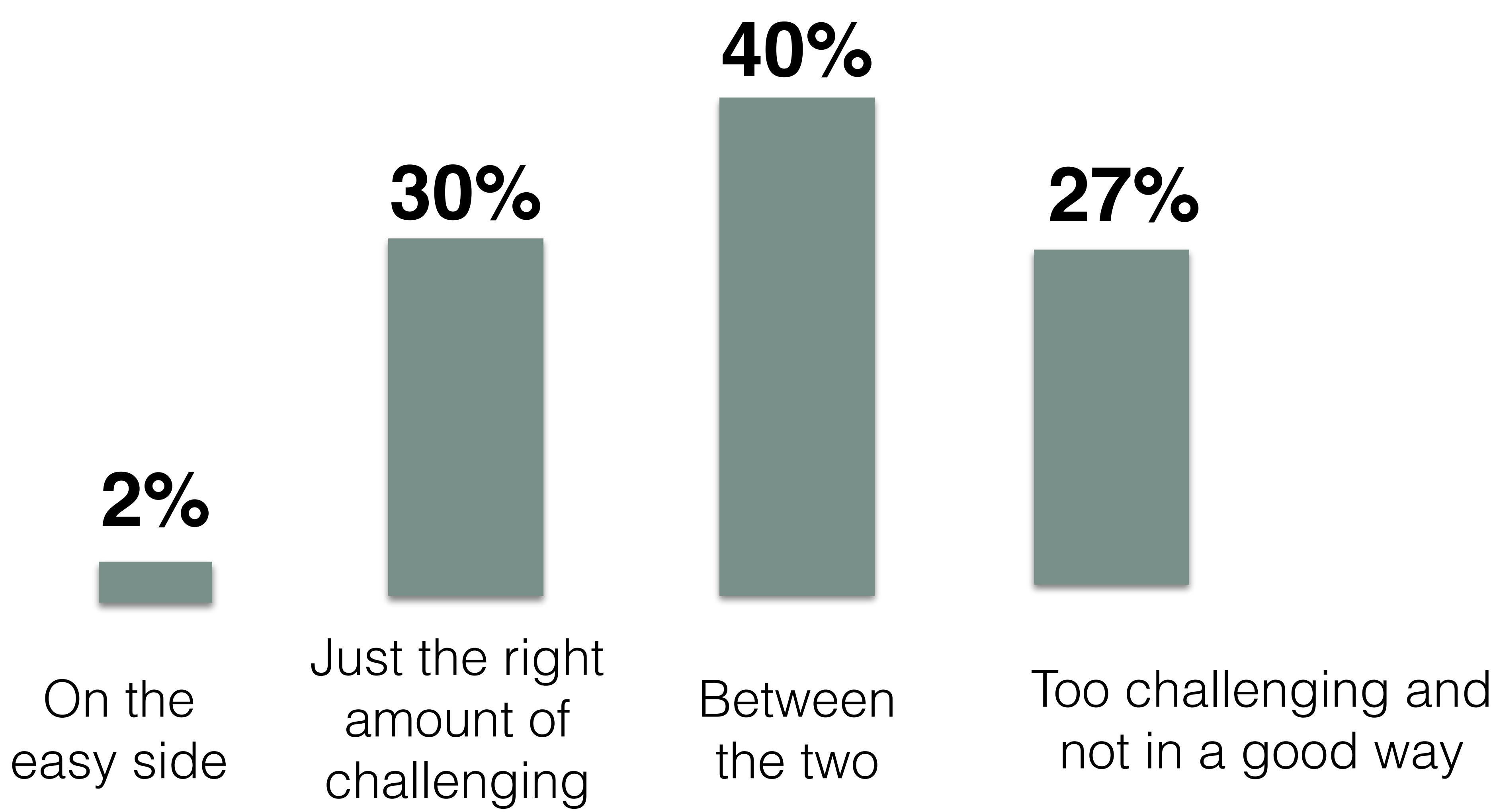
Greedy Algorithms: Minimizing Lateness

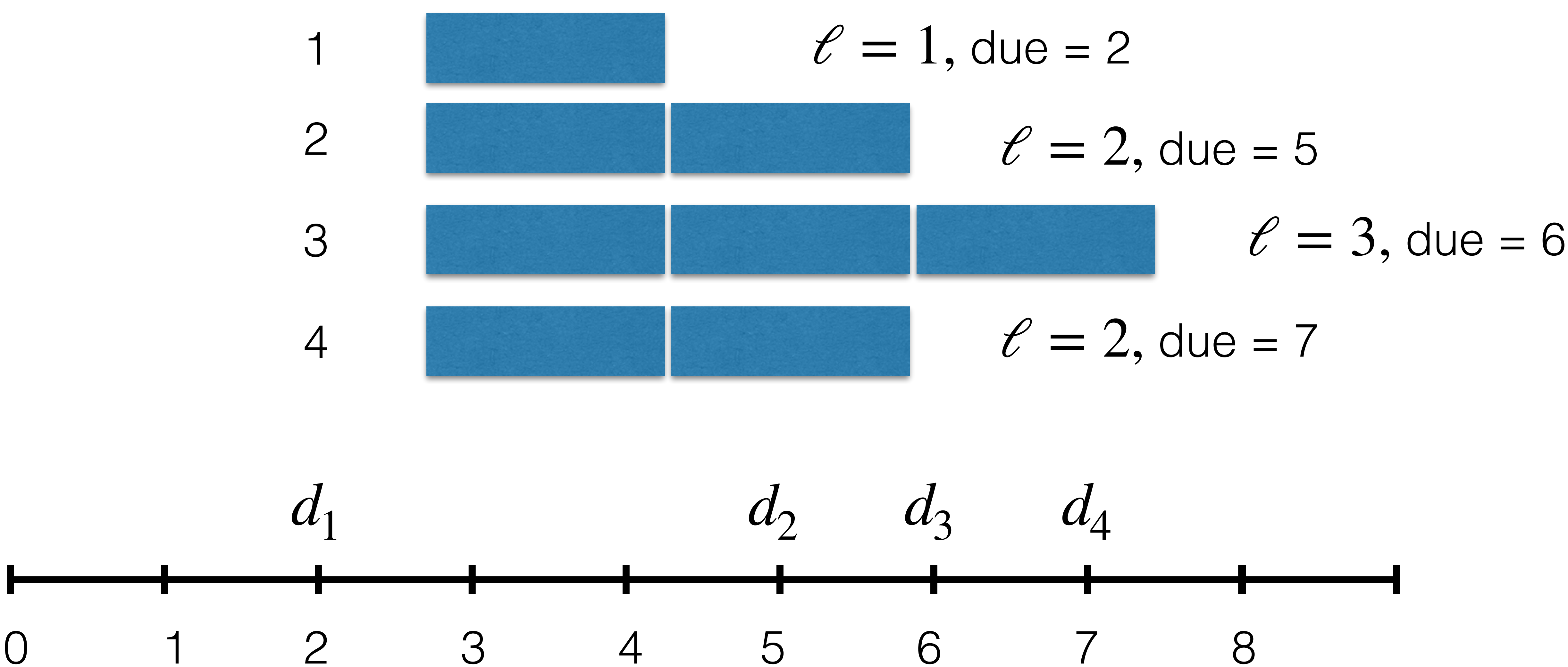
Reminders/Logistics

- CS256 URL: https://williams-cs.github.io/cs256-s21-www/
- Graded HW 1 will be returned today; solution on GLOW
- Feedback question response: Question 5 was a prime culprit



Minimizing Lateness: Problem

- You have n homework assignments of different lengths, with different deadlines
- How should you schedule your time to minimize lateness?
- Example:



Minimizing Lateness: Problem

Let's formalize the problem. The input is:

- A list of assignments or "jobs" that need to be scheduled
- Each job j has a **length** t_j and a **deadline** d_j

The output is a **schedule of all jobs**, what does it look like?

