CS 357: Algorithmic Game Theory **Spring 2022**

Assignment 5 (due 03/09/2022)

Instructor: Shikha Singh *Solution Template*

Note. This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of the answers. Reasoning must be provided with every answer, i.e., please show your work. All assignments are due **at 11 pm EST** on **Wednesday** March 9.

Bayes Nash and Revenue Equivalence

Problem 1. (All-Pay auction) Consider a single-item all-pay auction, in which the highest bidder wins the item and every bidder must pay their bid. Assume that the values of each bidder are drawn i.i.d. from the uniform distribution on $[0, 1]$.

In lecture 9, we guessed that the strategy $s_i(v_i) = \frac{n-1}{n} v_i^n$ by every bidder i is a symmetric Bayes-Nash equilibrium. Verify and prove that this is true.

Problem 2. (kth-price auction) Consider a single-item auction with n bidders where the highest bidder wins and pays the kth-highest bid, where $2 < k < n$. Myerson's lemma tells us that this auction is revenue equivalent to the DSIC second-price auction.

Solve for the symmetric Bayes' Nash equilibrium strategies in the kth-price auction when the agent values are drawn i.i.d. from $U(0, 1)$ using the approach we discussed in Lecture 9. Notice that the bidders bid *above* their value at equilibrium!

For this part, you only need solve for and state the BNE bidding strategy—no need to verify it is a BNE.[1]

VCG in General Auctions

The *Vickrey Charles Grove* (VCG) mechanism, described below, is a DSIC mechanism for any *multi-parameter* environment (where bidders may have different valuations for different allocations). Let A be the set of all feasible allocations.

- Collect sealed bids $\mathbf{b}_1, \ldots, \mathbf{b}_n$, where $\mathbf{b}_i$ is a vector (for all possible allocations A).
- Find the surplus maximization allocation $\mathbf{a}^*$, that is, $\mathbf{a}^*(\mathbf{b}) = \text{argmax}_{\mathbf{a} \in A} \sum_{i=1}^{n} \mathbf{b}_i(\mathbf{a})$.
- Charge each bidder their **externality**, that is, the surplus loss inflicted by their presence:

$$p_i(\mathbf{b}) = \underbrace{\text{argmax}_{\mathbf{a}_{-i} \in A_{-i}} \sum_{j \neq i} \mathbf{b}_j(a_{-i})}_{\text{without } i} - \underbrace{\sum_{j \neq i} \mathbf{b}_j(\mathbf{a}^*)}_{\text{with } i}$$

.

[1] You can verify your guess is indeed a symmetric Bayes' Nash equilibrium on your own, but since the process is identical to Problem 1, you do not need to turn this in for credit.

Problem 3. Consider the VCG mechanism in a *combinatorial auction* with two items A and B, where the possible allocation that can be given to a single bidder are AB (both items), A, B or $\emptyset$ (no item). Suppose we have 2 bidders whose valuations for each allocation as:

	$\emptyset$	A	B	AB
Bidder 1	0	2	0	2
Bidder 2	0	0	0	2

That is, bidder 1 only cares about receiving A and bidder 2 only cares about receiving the combination of A and B.

(a) What is the revenue generated by the VCG mechanism for this auction?

(b) One of the problems with the VCG mechanism is that it may generate worse revenue when the competition increases. Show that adding a third bidder to this example (with appropriate valuations) may actually decrease the revenue generated by the mechanism.

(c) Does this problem occur in a single-item Vickrey auction? Explain why or why not.