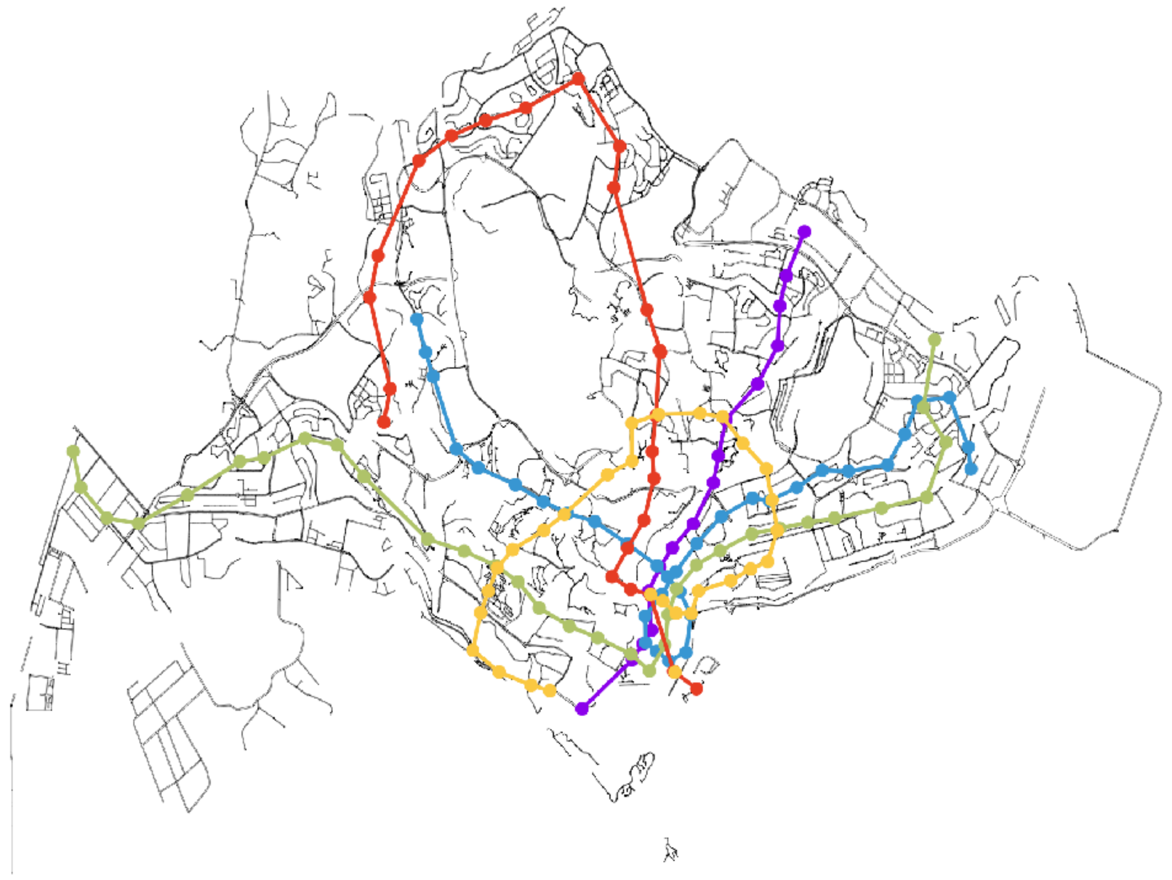
CSCI 357: Algorithmic Game Theory

Lecture 7: GSP & First Price Auctions

Shikha Singh

Announcements and Logistics

- **Assignment 4** out by and due Thurs 11 pm
    - Submit code via Github, latex answers and submit PDF
    - Assignment looks really long but a lot of it is just setup!
    - Based on lectures 6 and 7 on GSP vs VCG
- Feedback from HW 3:
    - Absorbing notations in AGT, esp auction theory can be a lot
        - Graduate level topic! Studying research from last two decades
    - Gets better in other topics of the course: promise!!!!
    - Happy to slow down, encourage interruptions and questions

Questions?

Last Time & Outline

- Wrapped up discussion on sponsored ad auctions
  - An example of how theory interacts with practice
- Talked briefly about first price auction and challenges
- This week: analyze first price auctions
  - Scratch the surface of Bayesian auction analysis
- Hope is to wrap up **direct-revelation auction design** this week!
- Next week is the last week on mechanism design with money:
  - Matching markets / ascending clock mechanisms
  - Application: spectrum auctions

Week 6: Matching Markets w/o Money

Week 5: Matching Markets w Money

Week 4: Bayesian Analysis & General Mechanism Design

Week 3: Application : Sponsored Ad Markets

Week 2: DSIC Auctions

Week 1: Game Theory

First Price Auctions

Bayesian Auction & Assumptions

- Game of incomplete information: bidders values (and thus utilities) are private
- No dominant strategy equilibrium, need to analyze using **Bayesian Nash**
- Assume bidders have **independent private value (IPV)** drawn independently and identically from the distribution G
    - We say values are drawn **i.i.d from** G
- The distribution G is **common knowledge**
    - Every bidder knows the distributions and knows that others know it as well
    - Often called "common prior"
- For first-price auction: we will further assume values are drawn **i.i.d from the uniform distribution** on $[0,1]$

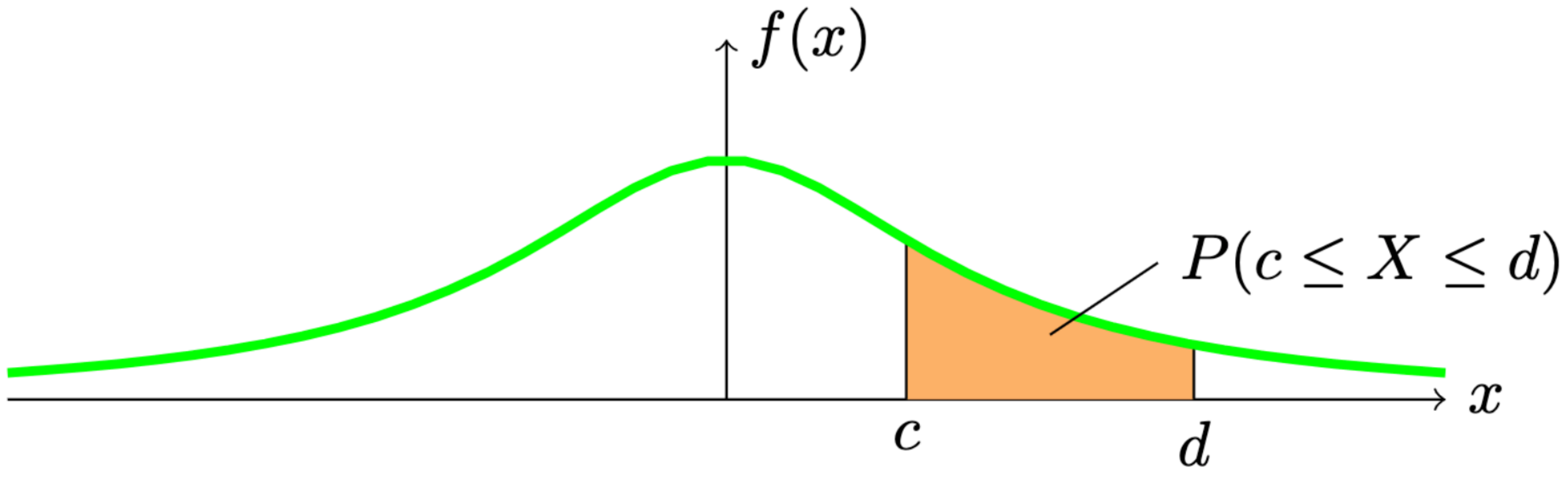
Continuous Probability Review

- A continuous random variable takes a range of values, which can be finite or infinite
- **(Definition)** A random variable X is continuous if there is a function $f(x)$ such that for any $c \leq d$ we have