- s_j = start time for job j (selected by the algorithm)
- $f_j = s_j + t_j$ finish time

- Restrictions on the schedule:
  - Only one job can be scheduled at a given time
  - A job must run to completion before another can be executed

Minimizing Lateness: Problem

What makes a schedule good?

- Let us define lateness of job j as

$$\ell_j = \begin{cases} 0 & \text{if } f_j \leq d_j \\ f_j - d_j & \text{if } f_j > d_j \end{cases}$$

- Maximum lateness $L = \max_j \ell_j$

Goal: Make maximum lateness as small as possible, minimize maximum lateness

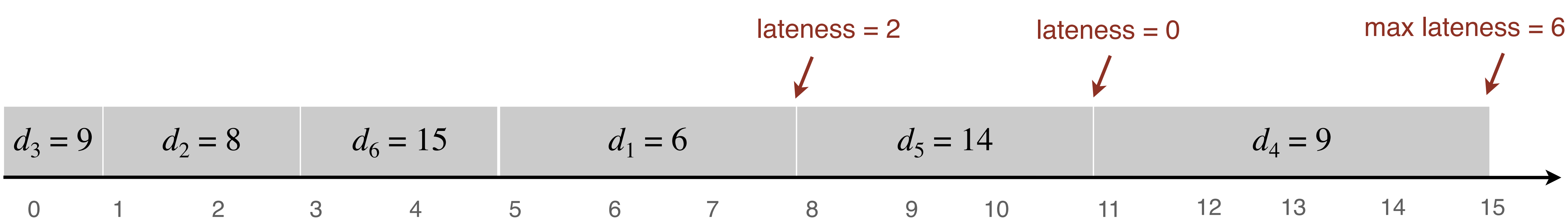
Minimizing Lateness: Problem

- Let us define lateness of job j as

$$\ell_j = \begin{cases} 0 & \text{if } f_j \leq d_j \\ f_j - d_j & \text{if } f_j > d_j \end{cases}$$

- Maximum lateness $L = \max_j \ell_j$

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Possible Greedy Approaches

- Observation:
  - Never hurts to schedule jobs consecutively with no "idle time" in between
  - Can start the first job at time 0
  - Schedule is then determined by order of jobs
  - What order should we pick?

Possible Greedy Approaches

- Possible strategies:
  - Shortest jobs first (get more done faster!)
  - Do jobs with shortest "slack" time first (slack of job i is $d_i - t_i$)
  - Earlier deadlines first (triage!)

Shortest job first (Counter-example)

	1	2
t_j	1	10
d_j	100	10

Gives max lateness: 1
OPT max lateness: 0

Possible Greedy Approaches

- Possible strategies:
  - Shortest jobs first (get more done faster!)
  - Do jobs with shortest "slack" time first (slack of job i is $d_i - t_i$)
  - Earlier deadlines first (triage!)

Shortest slack first (Counter-example)

	1	2
t_j	1	10
d_j	2	10

Gives max lateness: 9
OPT max lateness: 1

Possible Greedy Approaches

Earliest deadline first?

- How all computer scientists schedule their work
- Order jobs by their deadline $d_1 \leq d_2 \leq \ldots \leq d_n$ and schedule them in that order
    - **Intuition:** get the jobs due first done first
- This approach is optimal (We will show this)

Greedy Solution

Assuming jobs are ordered by deadline $d_1 \leq d_2 \leq \ldots \leq d_n$ the greedy ordering is simply $G = 1, 2, \ldots, n$

- All jobs are scheduled consecutively (no idle time)

Claim: G is optimal, that is, the earliest-deadline-first algorithm produces a schedule that minimizes maximum lateness

- Will prove this through proof by exchange argument
- High level idea: assume G is not optimal, then there exists an optimal solution $O \neq G$ that produces a different ordering or jobs; we will show that we can modify O to produce G one job at a time, without ever increasing lateness

Exchange Argument

Let O be an optimal solution, such that $O \neq G$, then we can modify O to produce a new solution O' that is:

- No worse than O
- Closer to G in some measurable way

Idea behind proof by exchange argument:

- Transform O into G one step at a time, without hurting solution (that is, preserving optimality)

O (optimal) $\rightarrow$ O' (optimal) $\rightarrow$ O'' (optimal) $\rightarrow \cdots \rightarrow$ G (optimal)

Exchange Argument

Let O be an optimal solution, such that $O \neq G$.

- Since $G = \{1, 2, \ldots, n\}$, where $d_1 \leq d_2 \leq \ldots \leq d_n$ and $O \neq G$, O must have an inversion
  - A pair of jobs (i, j) is an **inversion** if job j is scheduled before i but i's deadline is earlier ($d_j > d_i$)

a schedule with an inversion

recall: we assume the jobs are numbered so that $d_1 \leq d_2 \leq \ldots \leq d_n$

Structure of Optimal: Inversions

Observation. If an idle-free schedule has an inversion, then it has an adjacent inversion.

> Recall. i, j is an inversion if job j is scheduled before i but i's deadline is earlier ($d_i < d_j$)

Proof. [Contradiction]

- Let i, j be any two **closest** non-adjacent inversions
- Let k be element immediately to the right of j.
- Case 1. $[d_j > d_k]$ Then j, k is an adjacent and closer inversion ($\Rightarrow\!\Leftarrow$)
- Case 2. $[d_j < d_k]$ Since $d_i < d_j$, this means that i, k is a closer inversion ($\Rightarrow\!\Leftarrow$) ∎

	j	k			i		

Exchange Argument

Let O be an optimal solution, such that $O \neq G$.

- Since $G = \{1, 2, \ldots, n\}$, and $O \neq G$, O must have an inversion
- O must have at least one adjacent inversion
- **Claim**: We can swap adjacent inverted jobs without increasing maximum lateness.

a schedule with an inversion

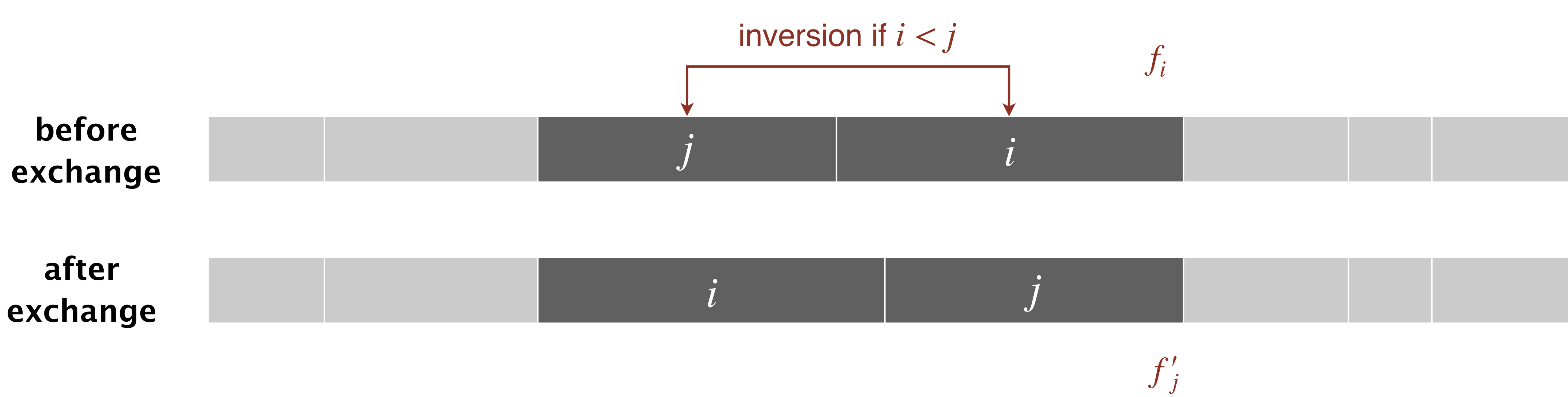
recall: we assume the jobs are numbered so that $d_1 \leq d_2 \leq \ldots \leq d_n$

Structure of Optimal: Inversions

Claim: We can swap adjacent inverted jobs without increasing maximum lateness.

