Approximating Set Cover

Admin

- Assignment 10 (Practical Problems) out; do not have to submit
- Review hours next week:
  - 2-3.30 pm Monday, **7-9 pm Tuesday**, 1.30-3 pm Wednesday
  - Ask questions in general or about practice problems
- 24-hour final will be available on Gradescope from Thursday May 20 8.30 am and must be submitted by May 28, 8.30 pm
- Open book, open notes, but nothing else (not allowed to google terms)

CS Colloquium Today: 3.15 pm

Rediet Abebe, UC Berkeley
“Modeling the Impact of Shocks on Poverty”

The dynamic nature of poverty presents a challenge in designing effective assistance policies. A significant gap in our understanding of poverty is related to the role of income shocks in triggering or perpetuating cycles of poverty. Such shocks can constitute unexpected expenses — such as a medical bill or a parking ticket — or an interruption to one’s income flow. Shocks have recently garnered increased public attention, in part due to prevalent evictions and food insecurity during the COVID-19 pandemic. However, shocks do not play a corresponding central role in the design and evaluation of poverty-alleviation programs.

To bridge this gap, we present a model of economic welfare that incorporates dynamic experiences with shocks and pose a set of algorithmic questions related to subsidy allocations. We then computationally analyze the impact of shocks on poverty using a longitudinal, survey-based dataset. We reveal insights about the multi-faceted and dynamic nature of shocks and poverty. We discuss how these insights can inform the design of poverty-alleviation programs and highlight directions at this emerging interface of algorithms, economics, and social work.

Very small incentive to attend: 2 pt **extra credit** on HW 9

Set Cover

- **Set Cover (Optimization version)**.
  Given a set U of n elements, a collection $\mathcal{S}$ of subsets of U, find the minimum number of subsets from $\mathcal{S}$ whose union covers U.

$$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$$

$$S_a = \{ 3, 7 \} \qquad S_b = \{ 2, 4 \}$$

$$S_c = \{ 3, 4, 5, 6 \} \qquad S_d = \{ 5 \}$$

$$S_e = \{ 1 \} \qquad S_f = \{ 1, 2, 6, 7 \}$$

a set cover instance

Greedy Algorithm

- Greedily pick sets that maximize coverage until done
- Greedy Cover($\mathscr{U}, \mathscr{S}$):
    - Initially all elements of $\mathscr{U}$ are marked uncovered
    - $C \leftarrow \varnothing$ (Initialize cover)
    - While there is an uncovered element in $\mathscr{U}$
        - Pick the set S_m from $\mathscr{S} \backslash C$ that maximizes the number of uncovered elements
        - $C \leftarrow C \cup \{S_m\}$
        - Mark elements of S_m as covered

Greedy Algorithm

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$\boxed{S_e = \{ 1, 8 \}}$ $\quad \boxed{S_f = \{ 2,4,6 \}}$

Analyzing Greedy

- **Claim**. Greedy set cover is a $\ln n$-approximation, that is, greedy uses at most $k \ln n$ sets where k is the size of the optimal set cover.

Main observations behind proof:

- If there exists k subsets whose union covers all n elements, then there exists a subset that covers $\geq 1/k$ fraction of elements
- Greedy always picks subsets that maximize remaining uncovered elements
- In each iteration, greedy's choice must cover at least $1/k$ fraction of the remaining elements
- Such a subset must always exist since the remaining elements can also be covered by at most k subsets

Analyzing Greedy

- **Claim**. Greedy set cover is a $\ln n$-approximation, that is, greedy uses at most $k \ln n$ sets where k is the size of the optimal set cover.
- **Proof.**
- Let E_t be the set of elements still uncovered after t th iteration.
- The optimal solution covers E_t with no more than k sets
- Greedy always picks the subset that covers most of E_t in step $t + 1$
- Selected subset must cover at least $|E_t|/k$ elements of E_t
- Thus $|E_{t+1}| \leq |E_t|\left(1 - 1/k\right)$ and as $E_0 = n$, inductively we have $|E_t| \leq n(1 - 1/k)^t$
- When $|E_t| < 1$, we are done, how many steps t until we get this?

