $c \in \sqrt{\mathfrak{a}+\mathfrak{b}}$. Then $c^n \in \mathfrak{a}+\mathfrak{b}$ for some $n \in \mathbb{N}$, so there exist $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ such that $c^n = x + y$. Since $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$ and $\mathfrak{b} \subseteq \sqrt{\mathfrak{b}}$, we have $x \in \sqrt{\mathfrak{a}}$ and $y \in \sqrt{\mathfrak{b}}$, so $c^n = x + y \in \sqrt{\mathfrak{a}}+\sqrt{\mathfrak{b}}$. Thus $a \in \sqrt{\sqrt{\mathfrak{a}}+\sqrt{\mathfrak{b}}}$.

Conversely, suppose $c \in \sqrt{\sqrt{\mathfrak{a}}+\sqrt{\mathfrak{b}}}$. Then $c^m \in \sqrt{\mathfrak{a}}+\sqrt{\mathfrak{b}}$ for some $m \in \mathbb{N}$, so there exist $x \in \sqrt{\mathfrak{a}}$ and $y \in \sqrt{\mathfrak{b}}$ such that $c^m = x + y$. By the definition of the radical, there exist $k, \ell \in \mathbb{N}$ such that $x^k \in \mathfrak{a}$ and $y^\ell \in \mathfrak{b}$. Then

$$c^{m(k+\ell)} = (x+y)^{k+\ell} = \sum_{i=1}^{k+\ell}\binom{k+\ell}{i}x^i y^{k+\ell-i} \in \mathfrak{a}+\mathfrak{b},$$

because $y^{k+\ell-i} \in \mathfrak{b}$ for the terms $i = 1, \ldots, k$ and $x^i \in \mathfrak{a}$ for the terms $i = k+1, \ldots, k+\ell$. Thus $c \in \sqrt{\mathfrak{a}+\mathfrak{b}}$. □

Lemma 3.3. *If $\sqrt{\mathfrak{a}} = A$, then $\mathfrak{a} = A$.*

Proof. If $\mathfrak{a}$ were a proper ideal, then it would be contained in some prime ideal $\mathfrak{p}$ by Zorn's lemma, and by definition of $\sqrt{\mathfrak{a}}$ as the intersection of all prime ideals containing $\mathfrak{a}$, we would have $\sqrt{\mathfrak{a}} \subseteq \mathfrak{p} \subsetneq A$. □

4. Saturation

Our third and final preparation concerns saturation.

Definition 4.1. A multiplicatively closed subset S of A is *saturated* if $xy \in S$ implies $x \in S$ and $y \in S$.

The following characterization of saturation is often useful.

Lemma 4.2. *S is saturated if and only if $A \setminus S$ is a union of prime ideals.*

Proof. ($\Longrightarrow$) Suppose S is saturated. Then every unit u in A is in S, since for any $x \in S$, $u(u^{-1}x) \in S$ implies $u \in S$.

We look at $S^{-1}A$. For any $x \in A \setminus S$, we know $\frac{x}{1}$ is not a unit in $S^{-1}A$, so (x) is a proper ideal of $S^{-1}A$. By Zorn's lemma, there exists a maximal ideal $\mathfrak{m}$ of $S^{-1}A$ containing (x). If $\phi : A \to S^{-1}A$ is the canonical map, then $\phi^{-1}(\mathfrak{m})$ is a prime ideal of A containing x and contained in $A \setminus S$. Thus $A \setminus S$ is a union of prime ideals.

($\Longleftarrow$) Conversely, suppose $A \setminus S = \bigcup_{\alpha \in I} \mathfrak{p}_\alpha$ is a union of prime ideals. If $xy \in S$, then $xy \notin A \setminus S = \bigcup_{\alpha \in I} \mathfrak{p}_\alpha$, so $xy \in A \setminus \bigcup_{\alpha \in I} \mathfrak{p}_\alpha = \bigcap_{\alpha \in I} A \setminus \mathfrak{p}_\alpha$. If either $x \in \mathfrak{p}_\beta$ or $y \in \mathfrak{p}_\beta$ for some $\beta \in I$, then $xy \in \mathfrak{p}_\beta$, a contradiction. So $x \notin \bigcup_{\alpha \in I} \mathfrak{p}_\alpha$ and $y \notin \bigcup_{\alpha \in I} \mathfrak{p}_\alpha$; that is, $x, y \in S$. Thus S is saturated. □

Lemma 4.3. *If S is a multiplicatively closed subset of A, then there exists a unique smallest saturated multiplicatively closed subset $\overline{S}$ of A containing S.*

Proof. Let $(\mathfrak{p}_\alpha)_{\alpha \in I}$ be the set of all prime ideals in A which do not intersect S. Let $\overline{S} = A \setminus \bigcup_{\alpha \in I} \mathfrak{p}_\alpha$. By Lemma 4.2, $\overline{S}$ is saturated. To see that it is the smallest saturated set containing S, suppose T is a saturated multiplicatively closed subset of A containing S. Then $A \setminus T$ is a union of prime ideals, say $\bigcup_{\beta \in J} \mathfrak{p}_\beta$. We wish to show that $\overline{S} \subseteq T$, or equivalently, $J \subseteq I$.

Suppose not; then there exists $\gamma \in J \setminus I$. So $\mathfrak{p}_\gamma$ intersects S; pick $x \in \mathfrak{p}_\gamma \cap S$. Then $x \in \bigcup_{\beta \in J} \mathfrak{p}_\beta$ and $x \in T = A \setminus \bigcup_{\beta \in J} \mathfrak{p}_\beta$, a contradiction.

Thus $\overline{S} \subseteq T$, showing that $\overline{S}$ is the smallest saturated multiplicatively closed subset of A containing S. □

The previous lemma allows us to make the following definition.