$$\text{Pr}(c \leq X \leq d) = \int_c^d f(x)dx$$

- Function $f(x)$ is called the **probability density function** (pdf)

$$\boxed{P(c \leq X \leq d) = \text{area under the graph between } c \text{ and } d.}$$



Continuous Probability Review

- **(Definition)** The **cumulative distribution function (cdf)** F of a continuous random variable X denotes the probability that it is at most a certain value

$$F(k) = \Pr(X \leq k) = \int_{-\infty}^{k} f(x)dx$$

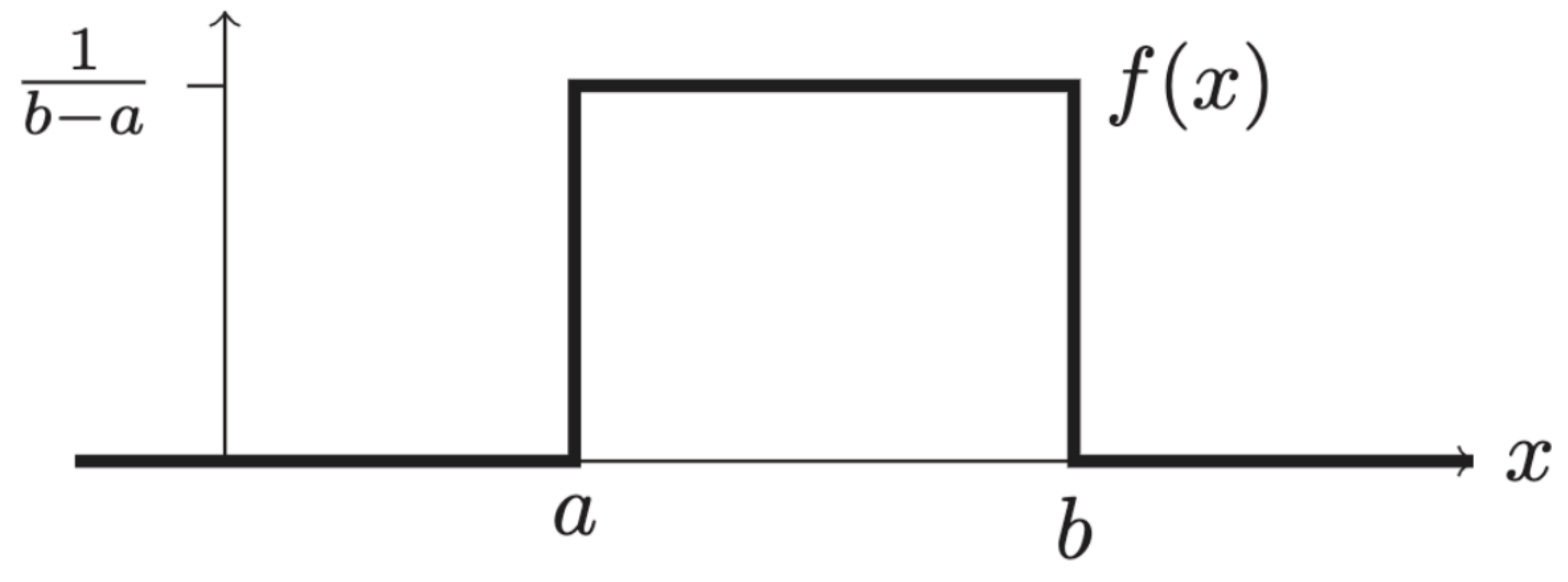
  where $f(x)$ is the probability density function of X

- In practice, we often say X has distribution or is drawn from distribution $F(x)$ rather than X has cumulative distribution function $F(x)$

Uniform Distribution

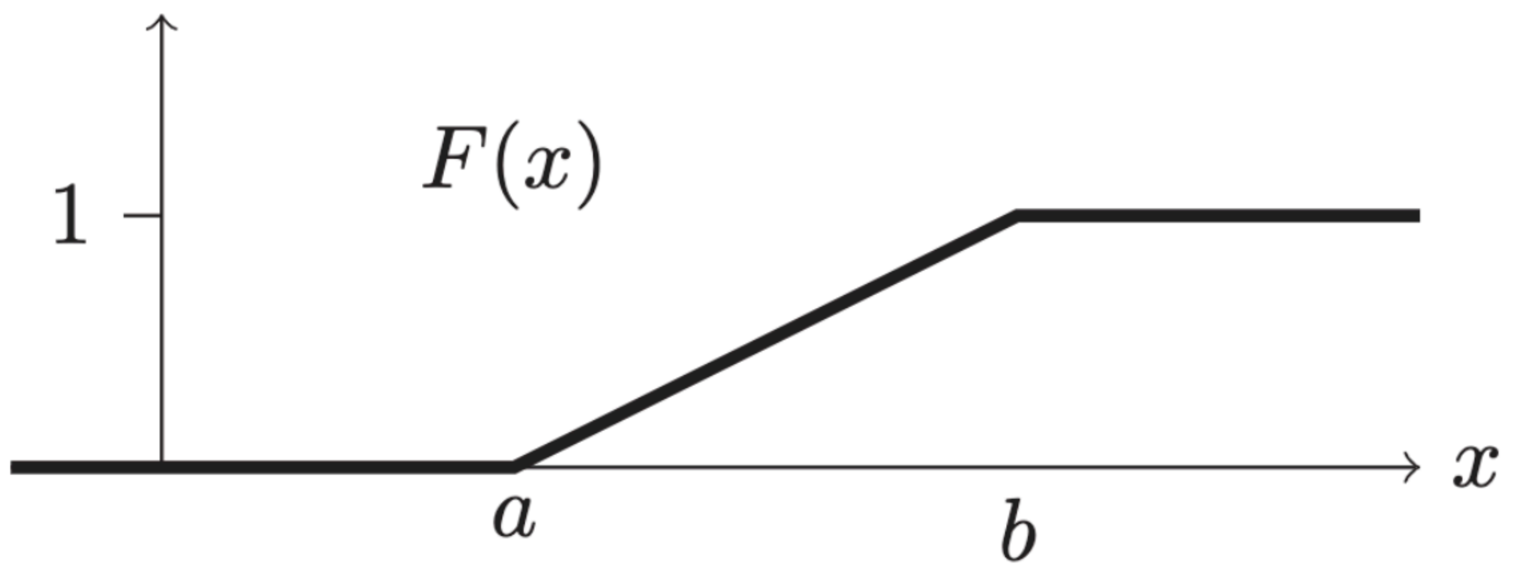
- Models situations where all outcomes in the range have equal probability
- Probability density function of a continuous uniform distribution on $[a, b]$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$