Analyzing Greedy

- **Claim.** Greedy set cover is a $\ln n$-approximation—greedy uses at most $k \ln n$ sets where k is the size of the optimal set cover.
- **Proof.** (Cont.)
- $|E_t| \leq n(1 - 1/k)^t$
- When $|E_t| < 1$, we are done
- Setting $t = k \ln n$, we get $|E_t| = n\left(1 - \frac{1}{k}\right)^{k \ln n} < n \cdot \frac{1}{n} = 1$
- Thus, greedy finishes in $k \ln n$ steps where k is the optimal-set cover size, so it uses at most $k \ln n$ sets
- If we stop a little earlier, can tighten analysis even more

$$\left(1 - \frac{1}{x}\right)^x < \frac{1}{e} \text{ for } x > 1$$

Analyzing Greedy

- **Claim**. If the optimal set cover has size k then the greedy set cover has size at most $k(1 + \ln(n/k))$.
- **Proof**. (Cont.)
- $|E_t| \leq n(1 - 1/k)^t$
- When $|E_t| \leq k$, we finish after selecting at most k more sets
- Setting $t = k\ln(n/k)$, we get $|E_t| = n\left(1 - \frac{1}{k}\right)^{k\ln(n/k)}$

$$\leq n \cdot k/n = k$$

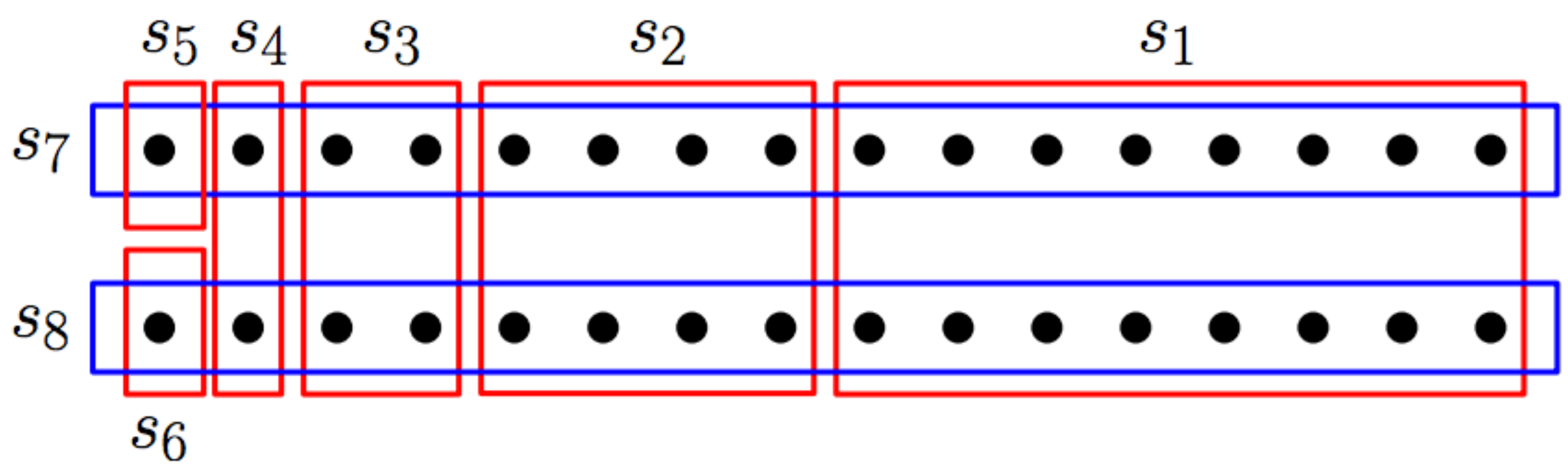
- Greedy uses at most $k + k\ln(n/k)$ sets in total