Definition 4.4. Let S be a multiplicatively closed subset of A. The *saturation* $\overline{S}$ of S is the unique smallest saturated multiplicatively closed subset of A containing S.

Lemma 4.5. *Let S, T be multiplicatively closed subsets of A such that $S \subseteq T$. Let $\phi : S^{-1}A \to T^{-1}A$ be the homomorphism mapping $\frac{a}{s} \in S^{-1}A$ to $\frac{a}{s} \in T^{-1}A$. The following are equivalent:*

(1) *ϕ is bijective,*
(2) *For each $t \in T$, $\frac{t}{1}$ is a unit in $S^{-1}A$,*
(3) *For each $t \in T$, there exists $x \in A$ such that $xt \in S$,*
(4) *$T \subseteq \overline{S}$,*
(5) *Every prime ideal which intersects T also intersects S.*

Proof. (1) $\implies$ (2). If ϕ is bijective, then for each $t \in T$ there exists $\frac{a}{s} \in S^{-1}A$ such that $\phi\left(\frac{a}{s}\right) = \frac{1}{t}$. Now

$$\phi\left(\frac{1}{1}\right) = \frac{1}{1} = \frac{t}{1} \cdot \frac{1}{t} = \phi\left(\frac{t}{1}\right)\phi\left(\frac{a}{s}\right) = \phi\left(\frac{t}{1} \cdot \frac{a}{s}\right)$$

implies $\frac{t}{1} \cdot \frac{a}{s} = 1$.

(2) $\implies$ (3). If $\frac{t}{1}$ is a unit in $S^{-1}A$, then there exists $\frac{a}{s} \in S^{-1}A$ such that $\frac{t}{1} \cdot \frac{a}{s} = 1$. By definition, there exists $u \in S$ such that $uat = us$, which is in S. Let $x = ua$.

(3) $\implies$ (4). For each $t \in T$, let $x \in A$ be such that $xt \in S$. Then $xt \in \overline{S}$, which is saturated, so $t \in \overline{S}$.

(4) $\implies$ (5). Let $\mathfrak{p}$ be a prime ideal that intersects T; pick $t \in \mathfrak{p} \cap T$. Then $t \in \overline{S}$. By Lemma 4.3, t is not contained in any prime ideal that does not intersect S, so $\mathfrak{p}$ must intersect S.

(5) $\implies$ (4). Let $t \in T$. Suppose, for the sake of contradiction, that $t \in A \setminus \overline{S}$. By Lemma 4.3, t is contained in some prime ideal $\mathfrak{p}$ that does not intersect S. But $\mathfrak{p}$ intersects T at t, a contradiction.

(4) $\implies$ (3). Let $S' = \{a \in A : \text{ there exists } x \in A \text{ such that } xa \in S\}$. We claim that $S' = \overline{S}$. Firstly, $S \subseteq S' \subseteq \overline{S}$. Indeed, for any $s \in S$, taking $x = 1$ shows that $s \in S'$, and for any $a \in S'$, $xa \in S \subseteq \overline{S}$ implies $a \in \overline{S}$.

It remains to show S' is saturated. If $ab \in S'$, then there exists $x \in A$ such that $x(ab) \in S$. We may rewrite this as $(xb)a \in S$ and $(xa)(b) \in S$, so $a, b \in S'$. Now $T \subseteq \overline{S} = S'$, as desired.

(3) $\implies$ (1). To see injectivity, suppose $\phi(\frac{a}{s}) = \frac{0}{1}$. Let $t \in T$ be such that $at = 0$. Let $x \in S$ be such that $xt \in S$. Then $(xt)a = 0$ shows that $\frac{a}{s} = \frac{0}{1}$ in $S^{-1}A$.

To see surjectivity, let $\frac{a}{t} \in T^{-1}A$. let $x \in S$ be such that $xt \in S$. Then $\frac{xa}{xt} \in S$ is such that $\frac{a}{t} = \frac{xa}{xt} = \phi\left(\frac{xa}{xt}\right)$. $\square$

5. The Basic Presheaf

We will start by defining $A(U)$ for basic open sets U. Recall that $X = \operatorname{Spec}(A)$ has basic open sets in the form $X_f = X \setminus V(f)$ for $f \in A$, where $V(f)$ is the set of prime ideals containing f.

A natural choice for $A(U)$ is the localization A_f at f, but we must verify this choice depends only on U and not on f. The following theorem makes this precise.

Lemma 5.1. *Let $f, g \in A$ be such that $X_f = X_g$. Then $A_f \cong A_g$.*

Proof. Since $X_f = X_g$, we have $V(f) = V(g)$. By the intersection of prime ideals definition of the radical, $\sqrt{(f)} = \sqrt{(g)}$. In terms of the first definition, $f \in \sqrt{(f)} = \sqrt{(g)}$ means that $f^k = ug$ for some $u \in A$ and $k \in \mathbb{N}$. Then $\phi_f(g)$ is a unit in A_f, since $\frac{g}{1} \cdot \frac{u}{f^k} = \frac{1}{1}$. It follows that the image of any element in $S_g = \{1, g, g^2, \dots\}$ under ϕ_f is a unit in A_f. Therefore, by the universal property of A_g, there exists a unique ring homomorphism $\psi_{gf} : A_g \to A_f$ such that $\phi_f = \psi_{gf} \circ \phi_g$. Symmetrically, we have a unique ring homomorphism $\psi_{fg} : A_f \to A_g$ such that

$\phi_g = \psi_{fg} \circ \phi_f$. We have the following commutative diagram:

$$\begin{tikzcd} & A \arrow[dl, "\phi_f"'] \arrow[dr, "\phi_g"] & \\ A_f \arrow[rr, shift left, "\psi_{fg}"] & & A_g \arrow[ll, shift left, "\psi_{gf}"] \end{tikzcd} \tag{5.1}$$

It remains to show that ψ_{gf} and ψ_{fg} are inverses of each other. Indeed, the diagram gives $\phi_f = \psi_{gf} \circ \psi_{fg} \circ \phi_f$, so the universal property of A_f implies $\psi_{gf} \circ \psi_{fg} = \mathrm{id}_{A_f}$. Symmetrically, $\psi_{fg} \circ \psi_{gf} = \mathrm{id}_{A_g}$, showing that $A_f \cong A_g$. □

Now we can define $A(U)$ for basic open sets U without worry:

Definition 5.2. For each basic open set U, let $f \in A$ be such that $U = X_f$. We define the *section* $A(U)$ over U to be A_f.

