Introduction to Network Flows

Admin

- Assignment 6 will be released today afternoon
  - More practice with dynamic programming
  - Shortest path with negative weights
- Some network flow questions (topic we'll start today)
- Due next week (Wed 11 pm, April 14)
  - In a week & a half
  - Utilize office & TA hours this week as well as next

Story So Far

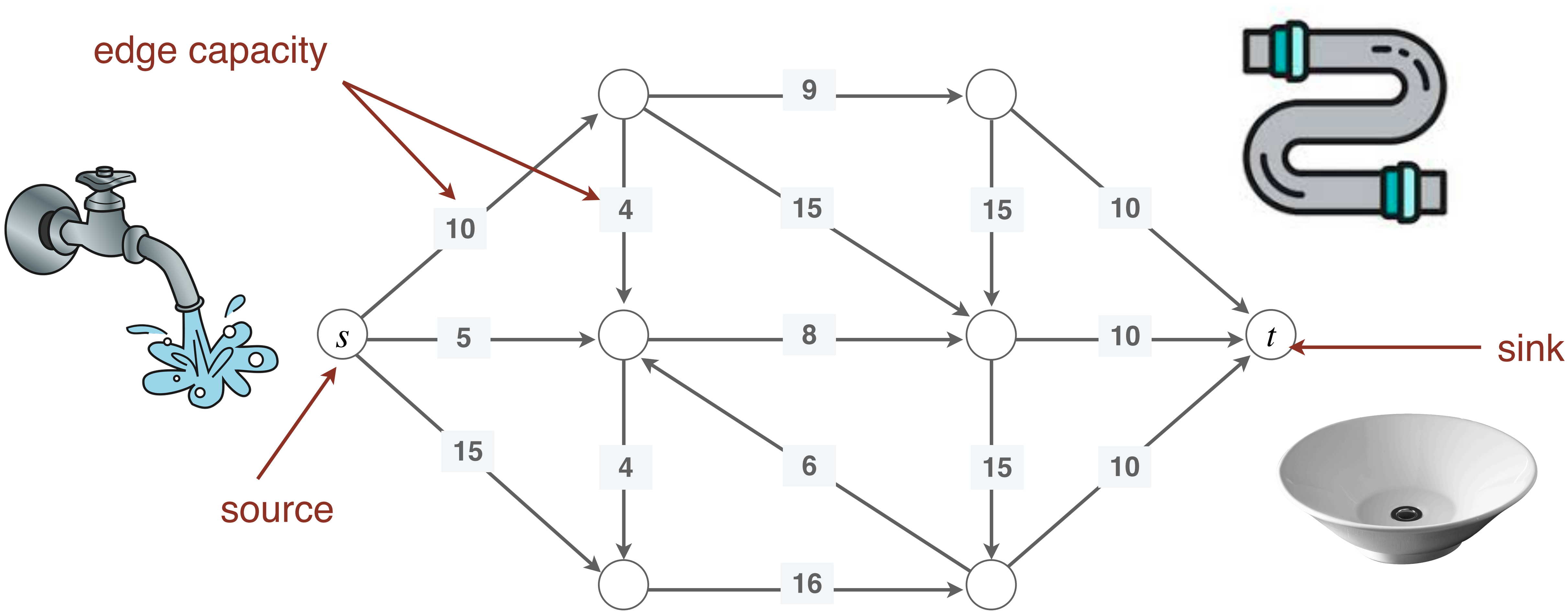
- Algorithmic design paradigms:
  - **Greedy**: simplest to design but works only for certain limited class of optimization problems
    - A good starting point for most problems but rarely optimal
  - **Divide and Conquer**
    - Solving a problem by breaking it down into smaller subproblems and recursing
  - **Dynamic programming**
    - Recursion with memoization: avoiding repeated work
    - Trading off space for time

New Algorithmic Paradigm

- **Network flows** model a variety of optimization problems
- These optimization problems look complicated with lots of constraints; on the face of it seem to have nothing to do with networks or flows
- Very powerful problem solving frameworks
- We'll focus on the concept of **problem reductions**
  - Problem **A** reduces to **B** if a solution to **B** leads to a solution to **A**
- Learn how to prove that our reductions are correct

What's a Flow Network?

- A flow network is a directed graph $G = (V, E)$ with a
  - A **source** is a vertex s with in degree 0
  - A **sink** is a vertex t with out degree 0
  - Each edge $e \in E$ has **edge capacity** $c(e) > 0$



Assumptions

- Assume that each node v is on some s-t path, that is, $s \rightsquigarrow v \rightsquigarrow t$ exists, for any vertex $v \in V$
    - Implies G is connected and $m \geq n - 1$
- Assume **capacities are integers**
    - Will revisit this assumption and what happens if its not
- Directed edge (u, v) written as $u \to v$
- For simplifying expositions, we will sometimes write $c(u \to v) = 0$ when $(u, v) \notin E$

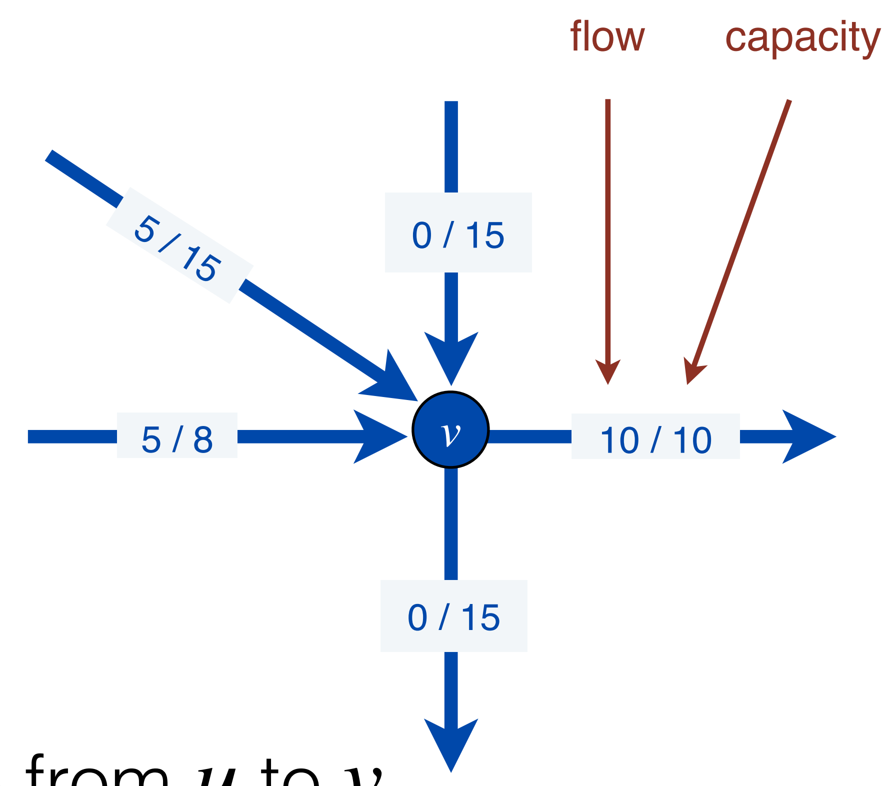
What's a Flow?

- Given a flow network, an (s, t)**-flow** or just **flow** (if source s and sink t are clear from context) $f : E \rightarrow \mathbb{Z}^+$ satisfies the following two constraints:

- **[Flow conservation]** $f_{in}(v) = f_{out}(v)$, for $v \neq s, t$ where

$$f_{in}(v) = \sum_u f(u \rightarrow v)$$

$$f_{out}(v) = \sum_w f(v \rightarrow w)$$

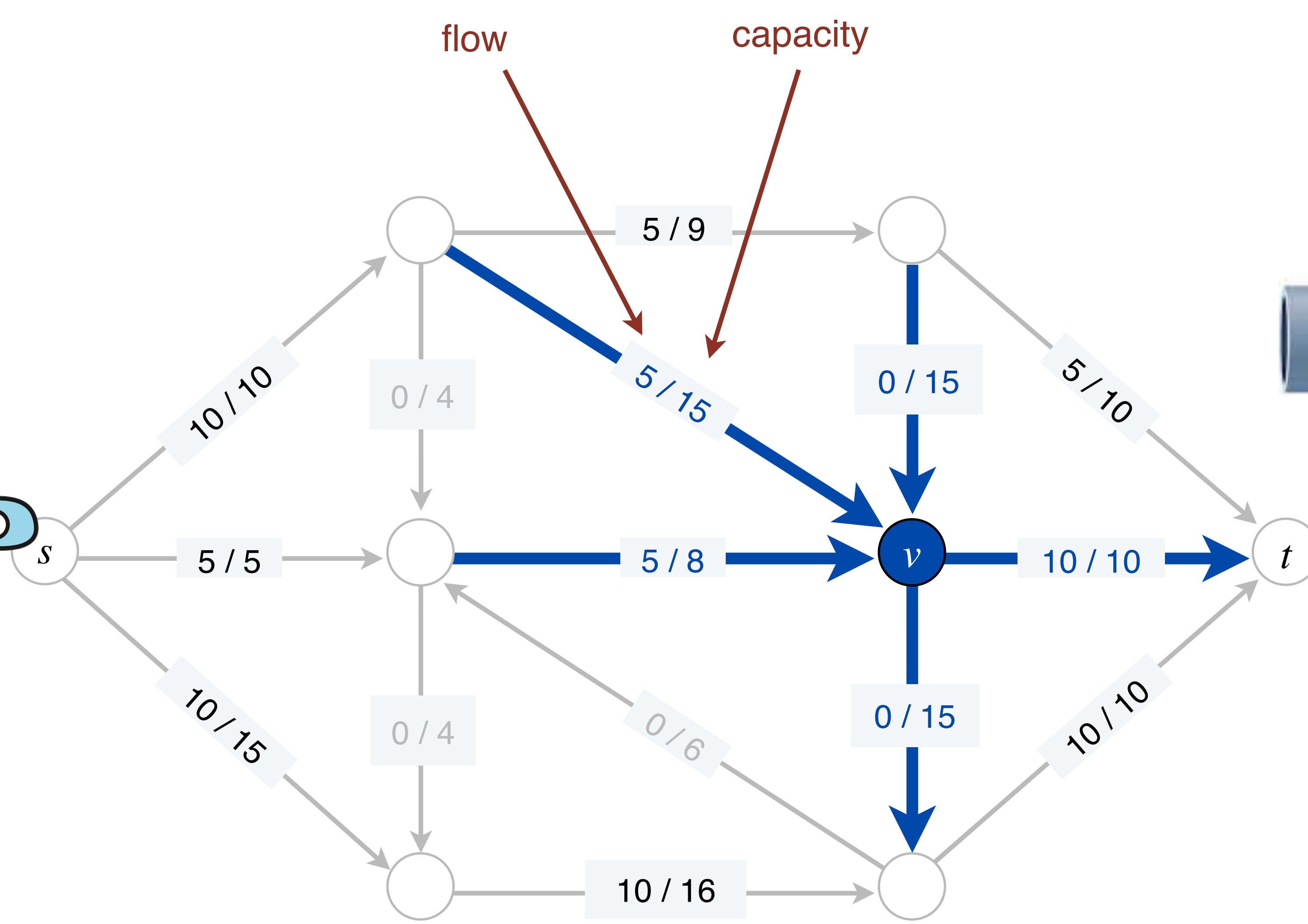


- To simplify, $f(u \rightarrow v) = 0$ if there is no edge from u to v

Feasible Flow

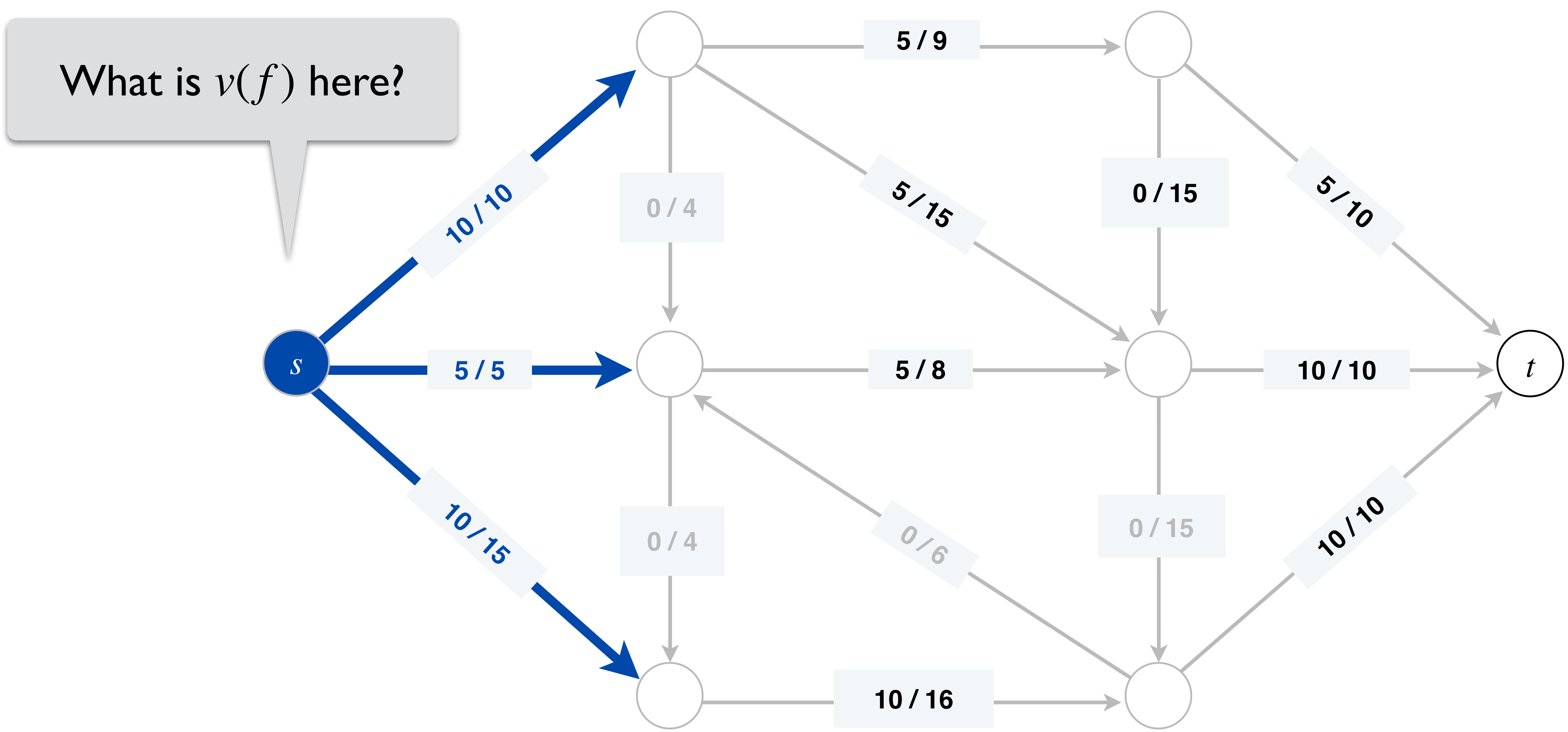
- And second, a feasible flow must satisfy the capacity constraints of the network, that is,

[Capacity constraint] for each $e \in E$, $0 \leq f(e) \leq c(e)$



Value of a Flow

- **Definition.** The **value** of a flow f, written $v(f)$, is $f_{out}(s)$.

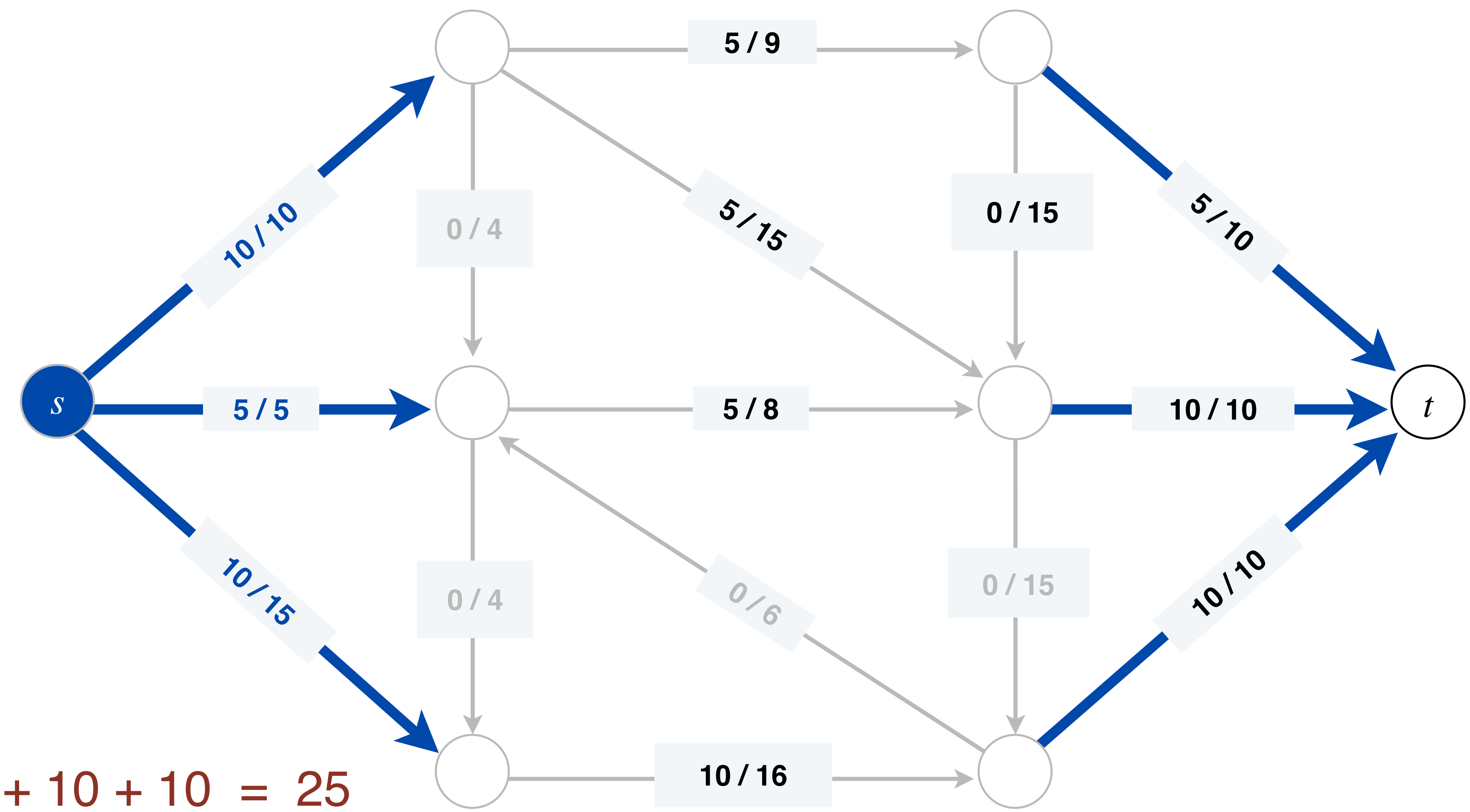


$v(f) = 5 + 10 + 10 = 25$

Value of a Flow

- **Definition.** The **value** of a flow f, written $v(f)$, is $f_{out}(s)$.
- **Lemma**. $f_{out}(s) = f_{in}(t)$

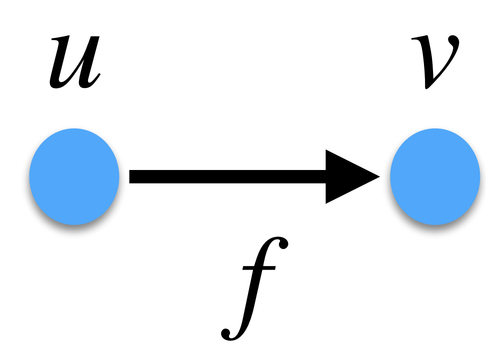
Intuitively, why do you think this is true?



value = 5 + 10 + 10 = 25

Value of a Flow

- **Lemma**. $f_{out}(s) = f_{in}(t)$
- **Proof**. Let $f(E) = \sum_{e \in E} f(e)$