- Cumulative density function of a continuous uniform distribution on $[a, b]$

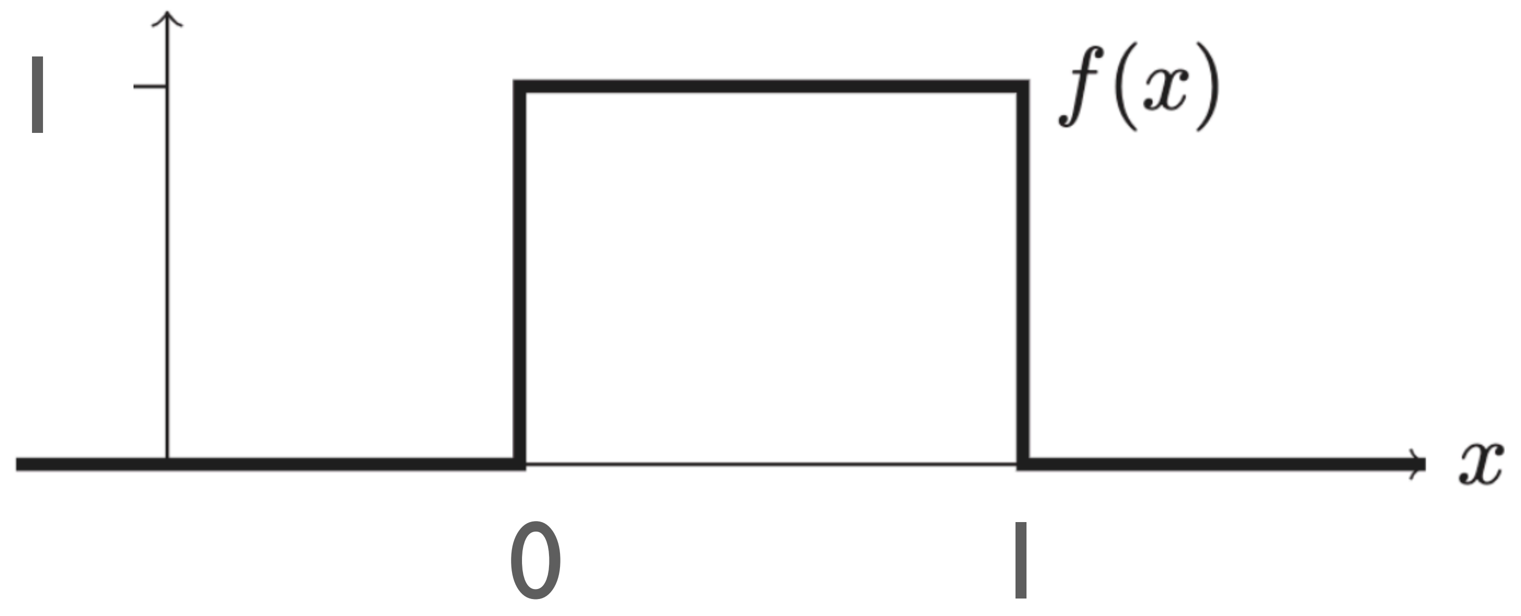
$$F(k) = \Pr(x \le k) = \begin{cases} 0 & \text{if } k \ge 0 \\ \frac{k-a}{b-a} & \text{if } a \le k \le b \\ 1 & \text{if} k > b \end{cases}$$



Uniform Distribution on [0, 1]

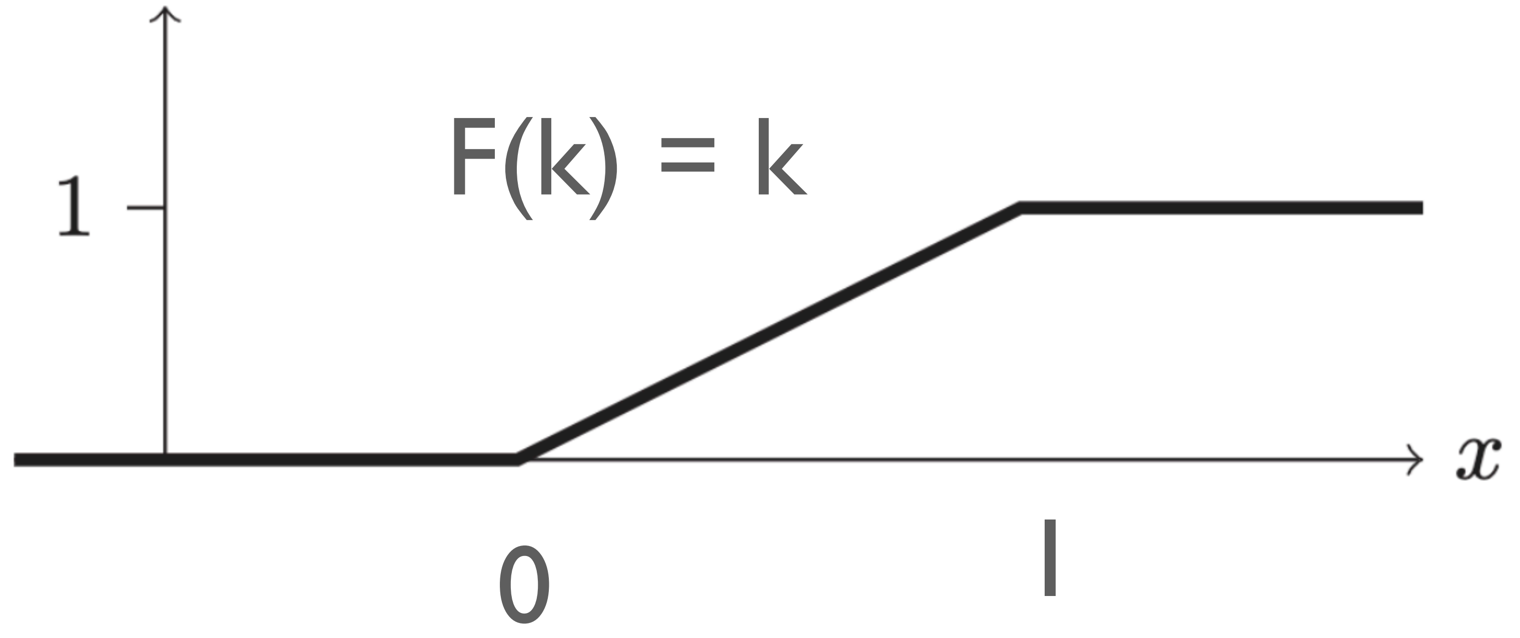
- Models situations where all outcomes in the range have equal probability
- Probability density function of a continuous uniform distribution on $[0,1]$