Moving on to the restriction maps between basic open sets before addressing sections over general open sets, suppose $X_g \subseteq X_f$. Then $V(f) \subseteq V(g)$, so every prime ideal containing f also contains g; that is, $g \in \sqrt{(f)}$. Then there exist $u \in A$ and $k \in \mathbb{N}$ such that $g^k = uf$. As in the proof of Lemma 5.1, the image of every element of $S_f = \{1, f, f^2, \dots\}$ under ϕ_g is a unit in A_g, so by the universal property of A_f, there exists a unique ring homomorphism $\rho^{X_f}_{X_g} : A_f \to A_g$ such that $\phi_g = \rho^{X_f}_{X_g} \circ \phi_f$.

We can abuse uniqueness to recover an explicit formula for $\rho^{X_f}_{X_g}$.

Theorem 5.3. *Let $X_g \subseteq X_f$ be an inclusion of basic open sets. Then the restriction map $\rho^{X_f}_{X_g} : A_f \to A_g$ is given by*

$$\rho^{X_f}_{X_g}\left(\frac{a}{f^m}\right) = \frac{au^m}{g^{mk}}.$$

Proof. By uniqueness, it suffices to show that this formula satisfies the universal property of A_f. Firstly, we verify $\rho^{X_f}_{X_g}$ is a ring homomorphism:

$$\begin{aligned} \rho^{X_f}_{X_g}\left(\frac{a}{f^m} + \frac{b}{f^n}\right) &= \rho^{X_f}_{X_g}\left(\frac{af^n + bf^m}{f^{m+n}}\right) \\ &= \frac{(af^n + bf^m)u^{m+n}}{g^{(m+n)k}} \\ &= \frac{au^m}{g^{mk}} + \frac{bu^n}{g^{nk}} \\ &= \rho^{X_f}_{X_g}\left(\frac{a}{f^m}\right) + \rho^{X_f}_{X_g}\left(\frac{b}{f^n}\right), \end{aligned}$$

and

$$\begin{aligned}\rho^{X_f}_{X_g}\left(\frac{a}{f^m}\cdot\frac{b}{f^n}\right) &= \rho^{X_f}_{X_g}\left(\frac{ab}{f^{m+n}}\right)\\ &= \frac{abu^{m+n}}{g^{(m+n)k}}\\ &= \frac{au^m}{g^{mk}}\cdot\frac{bu^n}{g^{nk}}\\ &= \rho^{X_f}_{X_g}\left(\frac{a}{f^m}\right)\cdot\rho^{X_f}_{X_g}\left(\frac{b}{f^n}\right).\end{aligned}$$

Now let $a \in A$. We have

$$\rho^{X_f}_{X_g}\left(\phi_f(a)\right) = \rho^{X_f}_{X_g}\left(\frac{a}{1}\right) = \frac{au^0}{g^{0n}} = \frac{a}{1} = \phi_g(a),$$

completing the proof. □

Once again, given an inclusion $V = X_g \subseteq X_f = U$ of basic open sets, we would like to define ρ^U_V as $\rho^{X_f}_{X_g}$. However, we must first show that this definition depends only on U and V, and not on f and g. But what does this even mean?

As before, let $f, f', g, g' \in A$ be such that $X_f = X_{f'}$ and $X_g = X_{g'}$. We cannot demand that $\rho^{X_f}_{X_g} = \rho^{X_{f'}}_{X_{g'}}$, since $A_f, A_{f'}$ are only isomorphic as rings, not identical as sets. Yet we have isomorphisms $\psi_{f'f} : A_{f'} \to A_f$ and $\psi_{gg'} : A_g \to A_{g'}$ from Lemma 5.1. At best, we can demand that the following diagram commutes:

$$\begin{tikzcd}
A_f \arrow[rr, "\rho^{X_f}_{X_g}"] & & A_g \arrow[dd, "\psi_{gg'}"] \\
& A \arrow[ul, "\phi_f"'] \arrow[ur, "\phi_g"] \arrow[dl, "\phi_{f'}"] \arrow[dr, "\phi_{g'}"'] & \\
A_{f'} \arrow[uu, "\psi_{f'f}"] \arrow[rr, "\rho^{X_{f'}}_{X_{g'}}"'] & & A_{g'}
\end{tikzcd} \tag{5.2}$$

Lemma 5.4. *The diagram (5.2) commutes.*

Proof. The only part left to show is

$$\rho^{X_{f'}}_{X_{g'}} = \psi_{gg'} \circ \rho^{X_f}_{X_g} \circ \psi_{f'f}.$$

By simple diagram-chasing, we have

$$\begin{aligned}\rho^{X_{f'}}_{X_{g'}} \circ \phi_{f'} &= \phi_{g'}\\ &= \psi_{gg'} \circ \phi_g\\ &= \psi_{gg'} \circ \rho^{X_f}_{X_g} \circ \phi_f\\ &= \psi_{gg'} \circ \rho^{X_f}_{X_g} \circ \psi_{f'f} \circ \phi_{f'}.\end{aligned}$$

