CSCI 357: Algorithmic Game Theory

Lecture 8: First Price Auction

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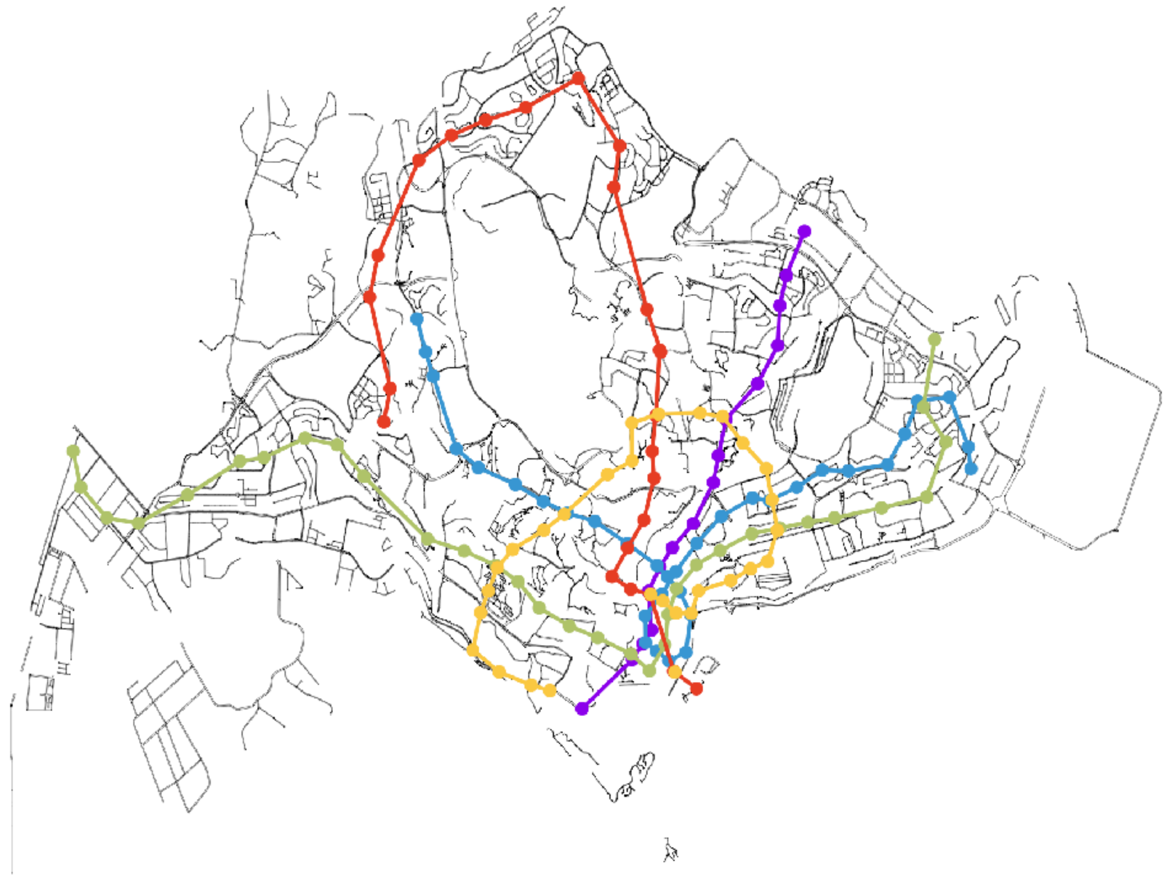


Announcements and Logistics

- **Assignment 4** out by and due Thurs 11 pm
  - Submit code via Github, latex answers and submit PDF
  - Assignment looks really long but a lot of it is just setup!
  - Based on lectures 6 and 7 on GSP vs VCG
- Feedback from HW 3:
  - Absorbing notations in AGT, esp auction theory can be a lot
    - Graduate level topic! Studying research from last two decades
  - Gets better in other topics of the course: promise!!!!
  - Happy to slow down, encourage interruptions and questions

Questions?

Proof Update

Want to show:

$$u_1 = \alpha_1 v_1 - \alpha_1 b_2 \geq \overbrace{\alpha_2 v_1 - \alpha_2 b_3}^{\text{utility for slot 2 at price } b_3}$$

Use BB condition for b_2:

$$\alpha_1 v_2 - \alpha_1 b_2 = \alpha_2 v_2 - \alpha_2 b_3$$

Substitute $\alpha_1 b_2$ in u_1

$$\alpha_1 v_1 + \alpha_2 v_2 - \cancel{\alpha_2 b_3} - \alpha_1 v_2 \geq \alpha_2 v_1 - \cancel{\alpha_2 b_3}$$

$$\alpha_1 (v_1 \cancel{- v_2}) \geq \alpha_2 (v_1 \cancel{- v_2})$$ since $v_1 \geq v_2$

$\alpha_1 \geq \alpha_2$ this is true! $\square$

Last Time & Outline

- Wrapped up discussion on sponsored ad auctions
  - An example of how theory interacts with practice
- Talked briefly about first price auction and challenges
- This week: analyze first price auctions
  - Scratch the surface of Bayesian auction analysis
- Hope is to wrap up **direct-revelation auction design** this week!
- Next week is the last week on mechanism design with money:
  - Matching markets / ascending clock mechanisms
  - Application: spectrum auctions

Week 6: Matching Markets w/o Money

Week 5: Matching Markets w Money

Week 4: Bayesian Analysis & General Mechanism Design

Week 3: Application : Sponsored Ad Markets

Week 2: DSIC Auctions

Week 1: Game Theory

First-Price vs Second Price

Both the first-price and second-price auction (at equilibrium) generate the same (expected) revenue!

