Probability and Recurrences

Reminders and Leftovers

- Assignment 7 is due tonight
  - Help hours today: 1.30-3pm (me), 3-5 pm and 5-11pm (TAs)
- Assignment 8 will be released today; due next Wed
- Where we are:
  - Last lecture we introduced basics of probability (sample space, events, independence, conditional probability)
  - Saw some examples
  - Today: we'll define random variable, expectation and see examples of analyzing expectation of probabilistic processes

The Birthday Paradox

- Suppose that there are m students in a lecture hall
- Assume for each student, any of the $n = 365$ possible days are equally likely as their birthday
- Assume birthday are mutually independent
- **Question**. What is the likelihood that no two students have the same birthday?
- Let A_i be the event that the ith persons birthday is different from the previous $i - 1$ people
- Pr (all m different birthdays)

$$= \Pr(A_1 \cap A_2 \cap \ldots \cap A_m)$$

$$= \Pr(A_1) \cdot \Pr(A_2 \mid A_1) \cdot \Pr(A_3 \mid A_1 \cap A_2) \ldots \Pr(A_n \mid A_1 \cap \ldots \cap A_{n-1})$$

The Birthday Paradox

- Pr (all m different birthdays)

$$= 1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdot \left(1 - \frac{3}{n}\right) \dots \left(1 - \frac{m-1}{n}\right)$$

$$= \prod_{j=1}^{m-1} \left(1 - \frac{j}{n}\right) \leq \prod_{j=1}^{m-1} e^{-j/n} = e^{-1/n(\sum_{j=1}^{m-1} j)} \approx e^{-m^2/2n}$$

- $m \approx \sqrt{2n \ln 2}$ for probability to be $1/2$
- For $n = 365$, we get $m = 22.49$
- Thus, with around 23 people in this class, we have a 50% chance of two people having the same birthday

Important Inequality:

$$(1 - x) \leq \left(\frac{1}{e}\right)^x \text{ for } x \geq 1$$

Random Variable

- Event either does or does not happen, what if we want to capture *magnitude* of a probabilistic event
- Suppose I flip n independent fair coins: the # of heads is a random variable
- Number that comes up when we roll a fair die is a random variable
- If an algorithm flips some coins then the running time of the algorithm is a random variable
- **Definition.** A random variable X is a function from a sample space S (with a probability measure) to some value set (e.g. real numbers, integers, etc.)

Random Variable: Example

- So for example I flip a coin 10 times. Let X be the number of heads
  - $\Pr[X = 0] = 1/2^{10}$
  - $\Pr[X = 10] = 1/2^{10}$
  - $\Pr[X = 4]$?
  - $\Pr[X = 4] = \binom{10}{4} \frac{1}{2^4} \frac{1}{2^6} = \frac{105}{512}$
- A random variable that is 0 or 1 (indicating if something happens or not) is called an *indicator random variable or Bernoulli random variable*

Expectation

- Every time you do the experiment, associated random variable takes a different value
- How can we characterize the average behavior of a random variable?
- **Definition**. Expected value of a random variable R defined on a sample space S is

$$\mathrm{E}(R) = \sum_{w \in S} R(w) \cdot \Pr(w)$$

- Let R be the number that comes up when we roll a fair, six-sided die, then the expected value of R is

$$\mathrm{E}(R) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$

To get the E to look good in latex, use `\mathrm{E}`

(We won't use it like $\mathbb{E}$ in this class, but if you really want to, it's `\mathbb`)

Expectation

- We can group together outcomes for which the random variable takes the same value
- **Alternate Definition**. Expected value of a random variable R defined on a sample space S is

$$E(R) = \sum_x x \cdot \Pr(R = x)$$

- If A is an arbitrary event with $\Pr[A] > 0$, the conditional expectation of X given A is

$$E[X \mid A] := \sum_x x \cdot \Pr[X = x \mid A]$$

- **(Law of total expectation)** If $\{A_1, A_2, \dots\}$ is a finite partition of the sample space:

$$E(X) = \sum_i E(X \mid A_i) \cdot \Pr(A_i)$$

Very useful !

Linearity of Expectation

- *Very important* tool in randomized algorithm
- Expectation of random variables obey a wonderful rule
- Informally, it says that the expectation of a sum is the sum of the expectations.
- Formally, for any random variables $X_1, X_2, \ldots, X_n$ and any coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$

$$E[\sum_{i=1}^{n} (\alpha_i \cdot X_i)] = \sum_{i=1}^{n} (\alpha_i \cdot E[X_i])$$

Very useful !

- **Note.** Always true! Linearity of expectation **does not require independence** of random variables.