Now $\rho^{X_{f'}}_{X_{g'}} \circ \phi_{f'} : A \to A_{g'}$ is a ring homomorphism such that for all $s \in S_{f'}$, $(\rho^{X_{f'}}_{X_{g'}} \circ \phi_{f'})(s)$ is a unit in $A_{g'}$. Indeed, $\phi_{f'}(s)$ is a unit in $A_{f'}$, so if it has inverse v then by the homomorphism property, $\rho^{X_{f'}}_{X_{g'}}(\phi_{f'}(s))$ has inverse $\rho^{X_{f'}}_{X_{g'}}(v)$. By the universal property of $A_{f'}$, there exists a unique ring homomorphism $\mu : A_{f'} \to A_{g'}$ such that $\rho^{X_{f'}}_{X_{g'}} \circ \phi_{f'} = \mu \circ \phi_{f'}$. By uniqueness of μ, we must have

$$\rho^{X_{f'}}_{X_{g'}} = \mu = \psi_{gg'} \circ \rho^{X_f}_{X_g} \circ \psi_{f'f}.$$

$\square$

Remark 5.5. In the last paragraph, we could replace $\rho^{X_{f'}}_{X_{g'}}$ and $\psi_{gg'} \circ \rho^{X_f}_{X_g} \circ \psi_{f'f}$ with any maps. This argument shows that the canonical map is an epimorphism in the category of rings; that is, it is right-cancellable.

At last, we can comfortably define the restriction map between basic open sets.

Definition 5.6. For each inclusion $V \subseteq U$ of basic open sets, let $f, g \in A$ be such that $U = X_f, V = X_g$. We define the *restriction* $\rho^U_V : A(U) \to A(V)$ to be $\rho^{X_f}_{X_g} : A_f \to A_g$.

6. Niceties

In this section we prove the two nice properties we desired in the introduction.

Theorem 6.1. *For all basic open sets U, $\rho^U_U = id_{A(U)}$.*

Proof. Let $U = X_f$. By construction, we have $\rho^U_U \circ \phi_f = \phi_f$. Since ϕ_f is an epimorphism, we cancel to obtain $\rho^U_U = \mathrm{id}_{A_f} = \mathrm{id}_{A(U)}$. $\square$

Theorem 6.2. *For all basic open sets $W \subseteq V \subseteq U$, $\rho^U_W = \rho^V_W \circ \rho^U_V$.*

Proof. Let $U = X_f, V = X_g, W = X_h$. From everything we have done, we know each third of the circle in the diagram below commutes:

$$\begin{tikzcd}
A_f \arrow[rr, bend left, "\rho^{X_f}_{X_h}"] \arrow[ddr, bend right, "\rho^{X_f}_{X_g}"'] & & A_h \\
& A \arrow[ul, "\phi_f"] \arrow[ur, "\phi_h"'] \arrow[d, "\phi_g"] & \\
& A_g \arrow[uur, bend right, "\rho^{X_g}_{X_h}"'] &
\end{tikzcd} \tag{6.1}$$

Now we can chase the diagram:

$$\rho^{X_f}_{X_h} \circ \phi_f = \phi_h = \rho^{X_g}_{X_h} \circ \phi_g = \rho^{X_g}_{X_h} \circ \rho^{X_f}_{X_g} \circ \phi_f.$$

Since ϕ_f is an epimorphism, we can cancel to obtain $\rho^{X_f}_{X_h} = \rho^{X_g}_{X_h} \circ \rho^{X_f}_{X_g}$. By definition, this means $\rho^U_W = \rho^V_W \circ \rho^U_V$. $\square$

We have thus constructed a presheaf of rings on the prime spectrum $X = \mathrm{Spec}(A)$.

7. Extension to Arbitrary Open Sets

We extend everything we have done so far to arbitrary open sets. Let $U \subseteq X$ be an open set. Let $(X_{f_\alpha})_{\alpha \in I}$ be the collection of basic open sets contained in U. For each $\alpha \in I$, let $\overline{S_{f_\alpha}}$ be the saturation of the multiplicatively closed subset $\{f_\alpha^n : n \in \mathbb{N}\}$ of A.

By the proof of Lemma 4.3, $\overline{S_{f_\alpha}} = A \setminus \bigcup_{\mathfrak{p} \in X_{f_\alpha}} \mathfrak{p}$. Indeed, if $\mathfrak{p} \in X_{f_\alpha}$, then $f_\alpha \notin \mathfrak{p}$, and since $\mathfrak{p}$ is prime, $f_\alpha^n \notin \mathfrak{p}$ for all $n \in \mathbb{N}$. Conversely, if $f_\alpha^n \notin \mathfrak{p}$ for all $n \in \mathbb{N}$, then $f_\alpha \notin \mathfrak{p}$, so $\mathfrak{p} \in X_{f_\alpha}$. Therefore X_{f_α} is precisely the set of prime ideals in A that do not intersect $\{f_\alpha^n : n \in \mathbb{N}\}$.

Let $S_U = \bigcap_{\alpha \in I} \overline{S_{f_\alpha}}$. This is saturated multiplicatively closed as an intersection of saturated multiplicatively closed sets. More explicitly,

$$S_U = \bigcap_{\alpha \in I} A \setminus \left(\bigcup_{\mathfrak{p} \in X_{f_\alpha}} \mathfrak{p} \right) = A \setminus \bigcup_{\alpha \in I} \left(\bigcup_{\mathfrak{p} \in X_{f_\alpha}} \mathfrak{p} \right) = A \setminus \bigcup_{\mathfrak{p} \in U} \mathfrak{p}. \tag{7.1}$$

We would like to define $A(U) = S_U^{-1}A$. First, we make the following sanity check.

Lemma 7.1. *Let $U = X_f$ be a basic open set. Then $S_U^{-1}A \cong A_f$.*

Proof. In this case, (7.1) becomes

$$S_U = A \setminus \bigcup_{\mathfrak{p} \in X_f} \mathfrak{p}.$$

We claim that $S_f \subseteq S_U$. Indeed, $f \notin \mathfrak{p}$ for all $\mathfrak{p} \in X_f$. So $f \in S_U$, and it follows that $f^m \in S_U$ for all $m \in \mathbb{N}$.