To show this, we need to analyze first-price auction, which is an incomplete-information or "Bayesian game"

First Price Auctions

Bayesian Auction & Assumptions

- Game of **incomplete information:** bidders values (and thus utilities) are private
- No dominant strategy equilibrium, need to analyze using **Bayesian Nash Eq**
- Assume bidders have **independent private value (IPV)** drawn independently and identically from the distribution G
  - We say values are drawn **i.i.d from** G
- The distribution G is **common knowledge**
  - Every bidder knows the distributions and knows that others know it as well
  - Often called "common prior"
- For first-price auction: we will further assume values are drawn **i.i.d from the uniform distribution** on $[0,1]$

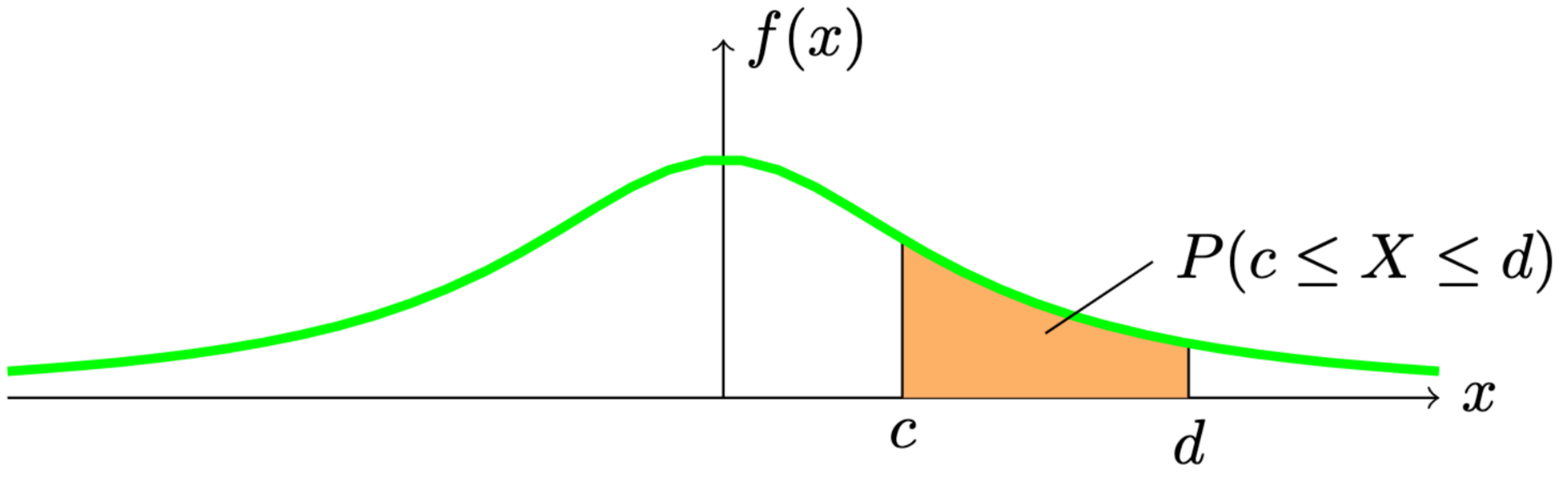
Continuous Probability Review

- A continuous random variable takes a range of values, which can be finite or infinite
- **(Definition)** A random variable X is continuous if there is a function $f(x)$ such that for any $c \leq d$ we have

$$\Pr(c \leq X \leq d) = \int_c^d f(x)dx$$

- Function $f(x)$ is called the **probability density function** (pdf)

$\boxed{P(c \leq X \leq d) = \text{area under the graph between } c \text{ and } d.}$



Continuous Probability Review

- **(Definition)** The **cumulative distribution function (cdf)** F of a continuous random variable X denotes the probability that it is at most a certain value

$$F(k) = \Pr(X \leq k) = \int_{-\infty}^{k} f(x)dx$$

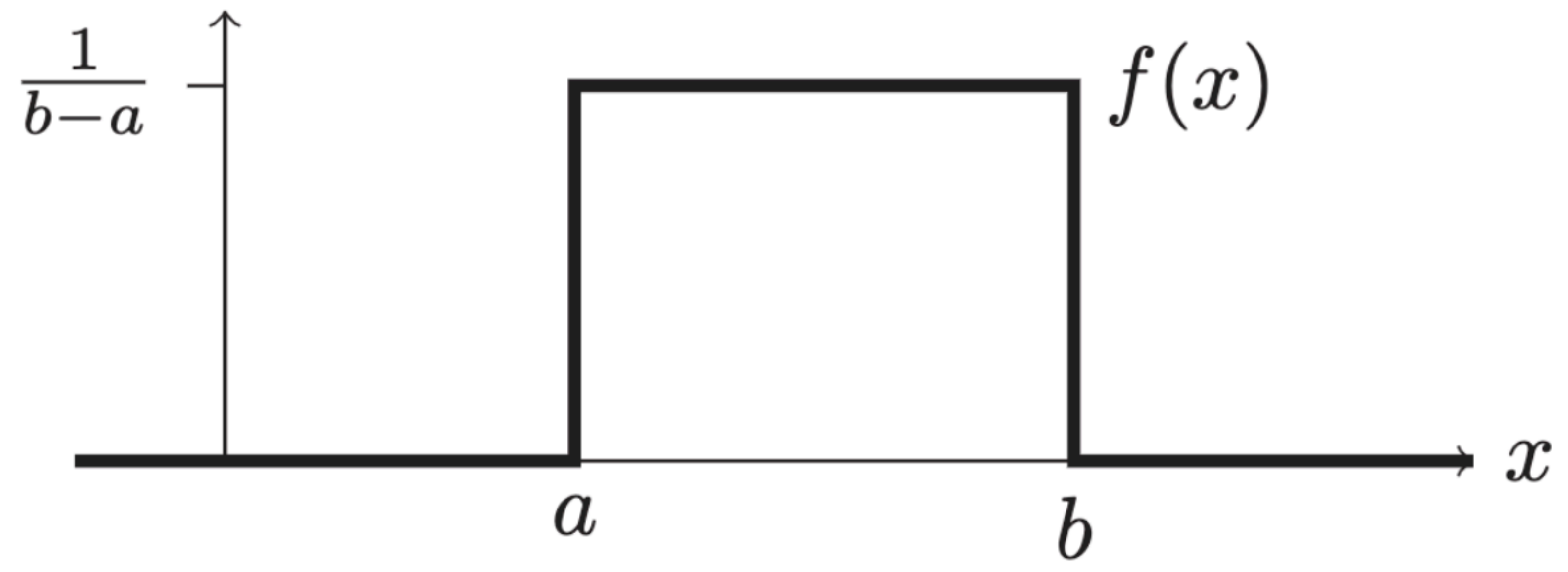
  where $f(x)$ is the probability density function of X

- In practice, we often say X has distribution or is drawn from distribution $F(x)$ rather than X has cumulative distribution function $F(x)$