Proof. Let i, j be adjacent inverted jobs with $i < j$. Let ℓ be the lateness before swapping them and ℓ' after the swap.

- $\ell_k = \ell'_k \qquad \forall k \neq i, j$ (swap doesn't affect other jobs)
- $\ell'_i \leq \ell_i \qquad$ (lateness of i improves after swap)
- $\ell'_j = \quad f'_j - d_j \quad = f_i - d_j \quad \leq f_i - d_i \quad \leq \ell_i$ ∎



Exchange Argument

Let O be an optimal solution, such that $O \neq G$.

- Since $G = \{1,2,\ldots,n\}$, and $O \neq G$, O must have an inversion
- O must have at least one adjacent inversion (i, j)
- **Claim**: We can swap adjacent inverted jobs (i, j) without increasing maximum lateness.
- Let O' be the new solution with adjacent inversion swapped, then
  - Max lateness of O' is no bigger than that of O (still optimal)
  - O' has one less inversion than O

O (optimal) $\rightarrow$ O' (optimal) $\rightarrow$ O'' (optimal) $\rightarrow \cdots \rightarrow$ G (optimal)

Exchange Argument

Summarizing and final proof.

- We started with an optimal solution O that is different than greedy solution G
- Without loss of generality assume O has no idle time
- If O has an inversion, must be adjacent, exchanging them
  - decreases # of inversions by 1 without increasing max lateness, we repeat until no inversions
- G is a schedule with no idle time and no inversions
- Thus, we have transformed O to G without ever increasing maximum lateness
- Greedy is thus optimal $\blacksquare$

Exchange Argument

General Pattern. An inductive exchange argument

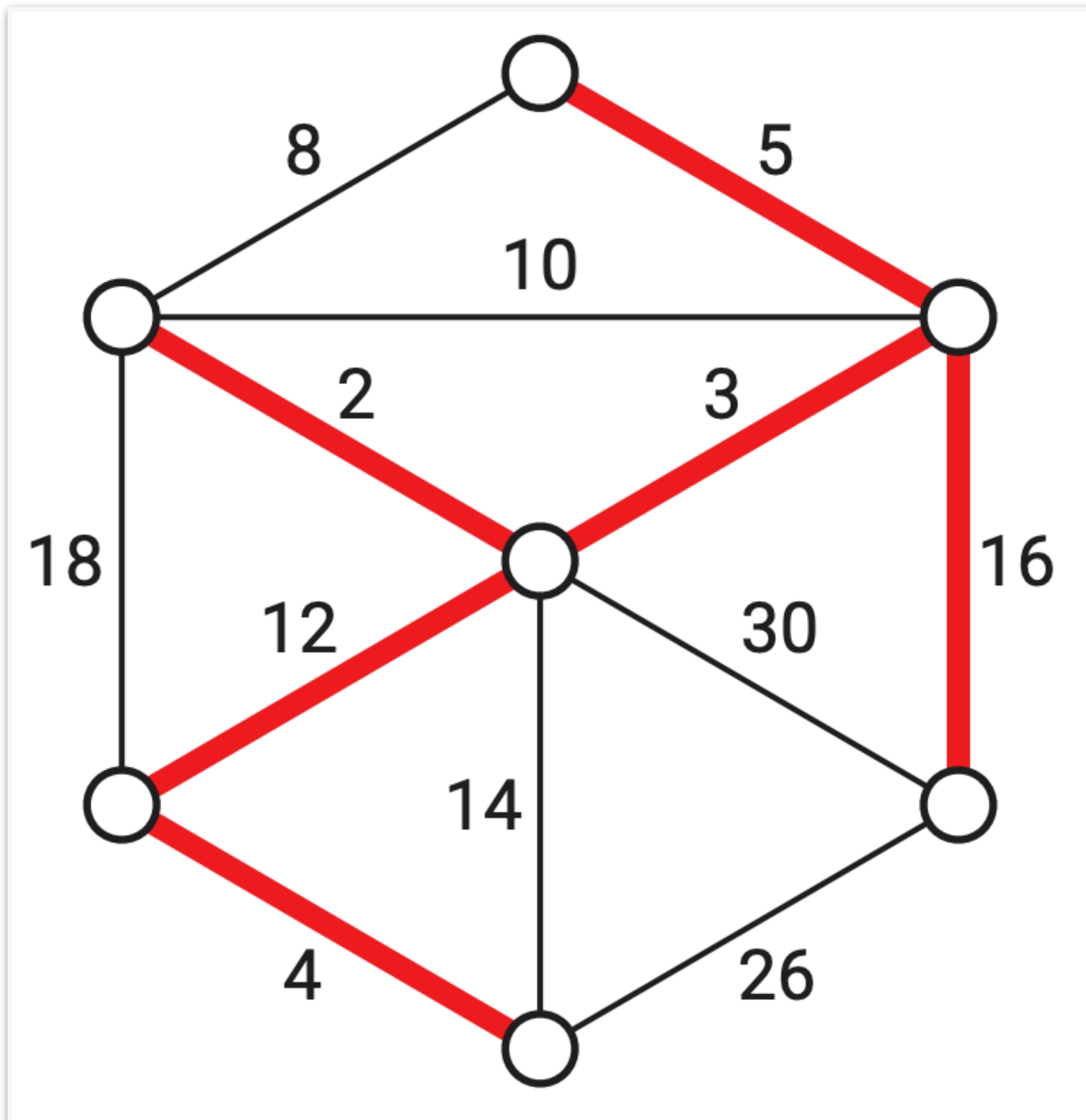
- You start with an arbitrary optimal solution, that is different from the greedy solution
- Find the “first” place where the two solutions differ
- Argue that we can exchange the optimal choice for the greedy choice without making the solution worse (although the exchange may make it better)
- Show that you can iteratively perform the exchange step until you get the greedy solution

Greedy: Takeaway

- The takeaway is that greedy algorithms do not usually work
- When greedy algorithms work, it is because the problem has structure that greedy can take advantage of

Greedy Graph Algorithms: Minimum Spanning Trees

Minimum Cost Spanning Trees



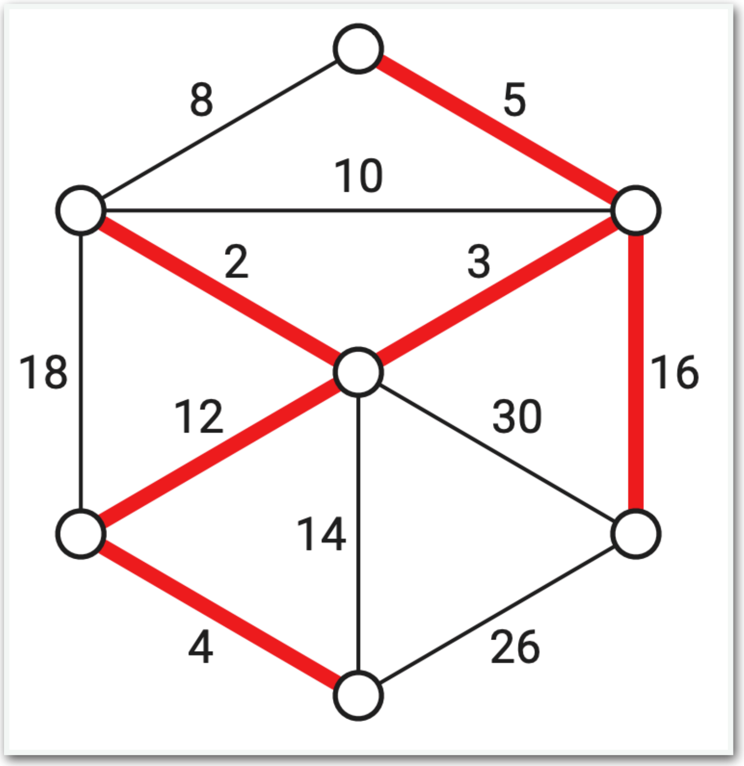
Minimum Spanning Trees

- Many applications!
  - Classic application:
    - Underground cable (Power, Telecom, etc)
  - Efficient broadcasting on a computer network (Note: different from shortest paths)
  - Approximate solutions to harder problems, such at TSP
  - Real-time face verification