Moreover, X_f is a basic open set contained in U; that is, $X_f \in (X_{f_\alpha})_{\alpha \in I}$ from the first paragraph of this section. Thus $S_U = \bigcap_{\alpha \in I} \overline{S_{f_\alpha}} \subseteq \overline{S_f}$. From (4) $\implies$ (1) in Lemma 4.5, ϕ is an isomorphism $S_f^{-1}A \cong S_U^{-1}A$. $\square$

Definition 7.2. The *section* $A(U)$ of an arbitrary open set U is $S_U^{-1}A$.

Let $V \subseteq U$ be an inclusion of arbitrary open sets. By (7.1), we have $S_U \subseteq S_V$, so we have a canonical homomorphism $\phi_V^U : S_U^{-1}A \to S_V^{-1}A$ mapping $\frac{a}{s} \in S_U^{-1}A$ to $\frac{a}{s}$ in $S_V^{-1}A$, as in Lemma 4.5.

Again, we need the following sanity check before we fossilize our definition.

Lemma 7.3. *Let $U = X_f, V = X_g$ be basic open sets such that $V \subseteq U$. Let $\phi_{f,U} : A_f \to S_U^{-1}A$ and $\phi_{g,V} : A_g \to S_V^{-1}A$ be the canonical isomorphisms as in Lemma 7.1. Then the following diagram commutes:*

$$\begin{tikzcd} A_f \arrow[r, "\rho_{X_g}^{X_f}"] \arrow[d, "\phi_{f,U}"] & A_g \arrow[d, "\phi_{g,V}"] \\ S_U^{-1}A \arrow[r, "\phi_V^U"] & S_V^{-1}A \end{tikzcd} \tag{7.2}$$

Proof. Let $\frac{a}{f^m} \in A_f$. On one hand, we have $(\phi_V^U \circ \phi_{f,U}) \left(\frac{a}{f^m} \right) = \phi_V^U \left(\frac{a}{f^m} \right) = \frac{a}{f^m}$ in $S_V^{-1}A$. On the other hand, we have

$$\phi_{g,V} \circ \rho_{X_g}^{X_f} \left(\frac{a}{f^m} \right) = \phi_{g,V} \left(\frac{au^m}{(uf)^m} \right) = \frac{au^m}{(uf)^m} = \frac{a}{f^m},$$

where the first equality is Theorem 5.3, the second equality is by definition of $\phi_{g,V}$, and the third is because $au^m f^m = au^m f^m$. $\square$

Definition 7.4. For each inclusion $V \subseteq U$ of open sets, the *restriction* $\rho_V^U : A(U) \to A(V)$ is the canonical map $\phi_V^U : S_U^{-1}A \to S_V^{-1}A$.

8. Extended Niceties

In ways more than one, the extended definition of the previous section produces a nicer structure than the original definition on basic open sets, which relied on clunky representatives. We generalize the two nice properties we proved for basic open sets to arbitrary open sets with relative ease.

Theorem 8.1. *For all open sets U, $\rho_U^U = id_{A(U)}$.*

Proof. For the canonical map $\phi_U^U : S_U^{-1}A \to S_U^{-1}A$, we immediately have any condition from (2) to (5) in Lemma 4.5. This implies ϕ_U^U is a bijection, and thus it must be the identity. $\square$

Theorem 8.2. *For all inclusions $W \subseteq V \subseteq U$ of open sets, $\rho_W^U = \rho_W^V \circ \rho_V^U$.*

Proof. This follows immediately from the definitions of ϕ_W^U, ϕ_W^V, and ϕ_V^U. $\square$

9. Stalks

Whenever we have a presheaf, we are interested in a piece of data called the stalk at a point. Given $\mathfrak{p} \in X$, the collection

$$I_\mathfrak{p} = \{U \subseteq X : U \text{ is open in } X \text{ and } \mathfrak{p} \in U\}$$

of open sets containing $\mathfrak{p}$ is partially ordered by reverse inclusion. We show that it is in fact a directed set.

Lemma 9.1. *$I_\mathfrak{p}$ is a directed set under reverse inclusion. More precisely, for any $U, V \in I_\mathfrak{p}$, there exists $W \in I_\mathfrak{p}$ such that $U \supseteq W$ and $V \supseteq W$.*

Proof. Let X_f and X_g be basic open sets containing $\mathfrak{p}$ contained in U and V, respectively. We must have $f, g \notin \mathfrak{p}$, and since $\mathfrak{p}$ is prime, $fg \notin \mathfrak{p}$. Thus $X_{fg} \in I_\mathfrak{p}$. We have

$$\begin{aligned} X_{fg} &= X \setminus V(fg) \\ &= X \setminus (V(f) \cup V(g)) \\ &= (X \setminus V(f)) \cap (X \setminus V(g)) \\ &= X_f \cap X_g. \end{aligned}$$

Thus $W = X_{fg}$ satisfies $U \supseteq W$ and $V \supseteq W$, so $I_\mathfrak{p}$ is a directed set. $\square$

It now follows from Theorem 8.1 and Theorem 8.2 that $(A(U), \rho_U^V)_{U,V \in I_\mathfrak{p}}$ is a directed system of A-modules [A-M69, Exercise 2.14]. We can thus make the following definition.

Definition 9.2. The *stalk* at $\mathfrak{p} \in X$ is defined as the direct limit

$$\varinjlim_{U \in I_\mathfrak{p}} A(U)$$

of the directed system $(A(U), \rho_U^V)_{U,V \in I_\mathfrak{p}}$.