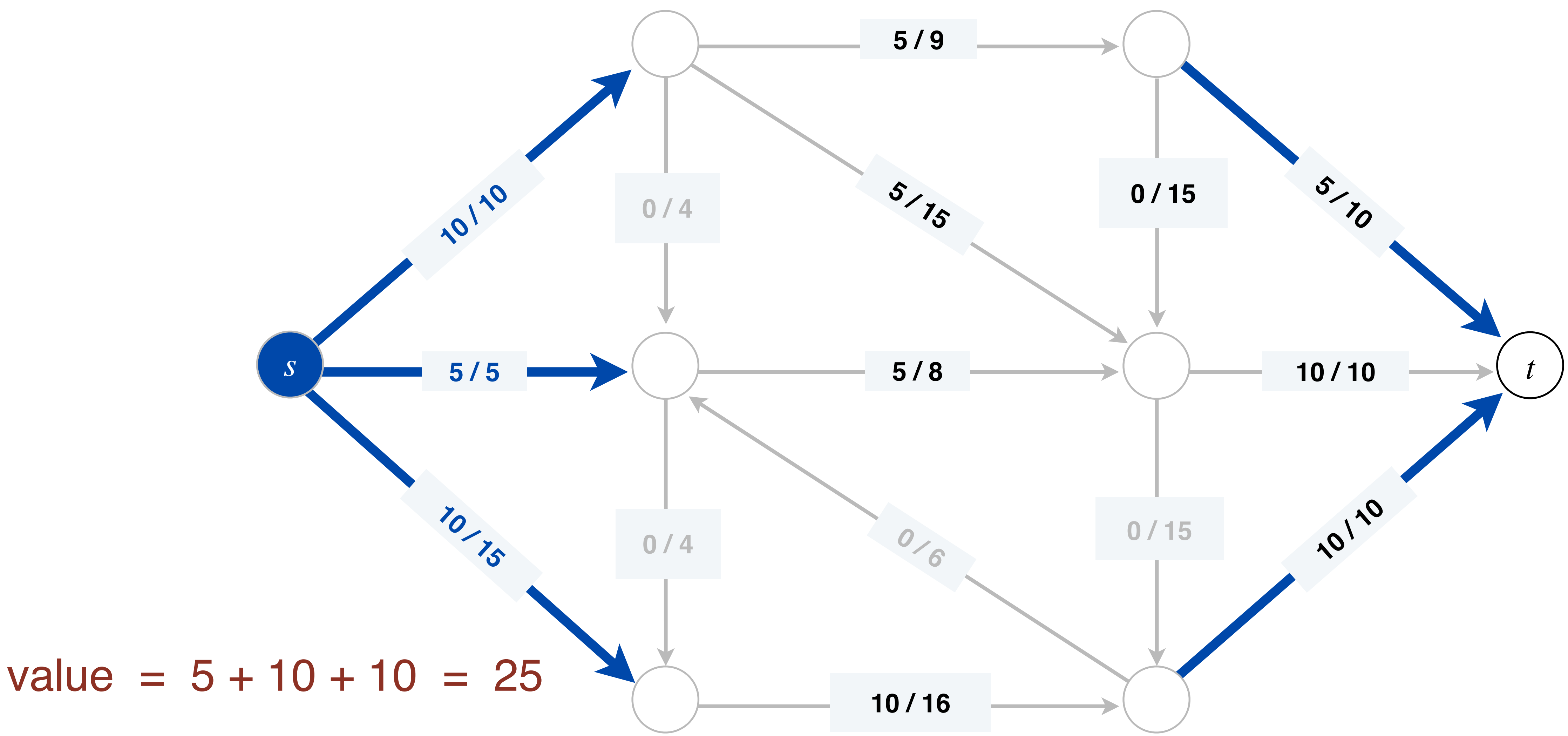


  - Then, $\sum_{v \in V} f_{in}(v) = f(E) = \sum_{v \in V} f_{out}(v)$
  - For every $v \neq s, t$ flow conversation implies $f_{in}(v) = f_{out}(v)$
  - Thus all terms cancel out on both sides except

    $f_{in}(s) + f_{in}(t) = f_{out}(s) + f_{out}(t)$
  - But $f_{in}(s) = f_{out}(t) = 0$ $\blacksquare$

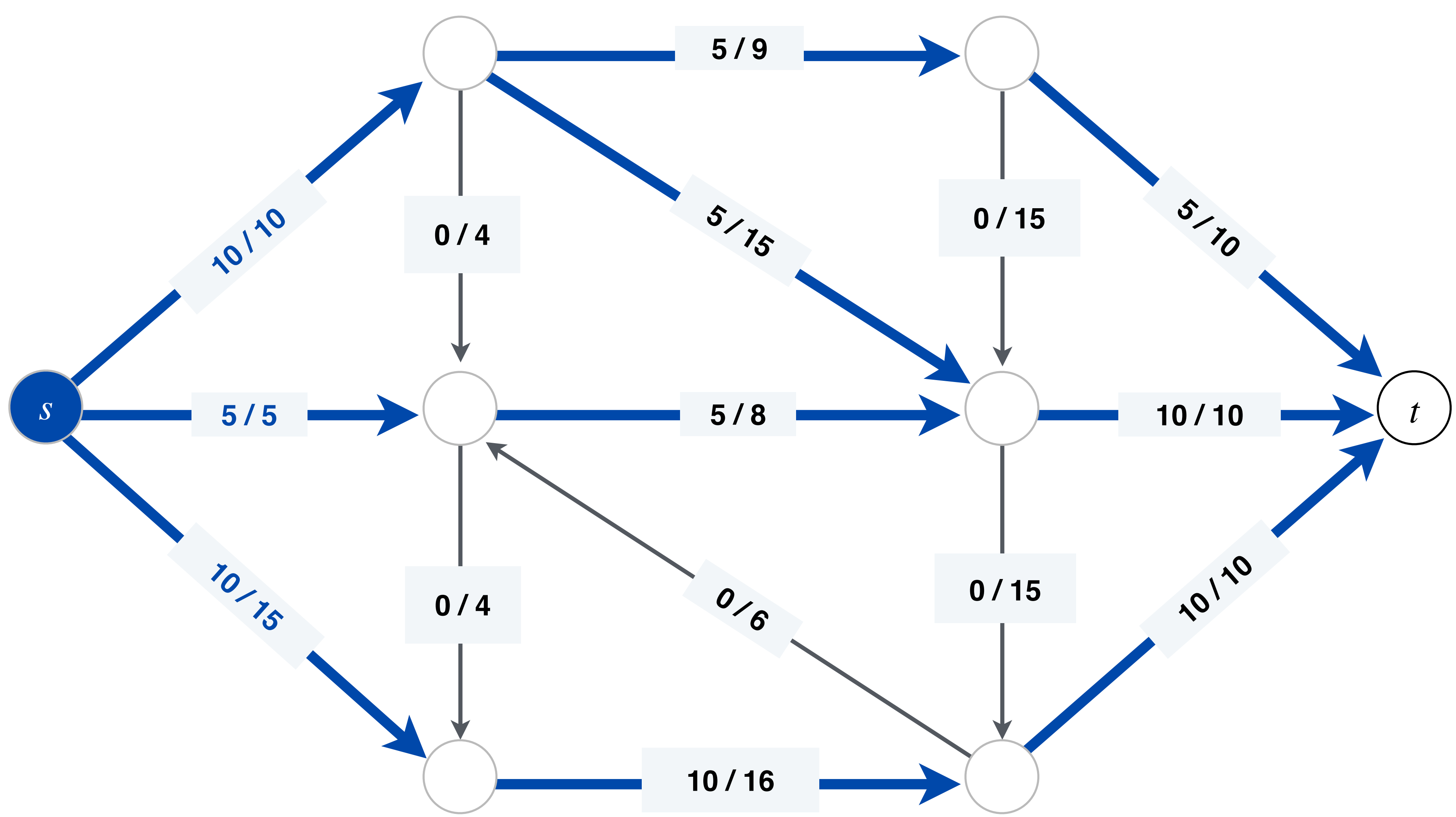
Value of a Flow

- **Lemma**. $f_{out}(s) = f_{in}(t)$
- **Corollary.** $v(f) = f_{in}(t)$.



Max-Flow Problem

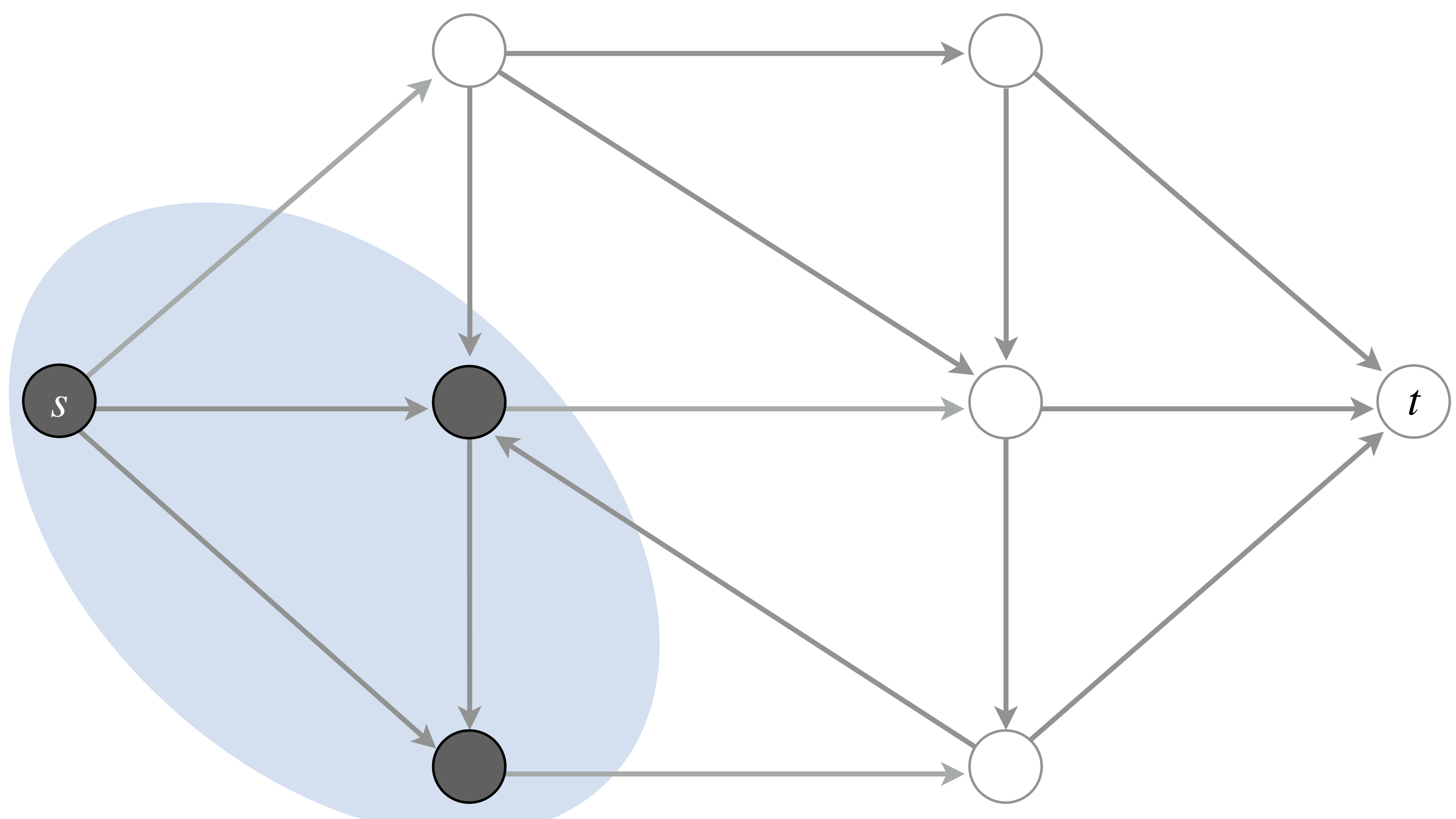
- **Problem**. Given an s-t flow network, find a feasible s-t flow of **maximum** value.



Minimum Cut Problem

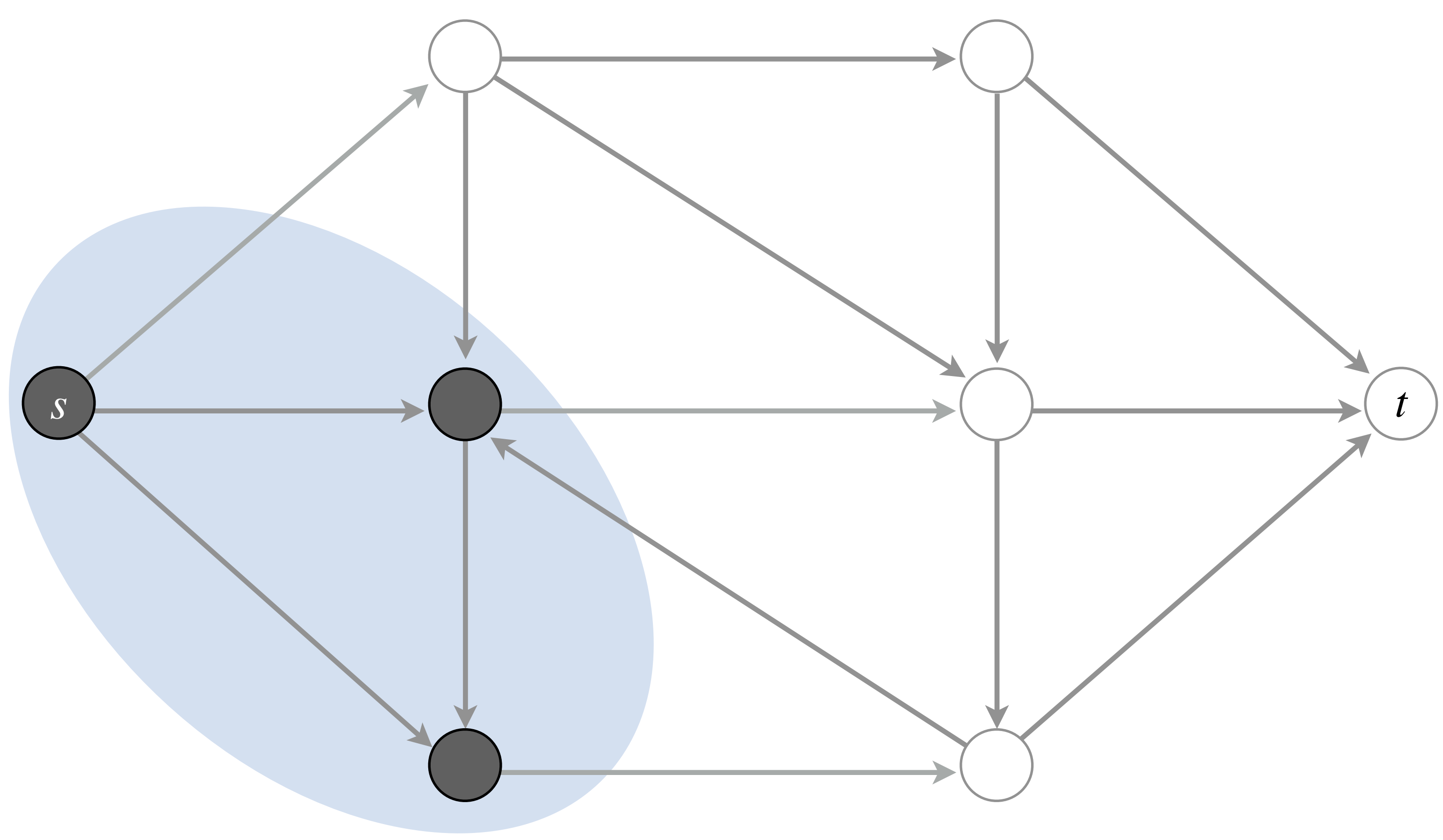
Cuts are Back!

- Cuts in graphs played a lead role when we were designing algorithms for MSTs
- What is the definition of a cut?



Cuts in Flow Networks

- **Recall**. A cut (S, T) in a graph is a partition of vertices such that $S \cup T = V$, $S \cap T = \emptyset$ and S, T are non-empty.
- **Definition**. An (s, t)-*cut* is a cut (S, T) s.t. $s \in S$ and $t \in T$.



Cut Capacity

- **Recall**. A cut (S, T) in a graph is a partition of vertices such that $S \cup T = V$, $S \cap T = \emptyset$ and S, T are non-empty.
- **Definition**. An (s, t)-*cut* is a cut (S, T) s.t. $s \in S$ and $t \in T$.
- **Capacity** of a (s, t)-cut (S, T) is the sum of the capacities of edges leaving S:
    - $$c(S, T) = \sum_{v \in S, w \in T} c(v \to w)$$

Quick Quiz

Question. What is the capacity of the s-t given by grey and white nodes?

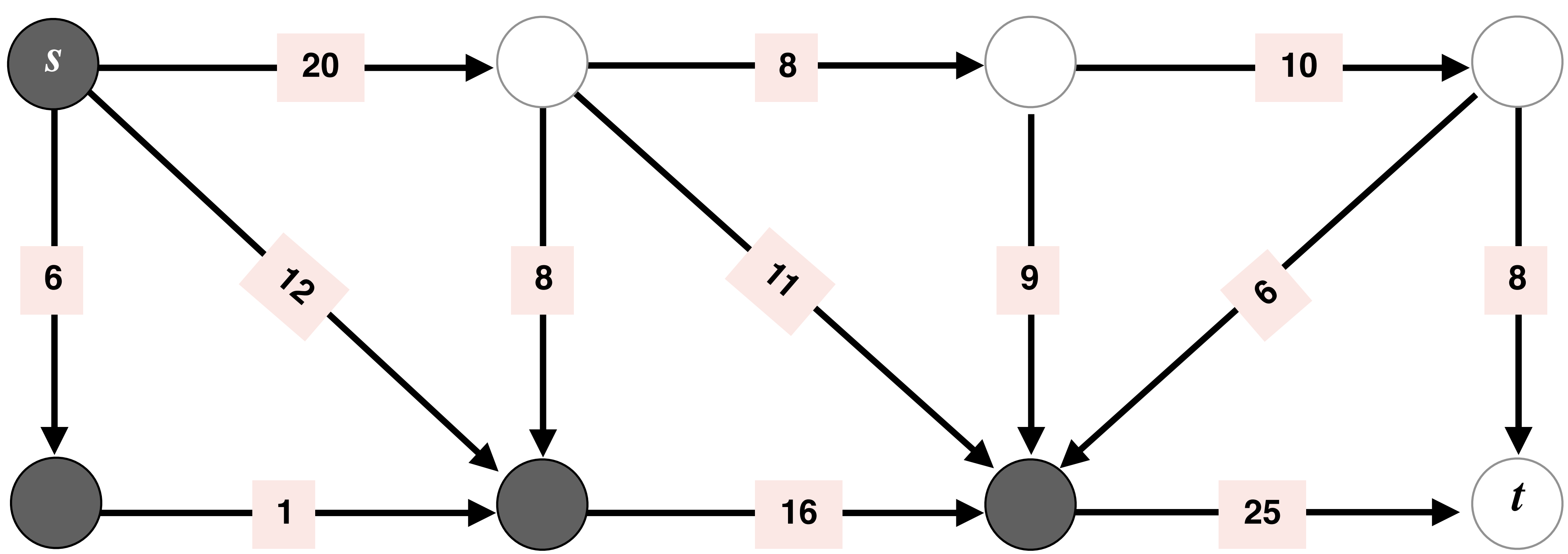
$$c(S,T) = \sum_{v \in S, w \in T} c(v \to w)$$