Minimum Spanning Trees

Problem. Given a connected, undirected graph $G = (V, E)$ with edge costs w_e, output **a minimum spanning tree**, i.e., set of edges $T \subseteq E$ such that

- (a spanning tree of G): T connects all vertices
- (has minimum weight): for any other spanning tree T' of G, we have $\sum_{e \in T} w_e \leq \sum_{e \in T'} w_e$



Distinct Edge Weights

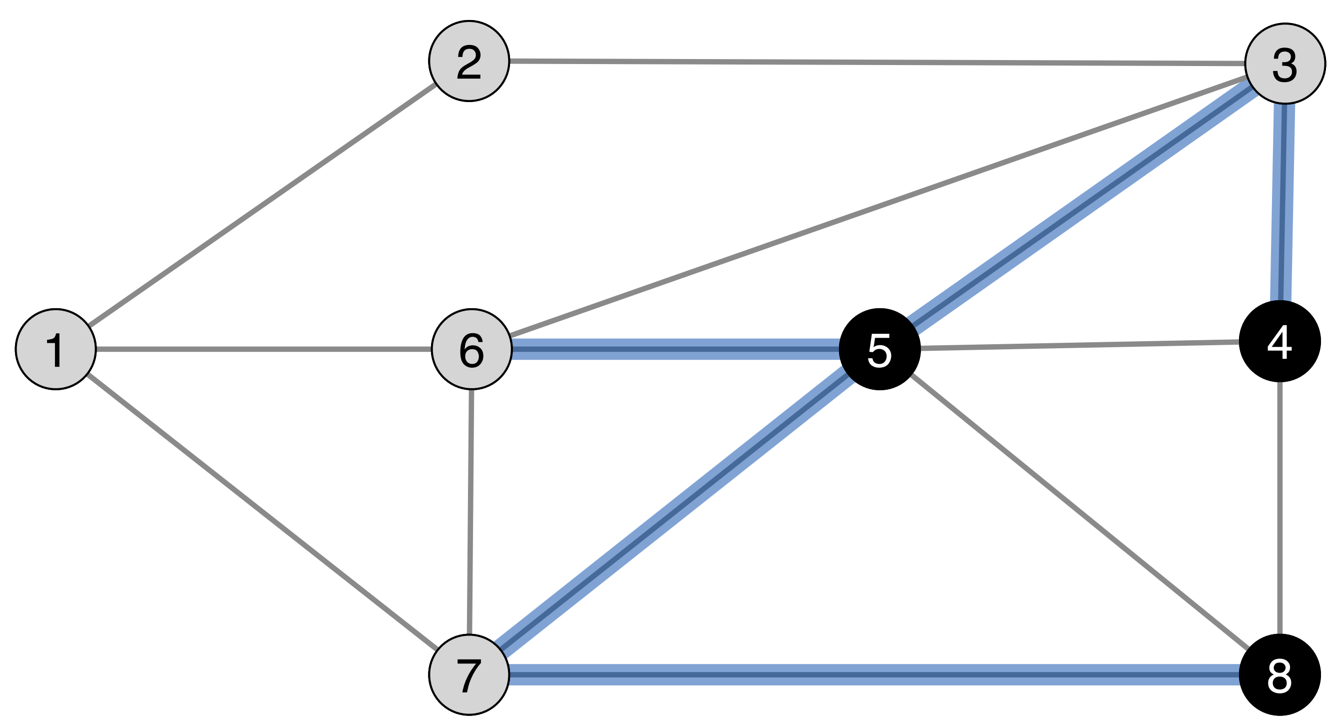
- Annoying subtlety in the problem statement is there may be multiple minimum spanning trees
    - If a graph has edges with same edge, e.g., all edges have weight 1: all spanning trees are min!
- To simplify discussion in our algorithm design, we will assume distinct edge weights

Lemma. If all edge weights in a connected graph are distinct, then it has a unique minimum spanning tree.

We will relax the distinct-edge-weight assumption later.

Spanning Trees and Cuts

A **cut** is a partition of the vertices into two **nonempty** subsets S and $V - S$. A **cut edge** of a cut S is an edge with one end point in S and another in $V - S$.



cut S = { 4, 5, 8 }

Cut edges = { (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) }

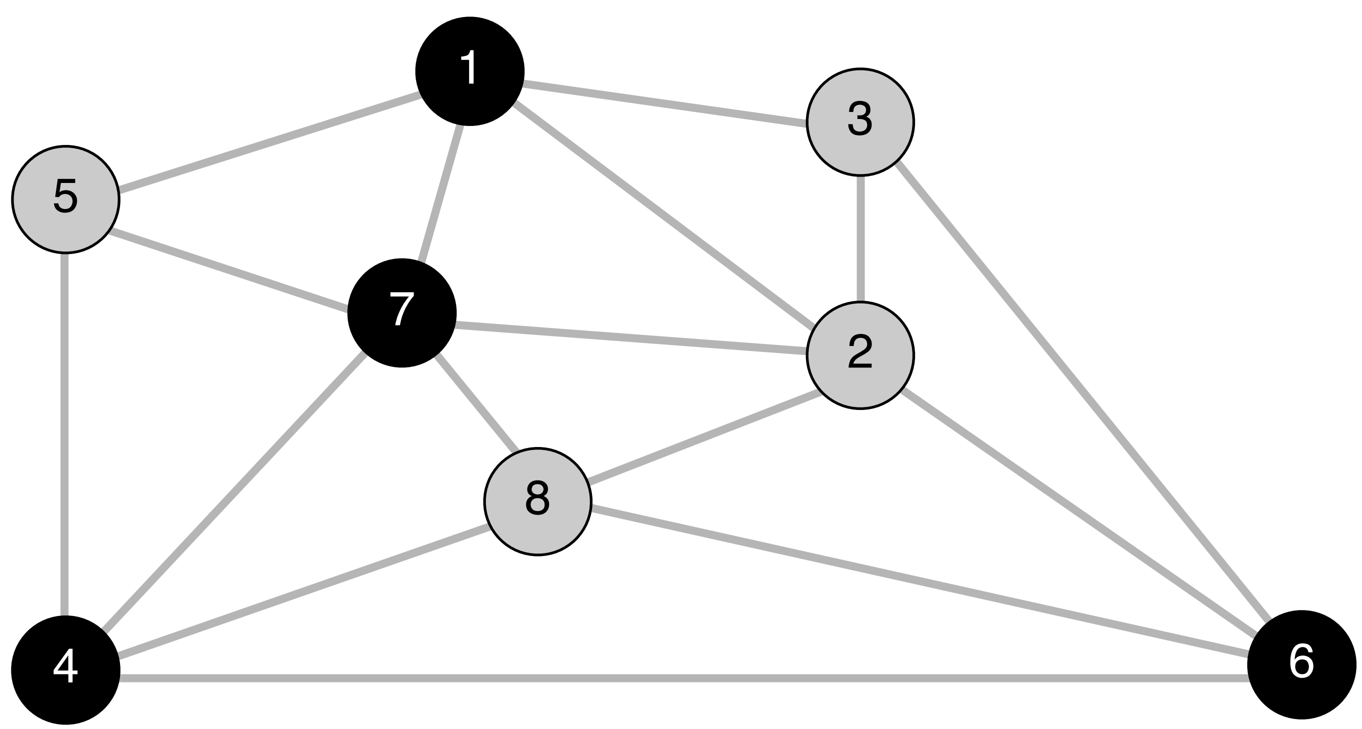
Spanning Trees and Cuts

Question. Consider the cut $S = \{1,4,6,7\}$. Which of the following edges are cut edges with respect to this cut?