Uniform Distribution

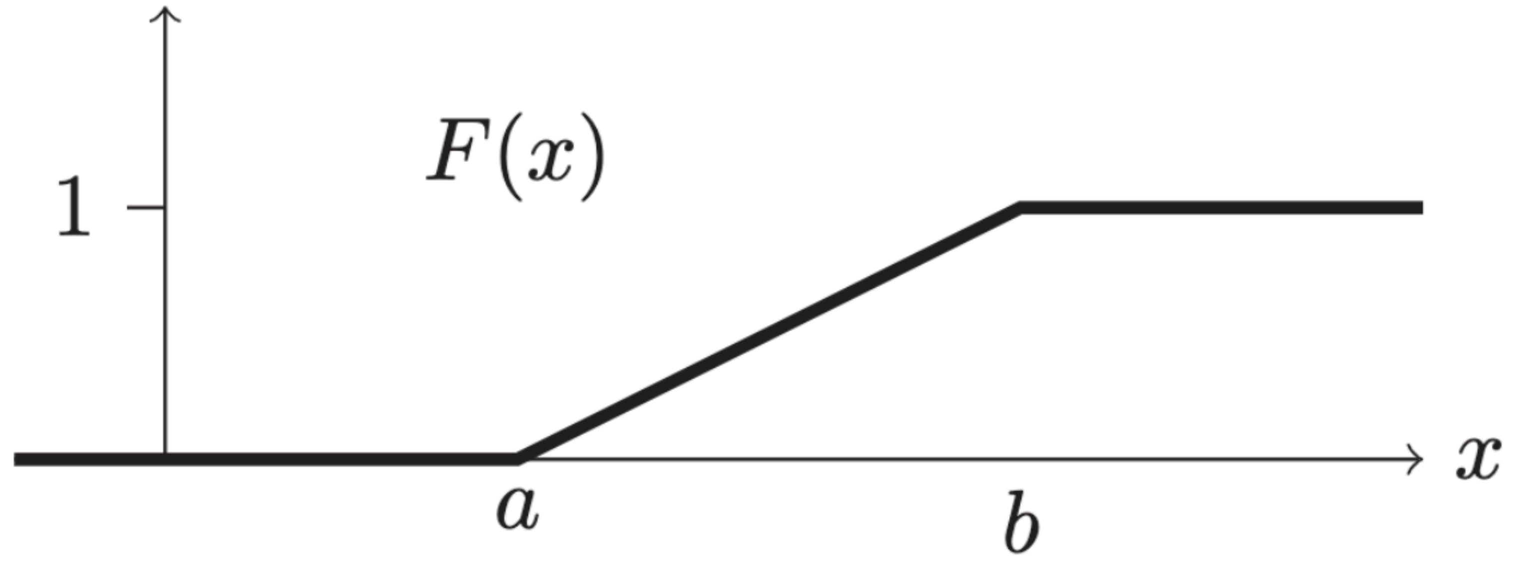
- Models situations where all outcomes in the range have equal probability
- Probability density function of a continuous uniform distribution on $[a, b]$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$



- Cumulative density function of a continuous uniform distribution on $[a, b]$

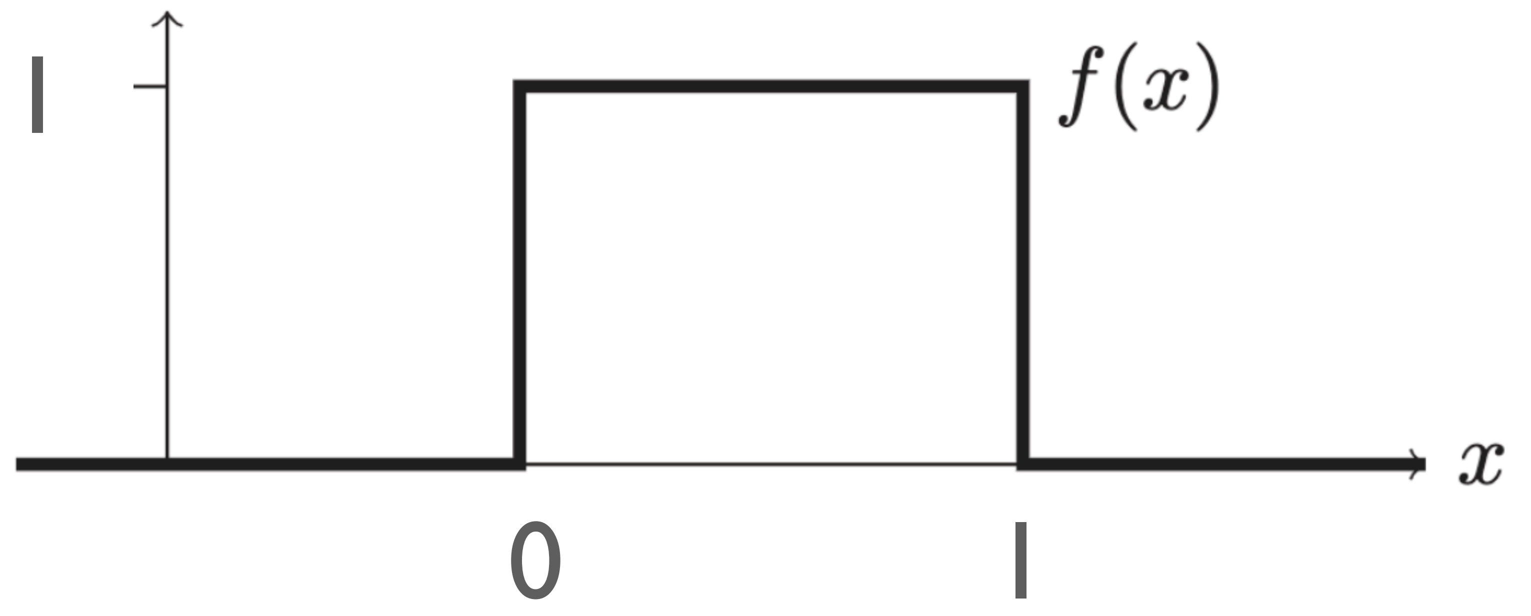
$$F(k) = \Pr(x \le k) = \begin{cases} 0 & \text{if } k \ge 0 \\ \frac{k-a}{b-a} & \text{if } a \le k \le b \\ 1 & \text{if} k > b \end{cases}$$



Uniform Distribution on [0, 1]