Bernoulli Distribution

- A probability distribution assigns a probability to each possible value of a random variable
- Suppose you run an experiment with probability of success p and failure $1 - p$.
- Example, coin toss where head is success.
- Let X be a Bernoulli or indicator random variable that is 1 if we succeed, and 0 otherwise. Then,

$$E[X] = \sum_x x \cdot \Pr[X = x] = 0 \cdot \Pr[X = 0] + 1 \cdot \Pr[X = 1] = p$$

- **Remember this:** expectation of an indicator random variable is exactly the probability of success!

Expected Success: n Bernoulli Trials

- Consider n independent Bernoulli trials (with success probability p). Let R denote the number of successes
  - R is said to follow a ***Binomial distribution*** (we'll revisit this)
- We want to know expected number of successes $\mathrm{E}(R)$
- Can write R as a sum of indicator random variables
  - $R = \sum_i R_i$ where $R_i = 0$ or $R_i = 1$
- Then $\mathrm{E}[R] = \mathrm{E}\left[\sum_i R_i\right]$, how can we simplify this by LoE?

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    - Then $\mathrm{E}[R] = \mathrm{E}\left[\sum_i R_i\right] = \sum_i \mathrm{E}[R_i] = \sum_{i=1}^{n} p = np$

Hat Check Problem

- There is a dinner party where n people check their hats. The hats get mixed up during dinner, so that afterward each person receives a random hat.
- What is the expected number of people who get their own hat?

Hat Check Problem

- There is a dinner party where n people check their hats. The hats get mixed up during dinner, so that afterward each person receives a random hat.
- What is the expected number of people who get their own hat?
- Let R be the random variable denoting the number of men who get their hat back. Goal: compute $\mathrm{E}(R)$.
- Usual trick. Express random variable R as a sum of indicator random variables R_i is 1 if ith person gets their hat back, else it is 0.
- Then, $R = \sum_{i=1}^{n} R_i$

Hat Check Problem

- What is $\mathrm{E}(R_i)$?
- $R_i = 1$ if i gets his hat; $R_i = 0$ otherwise
- By definition,
  $\mathrm{E}(R_i) = 1 \cdot \Pr(i \text{ gets } i\text{'s hat}) + 0 \cdot \Pr(i \text{ gets another hat})$
- $\mathrm{E}(R_i) = \Pr(i \text{ gets } i\text{'s hat})$
- Need $\Pr(i \text{ gets } i'\text{s hat})$
- Sample space: All orderings of hats $n!$ (returned in order)
- Number of outcomes where i's gets correct hat?
    - $(n-1)!$

Hat Check Problem

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- By definition,
  $\mathrm{E}(R_i) = 1 \cdot \Pr(i \text{ gets } i\text{'s hat}) + 0 \cdot \Pr(i \text{ gets another hat})$
- $\mathrm{E}(R_i) = \Pr(i \text{ gets } i\text{'s hat}) = \dfrac{(n-1)!}{n!} = \dfrac{1}{n}$
- $$\mathrm{E}(R) = E\left(\sum_{i=1}^{n} R_i\right) = \sum_{i=1}^{n} E(R_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$$
- In expectation, one person gets their hat back!

Uniform Distribution

- When every outcome is equally likely
- Let X be the random variable of the experiment and S be the sample space
- $\Pr[X = x] = \dfrac{1}{|S|}$
- $E[X] = \dfrac{1}{|S|} \cdot \sum_{x \in S} \Pr(X = x)$
- Example
    - fair coin toss: heads and tails are equally likely
    - fair die roll: all numbers are equally likely

Card Guessing: Memoryless

- To entertain your family you have them shuffle deck of n cards and then turn over one card at a time. Before each card is turned, you predict its identity. You have no psychic abilities or memory to remember cards
- Your strategy: guess uniformly at random
- How many predictions do you expect to be correct?
- Let X denote the r.v. equal to the number of correct predictions and X_i denote the indicator variable that the ith guess is correct
- Thus, $X = \sum_{i=1}^{n} X_i$ and $\mathrm{E}[X] = \mathrm{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathrm{E}[X_i]$
- $\mathrm{E}[X_i] = 0 \cdot \Pr(X_i = 0) + 1 \cdot \Pr(X_i = 1) = \Pr(X_i = 1) = 1/n$
- Thus, $\mathrm{E}[X] = 1$

Card Guessing: Memoryfull

- Suppose we play the same game but now assume you have the ability to remember cards that have already been turned
- Your strategy: guess uniformly at random among cards that have not been turned over
- Let X denote the r.v. equal to the number of correct predictions and X_i denote the indicator variable that the ith guess is correct
- Thus, $X = \sum_{i=1}^{n} X_i$ and $\mathrm{E}[X] = \mathrm{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathrm{E}[X_i]$
- $\mathrm{E}[X_i] = \Pr(X_i = 1) = \dfrac{1}{n-i+1}$
- Thus, $\mathrm{E}[X] = \sum_{i=1}^{n} \dfrac{1}{n-i+1} = \sum_{i=1}^{n} \dfrac{1}{i}$