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



- Cumulative density function of a continuous uniform distribution on $[a, b]$

$$F(k) = \Pr(x \leq k) = \begin{cases} 0 & \text{if } k \geq 0 \\ k & \text{if } a \leq k \leq b \\ 1 & \text{if} k > b \end{cases}$$



Bayesian Nash Equilibrium

- A strategy or plan of action for each player (as a function of types) should be such that it maximizes each players expected utility
  - expectation is over the private values of other players
- Given a Bayesian game with independent private values v_{-i} , i's interim **expected utility** for a strategy profile $s = (s_1, \ldots, s_n)$ is

$$\mathbb{E}[u_i(s)] = \sum_{v_{-i}} u_i(s) \cdot \mathrm{Pr}(v_{-i})$$

- A strategy profile s is a **pure strategy Bayes Nash equilibrium** if no player can increase their interim expected utility by **unilaterally changing** their strategy s_i

Strategy Assumptions

- Recall: strategy s_i is a function that maps their value to their bid b:
  - $s_i(v_i) = b_i$
- We assume that the strategy of all bidders in the auctions we study
  - Is a strictly increasing differentiable function: gives us that the bidder with higher value will also provide a higher bid (no ties)
  - $s_i(v_i) \leq v_i$ for all v_i and bidders i: that is, bidders can "shade" down their bids but will never bid above their true values
    - Also implies $s_i(0) = 0$
- These assumptions are just to simplify analysis

First-Price Auction: Two Bidders

- Suppose v_1, v_2 are both drawn i.i.d. from the uniform distribution on $[0,1]$

$\Pr(b_1 < b_2)$

v_1 loses & pays 0

Utility: 0

$\Pr(b_1 \geq b_2)$

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- Suppose both bidders bid symmetrically some factor of their value $s(v_i) = \alpha \cdot v_i$

$\Pr(b_1 < b_2)$

$\Pr(v_2\alpha \leq b_1)$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- Suppose both bidders bid symmetrically some factor of their value $s(v_i) = \alpha \cdot v_i$

Pr$(b_1 < b_2)$

Pr$(v_2 \leq (b_1/\alpha))$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- Suppose both bidders bid symmetrically some factor of their value $s(v_i) = \alpha \cdot v_i$

Pr$(b_1 < b_2)$

Pr $= b_1/\alpha$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- $\mathbb{E}[u_1] = (v_1 - b_1)(b_1/\alpha)$: how to set b_1 to maximize expected payment?

$\Pr(b_1 < b_2)$

$\Pr = b_1/\alpha$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- $\mathbb{E}'[u_1] = (1/\alpha)(v_1 - 2b_1) = 0$, that is, $b_1 = v_1/2$

$\Pr(b_1 < b_2)$

$\Pr = b_1/\alpha$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- **Theorem.** Assume two bidders with their values drawn i.i.d. from uniform distribution on $[0,1]$, then the strategy $s(v_i) = v_i/2$ is a symmetric Bayes Nash equilibrium of the sealed-bid first price auction.
- **Proof.** Assume agent 2 bids using $s(\,.\,)$, that is, $b_2 = v_2/2$
- We calculate agent 1's expected utility who has value v_1 and bid b_1
  - $E[u_1] = (v_1 - b_1) \cdot \Pr[1 \text{ wins with bid } b_1] \; +0 \cdot \Pr[1 \text{ loses with bid } b_1]$

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- We calculate agent 1's expected utility who has value v_1 and bid b_1
  - $E[u_1] = (v_1 - b_1) \cdot \text{Pr}[\text{1 wins with bid } b_1]$
    $= (v_1 - b_1) \cdot \text{Pr}[b_2 \leq b_1]$

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    $= (v_1 - b_1) \cdot \Pr[v_2/2 \leq b_1]$

    $= (v_1 - b_1) \cdot \Pr[v_2 \leq 2b_1]$

Here v_1, b_1 are fixed and v_2 is a random variable

First-Price Auction: Two Bidders

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$$= (v_1 - b_1) \cdot \text{Pr}[v_2/2 \leq b_1]$$

$$= (v_1 - b_1) \cdot \text{Pr}[v_2 \leq 2b_1]$$

$$= (v_1 - b_1) \cdot F(2b_1) = (v_1 - b_1) \cdot 2b_1$$

First-Price Auction: Two Bidders

- **Proof (Cont).** Assume agent 2 bids using $s(\,.\,)$, that is, $b_2 = v_2/2$
- Agent 1's expected utility who has value v_1 and bid b_1 when she wins
  - $E[u_1] = (v_1 - b_1) \cdot 2b_1 \; = 2v_1b_1 - 2b_1^2$

First-Price Auction: Two Bidders

- **Proof (Cont).** Assume agent 2 bids using $s(\,.\,)$, that is, $b_2 = v_2/2$
- Agent 1's expected utility who has value v_1 and bid b_1 when she wins
    - $E[u_1] = (v_1 - b_1) \cdot 2b_1 \; = 2v_1b_1 - 2b_1^2$
- Agent 1 with value v_1 should set b_1 to maximize $2v_1b_1 - 2b_1^2$ as a function of b_1
    - Differentiate and set derivate to zero (also check second order condition)