A. (1, 7)

B. (5, 7)

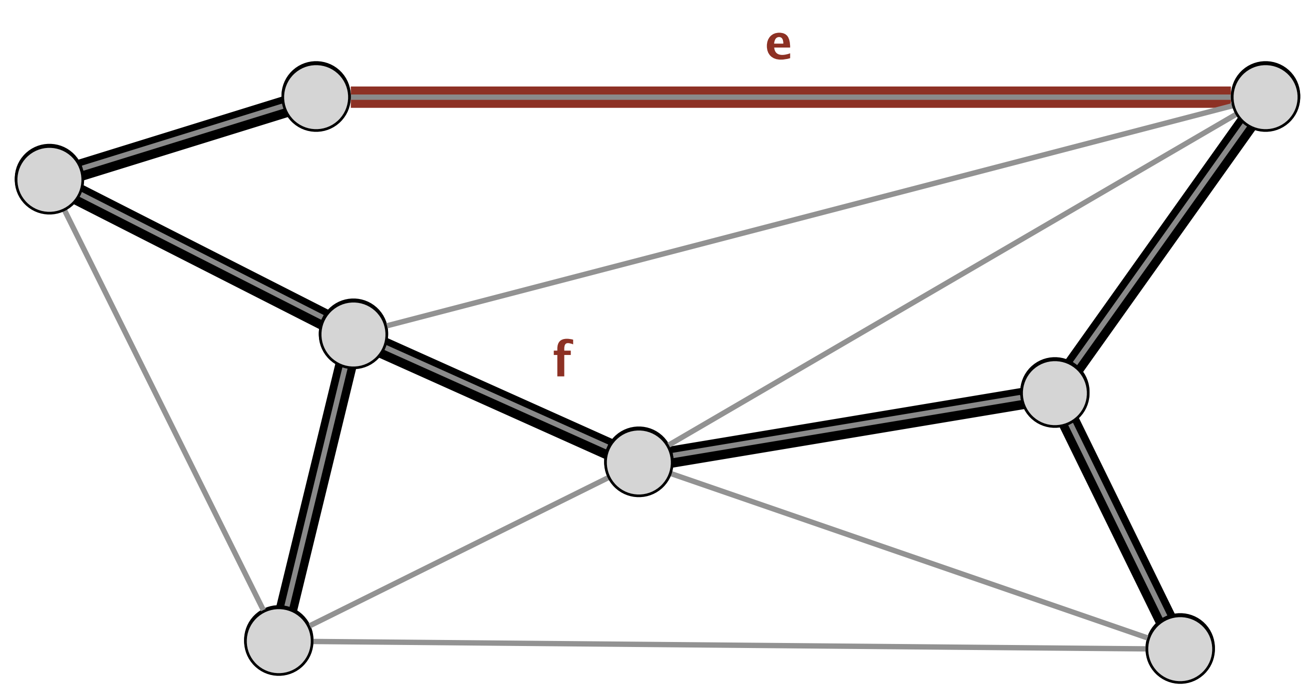
C. (2, 3)



Fundamental Cycle

Let T be a spanning tree of G.

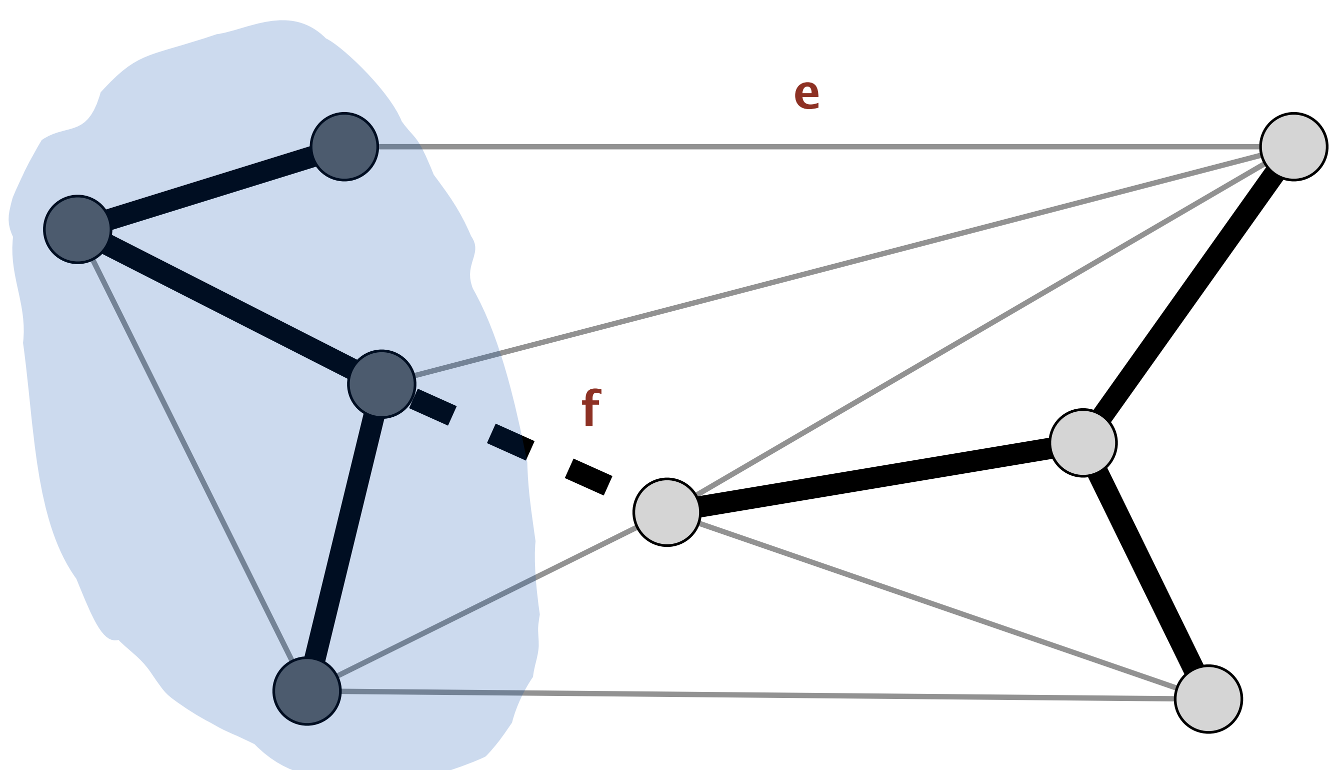
- For any edge $e \notin T$, $T \cup \{e\}$ creates a unique cycle C
- For any edge $f \in C : T \cup \{e\} - \{f\}$ is a spanning tree



Fundamental Cut

Let T be a spanning tree of G.

- For any edge $f \in T$, $T - \{f\}$ breaks the graph into two connected components, let D be the set of cut edges with end points in each component
- For any edge $e \in D : T - \{f\} \cup \{e\}$ is a spanning tree



Spanning Trees and Cuts

Lemma (Cut Property). For any cut $S \subset V$, let $e = (u, v)$ be the minimum weight edge connecting any vertex in S to a vertex in $V - S$, then every minimum spanning tree must include e.

Proof. (By contradiction)

Suppose T is a spanning tree that does not contain $e = (u, v)$.

Main Idea: We will construct another spanning tree $T' = T \cup e - e'$ with weight less than T ($\Rightarrow\!\Leftarrow$)

How to find such an edge e'?

Spanning Trees and Cuts

Proof (Cut Property).

Suppose T is a spanning tree that does not contain $e = (u, v)$.

- Adding e to T results in a unique cycle C
- C must "enter" and "leave" cut S, that is, $\exists e' = (u', v') \in C$ s.t. $u' \in S, v' \in V - S$
- $w(e') > w(e)$ (why?)
- $T' = T \cup e - e'$ is a spanning tree (why?)
- $w(T') < w(T)$

$(\Rightarrow\Leftarrow)$ ∎

Spanning Trees and Cycles

Lemma (Cycle Property). For any cycle C in G, its highest cost edge e is in no MST of G.