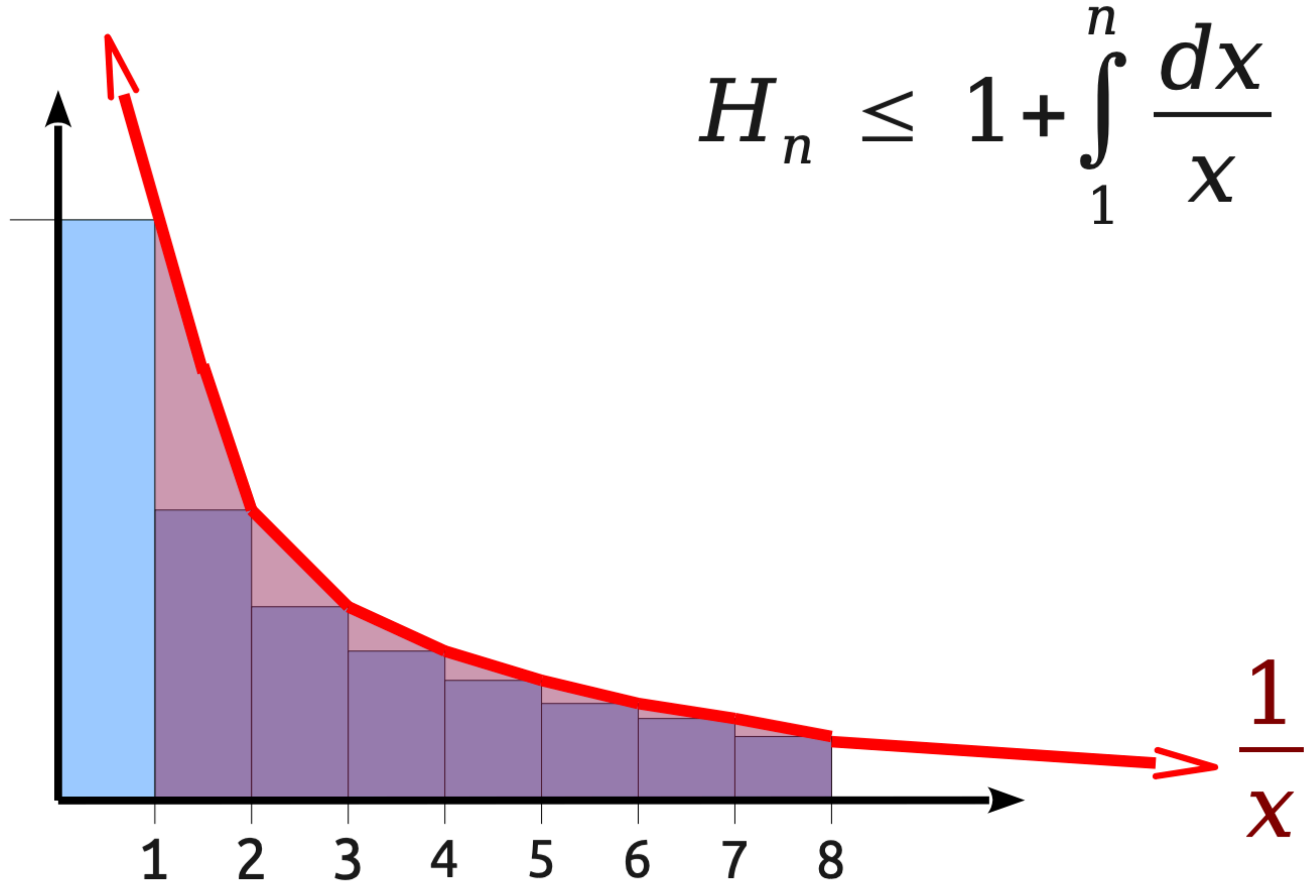
Harmonic Numbers

- The nth harmonic number, denoted H_n is defined as

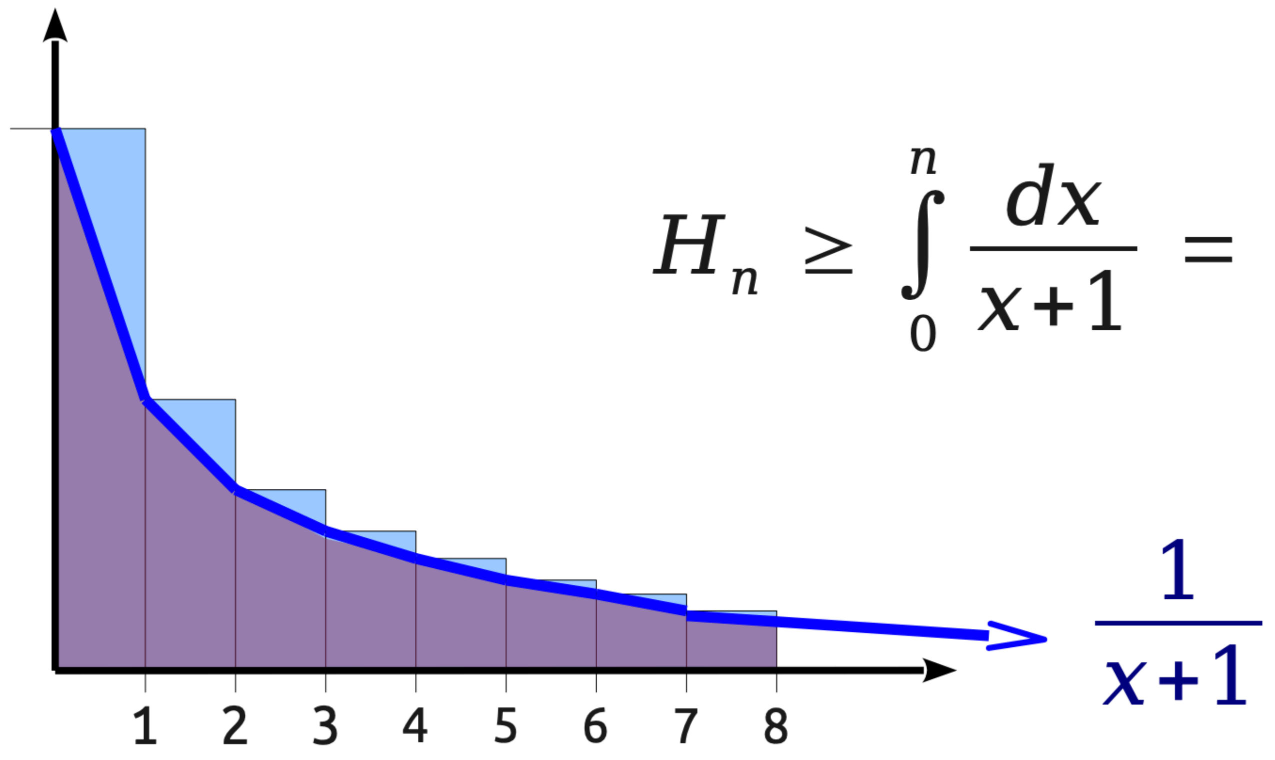
$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

- **Theorem**. $H_n = \Theta(\log n)$
- Proof Idea. Upper and lower bound area under the curve

$$H_n \leq 1 + \int_1^n \frac{dx}{x} = \ln n + 1$$



$$H_n \geq \int_0^n \frac{dx}{x+1} = \ln(n+1)$$



Card Guessing: Memoryfull

- Suppose we play the same game but now assume you have the ability to remember cards that have already been turned

- Your strategy: guess uniformly at random among cards that have not been turned over

- Let X denote the r.v. equal to the number of correct predictions and X_i denote the indicator variable that the ith guess is correct

- Thus, $X = \sum_{i=1}^{n} X_i$ and $E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$

- $E[X_i] = \Pr(X_i = 1) = \dfrac{1}{n-i+1}$

- Thus, $E[X] = \sum_{i=1}^{n} \dfrac{1}{n-i+1} = \sum_{i=1}^{n} \dfrac{1}{i} = \Theta(\log n)$

Geometric Distribution

- Let's say we do a sequence of Bernoulli trials $X_1, X_2, \ldots$ where X_i where each trial is successful ($X_i = 1$) with probability p, and fails ($X_i = 0$) with probability $1 - p$