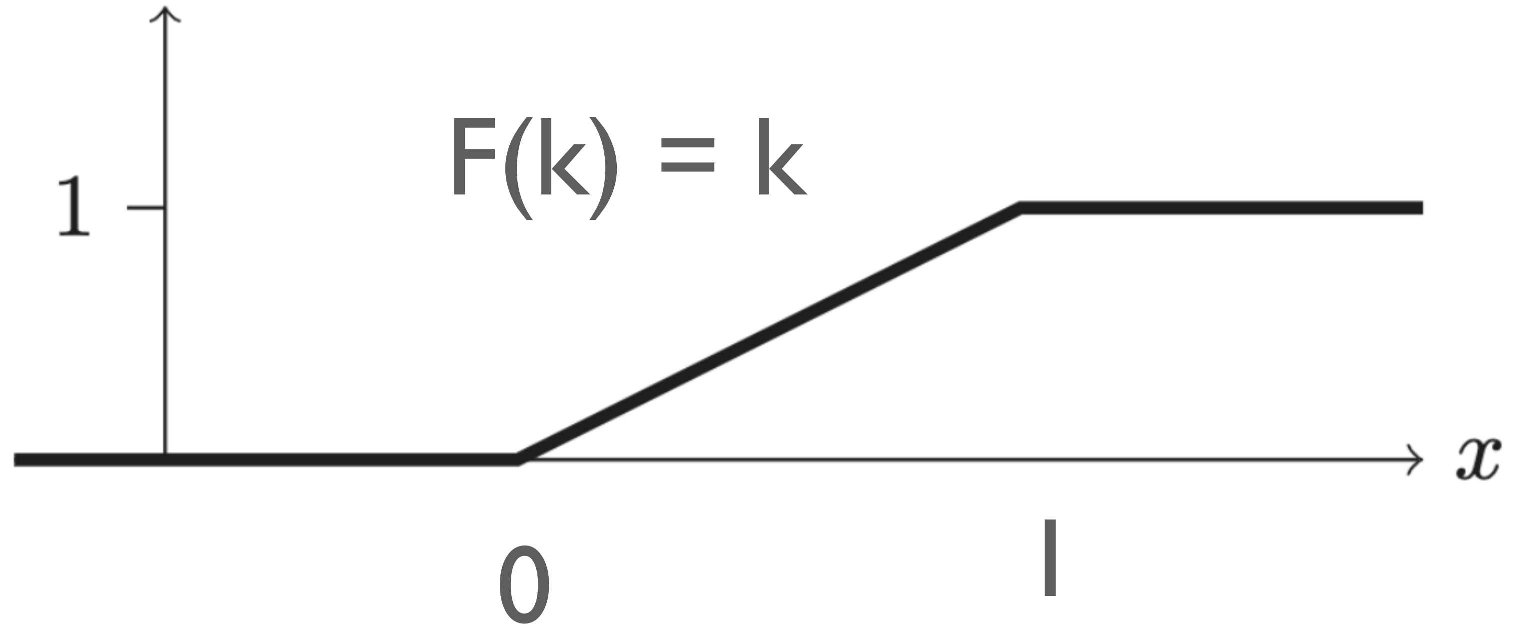
- Models situations where all outcomes in the range have equal probability
- Probability density function of a continuous uniform distribution on $[0,1]$

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



- Cumulative density function of a continuous uniform distribution on $[a, b]$

$$F(k) = \Pr(x \leq k) = \begin{cases} 0 & \text{if } k \geq 0 \\ k & \text{if } a \leq k \leq b \\ 1 & \text{if} k > b \end{cases}$$



Bayesian Nash Equilibrium

- A strategy or plan of action for each player (as a function of types) should be such that it maximizes each players expected utility
  - expectation is over the private values of other players
- Given a Bayesian game with independent private values v_{-i} , i's interim **expected utility** for a strategy profile $s = (s_1, \ldots, s_n)$ is

$$\mathbb{E}[u_i(s)] = \sum_{v_{-i}} u_i(s) \cdot \text{Pr}(v_{-i})$$

- A strategy profile s is a **pure strategy Bayes Nash equilibrium** if no player can increase their interim expected utility by **unilaterally changing** their strategy s_i

Strategy Assumptions

- Recall: strategy s_i is a function that maps their value to their bid b:
  - $s_i(v_i) = b_i$
- We assume that the strategy of all bidders in the auctions we study
  - Is a strictly increasing differentiable function: gives us that the bidder with higher value will also provide a higher bid (no ties)
  - $s_i(v_i) \leq v_i$ for all v_i and bidders i: that is, bidders can "shade" down their bids but will never bid above their true values
    - Also implies $s_i(0) = 0$
- These assumptions are just to simplify analysis

First-Price Auction: Two Bidders

- Suppose v_1, v_2 are both drawn i.i.d. from the uniform distribution on $[0,1]$

Pr$(b_1 < b_2)$

Pr$(b_1 \geq b_2)$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- Suppose both bidders bid symmetrically some factor of their value $s(v_i) = \alpha \cdot v_i$

Pr$(b_1 < b_2)$

Pr$(v_2 \alpha \leq b_1)$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- Suppose both bidders bid symmetrically some factor of their value $s(v_i) = \alpha \cdot v_i$

$\Pr(b_1 < b_2)$

v_1 loses & pays 0

Utility: 0

$\Pr(v_2 \leq (b_1/\alpha))$

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- Suppose both bidders bid symmetrically some factor of their value $s(v_i) = \alpha \cdot v_i$

$\text{Pr}(b_1 < b_2)$

v_1 loses & pays 0

Utility: 0

$\text{Pr} = b_1/\alpha$

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- $\mathbb{E}[u_1] = (v_1 - b_1)(b_1/\alpha)$: how to set b_1 to maximize expected payment?

$\text{Pr}(b_1 < b_2)$

v_1 loses & pays 0

Utility: 0

$\text{Pr} = b_1/\alpha$

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- $\mathbb{E}'[u_1] = (1/\alpha)(v_1 - 2b_1) = 0,$ that is, $b_1 = v_1/2$

$\Pr(b_1 < b_2)$

$\Pr = b_1/\alpha$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: Two Bidders

- **Theorem.** Assume two bidders with their values drawn i.i.d. from uniform distribution on $[0,1]$, then the strategy $s(v_i) = v_i/2$ is a symmetric Bayes Nash equilibrium of the sealed-bid first price auction.
- **Proof.** Assume agent 2 bids using $s(\,.\,)$, that is, $b_2 = v_2/2$
- We calculate agent 1's expected utility who has value v_1 and bid b_1
  - $E[u_1] = (v_1 - b_1) \cdot \Pr[1 \text{ wins with bid } b_1] \; + 0 \cdot \Pr[1 \text{ loses with bid } b_1]$

First-Price Auction: Two Bidders

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- We calculate agent 1's expected utility who has value v_1 and bid b_1
  - $E[u_1] = (v_1 - b_1) \cdot \text{Pr}[1 \text{ wins with bid } b_1]$
    $= (v_1 - b_1) \cdot \text{Pr}[b_2 \leq b_1]$