Proof. (By contradiction)

Suppose a MST T contains e.

- Main Idea: We will construct another spanning tree $T' = T - \{e\} \cup \{e'\}$ with weight less than T ($\Rightarrow\!\Leftarrow$)
- How to find such an e'?

Spanning Trees and Cycles

Lemma (Cycle Property). For any cycle C in G, its highest cost edge e is in no MST of G.

Proof. (By contradiction)

Suppose a MST T contains the heaviest edge $e = (u, v) \in C$.

- Removing e from T breaks the tree into two components: $S, V - S$ such that $u \in S$ and $v \in V - S$
- The cycle C in the graph must have another edge f going from S to $V - S$
- Adding f back to connect the components gives us another MST T'
- $w(T') < w(T)$, why?
- $\Rightarrow \Leftarrow$ $\blacksquare$

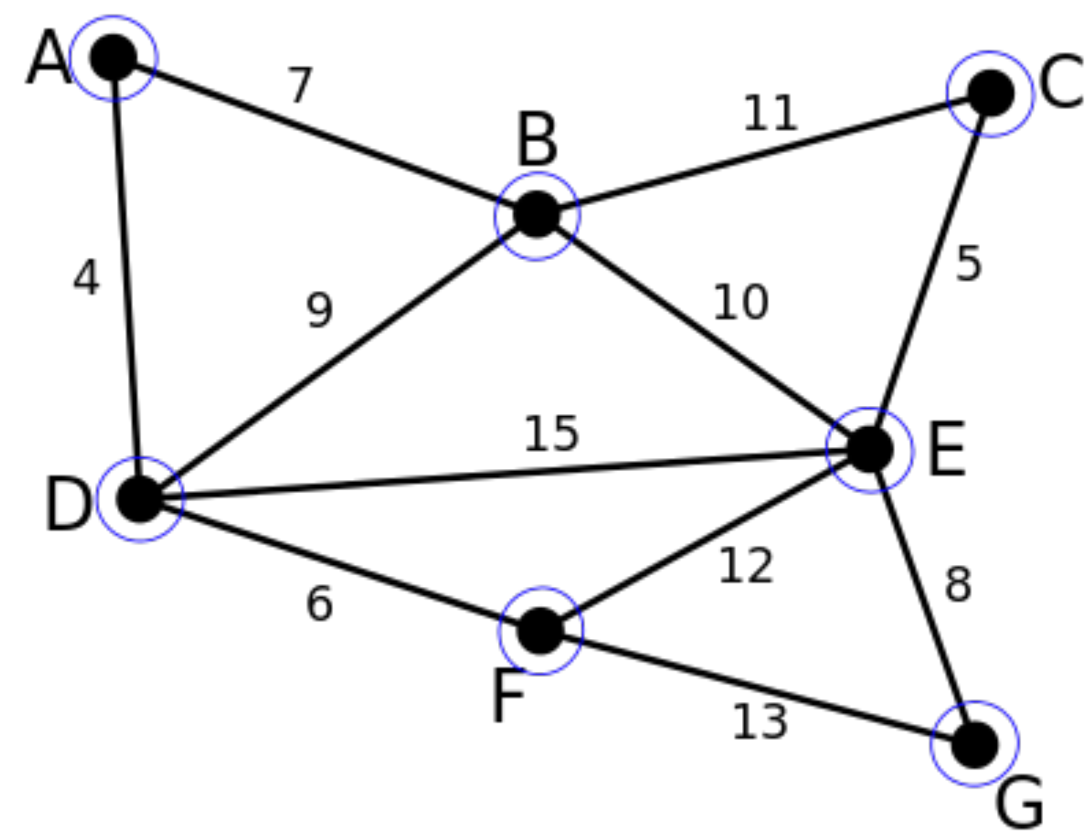
Designing an MST Algorithm

- What edges are always “safe” to add (does not violate optimality):
    - Min cost edges with respect to some cut
    - Why? Because of cut property these edges must be in every minimum spanning tree
- What edges should “never” be added?
    - Heaviest edge on any cycle
    - Follows from cycle property
- Correctness of our MST algorithms will follow from cut property
- Which MST algorithms have you heard about in 136?

“Prims” Algorithm

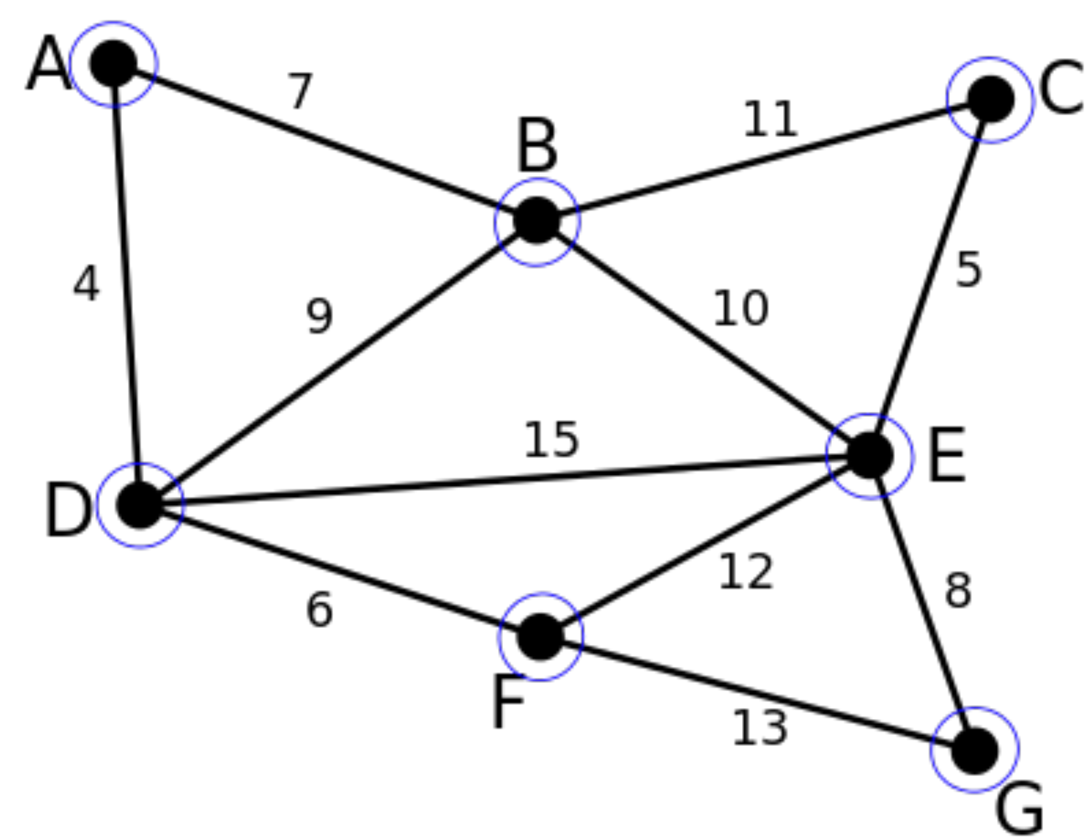
Jarník's ("Prims Algorithm")

- Initialize $S = \{u\}$ for any vertex $u \in V$ and $T = \emptyset$
- While $|T| \leq n - 1$:
  - Find the min-cost edge $e = (u, v)$ with one end $u \in S$ and $v \in V - S$
  - $T \leftarrow T \cup \{e\}$
  - $S \leftarrow S \cup \{v\}$



Jarník's ("Prims Algorithm")

- Initialize $S = \{u\}$ for any vertex $u \in V$ and $T = \emptyset$
- While $|T| \leq n - 1$:
    - Find the min-cost edge $e = (u, v)$ with one end $u \in S$ and $v \in V - S$
    - $T \leftarrow T \cup \{e\}$
    - $S \leftarrow S \cup \{v\}$
- Correctness:
    - Why is T a MST?