- **Question**: what is the expected number of trials until first success?
    - In expectation, what is the value of the first i such that $X_i = 1$?
    - E.g. number of coin flips until heads $(p = 1/2)$
    - E.g. number if times I roll a die until I get a 1 $(p = 1/6)$
- One way to solve it is to just do the sum:
    - $\sum_{i=1}^{\infty} i(1-p)^{i-1}p$

Geometric Expectation (using the sum)

$$\sum_{i=1}^{\infty} i(1-p)^{i-1}p = \sum_{i=1}^{\infty}\sum_{k=1}^{i}(1-p)^{i-1}p =$$

$$\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}(1-p)^{i-1}p = \sum_{k=1}^{\infty}p(1-p)^{k-1}\sum_{i=0}^{\infty}(1-p)^{i} =$$

$$\sum_{k=1}^{\infty}p(1-p)^{k-1}\frac{1}{1-(1-p)} = \sum_{k=1}^{\infty}(1-p)^{k-1} = \sum_{k=0}^{\infty}(1-p)^{k} = \frac{1}{p}$$

Geometric Expectation (using the sum)

- Want to know, how many tries in expectation until first success
- Let's think about this recursively

$$X \leftarrow \begin{cases} 1 \text{ with prob. } p \\ 0 \text{ with prob. } (1-p) \end{cases}$$

FindNumTries:

If $X = 1$

  Return 1

If $X = 0$

  Return $1+$ FindNumTries

If we fail in the first try, we start over from scratch!