Special Case

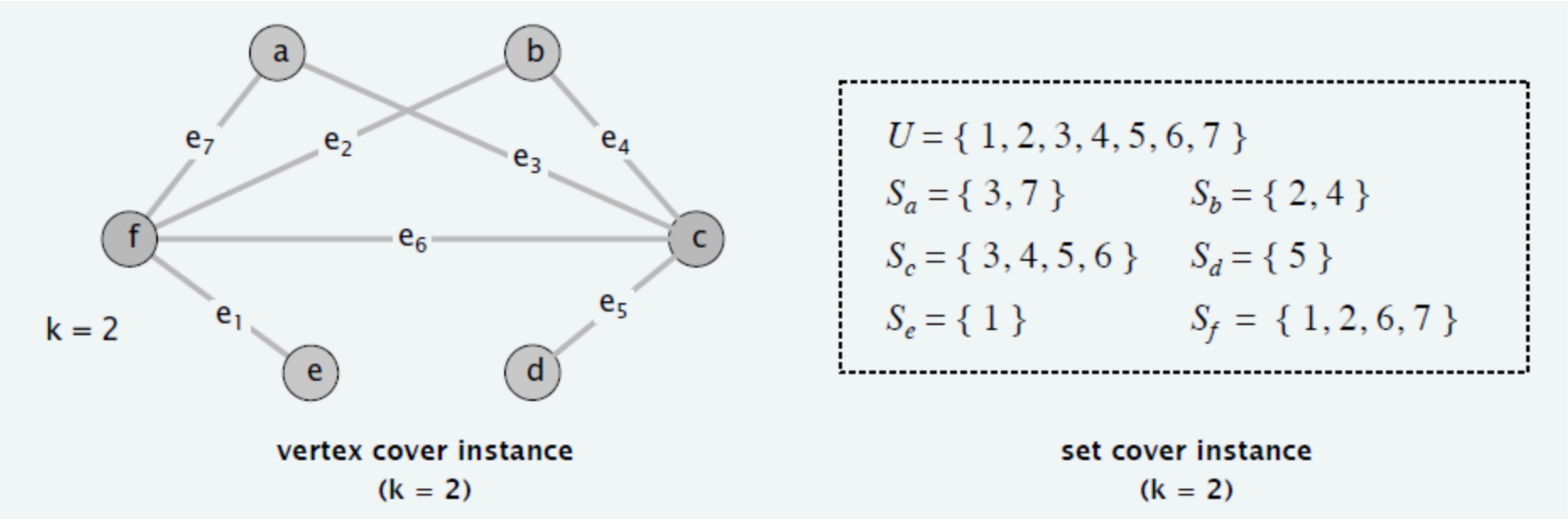
- We can do slightly better for special input
- **Claim**. If the maximum size of any subset in $\mathcal{S}$ is B then the greedy algorithm is $(\ln B + 1)$-approximation
- **Proof**.
- If each subset has almost B elements and the optimal set cover has k subsets then, then k must be at least
  - $k \geq n/B$
- Substituting $n/k \leq B$ shows that greedy is $(\ln B + 1)$ approximation

Tight Approximation

- This approximation is essentially tight
- Consider the following example with $n = 2^5$ elements
- s_1 has 16 elements, but s_7 and s_8 each have 15, so greedy chooses s_1
- Then, s_2 has 8 elements, but but s_7 and s_8 each have 7, so greedy chooses s_2 … this happens $\log_2(n/2) \; = \log_2 16 = 4$ times
- Only two sets s_7, s_8 are enough so, approximation ratio $(\log_2 \frac{n}{2})/2$



Approximating Vertex Cover



- We know that vertex cover reduces to set cover
- $\mathscr{U} = E$ and $\mathscr{S} = \{S_v \,|\, v \in V\}$ where $S_v = \{e \in E \mid e \text{ incident to } v\}$
- Thus the greedy approximation algorithm for set cover also gives an approximation algorithm for vertex cover
- Greedy picks vertices that cover maximum number of edges (i.e., vertices with max degrees w.r.t. uncovered edges)
- Greedy vertex cover is thus a $(\ln \Delta + 1)$ approximation where Δ is maximum degree of any vertex
- The seemingly stupider algorithm on assignment 9 is better than greedy—2-approximation is best known
- Finding a $(2 - \varepsilon)$-approximation of VC is a big open problem!
  - Can't be done under "unique games conjecture"

This won't work for all reductions

Approximate Weighted Set Cover

Weighted Set Cover

- In the weighted-version of the set cover problem, each subset $S_i \in \mathcal{S}$ has a weight w_i associated with it
- The goal is to find the a collection of subsets $C = \{S_1, \ldots, S_k\}$ such that they cover $\mathcal{U}$ and $\sum_{S_i \in C} w(S_i)$ is minimized

$U = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$

$\$5 \ S_a = \{ 1, 2, 3\}$ $\$4 \ S_b = \{ 4, 5 \}$

$\$13 \ S_c = \{ 3, 5, 7 \}$ $\$3 \ S_d = \{ 6 \}$

$\$1 \ S_e = \{ 1, 8 \}$ $\$15 \ S_f = \{ 2,4,6 \}$

$k = 2$

Weighted Set Cover

- We extend the greedy algorithm to the weighted case
- What should we be greedy about?
  - What could happen if we pick the largest?
  - What could happen if we pick the cheapest?

$U = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$

$\boxed{\$5 \;\; S_a = \{ 1, 2, 3 \}}$ $\boxed{\$4 \; S_b = \{ 4, 5 \}}$

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$\boxed{\$1 \; S_e = \{ 1, 8 \}}$ $\$15 \; S_f = \{ 2,4,6 \}$

$k = 2$

Weighted Case: Greedy

- In the weighted-version of the set cover problem, each subset $S_i \in \mathcal{S}$ has a weight w_i associated with it
- Each potential set that can be added to the solution has some "benefit" (elements it covers) and some "cost" (its weight)
- We can be greedy in terms of the cost/benefit or the "amortized cost" of choosing set S_i—how much are we spending per new item covered?