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$$= (v_1 - b_1) \cdot \text{Pr}[b_2 \le b_1]$$

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$$= (v_1 - b_1) \cdot \text{Pr}[b_2 \le b_1]$$

$$= (v_1 - b_1) \cdot \text{Pr}[v_2/2 \le b_1]$$

$$= (v_1 - b_1) \cdot \text{Pr}[v_2 \le 2b_1]$$

Here v_1, b_1 are fixed and v_2 is a random variable

First-Price Auction: Two Bidders

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  - $E[u_1] = (v_1 - b_1) \cdot \text{Pr}[\text{1 wins with bid } b_1]$

$$= (v_1 - b_1) \cdot \text{Pr}[b_2 \leq b_1]$$

$$= (v_1 - b_1) \cdot \text{Pr}[v_2/2 \leq b_1]$$

$$= (v_1 - b_1) \cdot \text{Pr}[v_2 \leq 2b_1]$$

$$= (v_1 - b_1) \cdot F(2b_1) = (v_1 - b_1) \cdot 2b_1$$

First-Price Auction: Two Bidders

- **Proof (Cont).** Assume agent 2 bids using $s(\,.\,)$, that is, $b_2 = v_2/2$
- Agent 1's expected utility who has value v_1 and bid b_1 when she wins
    - $E[u_1] = (v_1 - b_1) \cdot 2b_1 \; = 2v_1b_1 - 2b_1^2$

First-Price Auction: Two Bidders

- **Proof (Cont).** Assume agent 2 bids using $s(\,.\,)$, that is, $b_2 = v_2/2$
- Agent 1's expected utility who has value v_1 and bid b_1 when she wins
  - $E[u_1] = (v_1 - b_1) \cdot 2b_1 \; = 2v_1b_1 - 2b_1^2$
- Agent 1 with value v_1 should set b_1 to maximize $2v_1b_1 - 2b_1^2$ as a function of b_1
  - Differentiate and set derivate to zero (also check second order condition)

First-Price Auction: Two Bidders

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- Agent 1's expected utility who has value v_1 and bid b_1 when she wins
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- Agent 1 with value v_1 should set b_1 to maximize $2v_1b_1 - 2b_1^2$ as a function of b_1
  - Differentiate and set derivate to zero (also check second order condition)
  - $E'[u_1] = 2v_1 - 4b_1 = 0$, that is, $b_1 = v_1/2$

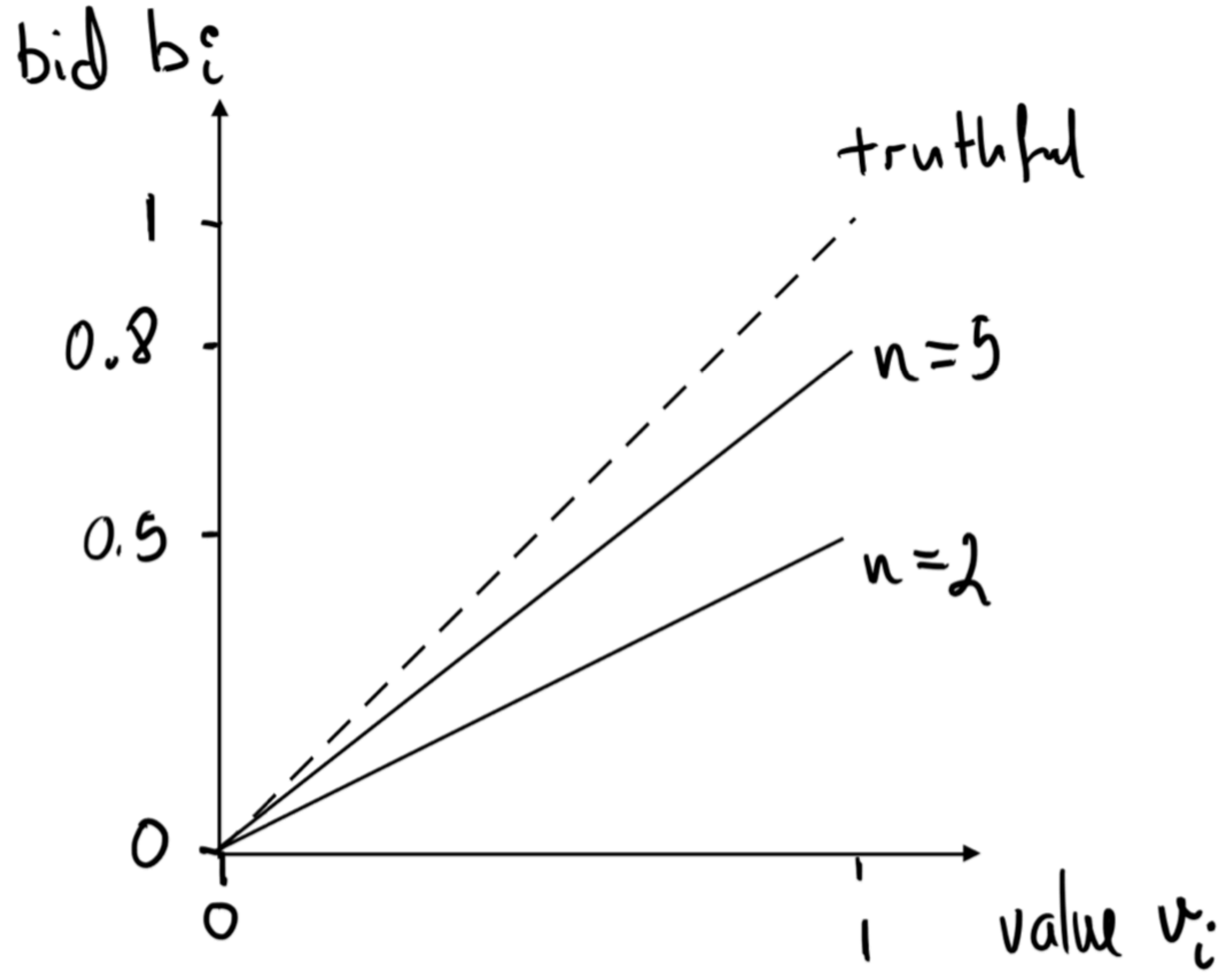
The analysis is symmetric for agent 2 as well.