- Let F be the number of times FindNumtries is called, what is $\mathrm{E}(F)$?

Geometric Expectation (using the sum)

- Let F be the number of times FindNumtries is called, what is $\mathrm{E}(F)$?
- $\mathrm{E}(F) = \mathrm{E}(F \mid X_1 = 1) \cdot \Pr(X_1 = 1) + \mathrm{E}(F \mid X_1 = 0) \cdot \Pr(X_1 = 0)$

  $= (1 + 0) \cdot p \quad + \quad (1 + E(F)) \cdot (1 - p)$
- $E(F) = 1/p$

FindNumTries:

If $X = 1$

    Return 1

If $X = 0$

    Return $1 +$ FindNumTries

If we fail in the first try, we start over from scratch!

Geometric Expectation: Formal Recursion

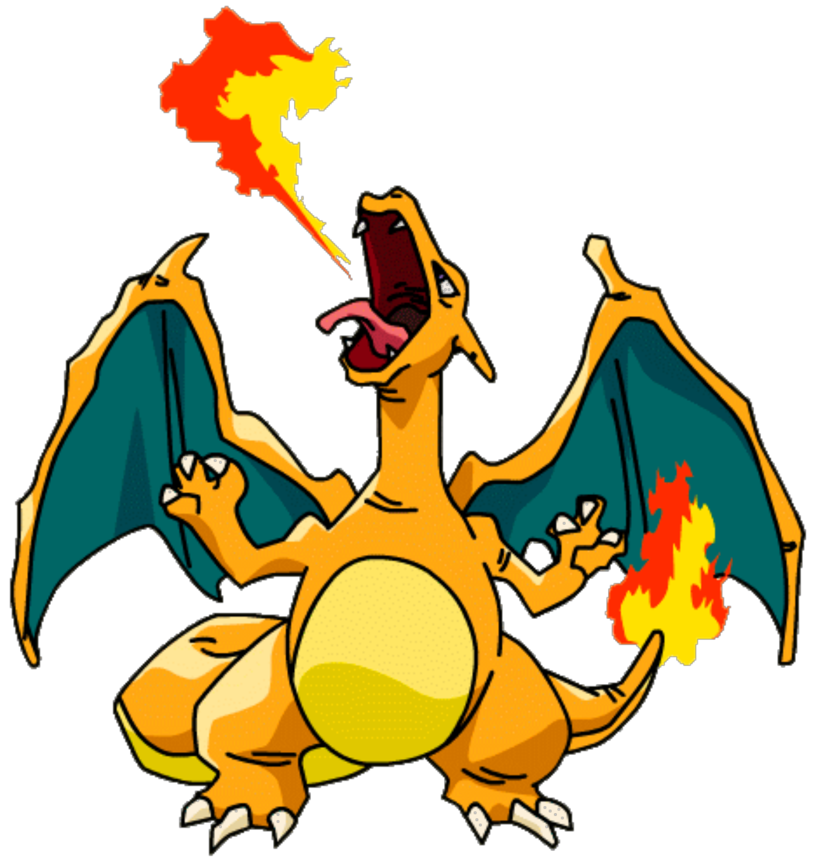
- Let X^p be a random variable indicating # flips until heads (with prob p)
- $$\mathrm{E}(X^p) = \sum_{i=1}^{\infty} i(1-p)^{i-1}p$$
- We can then write
- $$\mathrm{E}(X^p) = \sum_{i=1}^{\infty} i(1-p)^{i-1}p = p + \sum_{i=2}^{\infty} i(1-p)^{i-1}p = p + \sum_{i'=1}^{\infty} (1+i')(1-p)^{i'}p$$
- $$\mathrm{E}(X^p) = p + (1-p)\sum_{i'=1}^{\infty} (1+i')(1-p)^{i'-1}p = p + (1-p)\mathrm{E}(X^p+1)$$
- $\mathrm{E}(X^p) = p + (1-p)(\mathrm{E}(X^p)+1)$

You don't need to do this proof every time you use recursion. But, it can help if you're unsure of correctness

