CS256: Algorithm Design and Analysis **Spring 2021**

Assignment 8 (due 05/04/2021)

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Problem 1. A *tonian cycle* in an undirected graph G is a simple cycle that goes through at least half of the vertices of G.[1] Prove that the problem of determining whether or not a graph contains a tonian cycle is NP-hard.

Problem 2. Consider the process of throwing m balls into n bins, where each ball is thrown into a uniformly random bin, independent of other balls. What is the expected number of balls in a particular bin b?

Note. This balls and bins game models the expected number of collisions in a hash table with chaining, under uniformly random hash function.

Problem 3. Consider the following algorithm for generating a biased random coin. The subroutine FairCoin returns either 0 (heads) or 1 (tails) with equal probability; the random bits returned by two different calls to FairCoin are mutually independent.

```
BiasedCoin:
  if FairCoin = 0:
      return 0
  else:
      return 1 − BiasedCoin
```

(a) Prove that BiasedCoin returns 1 with probability 1/3. (*Hint.* Easier to show that it returns 0 with probability 2/3.)

(b) What is the expected number of times that BiasedCoin calls FairCoin?

Problem 4. Now suppose that you have access to a method BiasedCoin that in constant time returns 0 (heads) with probability p, and returns 1 (tails) with probability $1-p$, where $0 < p < 1$. However, you don't know what p is.

Write a method FairCoin that, uses BiasedCoin, to returns heads or tails with *equal probability*. Determine the expected running time of FairCoin and prove correctness (that is, show that it returns heads or tails with equal probability). *Hint.* We don't need Pr(heads) = Pr(tails) = 1/2, just that Pr(heads) = Pr(tails).

Problem 5. We roll a standard die over and over. What is the expected number of rolls until the first pair of consecutive sixes appear? (*Hint.* The answer is not 36.)

[1] Recall that a simple cycle does not visit a node more than once.

Problem 6. Consider a random walk on a path with vertices numbered $1, 2, \ldots, n$ from left to right. Starting at vertex 1, at each step, we flip a coin to decide which direction to walk, moving one step left or one step right with equal probability.

The random walk ends when we fall off one end of the path, either by moving left from vertex 1 or by moving right from vertex n.

(a) Prove that the probability that the walk ends by falling off the right end of the path is exactly $\frac{1}{n+1}$.

(b) (**Extra credit**) What is the expected number of steps before the random walk ends? You must provide a proof to justify your answer.