Jarník's ("Prims Algorithm")

- Initialize $S = \{u\}$ for any vertex $u \in V$ and $T = \emptyset$
- While $|T| \le n - 1$:
    - Find the min-cost edge $e = (u, v)$ with one end $u \in S$ and $v \in V - S$
    - $T \leftarrow T \cup \{e\}, \; S \leftarrow S \cup \{v\}$
- **Implementation crux.** Find and add min-cost edge for the cut $(S, V - S)$ and add it to the tree in each iteration, update cut edges
- How to implement? Naive implementation may take $O(nm)$
    - Need to maintain set of edges adjacent to nodes in T and extract min-cost cut edge from it each time
    - Which data structure from CS 136 can we use?

CS136 Review: Priority Queue

Managing such a set typically involves the following operations on S

- **Insert.** Insert a new element into S
- **Delete.** Delete an element from S
- **ExtractMin.** Retrieve highest priority element in S

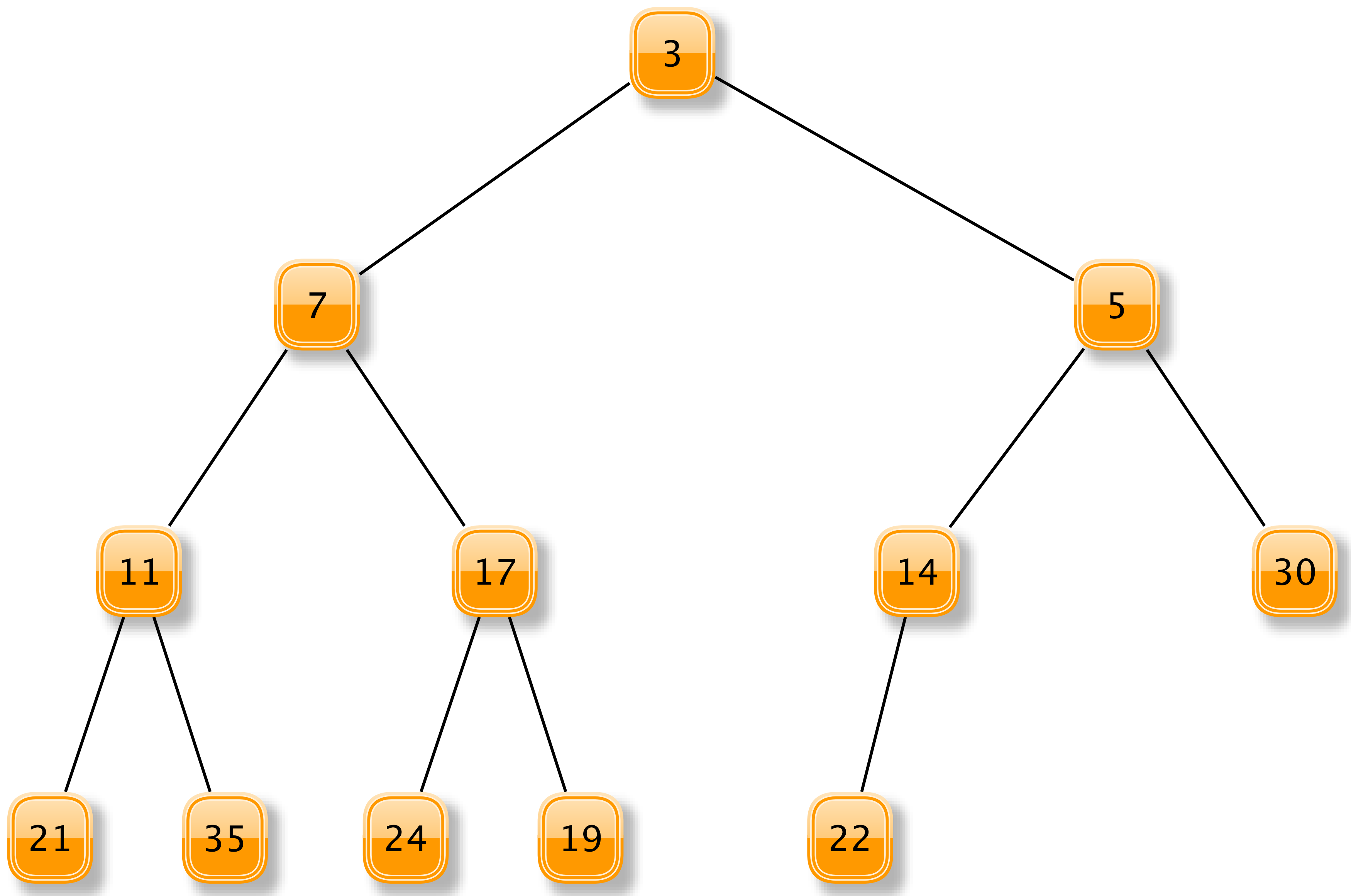
Priorities are encoded as a 'key' value

Typically: higher priority <—> lower key value

Heap as Priority Queue. Combines tree structure with array access

- Insert and delete: $O(\log n)$ time ('tree' traversal & moves)
- **Extract min.** Delete item with minimum key value: $O(\log n)$

Heap Example



H

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
X	3	7	5	11	17	14	30	21	35	24	19	22	-	-	-

“Prims” Implementation

- Use **Binary heaps**
  - Create a priority queue initially holding all edges incident to u.
  - At each step, dequeue edges from the priority queue until we find an edge (x, y) where $x \in S$ and $y \notin S$.
  - Add (x, y) to T.
  - Add to the queue all edges incident to y whose endpoints aren't in S.
  - Each edge is enqueued and dequeued at most once
  - Total runtime: $O(m \log m)$
    - In any graph, $m = O(n^2)$
    - So $O(m \log m) = O(m \log n)$

“Prims” Implementation

- Implementation using **Binary heaps**
  - Total runtime: $O(m \log n)$

- If a **Fibonacci heap** is used instead of binary heap:
  - Runs in $O(m + n \log n)$ **“amortized time”**
  - Support amortized $O(1)$-time inserts, $O(\log n)$ time extract min

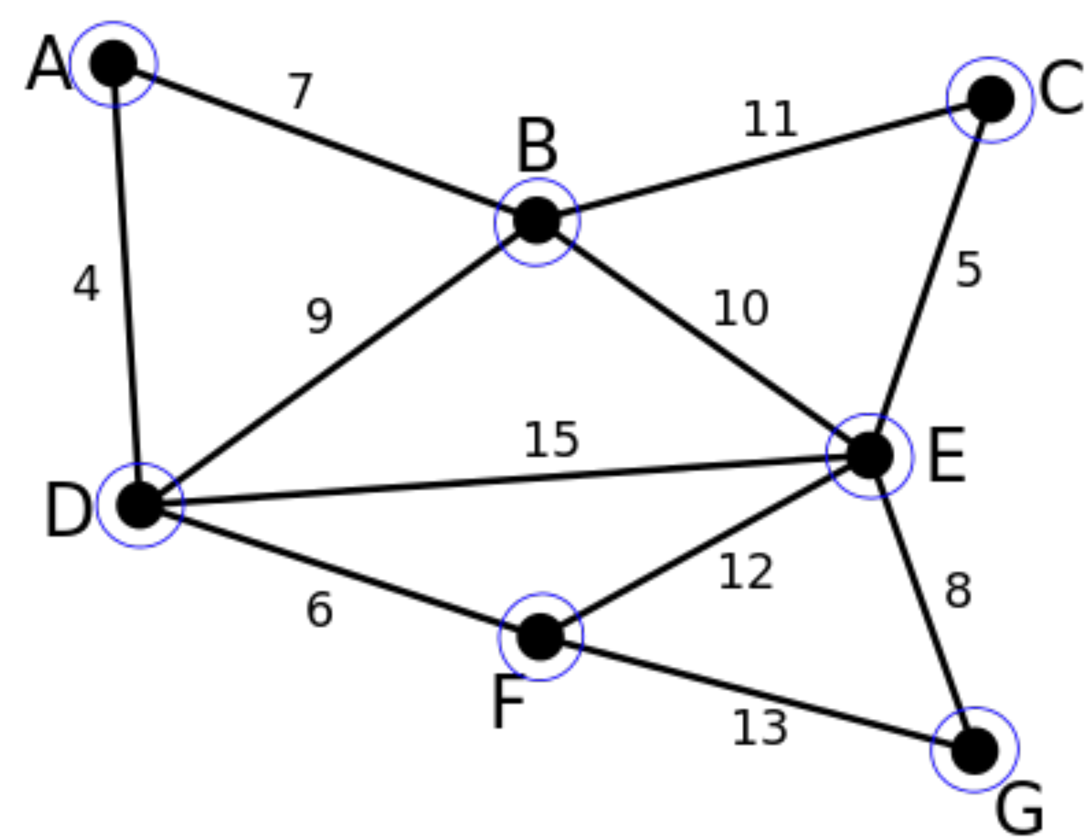
Definition. If k operations take total time $O(t \cdot k)$, then the amortized time per operation is $O(t)$.