Coupon/Pokemon Collector Problem

Gotta' Catch 'Em All

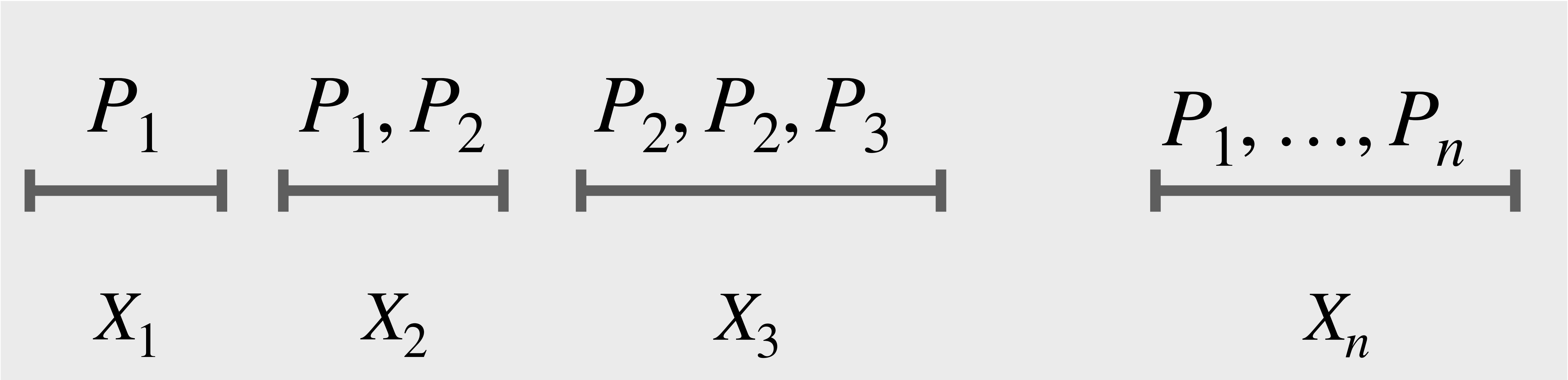
- Suppose there are n different types of Pokemon cards
- In each trial we purchase a pack that contains a Pokemon card
- We repeat until we have at least one of each type of card, how many packs does it take in expectation to collect all?
- Let X be the r.v. equal to the number of packs bought until you first have a card of each type. **Goal**: compute $E[X]$
- We break X into smaller random variables
- **Idea**: we make progress every time we get a card we don't already have



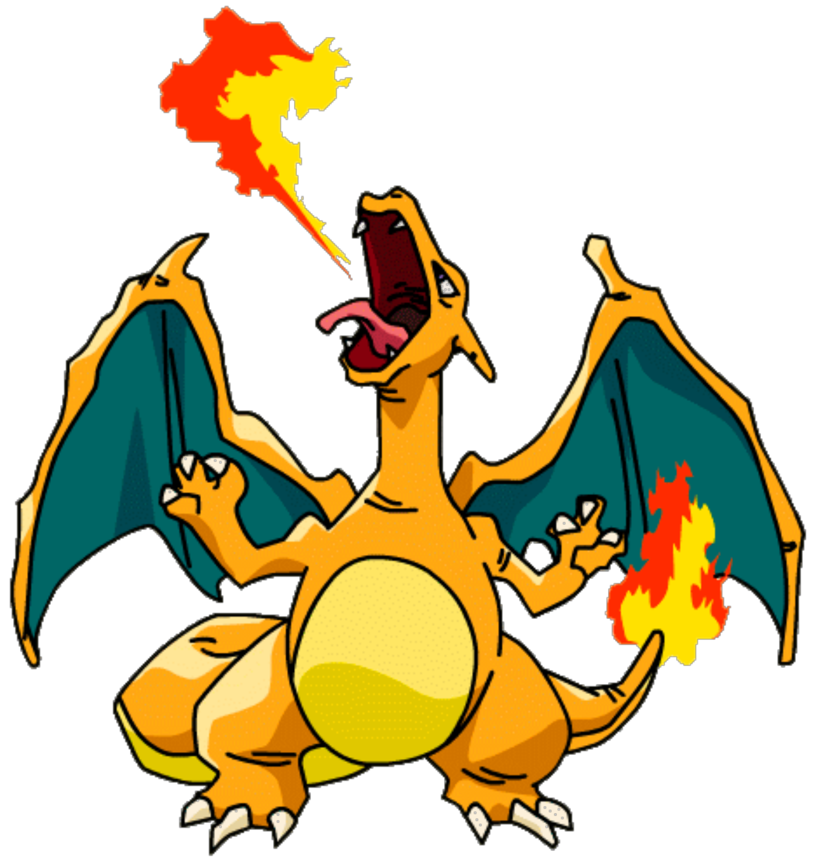
Pokemon Collector Problem

- Let X_i denote the "length of the ith phase", that is, the number of packs bought during the ith phase (ith phase ends as soon as we see the ith distinct card)