A. 11 (20 + 25 – 8 – 11 – 9 – 6)

B. 34 (8 + 11 + 9 + 6)

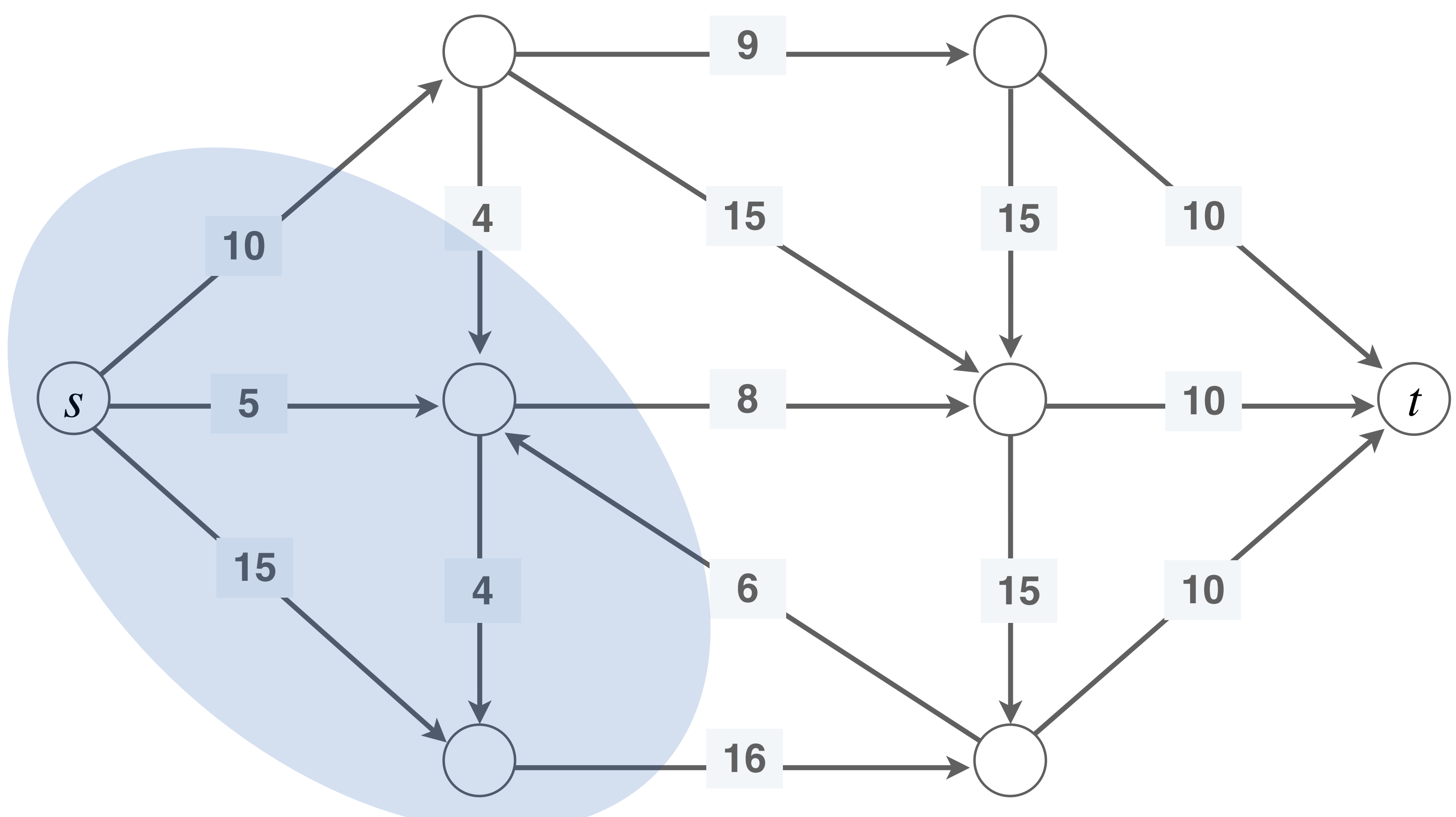
C. 45 (20 + 25)

D. 79 (20 + 25 + 8 + 11 + 9 + 6)



Min Cut Problem

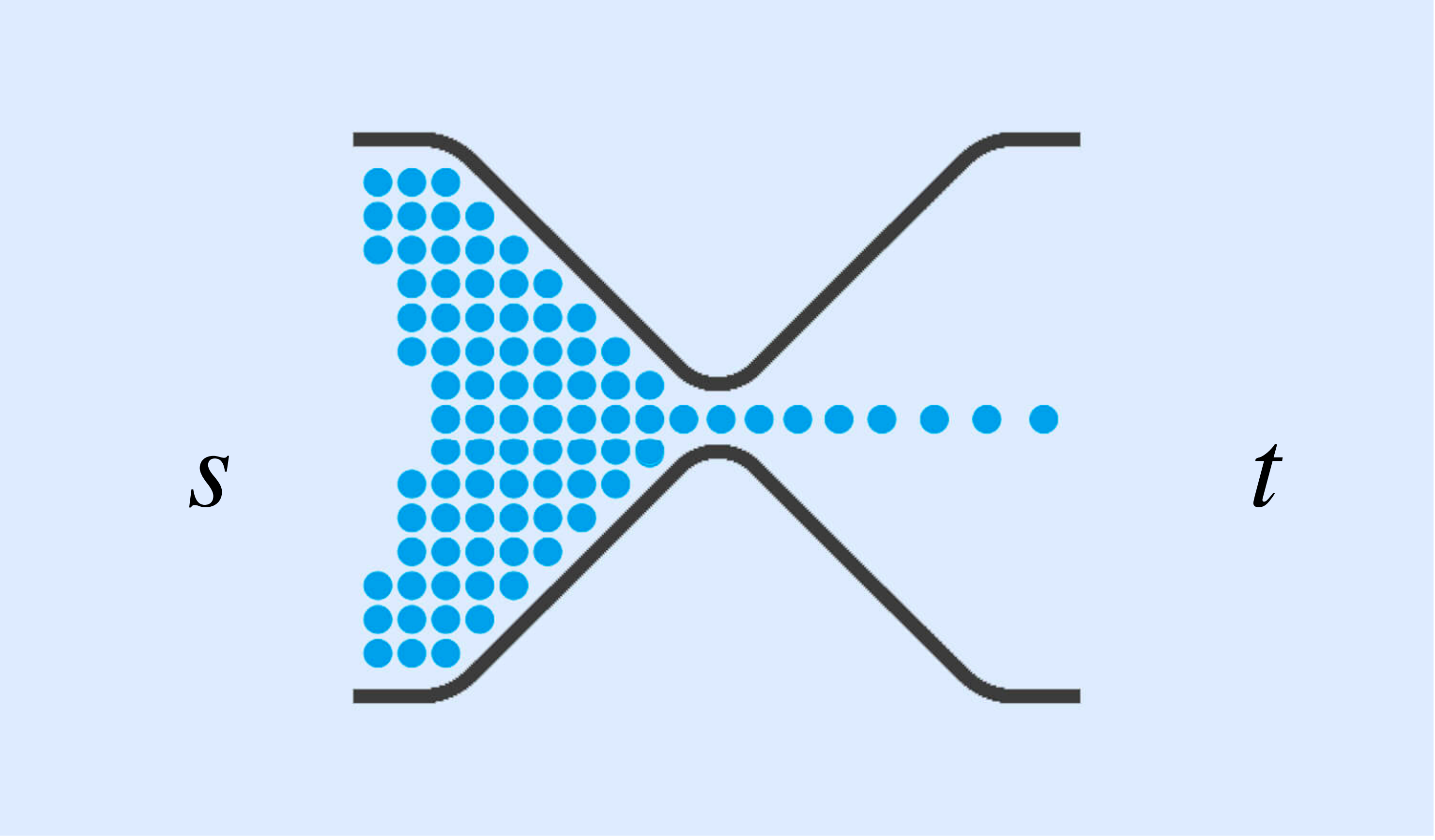
- **Problem**. Given an s-t flow network, find an s-t cut of **minimum** capacity.



Relationship between Flows and Cuts

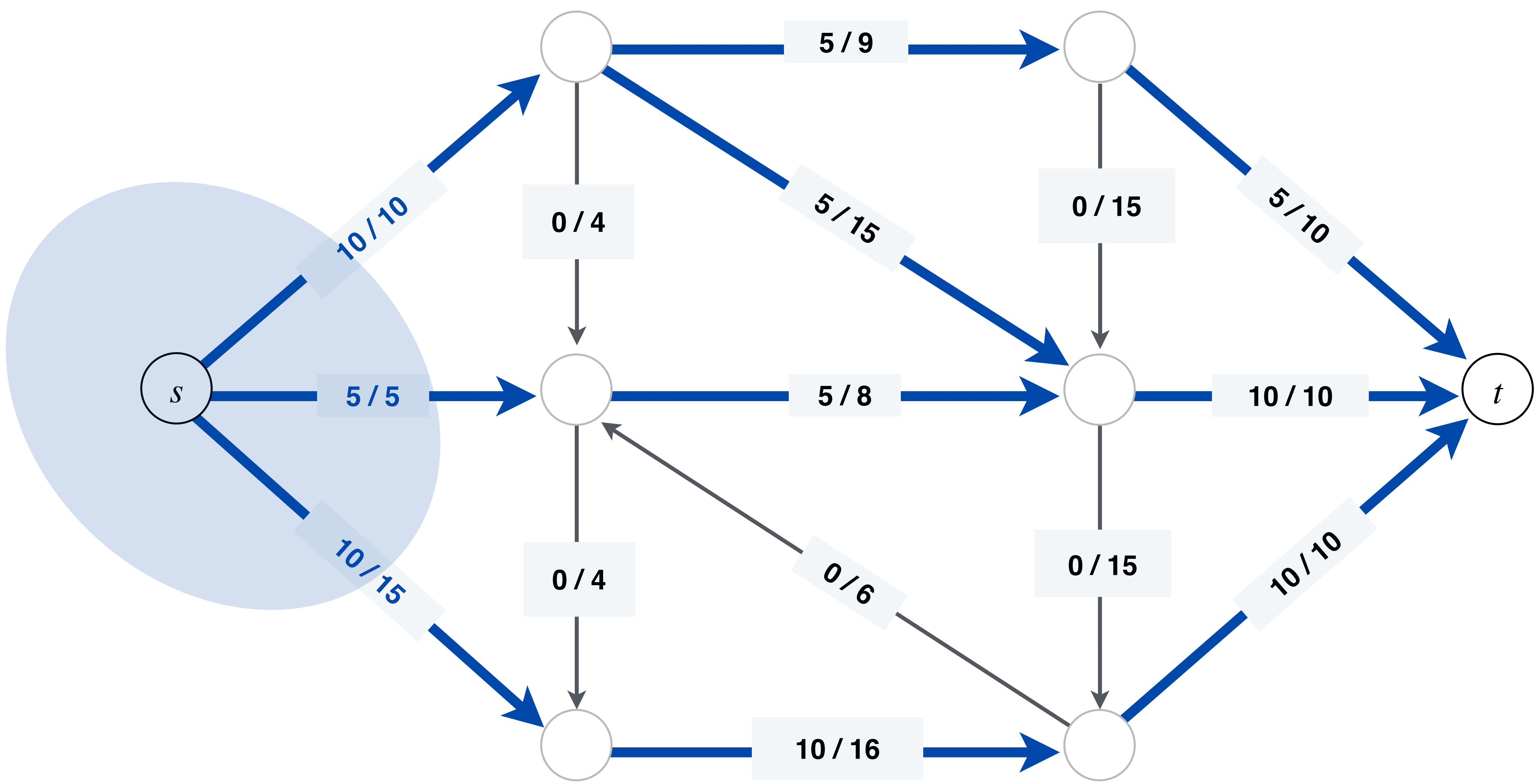
Flows and Cuts

- Cuts represent "**bottlenecks**" in a flow network
- For any cut, our flow needs to “get out” of that cut on its route from s to t
- Let us formalize this intuition



Flows and Cuts

- **Claim**. Let f be **any** s-t flow and (S, T) be **any** s-t cut then $v(f) \leq c(S, T)$
- There are two s-t cuts for which this is easy to see, which ones?



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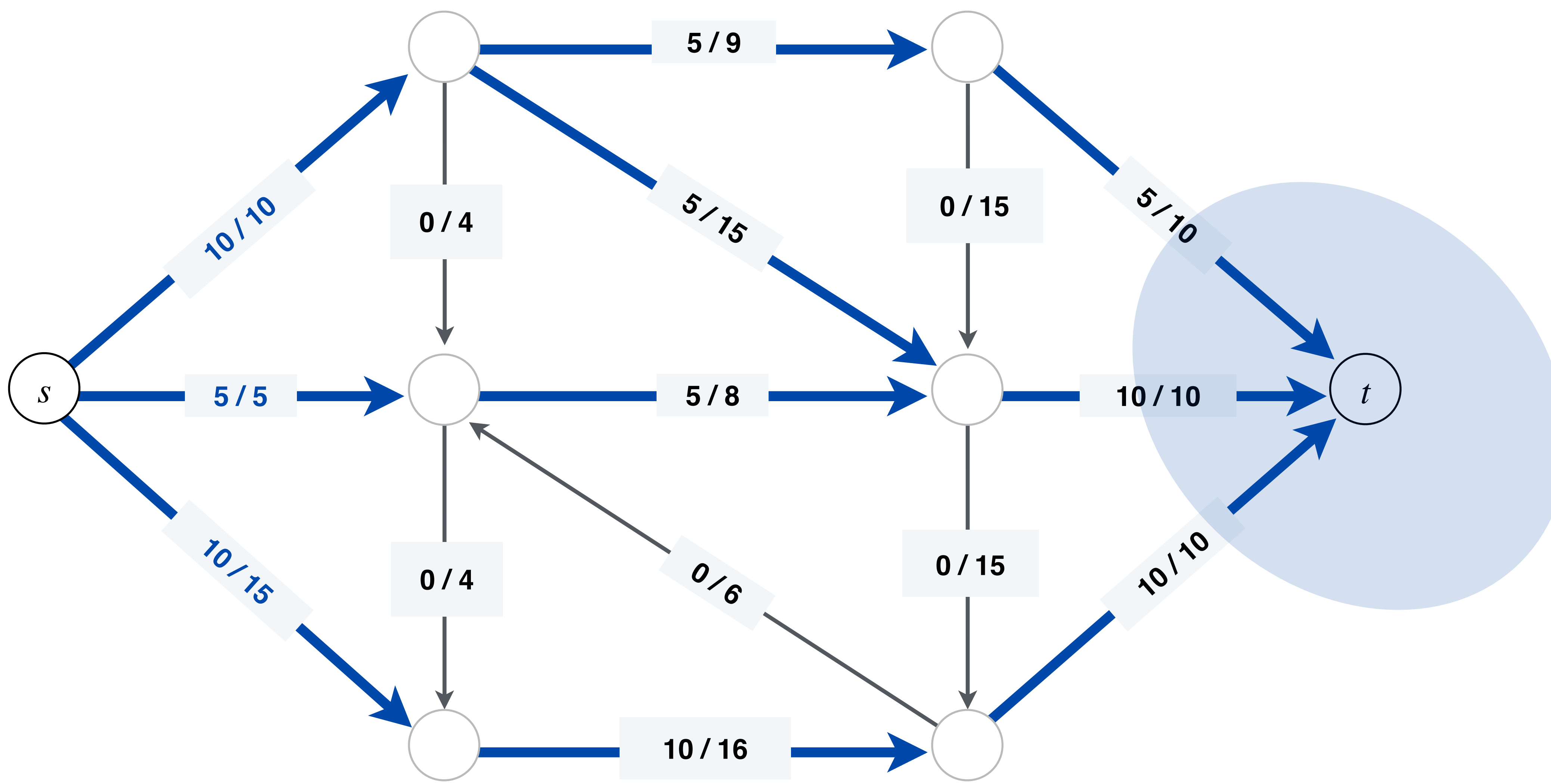


Flows and Cuts

- To prove this for any cut, we first relate the flow value in a network to the net flow leaving a cut
- **Lemma**. For any flow f on $G = (V, E)$ and any (s, t)-cut , then $v(f) = f_{out}(S) - f_{in}(S)$, where
  - $f_{out}(S) = \sum_{v \in S, w \in T} f(v \to w)$ (sum of flow 'leaving' S)
  - $f_{in}(S) = \sum_{v \in S, w \in T} f(w \to v)$ (sum of flow 'entering' S)
  - Note: $f_{out}(S) = f_{in}(T)$ and $f_{in}(S) = f_{out}(T)$

Flows and Cuts

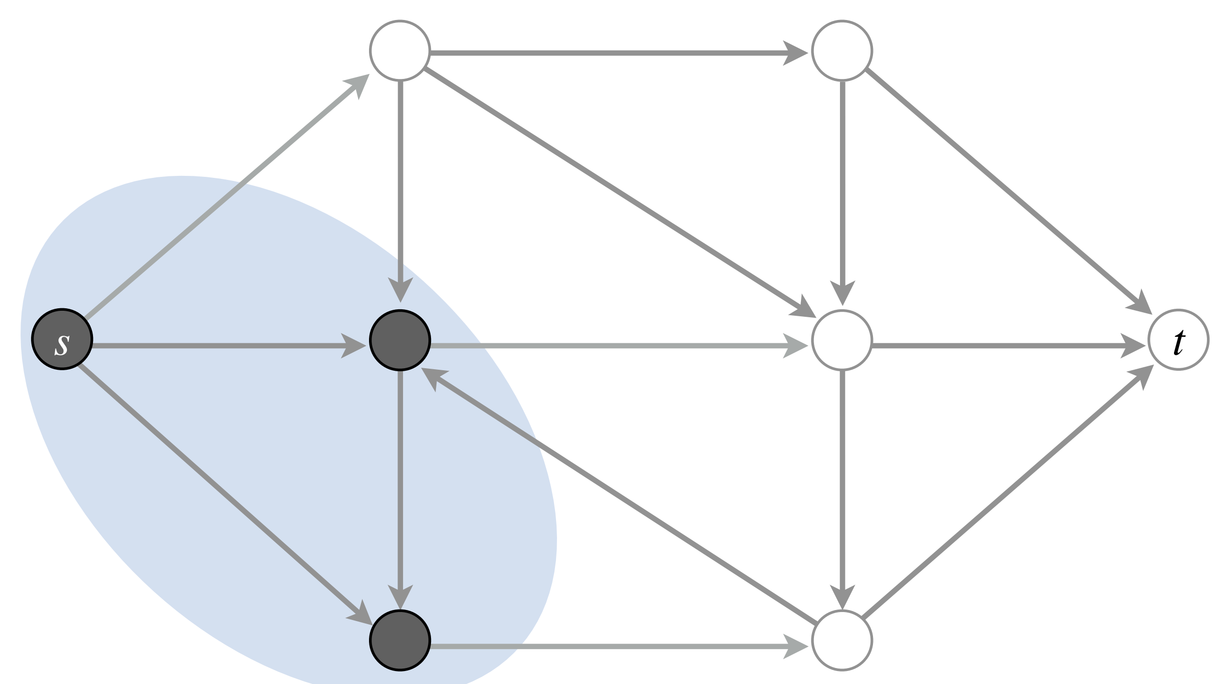
Proof. $f_{out}(S) - f_{in}(S)$

$$= \sum_{v \in S, w \in T} f(v \to w) - \sum_{v \in S, u \in T} f(u \to v) \quad \text{[by definition]}$$

Adding zero terms

$$= \left[\sum_{v,w \in S} f(v \to w) - \sum_{v,u \in S} f(u \to v) \right] + \sum_{v \in S, w \in T} f(v \to w) - \sum_{v \in S, u \in T} f(u \to v)$$

These are the same sum: they sum the flow of all edges with both vertices in S



Flows and Cuts

Rearranging terms

Proof. $f_{out}(S) - f_{in}(S)$

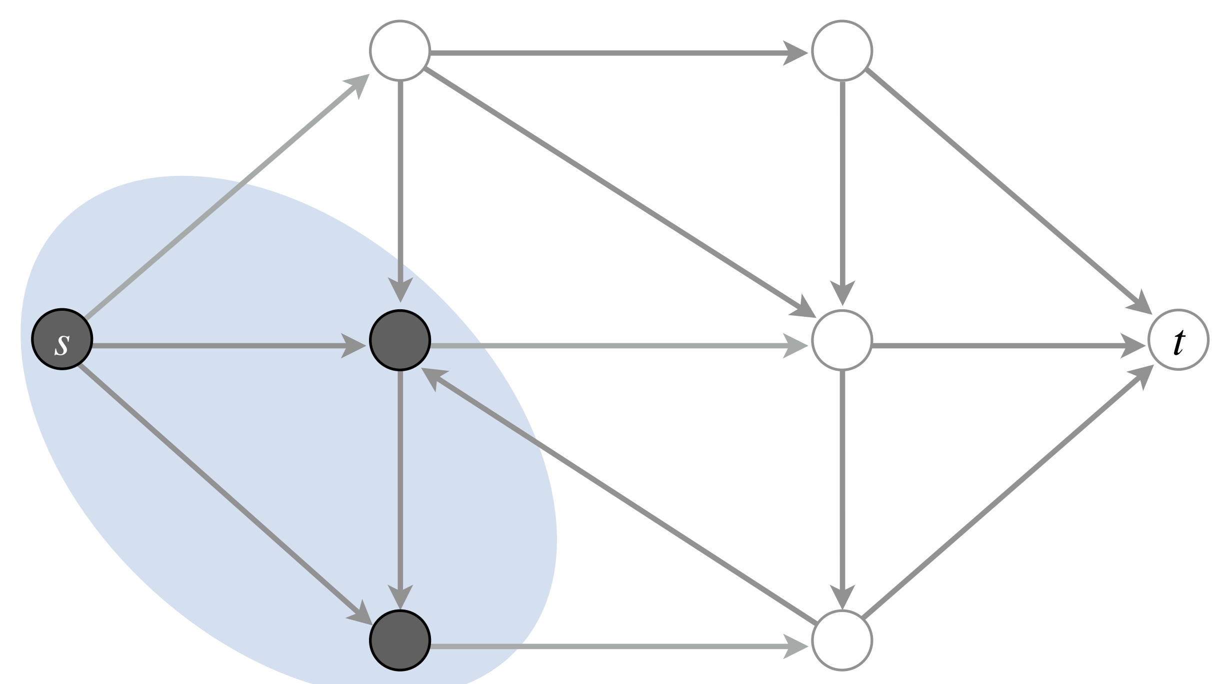
$$= \left[\sum_{v,w \in S} f(v \to w) - \sum_{v,u \in S} f(u \to v) \right] + \sum_{v \in S, w \in T} f(v \to w) - \sum_{v \in S, u \in T} f(u \to v)$$

$$= \sum_{v,w \in S} f(v \to w) + \sum_{v \in S, w \in T} f(v \to w) - \sum_{v,u \in S} f(u \to v) - \sum_{v \in S, u \in T} f(u \to v)$$

$$= \sum_{v \in S} \Big(\sum_{w} f(v \to w) - \sum_{u} f(u \to v) \Big)$$

$$= \sum_{v \in S} f_{out}(v) - f_{in}(v)$$

$$= f_{out}(s) = v(f) \quad \blacksquare$$



Flows and Cuts

- We use this result to prove that the value of a flow cannot exceed the capacity of any cut in the network
- **Claim**. Let f be any s-t flow and (S, T) be any s-t cut then $v(f) \leq c(S, T)$
- **Proof.** $v(f) = f_{out}(S) - f_{in}(S)$

$$\leq f_{out}(S) = \sum_{v \in S, w \in T} f(v \to w)$$

$$\leq \sum_{v \in S, w \in T} c(v, w) = c(S, T)$$

When is $v(f) = c(S, T)$?