Luckily, given an open set, we can always find a basic open set contained in it. More formally, let

$$J_{\mathfrak{p}} = \{X_f : f \in A \setminus \mathfrak{p}\}$$

be the collection of basic open sets containing $\mathfrak{p}$. Then $J_{\mathfrak{p}}$ inherits the partial order from $I_{\mathfrak{p}}$, and it possesses the following key property.

Lemma 9.3. *$J_{\mathfrak{p}}$ is cofinal in $I_{\mathfrak{p}}$. That is, for any $U \in I_{\mathfrak{p}}$, there exists $X_f \in J_{\mathfrak{p}}$ such that $X_f \subseteq U$.*

Proof. This is simply the definition of a basic open set. □

Corollary 9.4. *$J_{\mathfrak{p}}$ is a directed set under reverse inclusion, and*

$$\varinjlim_{U \in I_{\mathfrak{p}}} A(U) \cong \varinjlim_{X_f \in J_{\mathfrak{p}}} A(X_f).$$

Proof. We repeat the proof of Lemma 9.1. Given $X_f, X_g \in J_{\mathfrak{p}}$, $X_{fg} \in J_{\mathfrak{p}}$ is such that $X_{fg} \subseteq X_f$ and $X_{fg} \subseteq X_g$. Thus $J_{\mathfrak{p}}$ is a directed set under reverse inclusion.

It is intuitively clear that $\varinjlim_{U \in I_{\mathfrak{p}}} A(U) \cong \varinjlim_{X_f \in J_{\mathfrak{p}}} A(X_f)$. More rigorously, this is a diagram chase by uniqueness of the direct limit [Mac Lane, Theorem IX.2.1]. □

Thus, we have the following theorem to concretely describe the stalks of $\mathrm{Spec}(A)$.

Theorem 9.5. *Let $\mathfrak{p} \in X$. Then $\varinjlim_{U \in I_{\mathfrak{p}}} A(U) \cong A_{\mathfrak{p}}$.*

Proof. By Corollary 9.4, it suffices to compute the direct limit $\varinjlim_{X_f \in J_{\mathfrak{p}}} A(X_f)$. Given $f \in A \setminus \mathfrak{p}$, the definition of the localization map $\phi_{\mathfrak{p}} : A \to A_{\mathfrak{p}}$ implies $\phi_{\mathfrak{p}}(f) = \frac{f}{1}$ is a unit. Consequently, the image of S_f under $\phi_{\mathfrak{p}}$ consists entirely of units. Then by the universal property of A_f, there exists a unique ring homomorphism $\psi_f : A_f \to A_{\mathfrak{p}}$ such that $\phi_{\mathfrak{p}} = \psi_f \circ \phi_f$. That is, the thirds of the circle in the diagram below commute:

(9.1)
$$\begin{tikzcd}
A_f \arrow[rr, bend left, "\rho^{X_f}_{X_g}"] \arrow[dr, bend right=60, "\psi_f"'] & & A_g \arrow[dl, bend left=60, "\psi_g"] \\
& A \arrow[ul, "\phi_f"] \arrow[ur, "\phi_g"'] \arrow[d, "\phi_{\mathfrak{p}}"] & \\
& A_{\mathfrak{p}} &
\end{tikzcd}$$

Then by diagram chasing, we have

$$\psi_f \circ \phi_f = \phi_{\mathfrak{p}} = \psi_g \circ \phi_g = \psi_g \circ \rho^{X_f}_{X_g} \circ \phi_f.$$

Since ϕ_f is an epimorphism, we can cancel to obtain $\psi_f = \psi_g \circ \rho^{X_f}_{X_g}$. By the universal property of the direct limit [A-M69, Exercise 2.16], there exists a unique ring homomorphism $\psi : \varinjlim_{X_f \in J_{\mathfrak{p}}} A(X_f) \to A_{\mathfrak{p}}$ such that $\psi_f = \psi \circ \mu_f$ for all $f \in A \setminus \mathfrak{p}$, where $\mu_f : A_f \to \varinjlim_{X_f \in J_{\mathfrak{p}}} A(X_f)$ is the canonical injection of the direct limit.

It remains to show that ψ is an isomorphism. First, we claim that $\psi_f\left(\frac{a}{f^n}\right)=\frac{a}{f^n}$ for all $a \in A$ and $n \in \mathbb{N}$. This is obviously a homomorphism, and we verify that for any $a \in A$,

$$\phi_{\mathfrak{p}}(a)=\frac{a}{1}=\psi_f\left(\frac{a}{1}\right)=\psi_f \circ \phi_f(a).$$

By uniqueness of ψ_f, it must be given by this formula.

To see ψ is injective, suppose $b \in \varinjlim_{X_f \in J_{\mathfrak{p}}} A(X_f)$ is such that $\psi(b)=0$. Then there exists $f \in A \setminus \mathfrak{p}$ such that b has the form $\mu_f(\frac{a}{f^n})$ for some $\frac{a}{f^n} \in A_f$. Then

$$0=\psi(b)=(\psi \circ \mu_f)\left(\frac{a}{f^n}\right)=\psi_f\left(\frac{a}{f^n}\right)=\frac{a}{f^n}.$$

Thus $b=\mu_f(\frac{a}{f^n})=0$, showing that ψ is injective.

For surjectivity, suppose $\frac{a}{s} \in A_{\mathfrak{p}}$. Then $a \in A$ and $s \in A \setminus \mathfrak{p}$. Then $\frac{a}{s} \in A_s$, so let $b=\mu_s(\frac{a}{s}) \in \varinjlim_{X_f \in J_{\mathfrak{p}}} A(X_f)$. We have

$$\psi(b)=\psi \circ \mu_s\left(\frac{a}{s}\right)=\psi_s\left(\frac{a}{s}\right)=\frac{a}{s},$$

showing ψ is surjective. Therefore ψ is an isomorphism $\varinjlim_{X_f \in J_{\mathfrak{p}}} A(X_f) \cong A_{\mathfrak{p}}$. $\square$

10. Turning Sheaf

In this section, we will form a sheaf from the presheaf we have just constructed.