First-Price Auction: Two Bidders

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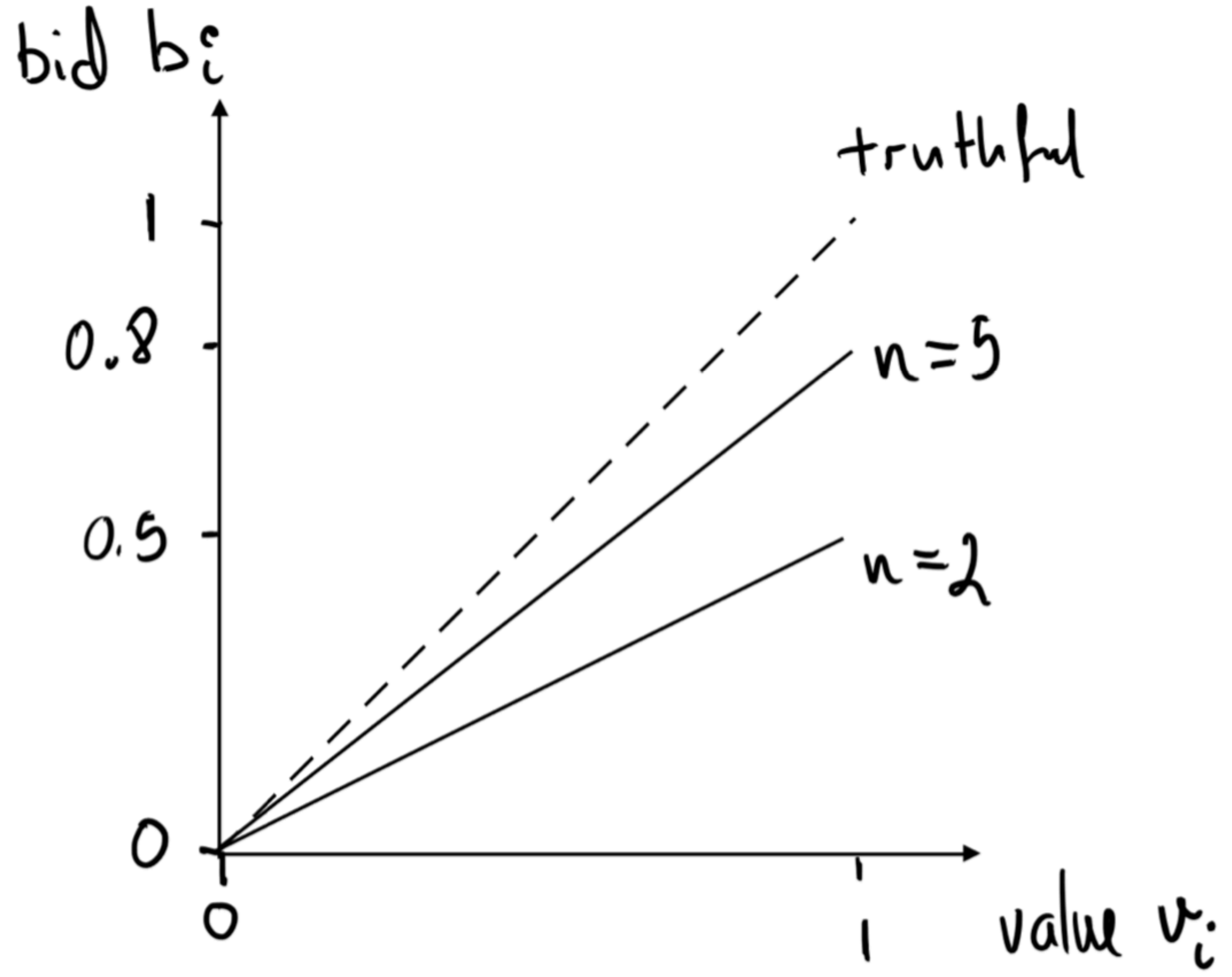
- Agent 1 with value v_1 should set b_1 to maximize $2v_1b_1 - 2b_1^2$ as a function of b_1
  - Differentiate and set derivate to zero (also check second order condition)
  - $E'[u_1] = 2v_1 - 4b_1 = 0$, that is, $b_1 = v_1/2$

The analysis is symmetric for agent 2 as well.

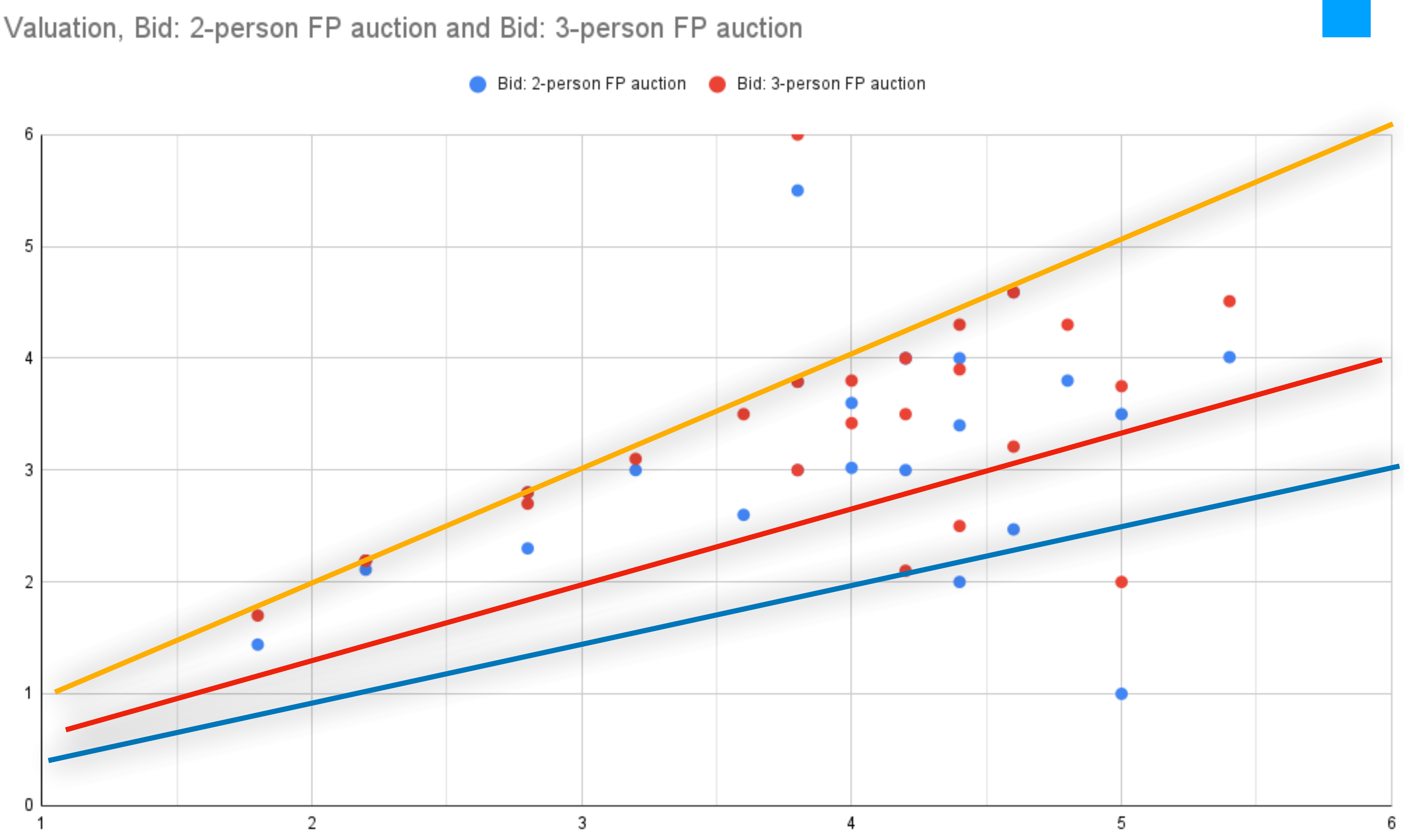
First-Price Auction: n Bidders

- Suppose we increase the number of bidders, how should the equilibrium strategy adjust to more competition?
- **Theorem.** Assume each of the n bidders have values drawn i.i.d. from uniform distribution on $[0,1]$. Then, the strategy $s(v_i) = \frac{n-1}{n} \cdot v_i$ is a symmetric Bayes Nash equilibrium of the sealed-bid first price auction.
- **Proof.** We can generalize the 2-bidder proof
  - On board. Also in Parkes and Seuven book.
- **Takeaway:** the more the competition, the more the bidders need to bid closer to their value if they want to win