$f_{in}(S) = 0,\ f_{out}(S) = c(S, T)$

Max-Flow & Min-Cut

- Suppose the $c_{\min}$ is the capacity of the minimum cut in a network
- What can we say about the feasible flow we can send through it
  - cannot be more than $c_{\min}$
- In fact, whenever we find any s-t flow f and any s-t cut (S, T) such that, $v(f) = c(S, T)$ we can conclude that:
  - f is the maximum flow, and,
  - (S, T) is the minimum cut
- The question now is, given any flow network with min cut $c_{\min}$, is it always possible to route a feasible s-t flow f with $v(f) = c_{\min}$

Max-Flow Min-Cut Theorem

- A beautiful, powerful relationship between these two problems in given by the following theorem
- **Theorem**. Given any flow network G, there exists a feasible (s, t)-flow f and a (s, t)-cut (S, T) such that,

$$v(f) = c(S, T)$$

- Informally, in a flow network, the max-flow = min-cut
- This will guide our algorithm design for finding max flow
- (Will prove this theorem by construction in a bit.)

Network Flow History

- In 1950s, US military researchers Harris and Ross wrote a classified report about the rail network linking Soviet Union and Eastern Europe
    - Vertices were the geographic regions
    - Edges were railway links between the regions
    - Edge weights were the rate at which material could be shipped from one region to next
- Ross and Harris determined:
    - Maximum amount of stuff that could be moved from Russia to Europe **(max flow)**
    - Cheapest way to disrupt the network by removing rail links **(min cut)**

Network Flow History

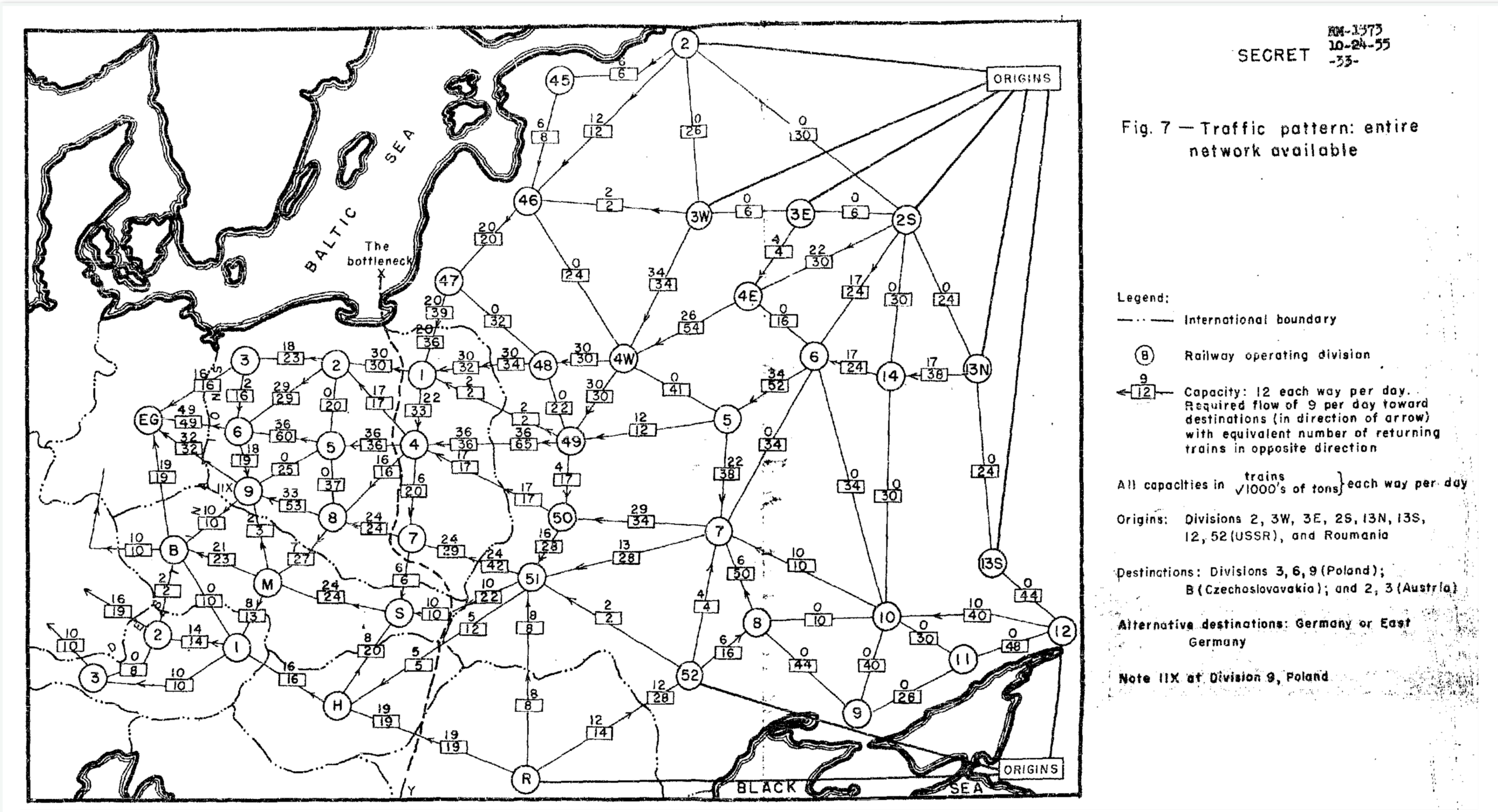


Image Credits: — Jeff Erickson's book and T[homas] E. Harris and F[rank] S. Ross. Fundamentals of a method for evaluating rail net capacities. The RAND Corporation, Research Memorandum RM-1517, October 24, 1955. United States Government work in the public domain. http://www.dtic.mil/dtic/tr/fulltext/u2/093458.pdf

Towards a Max-Flow Algorithm

- We will prove the max-flow min-cut theorem *constructively*
- Said differently, we will design a max-flow algorithm and prove its correctness by showing that there is a s-t cut s.t.
  - Value of flow computed by algorithm $=$ capacity of cut
- Let's start with a greedy approach
  - Push as much flow as possible down a s-t path
  - This won't actually work
  - But gives us a sense of what we need to keep track off to improve upon it

Towards a Max-Flow Algorithm

- Greedy strategy:
  - Start with $f(e) = 0$ for each edge
  - Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$
  - "Augment" flow (as much as possible) along path P
  - Repeat until you get stuck
- Let's take an example

Towards a Max-Flow Algorithm

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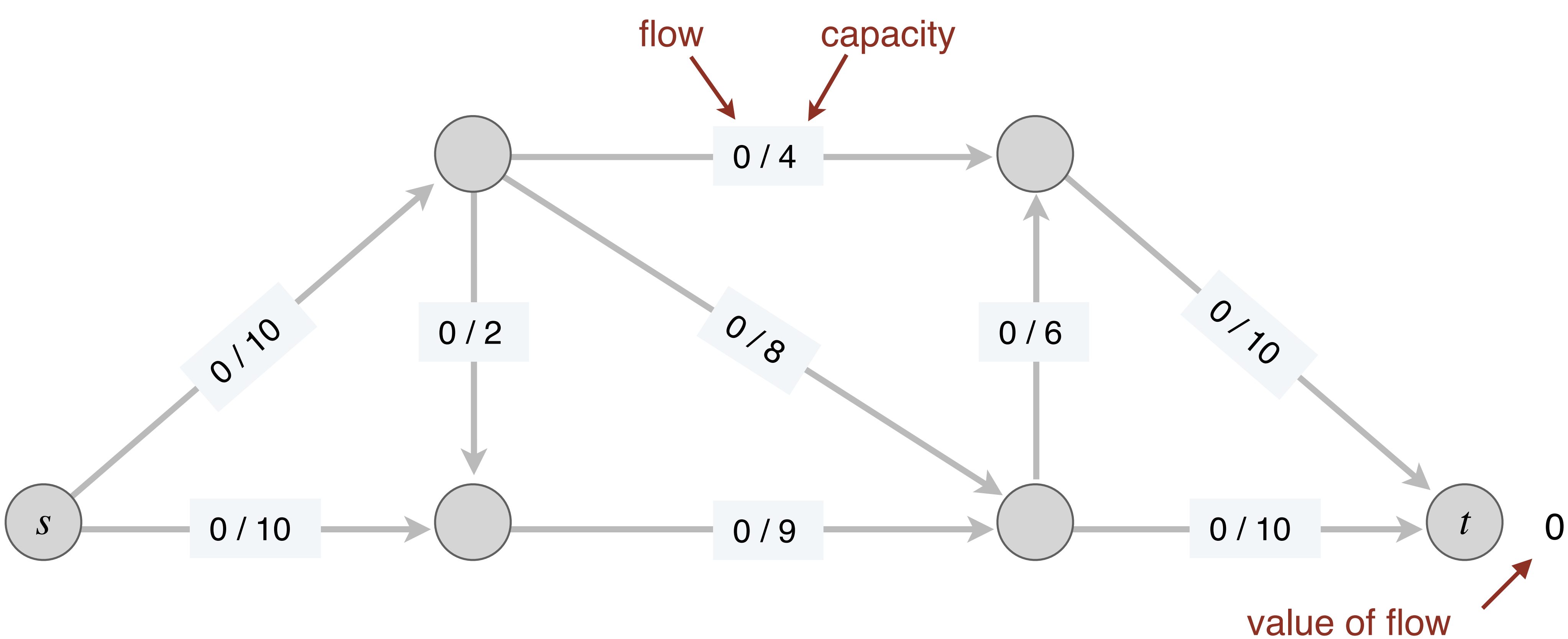

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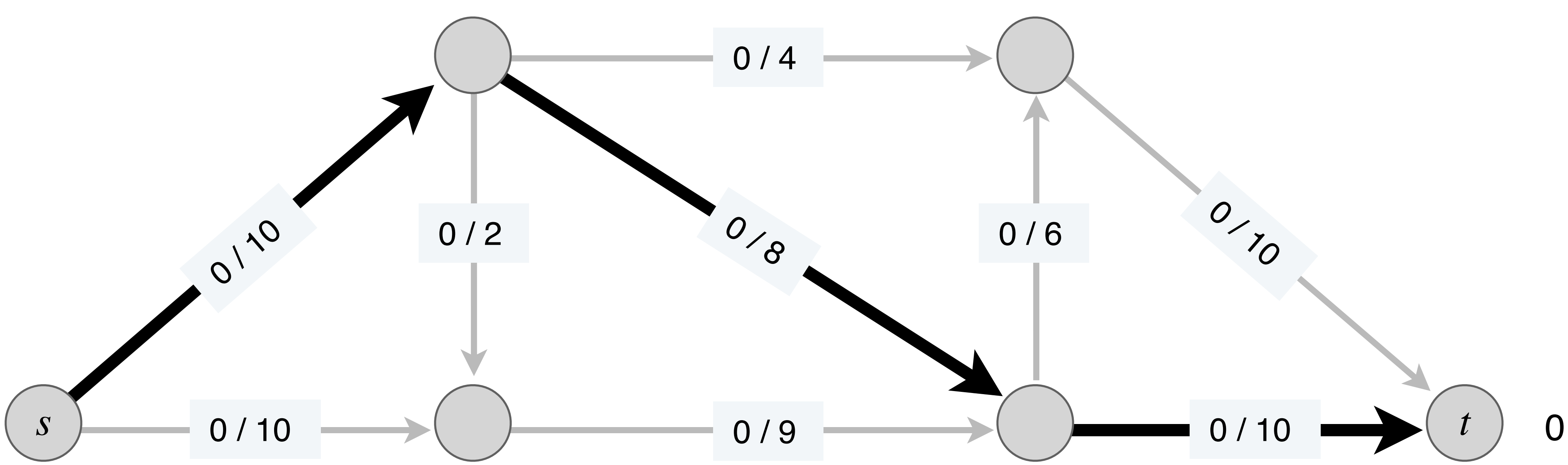

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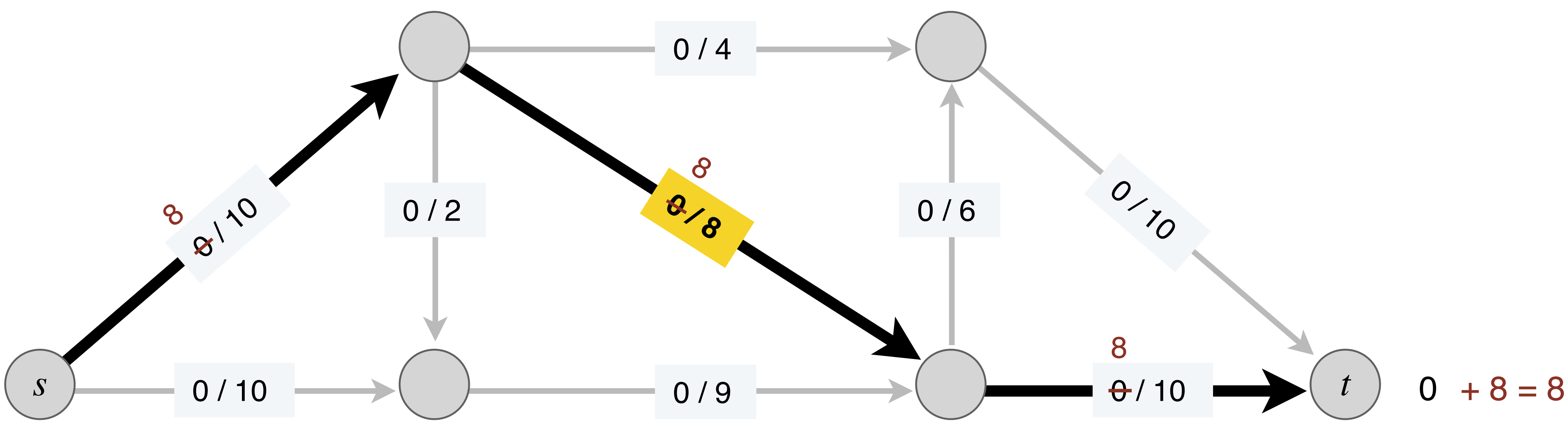

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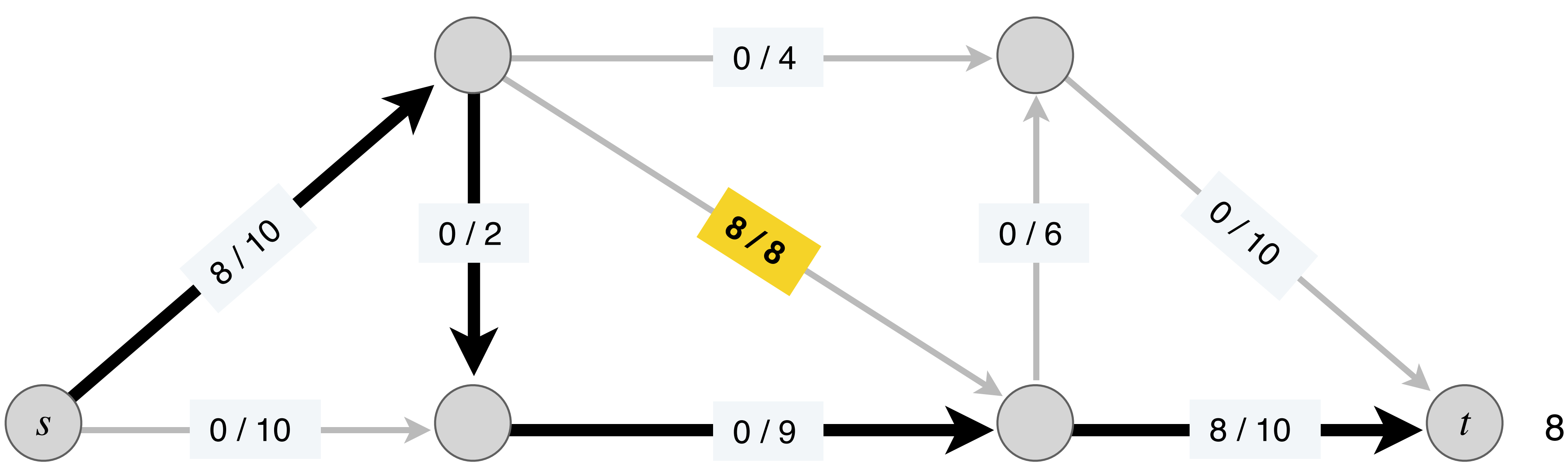