- Thus, $X = \sum_{1=1}^{n} X_i$

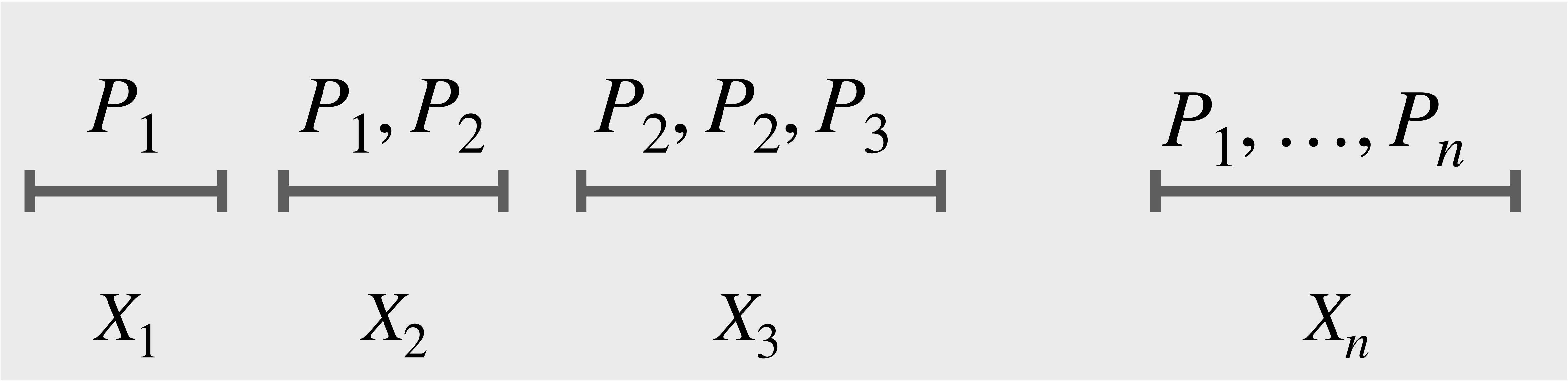


- Each phase can be though of as flipping a biased coin until we see a head, where seeing a head = getting a new card



Pokemon Collector Problem

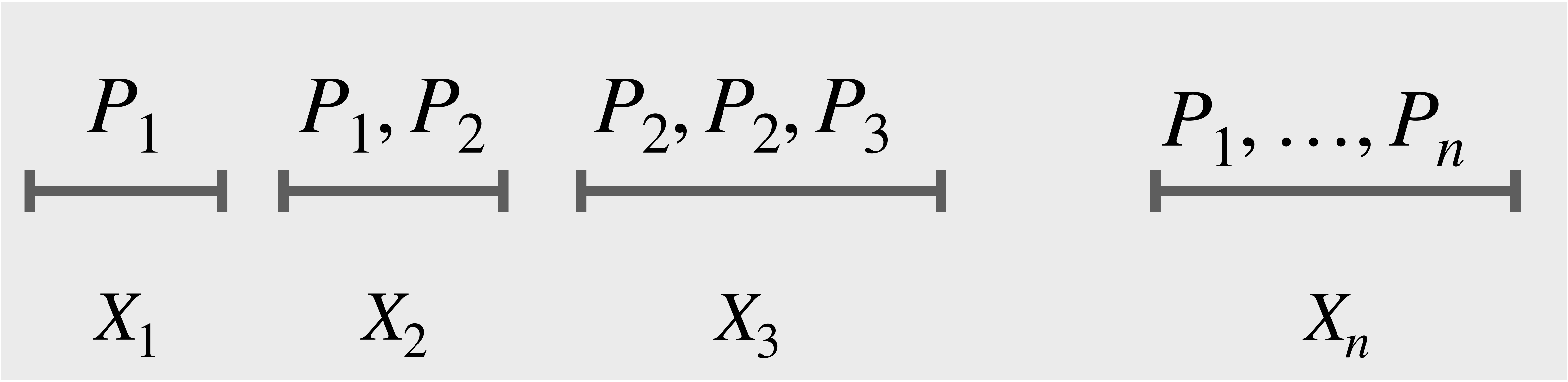
- $E[X_i]$ is the expected number of coin flips until success (expectation of a geometric r.v.)