First-Price Auction: n Bidders

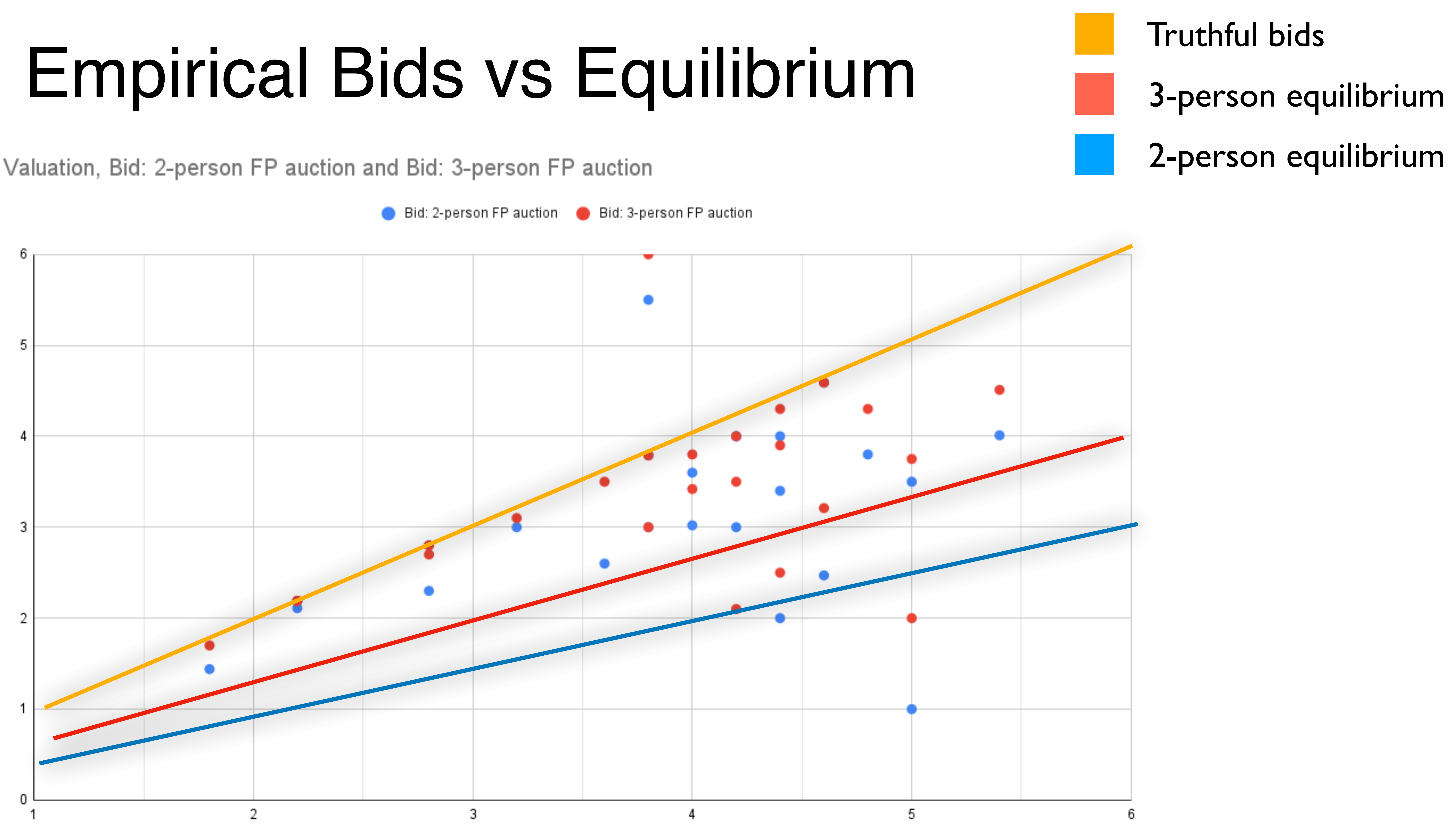
- Let us use the same approach to figure out the symmetric Bayes Nash equilibrium for n bidders
- Suppose every bidder $j \neq 1$ uses strategy $s_j = \alpha(n) \cdot v_j$
- **Class exercise.** Can you write the expression for expected utility of bidder 1 and figure out what value of b_1 maximizes it?
  - Fix b_1, v_1, write $\mathbb{E}(u_1)$ as a function of them
  - Each v_j for $j \neq 1$ is a random variable i.i.d. in uniform $[0,\ 1]$
- Deduce the value of $\alpha(n)$ from this

First-Price Auction: n Bidders

- Suppose we increase the number of bidders, how should the equilibrium strategy adjust to more competition?
- **Theorem.** Assume each of the n bidders have values drawn i.i.d. from uniform distribution on $[0,1]$. Then, the strategy $s(v_i) = \frac{n-1}{n} \cdot v_i$ is a symmetric Bayes Nash equilibrium of the sealed-bid first price auction.
- **Proof.** We can generalize the 2-bidder proof
  - On board. Also in Parkes and Seuven book.
- **Takeaway:** the more the competition, the more the bidders need to bid closer to their value if they want to win