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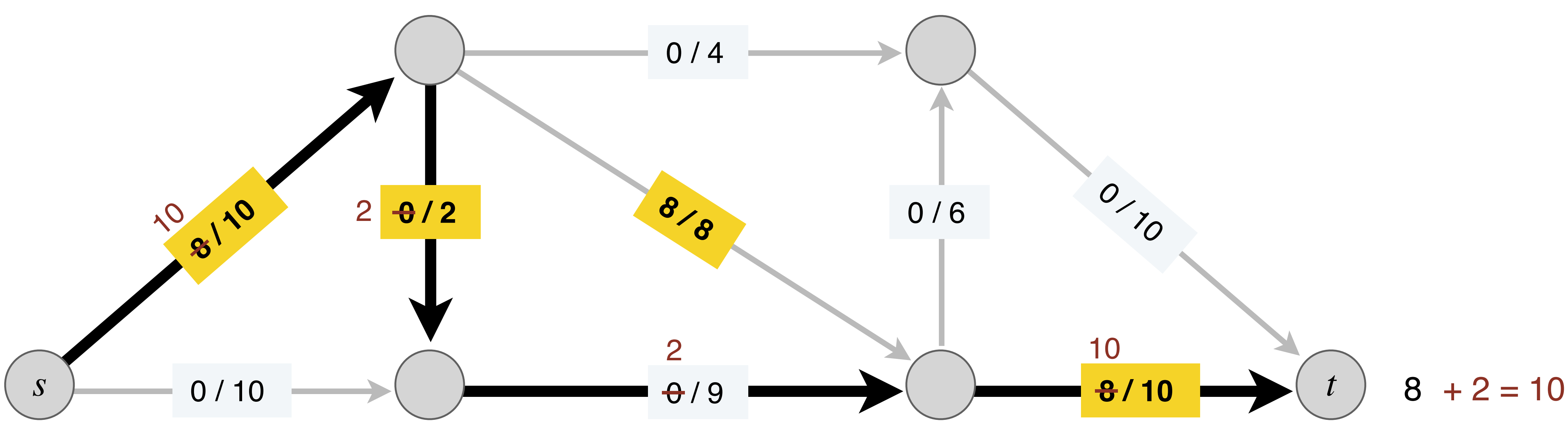

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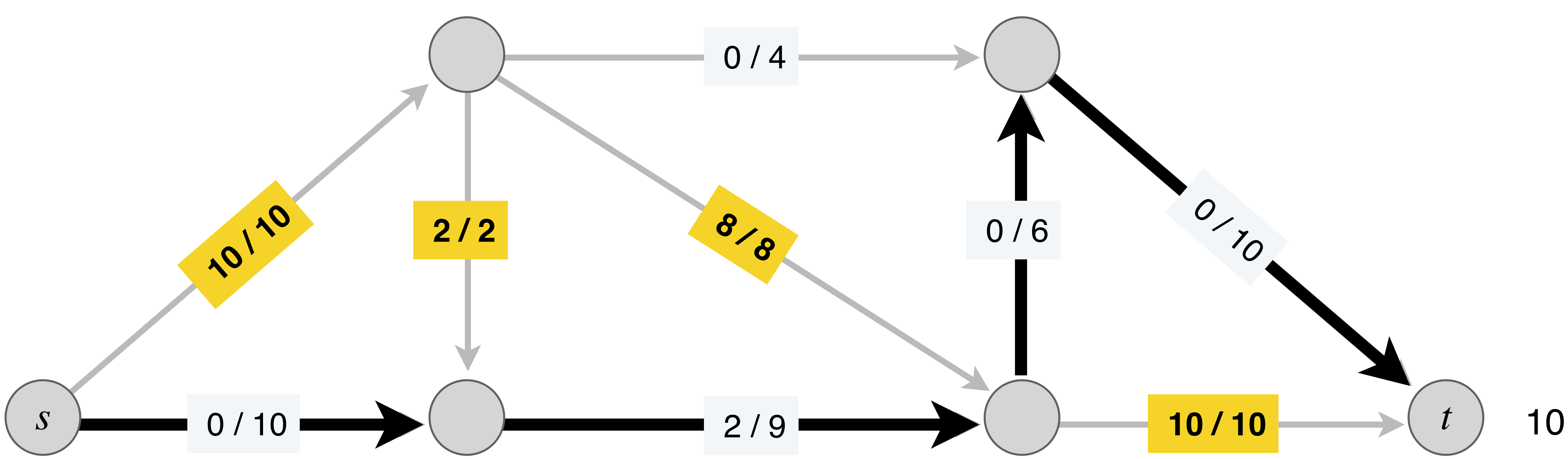

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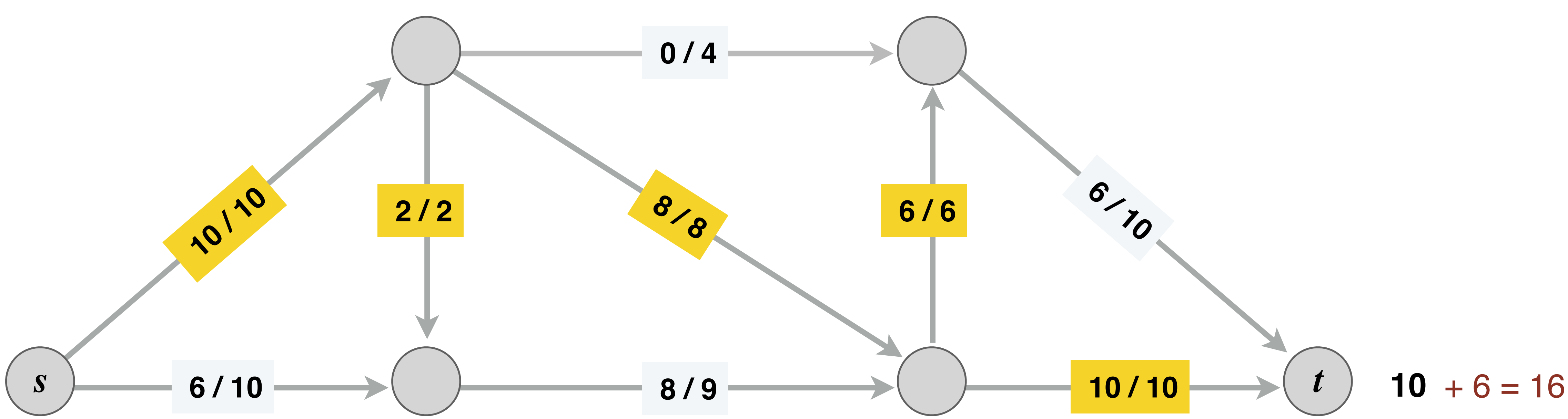
ending flow value = 16



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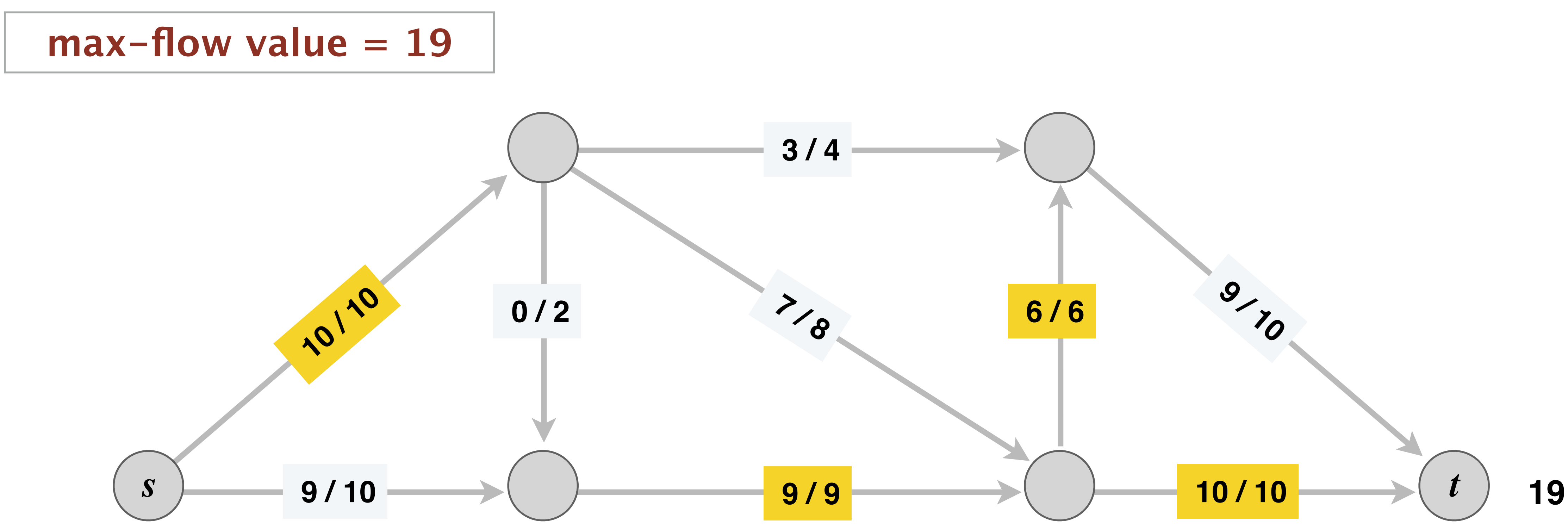
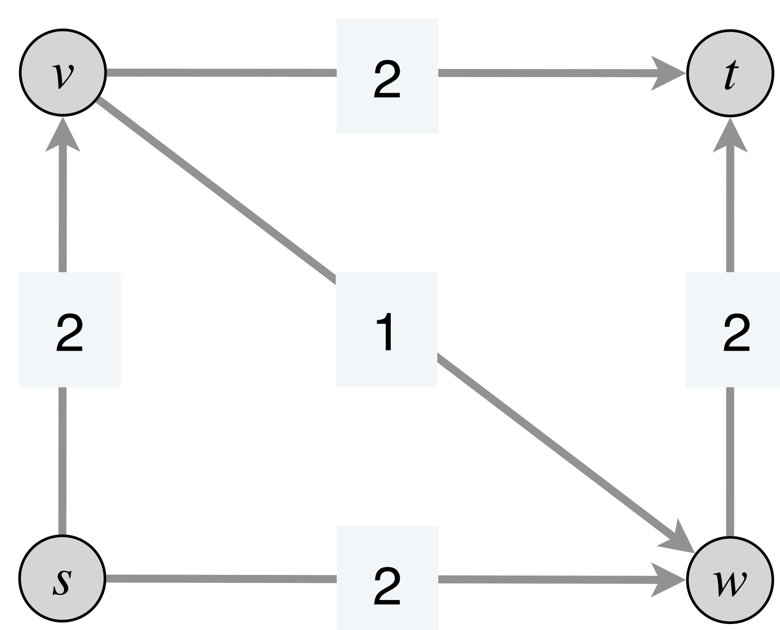


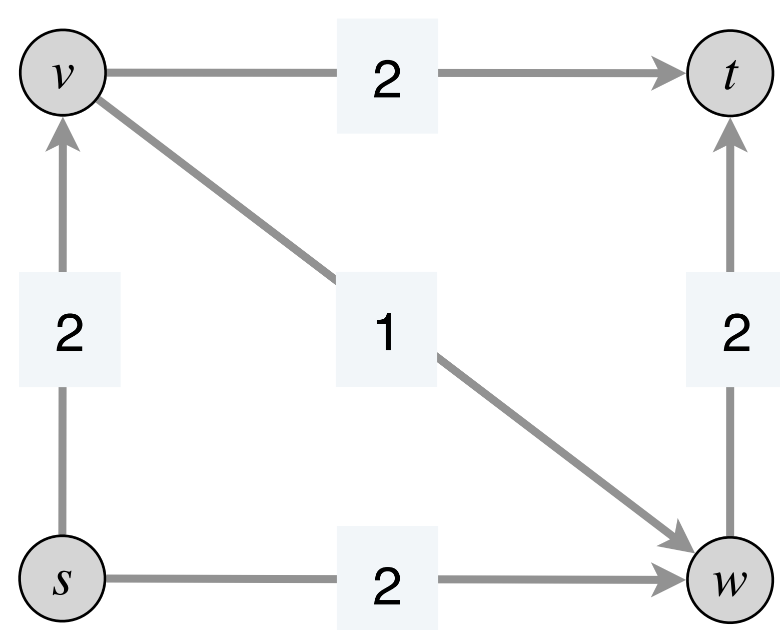
Why Greedy Fails

- **Problem**: greedy can never "undo" a bad flow decision
- Consider the following flow network



Why Greedy Fails

- **Problem**: greedy can never "undo" a bad flow decision
- Consider the following flow network
  - Unique max flow has $f(v \rightarrow w) = 0$
  - Greedy could choose $s \rightarrow v \rightarrow w \rightarrow t$ as first P



- **Summary**: Need a mechanism to "undo" bad flow decisions

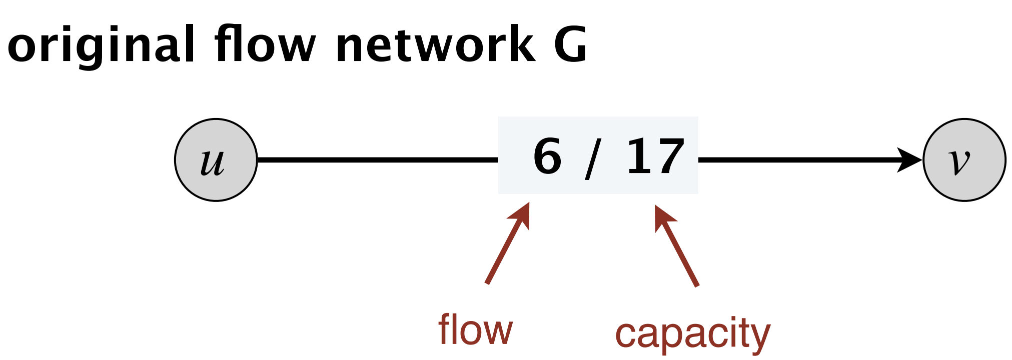
Ford-Fulkerson Algorithm

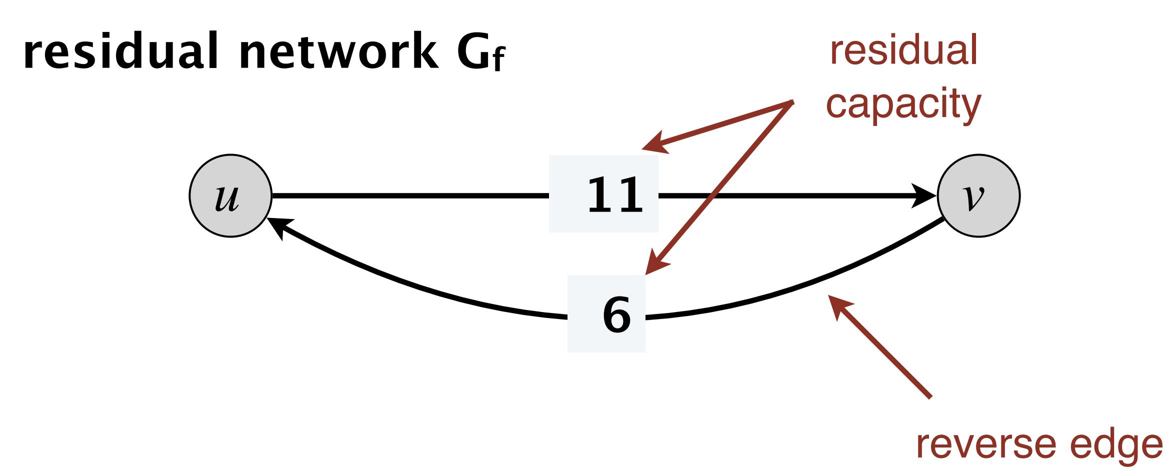
Ford Fulkerson: Idea

- Want to make “forward progress” while letting ourselves undo previous decisions if they’re getting in our way
- **Idea**: keep track of where we can push flow
    - Can push more flow along an edge with remaining capacity
    - Can also push flow “back” along an edge that already has flow down it
- Need a way to systematically track these decisions

Residual Graph

- Given flow network $G = (V, E, c)$ and a feasible flow f on G, **the residual graph** $G_f = (V, E_f, c_f)$ is defined as:
  - Vertices in G_f same as G
  - **(Forward edge)** For $e \in E$ with residual capacity $c_r = c(e) - f(e) > 0$ create $e \in E_f$ with capacity c_r
  - **(Backward edge)** For $e \in E$ with $f(e) > 0$, create $e_{\text{reverse}} \in E_f$ with capacity $f(e)$





Augmenting Path & Flow

- An **augmenting path** P is a simple $s \rightsquigarrow t$ path in the residual graph G_f
- The **bottleneck capacity** b of an augmenting path P is the minimum capacity of any edge in P.

```
AUGMENT(f, P)
---
b ← bottleneck capacity of augmenting path P.
FOREACH edge e ∈ P :
    IF (e ∈ E, that is, e is forward edge )
            Increase f(e) in G by b
    ELSE
            Decrease f(e) in G by b
RETURN f.
```

Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for each edge $e \in E$
- Find an $s \rightsquigarrow t$ path P in the residual network G_f
- Augment flow along path P
- Repeat until you get stuck

FORD–FULKERSON(G)

FOREACH edge $e \in E$: $f(e) \leftarrow 0$.

$G_f \leftarrow$ residual network of G with respect to flow f.

WHILE (there exists an s$\rightsquigarrow$t path P in G_f)

  $f \leftarrow$ AUGMENT(f, P).

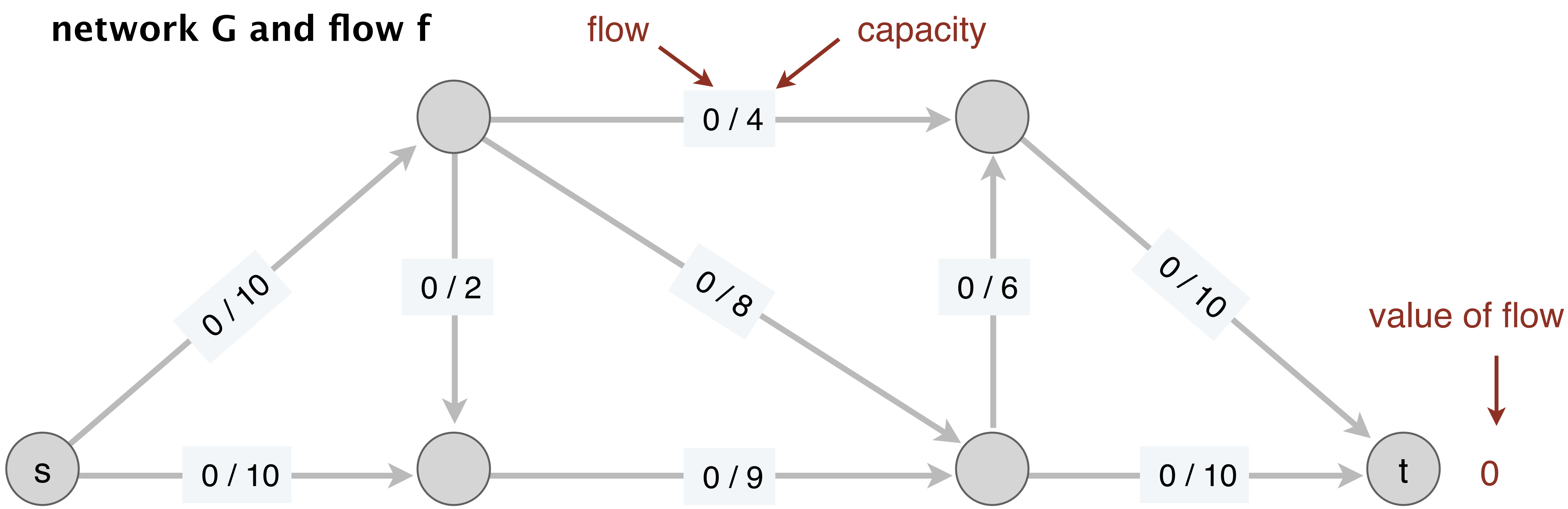
  Update G_f.