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

1.67	$\$5\ S_a = \{1, 2, 3\}$	$\$4\ S_b = \{4, 5\}$	2.00
4.33	$\$13\ S_c = \{3, 5, 7\}$	$\$3\ S_d = \{6\}$	3.00
.50	$\$1\ S_e = \{1, 8\}$	$\$15\ S_f = \{2,4,6\}$	5.00

$k = 2$

Weighted Case: Greedy

- In the weighted-version of the set cover problem, each subset $S_i \in \mathcal{S}$ has a weight w_i associated with it
- Each potential set that can be added to the solution has some "benefit" (elements it covers) and some "cost" (its weight)
- We can be greedy in terms of the cost/benefit or the "amortized cost" of choosing set S_i
- Greedy algorithm.
  - Begin with an empty cover and continue until all elements covered
  - In each iteration choose the set S_i that minimizes amortized cost w_i/e, where e is the # of new elements covered by S_i

Weighted Case: Greedy

- In the weighted-version of the set cover problem, each subset $S_i \in \mathcal{S}$ has a weight w_i associated with it
- Each potential set that can be added to the solution has some “benefit” (elements it covers) and some “cost” (its weight)
- We can be greedy in terms of the cost/benefit or the “amortized cost” of choosing set S_i
- How good is this strategy?

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.50	\$1 $S_e = \{ 1, 8 \}$	\$15 $S_f = \{ 2,4,6 \}$	5.00

$k = 2$

Weighted Case: Greedy

- How good is the greedy strategy for the weighted case?
- **Claim.** Greedy is a $(1 + \ln n)$-approximation for weighted set cover.
- We prove this by proving a **different claim**:
- Let $\mathcal{S}^*$ be the subsets picked by the optimal set cover
- For any $S_i \in \mathcal{S}^*$, greedy may not have picked S_i but
  - We show that greedy covers elements in S_i at cost no greater than $O(\log n) \cdot w(S_i)$ (than choosing S_i itself)
- Every element is in some subset of $\mathcal{S}^*$
- Thus, summing over subsets, we get that greedy covers all elements at cost no greater than $O(\log n) \cdot \sum_i w(S_i) = O(\log n) \cdot \text{OPT}$

Weighted Set Cover: Analysis

- **Claim**. For any subset $S_i \in \mathcal{S}$, the greedy algorithm covers the elements of S_i with a cost no greater than $O(\log n)$ times w_i (the cost of choosing S_i itself)
- **Proof**. Order the elements of $S_i = \{a_1, a_2, \ldots, a_d\}$ in the order in which they were covered by the greedy algorithm (if more than one are covered at the same time, break ties arbitrarily)
- Consider the time the element a_d is covered: the available sets to cover a_d include S_i itself
- Covering a_d with S_i would incur an amortized cost of w_i or less (if a_d is the only new element covered by S_i or less otherwise)
- Greedy picks the set with least amortized cost so its cost is at most w_i to cover a_d. Therefore $c_d \leq w_i$

Weighted Set Cover: Analysis

- **Claim**. For any subset $S_i \in \mathcal{S}$, the greedy algorithm covers the elements of S_i with a cost no greater than $O(\log n)$ times w_i (the cost of choosing S_i itself)
- **Proof**.
  Now look at when a_{d-1} is covered, at this time, it is possible to select S_i and cover both a_{d-1} and a_d incurring an amortized cost of $w_i/2$ or less (if more elements are covered)
- Greedy picks the set with least amortized cost so its cost to cover a_{d-1} is at most $w_i/2$, therefore $c_{d-1} \leq w_i/2$
- Similarly a_{d-2} is covered at amortized cost at most $w_i/3$. Each element a_j incurs an amortized cost at most $c_j \leq w_i/(d - j + 1)$ up until a_1 which is covered at amortized cost $c_1 \leq w_i/d$