Kruskal’s Algorithm

Kruskal's Algorithm

Idea: Add the cheapest remaining edge that does not create a cycle.

- Initialize $T = \varnothing$, $H \leftarrow E$
- While $|T| < n - 1$:
    - Remove cheapest edge e from H
    - If adding e to T does not create a cycle
        - $T \leftarrow T \cup \{e\}$
    - $H \leftarrow H - \{e\}$



Kruskal’s Algorithm

- Analysis:
  - Does it give us the correct MST?
    - Proof?
  - How quickly can we find the minimum remaining edge?
  - How quickly can we determine if an edge creates a cycle?

Kruskal's Implementation

- Sort edges by weight: $O(m \log m)$
    - Turns out this is the dominant cost
- Determine whether $T \cup \{e\}$ contains a cycle
    - Maintain a partition of V: components of T
    - Let $[u]$ denote component of u
    - Adding edge $e = (v, w)$ creates a cycle if and only if $[v] = [w]$
- Add an edge to T: update components

MST Algorithms History

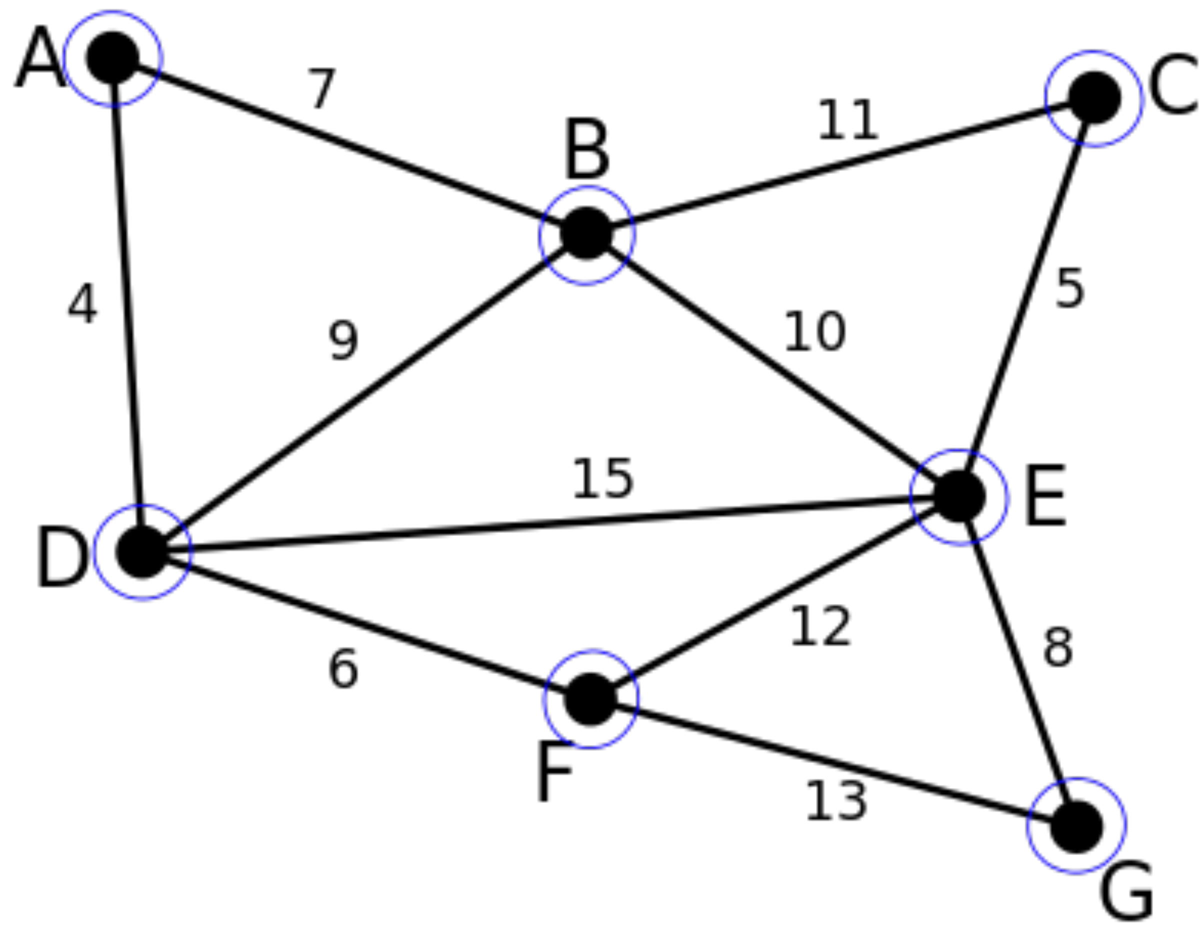
- **Borůvka's Algorithm** (1926)
    - The Borvka / Choquet / Florek-ukaziewicz-Perkal-Steinhaus-Zubrzycki / Prim / Sollin / Brosh algorithm
    - Oldest, most-ignored MST algorithm, most practical, often best!
- **Jarník's Algorithm** ("Prims Algorithm", 1929)
    - Published by Jarník, independently discovered by Kruskal in 1956, by Prims in 1957
- **Kruskal's Algorithm** (1956)
    - Kruskal designed this because he found Borůvka's algorithm "unnecessarily complicated"

Borůvka's Algorithm

- Start with F: a forest of n connected components (one for each vertex of G); $T = \emptyset$
- While F has more than one connected component:
  - **(Phase)** For each connected component $H \in F$
    - Find min-cost edge e connecting a vertex in H to a vertex outside H
    - $T \leftarrow T \cup \{e\}$
    - Add e to F: update connected components
- Each component "grows outward" using min cut edges, until we have one giant component

$$\boxed{\text{BORŮVKA: Add } \textbf{\textit{ALL}} \text{ the safe edges and recurse.}}$$

Borůvka's Algorithm Example



https://en.wikipedia.org/wiki/Bor%C5%AFvka%27s_algorithm

Borůvka's Analysis

- Each phase reduces the total # of components by how much?
  - In the worst case, the components of coalesce in pairs.
  - Number of phases $\leq \log_2 n$
- Each phase can be implemented as follows:
  - First: each edge is explored once, if both end points in different components, check to update min-cost-edge:
    - $O(m)$ time
  - Second: each component explored once to add min-cost edge and update component
    - At most $n \leq m + 1$ components
  - Each phase thus takes $O(m)$ time
- Overall takes $O(m \log n)$ time

Borůvka's Summary

- Each phase reduces the total # of components by how much?
  - In the worst case, the components of coalesce in pairs.
  - Number of phases $\leq \log_2 n$
- Each phase can be implemented as follows:
  - Each phase thus takes $O(m)$ time
- Overall takes $O(m \log n)$ time
- **Why it's the the MST algorithm you want:**
  - In reality, # of components go down much faster than half each time
  - Can be made $O(m)$ time for a broad class of graphs!
  - Allows for significant parallelism.

Acknowledgments

- The pictures in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)