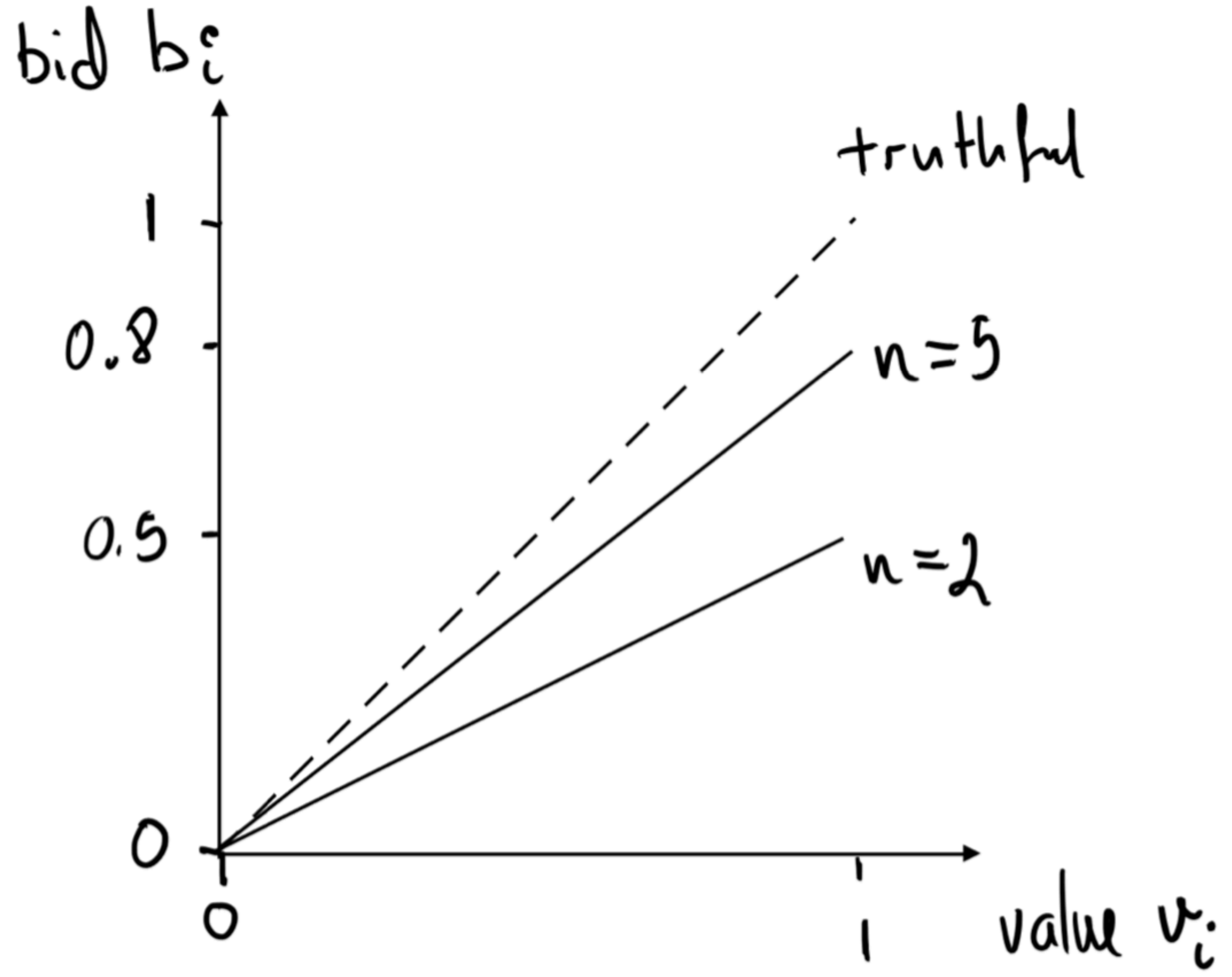


Empirical Bids vs Equilibrium



First-Price Auction: Guarantees

- Turns out this Bayes Nash equilibrium **is unique**
  - Generalizes to arbitrary i.i.d distributions
- Is linear time
- Does it maximize surplus?
  - Bids in Bayes Nash equilibrium are order-preserving: that is, for values $v_1 \geq v_2 \geq \ldots \geq v_n$, the equilibrium bids are $b_1 \geq b_2 \geq \ldots \geq b_n$
  - The item is allocated to the highest bidder, thus to the agent with the maximum valuation
  - Maximizes surplus (at equilibrium)
- Now, we want to compare the revenue of FP and SP auction

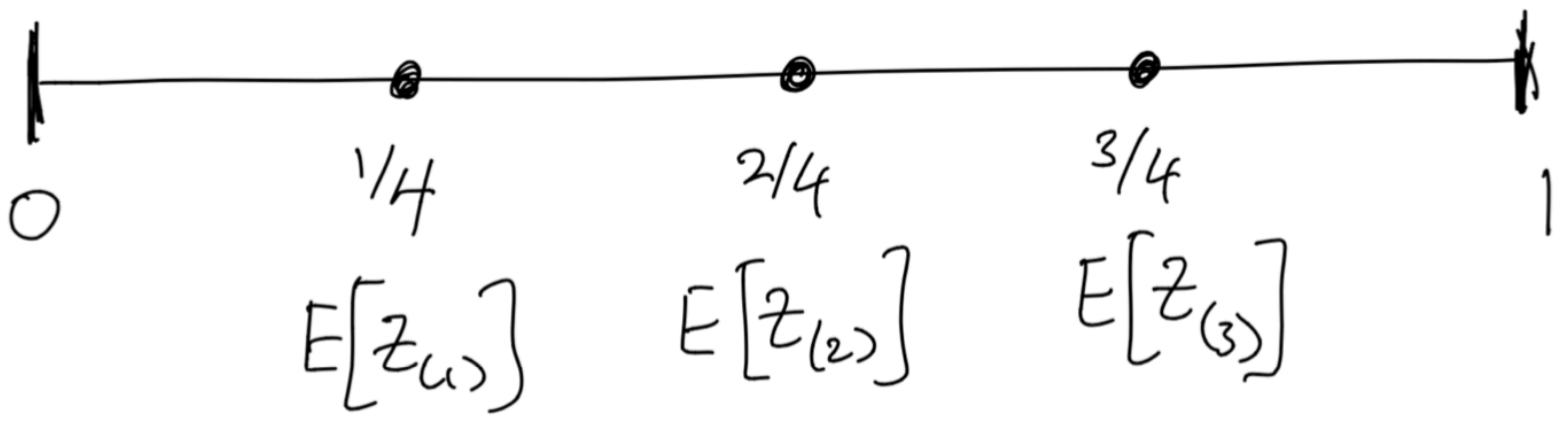


Order Statistics

- To do so, we need to define order statistics
- Let $X_1, X_2, \ldots X_n$ be n independent samples drawn identically from the uniform distribution on $[0,1]$
- The first-order statistic $X_{(1)}$ is the maximum value of the samples, the second-order statistic $X_{(2)}$is the second-maximum value of the samples, etc
- The expected value of the kth order statistic for n i.i.d samples from $U(a,\ b)$ is

$$E[X_{(k)}] = \frac{n-(k-1)}{n+1} \cdot (b-a)$$

- **Remember:** a uniform random variable evenly divides the interval it is over



Expected kth order statistic for 3 samples, uniform $[0,1]$

Revenue

- **Theorem.** If bidder's values are uniform i.i.d., then the expected revenue of the first-price auction is equal to that of the second-price auction at equilibrium.
- **Proof.** Let $E[R_1]$ and $E[R_2]$ be the expected revenues of the first and second-price auction.
- In second-price auction, the bidder with the highest value wins and pays second-highest value
    - $E[R_2] =$ expected value of second-order statistic
      $$= \frac{n-1}{n+1}$$
- In FP auction, bidders bid $s(v_i) = \dfrac{n-1}{n} \cdot v_i$ and highest bidder pays their bid

Revenue

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- **Proof.** Let $E[R_1]$ and $E[R_2]$ be the expected revenues of the first and second-price auction.
- In FP auction, bidders bid $s(v_i) = \dfrac{n-1}{n} \cdot v_i$ and highest bidder pays their bid
- $E[R_1] = E[b_{\max}] = E\left[\dfrac{n-1}{n} \cdot v_{\max}\right]$