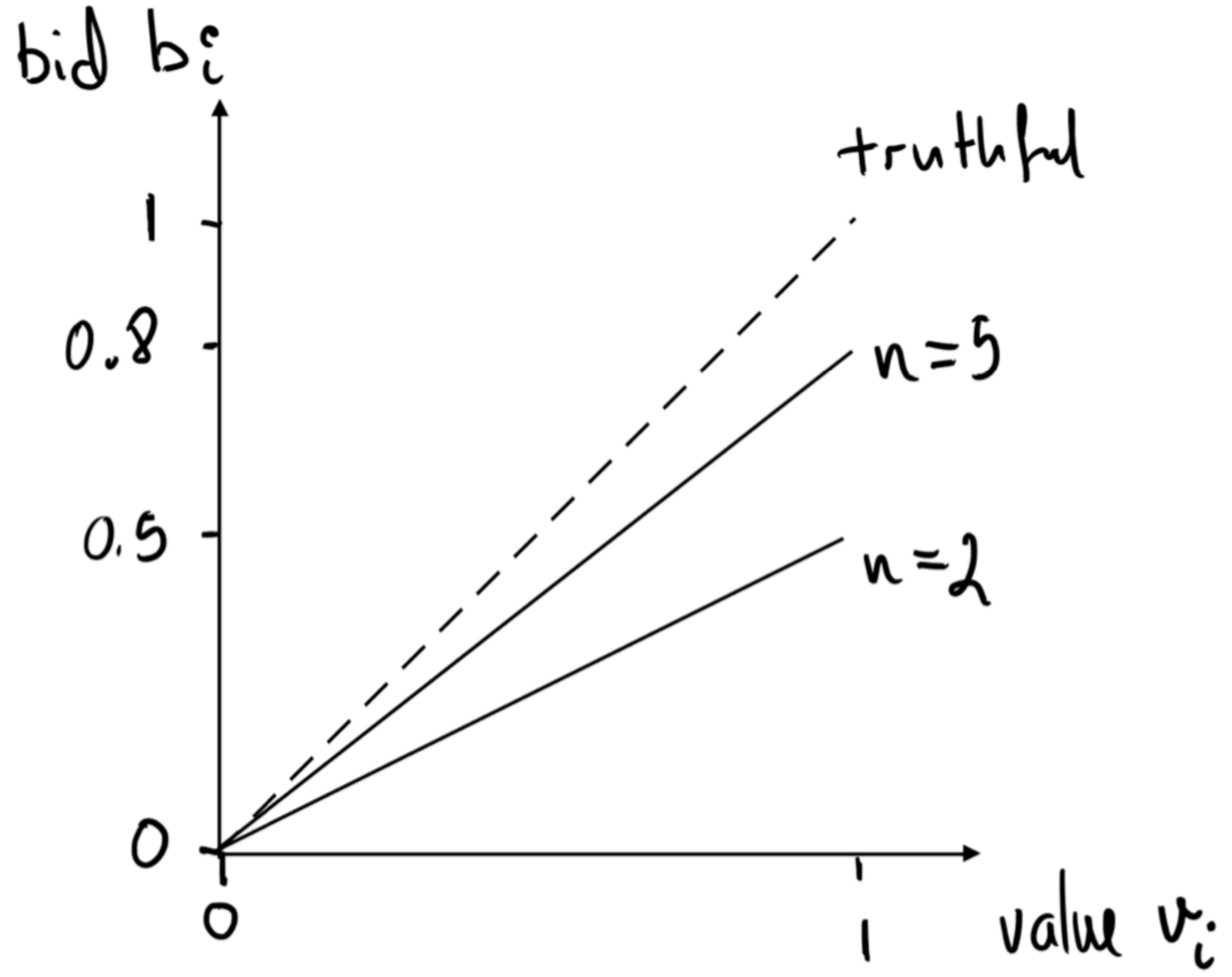
Empirical Bids vs Equilibrium

Truthful bids
3-person equilibrium
2-person equilibrium



First-Price Auction: Guarantees

- Turns out this Bayes Nash equilibrium **is unique**
    - Generalizes to arbitrary i.i.d distributions
- Is linear time
- Does it maximize surplus?
    - Bids in Bayes Nash equilibrium are order-preserving: that is, for values $v_1 \geq v_2 \geq \ldots \geq v_n$, the equilibrium bids are $b_1 \geq b_2 \geq \ldots \geq b_n$
    - The item is allocated to the highest bidder, thus to the agent with the maximum valuation
    - Maximizes surplus (at equilibrium)
- Now, we want to compare the revenue of FP and SP auction

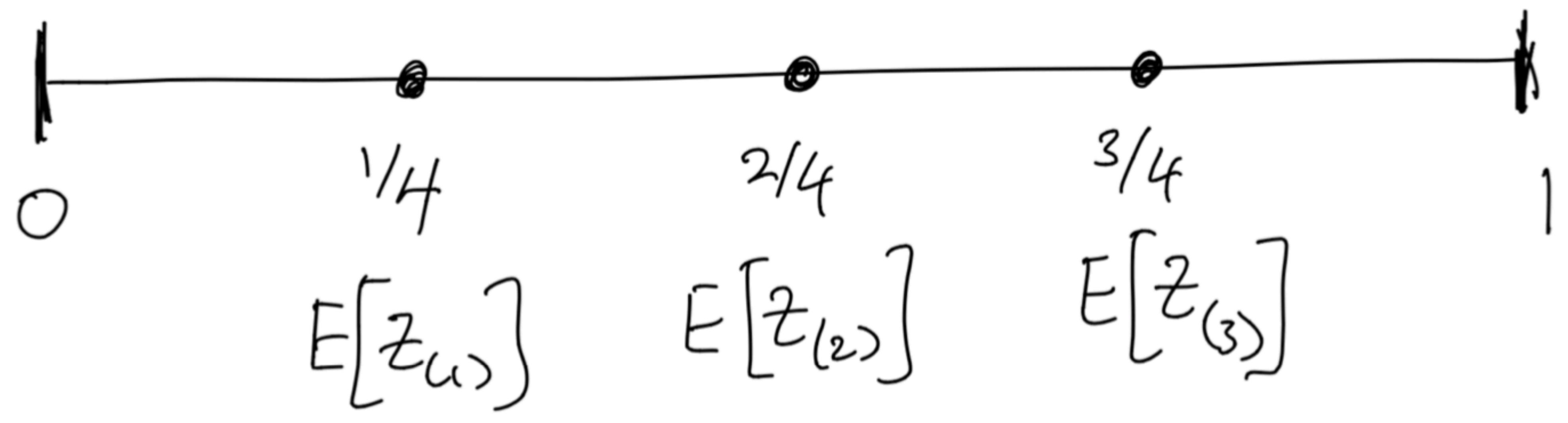


Order Statistics

- To do so, we need to define order statistics
- Let $X_1, X_2, \ldots X_n$ be n independent samples drawn identically from the uniform distribution on $[0,1]$
- The first-order statistic $X_{(1)}$ is the maximum value of the samples, the second-order statistic $X_{(2)}$is the second-maximum value of the samples, etc
- The expected value of the kth order statistic for n i.i.d samples from $U(a,\ b)$ is

$$E[X_{(k)}] = \frac{n-(k-1)}{n+1} \cdot (b-a)$$

- **Remember:** a uniform random variable evenly divides the interval it is over



Expected kth order statistic for 3 samples, uniform $[0,1]$

Revenue

- **Theorem.** If bidder's values are uniform i.i.d., then the expected revenue of the first-price auction is equal to that of the second-price auction at equilibrium.
- **Proof.** Let $E[R_1]$ and $E[R_2]$ be the expected revenues of the first and second-price auction.
- In second-price auction, the bidder with the highest value wins and pays second-highest value
    - $E[R_2] =$ expected value of second-order statistic $= \dfrac{n-1}{n+1}$
- In FP auction, bidders bid $s(v_i) = \dfrac{n-1}{n} \cdot v_i$ and highest bidder pays their bid