Definition 10.1. A *sheaf* of rings on X is a presheaf satisfying the following property. Let $(U_\alpha)_{\alpha \in I}$ be an open covering of X. For each $\alpha \in I$, let $s_\alpha \in A(U_\alpha)$ be such that for each pair of indices $\alpha, \beta \in I$, the images of s_α and s_β in $A(U_\alpha \cap U_\beta)$ under $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}$ and $\rho^{U_\beta}_{U_\alpha \cap U_\beta}$ are equal. Then there exists a unique $s \in A=A(X)$ whose image in $A(U_\alpha)$ is s_α, for each $\alpha \in I$.

To help show the presheaf satisfies this condition, we pick up a few lemmas.

Lemma 10.2. *$X=Spec(A)$ is compact.*

Proof. Let $(X_{f_\alpha})_{\alpha \in I}$ be a basic open covering of $X=\operatorname{Spec}(A)$. Then

$$X=\bigcup_{\alpha \in I} X_{f_\alpha}=\bigcup_{\alpha \in I} X \setminus V(f_\alpha)=X \setminus \bigcap_{\alpha \in I} V(f_\alpha)=X \setminus V((f_\alpha)_{\alpha \in I}).$$

Thus $V((f_\alpha)_{\alpha \in I})=\varnothing$, or $((f_\alpha)_{\alpha \in I})=(1)$. So there exist $\alpha_1, \ldots, \alpha_n \in I$ and $g_1, \ldots, g_n \in A$ such that

$$\sum_{i=1}^n g_i f_{\alpha_i}=1.$$

Then $((f_\alpha)_{\alpha \in I})=(1)$, and thus $(X_{f_{\alpha_i}})_{i=1}^n$ is a finite subcovering of $(X_{f_\alpha})_{\alpha \in I}$. $\square$

Definition 10.3. Two ideals $\mathfrak{a}, \mathfrak{b}$ in A are *coprime* if $\mathfrak{a}+\mathfrak{b}=A$.

Lemma 10.4. *Suppose $\mathfrak{a}, \mathfrak{b}$ are coprime in A. Then for any $m \in \mathbb{Z}^+$, $\mathfrak{a}^m$ and $\mathfrak{b}$ are coprime.*

Proof. By definition of the radical, $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}^m}$, and of course $\mathfrak{b} \subseteq \sqrt{\mathfrak{b}}$. Since $\mathfrak{a}+\mathfrak{b}=A$, we have $\sqrt{\mathfrak{a}^m}+\sqrt{\mathfrak{b}}=A$. Hence by Lemma 3.2

$$\sqrt{\mathfrak{a}^m+\mathfrak{b}}=\sqrt{\sqrt{\mathfrak{a}^m}+\sqrt{\mathfrak{b}}}=\sqrt{A}=A.$$

From here, Lemma 3.3 implies $\mathfrak{a}^m+\mathfrak{b}=A$. $\square$

Theorem 10.5. *The presheaf* $(A(U), \rho_V^U)$ *is a sheaf.*

Proof. Let $(U_\alpha)_{\alpha\in I}$ and $(s_\alpha)_{\alpha\in I}$ be as in Definition 10.1. Without loss of generality, we may assume that the U_α are basic open sets. If not, we can replace U_α by the collection $(U_\beta)_{\beta\in I_\alpha}$ of basic open sets contained in it, setting $s_\beta = \rho_{U_\beta}^{U_\alpha}(s_\alpha)$ for each $\beta \in I_\alpha$. By applying the following argument for uniqueness of s_α to the basic open covering $(U_\beta)_{\beta\in I_\alpha}$ of U_α, we see that the desired s uniquely exists for the original covering $(U_\alpha)_{\alpha\in I}$ if and only if it uniquely exists for the modified covering $((U_\beta)_{\beta\in I_\alpha})_{\alpha\in I}$.

Since $X = \operatorname{Spec}(A)$ is compact by Lemma 10.2, we have a finite subcover $U_{\alpha_1}, \dots, U_{\alpha_n}$. For $i = 1, \dots, n$, let $f_i \in A$ be such that $U_{\alpha_i} = X_{f_i}$. Write $s_{\alpha_i} \in A_{f_i}$ in the form $\frac{a_i}{f_i^{m_i}}$ for some $a_i \in A$ and $m_i \in \mathbb{N}$. Let $m = \max_{i=1}^n m_i$ and $b_i = a_i f_i^{m-m_i}$, so that

$$s_{\alpha_i} = \frac{a_i}{f_i^{m_i}} \cdot \left(\frac{f_i}{f_i}\right)^{m-m_i} = \frac{b_i}{f_i^m}.$$

Given any indices $j, k \in \{1, \dots, n\}$, the proof of Lemma 9.1 implies that $X_{f_j} \cap X_{f_k} = X_{f_j f_k}$. If $X_{f_j f_k} = \varnothing$, then every prime ideal contains $f_j f_k$, so $f_j f_k \in \mathfrak{N}$ and thus $f_j f_k$ is nilpotent. Consequently, $S_{f_j f_k}$ contains 0, so $A_{f_j f_k} = 0$ and the images of s_{α_j} and s_{α_k} agree trivially.