Revenue

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- **Proof.** Let $E[R_1]$ and $E[R_2]$ be the expected revenues of the first and second-price auction.
- In FP auction, bidders bid $s(v_i) = \dfrac{n-1}{n} \cdot v_i$ and highest bidder pays their bid
- $E[R_1] = E[b_{\max}] = E\left[\dfrac{n-1}{n} \cdot v_{\max}\right] = \dfrac{n-1}{n} E[v_{\max}] = \dfrac{n-1}{n} \cdot \dfrac{n}{n+1} = \dfrac{n-1}{n+1}$
- The last step uses linearity of expectation
  - $E(a \cdot X + b \cdot Y) = a \cdot E(X) + b \cdot E(Y)$ where a and b are constants

$\blacksquare$

Myerson's Lemma: DSE vs BNE

- Remember all DSE are BNE but not vice versa
- When characterizing DSE, the game was deterministic and so we can talk about the actual allocation and payment
- When characterizing BNE: $x_i(v_i)$ and $p_i(v_i)$ refer to the *probability of allocation* and *the expected payments*
    - Because a game played by agents with values drawn from a distribution will inherently, from agent i's perspective have a randomized outcome and payment
- Myerson's lemma also characterizes BNE in single-parameter mechanisms
- If two auctions have the same distribution of agent values and same way of allocation (at BNE), then Myerson's lemma tells us something amazing about them

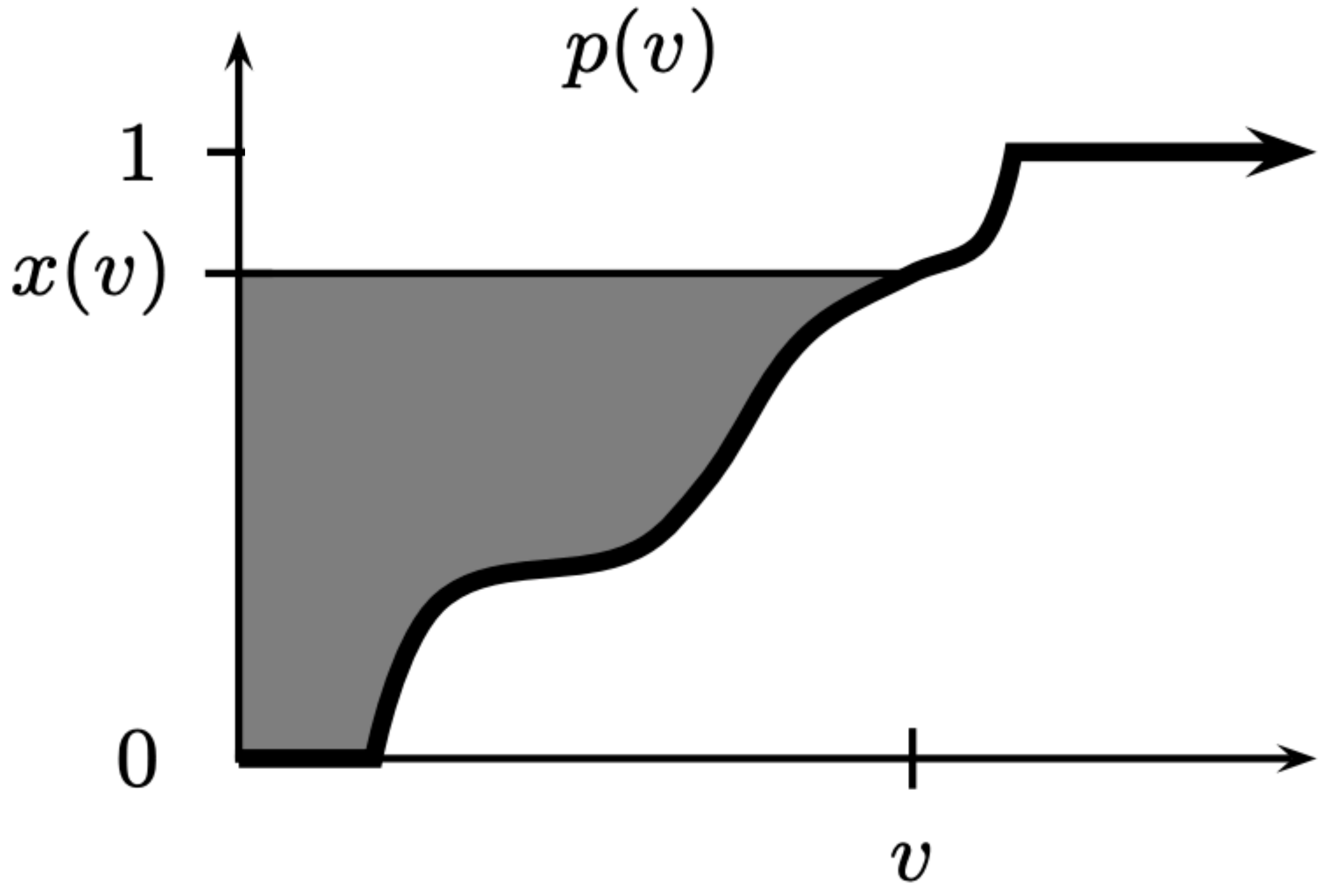
Myerson's Lemma for BNE

- Informal statement:
- A strategy profile s is a Bayes' Nash equilibrium in $(\mathbf{x}, \mathbf{p})$ if and only if for all i

(a) **(monotonicity)** the allocation probability $x_i(v_i)$ is monotone non decreasing

(b) **(payment identity)** agent i's expected payment is given by:

$$p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\ dz$$



Assuming that $p_i(0) = 0$.

Proof is analogous to the DSE case.

Credit: Hartline's Book on Mechanism Design

Revenue Equivalence

- Most significant observation in auction theory
- A mechanism with the same allocation in DSE (BNE) have the same (expected) revenue!
  - In fact, each agent has the same expected payment in each mechanism
- Direct corollary of Myerson's lemma
  - The interim expected payments depend only on the allocation probability!
- **Corollary (Revenue equivalence).**
  - For any two mechanisms in 0-1 single-parameter setting, if the mechanism have the same BNE allocation, then they have the same expected revenue (assuming 0-valued agents pay nothing)

If we want to increase the (expected) revenue, **changing payments or charging more won't do it!** You need to change how you allocate!

More Next Time!