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- In FP auction, bidders bid $s(v_i) = \frac{n-1}{n} \cdot v_i$ and highest bidder pays their bid
- $E[R_1] = E[b_{\max}] = E\left[\frac{n-1}{n} \cdot v_{\max}\right]$

Revenue

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- **Proof.** Let $E[R_1]$ and $E[R_2]$ be the expected revenues of the first and second-price auction.
- In FP auction, bidders bid $s(v_i) = \dfrac{n-1}{n} \cdot v_i$ and highest bidder pays their bid
- $E[R_1] = E[b_{\max}] = E\left[\dfrac{n-1}{n} \cdot v_{\max}\right] = \dfrac{n-1}{n} E[v_{\max}] = \dfrac{n-1}{n} \cdot \dfrac{n}{n+1} = \dfrac{n-1}{n+1}$
- The last step uses linearity of expectation
  - $E(a \cdot X + b \cdot Y) = a \cdot E(X) + b \cdot E(Y)$ where a and b are constants

$\blacksquare$

Myerson's Lemma: DSE vs BNE

- Remember all DSE are BNE but not vice versa
- When characterizing DSE, the game was deterministic and so we can talk about the actual allocation and payment
- When characterizing BNE, it is important that $x_i(v_i)$ and $p_i(v_i)$ refer to the *probability of allocation* and *the expected payments*
    - Because a game played by agents with values drawn from a distribution will inherently, from agent i's perspective have a randomized outcome and payment
- Myerson's lemma also characterizes BNE in single-parameter mechanisms
- If two auctions have the same distribution of agent values and same way of allocation (at BNE), then Myerson's lemma tells us something amazing about them

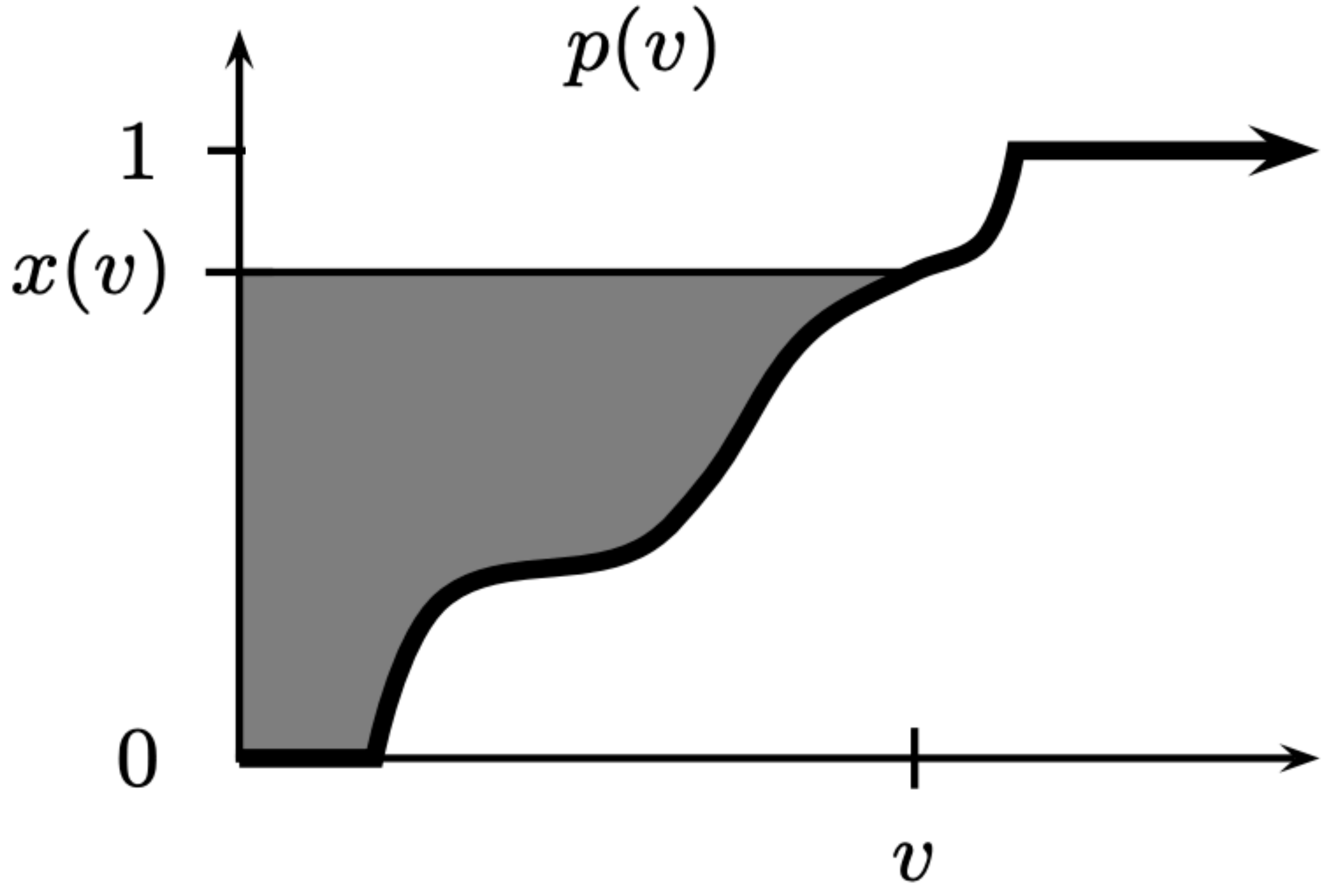
Myerson's Lemma for BNE

- Informal statement:
- A strategy profile s is a Bayes' Nash equilibrium in $(\mathbf{x}, \mathbf{p})$ if and only if for all i

(a) **(monotonicity)** the allocation probability $x_i(v_i)$ is monotone non decreasing

(b) **(payment identity)** agent i's expected payment is given by:

$$p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\ dz$$



Assuming that $p_i(0) = 0$.

Proof is analogous to the DSE case.