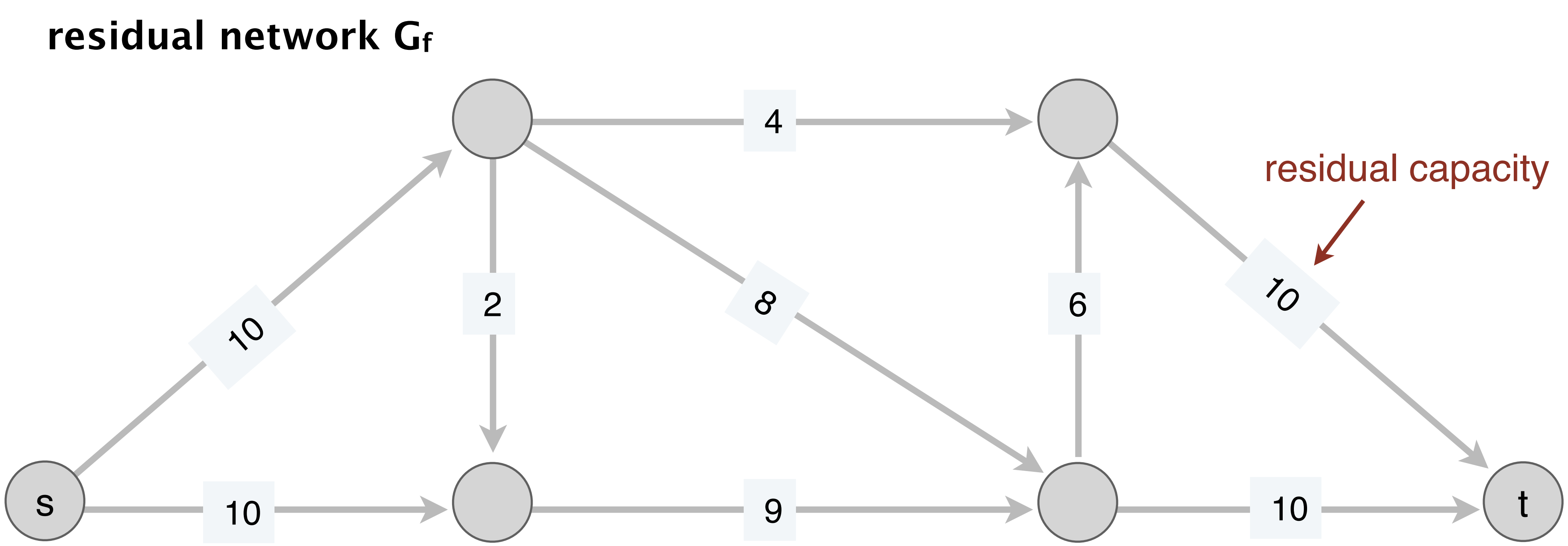
RETURN f.

Quick Check in

- Are we making forward progress here? (An augmenting path can "push flow back")
- Yes: cannot push flow back out of t or back into s
- Each augmenting path always increases the flow out of s and into t by at least 1 unit