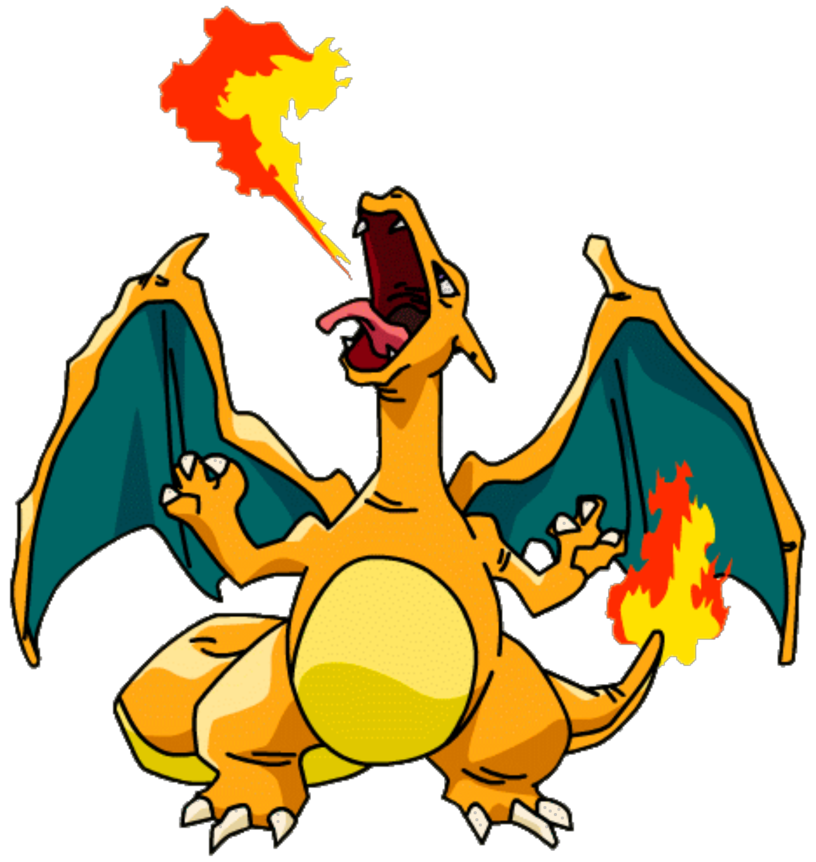


- We know, $E[X_i] = 1/p_i$ where p_i is the probability of success/ probability of seeing a heads during a coin flip in the ith phase
- Before the ith phase starts, we don't have $n - i + 1$ Pokemon
- Each of the n Pokemon are equally likely to be in a pack

Pokemon Collector Problem

- $E[X_i]$ is the expected number of coin flips until success (expectation of a geometric r.v.)

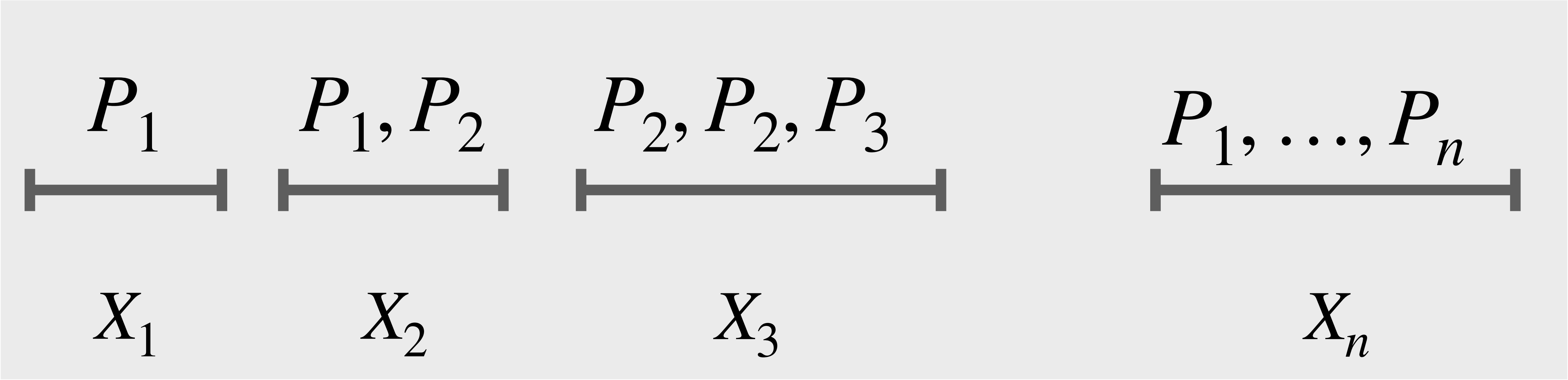




- We know, $E[X_i] = 1/p_i$ where p_i is the probability of success/ probability of seeing a heads during a coin flip in the ith phase
- $p_i = \dfrac{n - i + 1}{n}$

Pokemon Collector Problem

- We know, $E[X_i] = 1/p_i$ where p_i is the probability of success/ probability of seeing a heads during a coin flip in the ith phase



- $E[X_i] =$ Expected[number of flips until first heads] $= 1/p_i = \dfrac{n-i+1}{n}$
- $$E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{n}{n-i+1} = \sum_{i=1}^{n} \frac{n}{i} = nH_n = \Theta(n \log n)$$

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)
  - Hamiltonian cycle reduction images from Michael Sipser's Theory of Computation Book