1 Metric Spaces

We begin our first serious course in real analysis by revisiting a fact from first-year calculus which we will generalize throughout the course.

1.1 Completeness of $\mathbb{R}$

We take as axiom that every subset of $\mathbb{R}$ which is nonempty and bounded above has a supremum. A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers is said to be *Cauchy* if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n > N$ implies

$$|a_m - a_n| < \epsilon.$$

You should recall from MAT157 that a sequence converges if and only if it is Cauchy. For completeness (no pun intended), we will reprove it here.

Theorem 1.1.1

A sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ converges if and only if it is Cauchy.

Proof. ($\implies$) Suppose $a_n \to a$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \frac{1}{2}\epsilon$. If $m, n > N$, then

$$|a_m - a_n| \le |a_m - a| + |a - a_n| < \epsilon.$$

($\impliedby$) Suppose (a_n) is Cauchy. We first show that $\{a_n\}$ is bounded. Indeed, for $\epsilon = 1$ the Cauchy condition gives $N \in \mathbb{N}$ such that $|a_m - a_n| < 1$ for $m, n > N$. In particular

$$|a_m| < |a_{N+1}| + 1 \quad \text{for} \quad m > N,$$

so $M := \max\{|a_1|, \ldots, |a_N|, |a_{N+1}| + 1\}$ bounds $\{a_n\}$. Now define

$$A = \{x \in \mathbb{R} : \text{ there exists infinitely many } n \text{ such that } a_n \ge x\}.$$

This is nonempty since $\{a_n\}$ is bounded below; more specifically $-M \in A$ because $a_n \ge -M$ for all n. Moreover, A is bounded above since $\{a_n\}$ is bounded above; if $x \in A$ then $x \le a_n \le M$. By the supremum axiom of $\mathbb{R}$, there exists a supremum $a = \sup A$. We claim that $\lim_{n\to\infty} a_n = a$.

Given $\epsilon > 0$, let $N_1 \in \mathbb{N}$ be such that $|a_m - a_n| < \frac{1}{2}\epsilon$ for $m, n > N$, since (a_n) is Cauchy. Since a is an upper bound, $a + \frac{1}{2}\epsilon \notin A$, so $a_n \ge a + \frac{1}{2}\epsilon$ for only finitely many n. Let N_2 be sufficiently large that $a_n < a + \frac{1}{2}\epsilon$ for $n > N_2$. Observe that $a - \frac{1}{2}\epsilon \in B$; if not, then there are only finitely many n such that $a_n \ge a - \frac{1}{2}\epsilon$, so $a - \frac{1}{2}\epsilon$ would be an upper bound, contradicting the definition of $a = \sup A$. Now let $N \in \mathbb{N}$ be such that $N > \max\{N_1, N_2\}$ and $a_n > a - \frac{1}{2}\epsilon$ for $n > N$. Along with the definition of N_2, this implies $|a_n - a| < \frac{1}{2}\epsilon$, and combined with the definition of N_1, we conclude that

$$|a_n - a| \leq |a_n - a_m| + |a_m - a| < \epsilon.$$

□

Remark 1.1.2. A metric space satisfying Theorem 1.1.1 is called *complete*.

1.2 Metric spaces

A metric space is a set M with a metric; that is, a function $d\colon M \times M \to \mathbb{R}$ such that

(1) $d(x, y) \geq 0$ with equality if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Example 1.2.1. $\mathbb{R}$ is a metric space with $d(x, y) = |x - y|$. More generally, $\mathbb{R}^n$ has *Manhattan metric* $d(x, y) = \sum_{i=1}^n |x_i - y_i|$ and *Euclidean metric* $d(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}$.

Example 1.2.2. Let ℓ^1 be the set of absolutely convergent sequences in $\mathbb{R}$. We define a metric on ℓ^1 by

$$d((x_n), (y_n)) = \sum_{n=1}^{\infty} |x_n - y_n|.$$

Example 1.2.3. Let $C([0, 1]) = C^0([0, 1], \mathbb{R})$ be the set of continuous real-valued functions on $[0, 1]$. We define a metric on $C([0, 1])$ by

$$d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

A common way to obtain a metric is through a norm. A normed vector space is a vector space with a norm; that is, a function $\|\cdot\|\colon V \to \mathbb{R}$ satisfying

(1) $\|x\| \geq 0$ with equality if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\|x\|$.

(3) $\|x + y\| \leq \|x\| + \|y\|$.

A norm induces a metric $d(x,y) = \|x - y\|$. This is clearly positive-definite, symmetric, and satisfies the triangle inequality:

$$d(x,z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x,y) + d(y,z).$$

In fact, our example metrics on ℓ^1 and $C([0,1])$ arise in this way, with

$$\|x\|_{\ell^1} = \sum_{n=1}^{\infty} |a_n| \quad \text{and} \quad \|f\|_{\infty} = \sup_{x \in [0,1]} |f(x)|.$$