Weighted Set Cover: Analysis

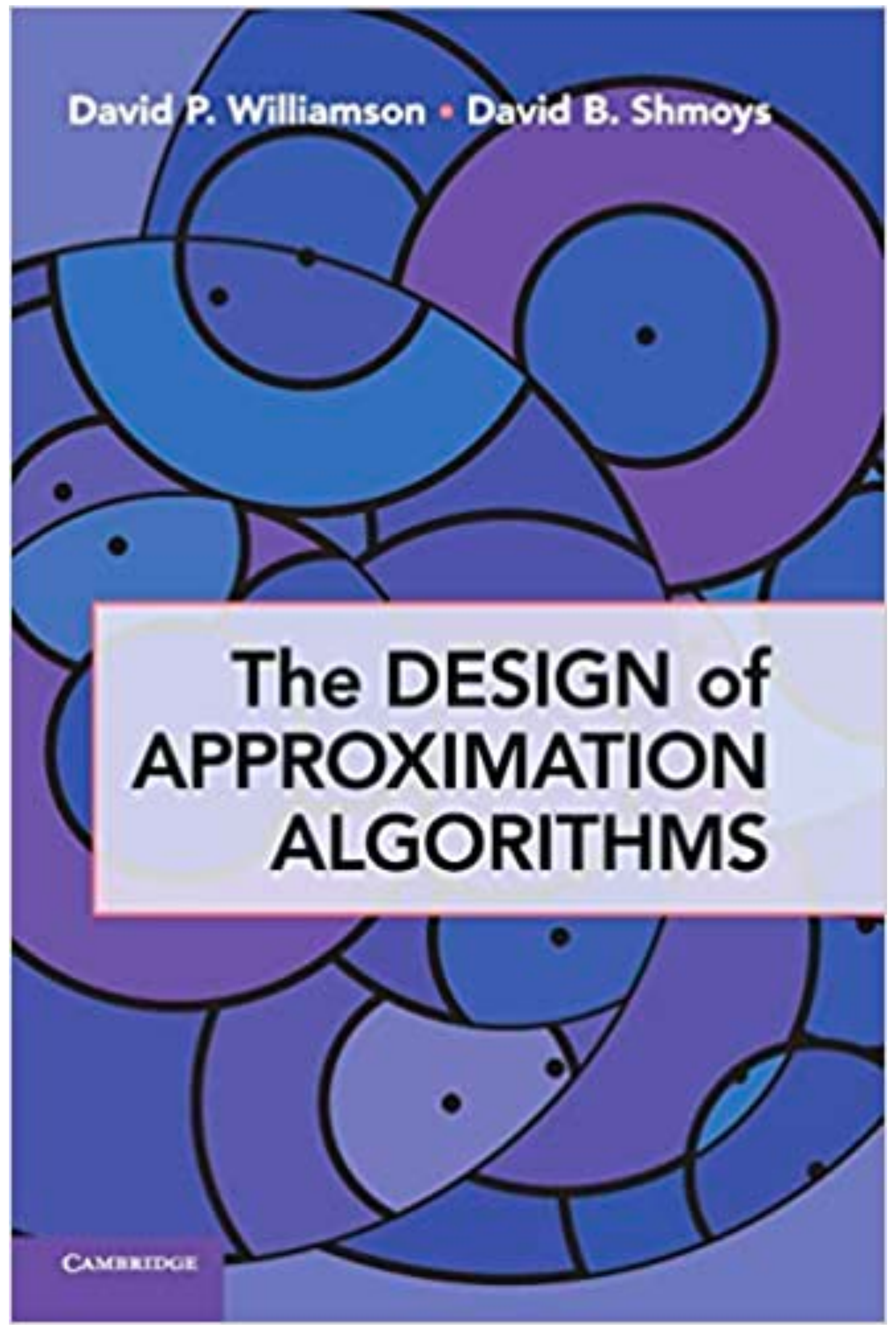
- **Claim**. For any subset $S_i \in \mathcal{S}$, the greedy algorithm covers the elements of S_i with a cost no greater than $O(\log n)$ times w_i (the cost of choosing S_i itself)
- **Proof**.
- Each element a_j incurs an amortized cost at most $w_i/(d-j+1)$
- Thus the total amortized cost of all elements in S_i is

$$\sum_{\ell \in S_i} c_\ell \le w_i \left(\sum_{j=1}^{d} \frac{1}{n-j+1} \right) = w_i H_d \le w_i H_n \le w_i(1+\ln n)$$

- This analysis can be shown to be essentially tight as well

Wrapping Up Approximations

- Set Cover. Can we do better than $O(\log n)$?
- [Raz & Safra 1997]. There exists a constant $c > 0$, there is no polynomial-time $c \ln n$-approximation algorithm, unless $\mathsf{P} = \mathsf{NP}$
- [Dinur & Steurer 2014] No polynomial time $(1-\epsilon)\ln n$ approximation for any constant $\epsilon > 0$ unless $\mathsf{P} = \mathsf{NP}$
- Other approximations of NP hard problems.
  - Knapsack and any weakly NP Hard problems we can approximate to $(1+\epsilon)$ factor with algorithms that are polynomial in both n and ϵ



Next Class: Skip Lists

- Next class we'll design a fun randomized data structure
  - Randomized binary search tree
  - Skip lists!

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)