Simplicial Homotopy Theory

I. Simplicial Sets.

Dreams: category $\mathcal{S}$ of simplicial sets $\rightsquigarrow$ three classes of morphisms
- cofibrations
- fibrations
- weak equivalences

"(closed) Model category axioms" – Quillen

s.t. CM1: $\mathcal{S}$ closed under finite lim & colims.

CM2:
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "h"'] & Y \arrow[d, "f"] \\ & Z \end{tikzcd}$$
2/3 weak equiv $\Downarrow$ 3/3 weak equiv.

CM3: f retracts g in the cat of maps in $\mathcal{S}$. Then g weak equiv [cofib] (fib) $\Downarrow$ f weak equiv [cofib] (fib).

CM4.
$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$
- i cofib
- p fib
- $\exists$. if i or p is weak eq.

CM5. any $f: X \to Y \rightsquigarrow f = \underbrace{p}_{\text{fib}} \circ \overbrace{i}^{\text{cofib + weak eq.}}$ and $f = \underbrace{q}_{\text{fib + weak eq.}} \circ \overbrace{j}^{\text{cofib}}$

fibrations in $\mathcal{S}$ will be Kan fibrations
cofibrations will be monomorphisms (left cancellable)
weak eq. will be morphisms giving homotopy equivalence as CW-complexes.

I.1. Basic definitions.

Δ := cat of finite ordinals and order-preserving fns. / "ordinal number cat."
- Obj = $\{0,\ 0\to 1,\ 0\to 1\to 2, \dots,\ 0\to 1\to 2\to \cdots \to n\}$
- $\theta \in \mathrm{Hom}_\Delta(m,n)$ is function $\theta: \{0 \dots m\} \longrightarrow \{0 \dots n\}$ s.t. $i<j \Rightarrow \theta(i) \le \theta(j)$.

equivalently, view the posets $0\to 1\to 2\to \cdots \to n$ as categories; morphisms in Δ are just functors.

Def. a simplicial set is a functor $X: \Delta^{op} \to \mathrm{Sets}$.

ex. $\exists$ standard functor $\Delta \longrightarrow \mathrm{Top}$, $n \longmapsto |\Delta^n|$ where $|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i = 1, t_i \ge 0\}$ is std n-simplex

$\theta: (0 \to \cdots \to n) \longrightarrow (0 \to \cdots \to m) \rightsquigarrow \theta_*: |\Delta^n| \to |\Delta^m|$
$(t_0 \dots t_n) \mapsto (s_0 \dots s_n)$ where $s_i = \sum_{j \in \theta^{-1}(i)} t_j$ $\hookrightarrow 0$ if $\emptyset$.

Functoriality of $\theta \mapsto \theta_*$:

- $\mathrm{id}_n : (0 \to \cdots \to n) \to (0 \to \cdots \to n)$

  $\rightsquigarrow$

  $(\mathrm{id}_n)_* = \mathrm{id}_{|\Delta^n|}$ clear.

- $f : (0 \to \cdots \to n) \to (0 \to \cdots \to m)$, $g : (0 \to \cdots \to m) \to (0 \to \cdots \to p)$

$\rightsquigarrow (g \circ f)_* (t_0 \ldots t_n) = (u_0 \ldots u_p)$ where $u_i = \sum_{k \in (g\circ f)^{-1}(i)} t_k$

$= \sum_{k \in f^{-1}(g^{-1}(i))} t_k$

$= \sum_{j \in g^{-1}(i)} s_j$ where $s_j = \sum_{k \in f^{-1}(j)} t_k$

i.e. $f_* (t_0 \ldots t_n) = (s_0 \ldots s_m)$

$= g_* (s_0 \ldots s_m)$

$= g_* \circ f_* (t_0 \ldots t_m)$ $\square$.

T top. space $\rightsquigarrow$ singular set $S(T) : \Delta^{op} \to \mathrm{Sets}$

given by $(0 \to \cdots \to n) \mapsto \hom_{\mathrm{Top}}(|\Delta^n|, T)$

functoriality: recall singular homology.

important morphisms in Δ:

- $d^i : n-1 \to n$ $\quad 0 \le i \le n$.

  $(0 \to 1 \to \cdots \to n-1) \mapsto (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$.

- $s^j : n+1 \to n$ $\quad 0 \le j \le n$

  $(0 \to 1 \to \cdots \to n+1) \mapsto (0 \to 1 \to \cdots \to j \xrightarrow{1} j \to \cdots \to n)$.

"cosimplicial identities":

$$\begin{cases} d^j d^i = d^i d^{j-1} & i < j \\ s^j d^i = d^i s^{j-1} & i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & i > j+1 \\ s^j s^i = s^i s^{j+1} & i \le j. \end{cases}$$