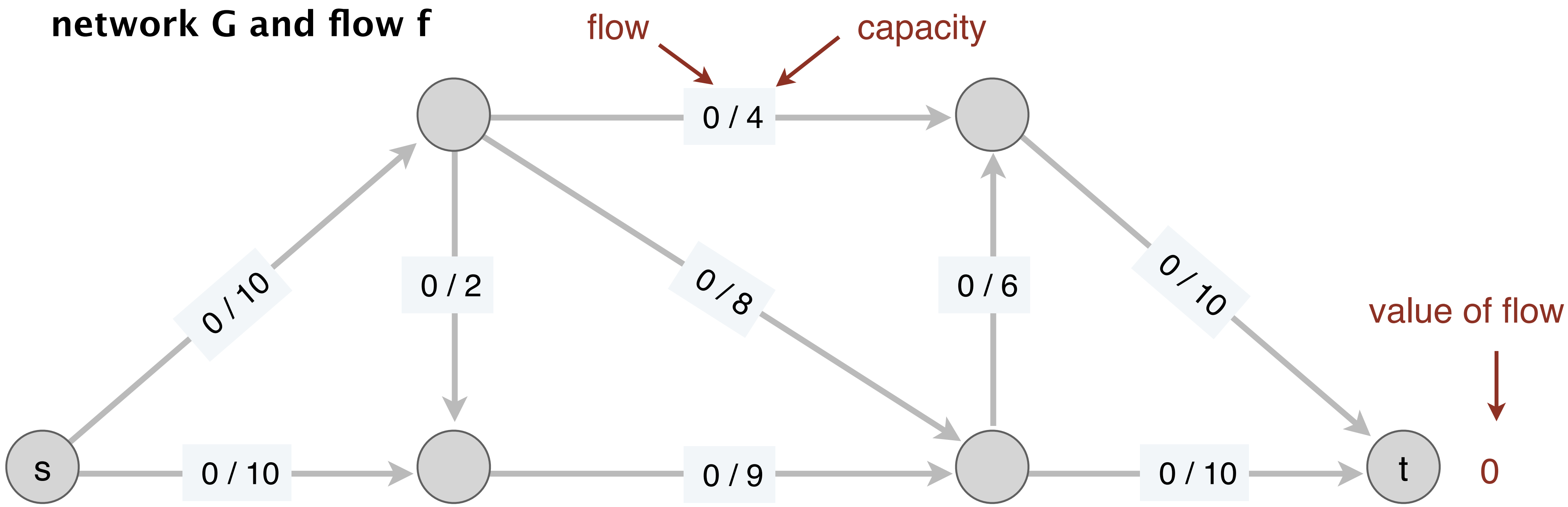
Ford-Fulkerson Example



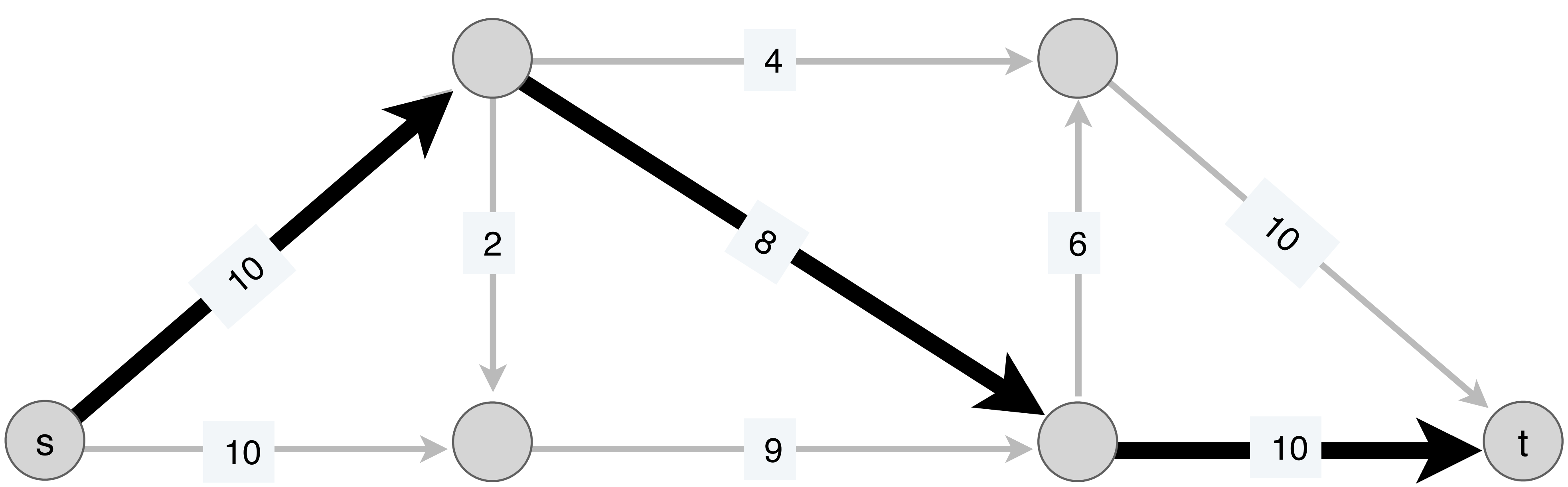


Ford-Fulkerson Example

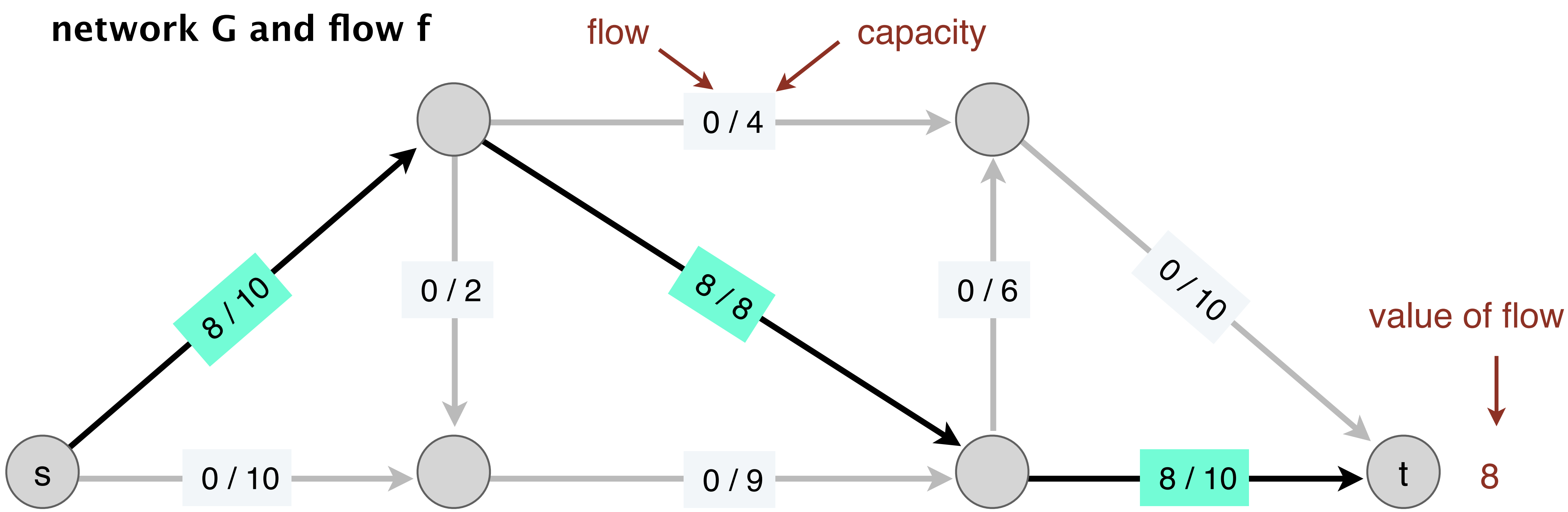
network G and flow f

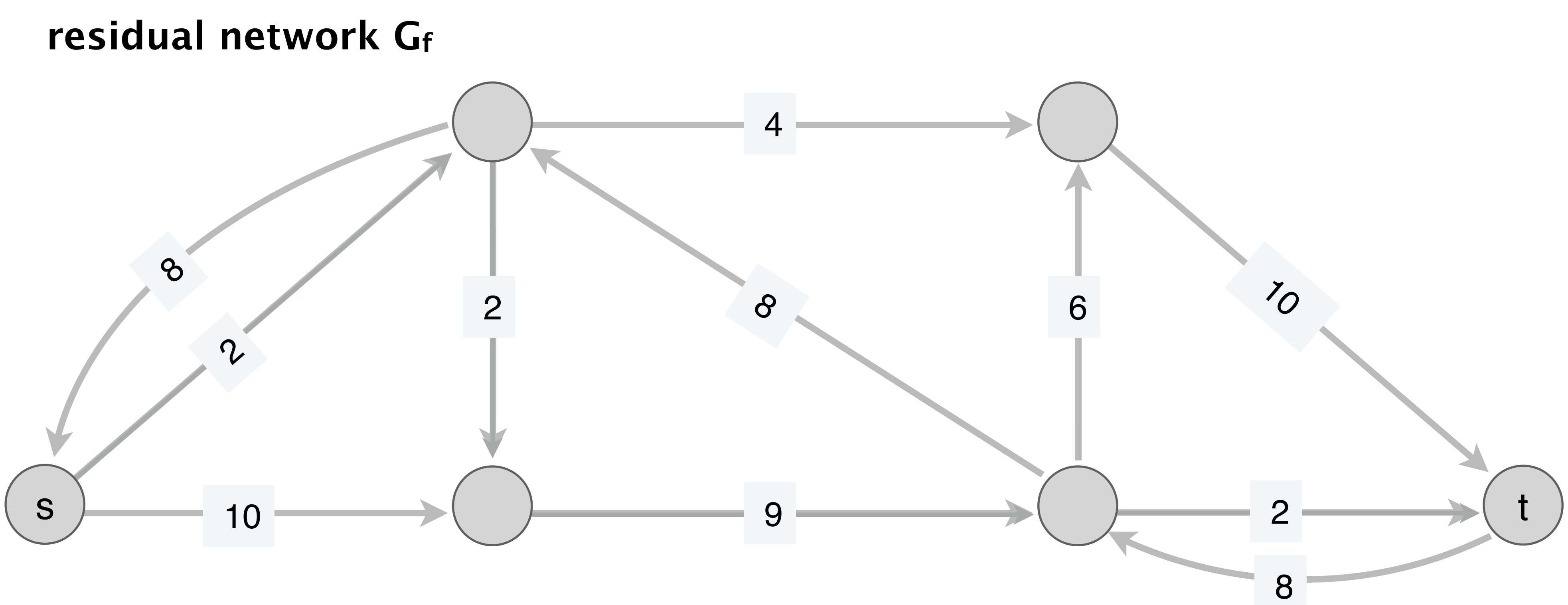


P in residual network G_f

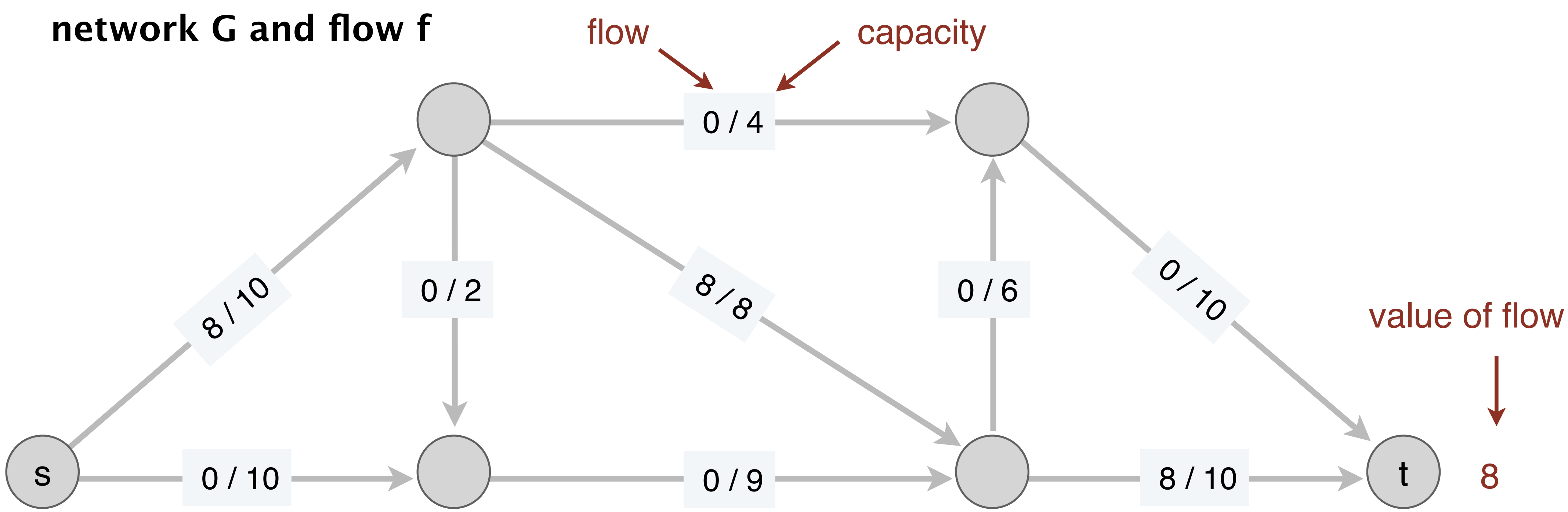


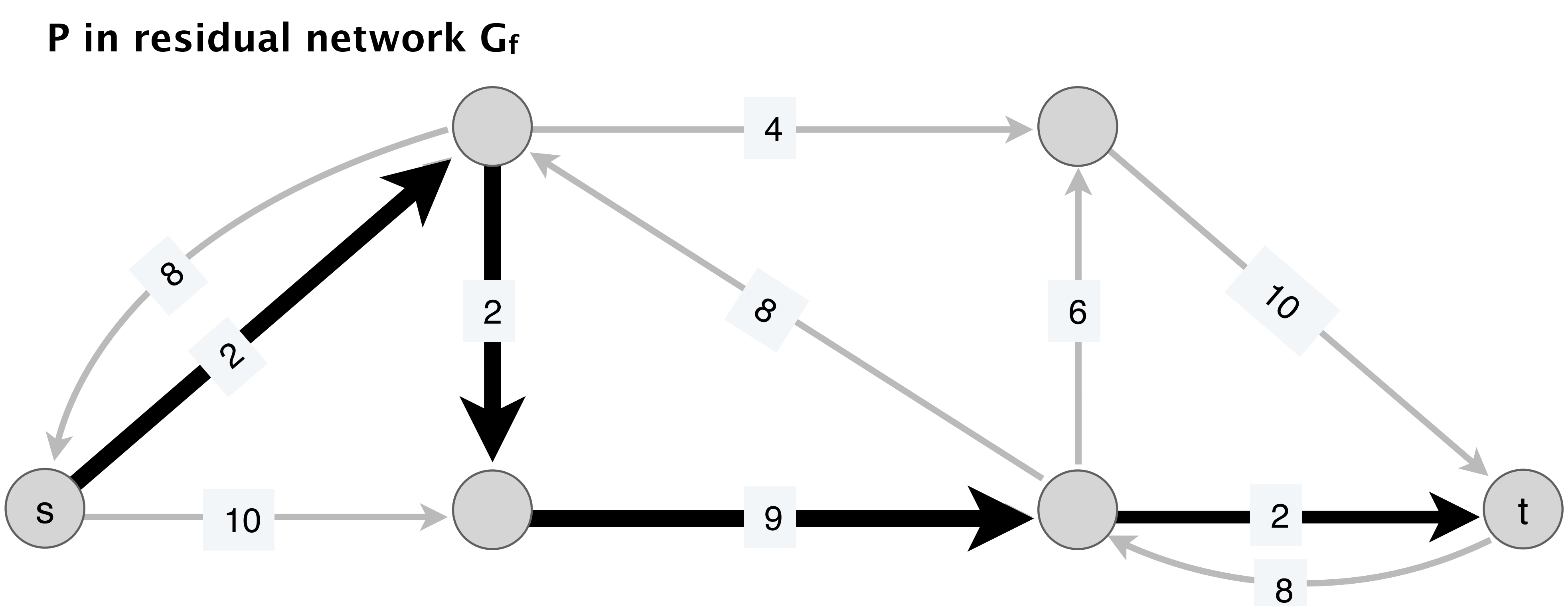
Ford-Fulkerson Example





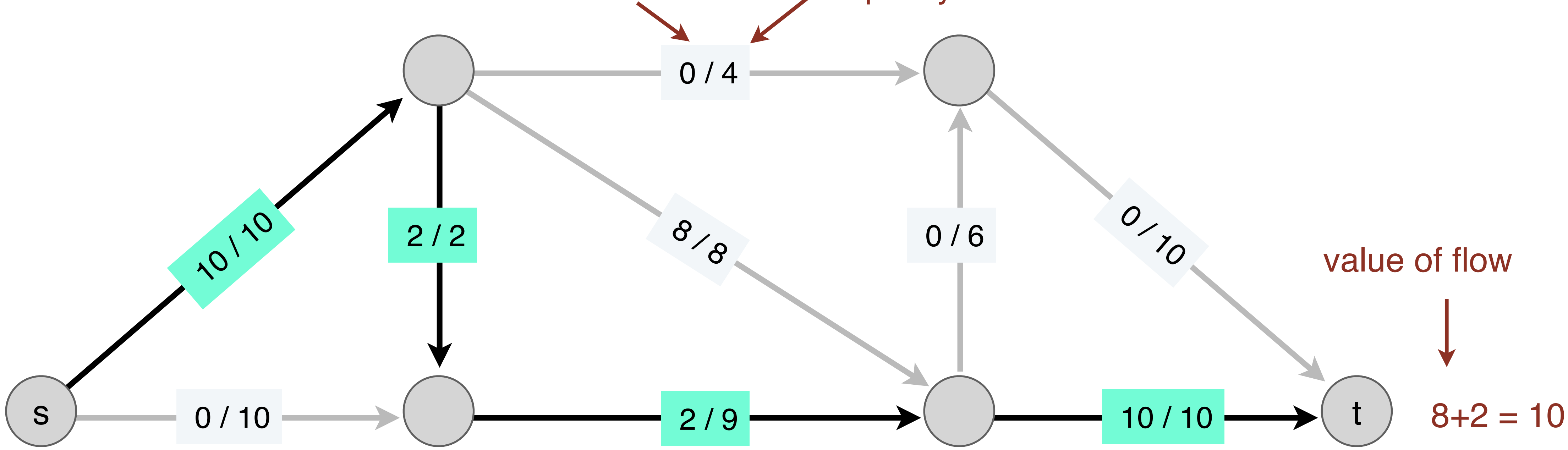
Ford-Fulkerson Example



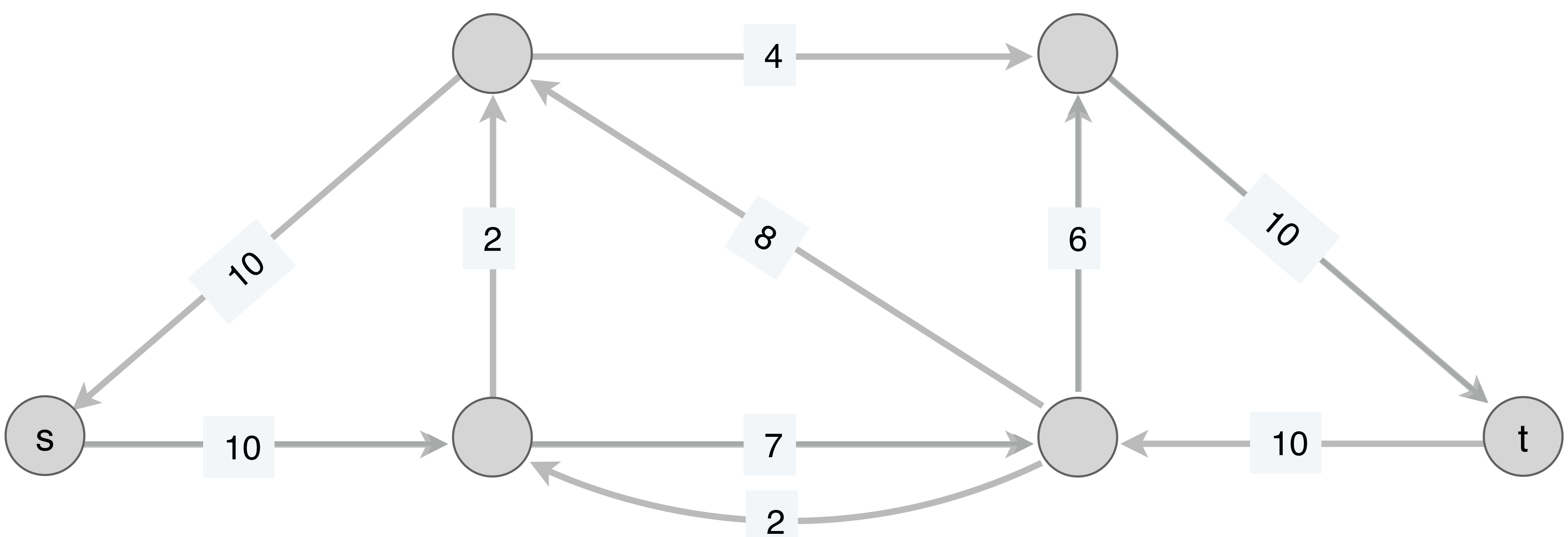


Ford-Fulkerson Example

network G and flow f

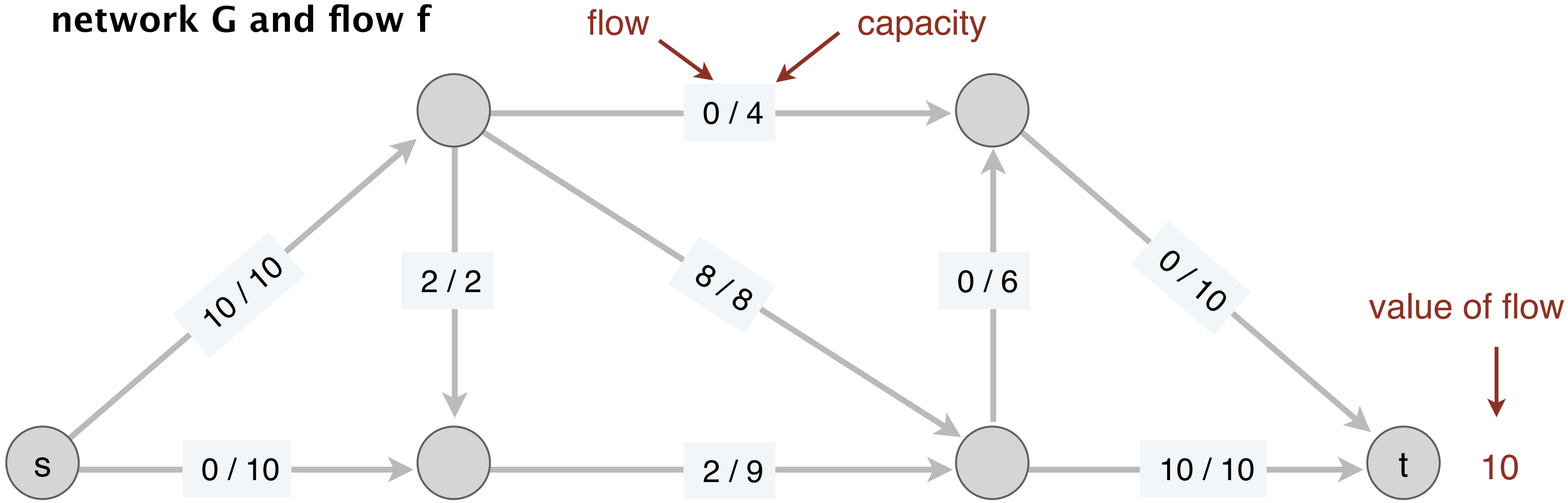


residual network G_f

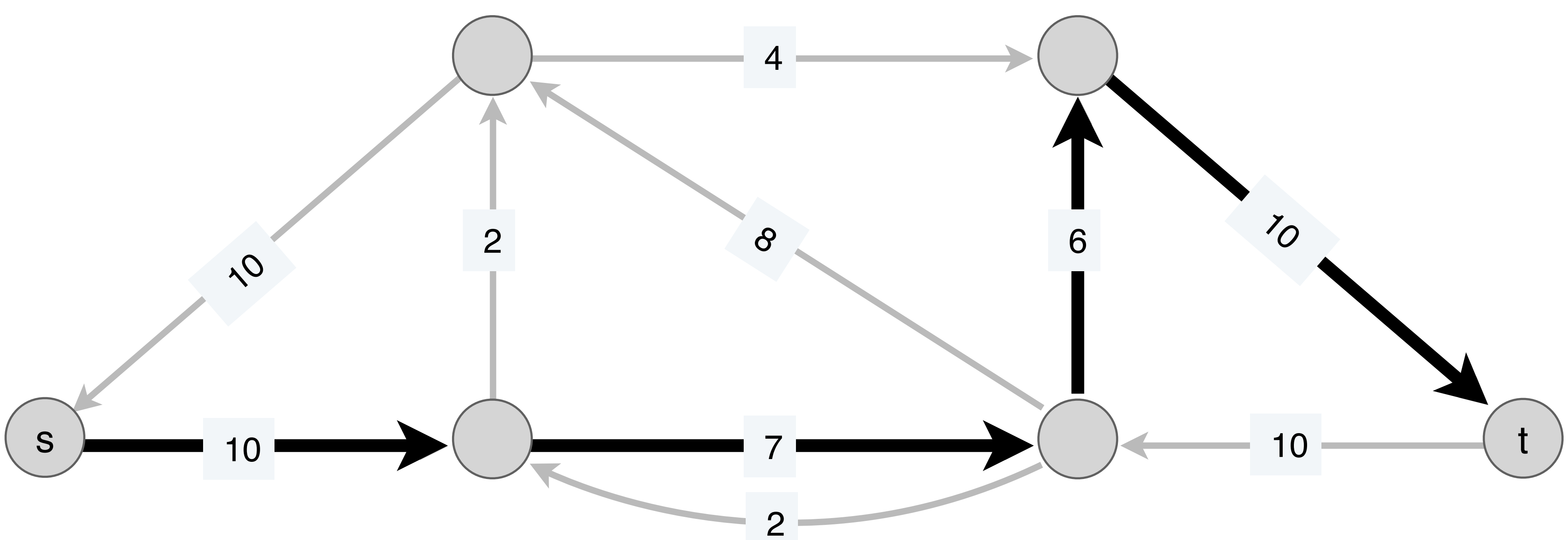


Ford-Fulkerson Example

network G and flow f

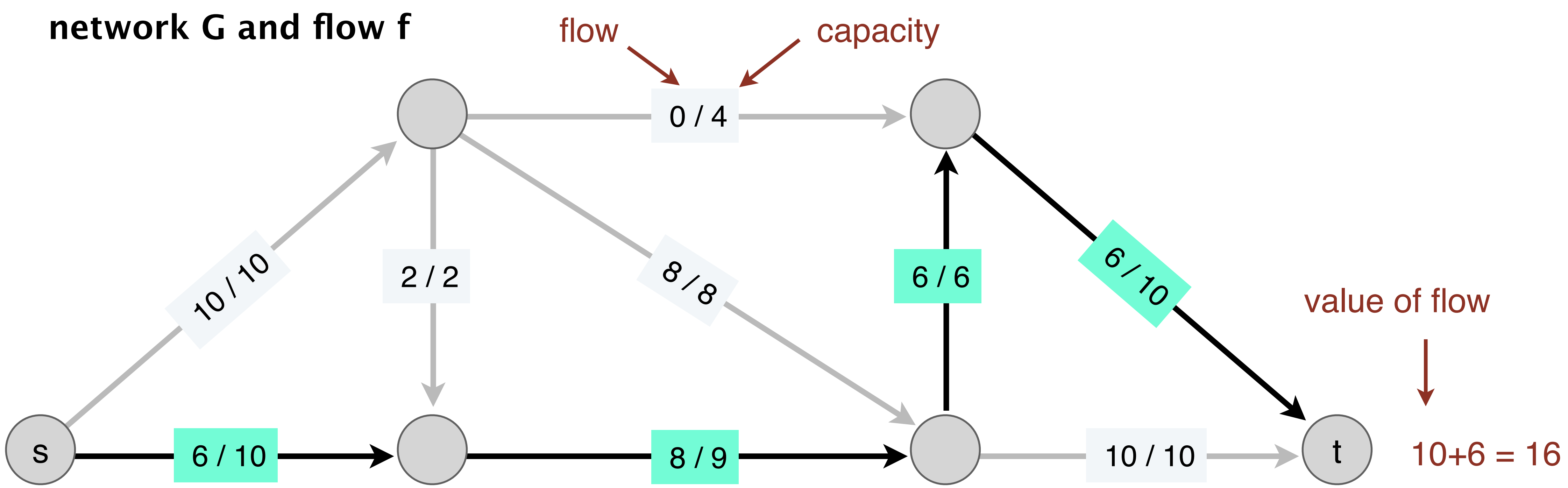


P in residual network G_f



Ford-Fulkerson Example

network G and flow f



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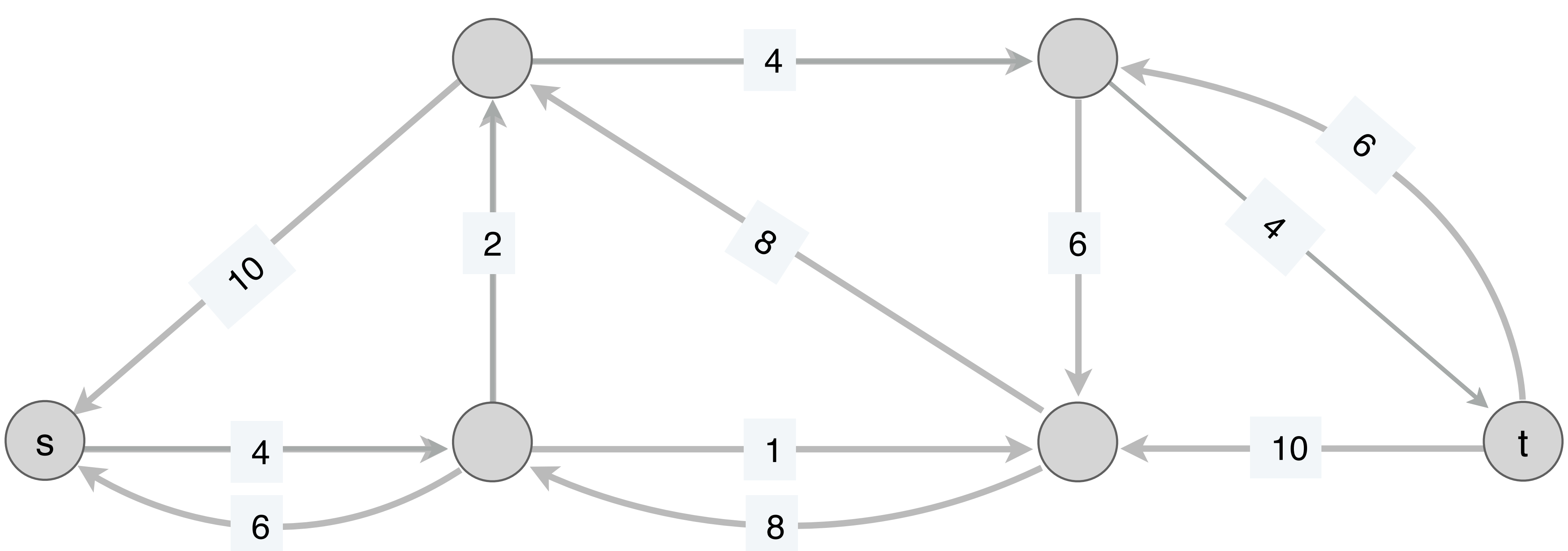
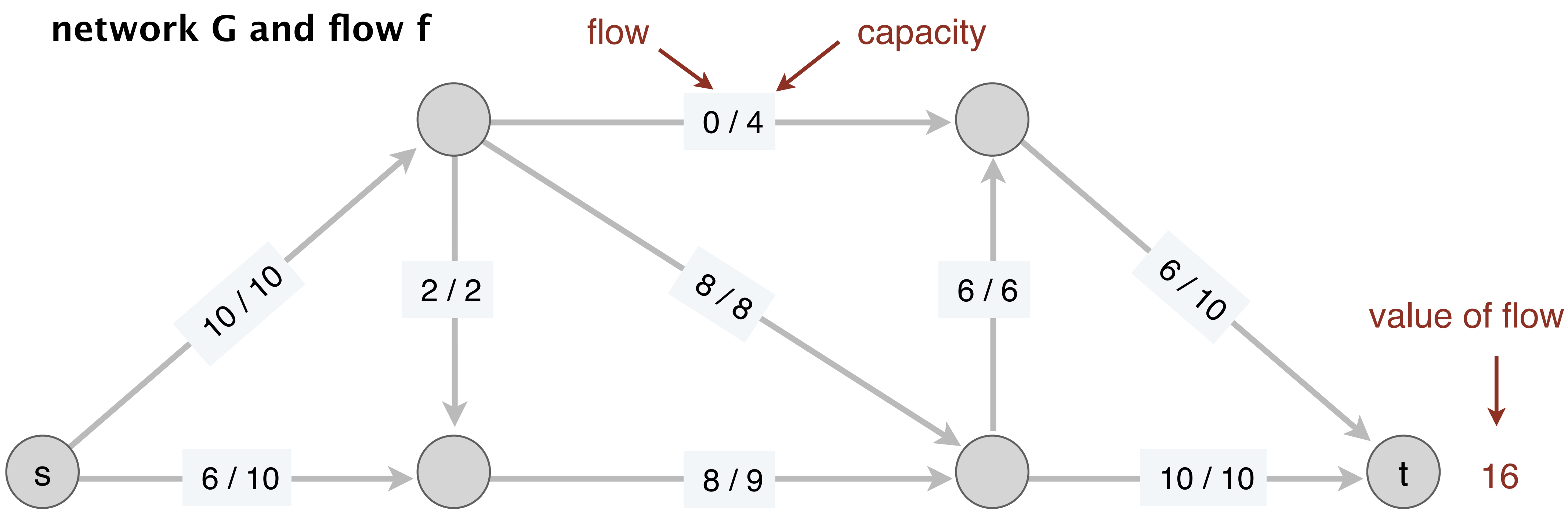
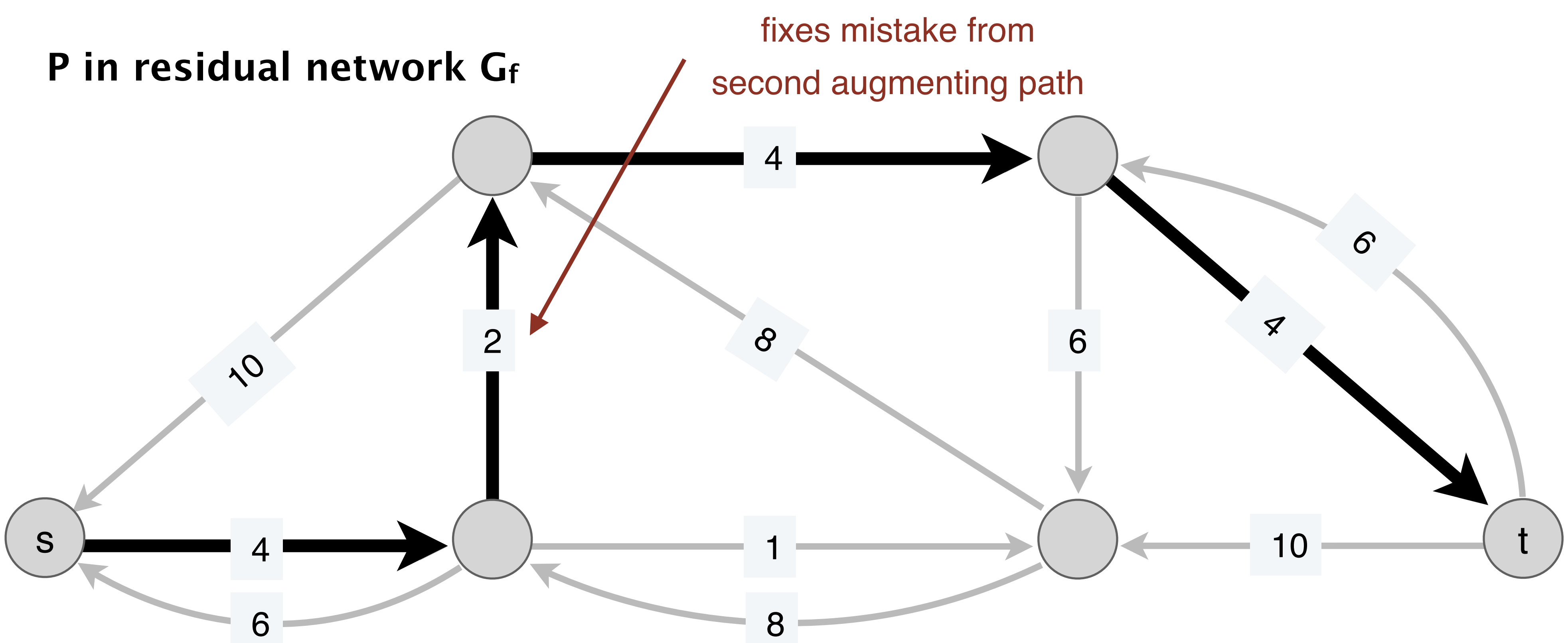