Credit: Hartline's Book on Mechanism Design

Revenue Equivalence

- Most significant observation in auction theory
- A mechanism with the same allocation in DSE (BNE) have the same (expected) revenue!
  - In fact, each agent has the same expected payment in each mechanism
- Direct corollary of Myerson's lemma
  - The interim expected payments depend only on the allocation probability!
- **Corollary (Revenue equivalence).**
  - For any two mechanisms in 0-1 single-parameter setting, if the mechanism have the same BNE allocation, then they have the same expected revenue (assuming 0-valued agents pay nothing)

If we want to increase the (expected) revenue, **changing payments or charging more won't do it!** You need to change how you allocate!

Solving for BNE of an Auction

- Myerson's lemma tells us what outcomes are possible in BNE, but not how to solve for BNE strategies
- Solving for BNE is important: if you are a bidder in an auction, you want to know how to bid!
- Can use revenue equivalence to solve for BNE strategies in symmetric environments
  - If two mechanisms have the same allocation rule, they must have the same expected payment
- To solve for BNE strategies in a mechanism M, then the approach is:
  - Express agents expected payment it terms of their strategy and value
  - Set it equal to the expected payment in a strategically-simpler revenue equivalent mechanism M' (usually a "second-price implementation")

Solving for BNE: Steps

- **Step 1.** *Guess* what the allocation might be in a Bayes-Nash equilibrium (usually a surplus-maximizing one)
    - E.g., in a first-price auction with values i.i.d. $U(0,1)$, guess that highest bidder wins is a BNE
- **Step 2.** *Calculate* the interim expected payment of an agent in terms of the strategy function
    - E.g., consider a 2-bidder first price auction with values i.i.d. in $U(0,1)$, then

$$E[p_1(v_1)] = E[p_1(v_1) \mid 1 \text{ wins}] \cdot \Pr[1 \text{ wins }] + E[p_1(v_1) \mid 1 \text{ loses}] \cdot \Pr[1 \text{ loses }]$$

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$$= s_1(v_1) \cdot \Pr[s_2(v_2) \leq s_1(v_1)]$$

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Assuming symmetric equilibria

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Assuming symmetric equilibria

$$= s_1(v_1) \cdot \Pr[v_2 \leq v_1]$$

Assuming strategies are monotone non-decreasing wrt value

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    - E.g., consider a 2-bidder first price auction with values i.i.d. in $U(0,1)$, then

Loser pays zero

$$E[p_1(v_1)] = E[p_1(v_1) \mid 1 \text{ wins}] \cdot \Pr[1 \text{ wins }] + \cancel{E[p_1(v_1) \mid 1 \text{ loses}] \cdot \Pr[1 \text{ loses }]}$$

$$= s_1(v_1) \cdot \Pr[s_2(v_2) \leq s_1(v_1)]$$

$$= s_1(v_1) \cdot \Pr[s(v_2) \leq s(v_1)]$$ Assuming symmetric equilibria

$$= s_1(v_1) \cdot \Pr[v_2 \leq v_1]$$ Assuming strategies are monotone non-decreasing wrt value

$$= s_1(v_1) \cdot v_1$$ Assuming v_2 is i.i.d in uniform [0, 1]

Solving for BNE: Steps

- **Step 3.** *Calculate* the interim expected payment of an agent in a strategically-simpler auction (usually the second-price version)
    - In a 2-bidder second-price auction with values i.i.d. in $U(0,1)$, we have

Loser pays zero

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$$= E[v_2 \mid 1 \text{ wins }] \cdot \Pr[1 \text{ wins }]$$

$$= E[v_2 \mid v_2 \leq v_1] \cdot \Pr[v_2 \leq v_1]$$

$$= \frac{v_1}{2} \cdot v_1$$

v_2 is a uniform random variable that evenly divides the interval it is on, so if $v_2 \in [0, v_1]$ then $E[v_2] = v_1/2$

$$= \frac{v_1^2}{2}$$

If we draw n samples i.i.d. from $U(0, a)$ then the expected value of kth highest draw is $\frac{n-(k-1)}{n+1} \cdot a$

Solving for BNE: Steps

- **Step 4.** Solve for the BNE strategy by setting the expected payment equal to each other
  - $s(v_1) \cdot v_1 = \dfrac{v_1^2}{2} \implies s(v_1) = v_1/2$
- **Step 5.** Finally, verify that the initial guess was correct and the strategy is a symmetric BNE of the mechanism
  - We have already verified for first-price auction
- Notice that we could have simplified our process
  - Probability an agent wins is the same in both auctions because of symmetric non-decreasing strategies and both auctions have the allocation
  - Since loser pays nothing, we can set

$$E^{SP}[p_1(v_1) \mid 1 \text{ wins}] = E^{FP}[p_1(v_1) \mid 1 \text{ wins}] = E[v_2 \mid v_2 \leq v_1] = s_1(v_1) = v_1/2$$

All Pay Auctions

- Revenue equivalence governs all kinds of auctions
- Consider a single-item all pay sealed-bid auction, where the highest bidder wins but everyone pays their bid
  - Thus, utility of participating can be negative in this case
  - Participating has a cost!
- Can you think of examples of this setting in practice?

Solving for BNE: All Pay

- **Example:** All-pay auction with 2 bidders i.i.d. $U(0,1)$

Solving for BNE: All Pay

- **Example:** All-pay auction with 2 bidders i.i.d. $U(0,1)$

$$\mathbf{E}[p_1(v_1)] = \underbrace{v_1^2/2}_{\text{second-price}} = \underbrace{s_1(v_i)}_{\text{all-pay}}.$$

Regardless of winning or losing you pay your bid

- Guess for n-bidder symmetric BNE?
  - $(\frac{n-1}{n})v_i^n$
- HW problem: verify that this is a symmetric BNE of all-pay auctions
- **Problem on HW 5.** Use this approach to solve for BNE in a kth price auction
  - Use the fact that if we draw n i.i.d. samples from $U(0,a)$ then expected value of kth highest is $\dfrac{n-(k-1)}{n+1} \cdot a$

Bayesian Analysis Challenges

- So far we considered symmetric settings:
  - Values are drawn i.i.d. from the same distribution
  - For these settings, Bayes Nash equilibrium are pretty well understood
  - They also lead to efficient outcomes: surplus maximizing outcomes
- For asymmetric settings:
  - Information asymmetry: e.g. different companies do different levels of "market research"
  - Can still study BNE but often more complicated: no closed form solutions and multiple (some inefficient) equilibrium
  - Near CS-driven approach: study approximation bounds (price of anarchy) of these auctions

Price of Anarchy: Auctions

- 2017 survey

The Price of Anarchy in Auctions

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Abstract

This survey outlines a general and modular theory for proving approximation guarantees for equilibria of auctions in complex settings. This theory complements traditional economic techniques, which generally focus on exact and optimal solutions and are accordingly limited to relatively stylized settings.

We highlight three user-friendly analytical tools: smoothness-type inequalities, which immediately yield approximation guarantees for many auction formats of interest in the special case of complete information and deterministic strategies; extension theorems, which extend such guarantees to randomized strategies, no-regret learning outcomes, and incomplete-information settings; and composition theorems, which extend such guarantees from simpler to more complex auctions. Combining these tools yields tight worst-case approximation guarantees for the equilibria of many widely-used auction formats.

Revenue Maximization