Otherwise $X_{f_i f_j}$ is nonempty. By Theorem 5.3, the condition $\rho_{U_{\alpha_j}\cap U_{\alpha_k}}^{U_{\alpha_j}}(s_{\alpha_j}) = \rho_{U_{\alpha_j}\cap U_{\alpha_k}}^{U_{\alpha_k}}(s_{\alpha_k})$ is equivalent to the following:

$$\begin{aligned}
\rho_{X_{f_j}\cap X_{f_k}}^{X_{f_j}}(s_{\alpha_j}) &= \rho_{X_{f_j}\cap X_{f_k}}^{X_{f_k}}(s_{\alpha_k}) \\
\rho_{X_{f_j f_k}}^{X_{f_j}}\left(\frac{b_j}{f_j^m}\right) &= \rho_{X_{f_j f_k}}^{X_{f_k}}\left(\frac{b_k}{f_k^m}\right) \\
\frac{b_j f_k^m}{f_j^m f_k^m} &= \frac{b_k f_j^m}{f_j^m f_k^m},
\end{aligned}$$

which holds if and only if there exists $m_{jk} \in \mathbb{N}$ such that

$$b_j f_k^m (f_j f_k)^{m+m_{jk}} = b_k f_j^m (f_j f_k)^{m+m_{jk}}.$$

Let $M = \max_{j=1}^n \max_{k=1}^n m_{jk}$. The following equation holds for all $j, k \in \{1, \dots, n\}$:

$$b_j f_k^m (f_j f_k)^{m+M} = b_k f_j^m (f_j f_k)^{m+M}.$$

Let $p \in \mathbb{N}$ be such that $m + M \le pm$. Multiplying the above equation by $(f_j f_k)^{pm-m-M}$, we obtain

$$b_j (f_k^m)^{p+1} (f_j^m)^p = b_k (f_j^m)^{p+1} (f_k^m)^p. \tag{10.1}$$

We will construct $s \in A$ such that for $\ell = 1, \dots, n$,

$$\rho_{U_{\alpha_\ell}}^X\left(\frac{s}{1}\right) = s_{\alpha_\ell}.$$

We remark that

$$X \setminus \bigcap_{i=1}^n V(f_i) = \bigcup_{i=1}^n X \setminus V(f_i) = \bigcup_{i=1}^n X_{f_i} = X.$$

This means

$$\varnothing = \bigcap_{i=1}^{n} V(f_i) = V\left(\sum_{i=1}^{n}(f_i)\right),$$

so $\sum_{i=1}^{n}(f_i) = A$. Repeatedly applying Lemma 10.4 to each (f_i),

$$\sum_{i=1}^{n}((f_i^m)^{p+1}) = \sum_{i=1}^{n}(f_i)^{m(p+1)} = A.$$

Thus there exist $c_1, \ldots, c_n \in A$ such that

$$1 = \sum_{i=1}^{n} c_i (f_i^m)^{p+1}.$$

Fix $\ell \in \{1, \ldots, n\}$. Multiplying the previous equation by $b_\ell (f_\ell^m)^p$, we have

$$b_\ell (f_\ell^m)^p = \sum_{i=1}^{n} c_i b_\ell (f_i^m)^{p+1} (f_\ell^m)^p.$$

Expanding the right-hand side with (10.1),

$$\begin{aligned} b_\ell (f_\ell^m)^p &= \sum_{i=1}^{n} c_i b_i (f_\ell^m)^{p+1} (f_i^m)^p \\ &= (f_\ell^m)^{p+1} \sum_{i=1}^{n} c_i b_(f_i^m)^p. \end{aligned}$$

Thus if we define $s = \sum_{i=1}^{n} c_i b_i (f_i^m)^p$, then we have

$$b_\ell (f_\ell^m)^p = (f_\ell^m)^{p+1} s.$$

By Theorem 5.3, $\rho^X_{U_{\alpha_\ell}} = \phi_{\alpha_\ell}$. So this precisely means that for $\ell = 1, \ldots, n$,

$$\rho^X_{U_{\alpha_\ell}}(s) = \frac{s}{1} = \frac{b_\ell}{f_\ell^m} = s_{\alpha_\ell},$$

and thus we have shown existence of the desired $s \in A$ for the finite subcovering.

Next we show that if s exists for the finite subcovering, then it is unique. Suppose $s, s' \in A$ are such that $\rho^X_{U_{\alpha_\ell}}(s) = s_{\alpha_\ell} = \rho^X_{U_{\alpha_\ell}}(s')$ for $\ell = 1, \ldots, n$. Then $\frac{s-s'}{1} = \rho^X_{U_{\alpha_\ell}}(s - s') = 0$ implies that $(s - s') f_\ell^{q_\ell} = 0$ for some $q_\ell \in \mathbb{N}$. Let $q = \max_{\ell=1}^{n} q_\ell$. Then for all $\ell = 1, \ldots, n$, we have $(s - s') f_\ell^q = 0$. As above, we have

$$\sum_{\ell=1}^{n}(f_\ell^q) = \sum_{\ell=1}^{n}(f_\ell)^q = A,$$

so there exist $d_1, \ldots, d_n \in A$ such that $1 = \sum_{\ell=1}^{n} c_\ell f_\ell^q$. Therefore

$$(s - s') = \sum_{\ell=1}^{n} c_\ell f_\ell^q (s - s') = \sum_{\ell=1}^{n} c_\ell \cdot 0 = 0.$$

Thus $s = s'$, showing uniqueness.

Now we extend our construction of s to the entire covering $(U_\alpha)_{\alpha \in I}$. Given $\alpha \in I$, define, for $i = 1, \ldots, n$, $V_i = U_\alpha \cap U_{\alpha_i}$. Then $(V_i)_{i=1}^{n}$ is a finite subcovering of U_α, so by everything we have done, we have a unique $s_\alpha \in U_\alpha$ with the desired property. That is, for $i = 1, \ldots, n$, we have

$$\rho^{U_\alpha}_{V_i}(s_\alpha) = \rho^{U_{\alpha_j}}_{V_i}(s_{\alpha_j}) = \rho^X_{V_i}(s) = (\rho^{U_\alpha}_{V_i} \circ \rho^X_{U_\alpha})(s).$$

By uniqueness of s_α, we conclude that $s_\alpha = \rho^X_{U_\alpha}(s)$, and we win. $\square$

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