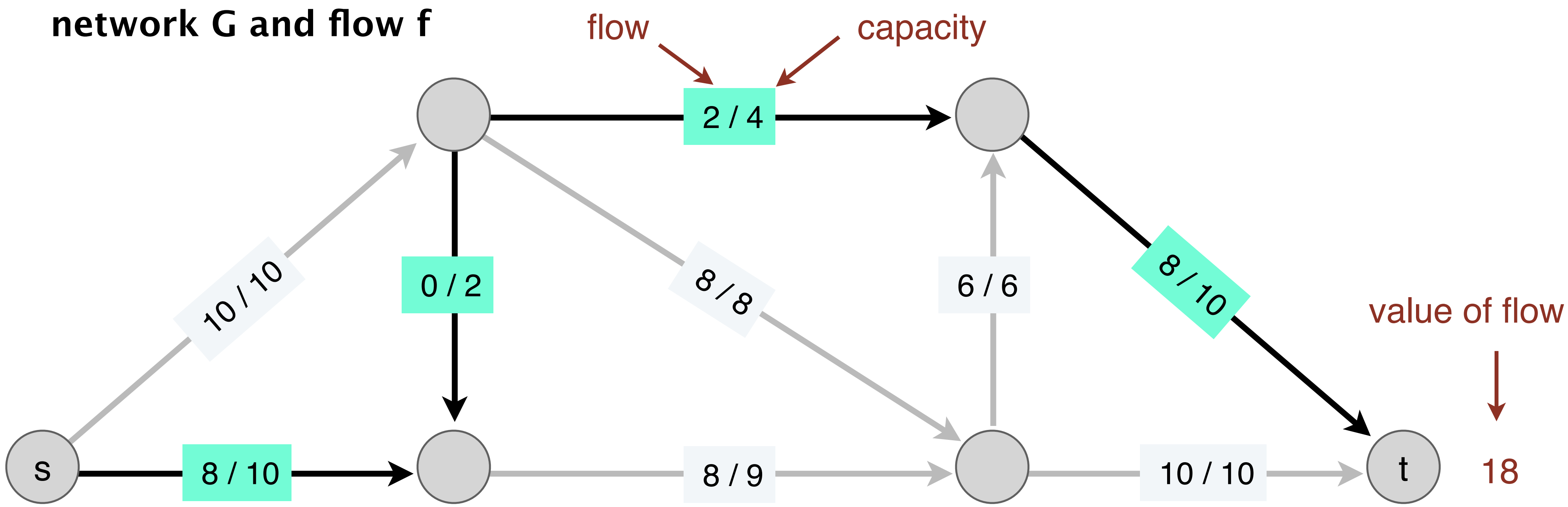
Ford-Fulkerson Example



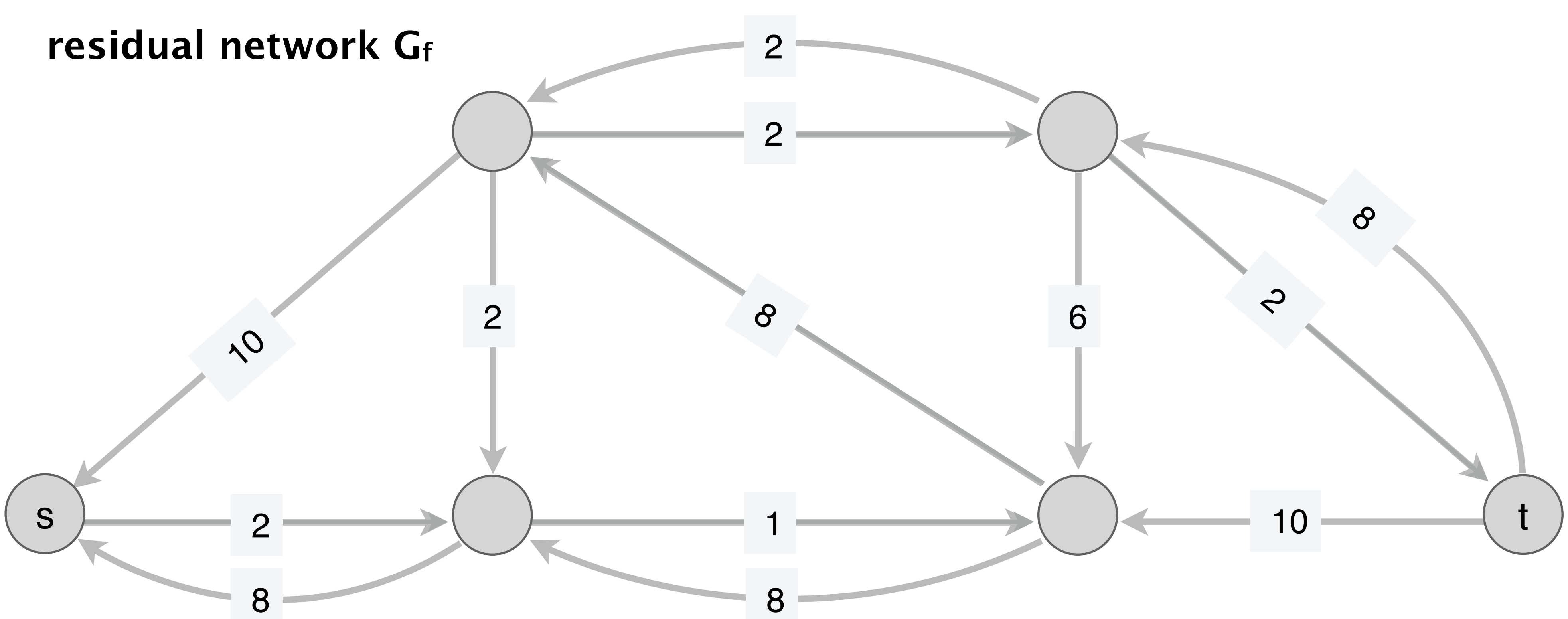


Ford-Fulkerson Example

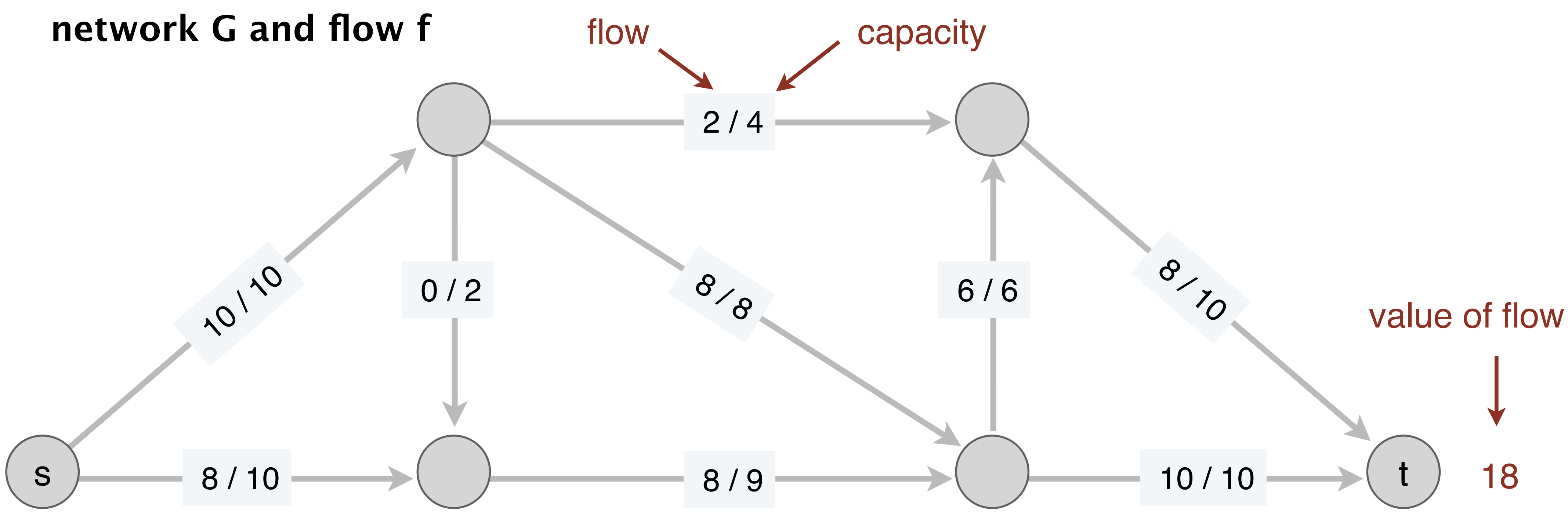
network G and flow f

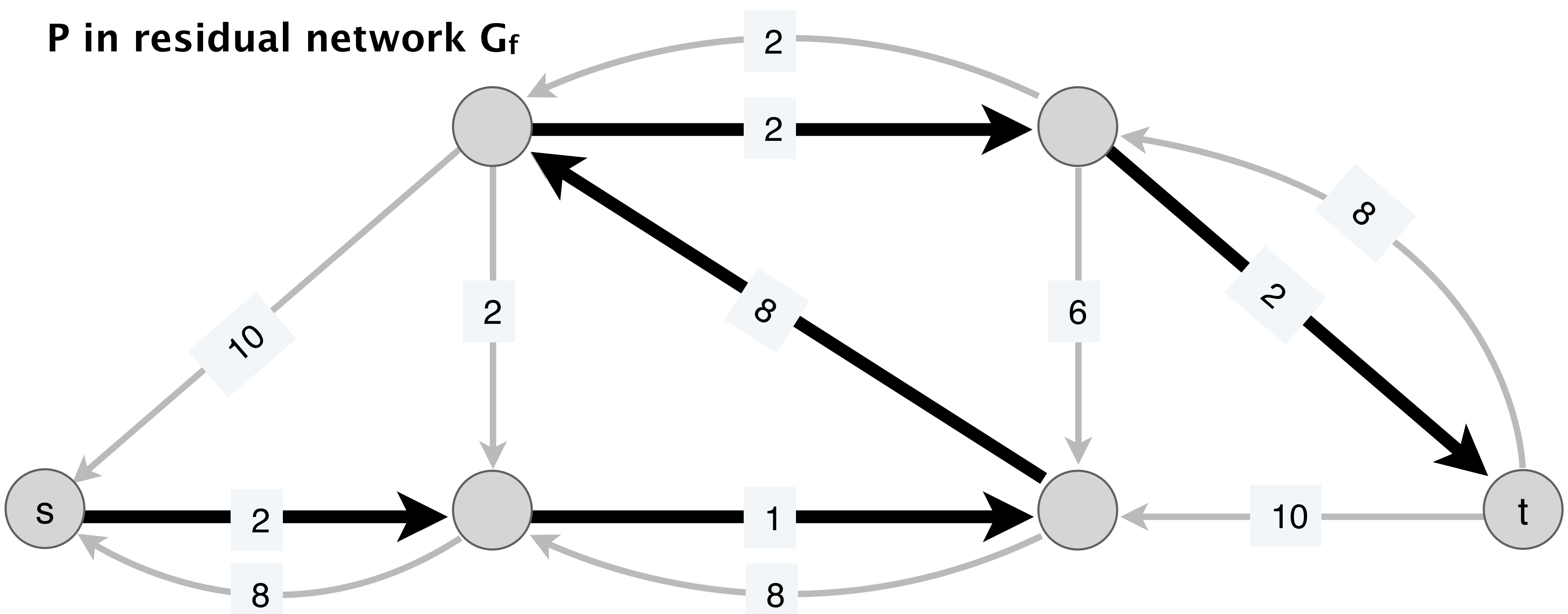


residual network G_f

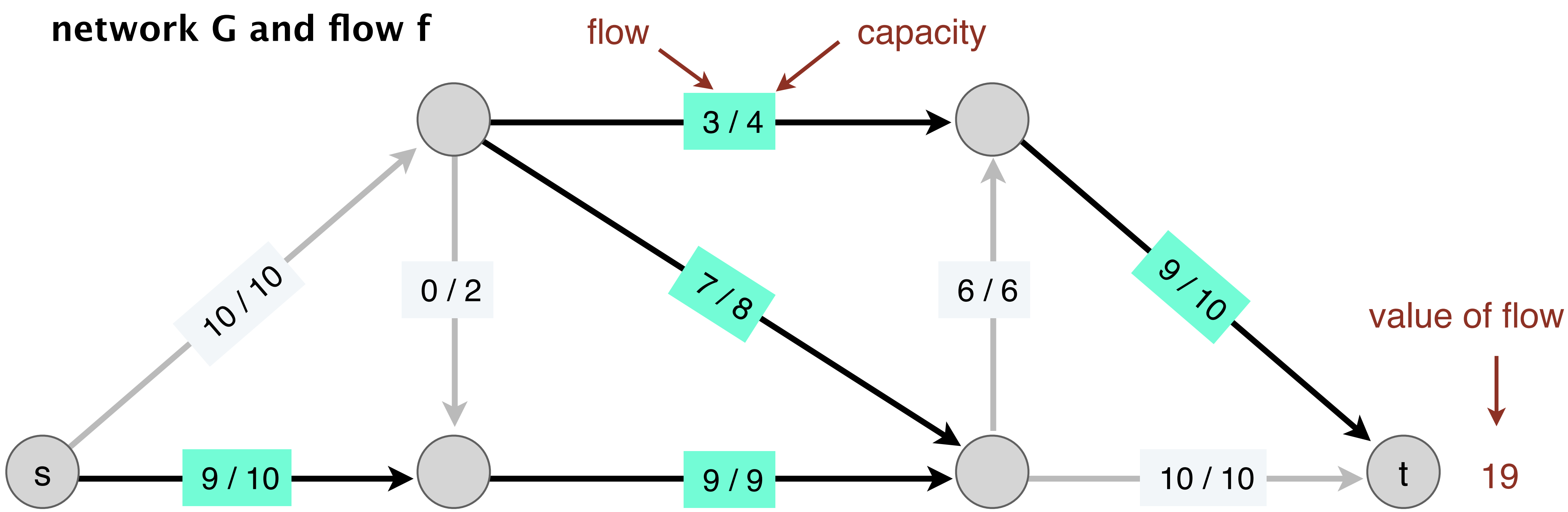


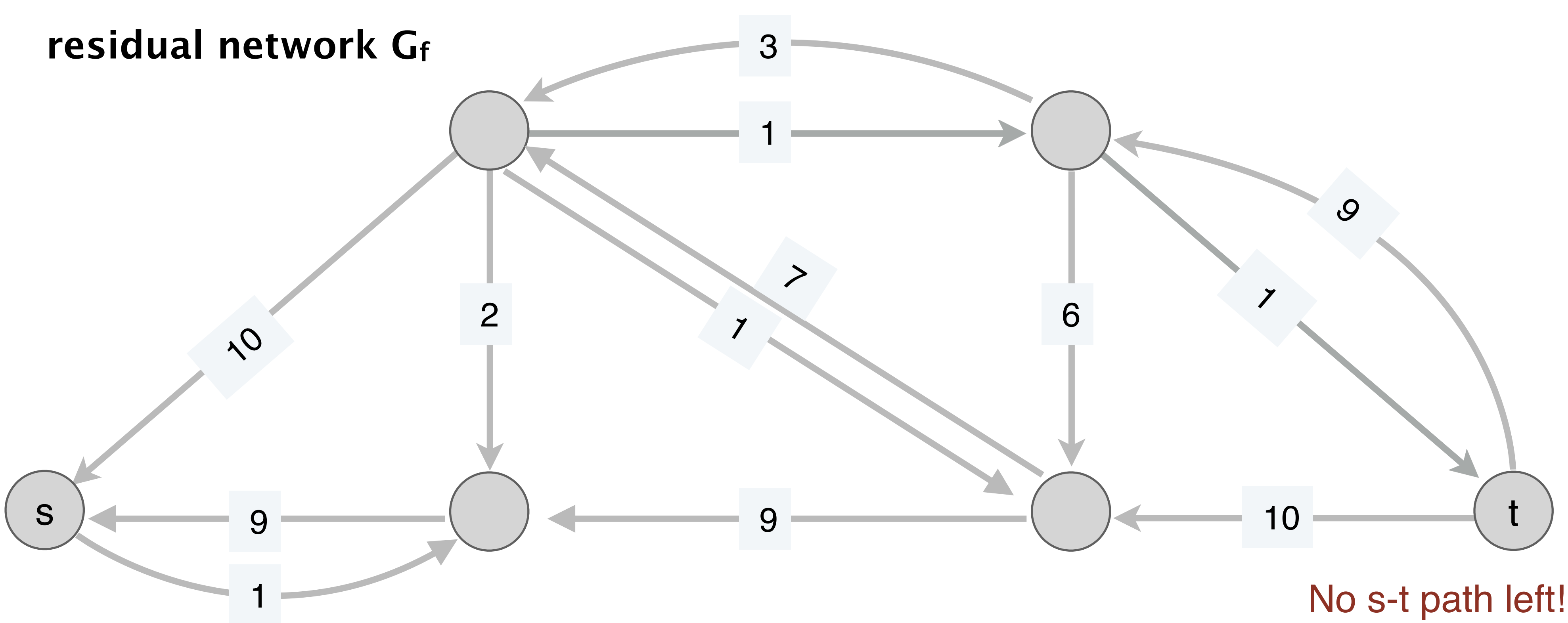
Ford-Fulkerson Example



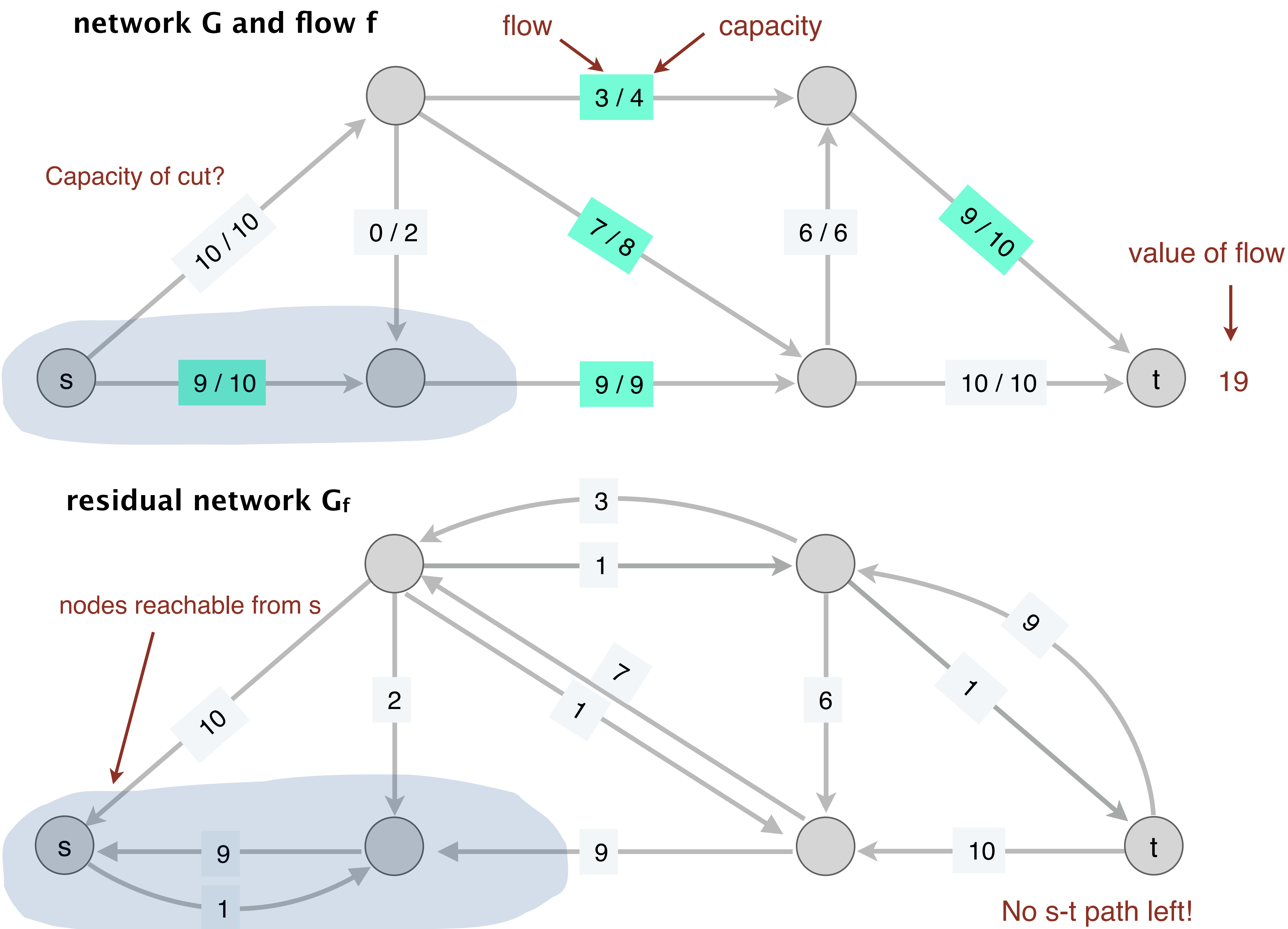


Ford-Fulkerson Example





Ford-Fulkerson Example



Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)